Algorithm Analysis

Textbook Ch 2,3

Outline

- Justification for analysis
- Landau symbols
- Run time of programs
- Best-, worst-, and average-case

Comparing algorithms

Suppose we have two algorithms, how can we tell which is better?

We could implement both algorithms, run them both

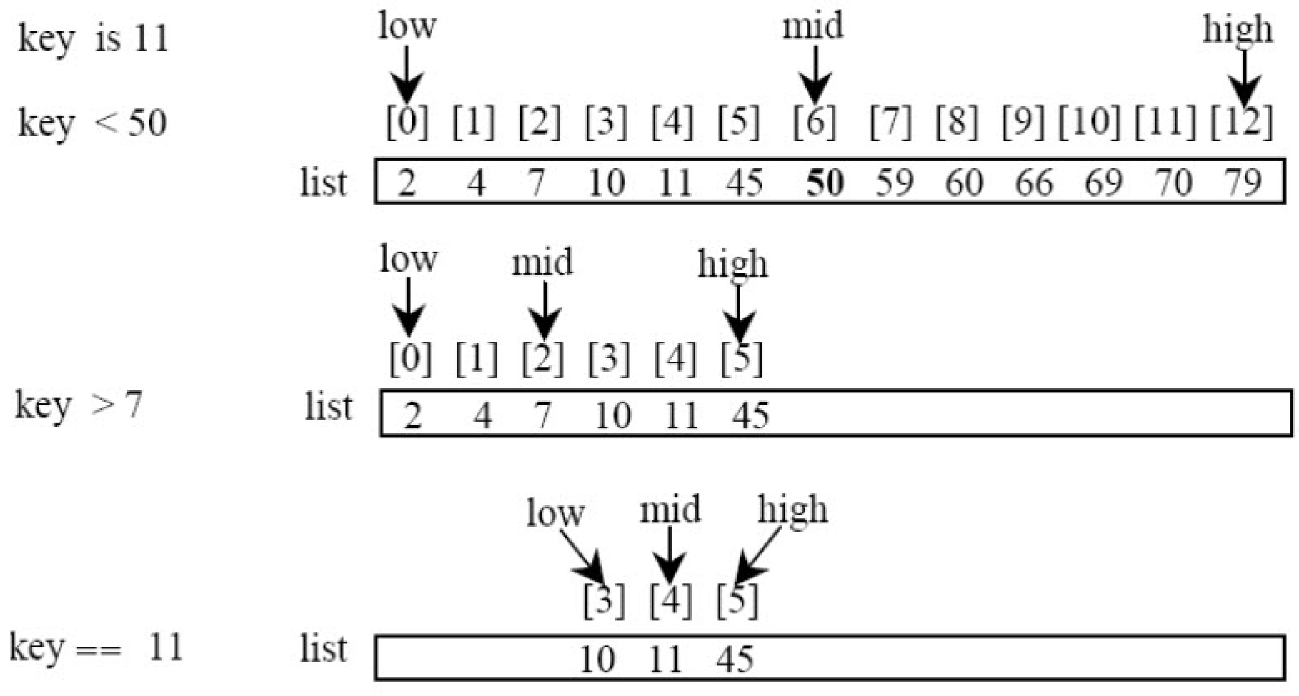
- Expensive and error prone

Preferably, we should analyze them mathematically

- *Algorithm analysis*

Example

- Find a item in a sorted array of length N
- Algorithm 1: Linear search (check each item from left to right)
  - Do you use this approach when looking up a word in a dictionary?
- Algorithm 2: Binary search



Example

- Find a item in a sorted array of length N
- Algorithm 1: Linear search
  - Check each item from left to right
- Algorithm 2: Binary search

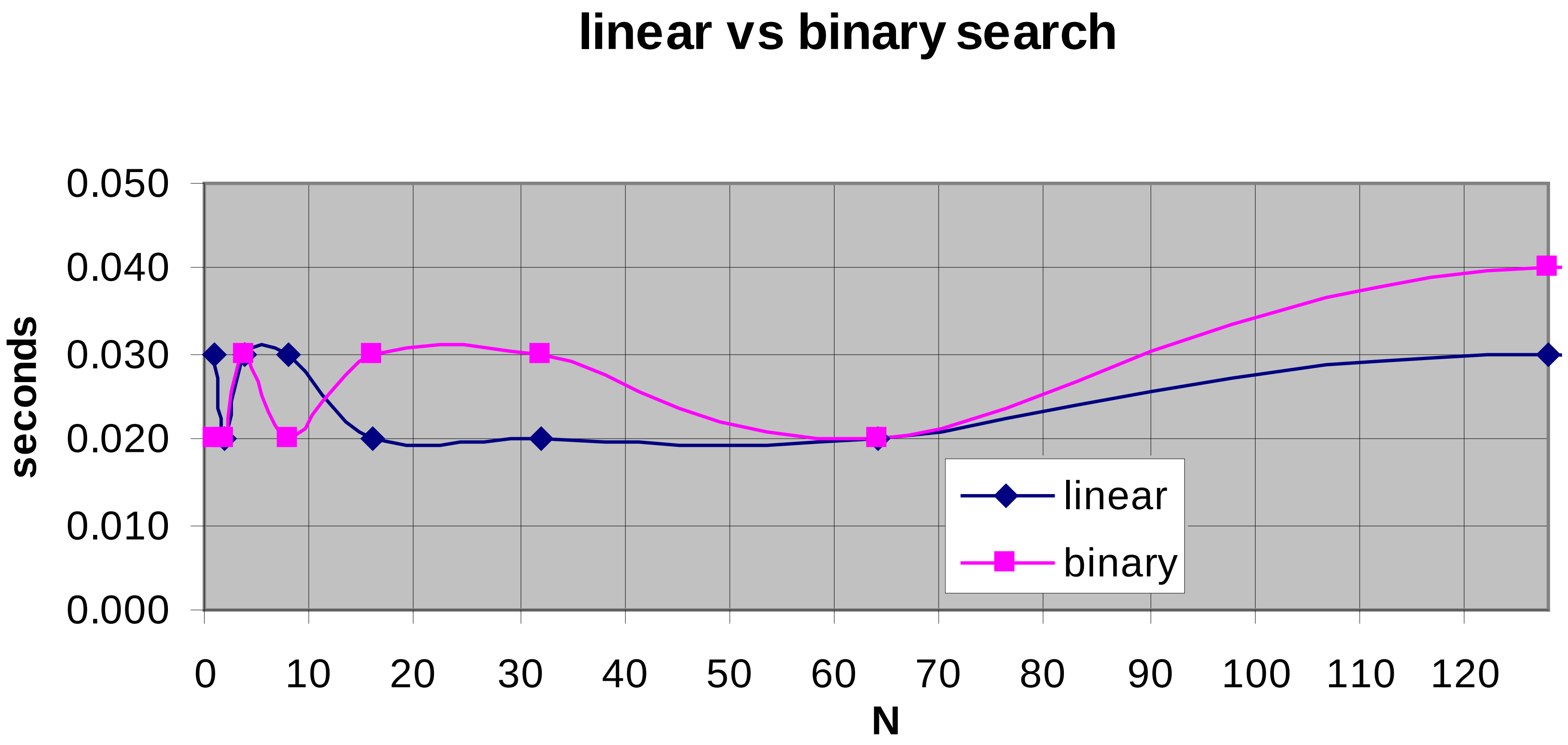
```
int bfind(int x, int a[], int left, int right)
{   if (left+1 == right) return –1;
    m = (left + right) / 2;
    if (x == a[m]) return m;
    if (x < a[m]) return bfind(x, a, left, m);
    else return bfind(x, a, m, right);
}
```

