VE477

Introduction to Algorithms

Homework 7

Manuel — UM-JI (Fall 2021)

Questions preceded by a * are optional. Although they can be skipped without any deduction, it is important to know and understand the results they contain.

Ex. 1 — *Karger-Stein's Algorithm*

In the lectures, although Karger-Stein's Algorithm was presented (5.28|5.243), only a sketch of proof was provided (5.29|5.244). In this exercise we want to prove the missing part, i.e. solve the recurrence relation

$$P(t) = 1 - \left(1 - \frac{1}{2}P\left(\frac{t}{\sqrt{2}}\right)\right)^2.$$

1. Prove that the probability of a cut to survive when $n \le 6$, at least $1/15$.

2. Using an appropriate change of variable, show that

$$\begin{cases} p_{k+1} &= p_k - \dfrac{1}{4}p_k^2, \\ p_0 &= 1/15. \end{cases}$$

3. Let $z_k = 4/p_k - 1$.

   a) Prove that

   $$\begin{cases} z_{k+1} &= z_k + 1 + \dfrac{1}{z_k}, \\ z_0 &= 59. \end{cases}$$

   * b) Show that for all $k \ge 1$, $k < z_k < 59 + 2k$.

4. Recalling that $t = n/\sqrt{2}$ and noting that the depth of the recursion is $2\log_2 n + \mathcal{O}(1)$, conclude that $P(n) = \Omega(1/\log n)$.

Ex. 2 — *Critical thinking*

1. Is it possible to design a stack supporting `push`, `pop`, and retrieving the minimum element in constant time? Explain.

2. A total of n ants are walking at speed 1 m/s on a thin 1 m long cable; if they collide they instantly reverse direction, and if an ant reaches the end of the cable it falls. How long will it take before all the ants have fallen?