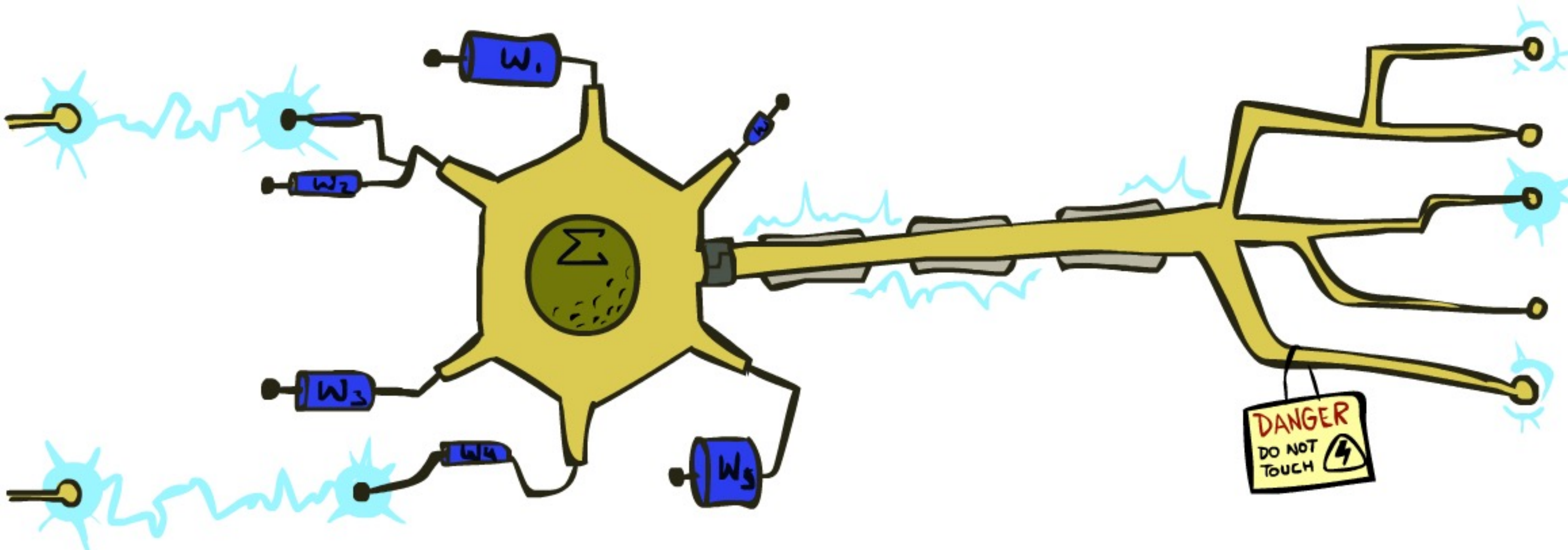
Ve492: Introduction to Artificial Intelligence

