Binary Tree

Textbook Ch B.5.3, 10.4

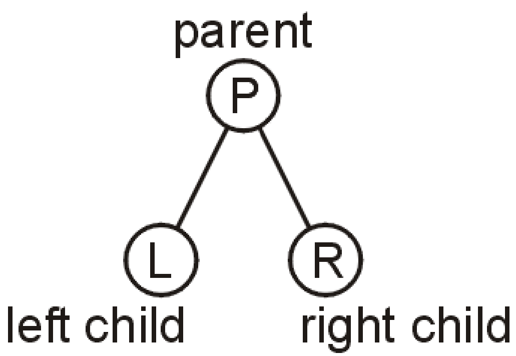
Outline

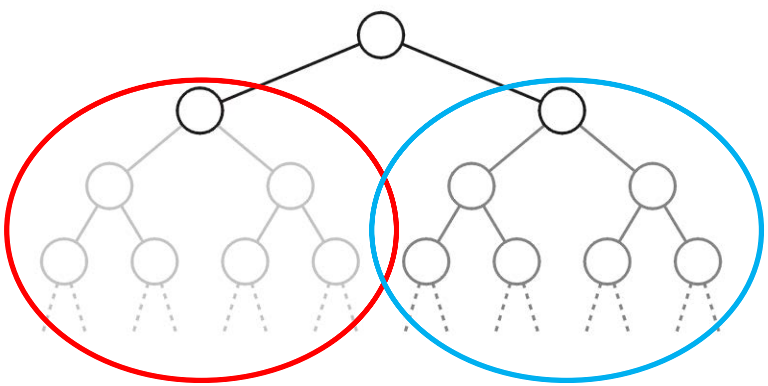
- Binary tree
- Perfect binary tree
- Complete binary tree
- Left-child right-sibling binary tree

Definition

A binary tree is a restriction where each node has exactly two children:

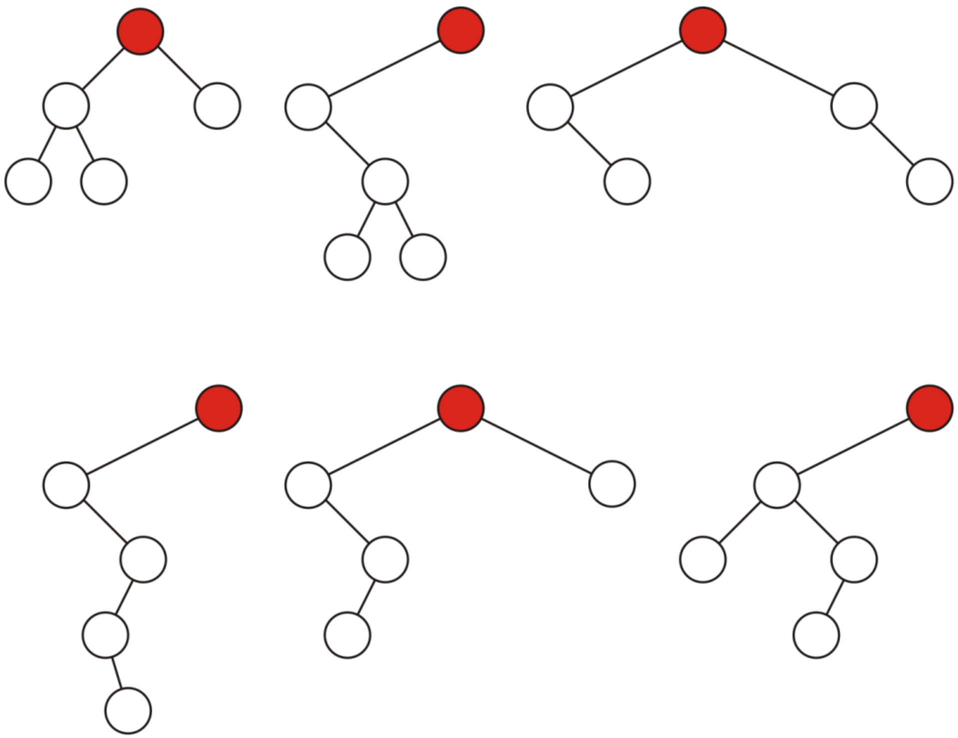
- Each child is either empty or another binary tree
- This restriction allows us to label the children as *left* and *right* subtrees





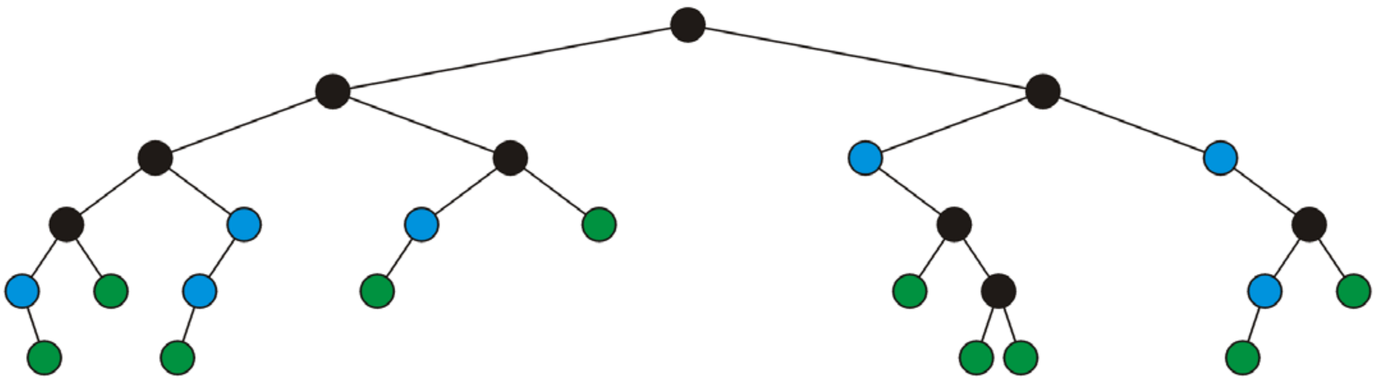
Definition

Some binary trees with five nodes:



Definition

A *full* node is a node where both the left and right sub-trees are non-empty trees

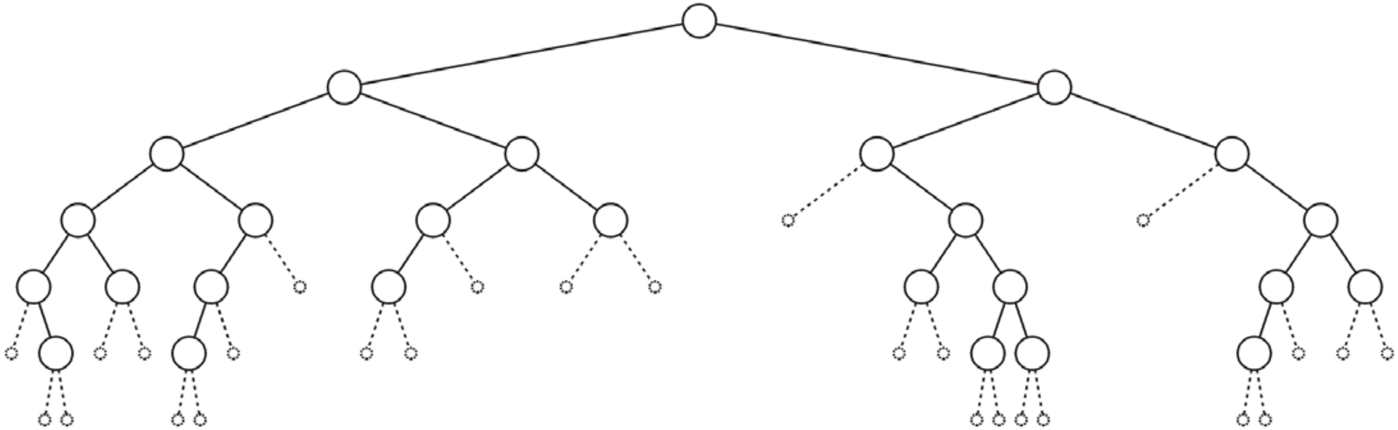


Legend:

full nodes neither leaf nodes

Definition

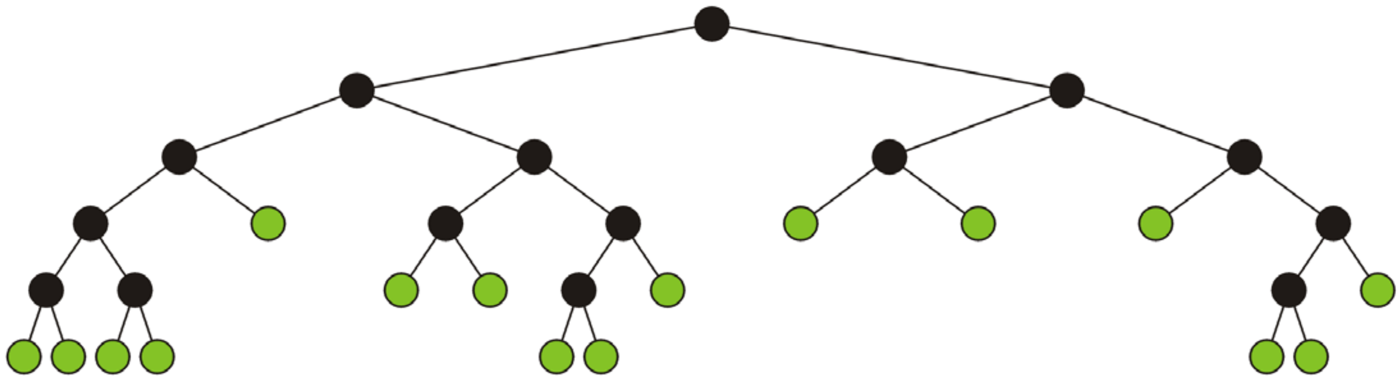
An *empty node* or a *null sub-tree* is any location where a new leaf node could be appended



Definition

A *full binary tree* is where each node is:

- A full node, or
- A leaf node



These have applications in

- Expression trees
- Huffman encoding

Binary Node Class

The binary node class is similar to the single node class:

```
template <typename Type>
class Binary_node {
    protected:
        Type element;
        Binary_node *left_tree;
        Binary_node *right_tree;

    public:
        Binary_node( Type const & );

        Type retrieve() const;
        Binary_node *left() const;
        Binary_node *right() const;

        bool is_leaf() const;
        int size() const;
}
```

Binary Node Class

We will usually only construct new leaf nodes

```
template <typename Type>
Binary_node<Type>::Binary_node( Type const &obj ):
element( obj ),
left_tree( nullptr ),
right_tree( nullptr ) {
    // Empty constructor
}
```

Binary Node Class

The accessors are similar to that of `Single_list`

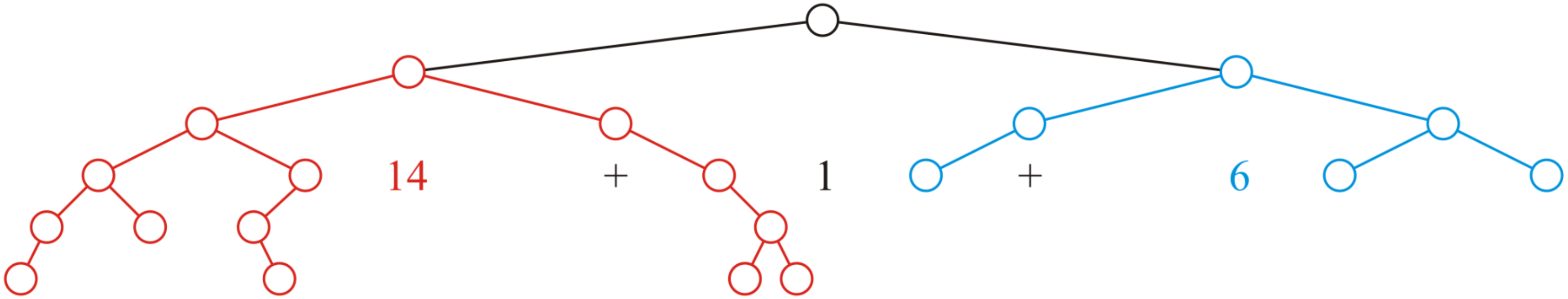
```
template <typename Type>
Type Binary_node<Type>::retrieve() const {
    return element;
}

template <typename Type>
Binary_node<Type> *Binary_node<Type>::left() const {
    return left_tree;
}

template <typename Type>
Binary_node<Type> *Binary_node<Type>::right() const {
    return right_tree;
}
```

Binary Node Class

```
template <typename Type>
bool Binary_node<Type>::is_leaf() const {
    return left() == nullptr && right() == nullptr;
}
```

Size

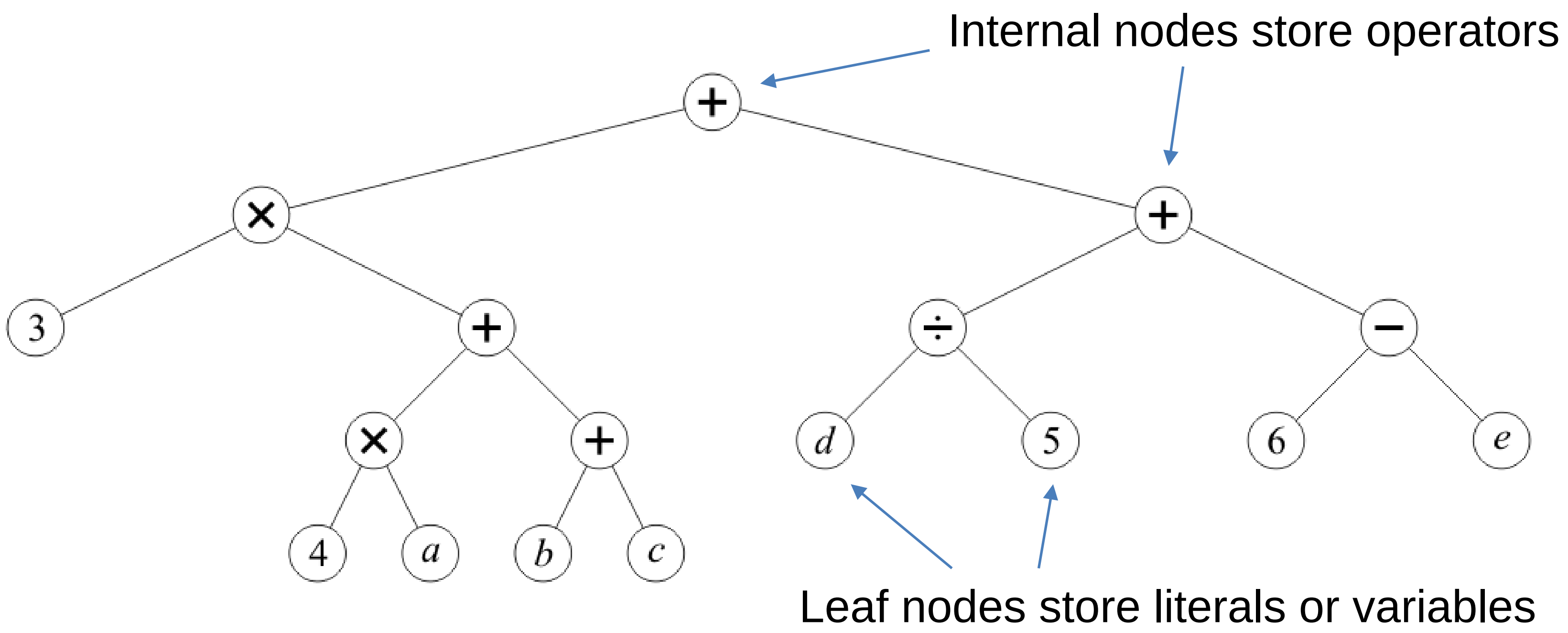
```
template <typename Type>
int Binary_node<Type>::size() const {
    return 1
        + left() == nullptr? 0 : left()->size()
        + right() == nullptr? 0 : right()->size();
}
```



