Week 01: Statistical Review

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Computational Methods in Statistics and Data Science (Stats 406)

Probability and Random Variables

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Some example events:

- A coin coming up heads.
- Picking a blue ball and then a red ball from an box.
- The value of a stock exceeding \$100.
- Going bust or making over \$100 at the roulette table after 1 play.

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Two events are **independent** if (and only if):

$$P(A, B) = P(A)P(B)$$

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Independence also implies

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

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Notation: uppercase X is the random variable, lower case x is a fixed value.

Distribution Functions

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- More generally, $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$.
- F is **non-decreasing**: $F(x_1) \leq F(x_2)$ for $x_1 < x_2$.
- F is **right continuous**: $\lim_{\epsilon \to 0^+} F(x + \epsilon) = F(x)$.

Continuous Distributions

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Note: we often suppress the fact that $f(x) = 0$ for $x \notin \mathcal{D}$

Example: $f(x) = 2(1-x)$

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$$P(X \leq 0.5) = \frac{3}{4}$$

Example: Normal Distribution

The **Normal distribution** ("Normal" is the name, not a descriptor) has PDF:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ \frac{-(x-\mu)^2}{2\sigma^2} \right\}$$

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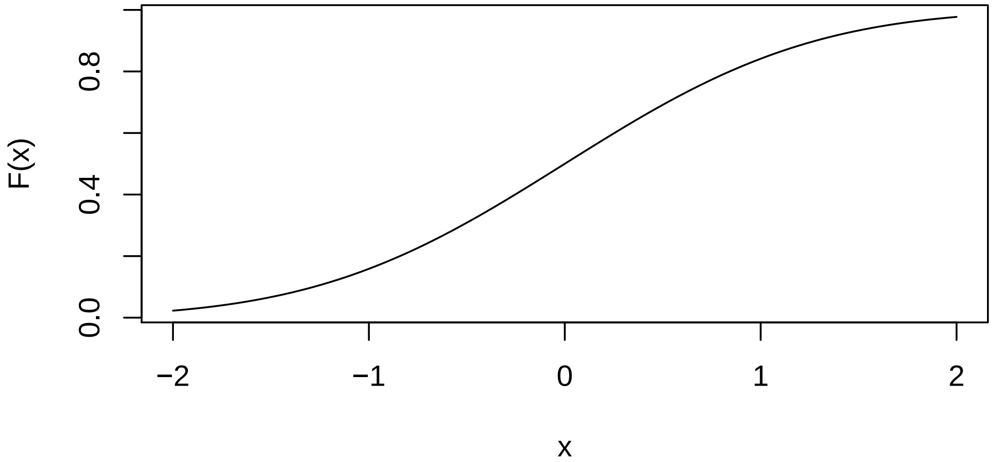
There is no closed form for F, so we need to use look up tables that have been pre-computed using numerical procedures.

$N(0, 1)$



Discrete Distributions

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As with the continuous case, we can build the CDF, from the PMF:

$$F(x) = \sum_{i=-\infty}^{x} P(X = i)$$

Bernoulli and Binomial Distributions

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The **sum of n independent** Bernoulli variables has **Binomial** distribution:

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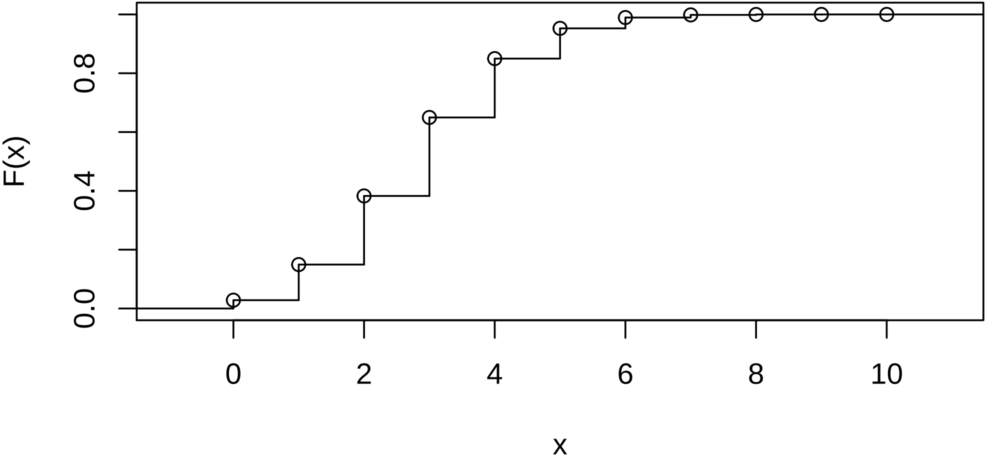
$$P(Y=y)=\binom{n}{y}\theta^{y}(1-\theta)^{n-y}$$