Empirical comparison

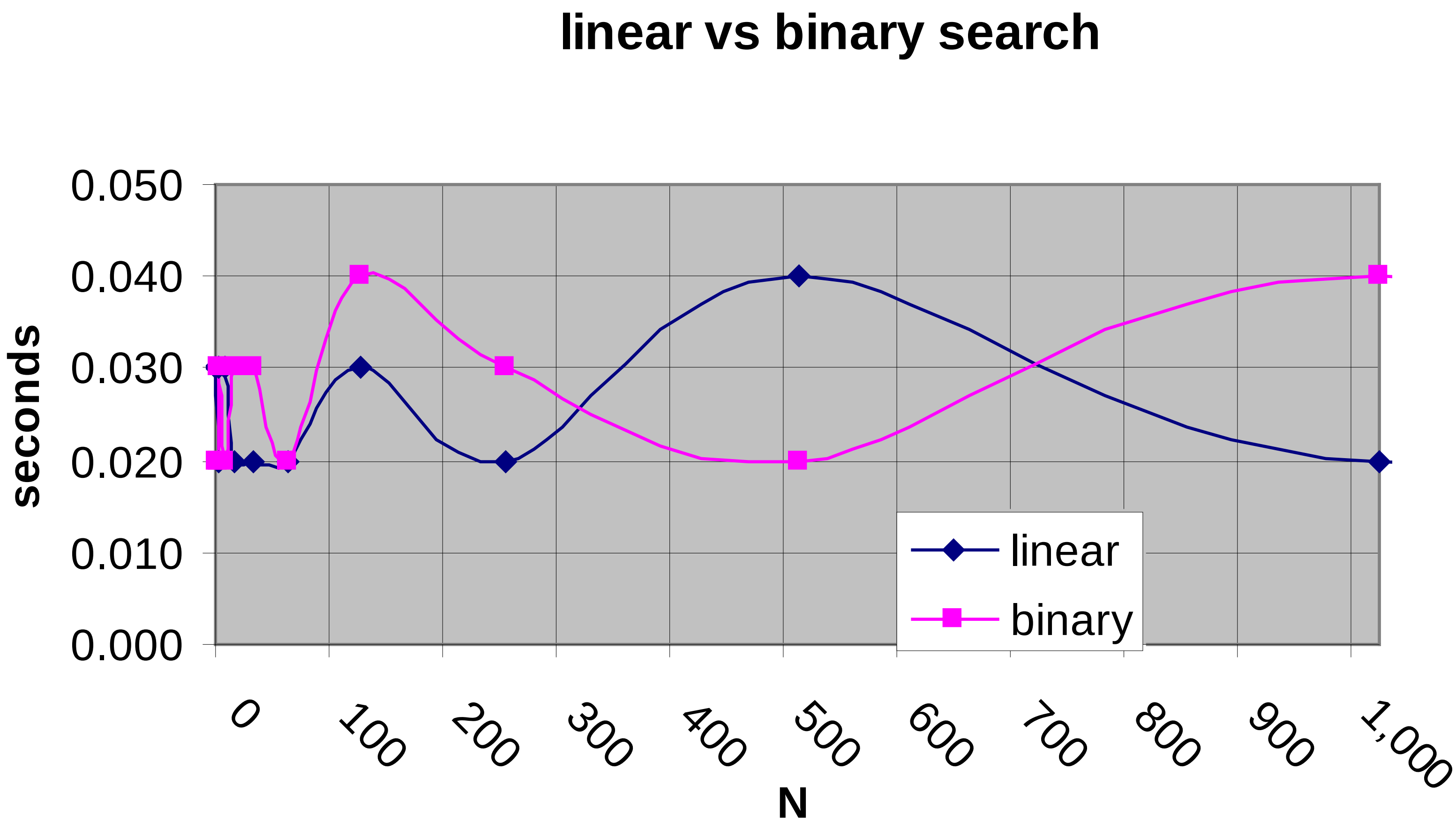
```
for (i=0; i<n; i++) a[i] = i;

for (i=0; i<n; i++) lfind(i,a,n);
```

```
or
bfind(i,a,-1,n)
```

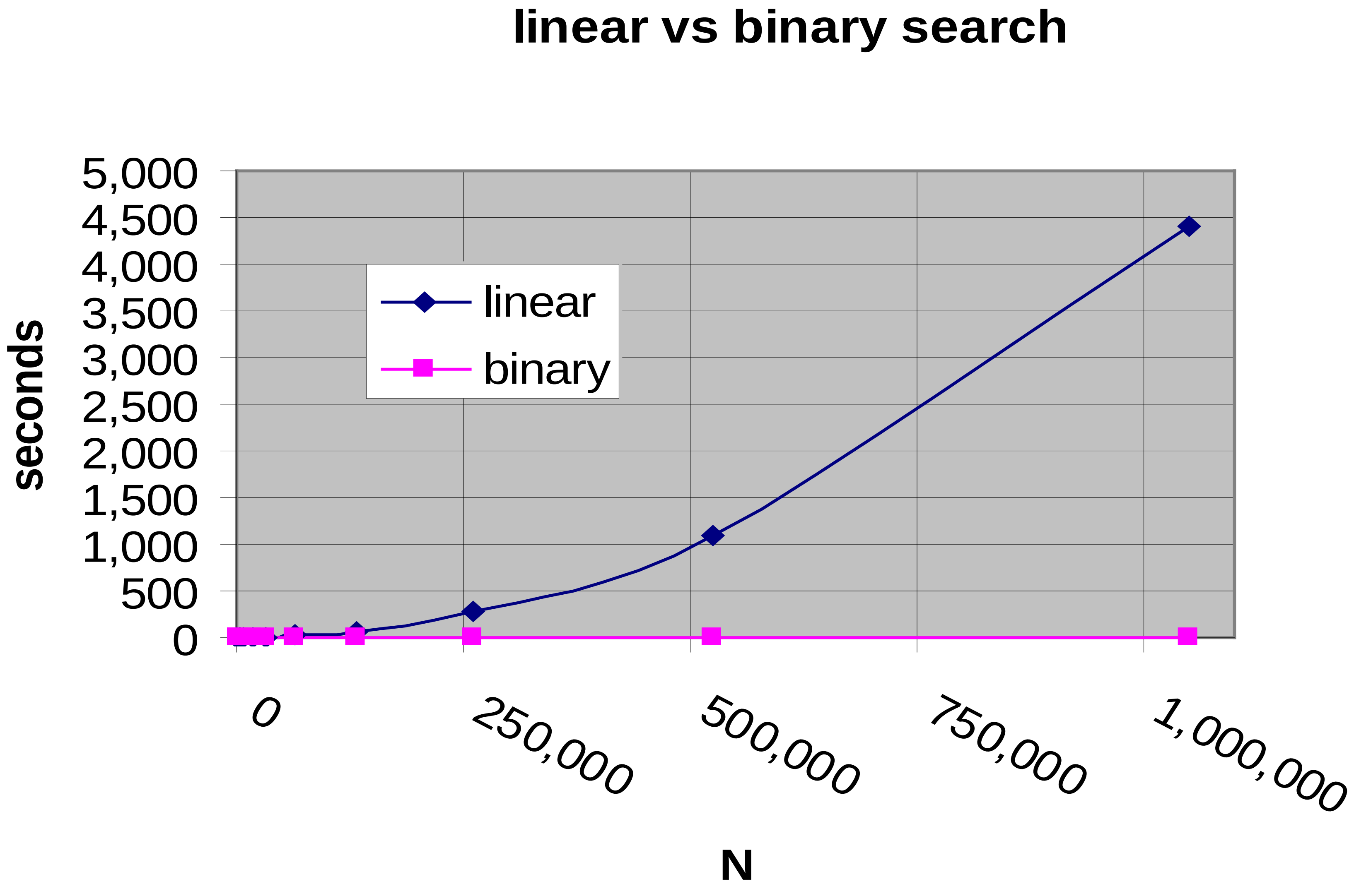
Empirical comparison



Empirical comparison



Empirical comparison



Analytical comparison

- Linear search
  - O(n)
- Binary search
  - O(log n)
- So binary search is better than linear search