Application: Expression Trees

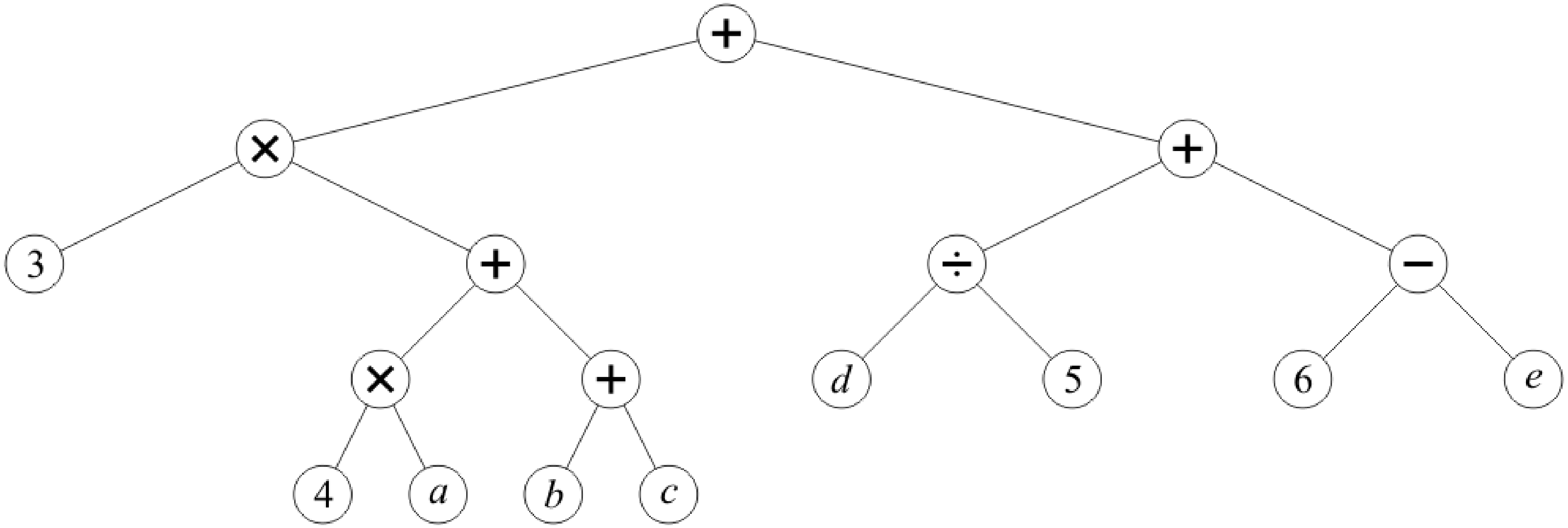
Any basic mathematical expression containing binary operators may be represented using a (full) binary tree

- For example, $3(4a + b + c) + d/5 + (6 - e)$



Application: Expression Trees

A post-order depth-first traversal converts such a tree to the reverse-Polish format



$$3\ 4\ a\ \times\ b\ c\ +\ +\ \times\ d\ 5\ \div\ 6\ e\ -\ +\ +$$

Application: Expression Trees

Computers think in post-order:

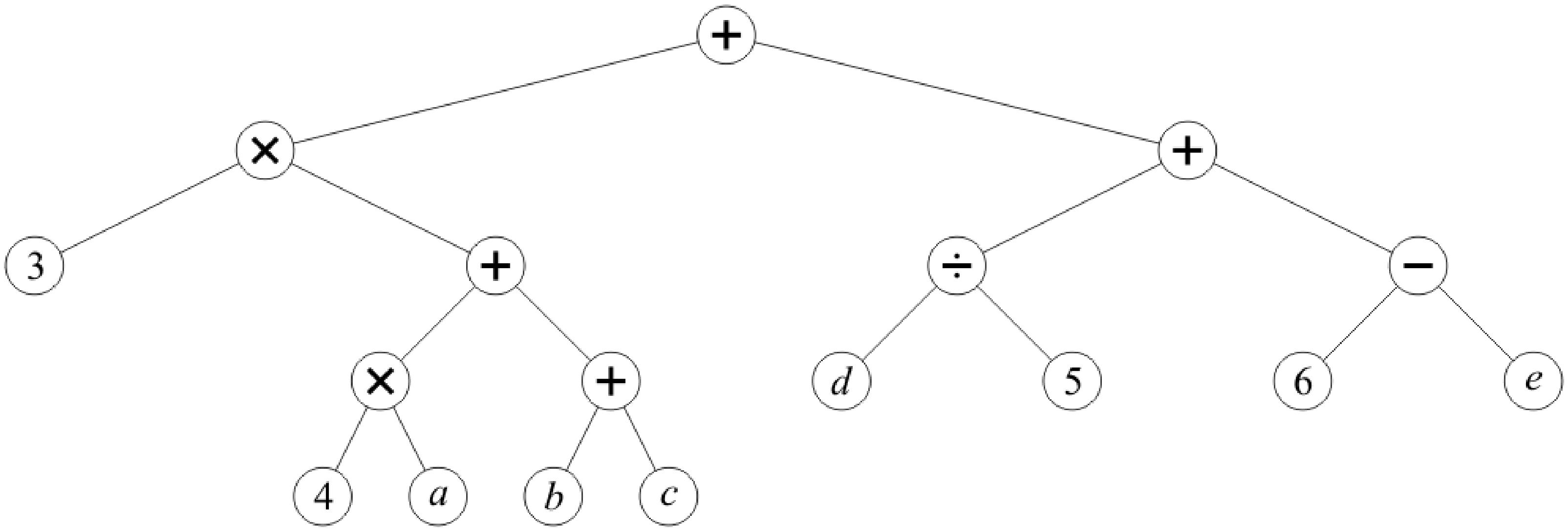
- Both operands must be loaded into registers
- The operation is then called on those registers

Humans think in in-order:

- First, the left sub-tree is traversed
- Then, the current node is visited
- Finally, the right-sub-tree is traversed

