Bootstrap For Complex Data

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Computational Methods in Statistics and Data Science (Stats 406)

Bootstrap Review

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- Estimate **sampling distribution** of T using draws from $T^* = T(X_1^*, \dots, X_n^*)$
- Use the bootstrap distribution to form **confidence intervals** (several methods)

Example: Student academic test scores

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Research in Social Stratification and Mobility 30 (2012) 97–112

Research in Social Stratification and Mobility

http://elsevier.com/locate/rssm

Differences between Hispanic and non-Hispanic families in social capital and child development: First-year findings from an experimental study

Adam Gamoran [a,*], Ruth N. López Turley [b], Alyn Turner [a], Rachel Fish [a]

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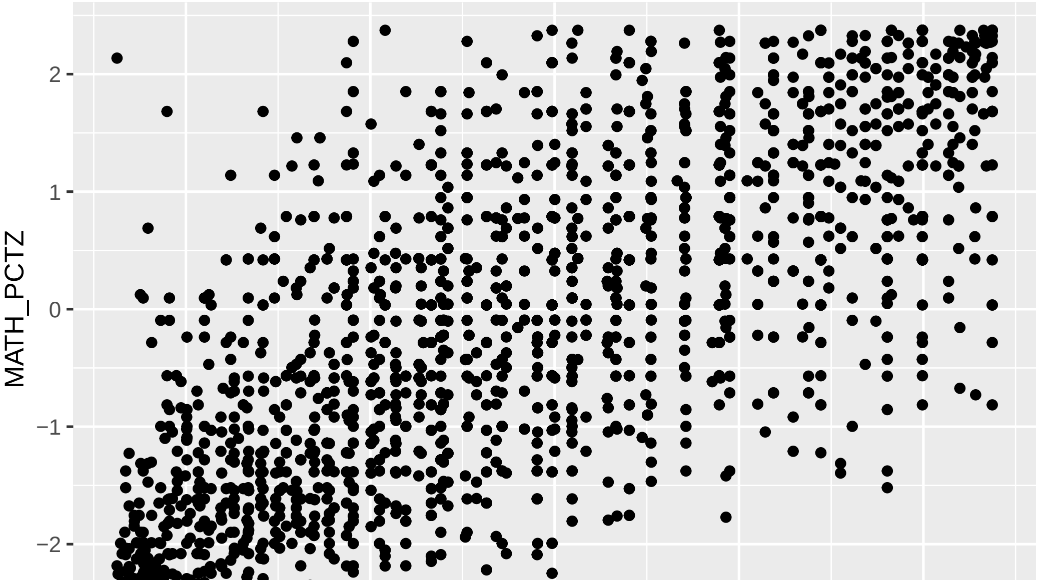
Received 15 March 2011; received in revised form 5 August 2011; accepted 9 August 2011

Reading Test Scores



Test Score Correlation

```
> library(boot)
> median_boot <- function(x, index) {
+   xstar <- x[index, ]
+   cor(xstar$READ_PCTZ, xstar$MATH_PCTZ)
+ }
> cor_boot <- boot(gamoran, statistic = median_boot, R = 100)
```

Confidence Intervals

```
> boot.ci(cor_boot, type = "basic")
BOOTSTRAP CONFIDENCE INTERVAL CALCULATIONS
Based on 100 bootstrap replicates

CALL :
boot.ci(boot.out = cor_boot, type = "basic")

Intervals :
Level      Basic
95%   ( 0.6924,  0.7443 )
Calculations and Intervals on Original Scale
Some basic intervals may be unstable
```

Bootstrap for Dependence and Structure

Bootstrap basics

Recall our usual setup for the bootstrap:

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We'll consider two cases:

- Bootstrap for **time series** (stochastic processes)
- **Stratified bootstrap** for different groups

Time series models

Suppose we are interested in the following **stochastic process**:

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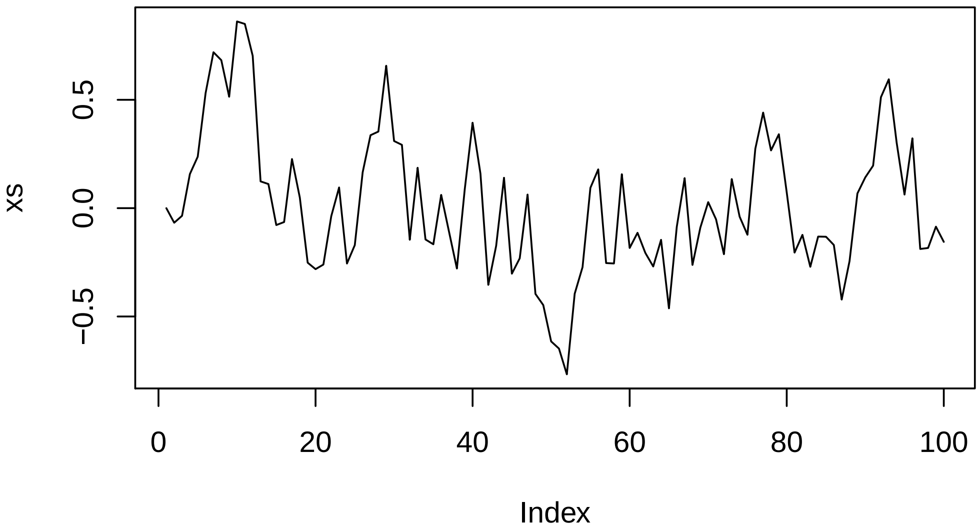
and we are interested in estimating ρ. Here is some simulated example data:

```
> rho <- 0.75
> k <- 100
> xs <- numeric(k)
> xs[1] <- 0
> for (i in 2:k) {
+     xs[i] <- rho * xs[i - 1] + (rbeta(1, 2, 2) - 0.5)
+ }
```


