Stack

Textbook Ch 10.1

Outline

- Stack ADT
- Implementation
- Example applications

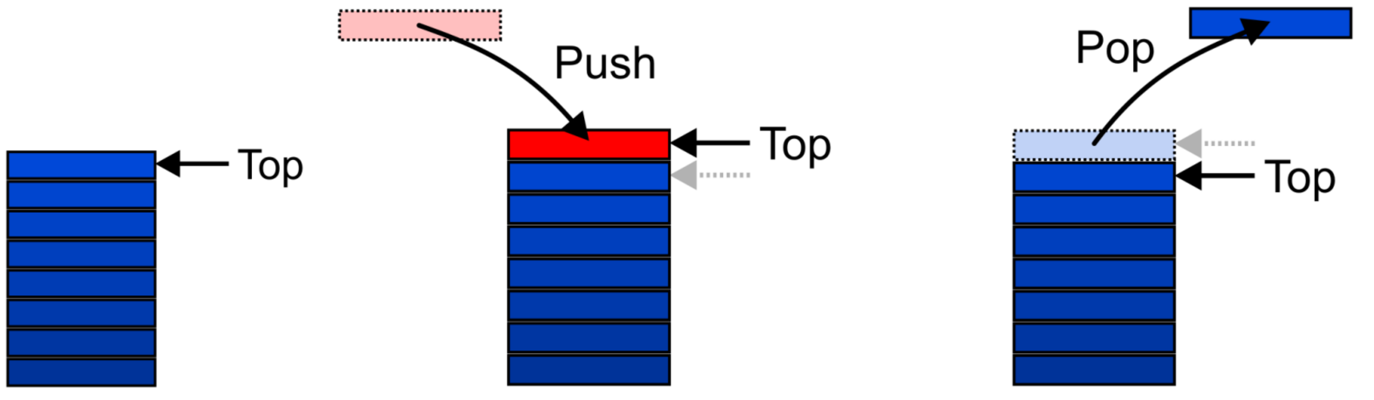
Stack ADT

- Uses a explicit linear ordering
- Two principal operations
  - *Push*: insert an object onto the top of the stack
  - *Pop*: erase the object on the top of the stack

Stack ADT

Also called a *last-in–first-out* (LIFO) behaviour

- Graphically, we may view these operations as follows:



Applications

Numerous applications:

- Parsing code:
  - Matching parenthesis
  - XML (e.g., XHTML)
- Tracking function calls
- Dealing with undo/redo operations
- Reverse-Polish calculators
- Assembly language

Outline

- Stack ADT
- Implementation
- Example applications

Implementations

We will look at two implementations of stacks:

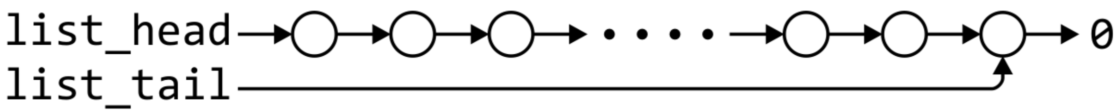
– Singly linked lists
– One-ended arrays

The optimal asymptotic run time of any algorithm is $\mathbf{\Theta}(1)$

– The run time of the algorithm is independent of the number of objects being stored in the container

Linked-List Implementation

Operations at the front of a singly linked list are all $\Theta(1)$



	Front/1^{st}	Back/n^{th}
Find	$\Theta(1)$	$\Theta(1)$
Insert	$\Theta(1)$	$\Theta(1)$
Erase	$\Theta(1)$	$\Theta(n)$

The desired behaviour of an Abstract Stack may be reproduced by performing all operations at the front

`Single_list` Definition

The definition of single list class:

```
template <typename Type>
class Single_list {
    public:
        Single_list();
         ~Single_list();

         int size() const;
         bool empty() const;
         Type front() const;
         Type back() const;
         Single_node<Type> *head() const;
         Single_node<Type> *tail() const;
         int count( Type const & ) const;

         void push_front( Type const & );
         void push_back( Type const & );
         Type pop_front();
         int erase( Type const & );
};
```

Stack-as-List Class

The stack class using a singly linked list has a single private member variable:

```
template <typename Type>
class Stack {
    private:
        Single_list<Type> list;
    public:
        bool empty() const;
        Type top() const;
        void push( Type const & );
        Type pop();
};
```

Stack-as-List Class

The empty and push functions just call the appropriate functions of the `Single_list` class

```
template <typename Type>
bool Stack<Type>::empty() const {
    return list.empty();
}

template <typename Type>
void Stack<Type>::push( Type const &obj ) {
    list.push_front( obj );
}
```

Stack-as-List Class

The top and pop functions, however, must check the boundary case:

```
template <typename Type>
Type Stack<Type>::top() const {
    if ( empty() ) {
        throw underflow();
    }

    return list.front();
}
```

```
template <typename Type>
Type Stack<Type>::pop() {
    if ( empty() ) {
        throw underflow();
    }

    return list.pop_front();
}
```

Stack-as-List Class

A constructor and destructor is not needed

- Because `list` is declared, the compiler will call the constructor of the `Single_list` class when the `Stack` is constructed

Array Implementation

For one-ended arrays, all operations at the back are $\Theta(1)$

	Front/1st	**Back/nth**
Find	$\Theta(1)$	$\Theta(1)$
Insert	$\Theta(n)$	$\Theta(1)$
Erase	$\Theta(n)$	$\Theta(1)$

Stack-as-Array Class

```
template <typename Type>
class Stack {
    private:
        int stack_size; //number of objects in the stack
        int array_capacity; //capacity of the array
        Type *array;
    public:
        Stack( int = 10 );
        ~Stack();
        bool empty() const;
        Type top() const;
        void push( Type const & );
        Type pop();
};
```

Constructor

The class is only storing the address of the array

- We must allocate memory for the array and initialize the member variables

```
#include <algorithm>
// ...

template <typename Type>
Stack<Type>::Stack( int n ):
stack_size( 0 ),
array_capacity( std::max( 1, n ) ),
array( new Type[array_capacity] ) {
    // Empty constructor
}
```

Destructor

The destructor must release the memory for the array

```
template <typename Type>
Stack<Type>::~Stack() {
    delete [] array;
}
```

Empty

The stack is empty if the stack size is zero:

```
template <typename Type>
bool Stack<Type>::empty() const {
    return ( stack_size == 0 );
}
```

Top

If there are n objects in the stack, the last is located at index $n - 1$

```
template <typename Type>
Type Stack<Type>::top() const {
    if ( empty() ) {
        throw underflow();
    }

    return array[stack_size - 1];
}
```

Pop

Removing an object simply involves reducing the size

- By decreasing the size, the previous top of the stack is now at the location `stack_size`

```
template <typename Type>
Type Stack<Type>::pop() {
    if ( empty() ) {
        throw underflow();
    }

    --stack_size;
    return array[stack_size];
}
```

Push

Pushing an object onto the stack can only be performed if the array is not full

```
template <typename Type>
void Stack<Type>::push( Type const &obj ) {
    if ( stack_size == array_capacity ) {
        throw overflow();     Best solution?????
    }

    array[stack_size] = obj;
    ++stack_size;
}
```

Array Capacity

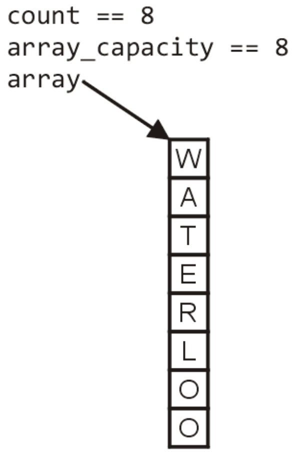
The best option is to increase the array capacity

If we increase the array capacity, the question is:

- How much?
- By a constant? `array_capacity += c;`
- By a multiple? `array_capacity *= c;`

Array Capacity

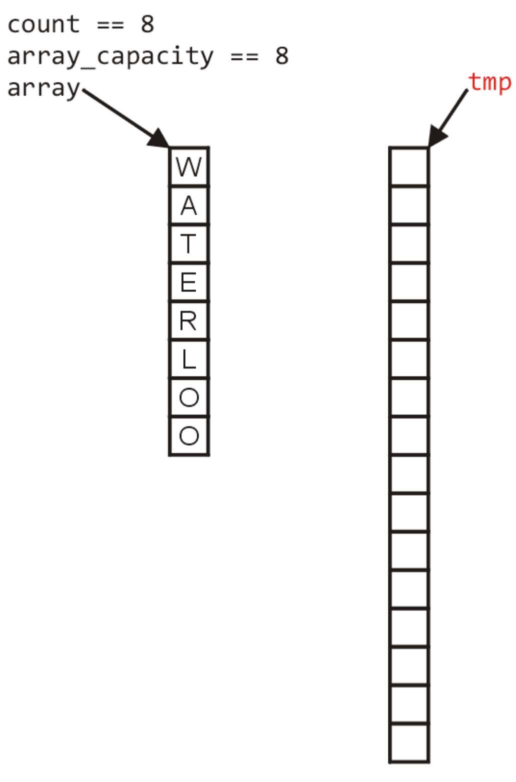
First, let us visualize what must occur to allocate new memory