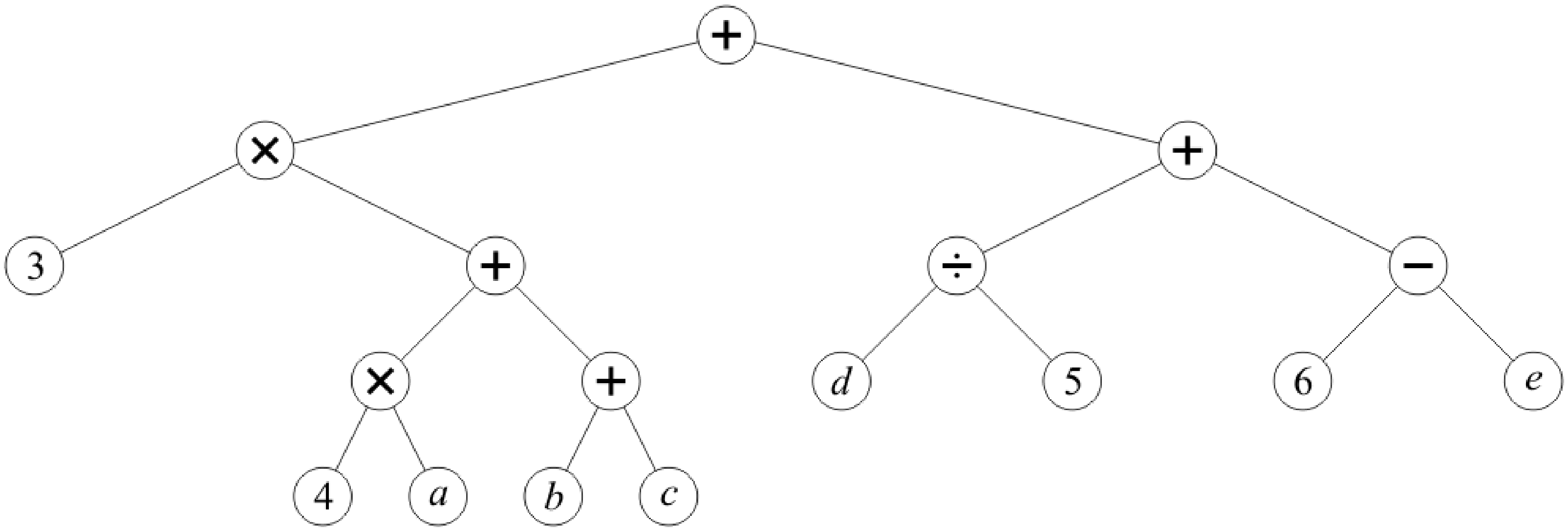
This is called an *in-order* traversal

In-order Traversal



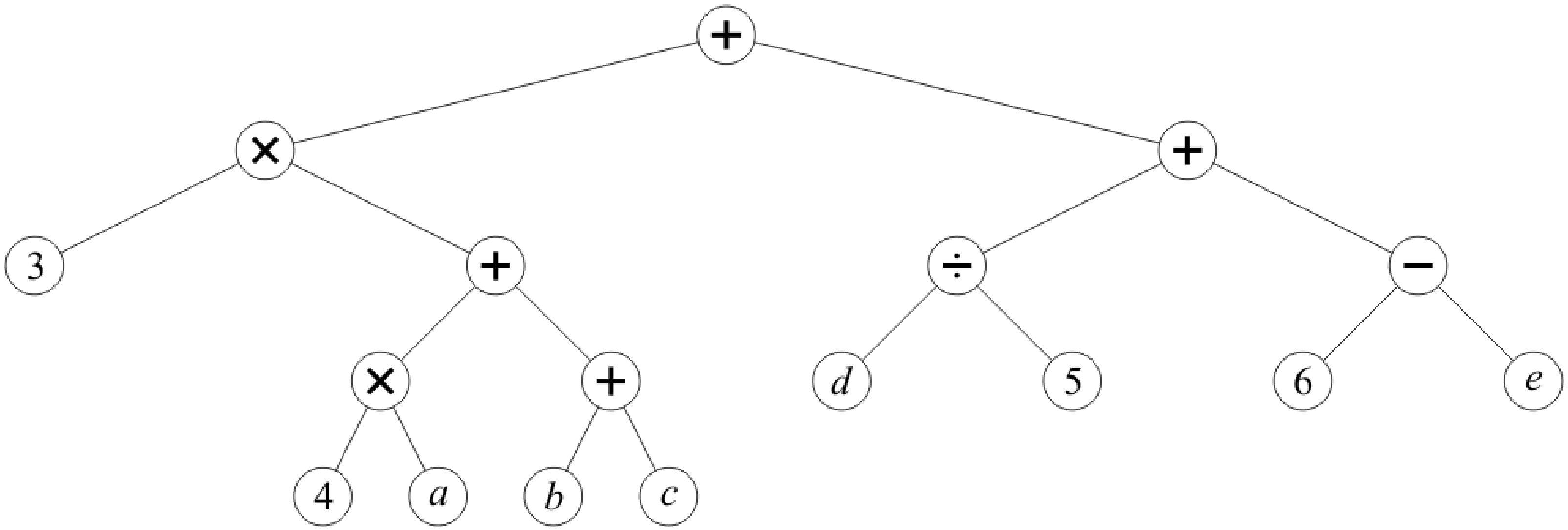
3

In-order Traversal



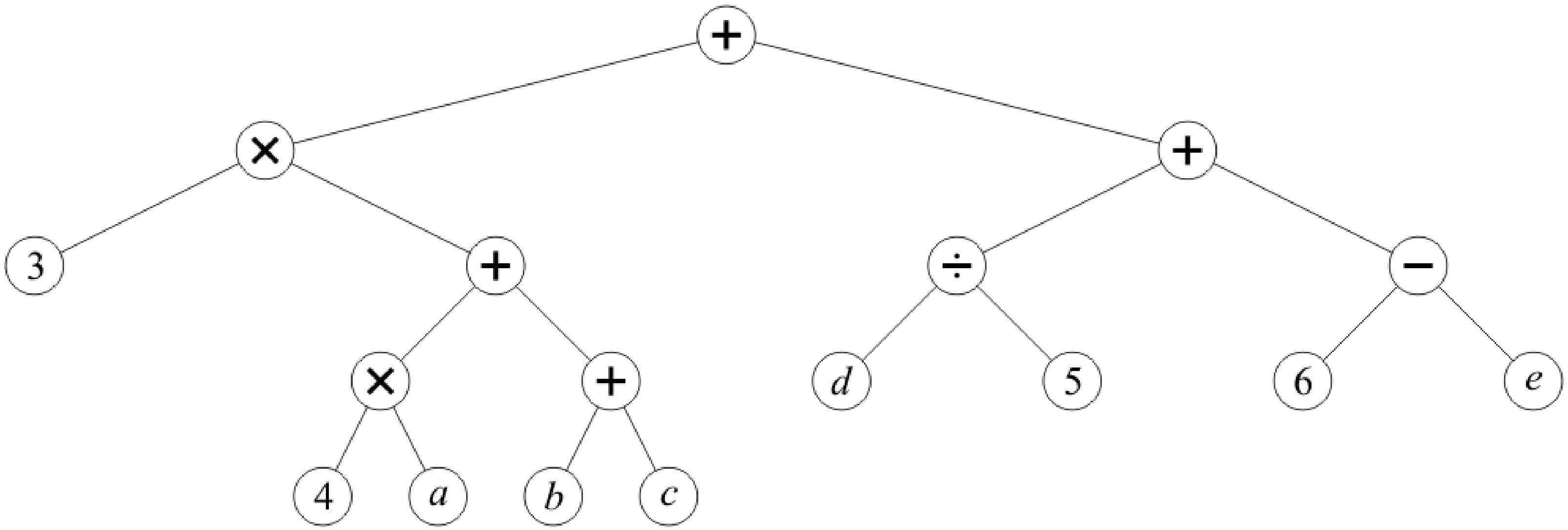
$3 \times$

In-order Traversal



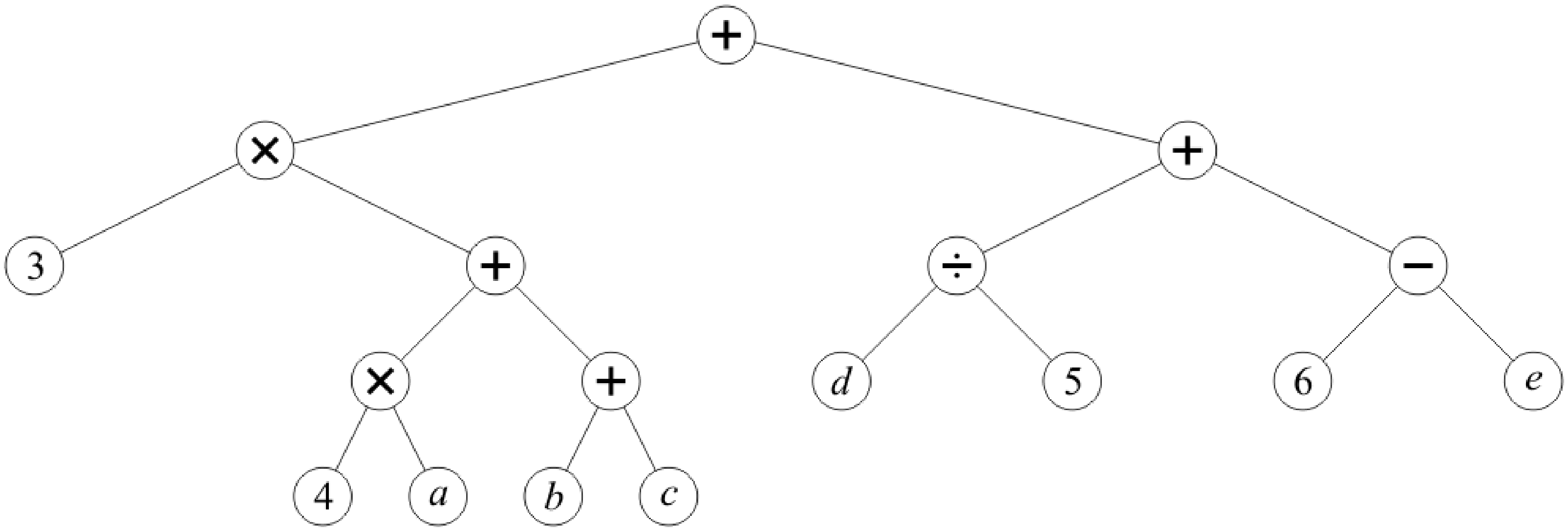
3×4

In-order Traversal



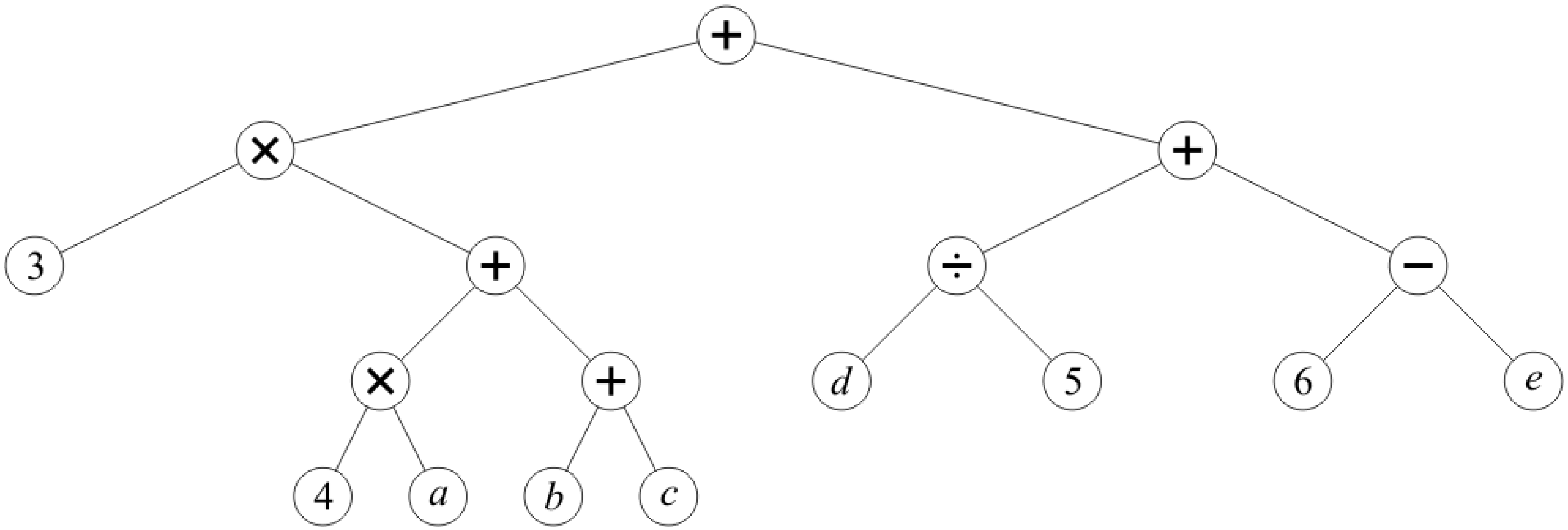
$3 \times 4 \times$

In-order Traversal



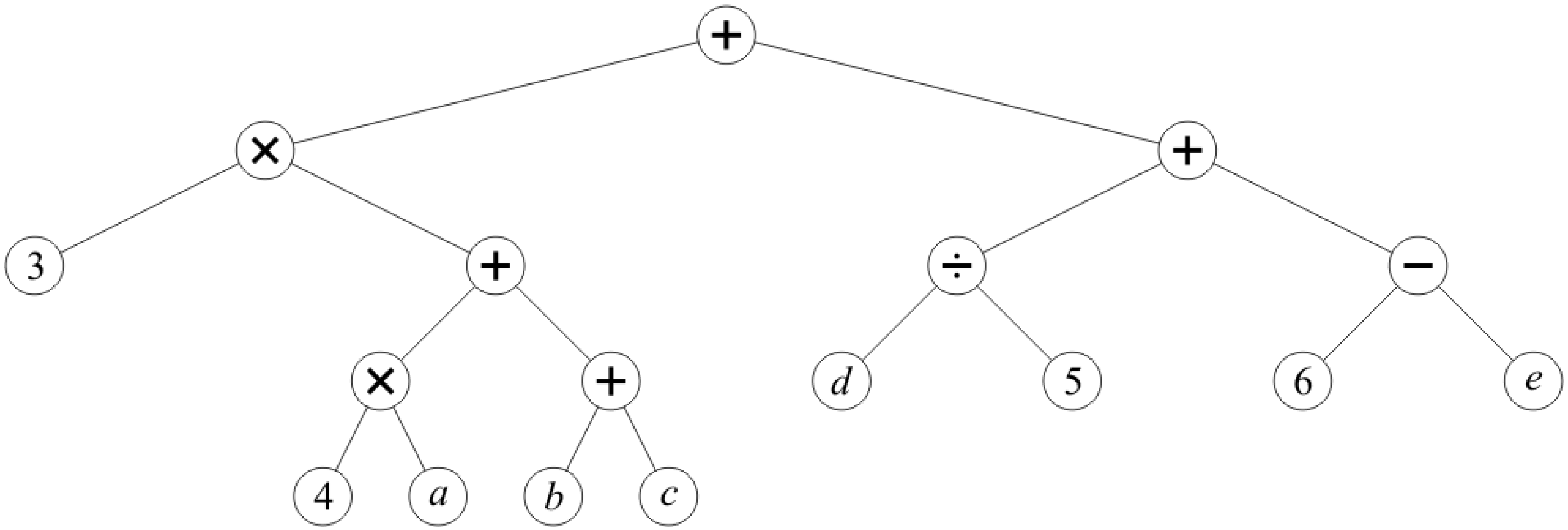
$3 \times 4 \times a$

In-order Traversal



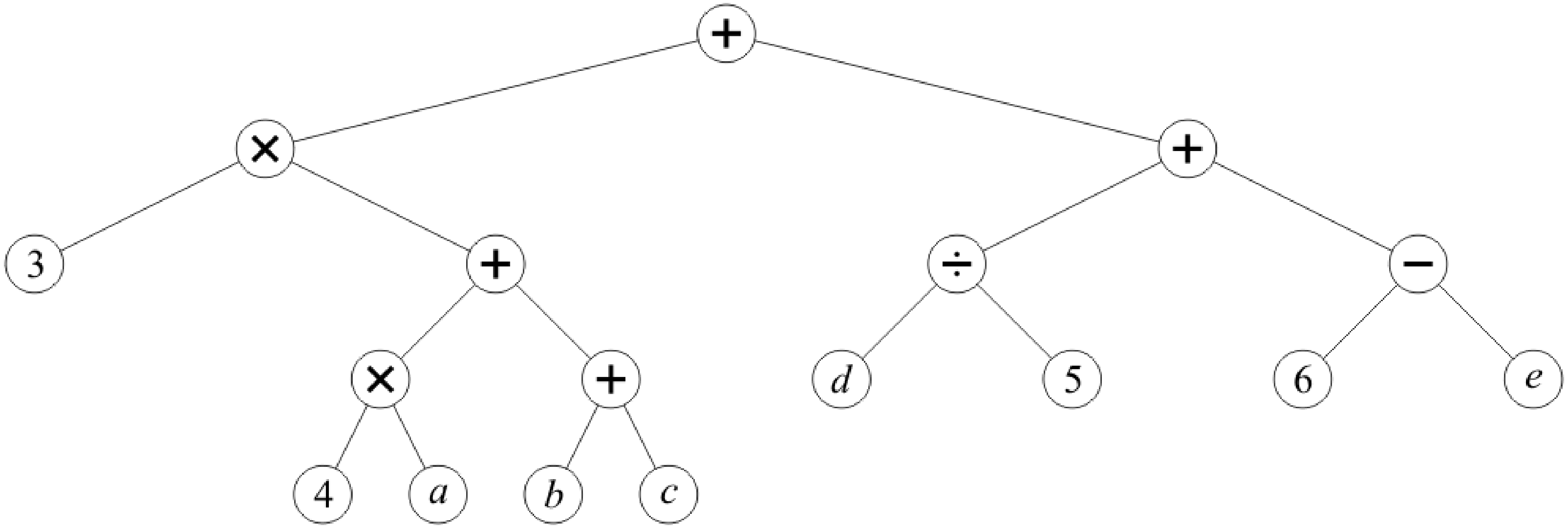
$3 \times 4 \times a +$

In-order Traversal



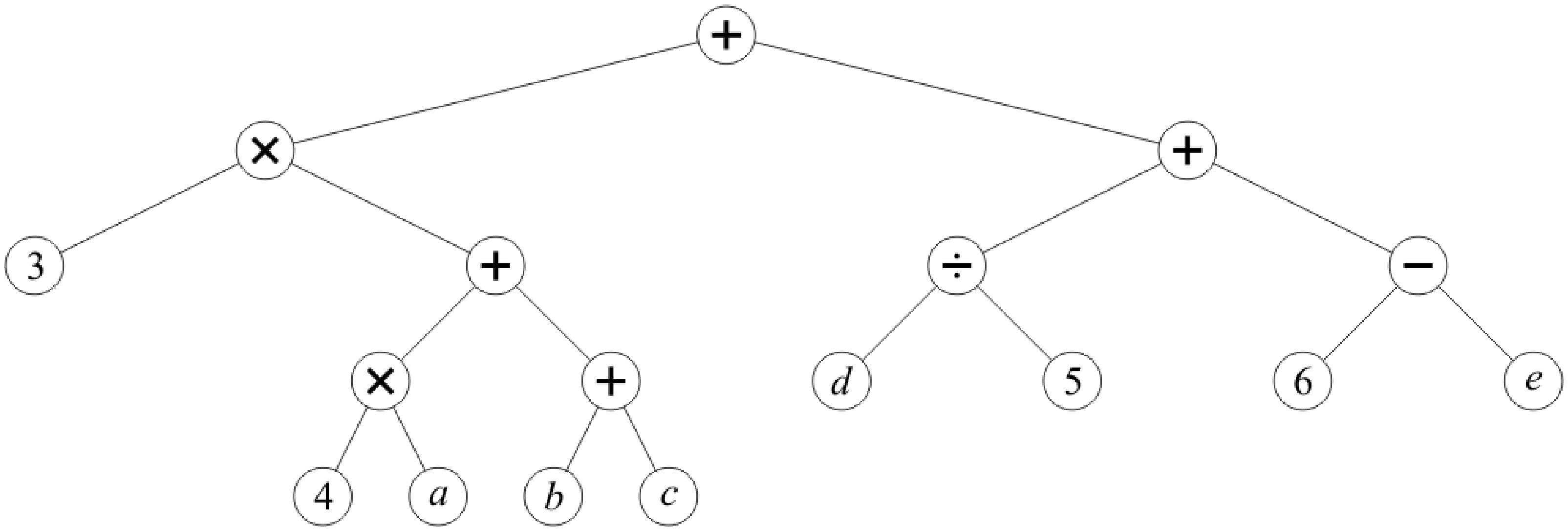
$3 \times 4 \times a + b$

In-order Traversal



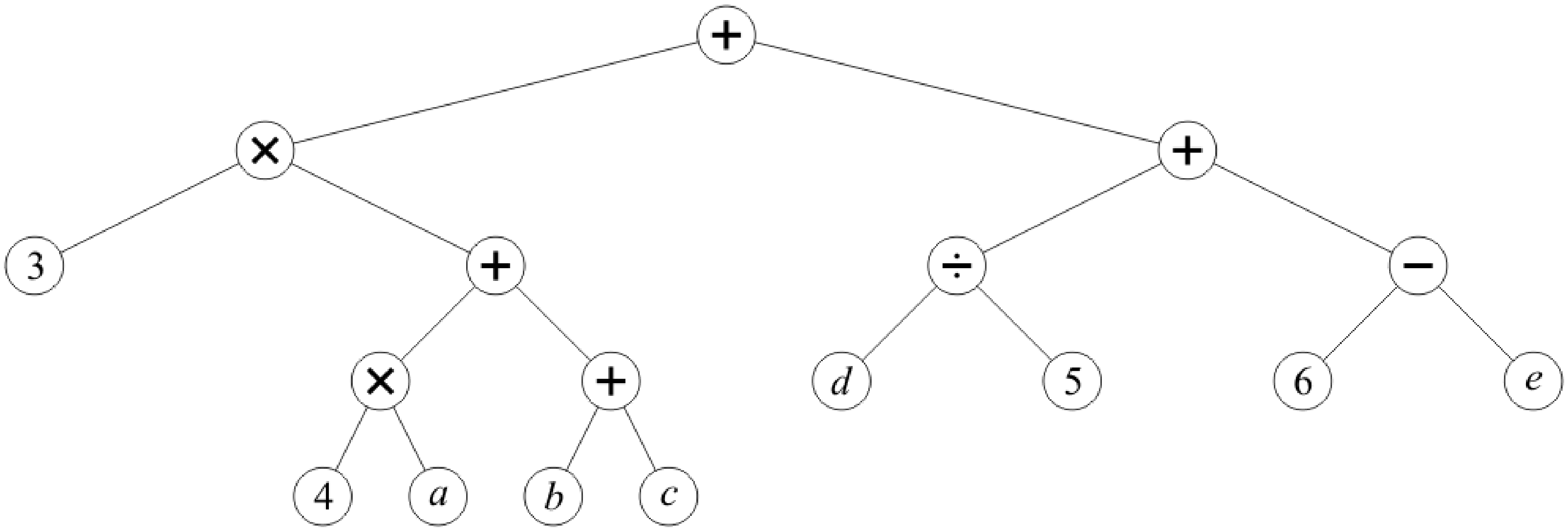
$3 \times 4 \times a + b +$

In-order Traversal



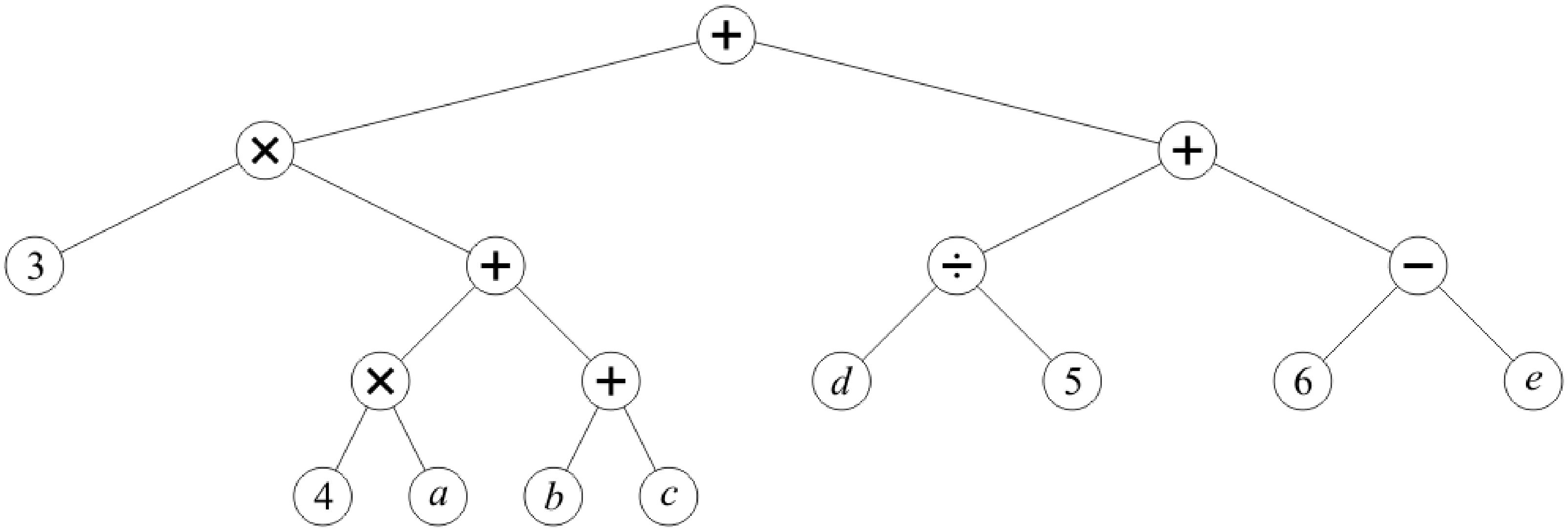
$3 \times 4 \times a + b + c$

In-order Traversal



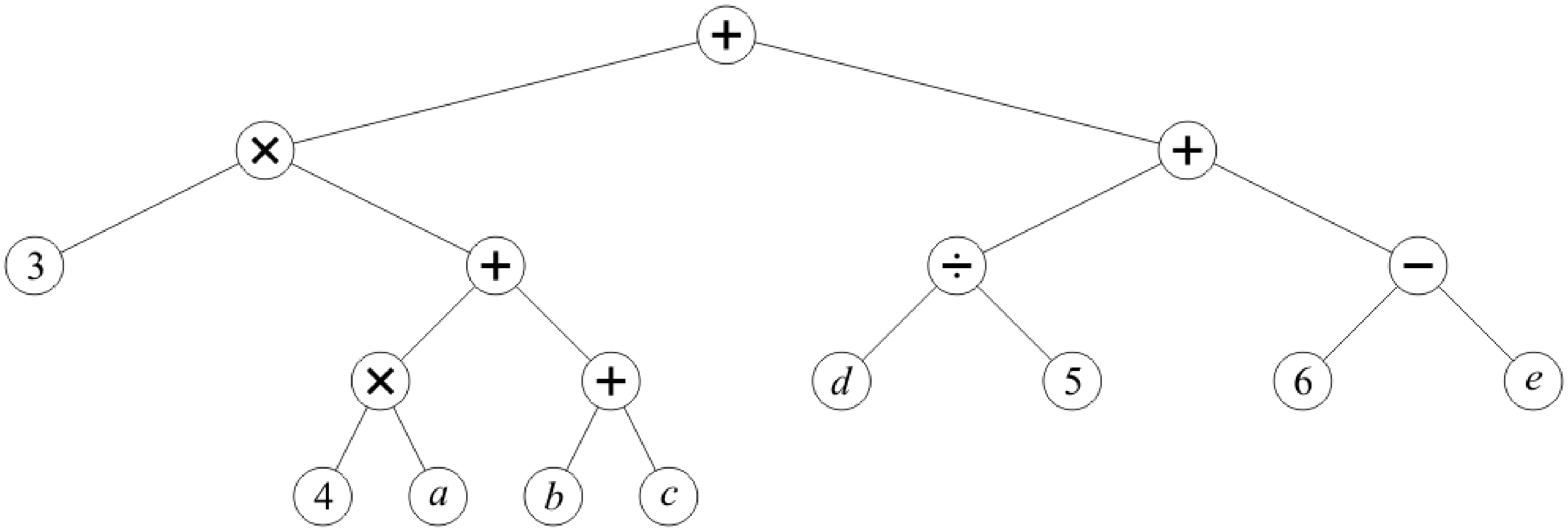
3 × 4 × a + b + c +

In-order Traversal



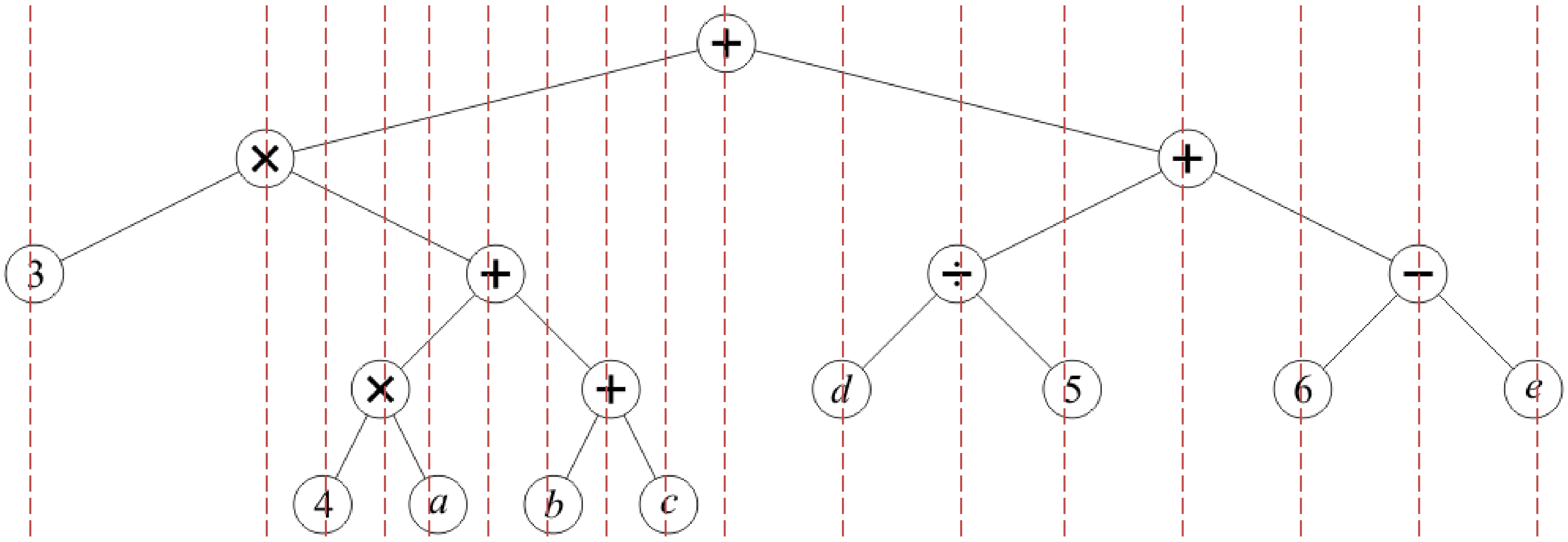
$3 \times 4 \times a + b + c + d \div 5 + 6 - e$

In-order Traversal



$$3 \times (4 \times a + (b + c)) + (d \div 5 + (6 - e))$$

In-order Traversal



$3 \times 4 \times a + b + c + d \div 5 + 6 - e$

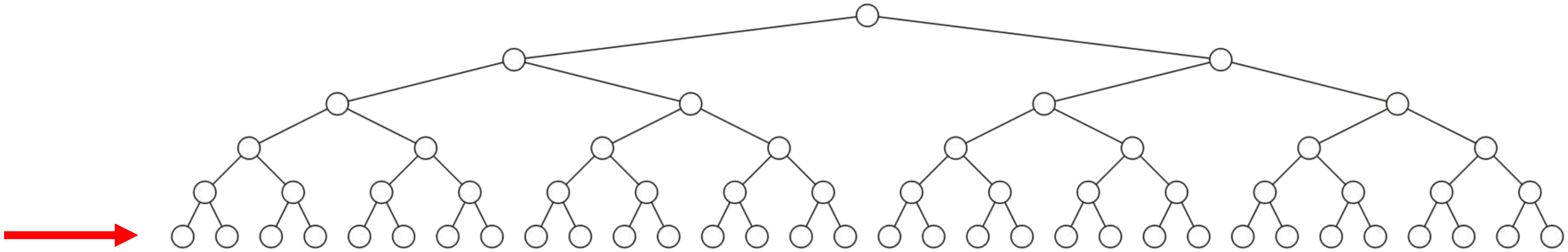
Outline

- Binary tree
- Perfect binary tree
- Complete binary tree
- Left-child right-sibling binary tree

Definition

Standard definition:

– A perfect binary tree of height h is a binary tree where
  - All leaf nodes have the same depth h
  - All other nodes are full



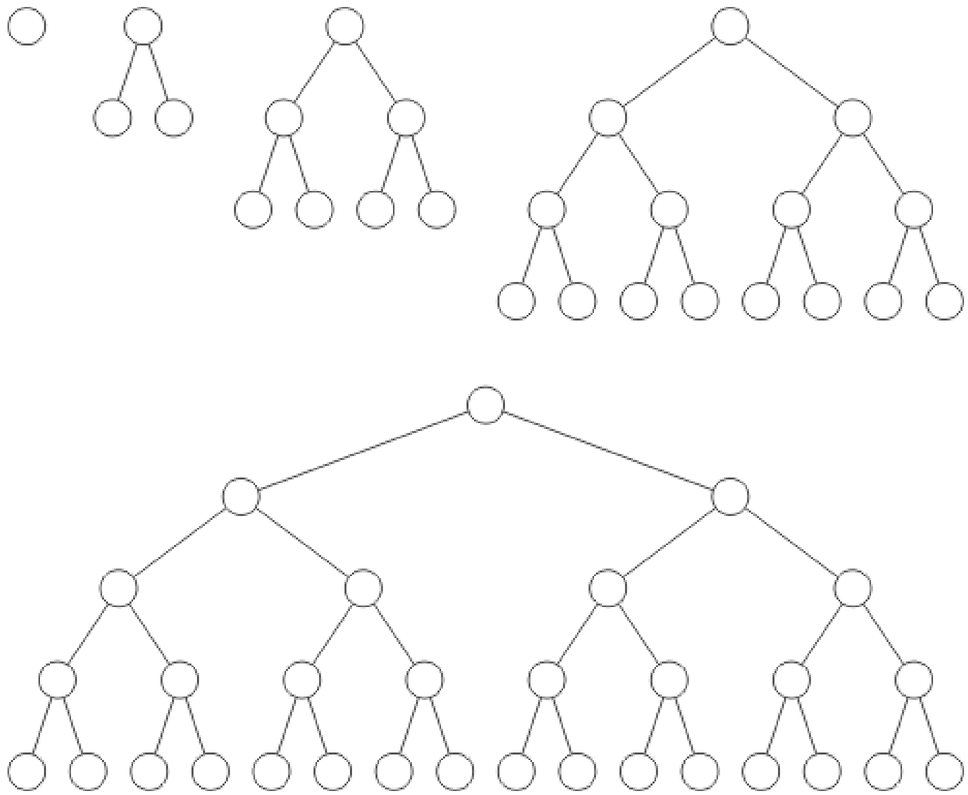
Definition

Recursive definition:

- A binary tree of height $h = 0$ is perfect
