Permutation and Randomization Tests

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Computational Methods in Statistics and Data Science (Stats 406)

Permutation Tests

Permutation invariant test statistics

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A close analog: assume $Y = \beta_0 + \beta_1 X + \epsilon$ and condition on $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$. The $\hat{\epsilon} = y - \hat{y}$ is **permutation invariant** (we can shuffle them around freely).

Permutation tests

We will consider a variety of **hypothesis tests** that exploit invariance. Suppose that Z is invariant conditional on the sample, then we need:

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Tests of this form are known as **permutation tests** when **sampling** and **randomization tests** when analyzing **randomized controlled trials**.

Example: Developing a test for symmetry

Suppose we have some data from

$$X_i \overset{\text{iid}}{\sim} F, -\infty < x < \infty, \text{continuous}$$

and we want to test the hypothesis:

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We'll develop a test in two parts: (a) a test for medians, and (b) a permutation test for symmetry

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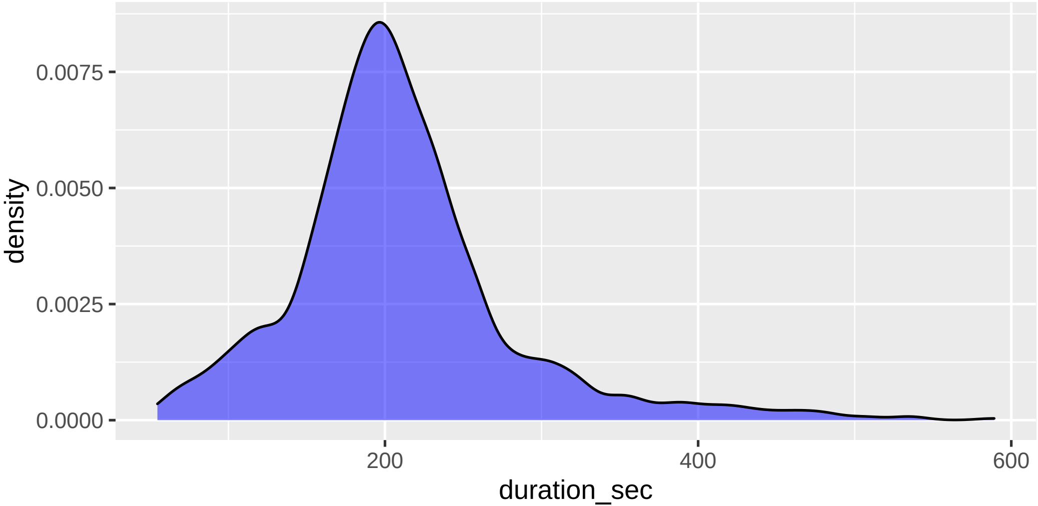
We can use `binom.test` to test this hypothesis.

Application: Length of songs on Spotify less than 10 minutes



Testing $\theta = 200$

```
> testMedian <- function(median0) {
+     y <- tracks10$duration_sec - median0 > 0
+     binom.test(sum(y), length(y), p = 0.5)$p.value
+ }
> testMedian(200)

[1] 0.385
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Confidence intervals

```
> medians <- unique(sort(c(
+               median(tracks10$duration_sec),
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> pvalues <- map_dbl(medians, testMedian)