Array Capacity

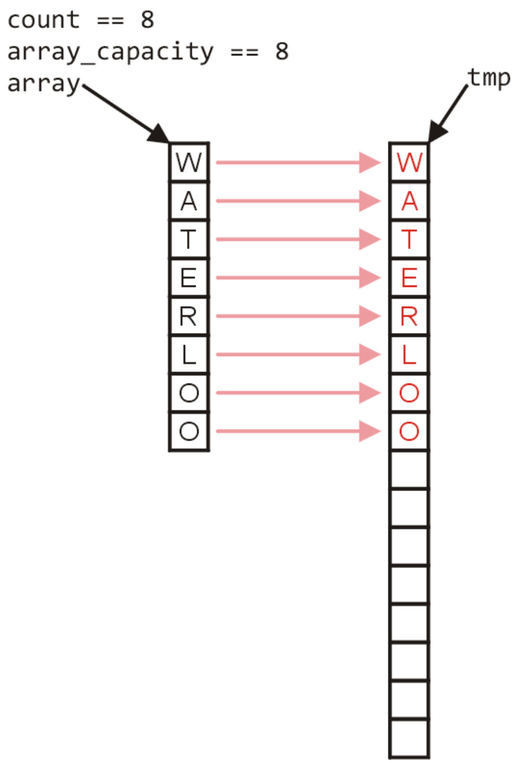
First, this requires a call to `new Type[`*N*`]` where *N* is the new capacity

- We must have access to this so we must store the address returned by new in a local variable, say `tmp`



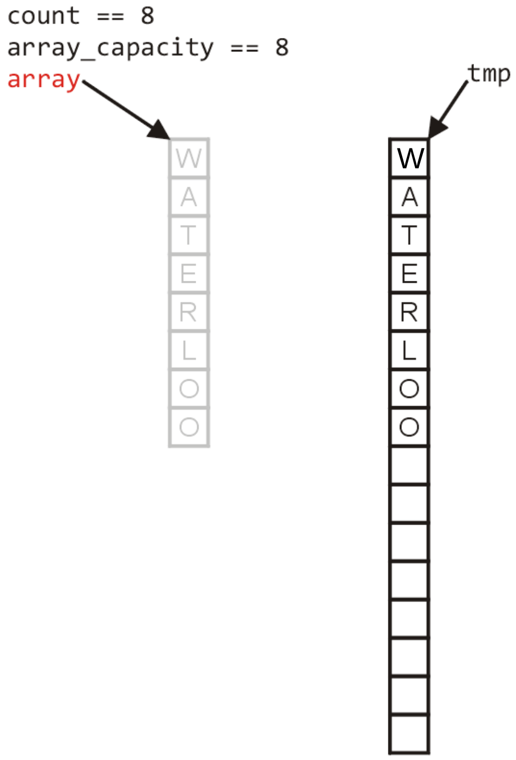
Array Capacity

Next, the values must be copied over



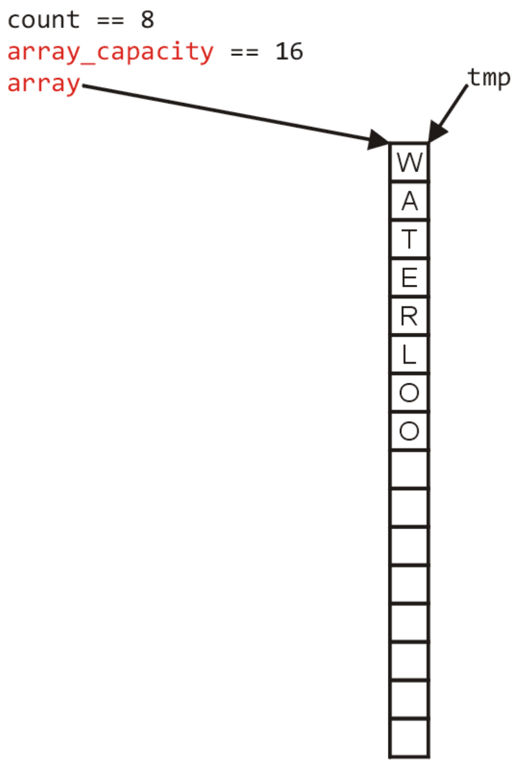
Array Capacity

The memory for the original array must be deallocated



Array Capacity

Finally, the appropriate member variables must be reassigned



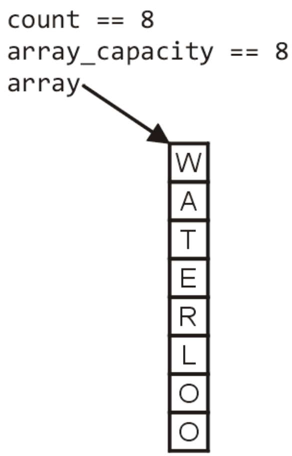
Array Capacity

The implementation:

```
void double_capacity() {



}
```

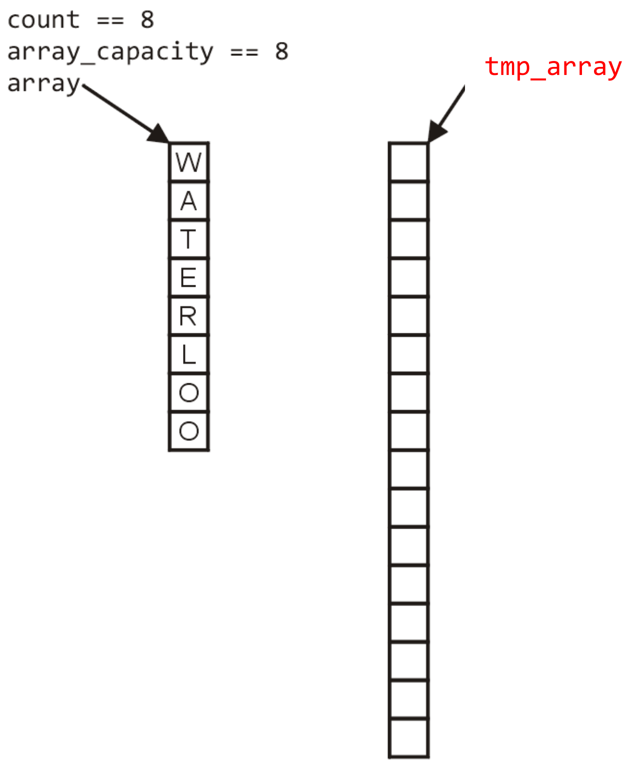


Array Capacity

The implementation:

```
void double_capacity() {
    Type *tmp_array = new Type[2*array_capacity];

}
```



Array Capacity

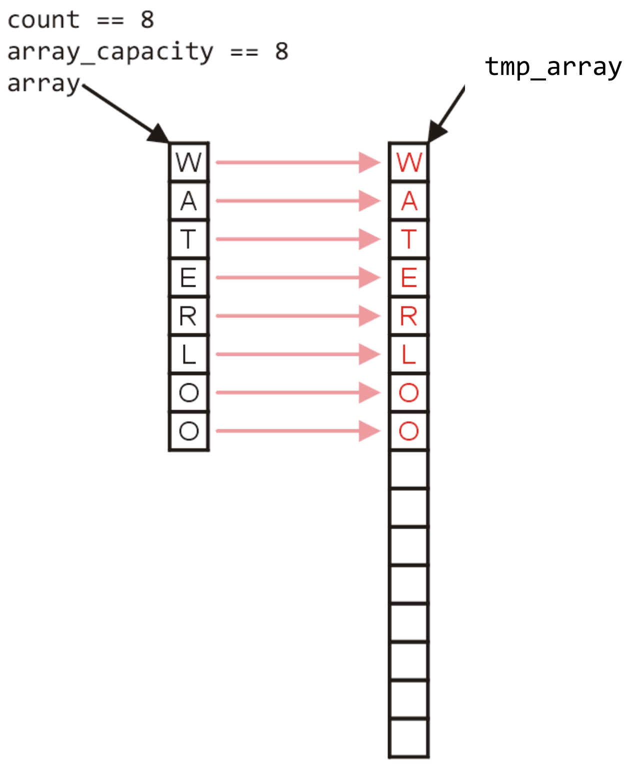
The implementation:

```
void double_capacity() {
    Type *tmp_array = new Type[2*array_capacity];

    for ( int i = 0; i < array_capacity; ++i ) {
        tmp_array[i] = array[i];
    }



}
```



Array Capacity

The implementation:

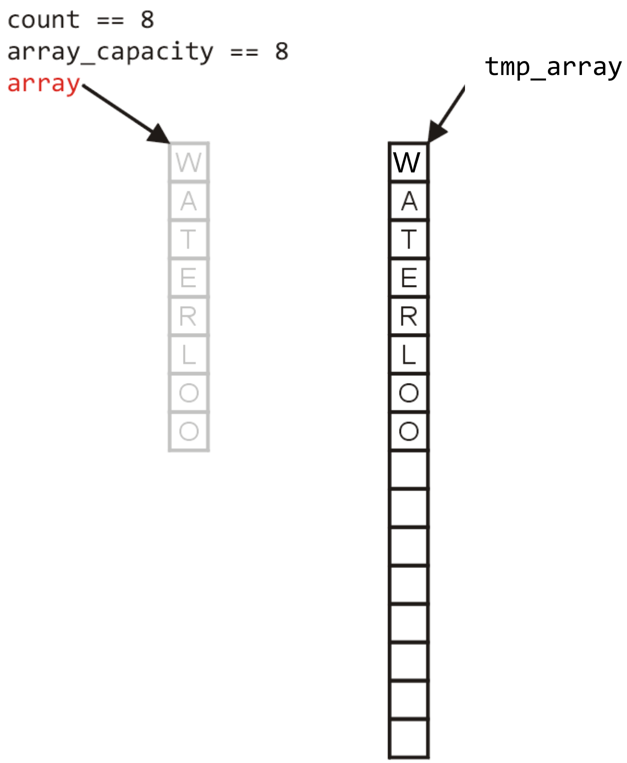
```
void double_capacity() {
    Type *tmp_array = new Type[2*array_capacity];

    for ( int i = 0; i < array_capacity; ++i ) {
        tmp_array[i] = array[i];
    }

    delete [] array;



}
```



