Transformations

Mark M. Fredrickson (`mfredric@umich.edu`)

Computational Methods in Statistics and Data Science (Stats 406)

Transformations

A new variable

Suppose we have some random variable X with a very complex $Q(u)$ so that the inversion method is difficult to implement.

A new variable

Suppose we have some random variable X with a very complex $Q(u)$ so that the inversion method is difficult to implement.

For example, the Normal distribution's $F(x)$ cannot be written in closed form, so we can't easily invert it (R uses extremely detailed look up tables).

A new variable

Suppose we have some random variable X with a very complex $Q(u)$ so that the inversion method is difficult to implement.

For example, the Normal distribution's $F(x)$ cannot be written in closed form, so we can't easily invert it (R uses extremely detailed look up tables).

Then we must search for **transformations** of random variables that we can sample from.

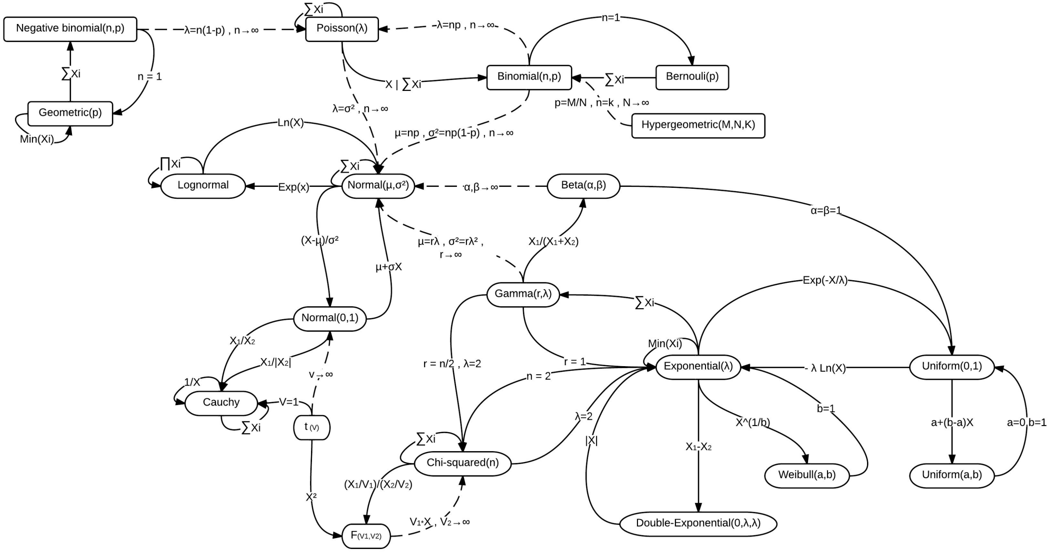
A new variable

Suppose we have some random variable X with a very complex $Q(u)$ so that the inversion method is difficult to implement.

For example, the Normal distribution's $F(x)$ cannot be written in closed form, so we can't easily invert it (R uses extremely detailed look up tables).

Then we must search for **transformations** of random variables that we can sample from.

General strategy, **show** $F(x) = G(x)$, where G is the CDF of the transformation.


Negative binomial(n,p)
λ=n(1-p) , n→∞
Poisson(λ)
∑Xi
λ=np , n→∞
n=1
Binomial(n,p)
∑Xi
Bernouli(p)
∑Xi
n = 1
Geometric(p)
Min(Xi)
X | ∑Xi
λ=σ² , n→∞
p=M/N , n=k , N→∞
Hypergeometric(M,N,K)
μ=np , σ²=np(1-p) , n→∞
Ln(X)
∏Xi
∑Xi
Lognormal
Exp(x)
Normal(μ,σ²)
α,β→∞
Beta(α,β)
α=β=1
(X-μ)/σ²
μ+σX
μ=rλ , σ²=rλ² ,
r→∞
X1/(X1+X2)
Gamma(r,λ)
Exp(-X/λ)
∑Xi
Normal(0,1)
X1/X2
X1/|X2|
v→∞
Min(Xi)
r = n/2 , λ=2
r = 1
Exponential(λ)
- λ Ln(X)
Uniform(0,1)
n = 2
1/X
Cauchy
V=1
t (V)
∑Xi
λ=2
b=1
X^(1/b)
|X|
X1-X2
a+(b-a)X
a=0,b=1
∑Xi
Chi-squared(n)
Weibull(a,b)
Uniform(a,b)
(X1/V1)/(X2/V2)
X²
V1*X , V2→∞
F(V1,V2)
Double-Exponential(0,λ,λ)

Creating variables from functions of other variables

Sometimes we define **RVs as functions of other random variables**.