Asymptotic Analysis

Given an algorithm, we want to describe its computational cost mathematically and in a machine-independent way

For this, we need Landau symbols (a.k.a. Big-O notation) and the associated asymptotic analysis

Outline

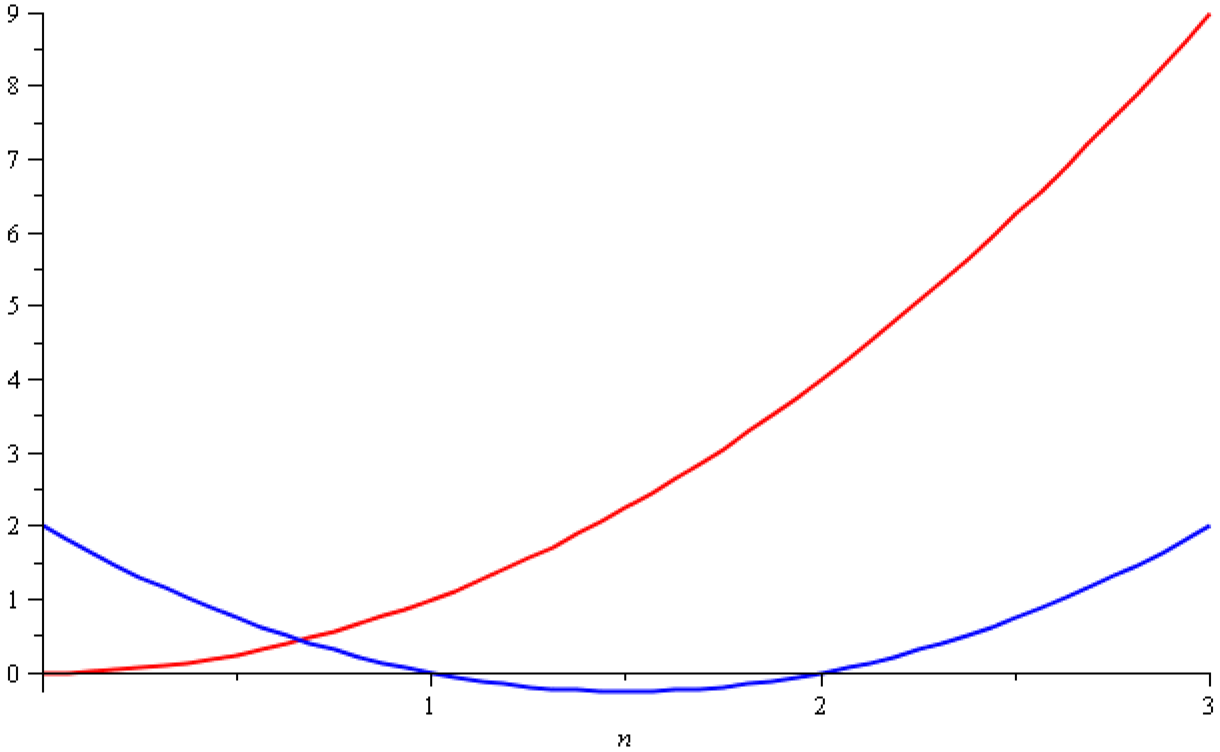
- Justification for analysis
- Landau symbols
- Run time of programs
- Best-, worst-, and average-case

Quadratic Growth

Consider the two functions

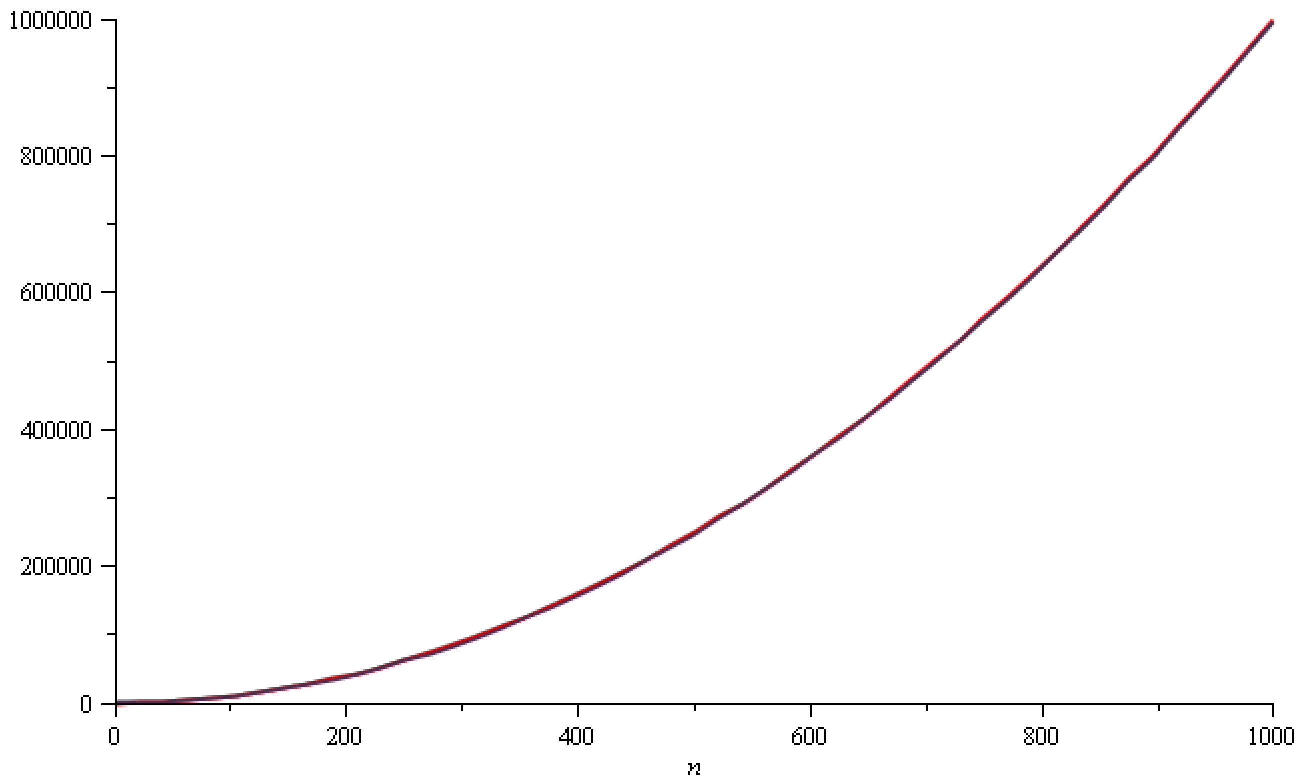
$f(n) = n^2$ and $g(n) = n^2 - 3n + 2$

Around $n = 0$, they look very different



Quadratic Growth

Yet on the range $n = [0, 1000]$, they are (relatively) indistinguishable:



Quadratic Growth

The absolute difference is large, for example,

$$f(1000) = 1\,000\,000$$

$$g(1000) = \quad 997\,002$$

but the relative difference is very small

$$\left|\frac{f(1000) - g(1000)}{f(1000)}\right| = 0.002998 < 0.3\%$$

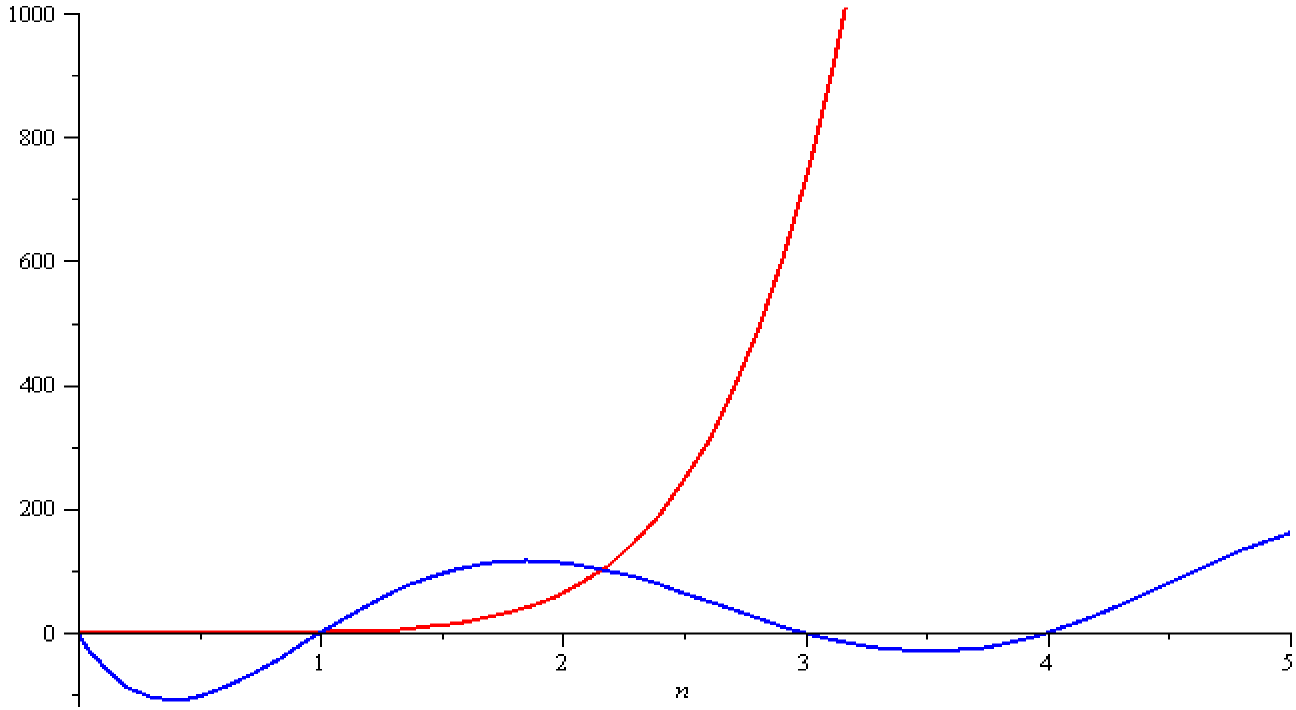
and this difference goes to zero as $n \to \infty$

Polynomial Growth

To demonstrate with another example,

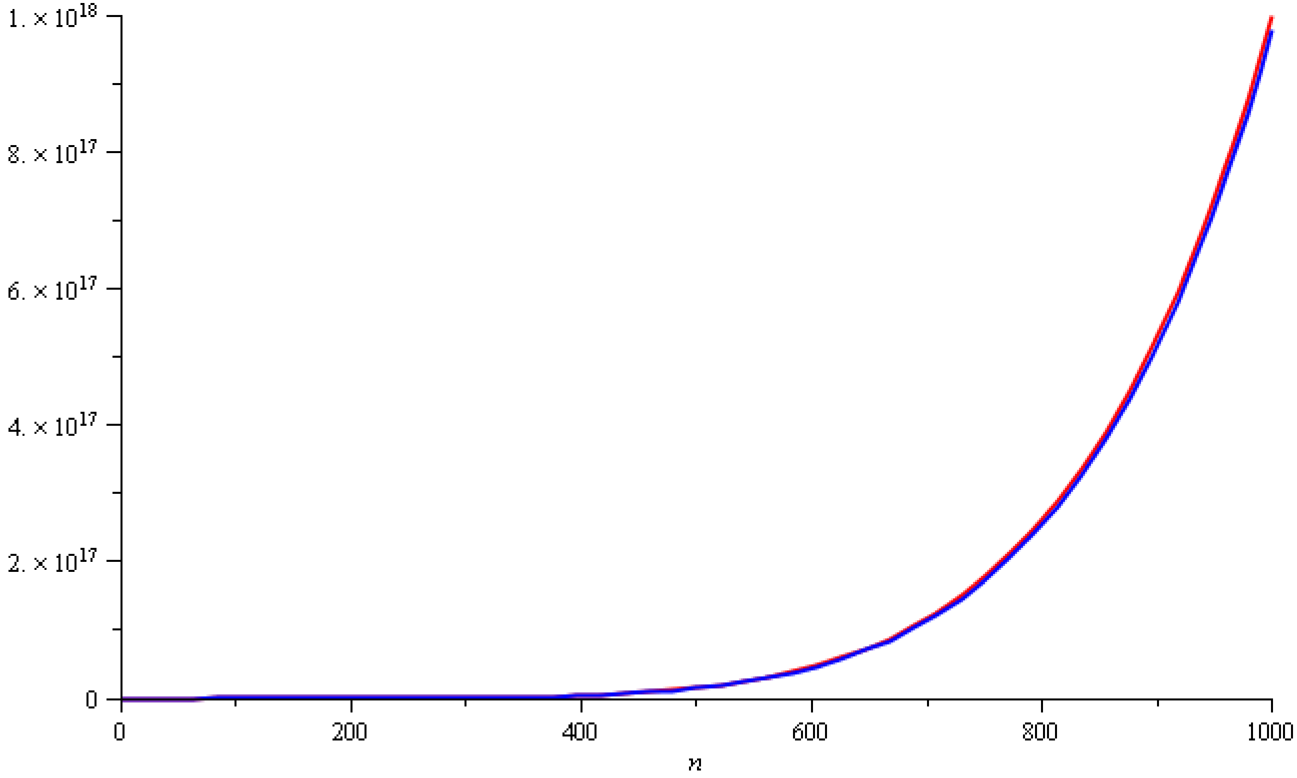
$f(n) = n^6$ and $g(n) = n^6 - 23n^5 + 193n^4 - 729n^3 + 1206n^2 - 648n$

Around $n = 0$, they are very different



Polynomial Growth

Still, around $n = 1000$, the relative difference is less than 3%



Polynomial Growth

The justification for both pairs of polynomials being similar is that, in both cases, they each had the same leading term:

n^2 in the first case, n^6 in the second

What if the coefficients of the leading terms were different?

– In this case, both functions would exhibit the same rate of growth, however, one would always be proportionally larger

However, if the two functions describe the run-time of two algorithms

– We can always run the slower algorithm on a faster computer to make them equally fast

In contrast: can we make linear search equally fast to binary search by using a faster computer (say, an Ultimate Laptop)?

Weak ordering

Consider the following definitions:

- We will consider two functions to be equivalent, $f \sim g$, if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c \quad \text{where} \quad 0 < c < \infty$$

- We will state that $f < g$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$

For functions we are interested in, these define a weak ordering

Weak ordering

Let $f(n)$ and $g(n)$ describe the run-time of two algorithms

- If $f(n) \sim g(n)$, then it is always possible to improve the performance of one function over the other by purchasing a faster computer
- If $f(n) < g(n)$, then you can never purchase a computer fast enough so that the second function always runs in less time than the first

Landau Symbols

Better known as big O notation

A function $\mathrm{f}(n) = \mathbf{O}(\mathrm{g}(n))$ if there exists N and c such that

$$\mathrm{f}(n) < c\,\mathrm{g}(n)$$

whenever $n > N$

– The function $\mathrm{f}(n)$ has a rate of growth no greater than that of $\mathrm{g}(n)$

Landau Symbols

Another Landau symbol is $\boldsymbol{\Theta}$

A function $\mathrm{f}(n) = \boldsymbol{\Theta}(\mathrm{g}(n))$ if there exist positive N, c_1, and c_2 such that

$$c_1\,\mathrm{g}(n) < \mathrm{f}(n) < c_2\,\mathrm{g}(n)$$

whenever $n > N$

– The function $\mathrm{f}(n)$ has a rate of growth equal to that of $\mathrm{g}(n)$