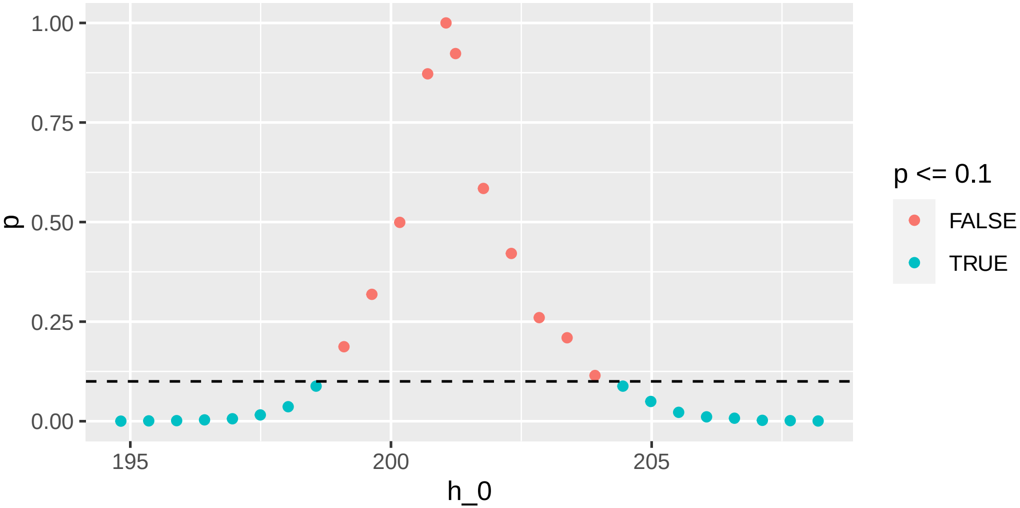
> medians[which.max(pvalues)] # point estimate

[1] 201.1

> range(medians[0.001 <= pvalues]) # 99.9% CI

[1] 195.9 207.7
```

Graphing p-values (90% CI)



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Recall the **decomposition of Laplace/Double Exponential** random variables that were **symmetric about θ**:

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To find the **conditional distribution of T**: enumerate all 2^n possible $\{-1, 1\}^n$ (vectors of ± 1).

It will be difficult to enumerate all 2^n possible values, but we can use **Monte Carlo sampling**:

```
> testSymmetry <- function(x, theta, k = 1000) {
+     n <- length(x)
+     s_0 <- sum(x - theta)
+     y <- abs(x - theta)
+     dist <- map_dbl(rerun(k, 2 * rbinom(n, size = 1, p = 0.5) - 1),
+                 ~ sum(y * .x))
+     2 * min(dist >= s_0, dist <= s_0) / k
+ }
> testSymmetry(tracks10$duration_sec, 201)
```

```
[1] 0
```

Example: Two Sample Problems

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We conceptualized this problem as **two samples**

$$X_1, \ldots, X_n \sim F, \quad \text{iid}, \quad \text{(Phonenix)}$$
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Previously, we found a confidence interval for $\theta = E(X) - E(Y)$, but we could also ask a more general question:

$$H_0 : F = G \quad \text{vs} \quad H_1 : F \neq G$$

Combined Sample Notation

It's often convenient to think about a combined sample of the form:

$$(W_i, Z_i), \quad i = 1, \ldots, n + m$$

Where

$$W_i = \begin{cases} X_i & i \leq n \\ Y_i & i > n \end{cases}$$

and

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The permutation approach then just requires shuffling the Z_i values.

Picking a Test Statistic

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A permutation test will permute the Z_i to get a conditional distribution for T.

Some Example Statistics

- Difference of means: $T(\mathbf{Z}) = \frac{1}{n}\sum_{i=1}^{n} Z_i w_i - \frac{1}{n}\sum_{i=1}^{n}(1 - Z_i)w_i$ (Welch's permutational t-test)

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- Comparisons of ECDFs (Kolmogorov-Smirnov, Andersong-Darling, Cramer-von Mises)

Phoenix and San Antonio