Discriminative Learning



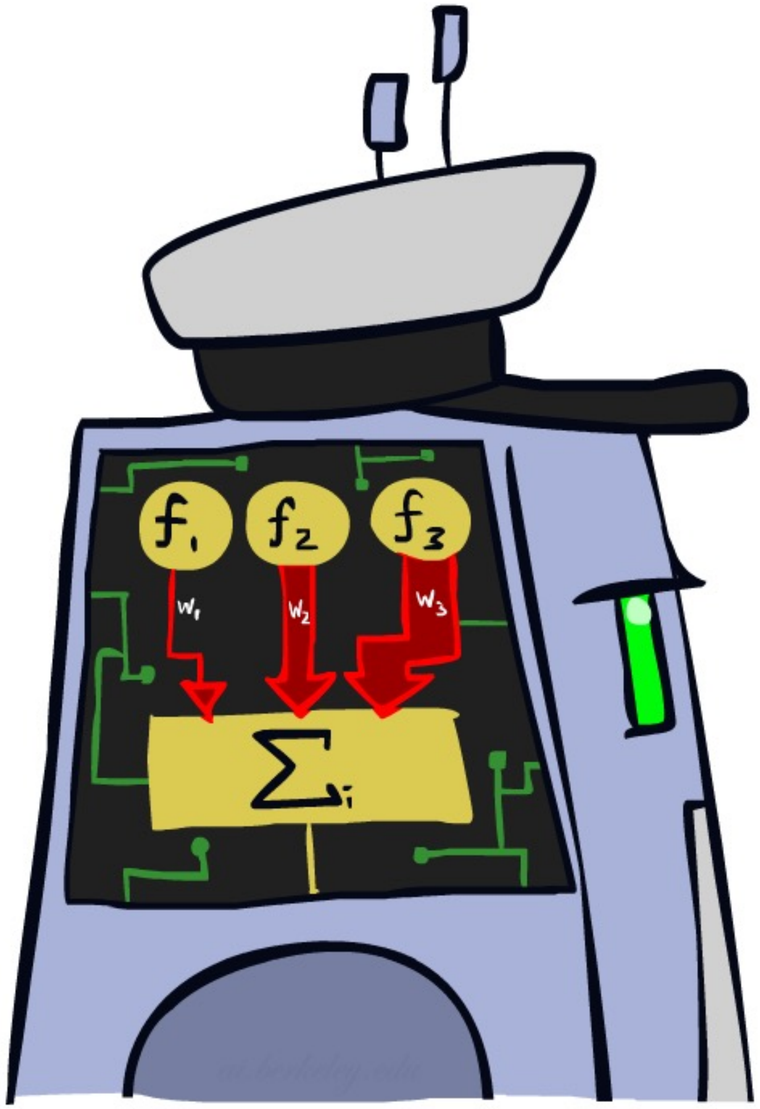
Paul Weng

UM-SJTU Joint Institute

Slides adapted from http://ai.berkeley.edu, AIMA, UM

Error-Driven Classification

Linear Classifiers



Feature Vectors

x $\varphi(x)$ y

```
Hello,

Do you want free
printr cartriges?
Why pay more when
you can get them
ABSOLUTELY FREE!
Just
```

```
# free       : 2
YOUR_NAME    : 0
MISSPELLED   : 2
FROM_FRIEND  : 0
     ...
```

