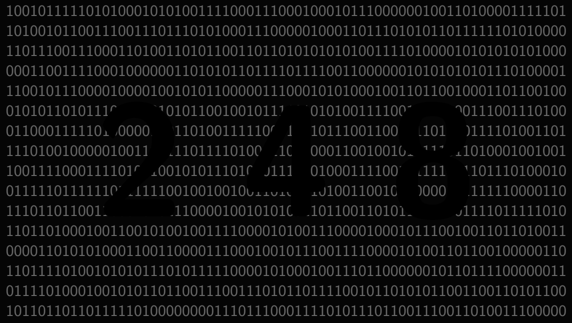
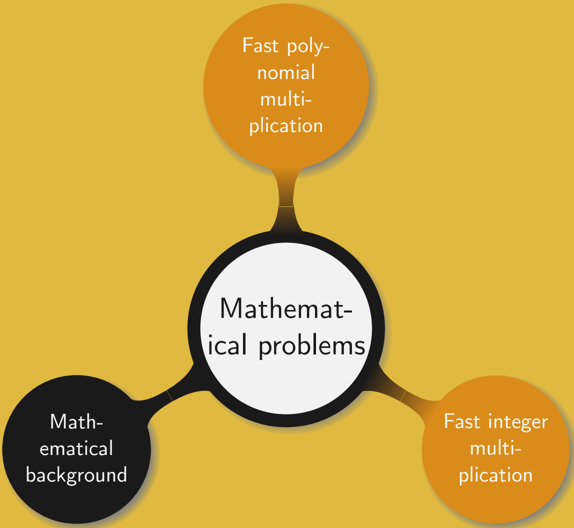


Introduction to Algorithms

6. Mathematical problems

Manuel – Fall 2021

Chapter organisation



Goal

Many applications require large numbers to be multiplied.

Common multiplication algorithms:

- Simple strategy: $\mathcal{O}(n^2)$
- Karatsuba: $\mathcal{O}(n^{\log_2 3})$

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Fast Fourier Transform (FFT):

- Fast polynomial and number multiplication
- One of the most important and used algorithms

Definitions

Let S and S' be two sets.

1. An *internal composition law* $(\circ)$ is an map from $S \times S$ into S such that
$$\begin{aligned} S \times S &\longrightarrow S \\ (x, y) &\longmapsto x \circ y. \end{aligned}$$
2. An *external composition law* $(*)$ is an map from $S' \times S$ into S such that
$$\begin{aligned} S' \times S &\longrightarrow S \\ (\alpha, x) &\longmapsto \alpha * x. \end{aligned}$$

Example. For a set S, the intersection $(\bigcap)$ and union $(\bigcup)$ define two internal composition laws for the class of subsets of S.

Definition (Group)

A *group* is a pair $(G, \circ)$ consisting of a set G and an internal composition law that verifies the following properties:

1. *Associativity:* $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in G$
2. *Existence of a unit element:* there exists an element $e \in G$ such that $a \circ e = e \circ a = a$ for all $a \in G$
3. *Existence of inverse:* for every $a \in G$ there exists an element $a^{-1} \in G$ such that $a \circ a^{-1} = a^{-1} \circ a = e$

A group is called *abelian* if in addition to the above properties

4. *Commutativity*: $a \circ b = b \circ a$ for all $a, b \in G$.

Definition (Ring)

A *ring* is a triple $(R, +, \cdot)$ consisting of a set R and two internal composition laws $(+)$ and $(\cdot)$, such that

i. $(R, +)$ is an abelian group
ii. *Multiplicative unit:* there exists an element $1 \in G$ such that
$$a \cdot 1 = 1 \cdot a = a \quad \text{for all } a \in R$$
iii. *Associativity:* for any $a, b, c \in R$,
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$
iv. *Distributivity:* for any $a, b, c \in R$,
$$a \cdot (b + c) = (a \cdot b) + (a \cdot c), \quad (b + c) \cdot a = (b \cdot a) + (c \cdot a)$$

