Mathematical background

Outline

This topic reviews the basic mathematics required in this course:

- A justification for a mathematical framework
- The ceiling and floor functions
- L’Hôpital’s rule
- Logarithms
- Arithmetic and other polynomial series
  - Mathematical induction
- Geometric series
- Recurrence relations
- Weighted averages
- Combinations

Justification

As engineers, you will not be paid to say:

Method A is *better* than Method B

or

Algorithm A is *faster* than Algorithm B

Such comparisons are said to be *qualitative*:

qualitative, *a.* Relating to, connected or concerned with, quality or qualities. Now usually in implied or expressed opposition to quantitative.

OED

Justification

Qualitative statements cannot guide engineering design decisions:

- Algorithm A may be “better” only under some circumstances. Such circumstances cannot be qualitatively specified.
- The advantage of algorithm A vs. algorithm B may be small, while implementing algorithm A may be very complicated. A tradeoff between benefit vs. cost shall be calculated.

Justification

Thus, we will look at a *quantitative* means of describing data structures and algorithms:

> **quantitative**, *a.* Relating to, concerned with, quantity or its measurement; ascertaining or expressing quantity. **OED**

This will be based on mathematics, and therefore we will look at a number of properties which will be used again and again throughout this course

Floor and ceiling functions

The *floor* function maps any real number x onto the greatest integer less than or equal to x:

$$\lfloor 3.2 \rfloor = \lfloor 3 \rfloor = 3$$

$$\lfloor -5.2 \rfloor = \lfloor -6 \rfloor = -6$$

– Consider it *rounding towards negative infinity*

The *ceiling* function maps x onto the least integer greater than or equal to x:

$$\lceil 3.2 \rceil = \lceil 4 \rceil = 4$$

$$\lceil -5.2 \rceil = \lceil -5 \rceil = -5$$

– Consider it *rounding towards positive infinity*

L'Hôpital's rule

If you are attempting to determine

$$\lim_{n\to\infty} \frac{f(n)}{g(n)}$$

but both $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = \infty$, it follows

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f^{(1)}(n)}{g^{(1)}(n)}$$

Repeat as necessary…

Logarithms

We will begin with a review of logarithms:

If $n = e^m$, we define

$$m = \ln(n)$$

It is always true that $e^{\ln(n)} = n$; however, $\ln(e^n) = n$ requires that n is real