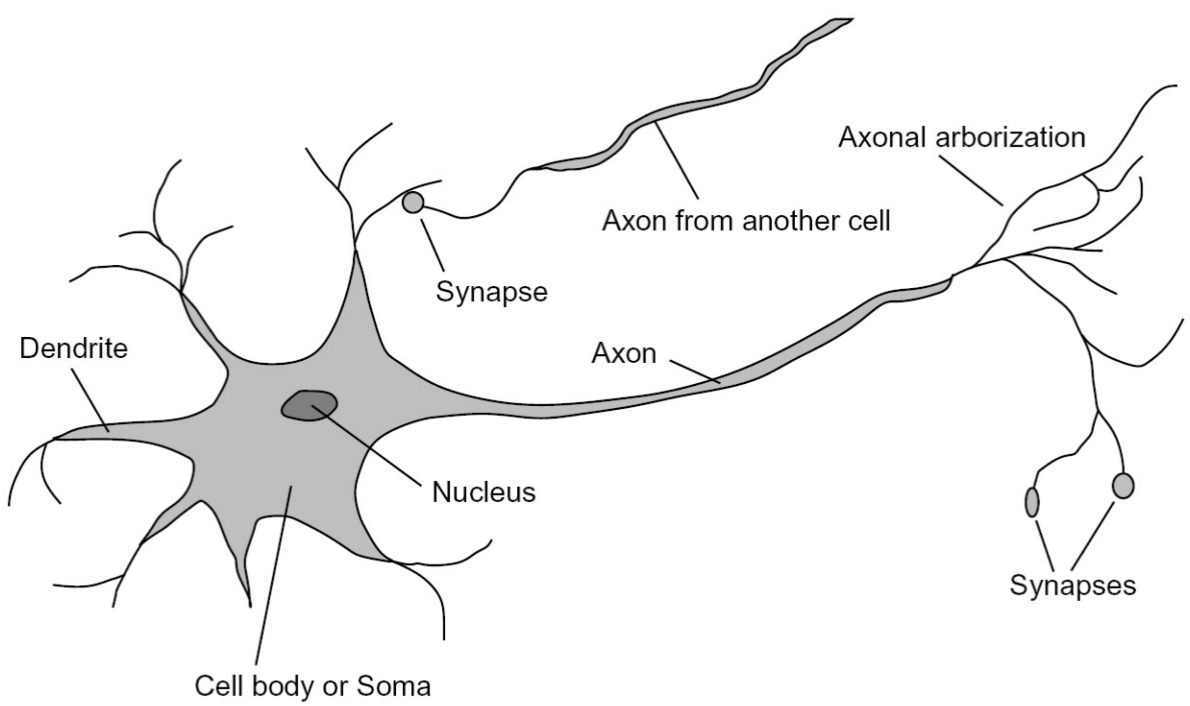
SPAM

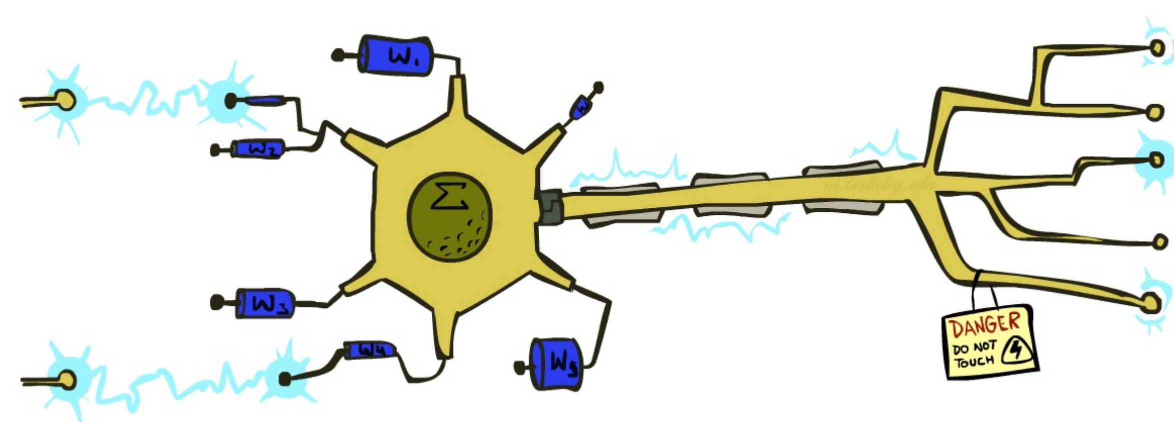
```
PIXEL-7,12   : 1
PIXEL-7,13   : 0
     ...
NUM_LOOPS    : 1
     ...
```

“2”