Array Capacity

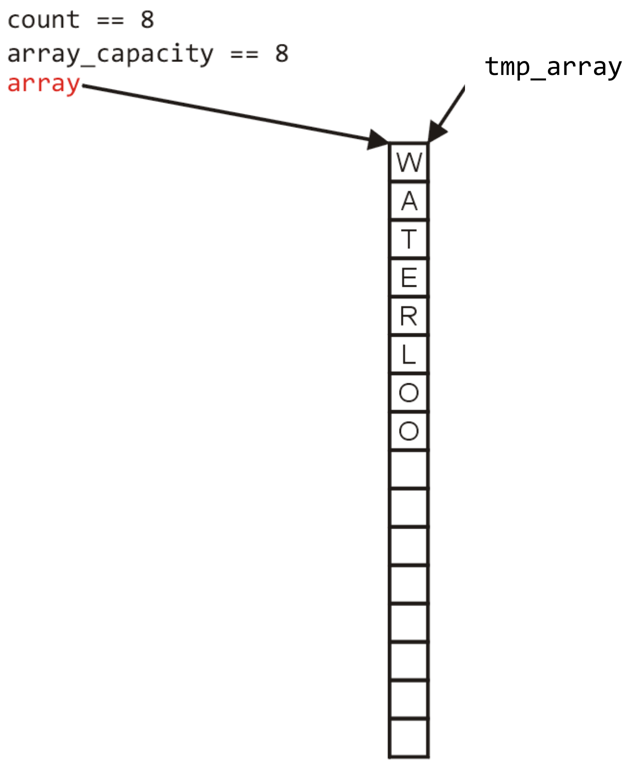
The implementation:

```
void double_capacity() {
    Type *tmp_array = new Type[2*array_capacity];

    for ( int i = 0; i < array_capacity; ++i ) {
        tmp_array[i] = array[i];
    }

    delete [] array;
    array = tmp_array;


}
```



Array Capacity

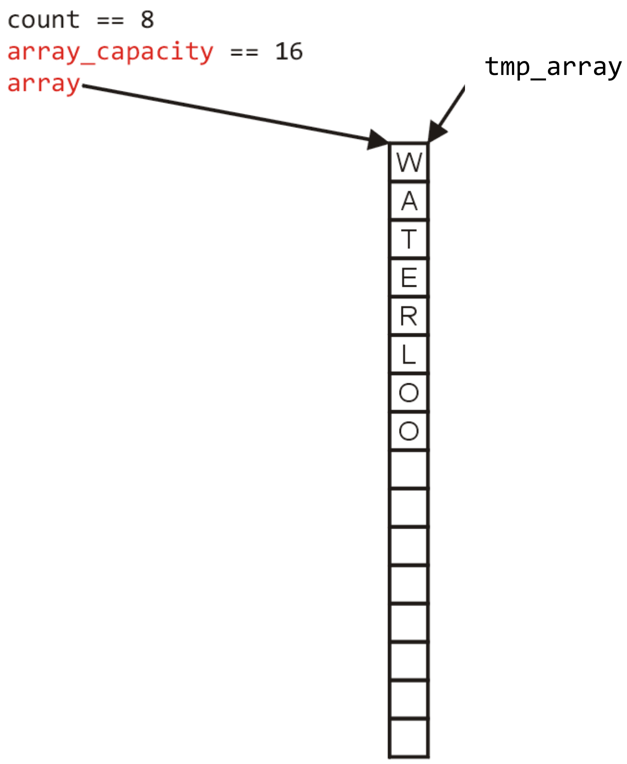
The implementation:

```
void double_capacity() {
    Type *tmp_array = new Type[2*array_capacity];

    for ( int i = 0; i < array_capacity; ++i ) {
        tmp_array[i] = array[i];
    }

    delete [] array;
    array = tmp_array;

    array_capacity *= 2;
}
```



Array Capacity

Back to the original question:

- How much do we change the capacity?
- Add a constant?
- Multiply by a constant?

First, we recognize that any time that we `push` onto a full stack, this requires n copies and the run time is $\Theta(n)$

Therefore, `push` is usually $\Theta(1)$ except when new memory is required

Array Capacity

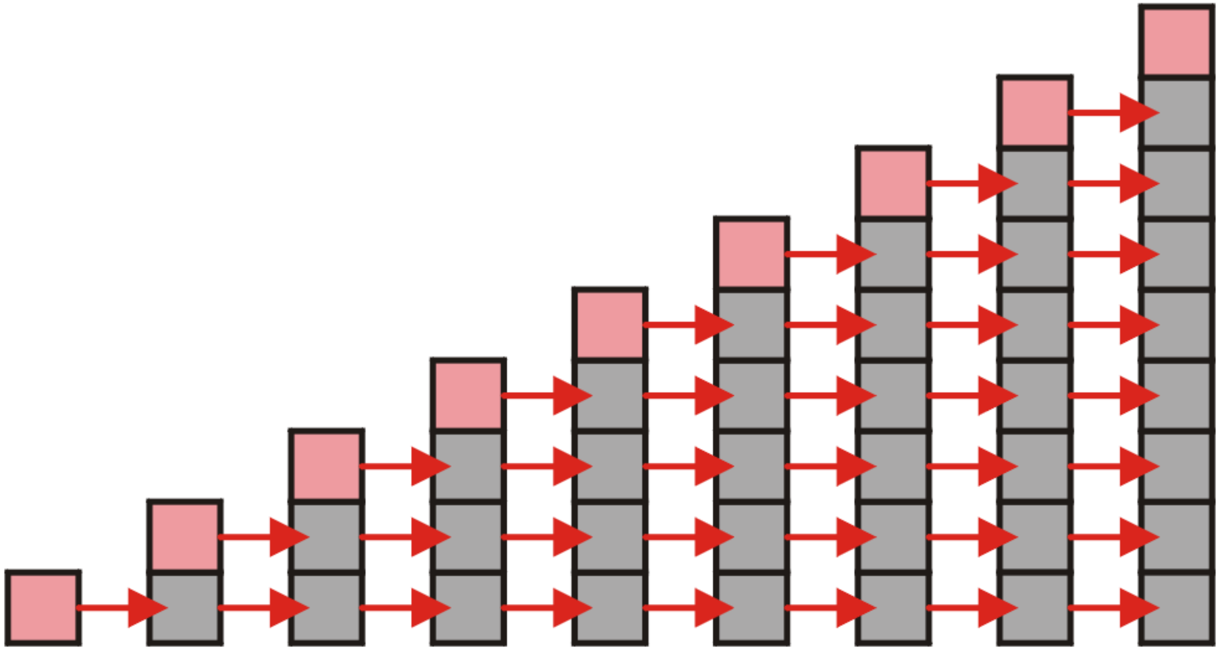
To state the average run time, we will introduce the concept of amortized time:

- If n operations requires $\Theta(\mathrm{f}(n))$, we will say that an individual operation has an amortized run time of $\Theta(\mathrm{f}(n)/n)$
- Therefore, if inserting n objects requires:
  - $\Theta(n^2)$ copies, the amortized time is $\Theta(n)$
  - $\Theta(n)$ copies, the amortized time is $\Theta(1)$

Array Capacity

Let us consider the case of increasing the capacity by 1 each time the array is full