- A binary tree with height $h > 0$ is a perfect if both sub-trees are prefect binary trees of height $h - 1$

Examples

Perfect binary trees of height $h = 0, 1, 2, 3$ and 4

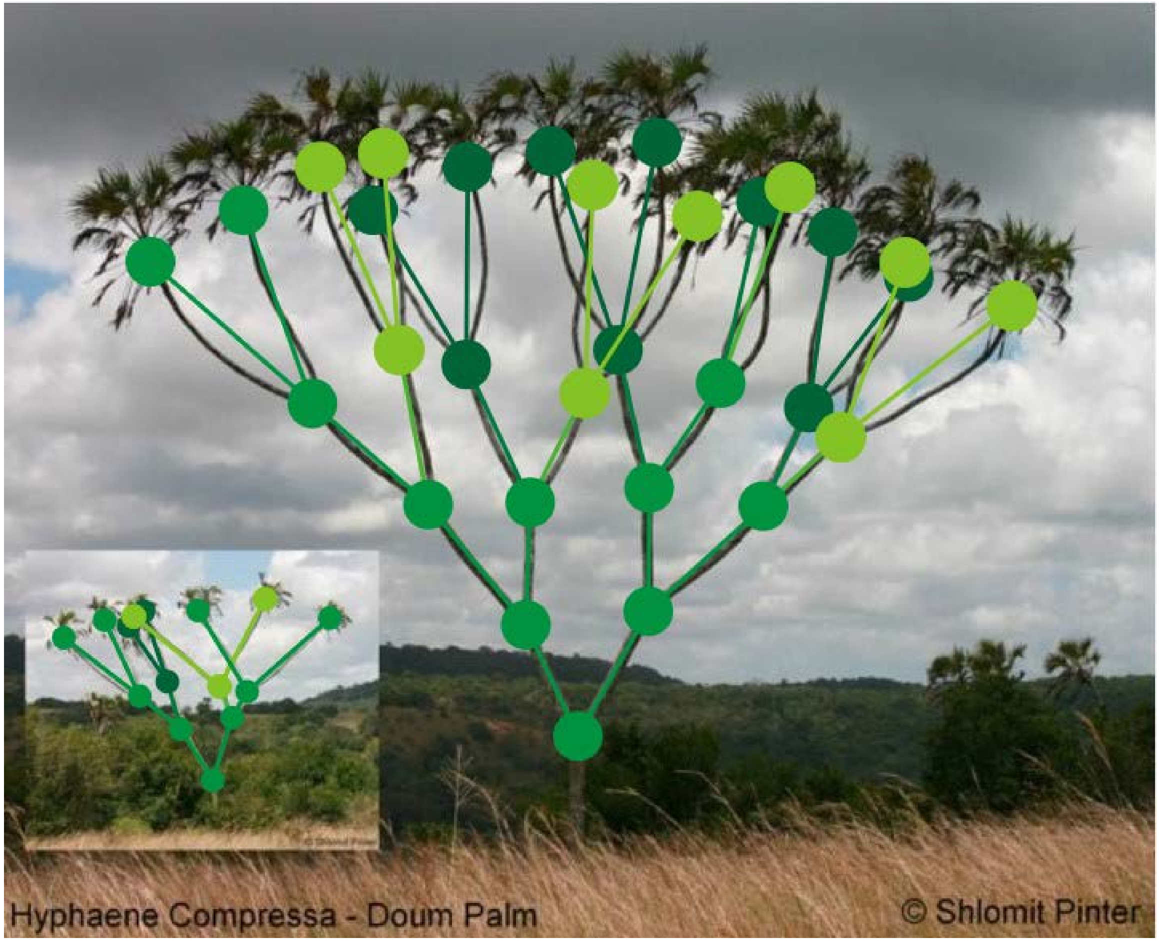


Examples

Perfect binary trees of height $h = 3$ and $h = 4$

Examples

Perfect binary trees of height $h = 3$ and $h = 4$



Theorems

Four theorems of perfect binary trees:

- A perfect binary tree of height h has $2^{h+1} - 1$ nodes
- The height is $\Theta(\ln(n))$
- There are 2^h leaf nodes
- The average depth of a node is $\Theta(\ln(n))$

These theorems will allow us to determine the optimal run-time properties of operations on binary trees

$2^{h+1} - 1$ Nodes

Theorem

A perfect binary tree of height h has $2^{h+1} - 1$ nodes

Proof:

We will use mathematical induction

$2^{h+1} - 1$ Nodes

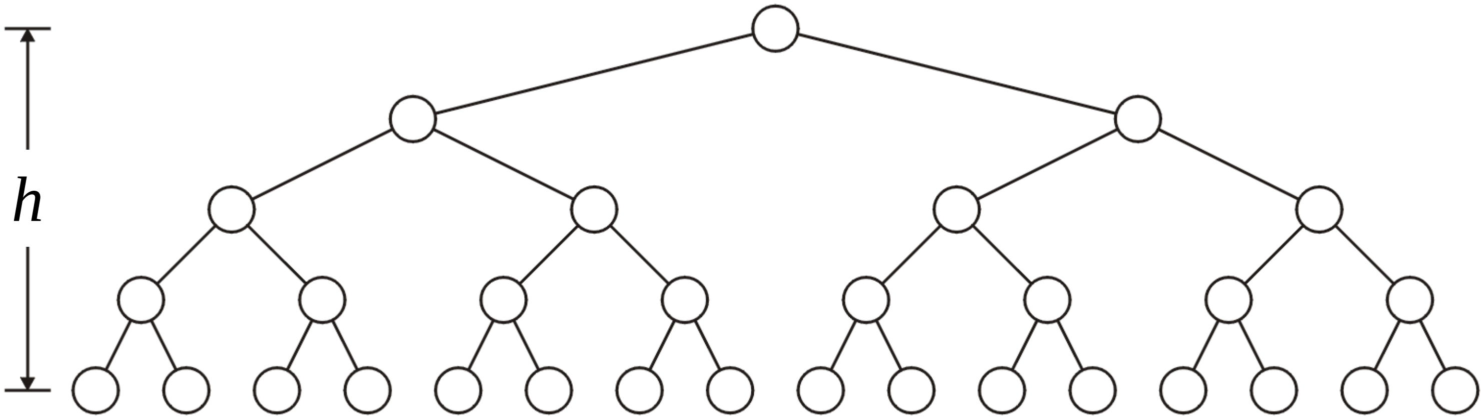
The base case:

- When $h = 0$ we have a single node $n = 1$
- The formula is correct: $2^{0+1} - 1 = 1$

$2^{h+1} - 1$ Nodes

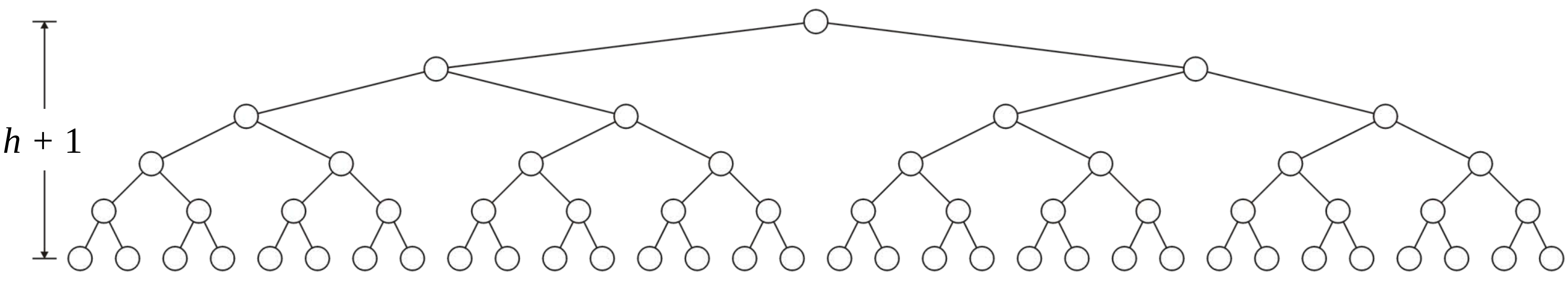
The inductive step:

– Assume that a tree of height h has $n = 2^{h+1} - 1$ nodes



$2^{h+1} - 1$ Nodes

We must show that a tree of height $h + 1$ has
$n = 2^{(h+1)+1} - 1 = 2^{h+2} - 1$ nodes

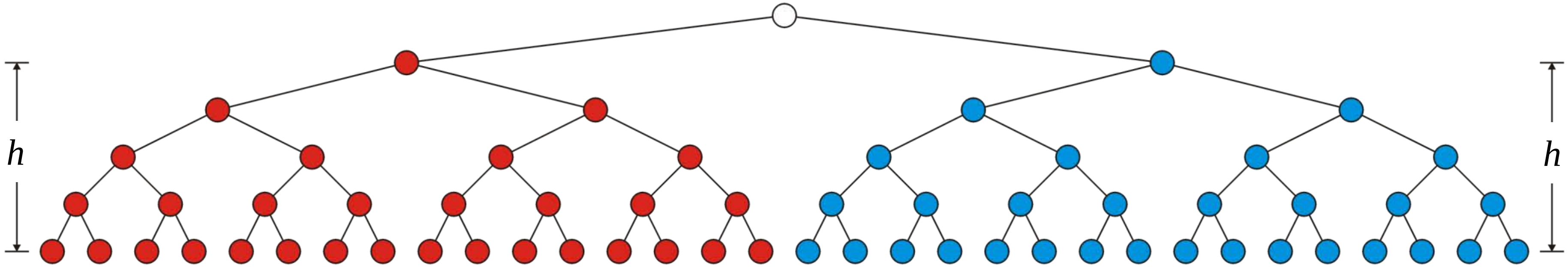


$2^{h+1} - 1$ Nodes

Using the recursive definition, both sub-trees are perfect trees of height h

- By assumption, each sub-tree has $2^{h+1} - 1$ nodes
- Therefore the total number of nodes is

$$(2^{h+1} - 1) + 1 + (2^{h+1} - 1) = 2^{h+2} - 1$$



$2^{h+1} - 1$ Nodes

Consequently

The statement is true for $h = 0$ and the truth of the statement for an arbitrary h implies the truth of the statement for $h + 1$.