```
> mean_diff <- function(w, z) {mean(w[z], na.rm = TRUE) -
+                                   mean(w[!z], na.rm = TRUE)}
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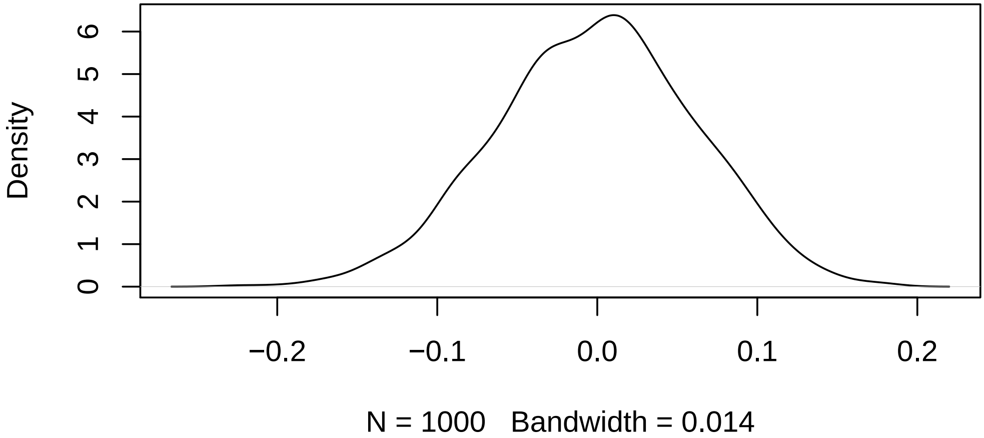
> n <- nrow(gamoran)
> dist.t <- replicate(1000, {
+     ## shuffle the "Z_i" values
+     permuted_label <- sample(gamoran$PH.AZ)
+     ## compute the test statistic
+     mean_diff(gamoran$READ_PCTZ, permuted_label)
+ })
```

Distribution under the null



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In other words, we can use created "two tailed" p-value:

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where t is the **observed value of the test statistic**.

```
> (t_observed <- mean_diff(gamoran$READ_PCTZ, gamoran$PH.AZ))
[1] 0.2568
> 2 * min(mean(dist.t <= t_observed), mean(dist.t >= t_observed))
[1] 0
```

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Like the bootstrap (or many other procedures we've seen), we can think about a general algorithm for permutation tests:

- Select a test statistic T that compares two samples
- Compute the observed value $\hat{T}$
- Randomly generate B permutations of the $n+m$ group labels and compute T_b
- Depending on the alternative, compute the p-value as

$$p^+ = \frac{1}{B}\sum_{b=1}^{B} I(T_b \geq \hat{T}), \quad p^- = \frac{1}{B}\sum_{b=1}^{B} I(T_b \leq \hat{T}), \quad p = 2 \times \min(p^+, p^-)$$

(Note: some sources add one to both numerator and denominator. For large B both approaches are about the same.)

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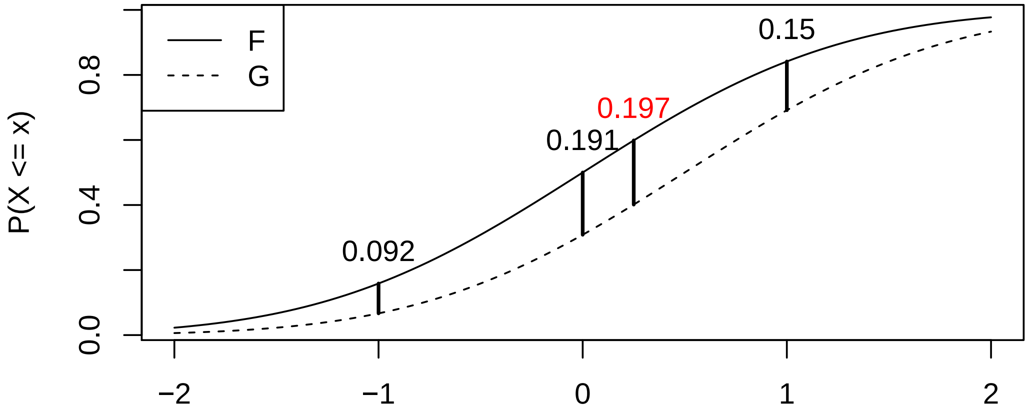
Next we'll look at a class of test statistics that have proved useful for many problems.

Kolmogorov-Smirnov



$$T(W,Z) = \max_{1 \leq i \leq n+m} \left| \widehat{F}_Z(W_i) - \widehat{G}_Z(W_i) \right|$$

Blood pressure for subjects taking aspirin vs. not

Previously we looked an example from the NHANES study where respondents had a **blood pressure exam** and answered a survey question about **taking aspirin**.

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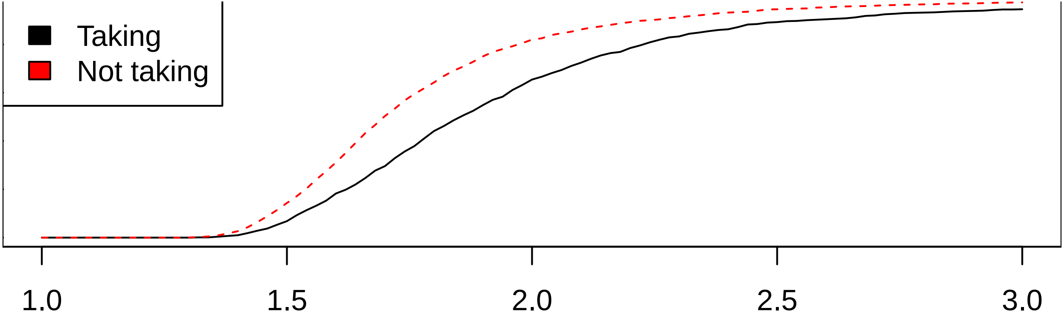
One way to summarize the BP measurements was to take the **ratio of systolic to diastolic** pressure:



Implementing KS