Logarithms

Exponentials grow faster than any non-constant polynomial

$$\lim_{n\to\infty} \frac{e^n}{n^d} = \infty$$

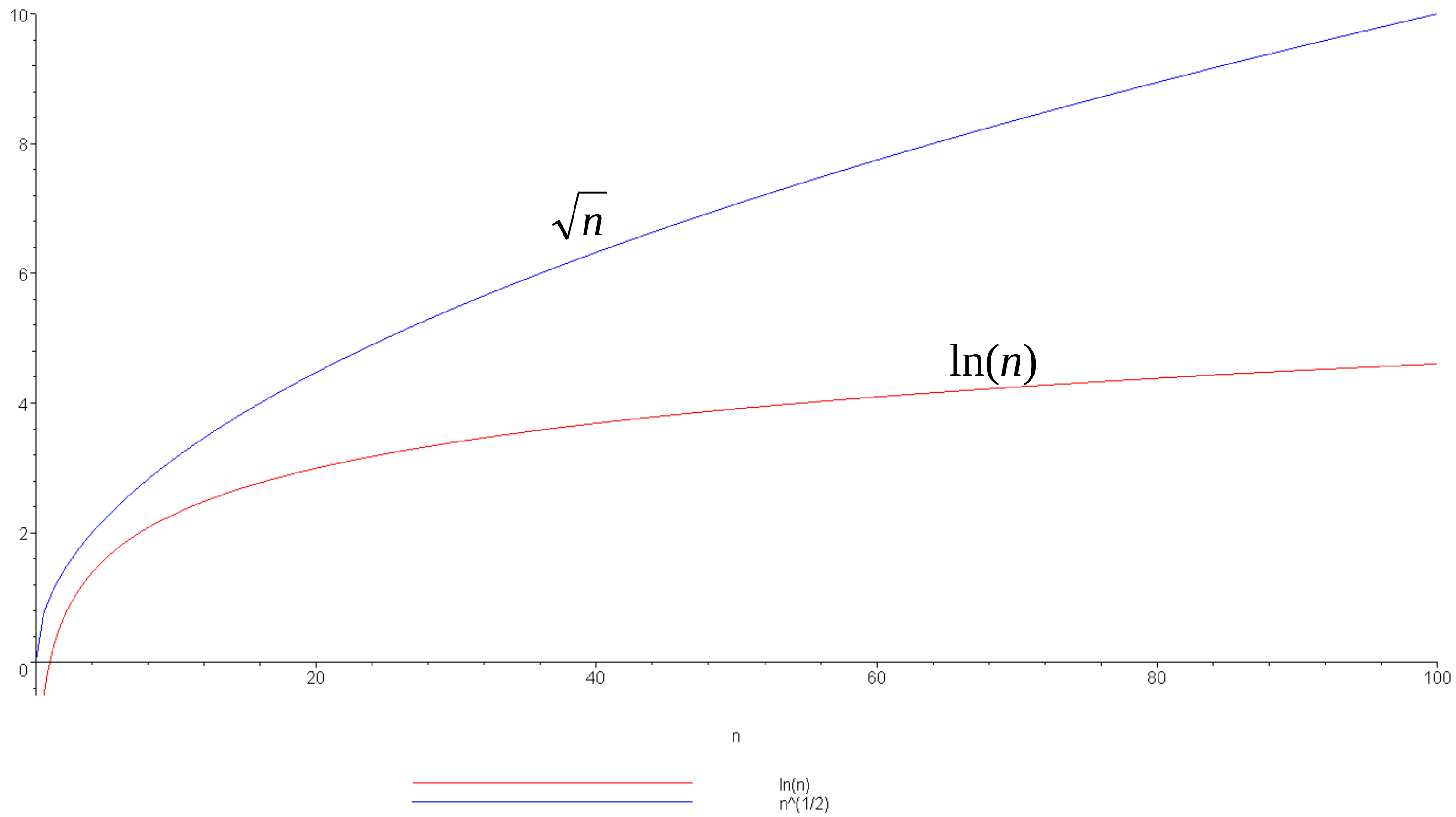
for any $d > 0$

Logarithms grow slower than any polynomial

$$\lim_{n\to\infty} \frac{\ln(n)}{n^d} = 0$$

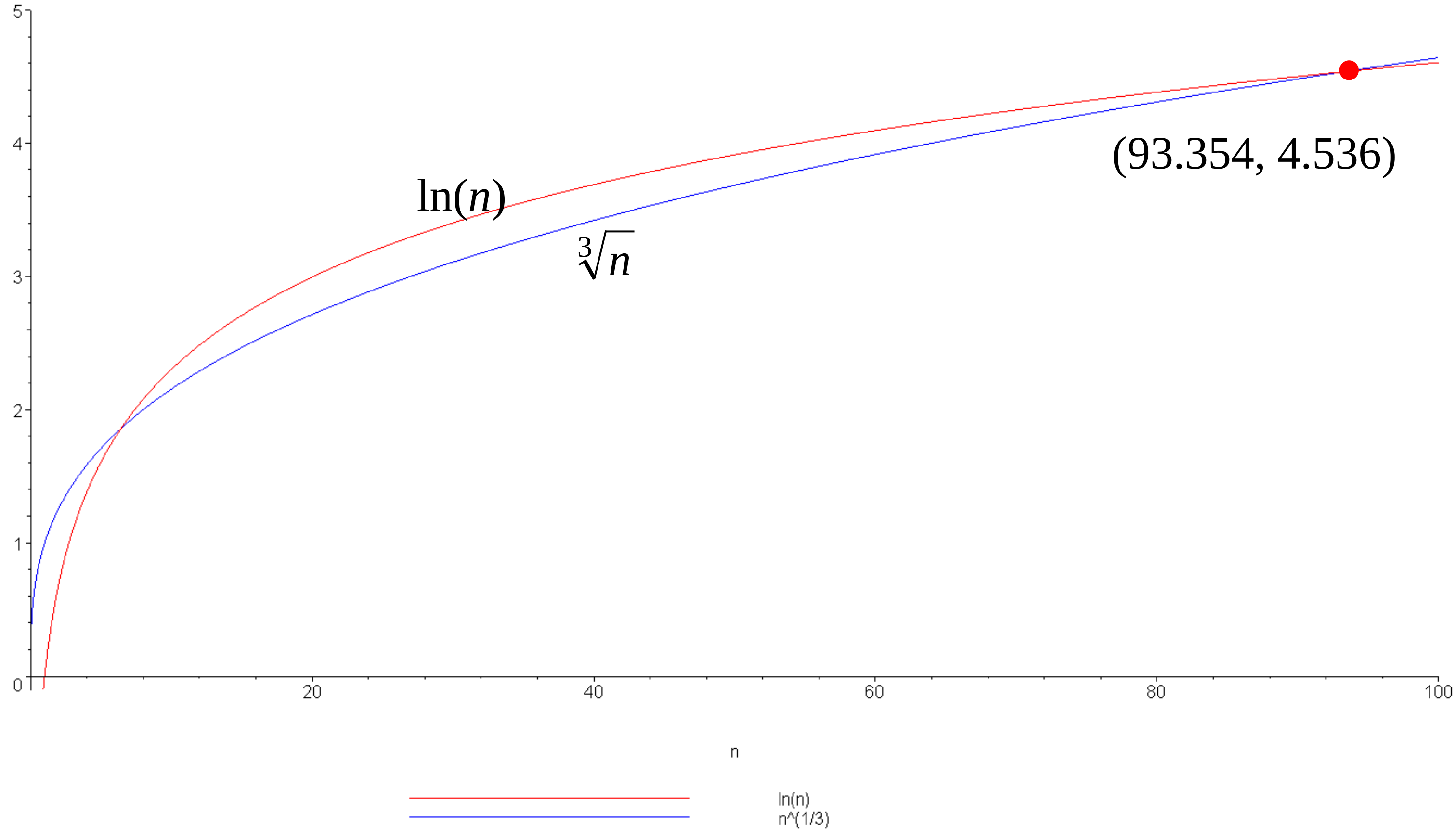
Logarithms

Example: $f(n) = n^{1/2} = \sqrt{n}$ is strictly greater than $\ln(n)$