Some (Simplified) Biology

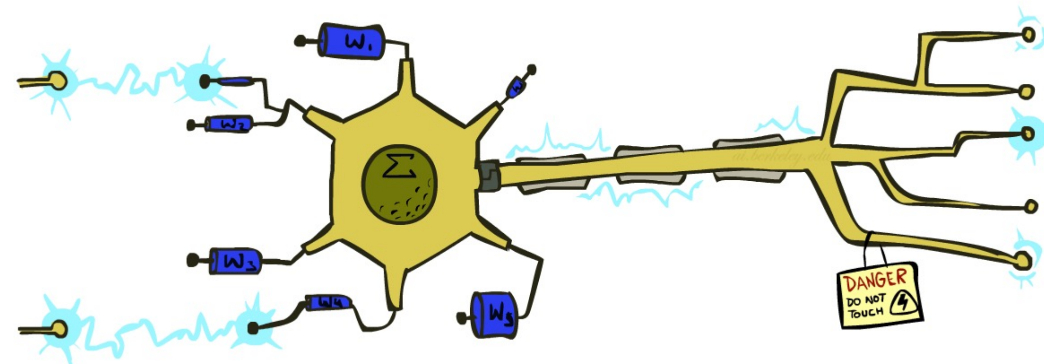
- Very loose inspiration: human neurons





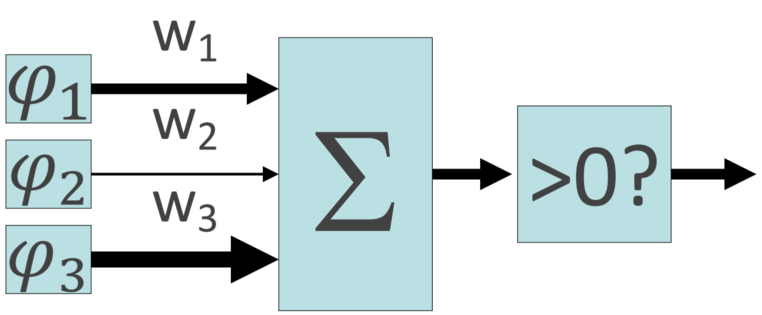
Linear Classifiers

- Inputs are feature values
- Each feature has a weight
- Sum is the activation



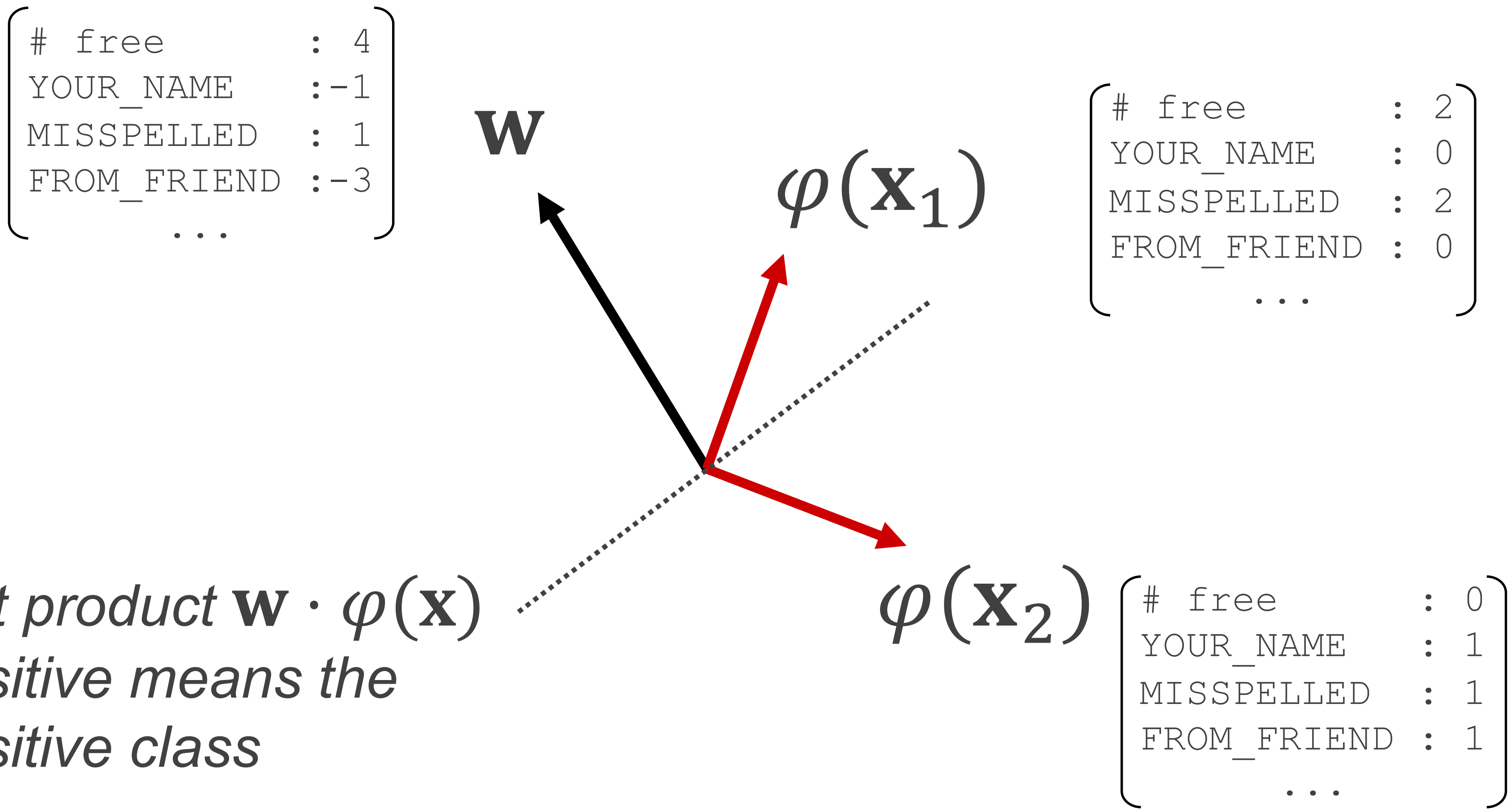
$$\text{activation}_{\mathbf{w}}(\mathbf{x}) = \sum_i w_i \varphi_i(\mathbf{x}) = \mathbf{w} \cdot \varphi(\mathbf{x})$$