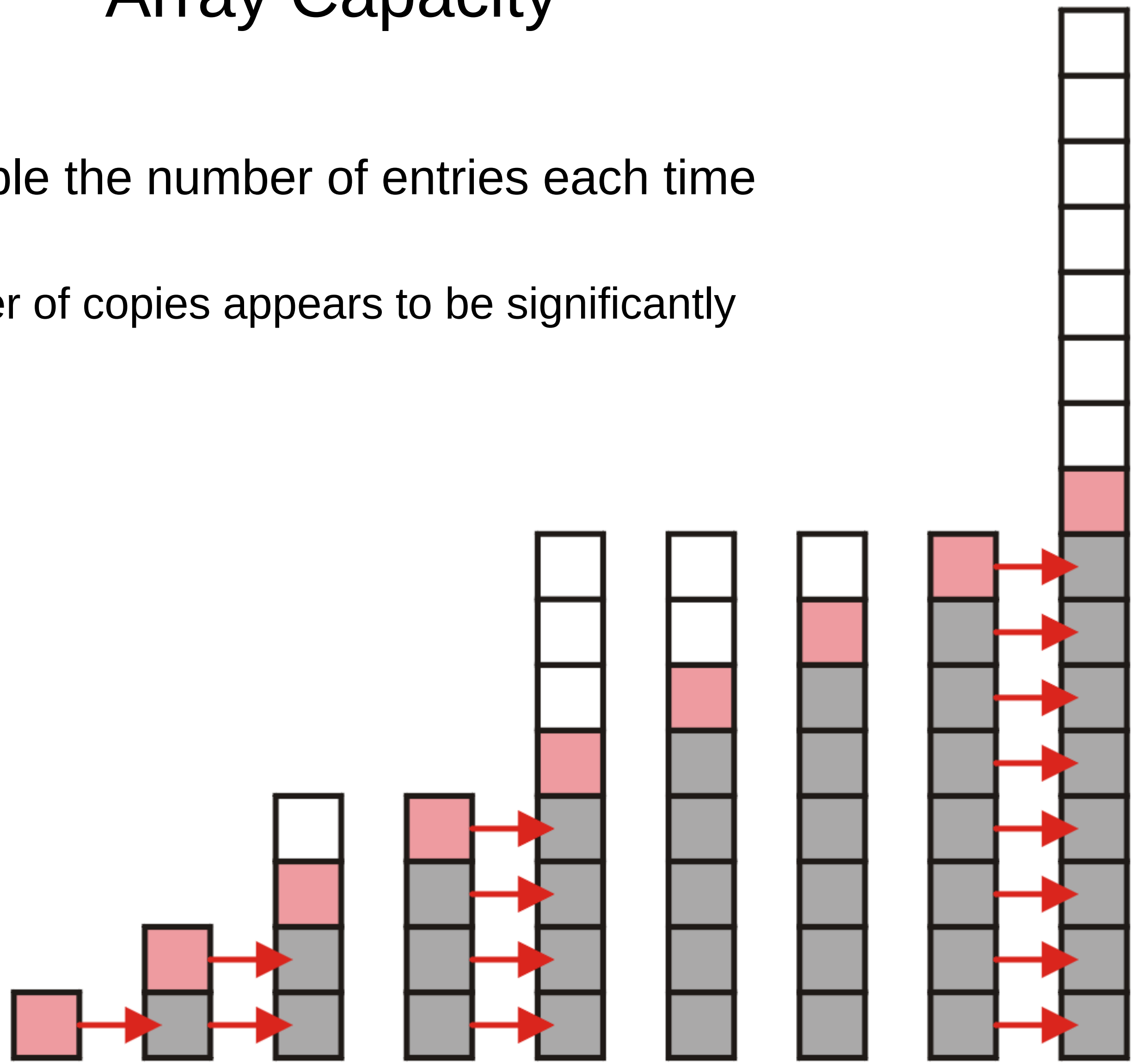
- With each insertion when the array is full, this requires all entries to be copied



Array Capacity

Suppose we double the number of entries each time the array is full

– Now the number of copies appears to be significantly fewer



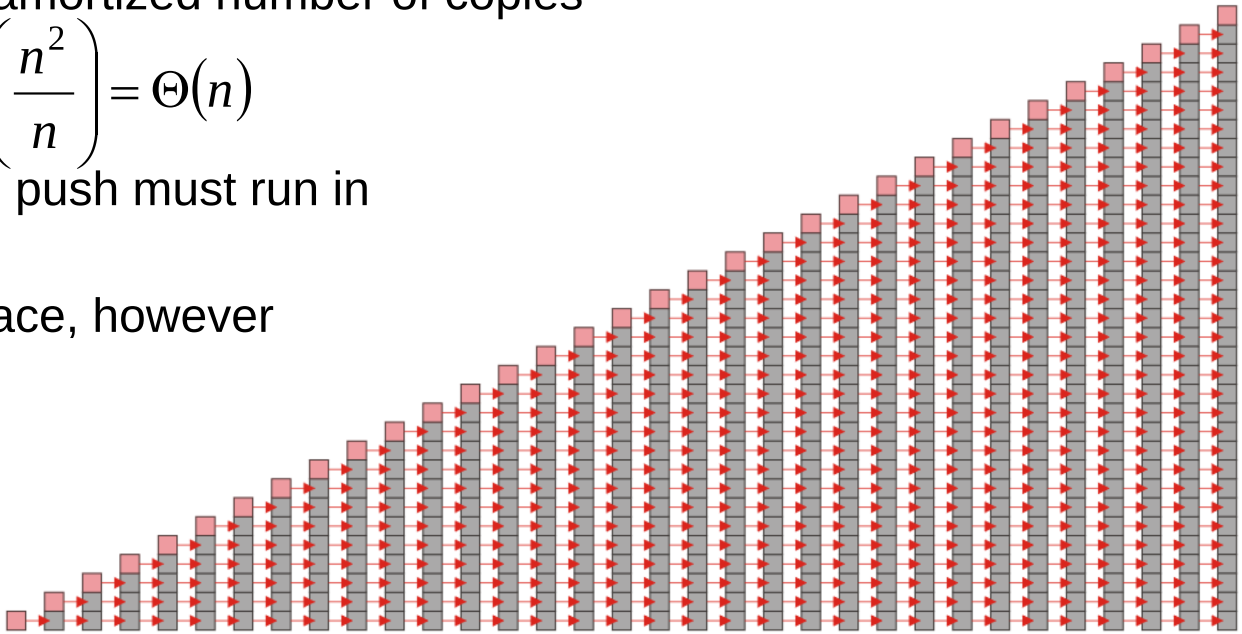
Array Capacity

Suppose we insert k objects

- The pushing of the k^{th} object on the stack requires k-1 copies
- The total number of copies is now given by:

$$\sum_{k=1}^{n}(k-1) = \left(\sum_{k=1}^{n} k\right) - n = \frac{n(n+1)}{2} - n = \frac{n(n-1)}{2} = \Theta\left(n^2\right)$$

- Therefore, the amortized number of copies is given by $$\Theta\left(\frac{n^2}{n}\right) = \Theta\left(n\right)$$
- Therefore each push must run in $\Theta(n)$ time
- The wasted space, however is $\Theta(1)$



Array Capacity

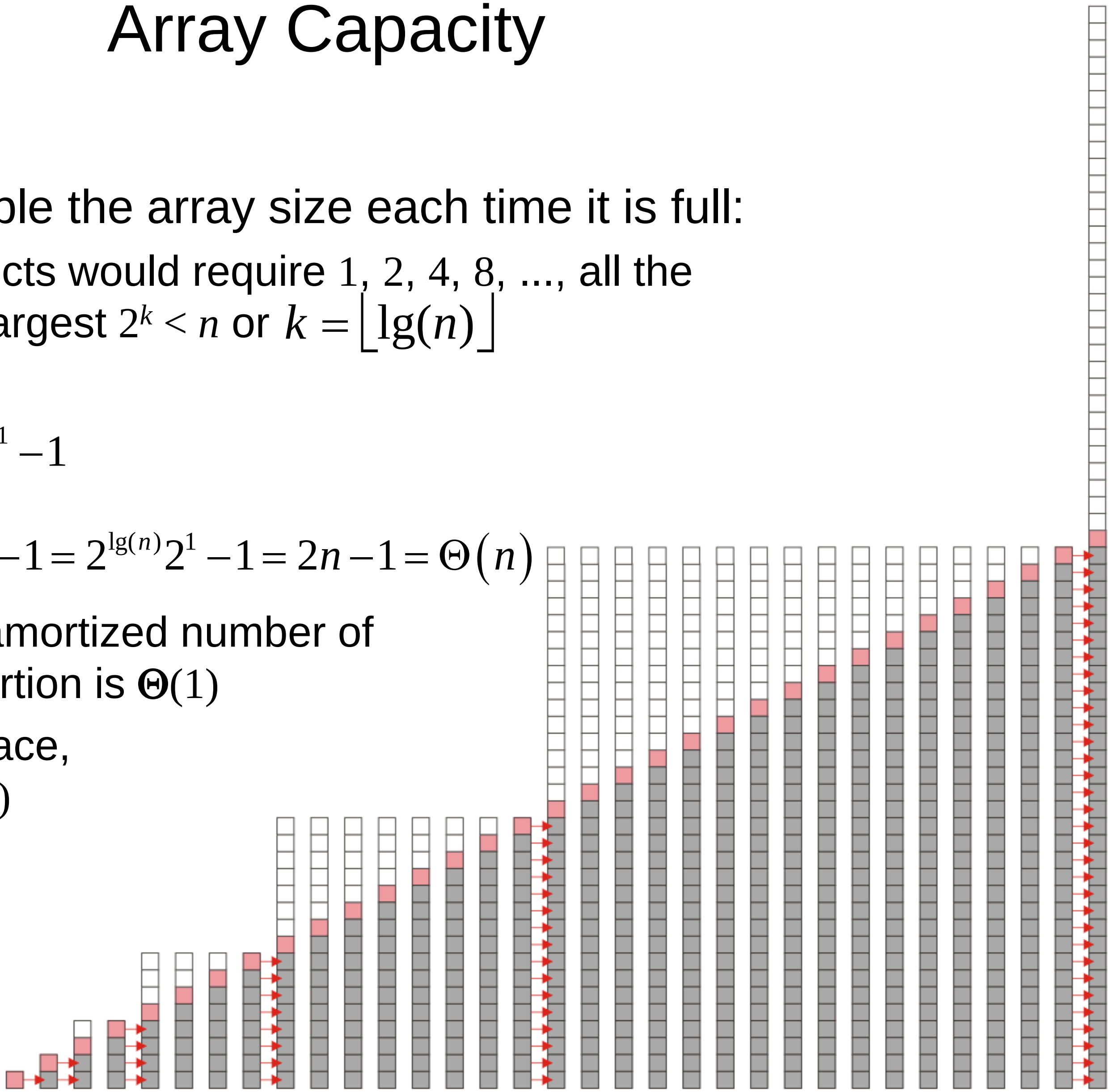
Suppose we double the array size each time it is full:

- Inserting n objects would require $1, 2, 4, 8, ...,$ all the way up to the largest $2^k < n$ or $k = \lfloor \lg(n) \rfloor$

$$\sum_{k=0}^{\lfloor \lg(n) \rfloor} 2^k = 2^{\lfloor \lg(n) \rfloor + 1} - 1$$

$$\le 2^{\lg(n)+1} - 1 = 2^{\lg(n)} 2^1 - 1 = 2n - 1 = \Theta(n)$$

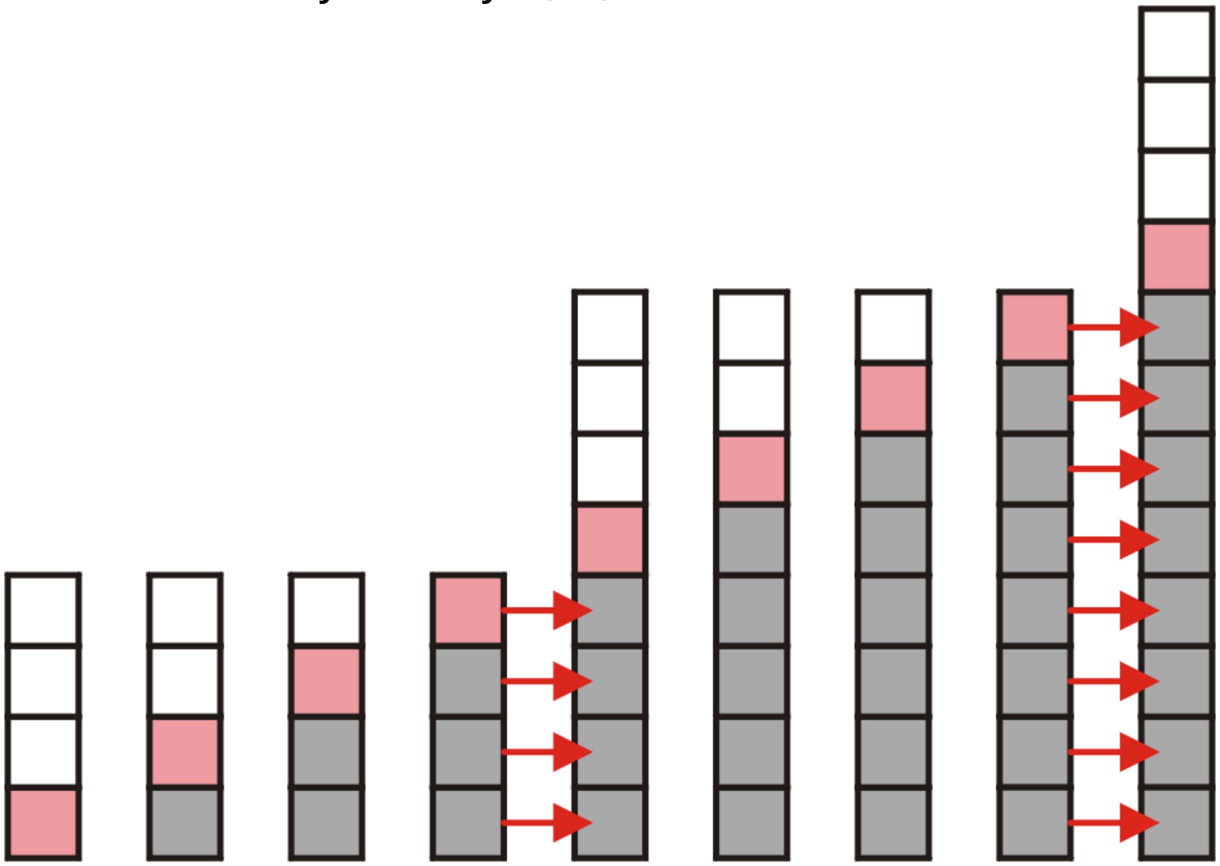
- Therefore the amortized number of copies per insertion is $\mathbf{\Theta}(1)$
- The wasted space, however is $\mathbf{O}(n)$



Array Capacity

What if we increase the array size by a larger constant?

– For example, increase the array size by 4, 8, 100?

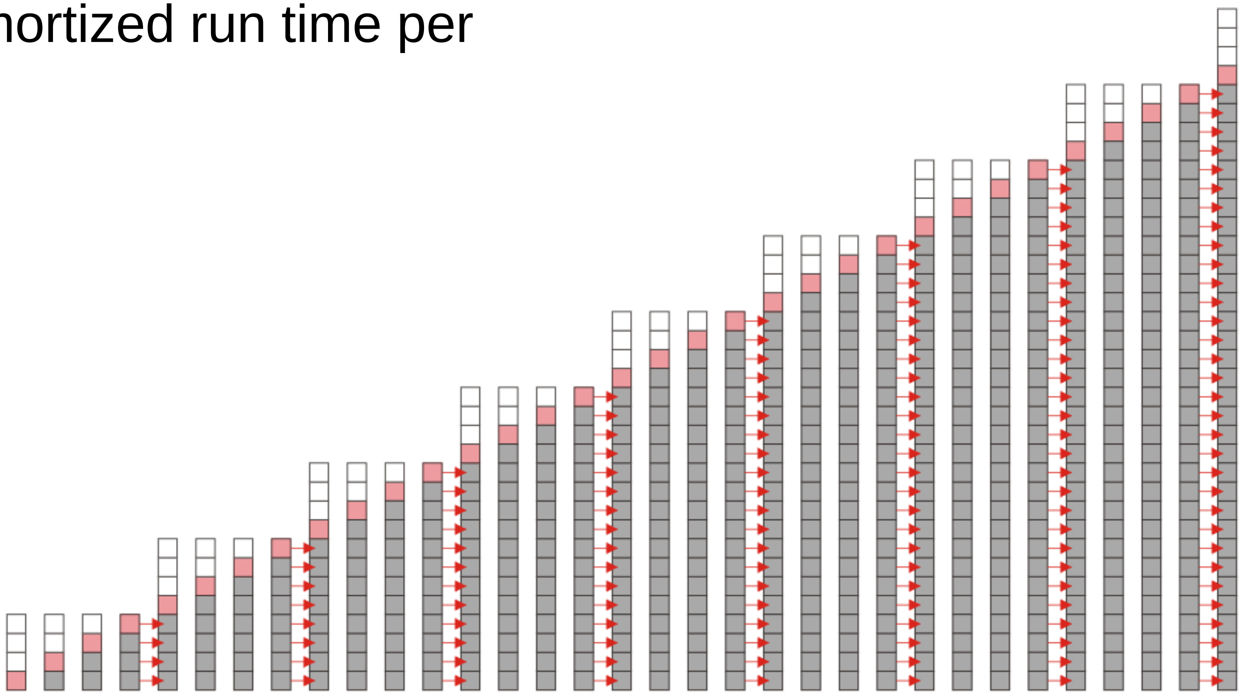


Array Capacity

Suppose we increase it by a **constant** value m

$$\sum_{k=1}^{n/m} mk = m\sum_{k=1}^{n/m} k = \frac{\frac{n}{m}\left(\frac{n}{m}+1\right)}{2} = \frac{n^2}{2m} + \frac{n}{2} = \Theta\left(n^2\right)$$

Therefore, the amortized run time per insertion is $\Theta(n)$



Array Capacity

Note the difference in worst-case amortized scenarios:

	Copies per Insertion	**Unused Memory**
Increase by 1	$n - 1$	0
Increase by m	n/m	$m - 1$
Increase by a factor of 2	1	n
Increase by a factor of $r > 1$	$1/(r - 1)$	$(r - 1)n$

Outline

- Stack ADT
- Implementation
- Example applications

Application: Parsing

Most parsing uses stacks

Examples includes:

- Matching tags in XHTML
- In C++, matching
  - parentheses **`( ... )`**
  - brackets, and **`[ ... ]`**
  - braces **`{ ... }`**

Parsing XHTML

A *markup language* is a means of annotating a document to given context to the text

– The annotations give information about the structure or presentation of the text

The best known example is HTML, or HyperText Markup Language

– We will look at XHTML

Parsing XHTML

XHTML is made of nested

- *opening tags*, e.g., `<some_identifier>`, and
- matching *closing tags*, e.g., `</some_identifier>`

```
<html>
  <head><title>Hello</title></head>
  <body><p>This appears in the <i>browser</i>.</p></body>
</html>
```

Nesting indicates that any closing tag must match the most <u>recent</u> opening tag

Parsing XHTML

Strategy for parsing XHTML:

- read though the XHTML linearly
- place the opening tags in a stack
- when a closing tag is encountered, check that it matches what is on top of the stack