Landau Symbols

If $\mathrm{f}(n)$ and $\mathrm{g}(n)$ are polynomials of the same degree with positive leading coefficients:

$$\lim_{n\to\infty}\frac{\mathrm{f}(n)}{\mathrm{g}(n)} = c \quad \text{where} \quad 0 < c < \infty$$

From the definition, this means given $c > \varepsilon > 0$ there exists an $N > 0$ such that $\left|\frac{\mathrm{f}(n)}{\mathrm{g}(n)} - c\right| < \varepsilon$ whenever $n > N$

That is,

$$c - \varepsilon < \frac{\mathrm{f}(n)}{\mathrm{g}(n)} < c + \varepsilon$$

$$\mathrm{g}(n)(c - \varepsilon) < \mathrm{f}(n) < \mathrm{g}(n)(c + \varepsilon)$$

Landau Symbols

If $\lim_{n \to \infty} \frac{\mathrm{f}(n)}{\mathrm{g}(n)} = c$ where $0 < c < \infty$, it follows that $\mathrm{f}(n) = \boldsymbol{\Theta}(\mathrm{g}(n))$

Landau Symbols

We will at times use five possible descriptions

$$\mathrm{f}(n) = \mathbf{o}(\mathrm{g}(n)) \qquad \lim_{n\to\infty} \frac{\mathrm{f}(n)}{\mathrm{g}(n)} = 0$$

$$\mathrm{f}(n) = \mathbf{O}(\mathrm{g}(n)) \qquad \lim_{n\to\infty} \frac{\mathrm{f}(n)}{\mathrm{g}(n)} < \infty$$

$$\mathrm{f}(n) = \mathbf{\Theta}(\mathrm{g}(n)) \qquad 0 < \lim_{n\to\infty} \frac{\mathrm{f}(n)}{\mathrm{g}(n)} < \infty$$

$$\mathrm{f}(n) = \mathbf{\Omega}(\mathrm{g}(n)) \qquad \lim_{n\to\infty} \frac{\mathrm{f}(n)}{\mathrm{g}(n)} > 0$$

$$\mathrm{f}(n) = \boldsymbol{\omega}(\mathrm{g}(n)) \qquad \lim_{n\to\infty} \frac{\mathrm{f}(n)}{\mathrm{g}(n)} = \infty$$

Landau Symbols

Graphically, we can summarize these as follows:

We say $\mathrm{f}(n) =$ $\mathbf{o}(\mathrm{g}(n))$, $\mathbf{O}(\mathrm{g}(n))$, $\Theta(\mathrm{g}(n))$, $\Omega(\mathrm{g}(n))$, $\omega(\mathrm{g}(n))$

if $\lim\limits_{n \to \infty} \frac{\mathrm{f}(n)}{\mathrm{g}(n)} =$ 0, $0 < c < \infty$, ∞

Landau Symbols

For the functions we are interested in, it can be said that

$\mathrm{f}(n) = \mathbf{O}(\mathrm{g}(n))$ is equivalent to $\mathrm{f}(n) = \mathbf{\Theta}(\mathrm{g}(n))$ or $\mathrm{f}(n) = \mathbf{o}(\mathrm{g}(n))$

and

$\mathrm{f}(n) = \mathbf{\Omega}(\mathrm{g}(n))$ is equivalent to $\mathrm{f}(n) = \mathbf{\Theta}(\mathrm{g}(n))$ or $\mathrm{f}(n) = \boldsymbol{\omega}(\mathrm{g}(n))$

Landau Symbols

Some other observations we can make are:

$$\mathrm{f}(n) = \mathbf{\Theta}(\mathrm{g}(n)) \Leftrightarrow \mathrm{g}(n) = \mathbf{\Theta}(\mathrm{f}(n))$$

$$\mathrm{f}(n) = \mathbf{O}(\mathrm{g}(n)) \Leftrightarrow \mathrm{g}(n) = \mathbf{\Omega}(\mathrm{f}(n))$$

$$\mathrm{f}(n) = \mathbf{o}(\mathrm{g}(n)) \Leftrightarrow \mathrm{g}(n) = \boldsymbol{\omega}(\mathrm{f}(n))$$

Big-Θ as an Equivalence Relation

If we look at the first relationship, we notice that $f(n) = \Theta(g(n))$ seems to describe an equivalence relation:

1. $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$
2. $f(n) = \Theta(f(n))$
3. If $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$, it follows that $f(n) = \Theta(h(n))$

Consequently, we can group all functions into equivalence classes, where all functions within one class are big-theta Θ of each other

Big-Θ as an Equivalence Relation

For example, all of

$$n^2 \qquad 100000\, n^2 - 4\, n + 19 \qquad n^2 + 1000000$$

$$323\, n^2 - 4\, n \ln(n) + 43\, n + 10 \qquad 42n^2 + 32$$

$$n^2 + 61\, n \ln^2(n) + 7n + 14 \ln^3(n) + \ln(n)$$

are big-Θ of each other

E.g., $42n^2 + 32 = \Theta(\, 323\, n^2 - 4\, n \ln(n) + 43\, n + 10\,)$

Big-Θ as an Equivalence Relation

We will select just one element to represent the entire class of these functions: n^2

– We could chose any function, but this is the simplest

Big-Θ as an Equivalence Relation

The most common classes are given names:

$\Theta(1)$	constant
$\Theta(\ln(n))$	logarithmic
$\Theta(n)$	linear
$\Theta(n \ln(n))$	"n log n"
$\Theta(n^2)$	quadratic
$\Theta(n^3)$	cubic
$2^n, e^n, 4^n, \ldots$	exponential

Logarithms and Exponentials

Recall that all logarithms are scalar multiples of each other

- Therefore $\log_b(n) = \boldsymbol{\Theta}(\ln(n))$ for any base b

On the other hand, there is no single equivalence class for exponential functions:

- If $1 < a < b$, $\lim_{n\to\infty} \frac{a^n}{b^n} = \lim_{n\to\infty} \left(\frac{a}{b}\right)^n = 0$
- Therefore $a^n = \mathbf{o}(b^n)$

But any exponentially growing function is almost universally undesirable to have!

Logarithms and Exponentials

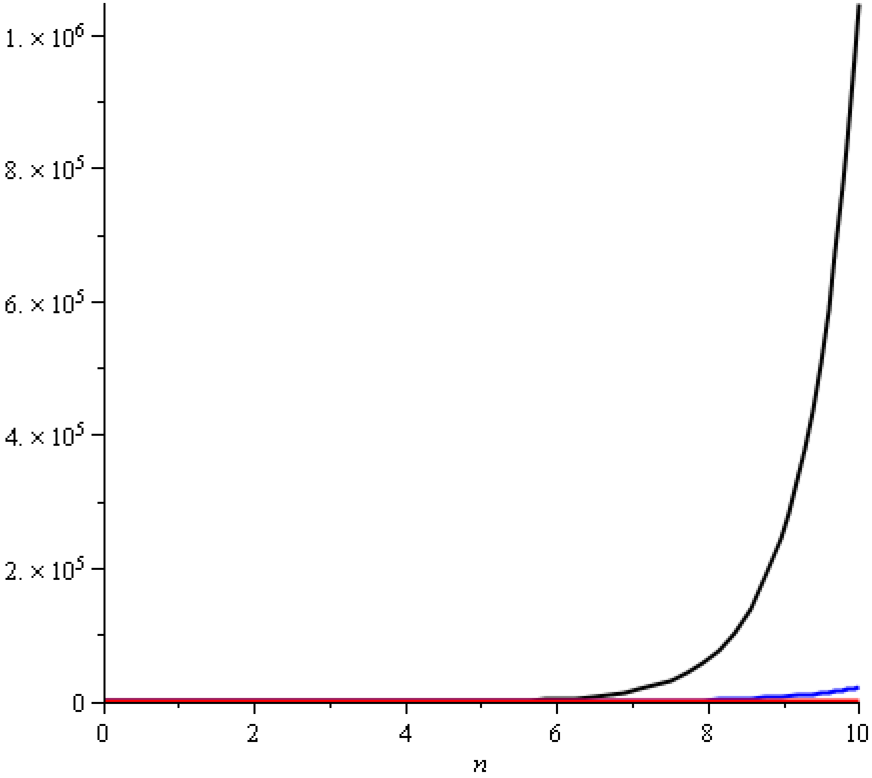
Plotting 2^n, e^n, and 4^n on the range $[1, 10]$ already shows how significantly different the functions grow