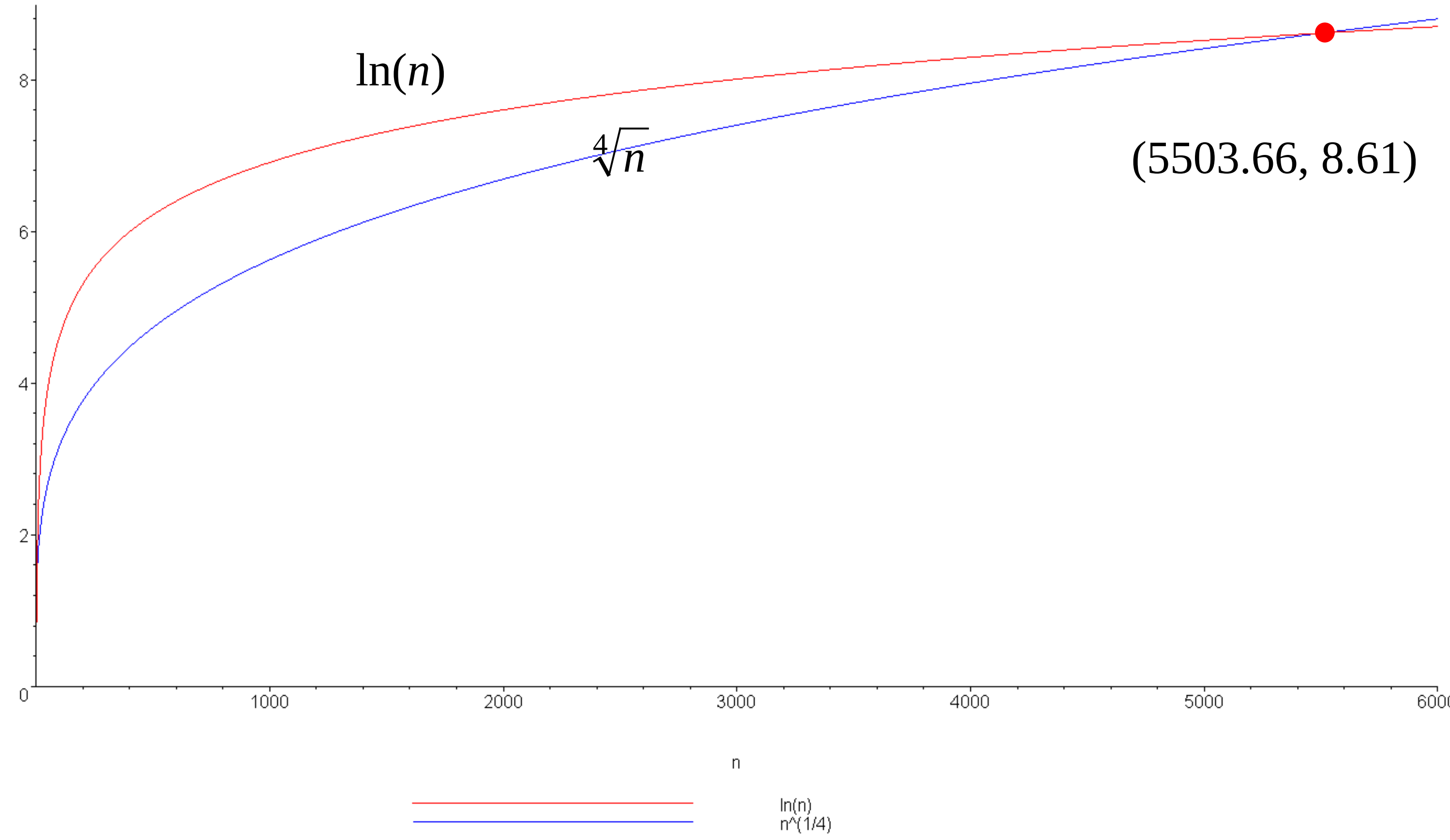
Logarithms

$f(n) = n^{1/3} = \sqrt[3]{n}$ grows slower but only up to $n = 93$



Logarithms

You can view this with any polynomial



Logarithms

We have compared logarithms and polynomials

- How about $\log_2(n)$ versus $\ln(n)$ versus $\log_{10}(n)$

You have seen the formula

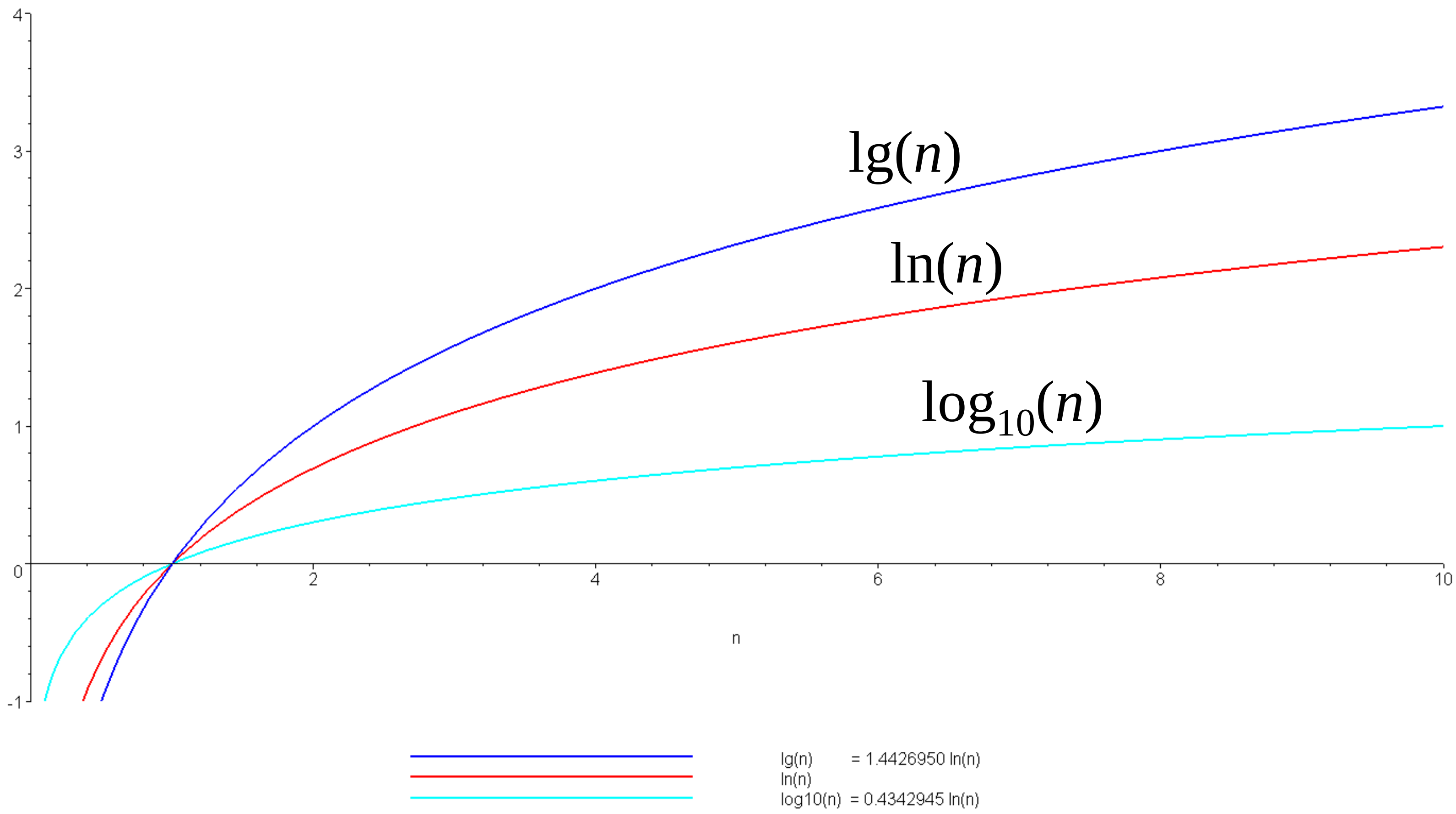
Constant

$$\log_b(n) = \frac{\ln(n)}{\ln(b)}$$

All logarithms are scalar multiples of each others

Logarithms

A plot of $\log_2(n) = \lg(n)$, $\ln(n)$, and $\log_{10}(n)$



Logarithms

Note: the base-2 logarithm $\log_2(n)$ is written as $\lg(n)$

It is an industry standard to implement the natural logarithm $\ln(n)$ as

```
double log( double );
```

The *common* logarithm $\log_{10}(n)$ is implemented as

```
double log10( double );
```