Parsing XHTML

```
<html>
  <head><title>Hello</title></head>
  <body><p>This appears in the
  <i>browser</i>.</p></body>
</html>
```

<html>			

Parsing XHTML

```
<html>
  <head><title>Hello</title></head>
  <body><p>This appears in the
  <i>browser</i>.</p></body>
</html>
```

<html>	<head>		

Parsing XHTML

```
<html>
  <head><title>Hello</title></head>
  <body><p>This appears in the
  <i>browser</i>.</p></body>
</html>
```

<html>	<head>	<title>	

Parsing XHTML

```
<html>
  <head><title>Hello</title></head>
  <body><p>This appears in the
  <i>browser</i>.</p></body>
</html>
```

<html>	<head>	<title>	

Parsing XHTML

```
<html>
  <head><title>Hello</title></head>
  <body><p>This appears in the
  <i>browser</i>.</p></body>
</html>
```

<html>	<head>		

Parsing XHTML

```
<html>
  <head><title>Hello</title></head>
  <body><p>This appears in the
  <i>browser</i>.</p></body>
</html>
```

<html>	<body>		

Parsing XHTML

```
<html>
  <head><title>Hello</title></head>
  <body><p>This appears in the
  <i>browser</i>.</p></body>
</html>
```

<html>	<body>	<p>	

Parsing XHTML

```
<html>
  <head><title>Hello</title></head>
  <body><p>This appears in the
  <i>browser</i>.</p></body>
</html>
```

<html>	<body>	<p>	<i>

Parsing XHTML

```
<html>
  <head><title>Hello</title></head>
  <body><p>This appears in the
  <i>browser</i>.</p></body>
</html>
```

<html>	<body>	<p>	<i>

Parsing XHTML

```
<html>
  <head><title>Hello</title></head>
  <body><p>This appears in the
  <i>browser</i>.</p></body>
</html>
```

<html>	<body>	<p>	

Parsing XHTML

```
<html>
  <head><title>Hello</title></head>
  <body><p>This appears in the
  <i>browser</i>.</p></body>
</html>
```

<html>	<body>		

Parsing XHTML

```
<html>
  <head><title>Hello</title></head>
  <body><p>This appears in the
  <i>browser</i>.</p></body>
</html>
```

<html>			

Parsing XHTML

We are finished parsing, and the stack is empty

Possible errors:

- a closing tag which does not match the opening tag on top of the stack
- a closing tag when the stack is empty
- the stack is not empty at the end of the document

Function calls

```
int a(){
  b();
  c();
  return 0;
}

int b(){ return 0; }

int c(){ return 0; }

int main(){
  a();
  return 0;
}
```

Function calls

	main			
main() calls a()	**main**	**a**		
a() calls b()	**main**	**a**	**b**	
b() returns	**main**	**a**		
a() calls c()	**main**	**a**	**c**	
c() returns	**main**	**a**		
a() returns	**main**			

Function calls

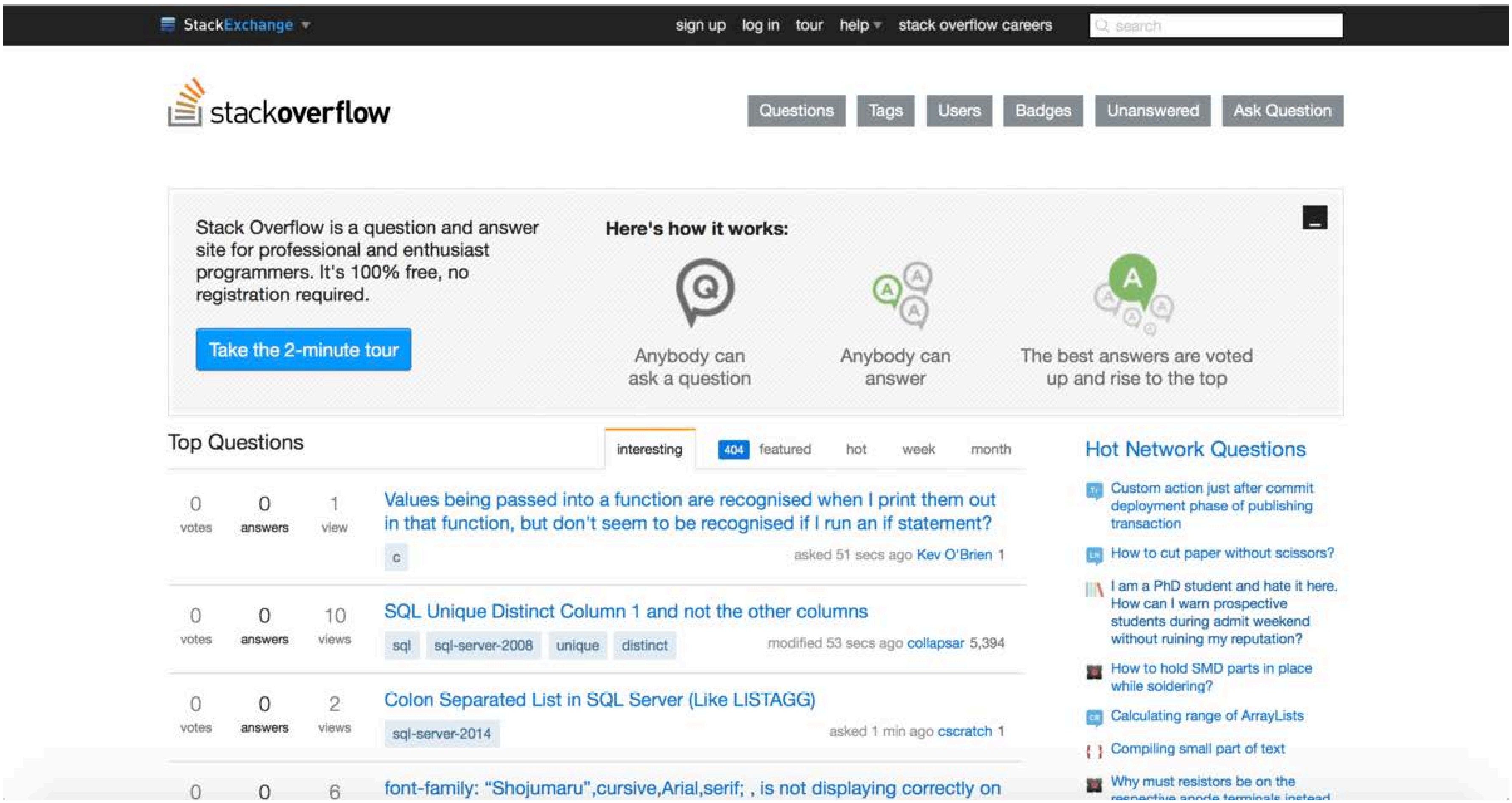
```
int a(){
  return a();
}
```

Function calls

calls a()	**a**			
a() calls a()	**a**	**a**		
a() calls a()	**a**	**a**	**a**	
a() calls a()	**a**	**a**	**a**	**a**
a() calls a()	**a**	**a**	**a**	**a** ?

Stack Overflow!

Stack Overflow



Reverse-Polish Notation

Normally, mathematics is written using what we call *in-fix* notation:

$$(3 + 4) \times 5 - 6$$

The operator is placed between to operands

One weakness: parentheses are required

$$(3 + 4) \times 5 - 6 = 29$$

$$3 + 4 \times 5 - 6 = 17$$

$$3 + 4 \times (5 - 6) = -1$$

$$(3 + 4) \times (5 - 6) = -7$$

Reverse-Polish Notation

Alternatively, we can place the operands first, followed by the operator:

$$(3 + 4) \times 5 - 6$$

$$3\ \ 4\ \ +\ \ 5\ \ \times\ \ 6\ \ -$$

Parsing reads left-to-right and performs any operation on the last two operands:

$$3\ \ 4\ \ +\ \ 5\ \ \times\ \ 6\ \ -$$

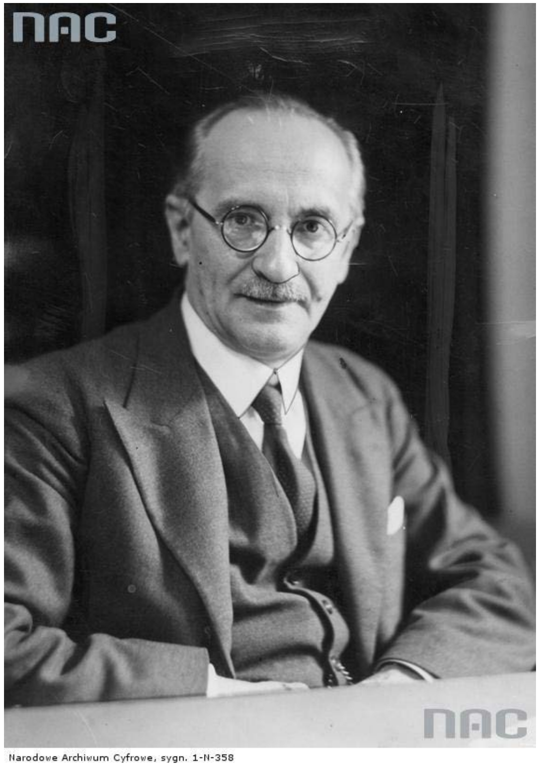
$$7\ \ \ \ 5\ \ \times\ \ 6\ \ -$$

$$35\ \ \ \ 6\ \ -$$

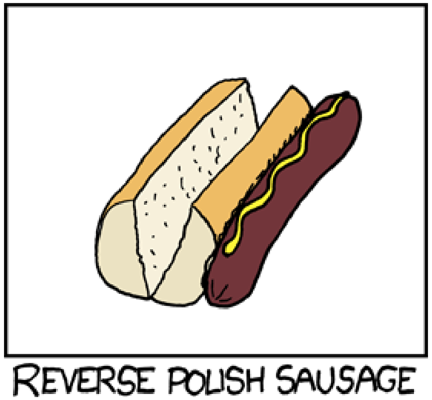
$$29$$

Reverse-Polish Notation

This is called *reverse-Polish* notation after the mathematician Jan Łukasiewicz



http://www.audiovis.nac.gov.pl/



http://xkcd.com/645/

Reverse-Polish Notation

Other examples:

$$3 \;\; 4 \;\; 5 \;\; \times \;\; + \;\; 6 \;\; -$$

$$3 \;\; 20 \;\; + \;\; 6 \;\; -$$

$$23 \;\; 6 \;\; -$$

$$17$$

$$3 \;\; 4 \;\; 5 \;\; 6 \;\; - \;\; \times \;\; +$$

$$3 \;\; 4 \;\; -1 \;\; \times \;\; +$$

$$3 \;\; -4 \;\; +$$

$$-1$$

Reverse-Polish Notation

Benefits:

- No ambiguity and no brackets are required
- It is the same process used by a computer to perform computations:
  - operands must be loaded into registers before operations can be performed on them

Reverse-Polish Notation

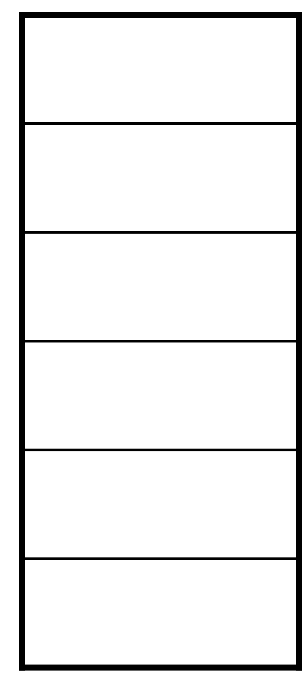
The easiest way to parse reverse-Polish notation is to use an operand stack:

- operands are processed by pushing them onto the stack
- when processing an operator:
  - pop the last two items off the operand stack,
  - perform the operation, and
  - push the result back onto the stack

Reverse-Polish Notation

Evaluate the following reverse-Polish expression using a stack:

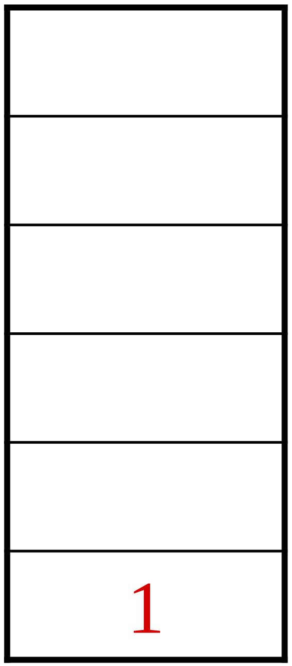
$$1 \;\; 2 \;\; 3 \;\; + \;\; 4 \;\; 5 \;\; 6 \;\; \times \;\; - \;\; 7 \;\; \times \;\; + \;\; - \;\; 8 \;\; 9 \;\; \times \;\; +$$



Reverse-Polish Notation

Push 1 onto the stack

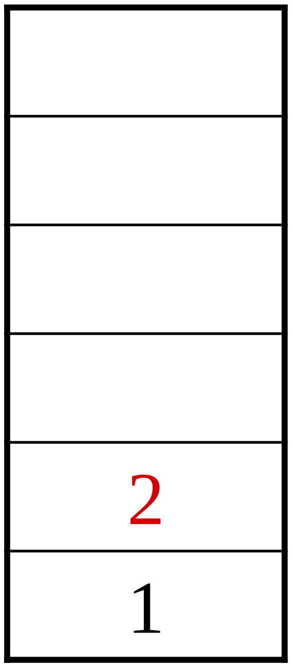
$1 \;\; 2 \;\; 3 \;\; + \;\; 4 \;\; 5 \;\; 6 \;\; \times \;\; - \;\; 7 \;\; \times \;\; + \;\; - \;\; 8 \;\; 9 \;\; \times \;\; +$



Reverse-Polish Notation

Push 1 onto the stack

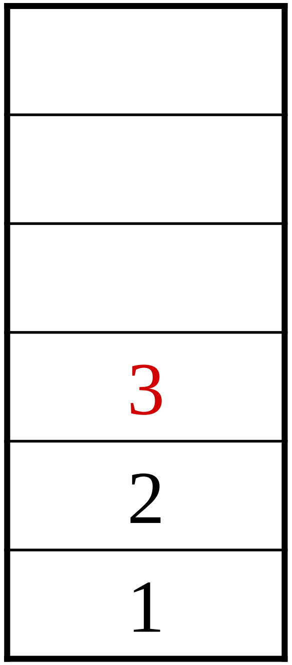
$$1 \quad 2 \quad 3 \quad + \quad 4 \quad 5 \quad 6 \quad \times \quad - \quad 7 \quad \times \quad + \quad - \quad 8 \quad 9 \quad \times \quad +$$



Reverse-Polish Notation

Push 3 onto the stack

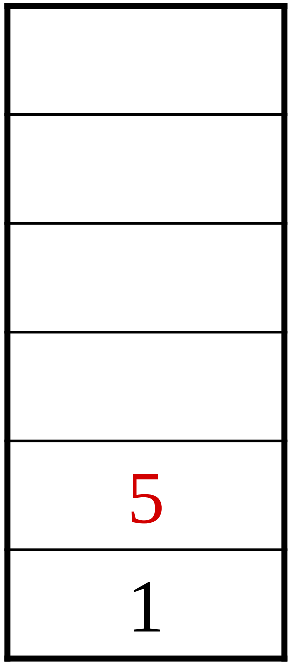
$$1 \quad 2 \quad 3 \quad + \quad 4 \quad 5 \quad 6 \quad \times \quad - \quad 7 \quad \times \quad + \quad - \quad 8 \quad 9 \quad \times \quad +$$



Reverse-Polish Notation

Pop 3 and 2 and push $2 + 3 = 5$

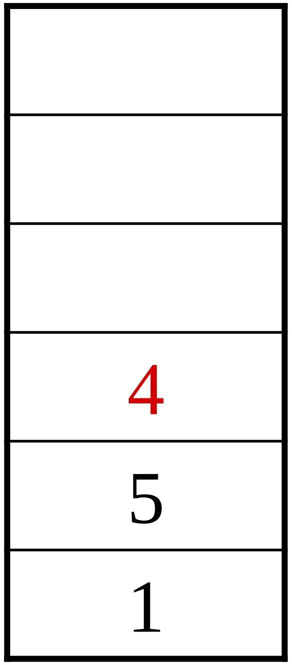
$$1 \;\; 2 \;\; 3 \;\; + \;\; 4 \;\; 5 \;\; 6 \;\; \times \;\; - \;\; 7 \;\; \times \;\; + \;\; - \;\; 8 \;\; 9 \;\; \times \;\; +$$



Reverse-Polish Notation

Push 4 onto the stack

$1 \quad 2 \quad 3 \quad + \quad \color{red}{4} \quad 5 \quad 6 \quad \times \quad - \quad 7 \quad \times \quad + \quad - \quad 8 \quad 9 \quad \times \quad +$



Reverse-Polish Notation

Push 5 onto the stack

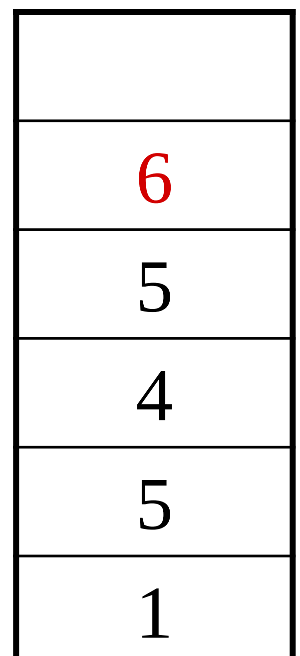
$1 \quad 2 \quad 3 \quad + \quad 4 \quad 5 \quad 6 \quad \times \quad - \quad 7 \quad \times \quad + \quad - \quad 8 \quad 9 \quad \times \quad +$

5
4
5
1

Reverse-Polish Notation

Push 6 onto the stack

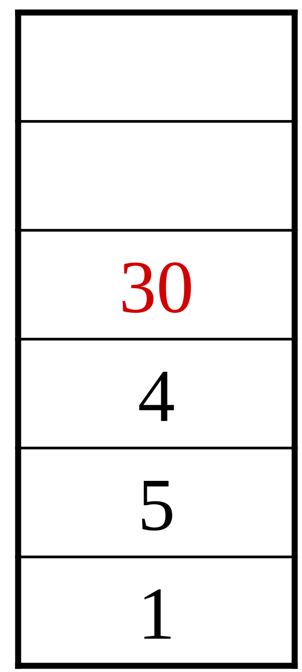
$$1 \quad 2 \quad 3 \quad + \quad 4 \quad 5 \quad {\color{red}6} \quad \times \quad - \quad 7 \quad \times \quad + \quad - \quad 8 \quad 9 \quad \times \quad +$$