Note:

$2^{10} = 1024$

$e^{10} \approx 22\,026$

$4^{10} = 1\,048\,576$



Little-$\mathbf{o}$ as a Weak Ordering

We can show that, for example

$$\ln(n) = \mathbf{o}(n^p)$$

for any $p > 0$

Proof: Using l'Hôpital's rule, we have

$$\lim_{n\to\infty}\frac{\ln(n)}{n^p} = \lim_{n\to\infty}\frac{1/n}{pn^{p-1}} = \lim_{n\to\infty}\frac{1}{pn^p} = \frac{1}{p}\lim_{n\to\infty} n^{-p} = 0$$

Conversely, $1 = \mathbf{o}(\ln(n))$

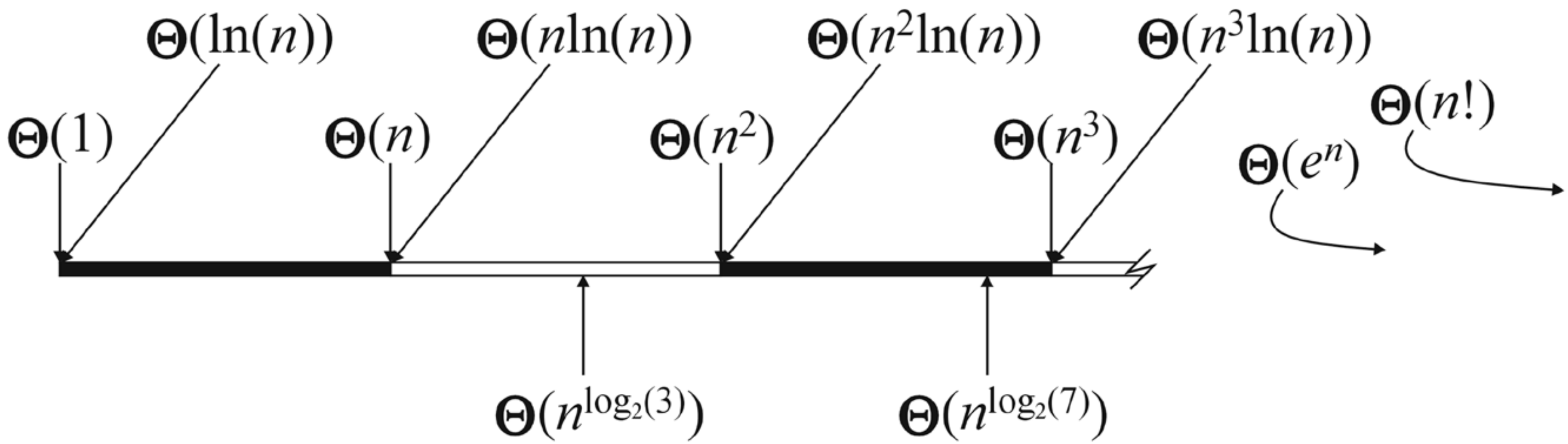
Little-$\mathbf{o}$ as a Weak Ordering

If p and q are real positive numbers where $p < q$

- It follows that $n^p = \mathbf{o}(n^q)$
- For example, matrix-matrix multiplication is $\Theta(n^3)$ but a refined algorithm is $\Theta(n^{\lg(7)})$ where $\lg(7) \approx 2.81$
- Also, $n^p = \mathbf{o}(\ln(n)n^p)$, but $\ln(n)n^p = \mathbf{o}(n^q)$
  - n^p has a slower rate of growth than $\ln(n)n^p$, but
  - $\ln(n)n^p$ has a slower rate of growth than n^q for $p < q$
  - Ex: $n \ln n = \mathbf{o}(n^{1.00000000001})$

Little-o as a Weak Ordering

Graphically, we can shown this relationship by marking these against the real line



Outline

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- Run time of programs
- Best-, worst-, and average-case

Algorithms Analysis

The goal of algorithm analysis is to determine the asymptotic run time or memory requirements based on various parameters

We will use Landau symbols to describe the complexity of algorithms. E.g.,

- Given an array of size n:
  - Selection sort requires $\Theta(n^2)$ time
  - Merge sort, quick sort, and heap sort all require $\Theta(n \ln(n))$ time
- However:
  - Merge sort requires $\Theta(n)$ additional memory
  - Quick sort requires $\Theta(\ln(n))$ additional memory
  - Heap sort requires $\Theta(1)$ additional memory

Algorithms Analysis

To properly investigate the determination of run times asymptotically:

- We will begin with machine instructions and basic operations
- Control statements
- Conditional-controlled loops
- Functions
- Recursive functions

Machine Instructions

Given any processor, it is capable of performing only a limited number of operations

These operations are called *instructions*

The collection of instructions is called the *instruction set*

- The exact set of instructions differs between processors
- MIPS, ARM, x86, 6800, 68k

Any instruction runs in a fixed amount of time (an integral number of CPU cycles)

Operators

There is a close relationship between basic operations and machine instructions, so we may assume each operation requires a fixed number of CPU cycles, i.e., $\Theta(1)$ time:

- Variable assignment `=`
- Integer operations `+ - * / % ++ --`
- Logical operations `&& || !`
- Bitwise operations `& | ^ ~`
- Relational operations `== != < <= >= >`
- Memory allocation and deallocation `new delete`

Operators

Of these, memory allocation and deallocation are the slowest by a significant factor

- A quick test on `eceunix` shows a factor of over 100
- They require communication with the operation system
- This does not account for the time required to call the constructor and destructor

Note that after memory is allocated, the constructor is run

- The constructor may not run in $\Theta(1)$ time

Blocks of Operations

Each operation runs in $\Theta(1)$ time and therefore any fixed number of operations also run in $\Theta(1)$ time, for example:

```
// Swap variables a and b
int tmp = a;
a = b;
b = tmp;
```

Blocks in Sequence

Suppose you have now analyzed a number of blocks of code run in sequence

```
template <typename T>
void update_capacity( int delta ) {
    T *array_old = array;
    int capacity_old = array_capacity;
    array_capacity += delta;
    array = new T[array_capacity];

    for ( int i = 0; i < capacity_old; ++i ) {
        array[i] = array_old[i];
    }

    delete[] array_old;
}
```

$\Theta(1)$

$\Theta(n)$

$\Theta(1)$

To calculate the total run time, add the entries: $\Theta(1 + n + 1) = \Theta(n)$

Blocks in Sequence

```
template <int M, int N>
Matrix<M, N> &Matrix<M, N>::operator= ( Matrix<M, N> const &A ) {
	if ( &A == this ) {
		return *this;
	}

	if ( capacity != A.capacity ) {
		delete [] column_index;
		delete [] off_diagonal;
		capacity = A.capacity;
		column_index = new int[capacity];
		off_diagonal = new double[capacity];
	}

	for ( int i = 0; i < minMN; ++i ) {
		diagonal[i] = A.diagonal[i];
	}

	for ( int i = 0; i <= M; ++i ) {
		row_index[i] = A.row_index[i];
	}

	for ( int i = 0; i < A.size(); ++i ) {
		column_index[i] = A.column_index[i];
		off_diagonal[i] = A.off_diagonal[i];
	}

	return *this;
}
```

$\mathbf{\Theta}(1)$

$\mathbf{\Theta}(1)$

$\mathbf{\Theta}(\min(M, N))$

$\mathbf{\Theta}(M)$

$\mathbf{\Theta}(n)$

$\mathbf{\Theta}(1)$

$$\mathbf{\Theta}(1 + 1 + \min(M, N) + M + n + 1) = \mathbf{\Theta}(M + n)$$

- Note that $\min(M, N) \leq M$
- We cannot say anything about M and n. As a convention, we keep both.