Logarithms

A more interesting observation we will repeatedly use:

$$n^{\log_b(m)} = m^{\log_b(n)},$$

a consequence of $n = b^{\log_b n}$:

$$
\begin{aligned}
n^{\log_b(m)} &= \left(b^{\log_b(n)}\right)^{\log_b(m)} \\
&= b^{\log_b(n)\log_b(m)} \\
&= \left(b^{\log_b(m)}\right)^{\log_b(n)} \\
&= m^{\log_b(n)}
\end{aligned}
$$

Arithmetic series

Next we will look various series

Each term in an arithmetic series is increased by a constant value (usually 1) :

$$0+1+2+3+\cdots+n=\sum_{k=0}^{n} k=\frac{n(n+1)}{2}$$

Arithmetic series

Proof 1: write out the series twice and add each column

$$
\begin{array}{rccccccccccccc}
 & 1 & + & 2 & + & 3 & + \cdots + & n-2 & + & n-1 & + & n \\
+ & \underline{n} & \underline{+} & \underline{n-1} & \underline{+} & \underline{n-2} & \underline{+ \cdots +} & \underline{3} & \underline{+} & \underline{2} & \underline{+} & \underline{1}
\end{array}
$$

$$(n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1)$$

$$= n\,(n+1)$$

Since we added the series twice, we must divide the result by 2

Arithmetic series

Proof 2 (by induction):

The statement is true for $n = 0$:

$$\sum_{i=0}^{0} k = 0 = \frac{0 \cdot 1}{2} = \frac{0(0+1)}{2}$$

Assume that the statement is true for an arbitrary n:

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

Arithmetic series

Then, for $n + 1$, we have:

$$\sum_{k=0}^{n+1} k = (n+1) + \sum_{i=0}^{n} k$$

By assumption, the second sum is known:

$$= (n+1) + \frac{n(n+1)}{2}$$

$$= \frac{(n+1)2 + (n+1)n}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

Arithmetic series

The statement is true for $n = 0$ and
the truth of the statement for n implies
the truth of the statement for $n + 1$.

Therefore, by the process of mathematical induction, the statement is true for all values of $n \geq 0$.

Other polynomial series

We could repeat this process, after all:

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

however, it is easier to see the pattern:

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2} \approx \frac{n^2}{2} \qquad \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \approx \frac{n^3}{3}$$

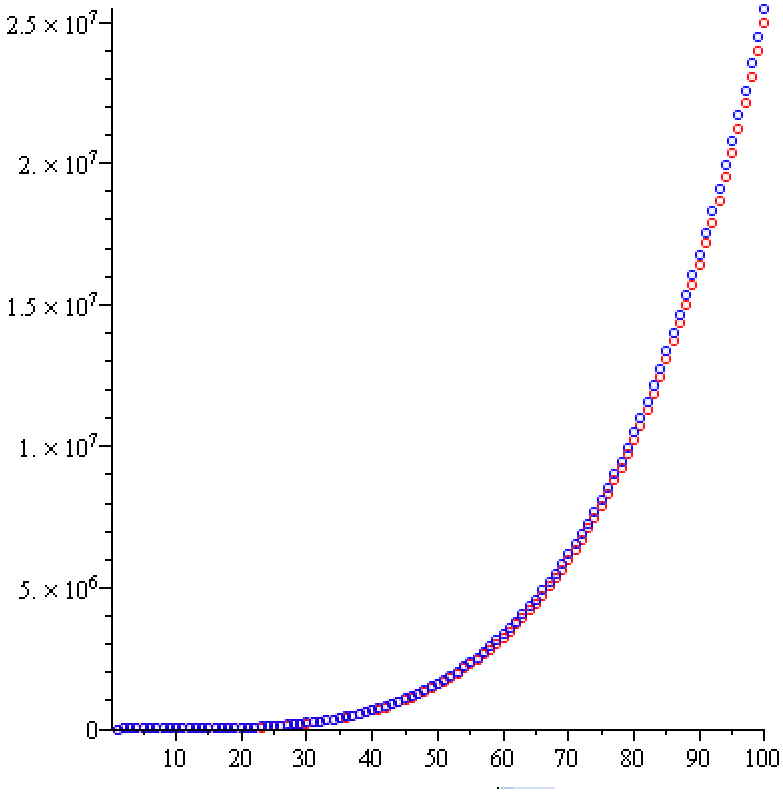
$$\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4} \approx \frac{n^4}{4}$$