A ring is called *commutative* if in addition to the above properties

v. *Commutativity:* $a \cdot b = b \cdot a$ for all $a, b \in R$

Definition (Field)

Let $(F, +, \cdot)$ be a commutative ring with unit element of addition 0 and unit element of multiplication 1. Then F is a *field* if

i. $0 \neq 1$
ii. For every $a \in F \setminus \{0\}$ there exists an element a^{-1} such that

$$a \cdot a^{-1} = 1.$$

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Remark. Another way of writing this definition is to say that $(F, +, \cdot)$ is a field if $(F, +)$ and $(F \setminus \{0\}, \cdot)$ are abelian groups, $0 \neq 1$, and $\cdot$ distributes over $+$.

$\mathbb{Z}$ is not a field, cannot form vector space

Example. Let n be an integer, and $\mathbb{Z}/n\mathbb{Z}$ be the set of the integers modulo n

- $(\mathbb{Z}/n\mathbb{Z}, +)$ also denoted $(\mathbb{Z}_n, +)$ is a group
- $(\mathbb{Z}/n\mathbb{Z}, +, \cdot)$ is a ring
- If n is prime then $(\mathbb{Z}/n\mathbb{Z}, +, \cdot)$ is the field $\mathbb{F}_n$
- The invertible elements of $\mathbb{Z}/n\mathbb{Z}$, with respect to '$\cdot$', form a group denoted $\mathrm{U}(\mathbb{Z}/n\mathbb{Z})$ or sometimes $\mathbb{Z}_n^\times$ or $\mathbb{Z}_n^*$
- $(\mathbb{Z}/n\mathbb{Z}[X], +, \cdot)$ is the ring of the polynomials over $\mathbb{Z}/n\mathbb{Z}$
- If n is prime and the polynomial $P(X)$ is irreducible then

$$(\mathbb{F}_n[X]/\langle P(X)\rangle, +, \cdot)$$

is a field; this is $\mathbb{F}_{n^{\deg P(x)}}$

Definitions

Let R be a ring and n be a strictly positive integer.

1. An element a of R is a *zero divisor* if there exists $b \in R \setminus \{0\}$ such that $ab = 0$.
2. If 0 is the only zero divisor in R, then R is an *integral domain*.
3. An element $\omega \in R$ is an *nth root of unity* if $\omega^n = 1$.
4. An element $\omega \in R$ is a *primitive nth root of unity* if $\omega^n = 1$ and for any $1 \leq k \leq n-1$, $\omega^k \neq 1$.

Example.

- In $\mathbb{C}$, $e^{2i\pi/8}$ is a primitive 8th root of unity;
- In $\mathbb{Z}_{17}$, 2 is not a primitive 16th root of unity;

Lemma

Let R be a ring, l, n be two integers such that $1 < l < n$, and ω be a primitive nth root of unity. Then (i) $\omega^l - 1$ is not a zero divisor in R, and (ii) $\sum_{j=0}^{n-1} \omega^{lj} = 0$.

Proof. First, note that for any $c \in R$ and m in $\mathbb{N}$

$$c^m - 1 = (c-1)(1 + c + c^2 + \cdots + c^{m-1}). \tag{6.1}$$

(i) Let d be the $\gcd(l, n)$. From the Extended Euclidean Algorithm we know the existence of $s, t \in \mathbb{Z}$ such that $sl + tn = d$. Recalling that $l < n$, we have $1 \leq d < n$, and we can cancel a prime factor r of n such that $d|(n/r)$.

For $c = \omega^d$ and $m = \frac{n}{rd}$ in (6.1) we get

$$\omega^{n/r} - 1 = (\omega^d - 1)(1 + \omega^d + \cdots + \omega^{d\left(\frac{n}{rd} - 1\right)}).$$

Hence if there exists $b \in R$ such that $b(\omega^d - 1) = 0$, then $b(\omega^{n/r} - 1)$ must also be zero. But as $\omega^{n/r} - 1$ is not a zero divisor, $b = 0$, and neither is $\omega^d - 1$ a zero divisor.