Therefore, by the process of mathematical induction, the statement is true for all $h \geq 0$

Logarithmic Height

Theorem

A perfect binary tree with n nodes has height $\lg(n + 1) - 1$

Proof

Solving $n = 2^{h+1} - 1$ for h:

$$n + 1 = 2^{h+1}$$

$$\lg(n + 1) = h + 1$$

$$h = \lg(n + 1) - 1$$

Logarithmic Height

Lemma

$\lg(n + 1) - 1 = \Theta(\ln(n))$

Proof

$$\lim_{n\to\infty}\frac{\lg(n+1)-1}{\ln(n)} = \lim_{n\to\infty}\frac{\frac{1}{(n+1)\ln(2)}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{(n+1)\ln(2)} = \lim_{n\to\infty}\frac{1}{\ln(2)} = \frac{1}{\ln(2)}$$

2^h Leaf Nodes

Theorem

A perfect binary tree with height h has 2^h leaf nodes

Proof (by induction):

When $h = 0$, there is $2^0 = 1$ leaf node.

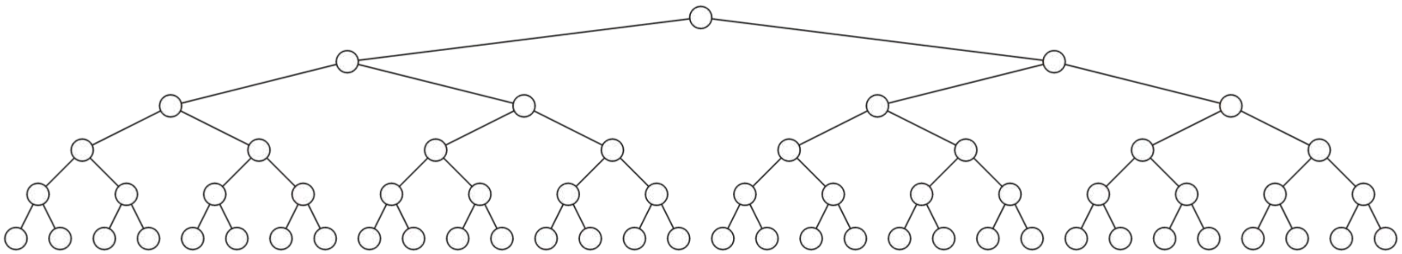
Assume that a perfect binary tree of height h has 2^h leaf nodes and observe that both sub-trees of a perfect binary tree of height $h + 1$ have 2^h leaf nodes.

Consequence: Over half of the nodes are leaf nodes:

$$\frac{2^h}{2^{h+1}-1} > \frac{1}{2}$$

The Average Depth of a Node

The average depth of a node in a perfect binary tree is



Depth	Count
0	1
1	2
2	4
3	8
4	16
5	32

Sum of the depths

$$\frac{\sum_{k=0}^{h} k2^k}{2^{h+1}-1} = \frac{h2^{h+1}-2^{h+1}+2}{2^{h+1}-1} = \frac{h(2^{h+1}-1)-(2^{h+1}-1)+1+h}{2^{h+1}-1}$$

$$= h-1+\frac{h+1}{2^{h+1}-1} \approx h-1 = \Theta\left(\ln(n)\right)$$

Number of nodes

Outline

- Binary tree
- Perfect binary tree
- Complete binary tree
- Left-child right-sibling binary tree

Background

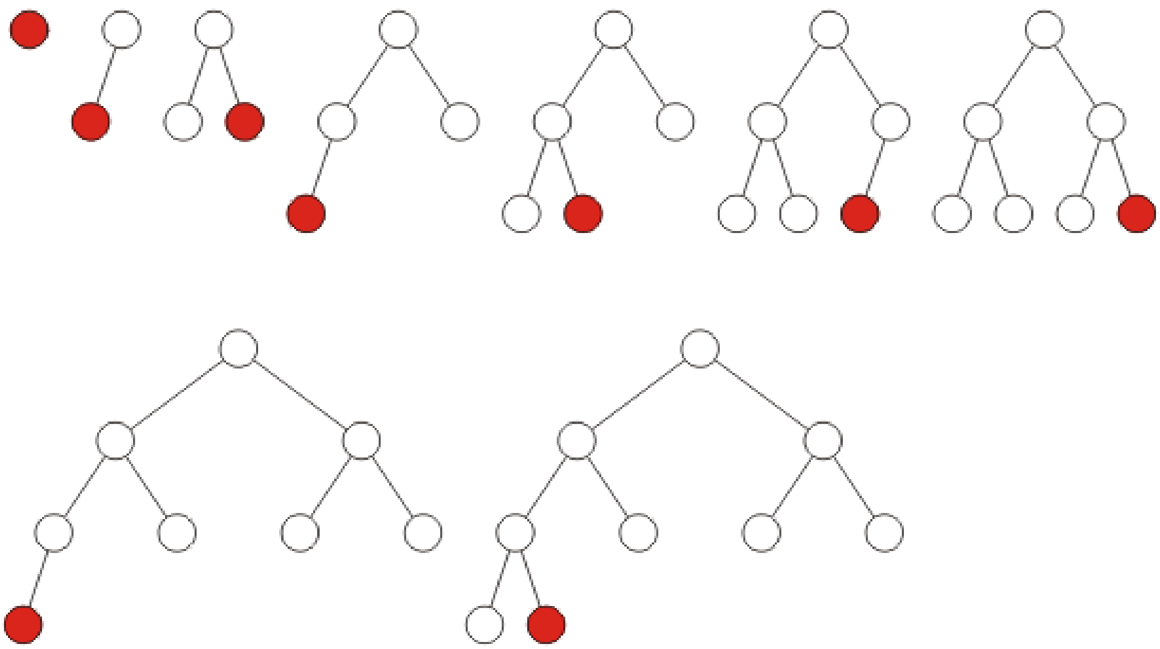
We require binary trees which are

- Similar to perfect binary trees, but
- Defined for any number of nodes

Definition

A complete binary tree filled at each depth from left to right

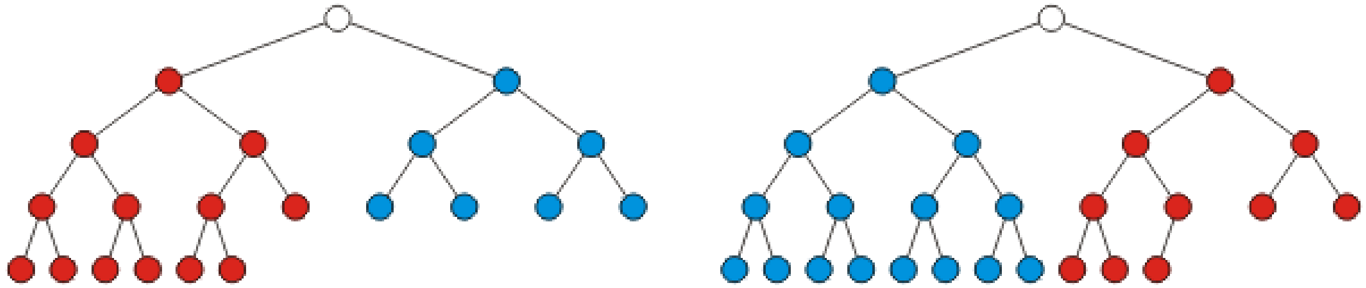
- Identical order to that of a breadth-first traversal



Recursive Definition

Recursive definition: a binary tree with a single node is a complete binary tree of height $h = 0$ and a complete binary tree of height h is a tree where either:

- The left sub-tree is a **complete tree** of height $h - 1$ and the right sub-tree is a **perfect tree** of height $h - 2$, or
- The left sub-tree is **perfect tree** with height $h - 1$ and the right sub-tree is **complete tree** with height $h - 1$



Height

Theorem

The height of a complete binary tree with n nodes is $h = \lfloor \lg(n) \rfloor$

Proof:

- Base case:
  - When $n = 1$ then $\lfloor \lg(1) \rfloor = 0$ and a tree with one node is a complete tree with height $h = 0$
- Inductive step:
  - Assume that a complete tree with n nodes has height $\lfloor \lg(n) \rfloor$
  - Must show that $\lfloor \lg(n + 1) \rfloor$ gives the height of a complete tree with $n + 1$ nodes
  - Two cases:
    - If the tree with n nodes is perfect, and
    - If the tree with n nodes is complete but not perfect

Height

Case 1 (the tree with n nodes is perfect):

- If it is a perfect tree then
  - It had $n = 2^{h+1} - 1$ nodes
  - Adding one more node must increase the height
- So the tree with $n+1$ nodes has height $h+1$ and we have:

$$\lfloor \lg(n+1) \rfloor = \lfloor \lg(2^{h+1} - 1 + 1) \rfloor = \lfloor \lg(2^{h+1}) \rfloor = h+1$$

Height

Case 2 (the tree with n nodes is complete but not perfect):

– If it is not a perfect tree then

$$2^h \le n < 2^{h+1} - 1$$

$$2^h + 1 \le n + 1 < 2^{h+1}$$

$$h < \lg\left(2^h + 1\right) \le \lg\left(n+1\right) < \lg\left(2^{h+1}\right) = h + 1$$

$$h \le \left\lfloor \lg\left(2^h + 1\right) \right\rfloor \le \left\lfloor \lg\left(n+1\right) \right\rfloor < h + 1$$

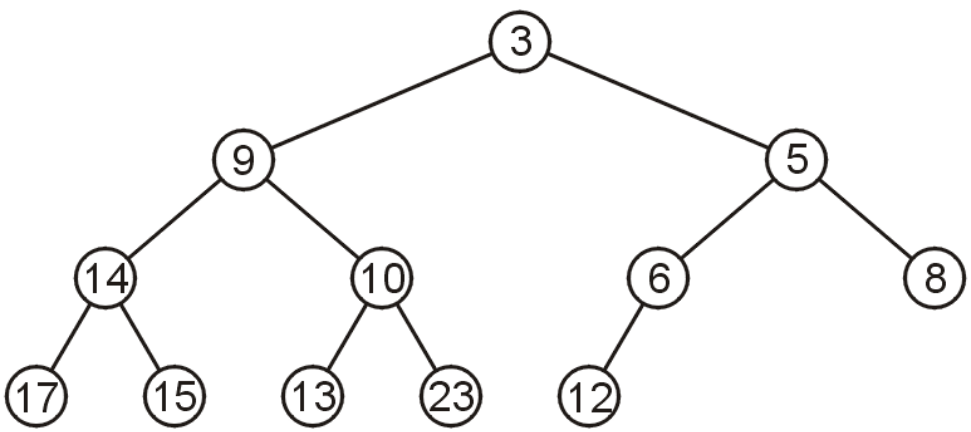
– So the tree with $n+1$ nodes has height h and we have $\lfloor \lg(n + 1) \rfloor = h$

By mathematical induction, the statement must be true for all $n \ge 1$

Array storage

We are able to store a complete tree as an array

- Traverse the tree in breadth-first order, placing the entries into the array



Array storage

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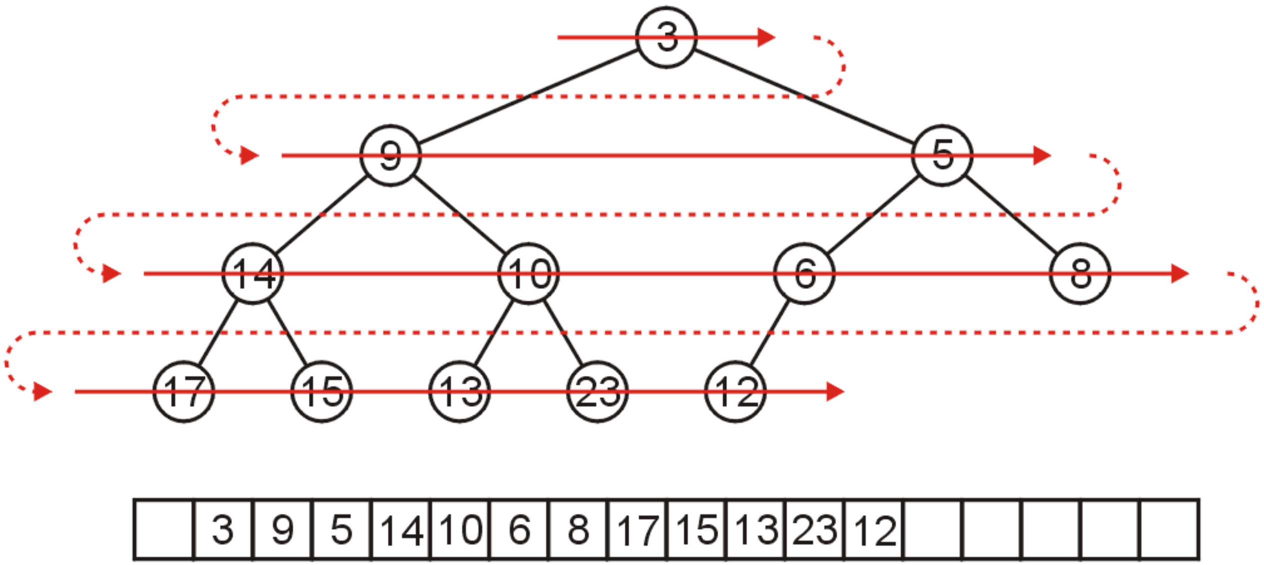
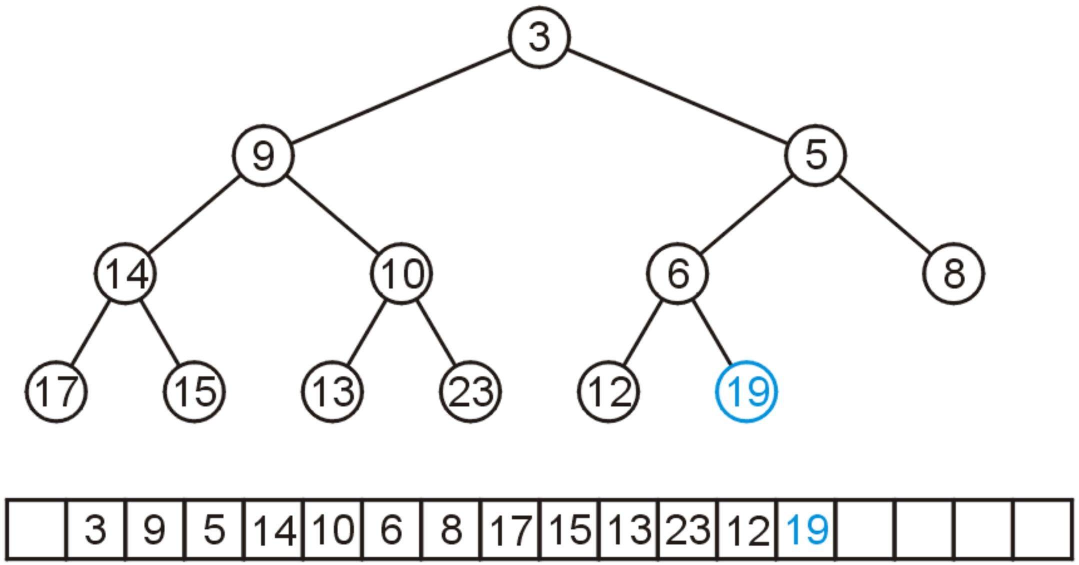