Blocks in Sequence

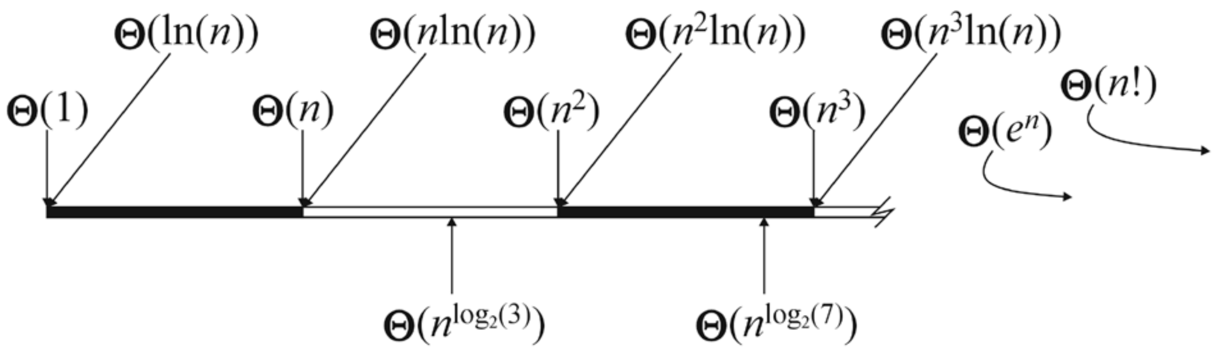
Other examples:

- Run three blocks of code which are $\Theta(1)$, $\Theta(n^2)$, and $\Theta(n)$

  Total run time $\Theta(1 + n^2 + n) = \Theta(n^2)$
- Run two blocks of code which are $\Theta(n \ln(n))$, and $\Theta(n^{1.5})$

  Total run time $\Theta(n \ln(n) + n^{1.5}) = \Theta(n^{1.5})$

Recall this linear ordering from the previous topic



- When considering a sum, take the dominant term

Blocks in Sequence

What if we have both big O and big Theta?

– if the leading term is big-$\mathbf{\Theta}$, then the result must be big-$\mathbf{\Theta}$, otherwise
– if the leading term is big-$\mathbf{O}$, we can say the result is big-$\mathbf{O}$

For example,

$$\mathbf{O}(n) + \mathbf{O}(n^2) + \mathbf{O}(n^4) = \mathbf{O}(n + n^2 + n^4) = \mathbf{O}(n^4)$$

$$\mathbf{O}(n) + \mathbf{\Theta}(n^2) = \mathbf{\Theta}(n^2)$$

$$\mathbf{O}(n^2) + \mathbf{\Theta}(n) = \mathbf{O}(n^2)$$

$$\mathbf{O}(n^2) + \mathbf{\Theta}(n^2) = \mathbf{\Theta}(n^2)$$

Control Statements

Next we will look at the following control statements

These are statements which potentially alter the execution of instructions

- Conditional statements

  `if`, **`switch`**
- Condition-controlled loops

  `for`, **`while`**, **`do-while`**
- Count-controlled loops

  ```
  for i from 1 to 10 do ... end do;        # Maple
  ```
- Collection-controlled loops

  ```
  foreach ( int i in array ) { ... }       // C#
  ```

Control Statements

Given

```
if ( condition ) {
    // true body
} else {
    // false body
}
```

The run time of a conditional statement is:

- the run time of the condition (the test), plus
- the run time of the body which is run

In most cases, the run time of the condition is $\Theta(1)$

Control Statements

In some cases, it is easy to determine which statement must be run:

```
int factorial ( int n ) {
        if ( n == 0 ) {
                return 1;
        } else {
                return n * factorial ( n – 1 );
        }
}
```

Control Statements

In others, it is less obvious

– Find the maximum entry in an array:

```
int find_max( int *array, int n ) {
    max = array[0];

    for ( int i = 1; i < n; ++i ) {
        if ( array[i] > max ) {
            max = array[i];
        }
    }

    return max;
}
```

Control Statements

If we had information about the distribution of the entries of the array, we may be able to determine it

- if the list is sorted (ascending) it will always be run
- if the list is sorted (descending) it will never be run
- if the list is randomly distributed, then??? We don’t know.

Control Statements

- Conditional

```
if C then S1 else S2
```

- Suppose you are doing a big O analysis

  Time(C) + Max(Time(S1),Time(S2)) or

  Time(C)+Time(S1)+Time(S2)

Condition-controlled Loops

The C++ for loop is a condition controlled statement:

```
for ( int i = 0; i < N; ++i ) {
        // ...
}
```

is identical to

```
int i = 0;                          // initialization
while ( i < N ) {                   // condition
    // ...
    ++i;                            // increment
}
```

Condition-controlled Loops

The initialization, condition, and increment usually are single statements running in $\Theta(1)$

```
for ( int i = 0; i < N; ++i ) {
        // ...
}
```

Condition-controlled Loops

If the body does not depend on the variable (in this example, **`i`**), then the run time of

```
for ( int i = 0; i < n; ++i ) {
    // code which is Theta(f(n))
}
```

is $\mathbf{\Theta}(n\ \mathrm{f}(n))$

If the body is $\mathbf{O}(\mathrm{f}(n))$, then the run time of the loop is $\mathbf{O}(n\ \mathrm{f}(n))$

Condition-controlled Loops

For example,

```
int sum = 0;
for ( int i = 0; i < n; ++i ) {
    sum += 1;     // Theta(1)
}
```

This code has run time

$$\Theta(n \cdot 1) = \Theta(n)$$

Condition-controlled Loops

Another example

```
int sum = 0;
for ( int i = 0; i < n; ++i ) {
    for ( int j = 0; j < n; ++j ) {
        sum += 1;     // Theta(1)
    }
}
```

The previous example showed that the inner loop is $\Theta(n)$, thus the outer loop is

$$\Theta(n \cdot n) = \Theta(n^2)$$

Condition-controlled Loops

Another example

```
int sum = 0;
for ( int i = 0; i < n; ++i ) {
    for ( int j = 0; j < i; ++j ) {
        sum += i + j;
    }
}
```

The inner loop is $\mathbf{\Theta}(i)$, hence the outer is

$$\mathbf{\Theta}\left(\sum_{i=0}^{n-1} i\right) = \mathbf{\Theta}\left(\frac{n(n-1)}{2}\right) = \mathbf{\Theta}\left(n^2\right)$$

Analysis of Repetition Statements

Final example:

```
int sum = 0;
for ( int i = 0; i < n; ++i ) {
    for ( int j = 0; j < i; ++j ) {
        for ( int k = 0; k < j; ++k ) {
            sum += i + j + k;
        }
    }
}
```

From inside to out:

$\mathbf{\Theta}(1)$

$\mathbf{\Theta}(j)$

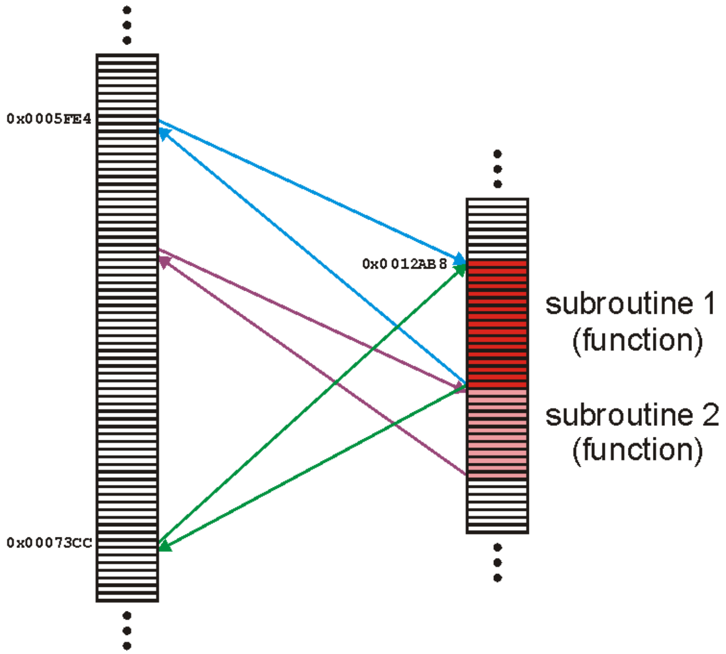
$\mathbf{\Theta}(i^2)$

$\mathbf{\Theta}(n^3)$

Functions

A function (or subroutine) is code which has been separated out:

- repeated operations
  - e.g., mathematical functions
- to group related tasks
  - e.g., initialization



Functions

Because a subroutine (function) can be called from anywhere, we must:

- prepare the appropriate environment
- deal with arguments (parameters)
- jump to the subroutine
- execute the subroutine
- deal with the return value
- clean up

We will assume that the overhead required to make a function call and to return is $\Theta(1)$.

Functions

Given a function $\mathrm{f}(n)$ (the run time of which depends on n) we will associate the run time of $\mathrm{f}(n)$ by some function $\mathrm{T}_{\mathrm{f}}(n)$

- We may write this as $\mathrm{T}(n)$
- This includes the time required to both call and return from the function

Functions

Consider this function:

$T_{set\_union} = 2T_{find} + \Theta(1)$

```
void Disjoint_sets::set_union( int m, int n ) {
        m = find( m );
        n = find( n );

        if ( m == n ) {
                return;
        }

        --num_disjoint_sets;

        if ( tree_height[m] >= tree_height[n] ) {
            parent[n] = m;

            if ( tree_height[m] == tree_height[n] ) {
                ++( tree_height[m] );
                max_height = std::max( max_height, tree_height[m] );
            }
        } else {
            parent[m] = n;
        }
}
```

$2T_{find}$

$\Theta(1)$

Recursive Functions

A function is relatively simple (and boring) if it simply performs operations and calls other functions

Most interesting functions designed to solve problems usually end up calling themselves

– Such a function is said to be *recursive*

Recursive Functions

As an example, we could implement the factorial function recursively:

```
int factorial( int n ) {
    if ( n <= 1 ) {
        return 1;
    } else {
        return n * factorial( n - 1 );
    }
}
```

$$\Theta(1)$$

$$T_!(n-1) + \Theta(1)$$

Recursive Functions

The analysis of the run time of this function yields a recurrence relation:

$$T_{!}(n) = T_{!}(n-1) + \Theta(1) \qquad T_{!}(1) = \Theta(1)$$

This recurrence relation has Landau symbols…

- Replace each Landau symbol with a representative function:

$$T_{!}(n) = T_{!}(n-1) + 1 \qquad T_{!}(1) = 1$$

- Then it is easy to prove that $T_{!}(n) = \Theta(n)$

Recursive Functions

Now consider binary search of a sorted list:

- Check the middle entry
- If we do not find it, check either the left- or right-hand side, as appropriate

Thus, $T(n) = T((n-1)/2) + \Theta(1)$

Also, if $n = 1$, then $T(1) = \Theta(1)$

Recursive Functions

Thus we have to solve:

$$\mathrm{T}(n) = \begin{cases} 1 & n = 1 \\ \mathrm{T}\left(\dfrac{n-1}{2}\right) + 1 & n > 1 \end{cases}$$

Assume $n = 2^k - 1$ where k is an integer

Then $(n - 1)/2 = (2^k - 1 - 1)/2 = 2^{k-1} - 1$

Recursive Functions

Thus, we can write

$$
\begin{aligned}
\mathrm{T}(n) &= \mathrm{T}(2^k - 1) \\
&= \mathrm{T}\left(\frac{2^k - 1 - 1}{2}\right) + 1 \\
&= \mathrm{T}(2^{k-1} - 1) + 1 \\
&= \mathrm{T}\left(\frac{2^{k-1} - 1 - 1}{2}\right) + 1 + 1 \\
&= \mathrm{T}(2^{k-2} - 1) + 2 \\
&\vdots
\end{aligned}
$$

Recursive Functions

Notice the pattern with one more step:

$$
\begin{aligned}
&= \mathrm{T}(2^{k-1}-1)+1 \\
&= \mathrm{T}\left(\frac{2^{k-1}-1-1}{2}\right)+1+1 \\
&= \mathrm{T}(2^{k-2}-1)+2 \\
&= \mathrm{T}(2^{k-3}-1)+3 \\
&\vdots
\end{aligned}
$$

Recursive Functions

Thus, in general, we may deduce that after $k - 1$ steps:

$$\begin{aligned}
\mathrm{T}(n) &= \mathrm{T}(2^k - 1) \\
&= \mathrm{T}(2^{k-(k-1)} - 1) + k - 1 \\
&= \mathrm{T}(1) + k - 1 = k
\end{aligned}$$

because $\mathrm{T}(1) = 1$

Recursive Functions

Thus, $\mathrm{T}(n) = k$, but $n = 2^k - 1$

Therefore $k = \lg(n + 1)$

Recall that $f(n) = \Theta(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$ for $0 < c < \infty$

$$\lim_{n\to\infty} \frac{\lg(n+1)}{\ln(n)} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)\ln(2)}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{(n+1)\ln(2)} = \frac{1}{\ln(2)}$$

Thus, $\mathrm{T}(n) = \boldsymbol{\Theta}(\lg(n + 1)) = \boldsymbol{\Theta}(\ln(n))$

Outline

- Justification for analysis
- Landau symbols
- Run time of programs
- Best-, worst-, and average-case

Cases

When determining the run time of an algorithm, because the data may not be deterministic, we may be interested in:

- Best-case run time
- Average-case run time
- Worst-case run time

In many cases, these will be significantly different

Cases

Searching a list linearly is simple enough

We will count the number of comparisons

- Best case:
  - The first element is the one we're looking for: $\mathbf{O}(1)$
- Worst case:
  - The last element is the one we're looking for, or it is not in the list: $\mathbf{O}(n)$
- Average case?
  - We need some information about the list...

Cases

Assume the item we are looking for is in the list and equally likely distributed

If the list is of size n, then there is a $1/n$ chance of it being in the ith location

Thus, we sum

$$\frac{1}{n}\sum_{i=1}^{n} i = \frac{1}{n}\frac{n(n+1)}{2} = \frac{n+1}{2}$$

which is $\mathbf{O}(n)$

Cases

Suppose we have a different distribution:

- there is a 50% chance that the element is the first
- for each subsequent element, the probability is reduced by ½

We could write:

$$\sum_{i=1}^{n} \frac{i}{2^i} < \sum_{i=1}^{\infty} \frac{i}{2^i} = 2$$

which is $\mathbf{O}(1)$

Cases

- Best-case run time
  - Not so useful

- Average-case run time
  - Need to choose a distribution over input instances
  - Average-case analysis may tell us more about the choice of distributions than about the algorithm itself.

- Worst-case run time
  - Most widely used to capture efficiency in practice.
  - Draconian view, but hard to find effective alternative.
  - Exceptions: some worst-case exponential-time algorithms are widely used because the worst-case instances seem to be rare.
    - E.g., the simplex algorithm

Summary

- Justification for analysis
- Landau symbols
  - $\mathbf{o}$ $\mathbf{O}$ $\mathbf{\Theta}$ $\mathbf{\Omega}$ $\boldsymbol{\omega}$
- Run time of programs
  - Basic operations
  - Control statements
  - Conditional-controlled loops
  - Functions
  - Recursive functions
- Best-, worst-, and average-case