We now set $c = \omega^l$ and $m = s$ in (6.1). In that case we can write $\omega^{sl} = \omega^{sl}\omega^{tn} - 1 = \omega^d$, and we see that $\omega^l - 1$ divides $\omega^d - 1$. Following a similar reasoning as in the previous case we obtain that $\omega^l - 1$ is a not a zero divisor.

(ii) Using (6.1) one more time, for $c = \omega^l$ and $m = n$ we obtain

$$\omega^{ln} - 1 = (\omega^l - 1)(1 + \omega^l + \cdots + \omega^{l(n-1)}).$$

As $\omega^{ln} = 1$, and $\omega^l - 1$ is not a zero divisor, $\sum_{j=0}^{n-1} \omega^{lj} = 0$. $\square$

Definition

Let R be a ring, and $\omega \in R$ be a primitive nth root of unity. We denote a polynomial $P(X)$ of degree less than n in $R[X]$ by its coefficients

$$P(X) = \sum_{i=0}^{n-1} a_i X^i = (a_0, \cdots, a_{n-1}).$$

The linear map

$$\begin{aligned} \mathrm{DFT}_\omega : R^n &\longrightarrow R^n \\ (a_0, \cdots, a_{n-1}) &\longmapsto (P(1), P(\omega), \cdots, P(\omega^{n-1})) \end{aligned}$$

evaluates P at the powers of ω and is called *Discrete Fourier Transform* (DFT).

As DFT_ω is a linear map it is expressed as a matrix transformation

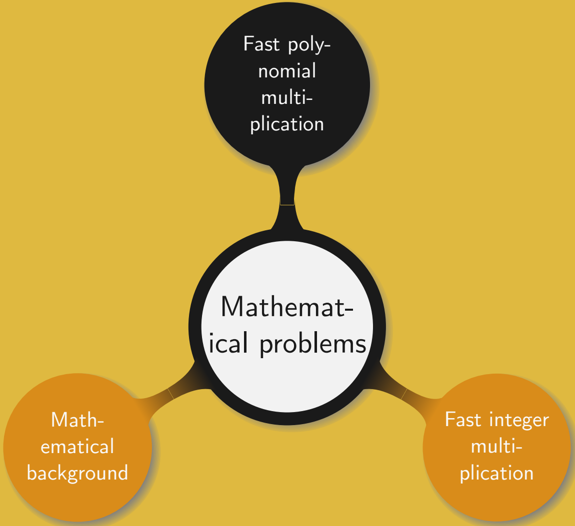
$$\begin{pmatrix} P(1) \\ P(\omega) \\ P(\omega^2) \\ \vdots \\ P(\omega^{n-1}) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{pmatrix}}_{V_\omega} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{pmatrix}.$$

Note that if ω is a primitive nth root of unity then so is ω^{-1}. Otherwise there would be a $1 \leq k < n$ such that $(\omega^{-1})^k$ is 1. And as $\omega^n = 1$ we would have $1 = \omega^{n+k}$ and $\omega^k = 1 \lightning$.

Then using lemma 6.10 observe that $V_\omega V_{\omega^{-1}} = n\,\mathrm{I}_n$, where I_n is the identity matrix of size $n \times n$. Thus the inverse of DFT_ω is

$$\mathrm{DFT}_\omega^{-1} = \frac{1}{n}\,\mathrm{DFT}_{\omega^{-1}}\,.$$

Chapter organisation



Two main cases depending on the structure of the polynomial:

- Dense: use an array where the coefficient of each monomial is stored at index "degree of the monomial"
- Sparse: use a structure composed of two arrays storing the degrees and the corresponding coefficients, respectively

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- Sparse: use a structure composed of two arrays storing the degrees and the corresponding coefficients, respectively

Alternative strategy: over an integral domain a polynomial of degree strictly less than n can be represented using its value at n distinct points

Let $P(X) = \sum_{i=0}^{n} a_i X^i$ be a polynomial of degree n. Evaluating P costs $\mathcal{O}(n^2)$ if naively computed. However note that $P(X)$ can be rewritten

$$P(X) = a_0 + X(a_1 + X(a_2 + \cdots + X(a_{n-1} + Xa_n))).$$

This remark dramatically decreases the complexity as it drops to $\mathcal{O}(n)$, and yields the following simple algorithm.