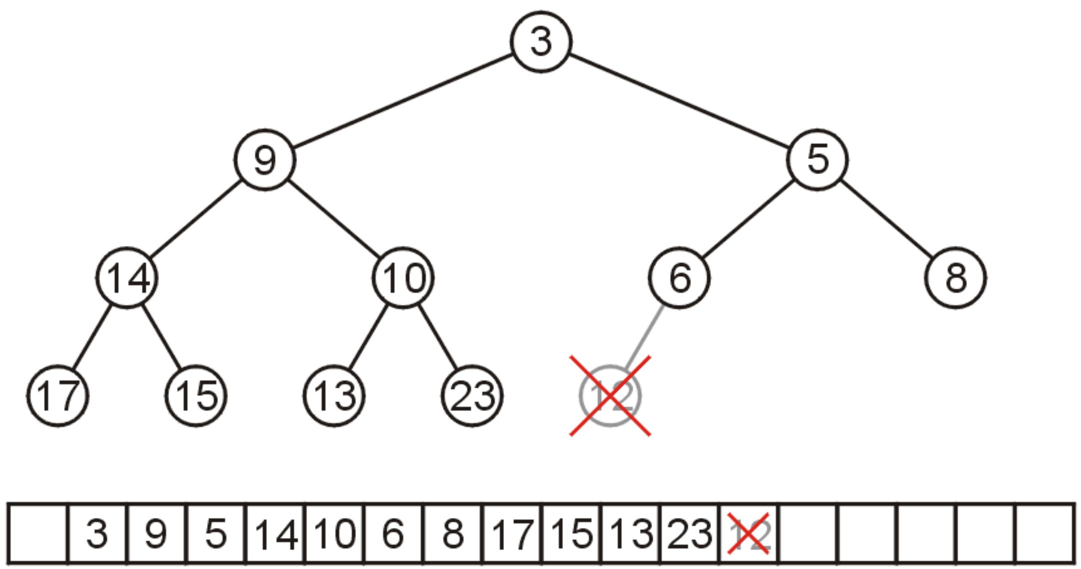
Array storage

To insert another node while maintaining the complete-binary-tree structure, we must insert into the next array location



Array storage

To remove a node while keeping the complete-tree structure, we must remove the last element in the array

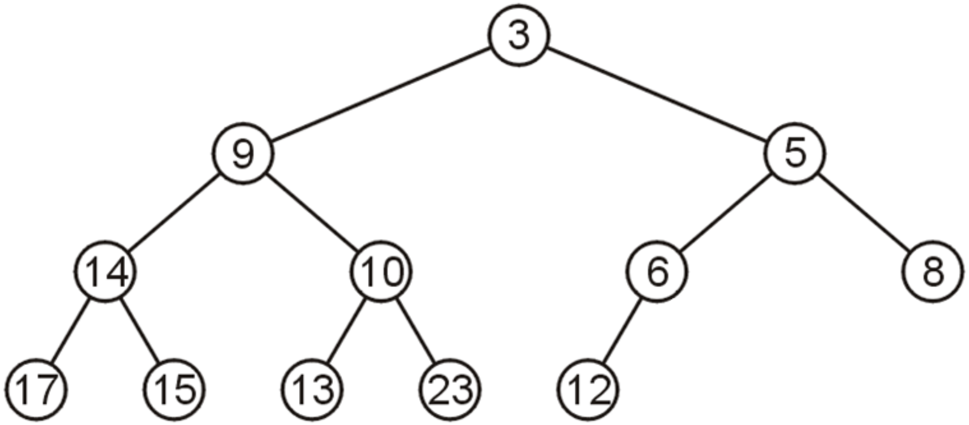


Array storage

Leaving the first entry blank yields a bonus:

– The children of the node with index k are in $2k$ and $2k + 1$

– The parent of node with index k is in $k \div 2$

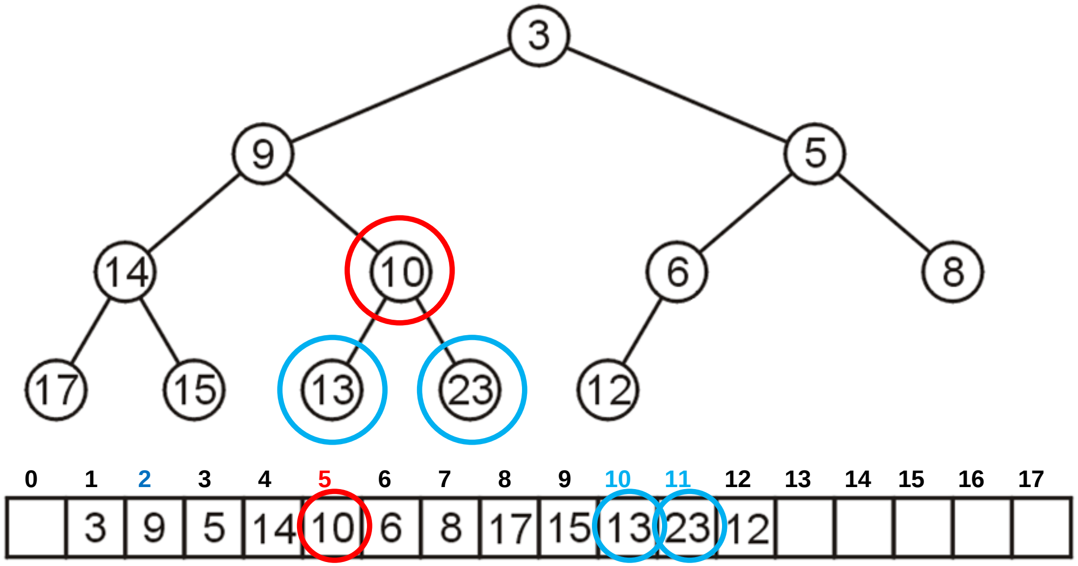


0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
	3	9	5	14	10	6	8	17	15	13	23	12					

Array storage

For example, node 10 has index 5:

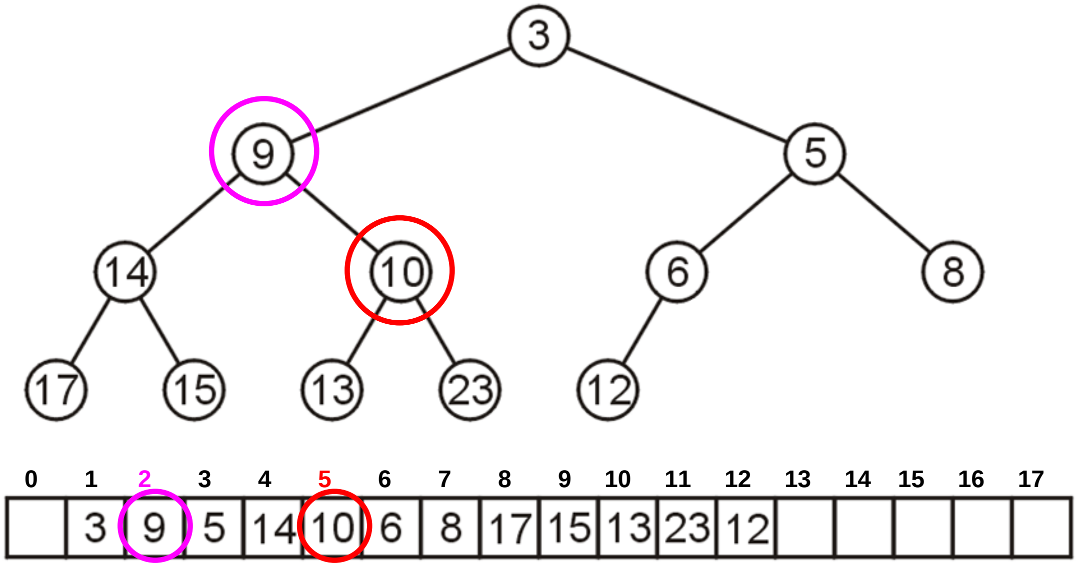
– Its children 13 and 23 have indices 10 and 11, respectively



Array storage

For example, node 10 has index 5:

- Its children 13 and 23 have indices 10 and 11, respectively
- Its parent is node 9 with index 5/2 = **2**



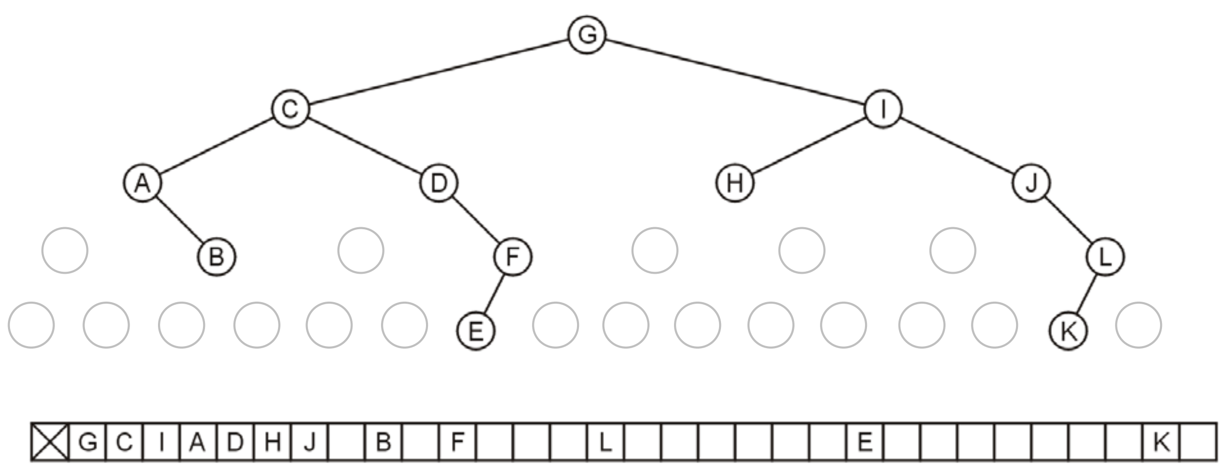
Array storage

Question: why not store any binary tree as an array in this way?

- There is a significant potential for a lot of wasted memory

Consider this tree with 12 nodes would require an array of size 32

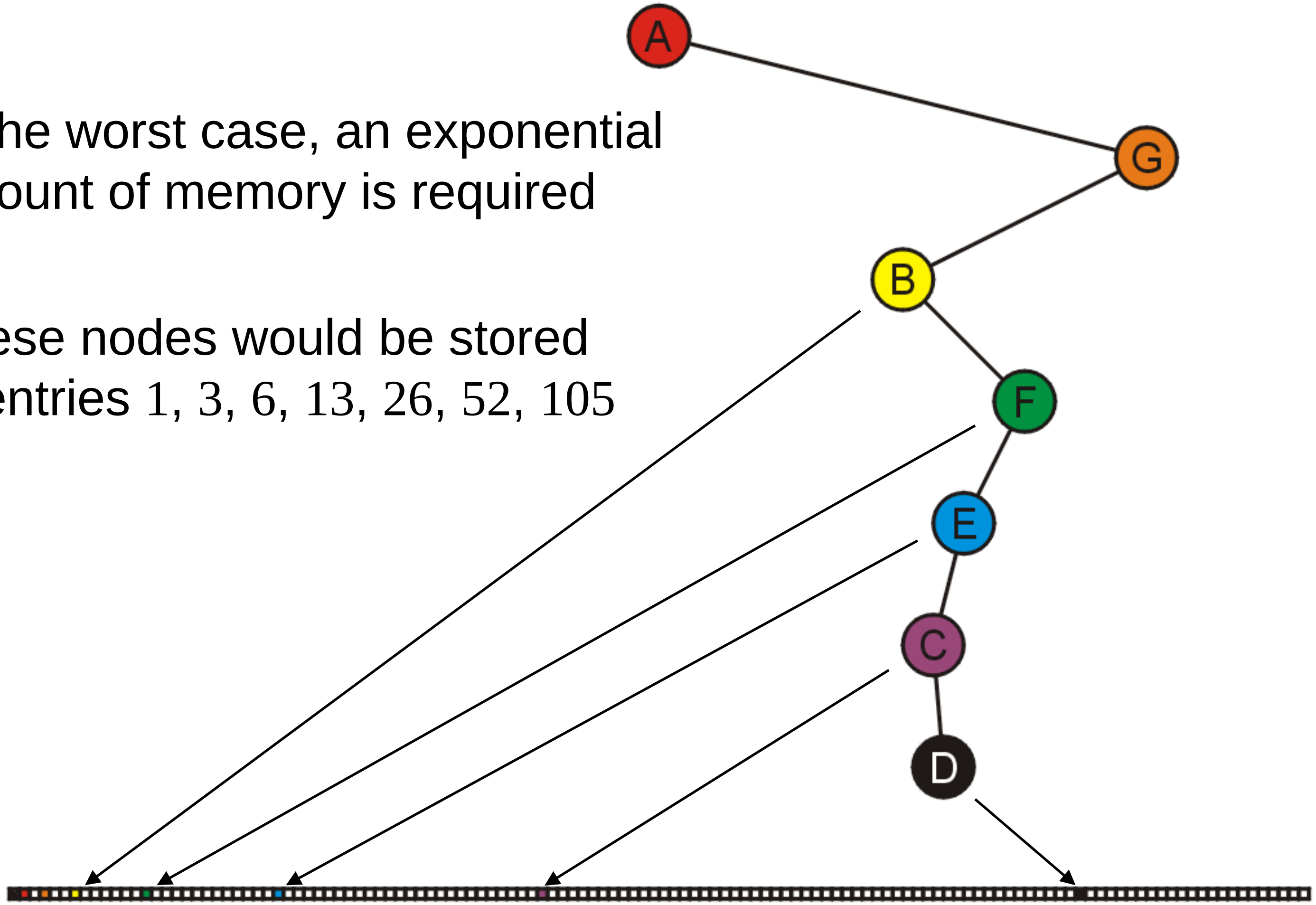
- Adding a child to node K doubles the required memory



Array storage

In the worst case, an exponential amount of memory is required

These nodes would be stored in entries 1, 3, 6, 13, 26, 52, 105



Outline

- Binary tree
- Perfect binary tree
- Complete binary tree
- Left-child right-sibling binary tree

Background

Our simple tree data structure is node-based where children are stored as a linked list

- Is it possible to store a general tree as a binary tree?

