## 0.1 Hitchcock Transport Problem

- *Algorithm:* Minimum Cost Flow (algo. 1)
- *Input:* an array of $m$ suppliers, an array of $n$ receivers, a two-dimensional array of cost from $m_i$ to $n_j$.
- *Complexity:* $\mathcal{O}(mn^2(\log m + n \log n))$.
- *Data structure compatibility:* N/A
- *Common applications:* Solve the minimum cost of transportation while meet the requests.

**Problem.** Hitchcock Transport Problem

Suppose there are $m$ sources of commodity $x_1, x_2, \ldots, x_m$ with $a(x_i)$ units of supply at $x_i$, and $n$ sinks $y_1, y_2, \ldots, y_n$, with demands $b(y_j)$ at $y_j$. Denote $a(x_i, y_j)$ as the unit cost of transportation from $x_i$ to $y_j$, find the solution that meet demands with supplies, while minimizing the transportation cost [3].

### Description

In the "Hitchcock Transport Problem"(abbreviated as HTP), the combination of suppliers and demanders, along with the transportation cost between them, can be seen as a bipartite graph. Therefore, we make some definitions below,

A *flow network* can be represented as a quadratuple $(G, b, u, c)$, where $G$ is a directed graph with $V$, the vertex set, and $E$, the edge set. $b$ is a mapping $V \to \mathbb{R}$ with $\sum_{v \in V} b(v) = 0$, which means that the total flow leaving or entering the network should be none. $u$ is the a *capacity function* with the mapping $E \to \mathbb{R}^+$. $c$ is a *cost function* $E \to \mathbb{R}^+ \cup \{0\}$ [1].

By this construction, our goal is to find the flow $f$ which minimizes $c(f) := \sum_{e \in E} f(e)c(e)$, given the flow network $(G, b, u, c)$.

Also define the set of supplier nodes as $X$, set of demanders as $Y$. Since it would be a necessity to control the nodes $x \in X$ with multiple outgoing edges with positive flow, with setting $f$ as the solution to the HTP, define $\tau_f(x) = |\{y \in Y \mid f(x, y) > 0\}|$ be the number of outgoing edges for some $x$, and $F_f = \{x \in X \mid \tau_f(x) > 1\}$ be the number of supplier nodes $x$ that have multiple outgoing edges.

Vygen [4] has shown that for constant $n$, an optimal solution $f$ can be transformed into an optimal solution $g$ in linear time with $|F_g| \leq n - 1$.

Then given a flow network $(G, b, u, c)$, an optimal solution $f$, and $F_f$, we can transform $f$ in $\mathcal{O}(n^2|F_f|)$ time into an optimum solution $g$.

Moreover, according to Ford and Fulkerson [2], if we have an instance $(G, b, u, c)$ with finite capacities on $\sigma$ edges, there exists an equivalent "uncapacitated minimum cost flow instance" [1] with $|V| + \sigma$ nodes and $|E| + \sigma$ edges. Also, denote $M_i(y) = \{x \in X \mid f((x, y)) = b(x)\}$.

Then applying the **Minimum Cost Flow** 1, we can get the minimum cost in $\mathcal{O}(mn^2(\log m + n \log n))$ time.

**Algorithm 1:** Minimum Cost Flow

**Input :** A flow network instance $(G, b, u, c)$
**Output:** The minimum cost flow $f$

**foreach** *edge e in E* **do**
    $f(e) \leftarrow 0$;
**end foreach**
Sort the nodes in $X$ such that $X = x_1, x_2, \dots, x_m$ with $b(x_1) \geq b(x_2) \geq \cdots \geq b(x_n)$;
**for** $i = 1$ *to* $n$ **do**
    Construct a new flow network $NW_i = (G^i, b^i, u^i, c^i)$ as stated above;
    Compute a minimum cost flow $g$ in $NW_i$ such that the edges with positive flow do not form a cycle;
    **for** $e = (v, w) \in E(G^i)$ *and* $g(e) > 0$ **do**
        /* $e \in X \times Y$ means that $v \in X$ and $w \in Y$ */
        **if** $e \in X \times Y$ **then**
            $f(e) \leftarrow f(e) + g(e)$;
        **end if**
        **if** $e \in Y \times F_f$ **then**
            /* update the backward edges */
            $f((w, v)) \leftarrow f((w, v)) - g(e)$;
        **end if**
        **if** $e \in Y \times Y$ **then**
            Select $x \in M_i(x)$ with $c^i(e) = c((x, w)) - cost((x, v))$;
            $f((x, v)) \leftarrow f((x, v)) - g(e)$;
            $f((x, w)) \leftarrow f((X, y)) + g(e)$;
        **end if**
    **end for**
    Retransform $f$ according to Ford and Fulkerson;
**end for**
**return** $f$;

## References.


[1] Ulrich Brenner. "A faster polynomial algorithm for the unbalanced Hitchcock Transportation Problem". In: *Operations Research Letters* 36.4 (2008), pp. 408–413. DOI: 10.1016/j.orl.2008.01.011 (cit. on p. 1).

[2] Lester Randolph Ford and Delbert Ray Fulkerson. *Flows in networks*. Princeton University Press, 1962 (cit. on p. 1).

[3] Frank L. Hitchcock. "The distribution of a product from several sources to numerous localities". In: *Journal of Mathematics and Physics* 20.1-4 (1941), pp. 224–230. DOI: 10.1002/sapm1941201224 (cit. on p. 1).

[4] Jens Vygen. "Geometric quadrisection in linear time, with application to VLSI Placement". In: *Discrete Optimization* 2.4 (2005), pp. 362–390. DOI: 10.1016/j.disopt.2005.08.007 (cit. on p. 1).