Algorithm. (*Horner*)

Input : a polynomial P and x the value to evaluate P at
Output: Px the evaluation of P at x

Function `Horner`(P, x)**:**
  $Px \leftarrow 0$;
  for $i \leftarrow \deg P$ **to** *0* **do** $Px \leftarrow Px \cdot x +$ coeff$[i]$;
  return Px;
end

Two main cases depending on the polynomial representation:

- Dense: usual approach, multiply the various coefficients together; complexity $\mathcal{O}(n^2)$
- Evaluation: evaluate the polynomials in n points, multiply them two by two as

$$PQ(x_i) = P(x_i)Q(x_i).$$

  Complexity is $\Omega(n^2)$ since n evaluations are necessary, to which have to be added the cost of the multiplications and of the interpolation to get the final polynomial (usually $\mathcal{O}(n^2)$).

Looking back at DFT_ω, it can be viewed as a special multipoint evaluation of a polynomial in the powers $1, \omega, \cdots, \omega^{n-1}$ of a primitive nth root of unity ω. Then its inverse DFT_ω^{-1}, which given n evaluations of a polynomial allows to recover its coefficients, is just the interpolation at the powers of ω.

From the previous discussion on the DFT (6.13), it is clear that knowing how to compute it efficiently means being able to also compute its inverse efficiently.

For the sake of simplicity assume $n = 2^k$, $k \in \mathbb{N}$, and observe that

$$\begin{aligned} P(X) &= \sum_{i=0}^{n-1} a_i X^i \\ &= (a_0 + a_2X^2 + \cdots + a_{n-2}X^{n-2}) + (a_1X + a_3X^3 + \cdots + a_{n-1}X^{n-1}) \\ &= P_1(X^2) + XP_2(X^2) \end{aligned} \tag{6.2}$$

with both P_1 and P_2 of degree less than $(n-2)/2 < n/2$.

The structure of equation (6.2) suggests a "divide and conquer" approach in order to determine

$$P(\omega^i) = P_1(\omega^{2i}) + \omega^i P_2(\omega^{2i}), \quad 0 \leq i < n. \tag{6.3}$$

This formulation can be further rewritten by noticing that

$$\begin{aligned} 0 &= \omega^n - 1 \\ &= (\omega^{n/2} - 1)(\omega^{n/2} + 1). \end{aligned}$$

By lemma 6.10 none of the two factors is a zero divisor and as $\omega^{n/2} \neq 1$, ω being a primitive nth root of the unity, we conclude that $\omega^{n/2} = -1$. Hence, for all $0 \leq i < n/2$, $\omega^i = -\omega^{n/2+i}$, and (6.3) can be rewritten

$$\begin{aligned} P(\omega^i) &= P_1(\omega^{2i}) + \omega^i P_2(\omega^{2i}), \quad 0 \leq i < n/2, \\ P(\omega^{n/2+i}) &= P_1(\omega^{2i}) - \omega^i P_2(\omega^{2i}), \quad 0 \leq i < n/2. \end{aligned} \tag{6.4}$$

Algorithm. (*Fast Fourier Transform (FFT)*)

Input : a polynomial P of degree $< n$, with n a power of 2, and ω a primitive nth root of unity

Output : $\mathrm{DFT}_\omega(P)$

Function $\texttt{FFT}(P, \omega)$**:**

  $n \leftarrow \deg P + 1$;

  if $n = 1$ **then return** P;

  $P_1 \leftarrow \texttt{FFT}(\sum_{j=0}^{n/2-1} a_{2j} X^{2j}, \omega^2)$;

  $P_2 \leftarrow \texttt{FFT}(\sum_{j=0}^{n/2-1} a_{2j+1} \omega X^{2j}, \omega^2)$;

  for $i \leftarrow 0$ **to** $n/2$ **do**

    $P_\omega[i] \leftarrow P_{1,\omega^2}[i] + \omega^i P_{2,\omega^2}[i]$;

    $P_\omega[n/2 + i] \leftarrow P_{1,\omega^2}[i] - \omega^i P_{2,\omega^2}[i]$;

  end for

  return P_ω;

end

Theorem

Given a polynomial P over a commutative ring and ω a primitive nth root of unity, the FFT algorithm correctly computes $\mathrm{DFT}_\omega(P)$ in time $\mathcal{O}(n \log n)$.