Reverse-Polish Notation

Pop 6 and 5 and push $5 \times 6 = 30$

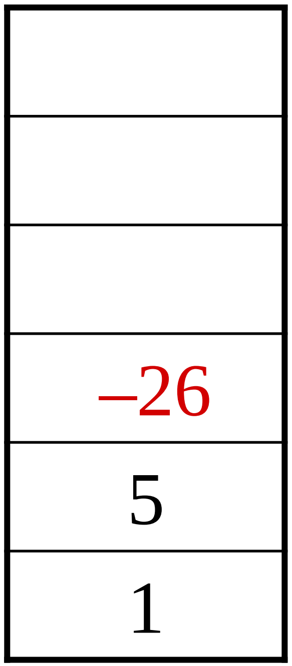
$$1 \quad 2 \quad 3 \quad + \quad 4 \quad 5 \quad 6 \quad \times \quad - \quad 7 \quad \times \quad + \quad - \quad 8 \quad 9 \quad \times \quad +$$



Reverse-Polish Notation

Pop 30 and 4 and push 4 – 30 = –26

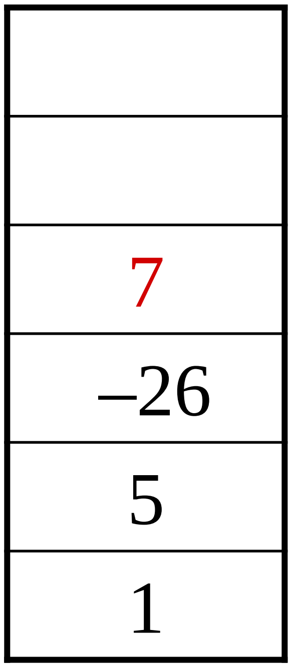
1 2 3 + 4 5 6 × – 7 × + – 8 9 × +



Reverse-Polish Notation

Push 7 onto the stack

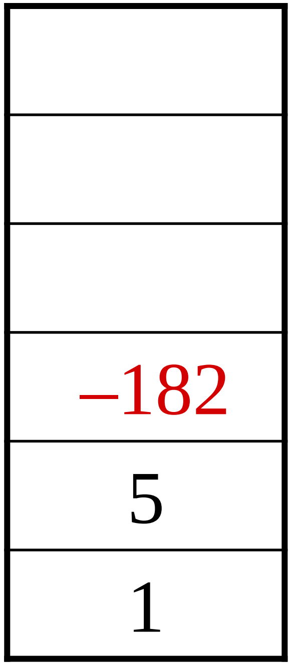
$$1\ 2\ 3\ +\ 4\ 5\ 6\ \times\ -\ 7\ \times\ +\ -\ 8\ 9\ \times\ +$$



Reverse-Polish Notation

Pop 7 and –26 and push –26 × 7 = –182

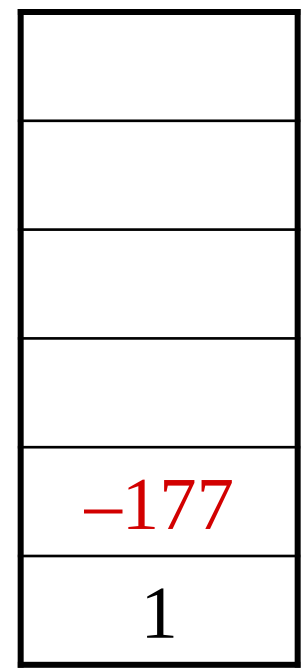
$1\ \ 2\ \ 3\ \ +\ \ 4\ \ 5\ \ 6\ \ \times\ \ -\ \ 7\ \ \times\ \ +\ \ -\ \ 8\ \ 9\ \ \times\ \ +$



Reverse-Polish Notation

Pop –182 and 5 and push –182 + 5 = –177

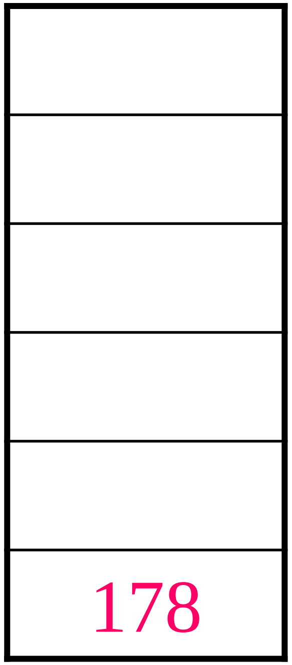
1 2 3 + 4 5 6 × – 7 × + – 8 9 × +



Reverse-Polish Notation

Pop -177 and 1 and push $1 - (-177) = 178$

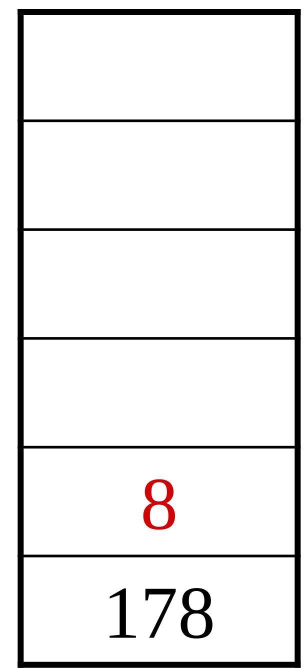
$$1 \;\; 2 \;\; 3 \;\; + \;\; 4 \;\; 5 \;\; 6 \;\; \times \;\; - \;\; 7 \;\; \times \;\; + \;\; - \;\; 8 \;\; 9 \;\; \times \;\; +$$



Reverse-Polish Notation

Push 8 onto the stack

$1 \;\; 2 \;\; 3 \;\; + \;\; 4 \;\; 5 \;\; 6 \;\; \times \;\; - \;\; 7 \;\; \times \;\; + \;\; - \;\; {\color{red}8} \;\; 9 \;\; \times \;\; +$



Reverse-Polish Notation

Push 1 onto the stack

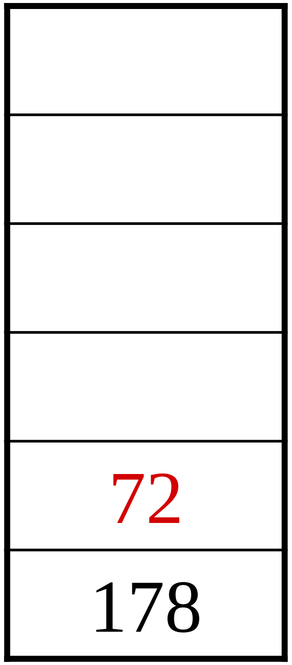
$$1 \quad 2 \quad 3 \quad + \quad 4 \quad 5 \quad 6 \quad \times \quad - \quad 7 \quad \times \quad + \quad - \quad 8 \quad {\color{red}9} \quad \times \quad +$$

9
8
178

Reverse-Polish Notation

Pop 9 and 8 and push $8 \times 9 = 72$

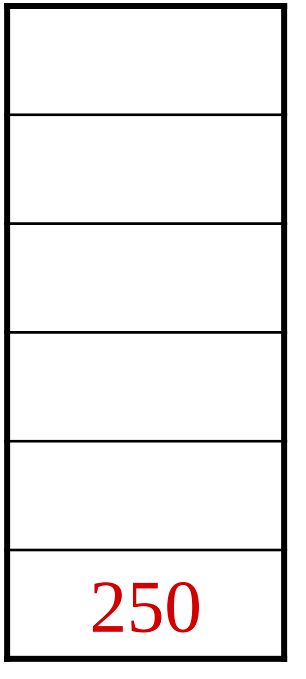
$$1 \quad 2 \quad 3 \quad + \quad 4 \quad 5 \quad 6 \quad \times \quad - \quad 7 \quad \times \quad + \quad - \quad 8 \quad 9 \quad \times \quad +$$



Reverse-Polish Notation

Pop 72 and 178 and push 178 + 72 = 250

$$1\ 2\ 3\ +\ 4\ 5\ 6\ \times\ -\ 7\ \times\ +\ -\ 8\ 9\ \times\ +$$



Reverse-Polish Notation

Thus

$$1\ \ 2\ \ 3\ \ +\ \ 4\ \ 5\ \ 6\ \ \times\ \ -\ \ 7\ \ \times\ \ +\ \ -\ \ 8\ \ 9\ \ \times\ \ +$$

evaluates to the value on the top: 250

The equivalent in-fix notation is

$$((1 - ((2 + 3) + ((4 - (5 \times 6)) \times 7))) + (8 \times 9))$$

We reduce the parentheses using order-of-operations:

$$1 - (2 + 3 + (4 - 5 \times 6) \times 7) + 8 \times 9$$

Summary

- Stack ADT
  - Push, pop, LIFO
- Implementation
  - Linked list
  - Array
    - How to increase the array capacity
- Applications
  - Parsing XHTML
  - Function calls
  - Reverse-Polish Notation