Examples:

- The "log Normal" distribution:

$$X = \exp(Y), \quad Y \sim N(\mu, \sigma^2)$$

Creating variables from functions of other variables

Sometimes we define **RVs as functions of other random variables**.

Examples:

- The "log Normal" distribution:

$$X = \exp(Y), \quad Y \sim N(\mu, \sigma^2)$$

- The χ^2_k distribution: $\sum_{i=1}^{n} Z_i^2, Z_i \overset{\text{iid}}{\sim} N(0,1)$

Creating variables from functions of other variables

Sometimes we define **RVs as functions of other random variables**.

Examples:

- The "log Normal" distribution:

$$X = \exp(Y), \quad Y \sim N(\mu, \sigma^2)$$

- The χ_k^2 distribution: $\sum_{i=1}^n Z_i^2, Z_i \overset{\text{iid}}{\sim} N(0,1)$
- Test statistics/estimators: $f(X_1, X_2, \ldots, X_n)$

More difficult case: Working backwards

Suppose we want to sample X, can we find transformations that lead to it?

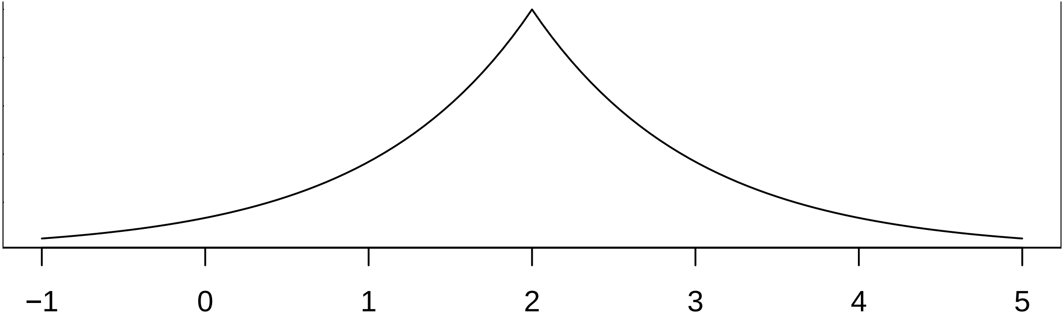
Tips and tricks:

- Reduce to a simpler case
- Write down $f(x)$ or $F(x)$ and find patterns.
- Try conditioning on some event, then randomly generate that event.
- Calculus techniques on $f(x)$ (change of variables)

Here is the PDF of the **Laplace distribution**:

$$f(x) = \frac{1}{2} \exp\{-|x-\theta|\}$$

For example, when $\theta = 2$



Example: Laplace distribution

To generate:

$$X \sim \text{Laplace}(\theta)$$

Claim: X can be generated by a transformation:

$$X = SM + \theta$$

where:

$$P(S = 1) = 0.5, P(S = -1) = 0.5, \quad M \sim \text{Exp}(1)$$

Example: Laplace distribution

To generate:

$$X \sim \text{Laplace}(\theta)$$

Claim: X can be generated by a transformation:

$$X = SM + \theta$$

where:

$$P(S = 1) = 0.5, P(S = -1) = 0.5, \quad M \sim \text{Exp}(1)$$

Where does this come from?

Reducing to a simpler case

First, let's deal with the **parameter θ**.

Let $Y \sim \text{Laplace}(\theta)$ with CDF:

$$P(Y \leq t) = \int_{-\infty}^{t} \frac{1}{2} e^{-|y-\theta|} \, dy$$

Reducing to a simpler case

First, let's deal with the **parameter θ**.