- If the activation is:
  - Positive, output +1
  - Negative, output -1



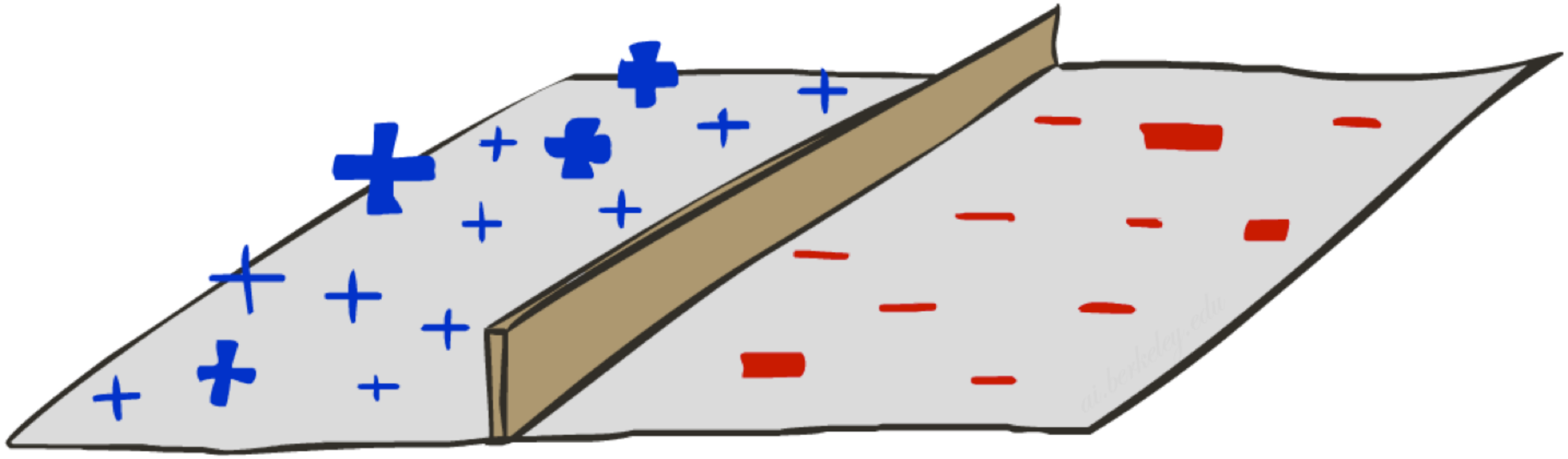
Weights

- Binary case: compare features to a weight vector
- Learning: figure out the weight vector from examples



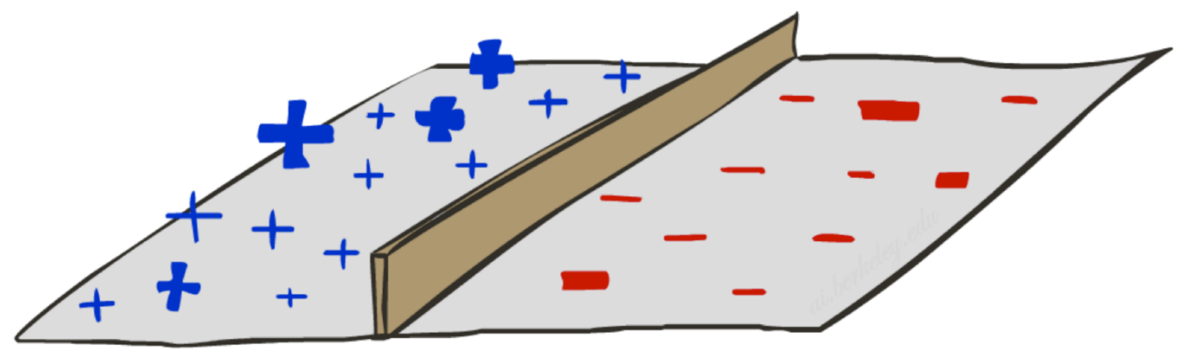
Dot product $\mathbf{w} \cdot \varphi(\mathbf{x})$ positive means the positive class