```
> ks <- function(w, z) {
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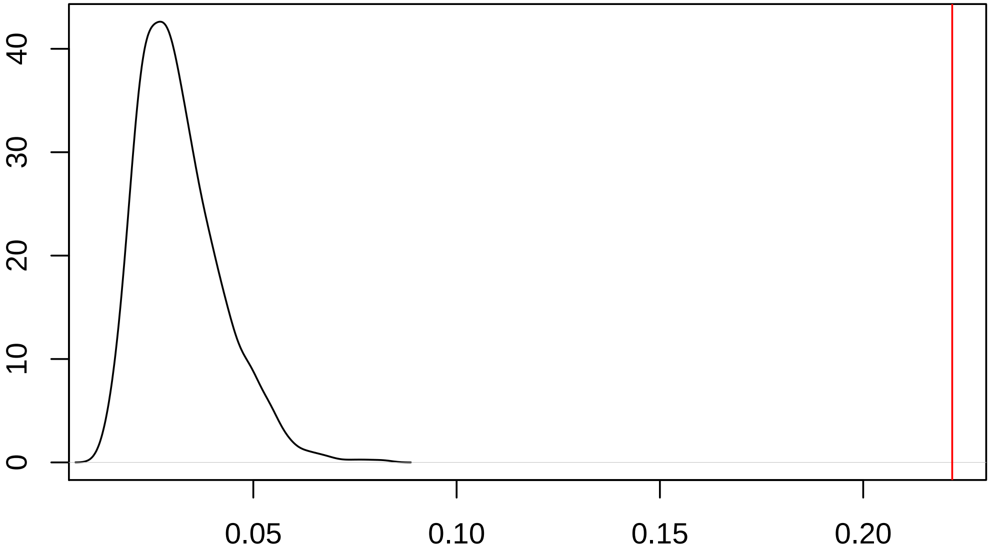
> perms <- replicate(1000, sample(nhanes$taking_aspirin))
> ts <- apply(perms, 2, function(zstar) {
+     ks(nhanes$ratio, zstar)
+ })
> observed_ks <- ks(nhanes$ratio, nhanes$taking_aspirin)
> (ksp <- 2 * min(mean(ts >= observed_ks), mean(ts <= observed_ks)))