Proof. The correctness clearly follows from the previous discussion, more particularly from the recurrence relation (6.4).

Let $T(n)$ be the time to compute a DFT. Then in relation (6.4) two smaller DFT are computed, plus $n/2$ multiplications and n additions, that is

$$\begin{aligned} T(n) &\leq 2T(n/2) + n/2 + n \\ &\leq 2T(n/2) + 3n/2. \end{aligned}$$

By the Master theorem (2.31) the time complexity of a DFT is $\mathcal{O}(n \log n)$. $\square$

Polynomial multiplication

Let R be a ring containing a primitive nth root of unity. Given two polynomials P and Q with degrees less than $n/2$, defined over $R[X]$, we want to determine $S = PQ$. Note that n is still taken to be a power of 2.

As both P and Q are of degrees less than n they can be efficiently evaluated using the FFT algorithm (6.20). Then n multiplications in R are enough to determine S, represented using its evaluation in n points.

Applying DFT_ω^{-1} to the n evaluations of S, is achieved through the calculation of $1/n\,\mathrm{DFT}_{\omega^{-1}}\,S$ (6.13). This computation returns the interpolation of S in n points (6.18), that is it determines the unique polynomial of degree less than n passing through the n points. Hence we obtain a fast strategy to compute the product of two polynomials.

Algorithm. (*Fast polynomial multiplication*)

Input : P and Q two polynomials of degree $< n/2$, with n a power of 2, a primitive nth root of unity ω

Output : $S = PQ$

Function `FPMult`(P, Q, ω):

- $P_\omega \leftarrow \texttt{FFT}(P, \omega)$;
- $Q_\omega \leftarrow \texttt{FFT}(Q, \omega)$;
- $S_\omega \leftarrow P_\omega Q_\omega$;
- $S \leftarrow \frac{1}{n}\texttt{FFT}(S_\omega, \omega^{-1})$;
- **return** S

end

Algorithm. (*Fast polynomial multiplication*)

Input : P and Q two polynomials of degree $< n/2$, with n a power of 2, a primitive nth root of unity ω

Output : $S = PQ$

```
Function FPMult(P, Q, ω):
    P_ω ← FFT(P, ω);
    Q_ω ← FFT(Q, ω);
    S_ω ← P_ω Q_ω;
    S ← (1/n) FFT(S_ω, ω^{-1});
    return S
end
```

Definition

A commutative ring containing a primitive 2^kth root of unity for any k in $\mathbb{N}^*$ is said to *support the FFT*.

Theorem

Let R be a ring supporting the FFT and n be 2^k, with k in $\mathbb{N}^*$. Then for two polynomials P and Q in $R[X]$, with $\deg PQ < n$, the fast polynomial multiplication algorithm computes their product in time $\mathcal{O}(n \log n)$.

Proof. The correctness of the algorithm is clear when considering the previous discussion (6.22).

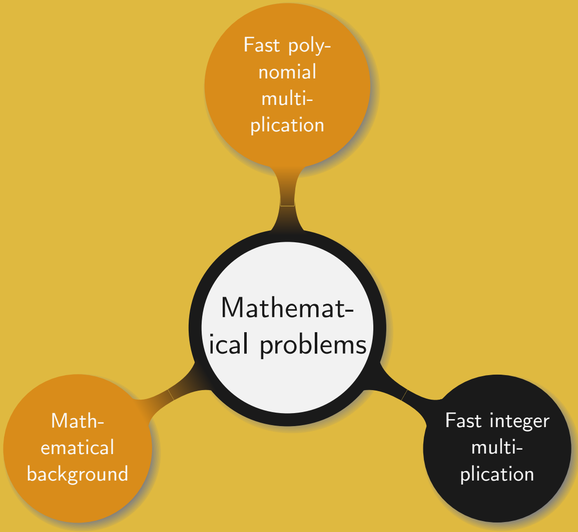
The algorithm computes three DFT, n component-wise products in R, as well as n multiplications by the inverse of n in R. Therefore the overall complexity is dominated by $\mathcal{O}(n \log n)$. $\square$