Let $Y \sim \text{Laplace}(\theta)$ with CDF:

$$P(Y \leq t) = \int_{-\infty}^{t} \frac{1}{2} e^{-|y-\theta|}\, dy$$

Perform a **change of variables**, $x = y - \theta$ so $y = x + \theta$ and $dy = dx$, so that

$$P(Y \leq t) = \int_{-\infty}^{t-\theta} \frac{1}{2} e^{-|x|}\, dx$$

Reducing to a simpler case

First, let's deal with the **parameter** θ.

Let $Y \sim \text{Laplace}(\theta)$ with CDF:

$$P(Y \leq t) = \int_{-\infty}^{t} \frac{1}{2} e^{-|y-\theta|}\, dy$$

Perform a **change of variables**, $x = y - \theta$ so $y = x + \theta$ and $dy = dx$, so that

$$P(Y \leq t) = \int_{-\infty}^{t-\theta} \frac{1}{2} e^{-|x|}\, dx = P(X \leq t - \theta)$$

Reducing to a simpler case

First, let's deal with the **parameter** θ.

Let $Y \sim$ Laplace(θ) with CDF:

$$P(Y \leq t) = \int_{-\infty}^{t} \frac{1}{2} e^{-|y-\theta|}\, dy$$

Perform a **change of variables**, $x = y - \theta$ so $y = x + \theta$ and $dy = dx$, so that

$$P(Y \leq t) = \int_{-\infty}^{t-\theta} \frac{1}{2} e^{-|x|}\, dx = P(X \leq t - \theta) = P(X + \theta \leq t)$$

where $X \sim \mathit{Laplace}(0)$. In other words, Y is a **Laplace(0) added to** θ.

Using symmetry of X

Notice that X (with $\theta = 0$) is **symmetric about 0**:

$$f(x) = \frac{1}{2}\exp\{-|x|\} = f(-x)$$

Using symmetry of X

Notice that X (with $\theta = 0$) is **symmetric about 0**:

$$f(x) = \frac{1}{2}\exp\{-|x|\} = f(-x) \Rightarrow P(X < 0) = P(X > 0) = \frac{1}{2}$$

Using symmetry of X

Notice that X (with $\theta = 0$) is **symmetric about 0**:

$$f(x) = \frac{1}{2}\exp\{-|x|\} = f(-x) \Rightarrow P(X < 0) = P(X > 0) = \frac{1}{2}$$

We can use the **disjoint events** $X < 0$ and $X > 0$ to write:

$$P(X \leq t) = P(X \leq t \mid X < 0)P(X < 0) + P(X \leq t \mid X > 0)P(X > 0)$$

Using symmetry of X

Notice that X (with $\theta = 0$) is **symmetric about 0**:

$$f(x) = \frac{1}{2}\exp\{-|x|\} = f(-x) \Rightarrow P(X < 0) = P(X > 0) = \frac{1}{2}$$

We can use the **disjoint events** $X < 0$ and $X > 0$ to write:

$$P(X \leq t) = P(X \leq t \mid X < 0)P(X < 0) + P(X \leq t \mid X > 0)P(X > 0)$$

As we did with the truncated Normal, notice that (for $t > 0$):

$$P(X \leq t \mid X > 0) = \int_0^t \frac{f(x)}{P(X > 0)}\,dx$$

Using symmetry of X

Notice that X (with $\theta = 0$) is **symmetric about 0**:

$$f(x) = \frac{1}{2}\exp\{-|x|\} = f(-x) \Rightarrow P(X < 0) = P(X > 0) = \frac{1}{2}$$

We can use the **disjoint events** $X < 0$ and $X > 0$ to write:

$$P(X \le t) = P(X \le t \mid X < 0)P(X < 0) + P(X \le t \mid X > 0)P(X > 0)$$

As we did with the truncated Normal, notice that (for $t > 0$):

$$P(X \le t \mid X > 0) = \int_0^t \frac{f(x)}{P(X > 0)}\,dx = \int_0^t \frac{(1/2)\exp\{-|x|\}}{1/2}\,dx$$

Using symmetry of X

Notice that X (with $\theta = 0$) is **symmetric about 0**:

$$f(x) = \frac{1}{2}\exp\{-|x|\} = f(-x) \Rightarrow P(X < 0) = P(X > 0) = \frac{1}{2}$$