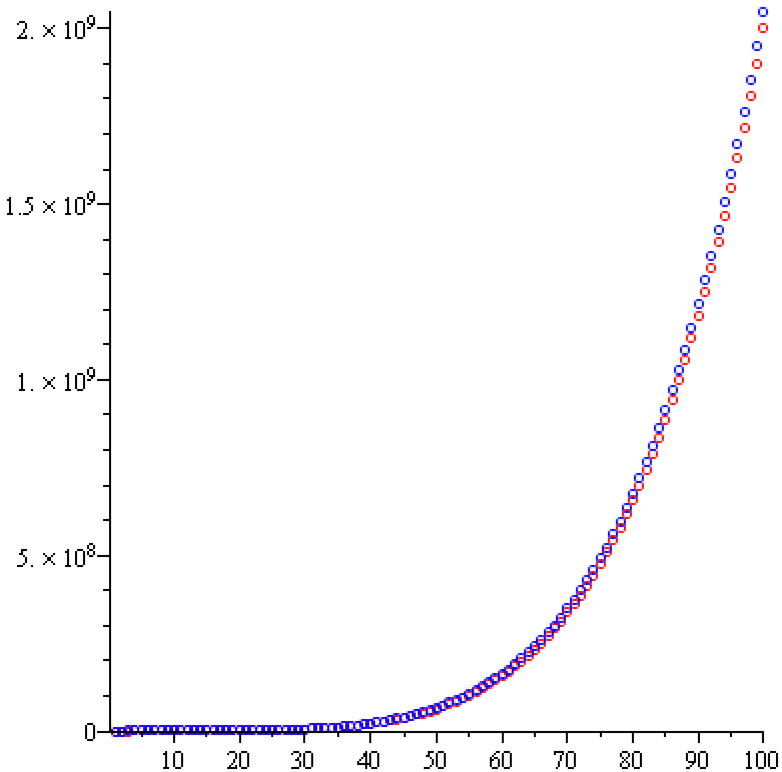
Other polynomial series

We can generalize this formula

$$\sum_{k=0}^{n} k^d \approx \frac{n^{d+1}}{d+1}$$

Demonstrating with $d = 3$ and $d = 4$:



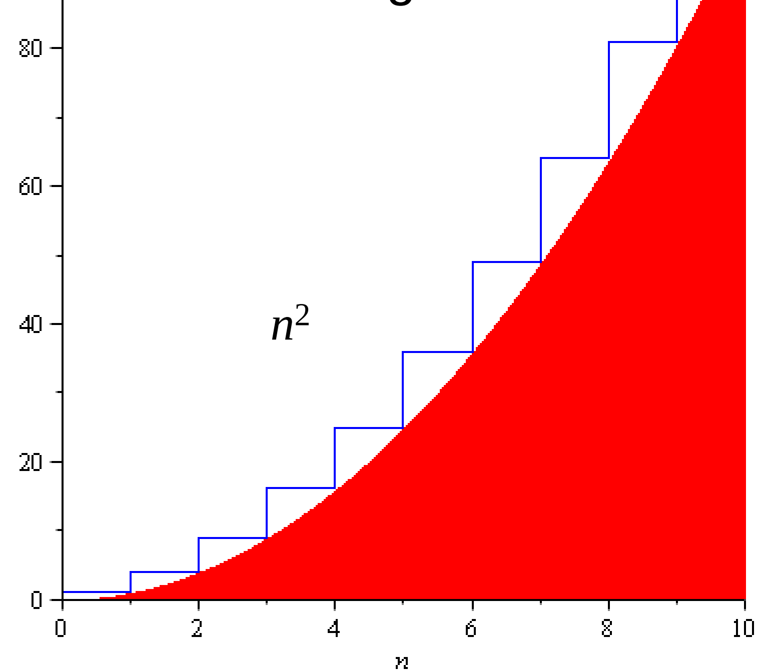


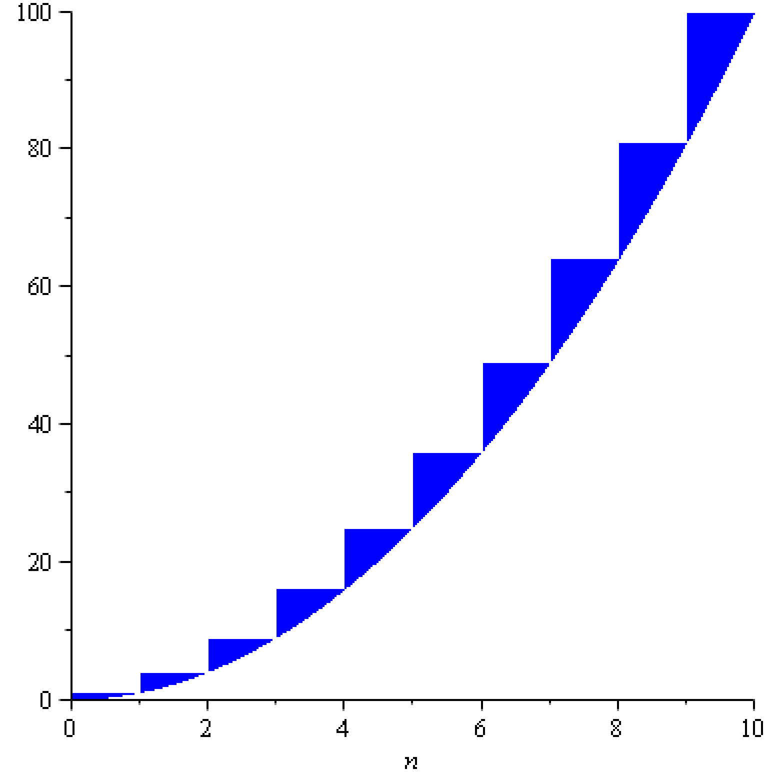
Other polynomial series

The justification for the approximation is that we are approximating the sum with an integral:

$$\sum_{k=0}^{n} k^d \approx \int_0^n x^d \, dx = \left. \frac{x^{d+1}}{d+1} \right|_{x=0}^{n} = \frac{n^{d+1}}{d+1} - 0$$

However, there is an accumulating error:





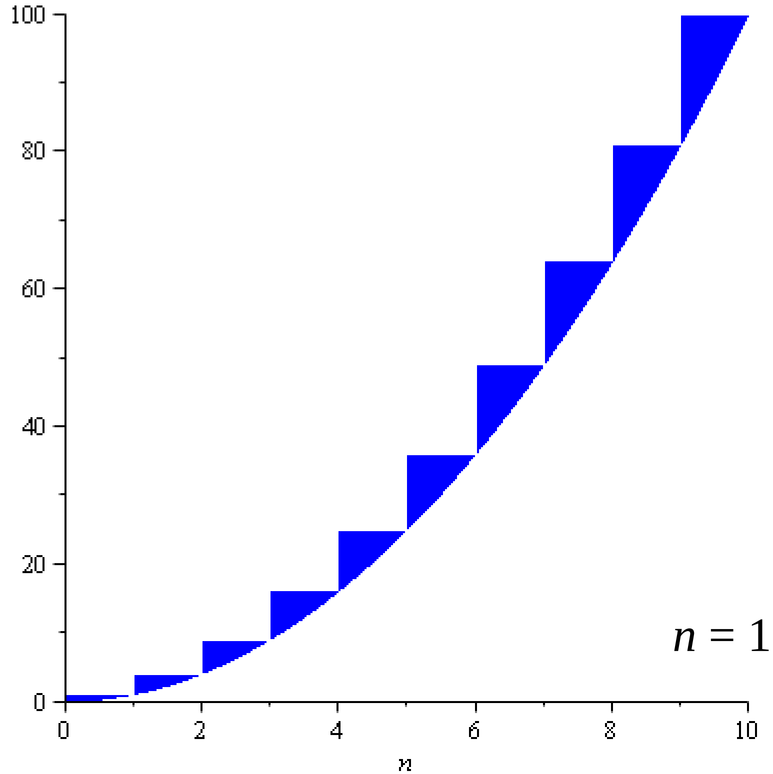
Other polynomial series

How large is the error?

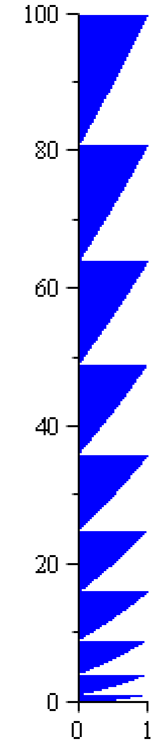
– Assuming $d > 1$, shifting the errors, we see that they would be

$$\frac{n^d}{2} \le \sum_{k=0}^{n} k^d - \frac{n^{d+1}}{d+1} < n^d \ll n^{d+1}$$

$n^2 = 100$



$n = 10$



Other polynomial series

In the limit, as $n \to \infty$

- The ratio between the error and the approximation goes to 0

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} k^d - \frac{n^{d+1}}{d+1}}{\frac{n^{d+1}}{d+1}} = 0$$

- Therefore, the ratio between the sum and the approximation goes to 1

$$\lim_{n\to\infty} \frac{\frac{n^{d+1}}{d+1}}{\sum_{k=0}^{n} k^d} = 1$$

Geometric series

The next series we will look at is the geometric series with common ratio r: $1, r, r^2, r^3, \cdots, r^n$

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

and if $|r| < 1$ then it is also true that

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

Geometric series

Elegant proof: multiply by $1 = \dfrac{1-r}{1-r}$

all but the first and last terms cancel

$$\sum_{k=0}^{n} r^k = \frac{(1-r)\sum_{k=0}^{n} r^k}{1-r}$$

$$= \frac{\sum_{k=0}^{n} r^k - r\sum_{k=0}^{n} r^k}{1-r}$$

$$= \frac{(1 + r + r^2 + \cdots + r^n) - (r + r^2 + \cdots + r^n + r^{n+1})}{1-r}$$

$$= \frac{1-r^{n+1}}{1-r}$$

Ref: Bret D. Whissel, *A Derivation of Amortization*

Geometric series

Proof by induction:

The formula is correct for n = 0: $\sum_{k=0}^{0} r^k = r^0 = 1 = \frac{1-r^{0+1}}{1-r}$

Assume the formula $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$ is true for an arbitrary n; then

$$\sum_{k=0}^{n+1} r^k = r^{n+1} + \sum_{k=0}^{n} r^k = r^{n+1} + \frac{1-r^{n+1}}{1-r} = \frac{(1-r)r^{n+1} + 1 - r^{n+1}}{1-r}$$

$$= \frac{r^{n+1} - r^{n+2} + 1 - r^{n+1}}{1-r} = \frac{1-r^{n+2}}{1-r} = \frac{1-r^{(n+1)+1}}{1-r}$$

and therefore, by the process of mathematical induction, the statement is true for all $n \geq 0$.