Rings not supporting the FFT

In order to run the fast polynomial multiplication algorithm (6.23) the underlying ring R must support the FFT. In the case where R does not contain any primitive 2^kth root of unity a “virtual” one can be attached to the ring.

The Schönage-Strassen algorithm handles this special case at the cost of a slightly worse complexity. In fact their result states that over any ring the product of two polynomials of degree less than n can be computed in $\mathcal{O}(n \log n \log \log n)$ operations.

Chapter organisation



Let a and b be two N-bit long integers. For the sake of simplicity we assume N to be of the form 2^{2^l} for some integer $l > 0$. Let $a_N, \cdots, a_0$ and $b_N, \cdots, b_0$ denote the binary representations of a and b, respectively.

Since N was chosen to be a square it is easy to split a and b into blocks of size $\sqrt{N}$ and write them

$$a = \sum_{i=0}^{\sqrt{N}-1} A_i 2^{i\sqrt{N}}, \text{ and } b = \sum_{i=0}^{\sqrt{N}-1} B_i 2^{i\sqrt{N}}, \quad 0 \leq A_i, B_i \leq 2^{\sqrt{N}} - 1.$$

If we consider the polynomials $A(X) = \sum_{i=0}^{\sqrt{N}-1} A_i X^i$ and $B(X) = \sum_{i=0}^{\sqrt{N}-1} B_i X^i$ then the product AB evaluated at $2^{\sqrt{N}}$ is exactly the product ab. Therefore integer multiplication can be performed via polynomial multiplication: (i) write the two integers as polynomials, (ii) apply the fast polynomial multiplication on them, and (iii) finally evaluate the product at $2^{\sqrt{N}}$.

Attaching a new element

While the first step is simple to achieve the second one requires more technical considerations. In fact the two polynomials A and B are both of degree less than $2\sqrt{N}$ and as such R must contain a primitive $2\sqrt{N}$th root of unity.

As $\mathbb{Z}$ does not support FFT some extra work is needed in order to attach a new "virtual" element to the ring without altering the final result.

Consider the ring $\mathbb{Z}_{2^{\sqrt{N}}+1}$ and observe that 2 is a primitive $2\sqrt{N}$th root of unity. This is clear as $2^{\sqrt{N}}$ is -1 modulo $2^{\sqrt{N}}+1$. Thus $t = 2\sqrt{N}$ is the smallest power for which 2^t is 1.

An obvious idea is then to perform the computation in the ring $\mathbb{Z}_{2^{\sqrt{N}}+1}[X]$. However as both A and B can feature coefficients as large as $2^{\sqrt{N}} - 1$, the polynomial C resulting from their product can have coefficients up to $\sqrt{N}2^{2\sqrt{N}}$, which is larger than $2^{\sqrt{N}} + 1$.

As a result, if coefficients in C happen to be too large they will be reduced modulo $2^{\sqrt{N}}+1$, ruining the whole calculation. A simple solution consists in performing all the computation in a larger ring. At that stage two points must be taken into consideration: (i) the use of a large ring increases the computational cost and (ii) the ring must contain a primitive $2\sqrt{N}$th root of unity.

Note that for any $N \geq 1$, $2^{3\sqrt{N}} > \sqrt{N}2^{2\sqrt{N}}$, while 8 is a primitive $2\sqrt{N}$th root of unity in $\mathbb{Z}_{2^{3\sqrt{N}}+1}$. The latter being a consequence of 2 being a primitive $6\sqrt{N}$th root of unity.