Bernoulli(10, 0.3)



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Generally, the same properties hold for joint distributions as for univariate distributions. E.g.,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$$

Example: Constrained support

Here is a density for RVs X and Y:

$$f(x,y) = \begin{cases} cx^2y & x^2 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- Compute c to make this a valid distribution
- Compute $P(X \geq Y)$

Finding c

By the **laws of total probability**,

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$$= \int_{-1}^{1}\int_{x^2}^{1} cx^2 y\,dy\,dx$$

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So $c = 21/4$.

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Notice: since $y \geq x^2$, it is also the case that $y \geq 0$. Therefore $x \geq 0$ for this set.

$$\int_0^1 \int_{x^2}^{x} \frac{21}{4} x^2 y \, dy \, dx = \frac{3}{20}$$

Marginal Distributions and Independence

We can **integrate out** one variable to get the **marginal distribution of the other**:

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E.g., immediately from previous example $f(x, y) = (21/4)x^2y$,

$$f(x) = \frac{21}{8}\left(x^2 - x^6\right), -1 \leq x \leq 1$$

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Taking the limit as $\epsilon \to 0$, we get the **conditional density (or mass) function for** $X \mid Y = u$:

$$f(x \mid y = u) = \frac{f_{xy}(x, u)}{f_y(u)}$$

Example: Conditional Distribution

Suppose

$$f(x, y) = \theta^2 (xy)^{\theta-1}, \quad 0 \leq x \leq 1, 0 \leq y \leq 1, \theta \geq 0$$

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$$f(y \mid x) = \frac{f(x, y)}{f(x)} = \theta y^{\theta-1}$$

Factoring Independent RVs

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Factorizing the joint density (mass) function is both **necessary and sufficient** for independence.

This result also applies to CDFs: $F_{xy}(a,b) = F_x(a)F_y(b)$

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We often want to "average over" X to get a sense of a typical value for $g(X)$. We define the **expectation** of $g(X)$ as:

$$E(g(X)) = \sum_{i=-\infty}^{\infty} P(X = x)g(x) \qquad \text{(discrete)}$$

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)\,dx \qquad \text{(continuous)}$$

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We call the **expectation of the identity function**, $E(X)$ the **mean of X**.

We call $E\left[(X - E(X))^2\right]$ the **variance**.

Example: Computing $E(Y)$ for Bernoulli(10, 0.3)

Recall that $f(y)$ is

$$P(Y = y) = \binom{10}{y} (0.3)^y (0.7)^{10-y}$$

and the support is the integers from zero to ten.

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```
> terms <- map_dbl(0:10, function(i) {
+   choose(10, i) * 0.3^i * 0.7^(10 - i) * i
+ })
> sum(terms)

[1] 3
```

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[1] 1.311
```

Example: Expectation for a Continuous RV

Suppose we have

$$f(x) = \theta x^{\theta-1}, \quad 0 \leq x \leq 1, \theta > 0$$

and we want to find **the variance of X**, $\mathrm{E}\left(X^2\right) - \mathrm{E}\left(X\right)^2$.