The Idea

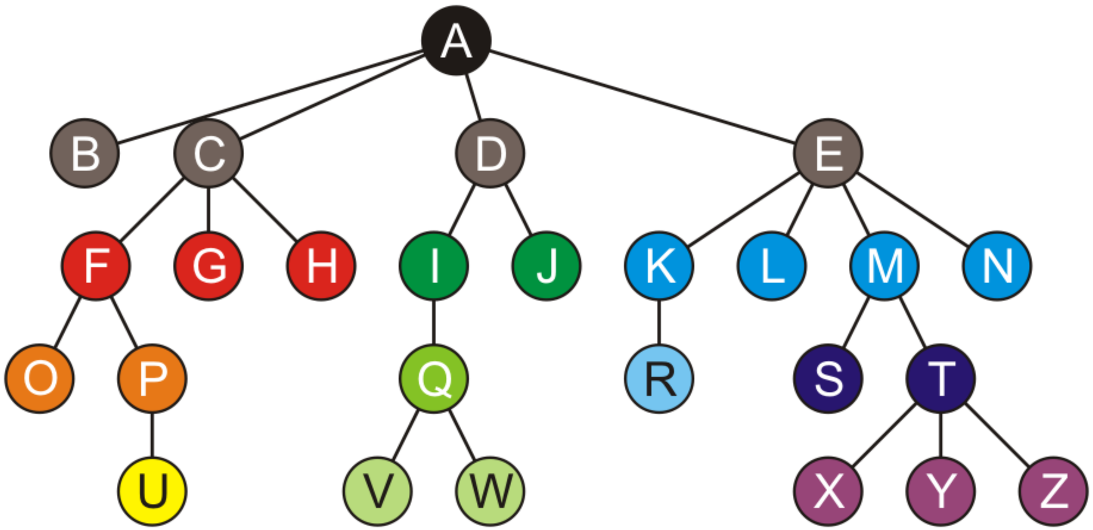
Consider the following:

- The first child of each node is its left sub-tree
- The next sibling of each node is in its right sub-tree

This is called a left-child—right-sibling binary tree

Example

Consider this general tree

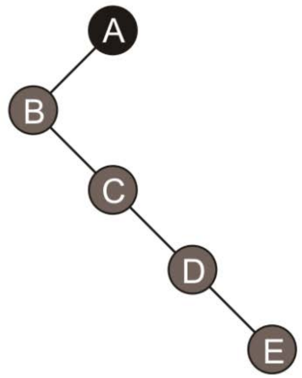


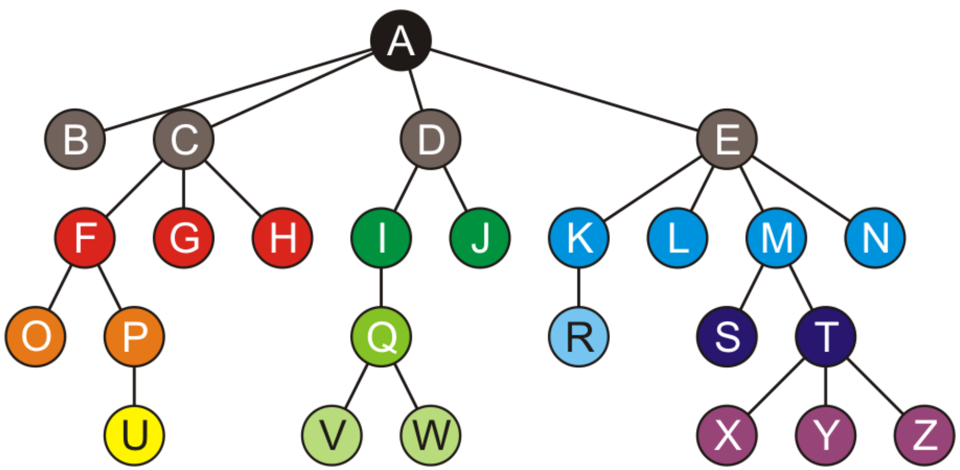
Example

B, the first child of A, is the left child of A

For the three siblings C, D, E:

- C is the right sub-tree of B
- D is the right sub-tree of C
- E is the right sub-tree of D





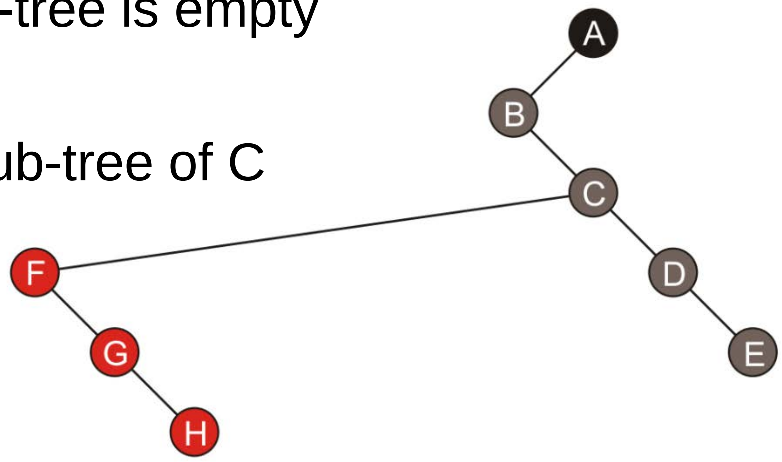
Example

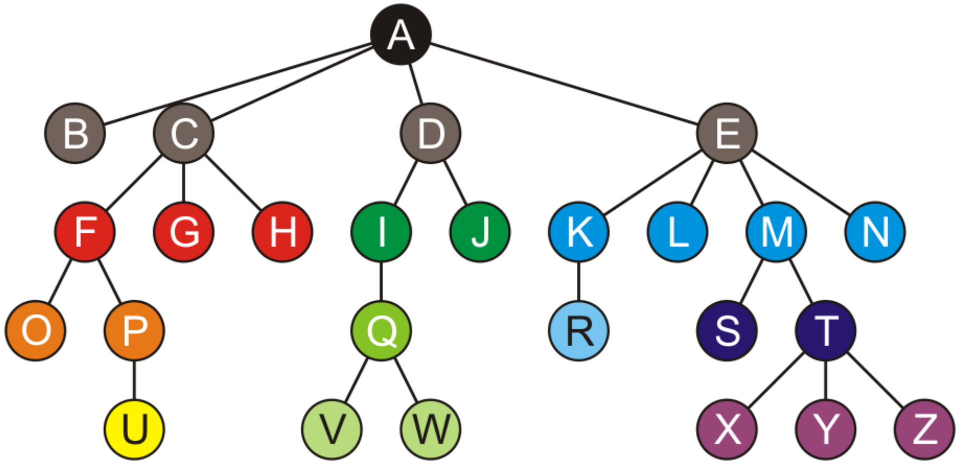
B has no children, so it’s left sub-tree is empty

F, the first child of C, is the left sub-tree of C

For the next two siblings:

- G is the right sub-tree of F
- H is the right sub-tree of G

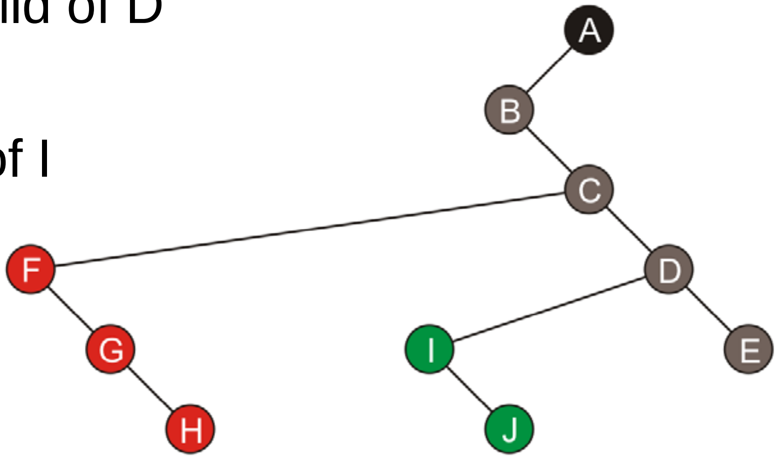


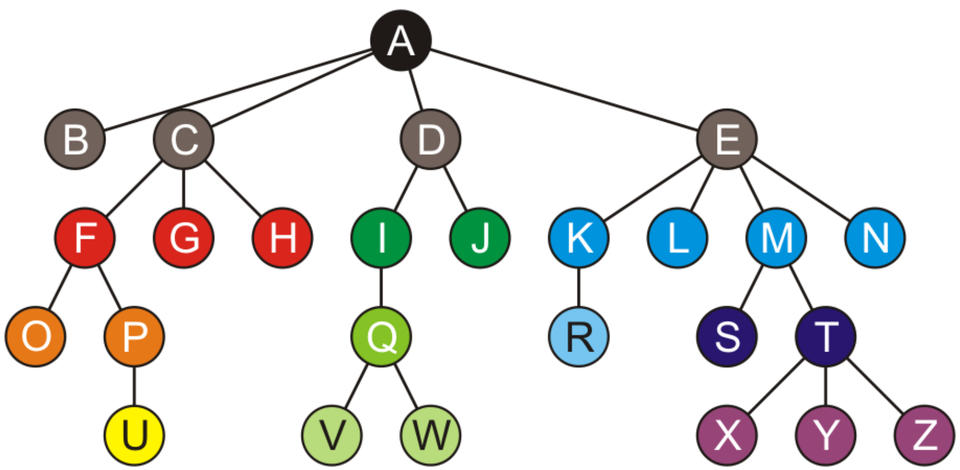


Example

I, the first child of D, is the left child of D

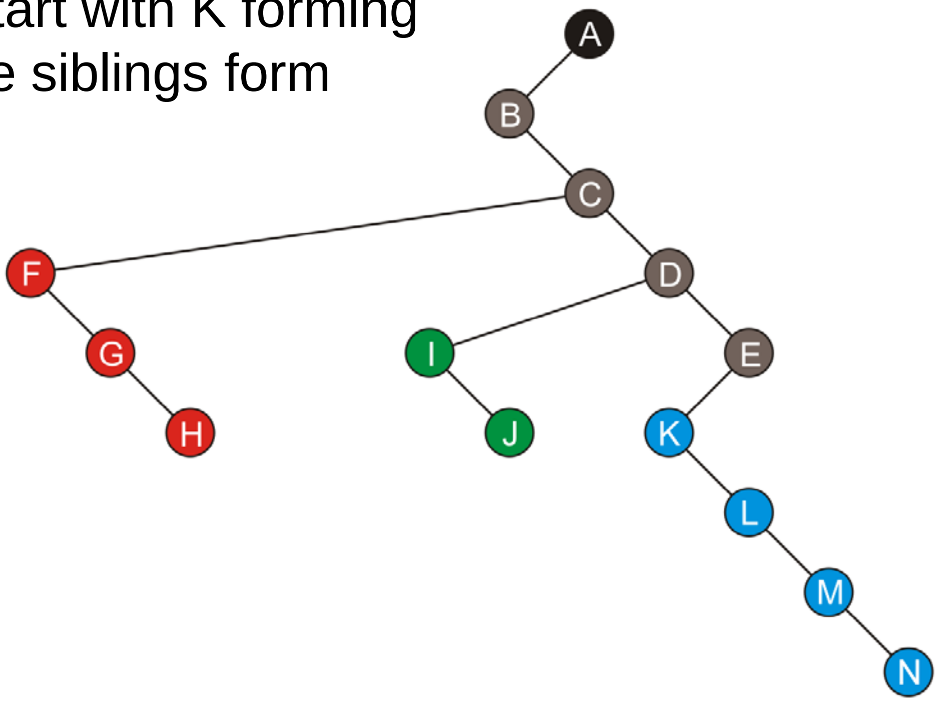
Its sibling J is the right sub-tree of I

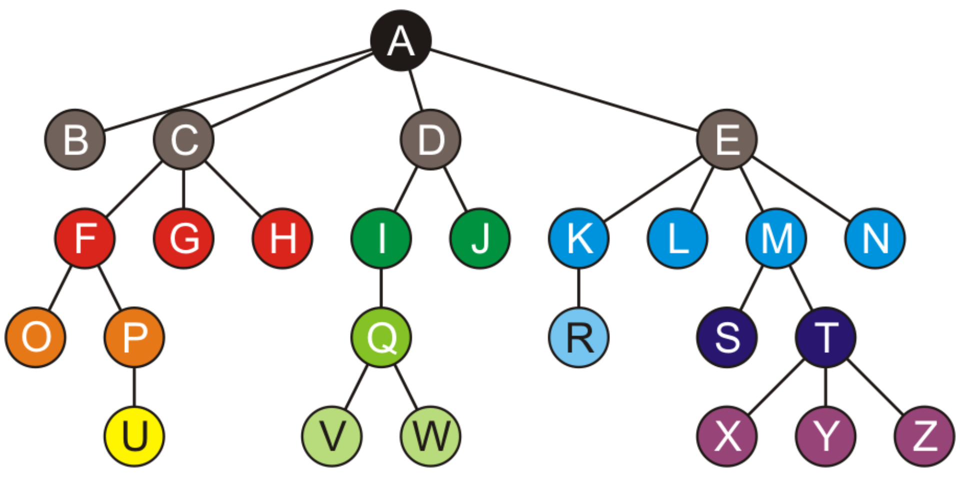




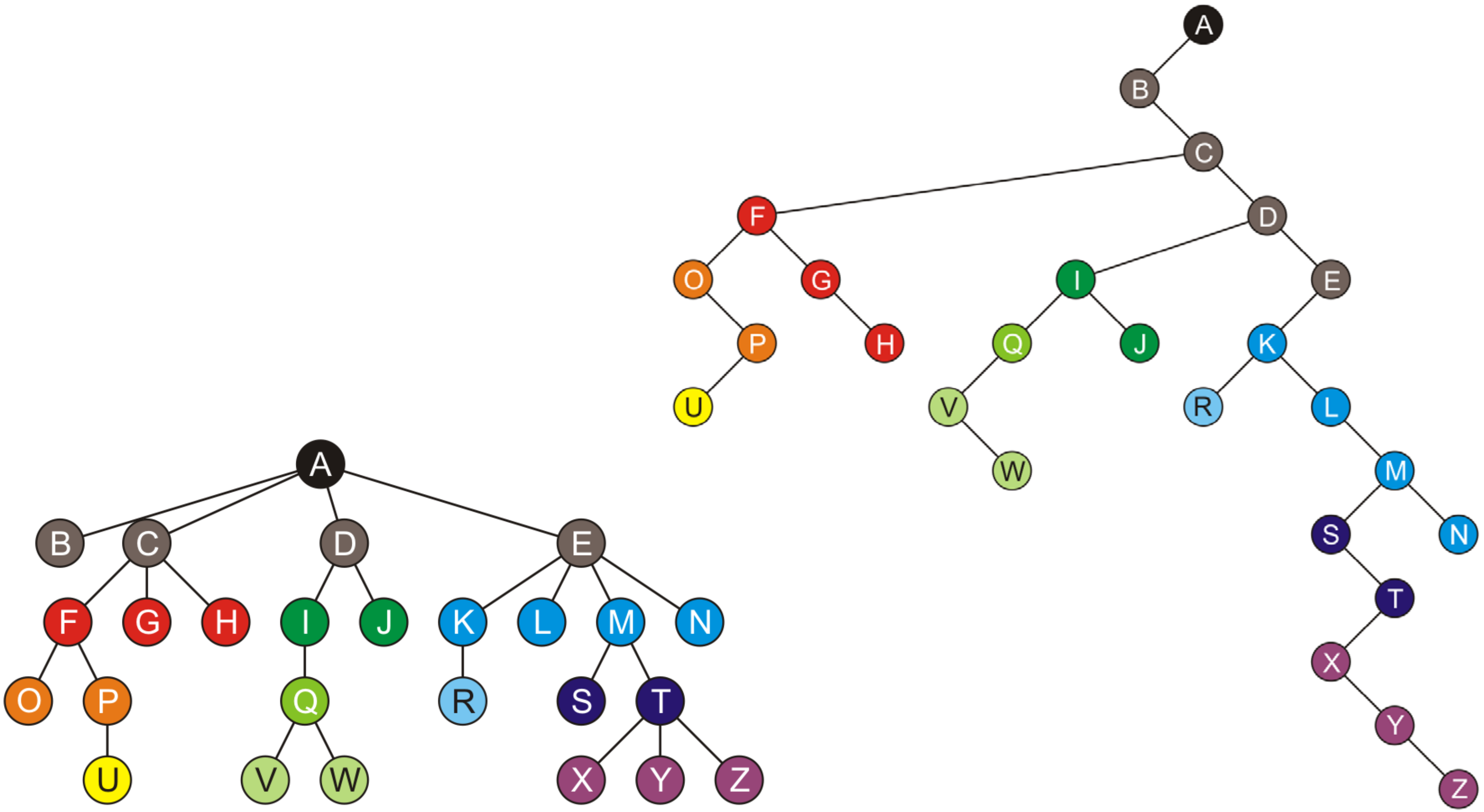
Example

Similarly, the four children of E start with K forming the left sub-tree of E and its three siblings form a chain along the right sub-trees