Geometric series

A common geometric series will involve the ratios r = ½ and r = 2:

$$\sum_{i=0}^{n}\left(\frac{1}{2}\right)^{i} = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = 2 - 2^{-n}$$

$$\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^{i} = 2$$

$$\sum_{k=0}^{n} 2^{k} = \frac{1-2^{n+1}}{1-2} = 2^{n+1} - 1$$

Recurrence relations

Finally, we will review recurrence relations:

- A recurrence relationship is a means of defining a sequence based on previous values in the sequence
- Such definitions of sequences are said to be *recursive*

Recurrence relations

Define an initial value: *e.g.*, $x_1 = 1$

Defining x_n in terms of previous values:

– For example,

$$x_n = x_{n-1} + 2$$

$$x_n = 2x_{n-1} + n$$

$$x_n = x_{n-1} + x_{n-2}$$

Recurrence relations

We will use a functional form of recurrence relations:

Calculus	**Functional form**
$x_1 = 1$	$\mathrm{f}(1) = 1$
$x_n = x_{n-1} + 2$	$\mathrm{f}(n) = \mathrm{f}(n-1) + 2$
$x_n = 2x_{n-1} + n$	$\mathrm{f}(n) = 2\,\mathrm{f}(n-1) + n$

Recurrence relations

In some cases, given the recurrence relation, we can find the explicit formula:

- Consider the Fibonacci sequence:

  $$f(n) = f(n-1) + f(n-2)$$

  $$f(0) = f(1) = 1$$

  that has the solution

  $$f(n) = \frac{2+\phi}{5}\phi^{n} + \frac{3-\phi}{5}\phi^{-n}$$

  where ϕ is the golden ratio:

  $$\phi = \frac{\sqrt{5}+1}{2} \approx 1.6180$$

Weighted averages

Given n objects $x_1, x_2, x_3, ..., x_n$, the *average* is

$$\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$

Given a sequence of coefficients $c_1 , c_2 , c_3 , \ldots , c_n$ where

$$c_1 + c_2 + c_3 + \cdots + c_n = 1$$

then we refer to

$$c_1 x_1 + c_2 x_2 + c_3 x_3 + \cdots + c_n x_n$$

as a *weighted* average

For an average, $c_1 = c_2 = c_3 = \cdots = c_n = \dfrac{1}{n}$

Combinations

Given n distinct items, in how many ways can you choose k of these?

- I.e., "In how many ways can you combine k items from n?"
- For example, given the set $\{1, 2, 3, 4, 5\}$, I can choose three of these in any of the following ways:

$$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\},$$
$$\{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}$$

The number of ways such items can be chosen is written

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

where $\binom{n}{k}$ is read as "n choose k"s

There is a recursive definition: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

Combinations

The most common question we will ask in this vein:

- Given n items, in how many ways can we choose two of them?
- In this case, the formula simplifies to:

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$

For example, given $\{0, 1, 2, 3, 4, 5, 6\}$, we have the following 21 pairs:

$$\begin{array}{r}
\{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 4\}, \{0, 5\}, \{0, 6\}, \\
\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}, \\
\{2, 3\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \\
\{3, 4\}, \{3, 5\}, \{3, 6\}, \\
\{4, 5\}, \{4, 6\}, \\
\{5, 6\}
\end{array}$$

Combinations

You have also seen this in expanding polynomials:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

For example,

$$\begin{aligned}(x+y)^4 &= \sum_{k=0}^{4} \binom{4}{k} x^k y^{4-k} \\ &= \binom{4}{0} y^4 + \binom{4}{1} xy^3 + \binom{4}{2} x^2 y^2 + \binom{4}{3} x^3 y + \binom{4}{4} x^4 \\ &= y^4 + 4xy^3 + 6x^2 y^2 + 4x^3 y + x^4\end{aligned}$$

Summary

In this topic, we have discussed:

– A review of the necessity of quantitative analysis in engineering

We reviewed the following mathematical concepts:

– The floor and ceiling functions
– L'Hôpital's rule
– Logarithms
– Arithmetic and other polynomial series
  • Mathematical induction
– Geometric series
– Recurrence relations
– Weighted average
– Combinations