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Conditional Expectation

Recall that $X \mid Y = y$ is a random variable, so it we can consider **the conditional expectation of X given $Y = y$**:

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The result will be a function of y, i.e., $h(y) = \int_{-\infty}^{\infty} x\, f(x \mid y)\, dx$. This leads to the useful result of the **law of iterated expectation**:

$$\mathrm{E}\left(h(Y)\right) = \mathrm{E}\left(\mathrm{E}\left(X \mid Y\right)\right) = \mathrm{E}\left(X\right)$$

Properties of Expectations

Some useful facts (which also apply to $g(X)$, $h(Y)$, etc):

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Analytic solution for Binomial mean

Recall we can think of $Y \sim \text{Binomial}(n, \theta)$:

$$Y = \sum_{i=1}^{n} X_i, \quad X_1, X_2, \dots X_n \overset{\text{iid}}{\sim} \text{Bernoulli}(\theta)$$

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Finally,

$$E(X_1) = \theta \times 1 + (1 - \theta) \times 0 = \theta \Rightarrow E(Y) = n\theta$$

- **Random variables** are random outcomes described by **real numbers**
- All RVs have **(cumulative) distribution function**: $F(x) = \Pr X \leq x$
- **Continuous** RV: (a) probability density functions $f(x)$, (b) **expectation** is $E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\, dx$.
- **Discrete** RVs: (a) probability mass functions $p(x)$, (b) expectation is $E(g(X)) = \sum_{x \in \Omega} p(x) x$
- **Independence**: the joint distribution is the production of the marginal distributions.

Inference

What is inference?

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We call these process **inference**. We want to tools that **behave well** when performing inference (i.e., operating characteristics).

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Important: statistics are **random variables** too!

Estimation

If $X_1, X_2, \ldots X_n$ are from the **same distribution**, we say that they are **identical**. Often, we also assume **independence** (IID):

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We often wish to estimate θ for the population using an **estimator** (a statistic):

$$\hat{\theta}(X_1, X_2, \dots, X_n) = \text{ do something with the } X_i \text{ values}$$

Sampling Distributions

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- **Variance**: $\text{Var}(\hat{\theta})$
- **Mean Squared Error (MSE)**:

$$E\left[(\hat{\theta} - \theta)^2\right] = \text{Bias}^2 + \text{Var}(\hat{\theta})$$

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Conclusion, the **MSE of a single observation is n times larger than the MSE of $\bar{X}$**.

Method of Moments Estimation

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Substituting $\bar{X}$ for $\mathrm{E}(X)$, and solving for θ we get:

$$\bar{X} = \frac{\hat{\theta}}{\hat{\theta} + 1} \Rightarrow \hat{\theta} = \frac{\bar{X}}{1 - \bar{X}}$$

Likelihood Functions

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Note: When X_i **are IID**,

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MLEs have many nice properties including **invariance** and **low variance**.

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Some useful things to remember:

- $\log(e^x) = x$
- $\exp(x + y) = \exp(x)\exp(y)$, so $\log(xy) = \log(x) + \log(y)$,
- $\log(x^y) = y \log(x)$, with previous we get $\log(x/y) = \log(x) - \log(y)$
- $$\frac{d}{dx}\log(x) = \frac{1}{x}$$

Example: Normal mean, $\sigma^2 = 1$

The likelihood for μ in $N(\mu, 1)$ is

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Our standard calculus strategy is to take the derivative and set to zero:

$$0 = \sum_{i=1}^{n} -(x_i - \mu) \Rightarrow n\mu = \sum_{i=1}^{n} x_i \Rightarrow \hat{\mu} = \bar{X}$$

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NB: notice that for $x \leq 1$, $\log(x) \leq \log(1) = 0$, so $\hat{\theta} \geq 0$.

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H_0 :& E(X) \leq \mu_0 && \text{vs. } H_1 : E(X) > \mu_0 \\
H_0 :& F(x, y) = F_x(x) F_x(y) && \text{v.s. } H_1 : F(x, y) \neq F_x(x) F_y(y)
\end{aligned}
$$

Hypothesis Tests

Another place we use statistics is for **hypothesis tests**.