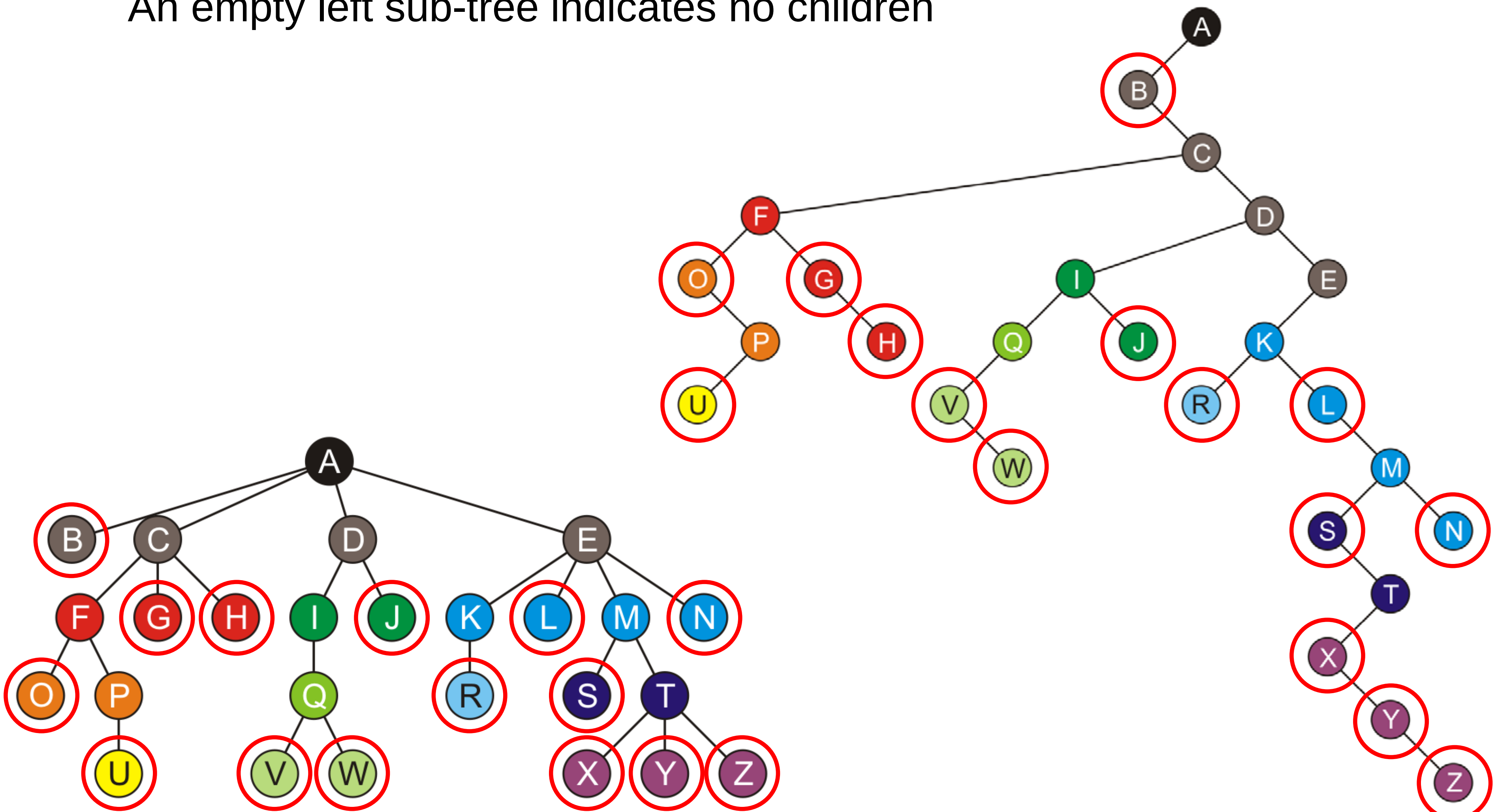


Example



Example

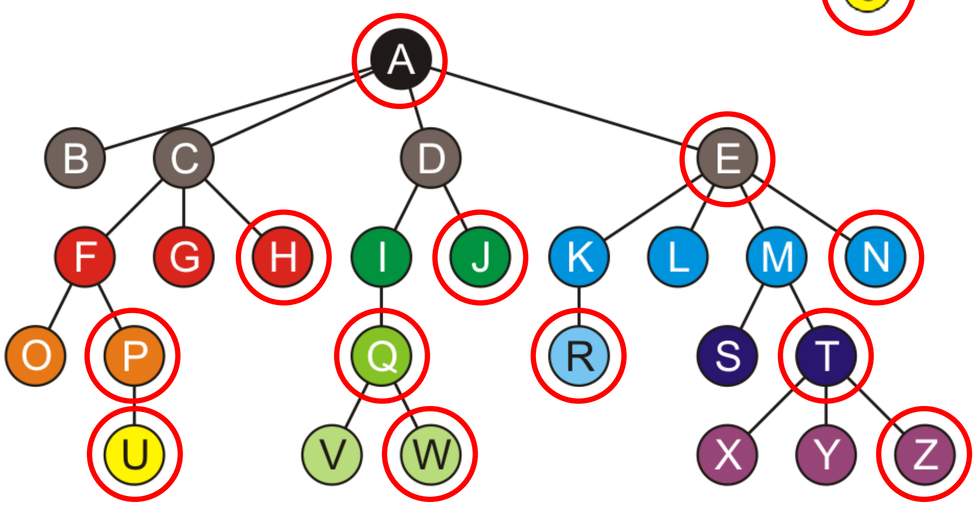
An empty left sub-tree indicates no children

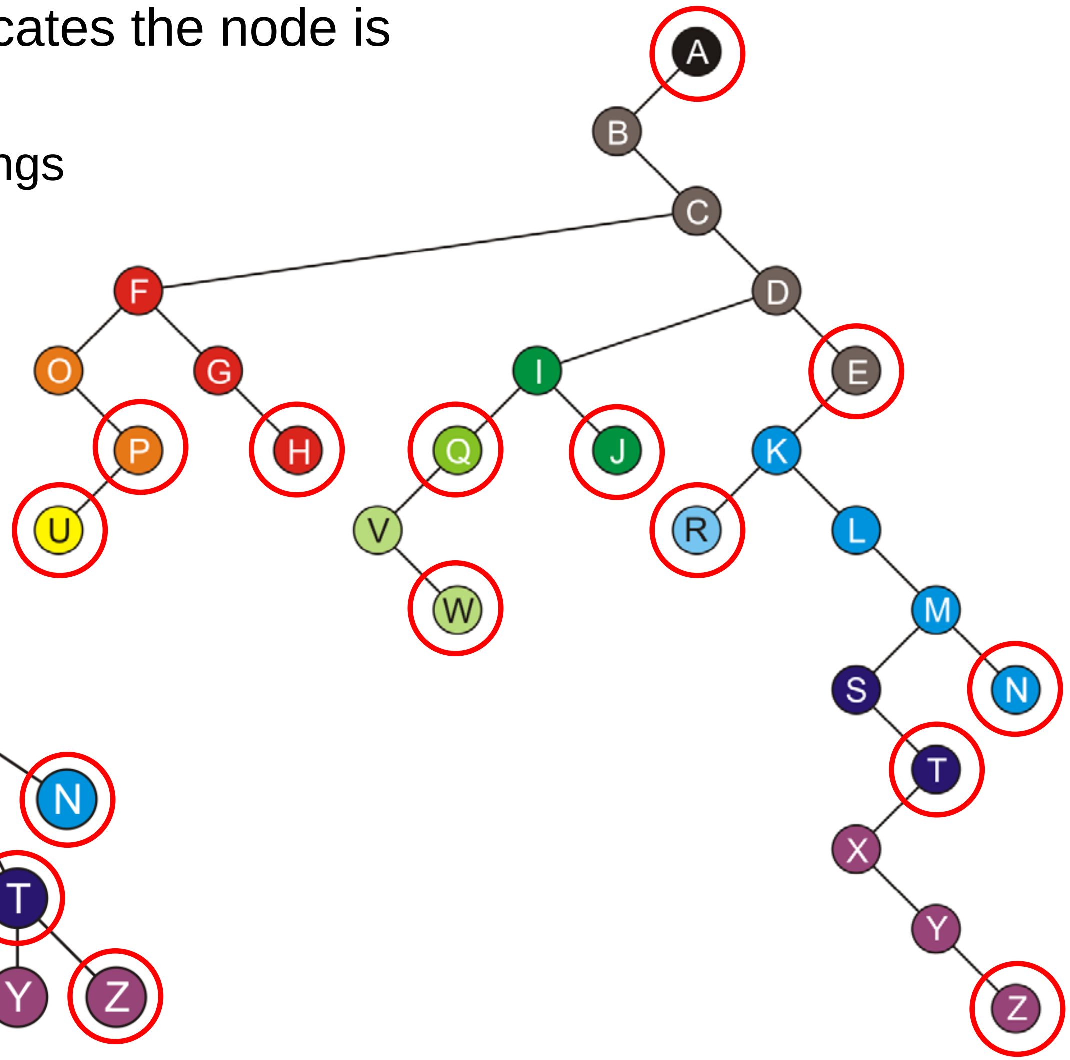


Example

An empty right sub-tree indicates the node is the last of its siblings

- The root node, has no siblings





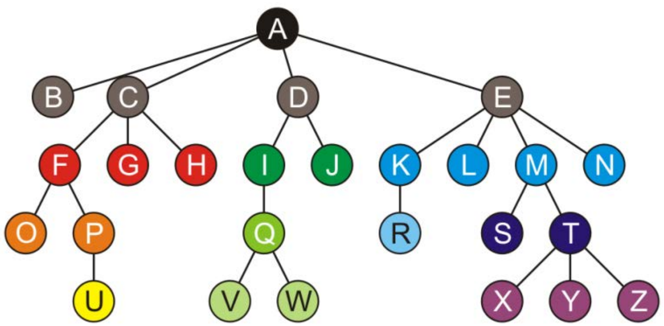
Transformation

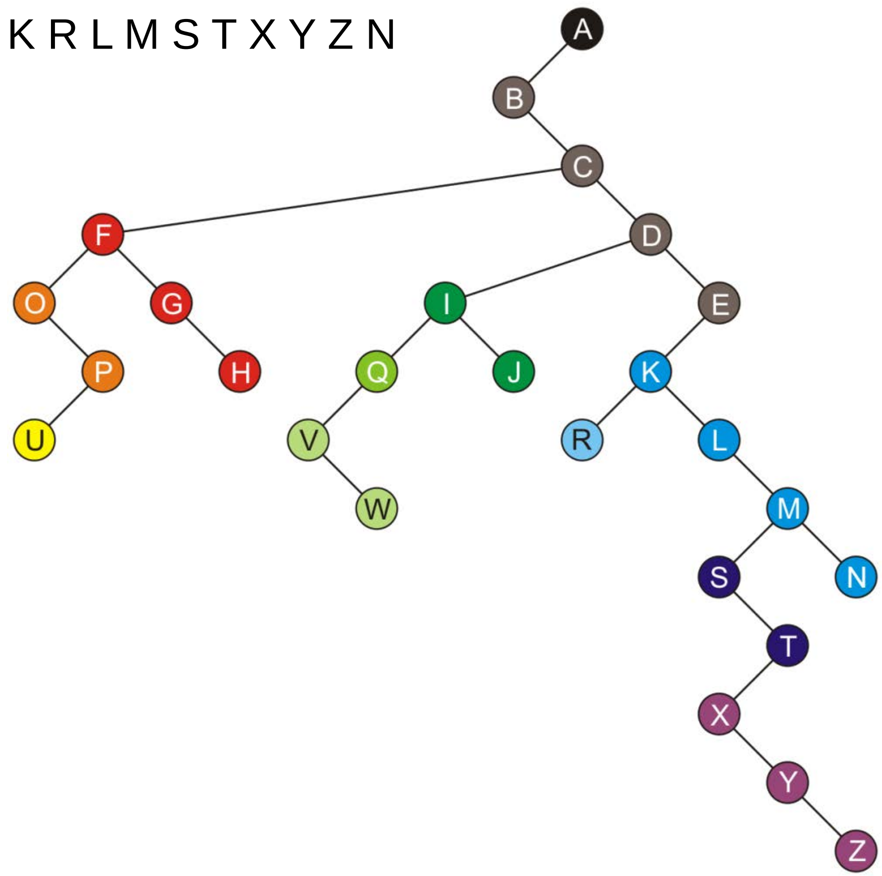
The transformation of a general tree into a left-child right-sibling binary tree has been called the Knuth transform

Traversals

A pre-order traversal of the original tree is identical to the pre-order traversal of the Knuth transform

A B C F O P U G H D I Q V W J E K R L M S T X Y Z N

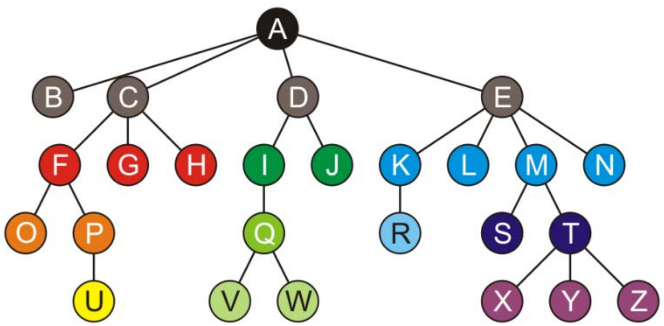


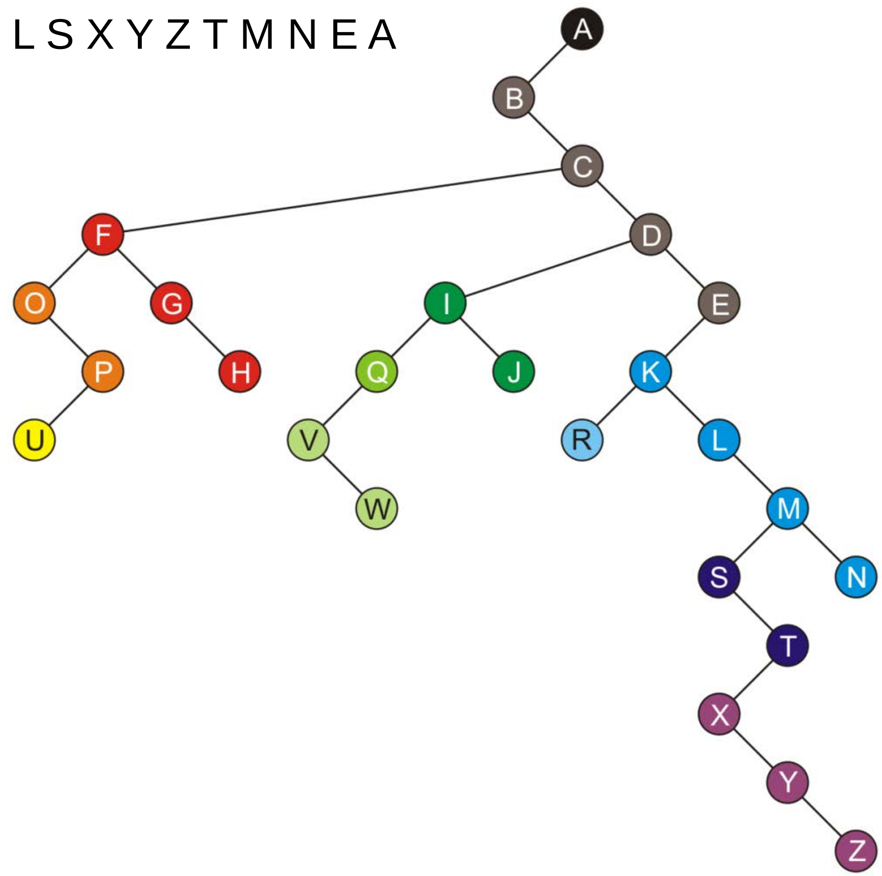


Traversals

A post-order traversal of the original tree is identical to the in-order traversal of the Knuth transform

B O U P F G H C V W Q I J D R K L S X Y Z T M N E A





Implementation

The class is similar to that of a binary tree

```
template <typename Type>
class LCRS_tree {
    private:
        Type element;
        LCRS_tree *first_child_tree;
        LCRS_tree *next_sibling_tree;

    public:
        LCRS_tree();
        LCRS_tree *first_child();
        LCRS_tree *next_sibling();
        // ...
};
```

Implementation

The implementation of various functions now differs

```
template <typename Type>
int LCRS_tree<Type>::degree() const {
    int count = 0;

    for (
        LCRS_tree<Type> *ptr = first_child();
        ptr != nullptr;
        ptr = ptr->next_sibling()
    ) {
        ++count;
    }

    return count;
}
```

Implementation

The implementation of various functions now differs

```
template <typename Type>
bool LCRS_tree<Type>::is_leaf() const {
    return ( first_child() == nullptr );
}
```

Implementation

The implementation of various functions now differs

```
template <typename Type>
LCRS_tree<Type> *LCRS_tree<Type>::child( int n ) const {
    if ( n < 0 || n >= degree() ) {
        return nullptr;
    }

    LCRS_tree<Type> *ptr = first_child();

    for ( int i = 0; i < n; ++i ) {
        ptr = ptr->next_sibling();
    }

    return ptr;
}
```

Implementation

The implementation of various functions now differs

```
template <typename Type>
void LCRS_tree<Type>::append( Type const &obj ) {
    if ( first_child() == nullptr ) {
        first_child_tree = new LCRS_tree<Type>( obj );
    } else {
        LCRS_tree<Type> *ptr = first_child();

        while ( ptr->next_sibling() != nullptr ) {
            ptr = ptr->next_sibling();
        }

        ptr->next_sibling_tree = new LCRS_tree<Type>( obj );
    }
}
```

Implementation

The implementation of various functions now differs

– The size doesn't care that this is a general tree…

```
template <typename Type>
int LCRS_tree<Type>::size() const {
    return 1
       + (  first_child() == nullptr ? 0 :  first_child()->size() )
       + ( next_sibling() == nullptr ? 0 : next_sibling()->size() );
}
```

Implementation

The implementation of various functions now differs

– The height member function is closer to the original implementation

```
template <typename Type>
int LCRS_tree<Type>::height() const {
    int h = 0;

    for (
        LCRS_tree<Type> *ptr = first_child();
        ptr != nullptr;
        ptr = ptr->next_sibling()
    ) {
        h = std::max( h, 1 + ptr->height() );
    }

    return h;
}
```

Summary

- Binary tree
  - Each node has two children
  - In-order traversal
- Perfect binary tree
  - Number of nodes, height, number of leaf nodes, average depth
- Complete binary tree
  - Height, array storage
- Left-child right-sibling binary tree