[1] 0
```


0
10
20
30
40
0.05
0.10
0.15
0.20

Distribution Free Statistics

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$$W = (3, 9, 2, 4) \to R = (2, 4, 1, 3)$$

Claim: Let

$$D^+ = \max_i \hat{F}(W_i) - \hat{G}(W_i), \quad D^- = \max_i \hat{G}(W_i) - \hat{F}(W_i)$$

Then the statistic $\max(D^+, D^-)$ is **distribution free**.

Proof

For all $i = 1, \ldots, n+m$, let R_i be the **rank** of W_i in the combined sample. We'll show that D^+ only depends on the ranks, and the rest of the claim follows similarly.

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Remembering that $Z_i = 1$ for $i \leq n$ and $Z_i = 0$ for $n \leq i \leq n+m$, notice that for any W_i,

$$\hat{F}(W_i) = \frac{1}{n} \sum_{j=1}^{n} I(W_j \leq W_i)$$

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$$\begin{aligned}\hat{F}(W_i) &= \frac{1}{n}\sum_{j=1}^{n} I(W_j \leq W_i) \\ &= \frac{1}{n}\sum_{j=1}^{n+m} Z_j I(W_j \leq W_i)\end{aligned}$$

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\begin{aligned}
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&= \frac{1}{n} \sum_{j=1}^{n+m} Z_j I(W_j \leq W_i) \\
&= \frac{1}{n} \sum_{j=1}^{n+m} Z_j I(R_j \leq R_i)
\end{aligned}
$$

Proof, cont.: Replace ranks with integers

So few have $\hat{F}(W_i) = (1/n)\sum_{j=1}^{n+m} Z_j I(R_j \leq R_i)$ and, likewise, $\hat{G}(W_i) = (1/m)\sum_{j=1}^{n+m}(1 - Z_j) I(R_j \leq R_i)$.

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&= \max_{1\leq i\leq n+m} \frac{1}{n}\sum_{j=1}^{n+m} Z_j I(R_j \leq i) - \frac{1}{m}\sum_{j=1}^{n+m}(1 - Z_j)I(R_j \leq i)
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Distribution free tests

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More generally, so we can **compute the distribution of any statistic $T(R, Z)$ before we see any data**. (provided T doesn't depend on w otherwise.)

This is precisely what we call **distribution free**. (Technically, we have parameters n and m, but nothing that depends on the data.)

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This test is known as the **Wilcoxon-Mann-Whitney** test.

- Kolmogorov-Smirnov: `ks.test`
- Wilcoxon-Mann-Whitney: `wilcox.test`
- Normal Scores: for $H_i = \Phi^{-1}(R_i/(n+m+1))$:

$$T(Z, H) = \frac{1}{n}\sum_{i=1}^{n+m} Z_i H_i - \frac{1}{m}\sum_{i=1}^{n+m}(1 - Z_i)H_i$$

  is implemented in the `SuppDist` package.

And we can always estimate any other test statistic distribution using a Monte Carlo approach.

```
> with(nhanes, ## creates variables ratio, taking_apsirin
+      ks.test(x = ratio[taking_aspirin],
+              y = ratio[!taking_aspirin]))

        Two-sample Kolmogorov-Smirnov test

data:  ratio[taking_aspirin] and ratio[!taking_aspirin]
D = 0.22, p-value <2e-16
alternative hypothesis: two-sided
```

```
> with(nhanes,
+       wilcox.test(x = ratio[taking_aspirin],
+                   y = ratio[!taking_aspirin]))

        Wilcoxon rank sum test with continuity correction

data:  ratio[taking_aspirin] and ratio[!taking_aspirin]
W = 1645455, p-value <2e-16
alternative hypothesis: true location shift is not equal to 0
```

Summary

Permutation tests are a large class of hypothesis tests based on **permutation invariant test statistics**: by conditioning on some aspect of the sample, the test statistic has **the same distribution under any permutation of the remaining random data**.

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Very common setting is the **two sample problem** where we are testing that groups have the same distribution: $H_0 : F = G$.

Distribution free tests replace data with ranks (or similar) to make test statistics not depend on the underlying distribution of the data.