Therefore it suffices to consider A and B as polynomials over the ring $\mathbb{Z}_{2^{3\sqrt{N}}+1}[X]$. Then applying the fast polynomial multiplication algorithm (6.23) and evaluating the product at $2^{\sqrt{N}}$ yields the result.

Algorithm. (*Fast integer multiplication*)

Input : two N bit integers a and b, a ring $R = \mathbb{Z}_{2^{3\sqrt{N}}+1}$, $\omega = 8$ a primitive $2\sqrt{N}$th root of unity

Output: $c = a \cdot b$

```
A ← poly(a);                  /* encode a as a polynomial */
B ← poly(b);                  /* encode b as a polynomial */
C ← FPMult (A, B, ω);
c ← Horner (C, 2^√N);
return c;
```

Theorem

Given two integers of length N in a ring R, the fast integer multiplication algorithm computes their product in $\mathcal{O}(\sqrt{N}\log\sqrt{N})$ arithmetic operations in R.

Proof. The correctness results from the previous discussion (6.27).

The pre-dominant computation in the algorithm is the fast polynomial multiplication of A and B, which by theorem 6.24 takes $\mathcal{O}(\sqrt{N}\log\sqrt{N})$. □

Remark. With a bit more work it is possible to determine the complexity in term of bit operations, instead of arithmetic operations. In that case the complexity becomes

$$\mathcal{O}(N\log^{2+\log_2 3-1} N).$$

Other interesting remarks:

- At the bit level, most operations can be performed using "shifts", incurring a linear cost in the length of the integers. This is, in particular the main reason for choosing ω to be a power of 2.
- The integer $2\sqrt{N}$ has inverse $2^{6\sqrt{N}-\log_2\sqrt{N}-1}$ in $\mathbb{Z}_{2^{3\sqrt{N}}+1}$. Observe that $2^{6\sqrt{N}} \equiv 1 \bmod 2^{3\sqrt{N}+1}$, and $2^{\log_2\sqrt{N}} = \sqrt{N}$.
- Although in the algorithm no coefficient reaches $2^{3\sqrt{N}}+1$ all the calculations are performed in the ring $\mathbb{Z}_{2^{3\sqrt{N}}+1}$ and the special case of adding two $3\sqrt{N}$ bits long elements has to be considered when defining addition.
- To date the asymptotically fastest integer multiplication algorithm, due to Fürer, takes $N \log N 2^{\mathcal{O}(\log^* N)}$ bit operations.

Definition

The complexity of multiplying two polynomials of degree less than n is denoted $\mathsf{M}(n)$, while the complexity of multiplying two n-bit integer is denoted $\mathsf{M}_I(n)$.

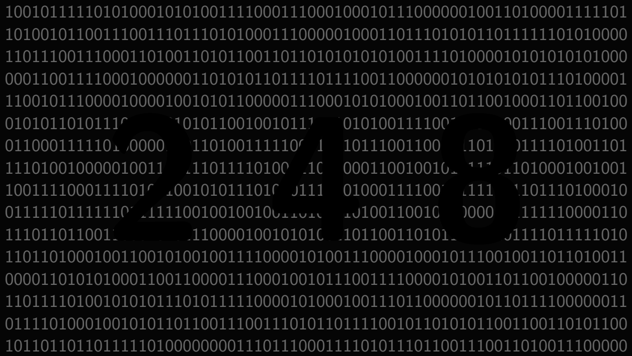
For all n and m in $\mathbb{N}$,

$$\mathsf{M}(n+m) \geq \mathsf{M}(m) + \mathsf{M}(n) \quad \text{and } \mathsf{M}_I(n+m) \geq \mathsf{M}_I(n) + \mathsf{M}_I(m).$$

These simpler notations allows an easier complexity study of more advanced algorithms such as fast multi-point evaluation and fast interpolation.

Key points

- Define all the basic mathematical structures.
- What is the function DFT_ω?
- How to represent polynomials?
- Describe the basic idea behind the FFT.
- How to view integers if one wants to run fast multiplication?



Thank you!