A hypothesis test requires stating a **null hypothesis** H_0 and an **alternative hypothesis** H_1. Some examples:

$$\begin{aligned} H_0 :& X \overset{\text{iid}}{\sim} F_0 && \text{vs. } H_1 : X \overset{\text{iid}}{\sim} F_1 \\ H_0 :& E(X) \leq \mu_0 && \text{vs. } H_1 : E(X) > \mu_0 \\ H_0 :& F(x, y) = F_x(x)F_x(y) && \text{v.s. } H_1 : F(x, y) \neq F_x(x)F_y(y) \end{aligned}$$

Goal: Either **accept** the null hypothesis or **reject** the null hypothesis in favor of the alternative.

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Useful framework: pick a **maximum Type I error** α and then pick a test that has **good power**.

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Usually, we pick $\mathcal{R}$ so that we maintain our α-level:

$$P(T \in \mathcal{R} \,|\, H_0) \leq \alpha \quad \text{(size less than level)}$$

and generates high power:

$$P(T \in \mathcal{R} \,|\, H_1) \text{ is large}$$

Example: Testing $\mu_0 = 0$ vs. $\mu_0 = 1$

Suppose we assume that

$$X_i \overset{\text{iid}}{\sim} N(\mu, 1) \quad (\mu \text{ unknown})$$

and want to test:

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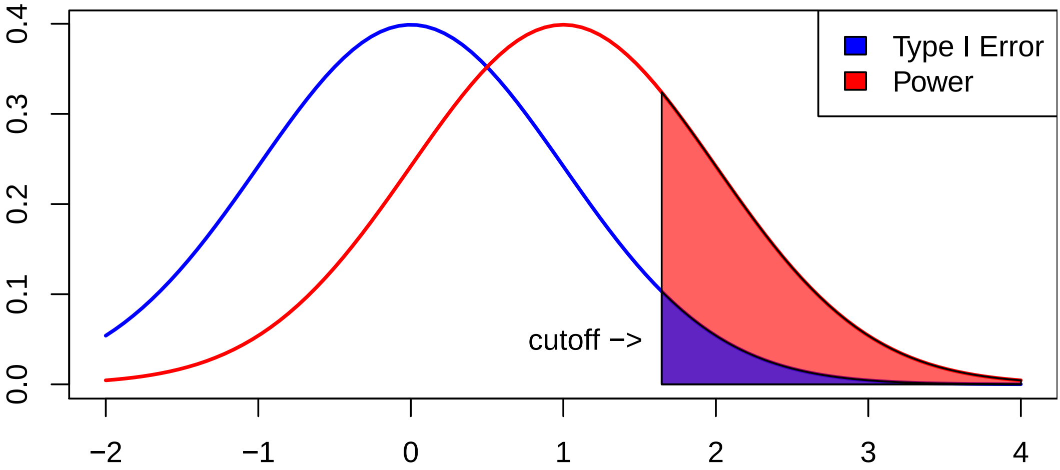

Type I Error
Power
cutoff ->
0.0
0.1
0.2
0.3
0.4
−2
−1
0
1
2
3
4

Computing rejection region and power ($n = 2$)

Computing the rejection region when $H_0 : \mu = 0$:

```
> n <- 2
> (cutoff <- qnorm(0.95, mean = 0, sd = 1/n))

[1] 0.8224
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```

```
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Computing the power of the test when $H_1 : \mu = 1$:

```
> 1 - pnorm(cutoff, mean = 1, sd = 1/n)
```

```
[1] 0.6388
```

Summary: Inference

- Write down quantities of interest as **population parameters**.
- Use **sample statistics** make decisions about parameters.
- **Estimation**: make reasonable guess, **sampling distribution** defines uncertainty
- **Hypothesis tests**: see if data conform to hypothesis, **null** and **alternative** distributions define uncertainty.
- **Method of moments** and **maximum likelihood** will be our two main estimation techniques.