xs
0.5
0.0
−0.5
0
20
40
60
80
100
Index

Estimating ρ

One way we can estimate ρ is to perform **OLS on $X(t)$ given $X(t-1)$**:

```
> (est <- lm(xs[2:k] ~ xs[1:(k - 1)] - 1)) # no intercept term

Call:
lm(formula = xs[2:k] ~ xs[1:(k - 1)] - 1)


Coefficients:
xs[1:(k - 1)]
         0.71
```

What is the variation? Can we get bootstrap confidence intervals?

Parametric bootstrap

Here we need **bootstrap samples with dependence**:

$$X(0)^*, X(1)^*, \ldots, X(t)^* \quad \text{such that: } X(s)^* = \rho X(s-1)^* + e(s)^*$$

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We have already seen **parametric bootstrapping** in which we use **a model** to generate the bootstrap samples.

Idea: starting from the observed $X(0)$, draw from the **estimated distribution of e** to create a **bootstrap series**:

$$X(1)^* = \hat{\rho} X(0) + e(1)^*, \ldots, X(t)^* = \hat{\rho} X(t-1)^* + e(t)^*$$

```
> rhohat <- coef(est)
> et_hat <- xs[2:k] - predict(est) # estimate residuals
> est_rho_stat <- function(x, index) {
+   estar <- x[index]
+   new_series <- numeric(k)
+   new_series[1] <- xs[1] # starts at the same point
+   for (i in 2:k) {
+     new_series[i] <- rhohat * new_series[i - 1] + estar[i]
+   }
+   coef(lm(new_series[2:k] ~ new_series[1:(k - 1)] - 1))
+ }
```

CIs for ρ

```
> boot_rho <- boot(et_hat, est_rho_stat, R = 1000)
> boot.ci(boot_rho, type = "perc")
```

```
BOOTSTRAP CONFIDENCE INTERVAL CALCULATIONS
Based on 1000 bootstrap replicates

CALL :
boot.ci(boot.out = boot_rho, type = "perc")

Intervals :
Level     Percentile
95%   ( 0.5263,  0.8143 )
Calculations and Intervals on Original Scale
```

Two Sample Problems

In the previous example, all the $e(t)$ were **identical**, but the $X(t)$ were not **independent**.

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Two sample problems consider the situation of having two independent samples

$$X_1, \ldots, X_n \overset{\text{iid}}{\sim} F \quad Y_1, \ldots, Y_m \overset{\text{iid}}{\sim} G$$

and estimating quantities such as the **difference of means**:

$$\Delta = \mathrm{E}(X) - \mathrm{E}(Y)$$

(notice: if $F = G$, $\Delta = 0$, though not necessarily the converse.)

Students in Cities

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In fact, we might be interested in knowing if the two populations have the same distribution (or same mean, etc). In particular, we'll focus on $\Delta = \mathrm{E}\,(\text{San Antonio}) - \mathrm{E}\,(\text{Phoenix})$.

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We'll bootstrap in the two groups separately to get estimates of means and then combine them.

Estimating the difference of means

The boot function has a strata (groups) argument we can use.

```
> mean_diff <- function(x, index) {
+     xstar <- x[index, ] # boot will handle stratification for us
+     mean(xstar$READ_PCTZ[xstar$PH.AZ], na.rm = TRUE) -
+         mean(xstar$READ_PCTZ[!xstar$PH.AZ], na.rm = TRUE)
+ }
> gam.boot <- boot(gamoran,
+                  statistic = mean_diff,
+                  strata = gamoran$PH.AZ,
+                  R = 1000)
```

CI

```
> (gbci <- boot.ci(gam.boot, type = "basic"))

BOOTSTRAP CONFIDENCE INTERVAL CALCULATIONS
Based on 1000 bootstrap replicates

CALL :
boot.ci(boot.out = gam.boot, type = "basic")

Intervals :
Level      Basic
95%   ( 0.1398,  0.3782 )
Calculations and Intervals on Original Scale
```

Interpreting Results

We saw that the **95% confidence interval for Δ did not include zero**:

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Additionally, we can **reject** any hypothesis that says that **San Antonio has a higher average reading score**. (All at $\alpha = 0.05$ level).

Let's be clear that this is not a **causal conclusion**, merely that on average San Antonio students tend to score lower.

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- For dependent data, generate samples following dependence.
- For structured data, incorporate the structure.