We can use the **disjoint events** $X < 0$ and $X > 0$ to write:

$$P(X \le t) = P(X \le t \mid X < 0)P(X < 0) + P(X \le t \mid X > 0)P(X > 0)$$

As we did with the truncated Normal, notice that (for $t > 0$):

$$P(X \le t \mid X > 0) = \int_0^t \frac{f(x)}{P(X > 0)}\,dx = \int_0^t \frac{(1/2)\exp\{-|x|\}}{1/2}\,dx = \int_0^t \exp\{-|x|\}\,dx$$

Recognizing CDF

We saw that $X \mid X > 0$ had the CDF:

$$P(X \leq t \mid X > 0) = \int_0^t \exp\{-|x|\}\, dx$$

Recognizing CDF

We saw that $X \mid X > 0$ had the CDF:

$$P(X \leq t \mid X > 0) = \int_0^t \exp\{-|x|\}\, dx = \int_0^t \exp\{-x\}\, dx$$

Recognizing CDF

We saw that $X \mid X > 0$ had the CDF:

$$P(X \leq t \mid X > 0) = \int_0^t \exp\{-|x|\}\, dx = \int_0^t \exp\{-x\}\, dx$$

We recognize that $X \mid X > 0 = M \sim \text{Exp}(1)$ and by symmetry we see that $X \mid X < 0$ is $-M$.

Recognizing CDF

We saw that $X \mid X > 0$ had the CDF:

$$P(X \leq t \mid X > 0) = \int_0^t \exp\{-|x|\}\, dx = \int_0^t \exp\{-x\}\, dx$$

We recognize that $X \mid X > 0 = M \sim \text{Exp}(1)$ and by symmetry we see that $X \mid X < 0$ is $-M$.

Let A be the event that $X > 0$, which has $P(A) = 1/2$.

- When A occurs $X = M$.

Recognizing CDF

We saw that $X \mid X > 0$ had the CDF:

$$P(X \leq t \mid X > 0) = \int_0^t \exp\{-|x|\}\, dx = \int_0^t \exp\{-x\}\, dx$$

We recognize that $X \mid X > 0 = M \sim \text{Exp}(1)$ and by symmetry we see that $X \mid X < 0$ is $-M$.

Let A be the event that $X > 0$, which has $P(A) = 1/2$.

- When A occurs $X = M$.
- When A^c occurs, $X = -M$

Recognizing CDF

We saw that $X \mid X > 0$ had the CDF:

$$P(X \leq t \mid X > 0) = \int_0^t \exp\{-|x|\}\, dx = \int_0^t \exp\{-x\}\, dx$$

We recognize that $X \mid X > 0 = M \sim \text{Exp}(1)$ and by symmetry we see that $X \mid X < 0$ is $-M$.

Let A be the event that $X > 0$, which has $P(A) = 1/2$.

- When A occurs $X = M$.
- When A^c occurs, $X = -M$
- In other words, if $P(S = 1) = P(S = -1) = 1/2$, then $X = SM$

Putting it all together

We first saw that if

$$Y \sim \text{Laplace}(\theta)$$

Putting it all together

We first saw that if

$$Y \sim \text{Laplace}(\theta) \Rightarrow Y = X + \theta, X \sim \text{Laplace}(0)$$

Putting it all together

We first saw that if

$$Y \sim \text{Laplace}(\theta) \Rightarrow Y = X + \theta, X \sim \text{Laplace}(0)$$

How to generate Laplace(0)? We found that could be represented as:

$$X = SM, M \sim \text{Exp}(1), P(S = 1) = P(S = -1) = 1$$

Putting it all together

We first saw that if

$$Y \sim \text{Laplace}(\theta) \Rightarrow Y = X + \theta, X \sim \text{Laplace}(0)$$

How to generate Laplace(0)? We found that could be represented as:

$$X = SM, M \sim \text{Exp}(1), P(S = 1) = P(S = -1) = 1$$

In R:

```
> rlaplace <- function(n, theta) {
+     sample(c(-1, 1), n, replace = T) * rexp(n) + theta }
```

Checking results