Decision Rules



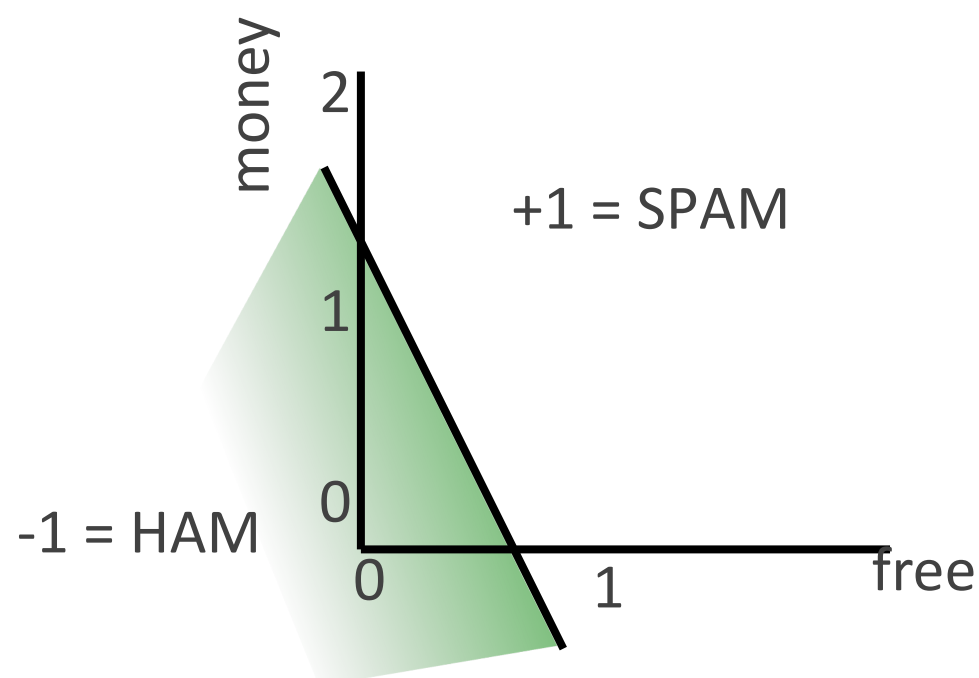
Binary Decision Rule

- In the space of feature vectors
  - Examples are points
  - Any weight vector is a hyperplane
  - One side corresponds to Y=+1
  - Other corresponds to Y=-1



w

```
BIAS  : -3
free  :  4
money :  2
...
```

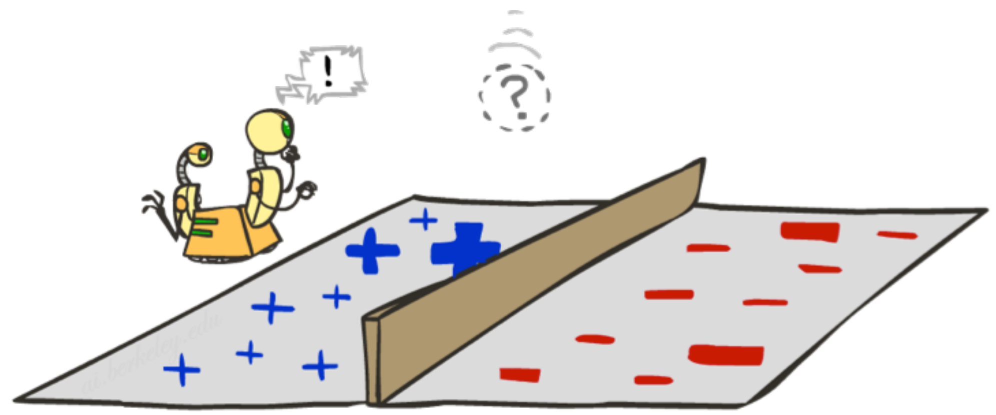


$$\mathbf{w} \cdot \varphi(\mathbf{x}) = 0$$

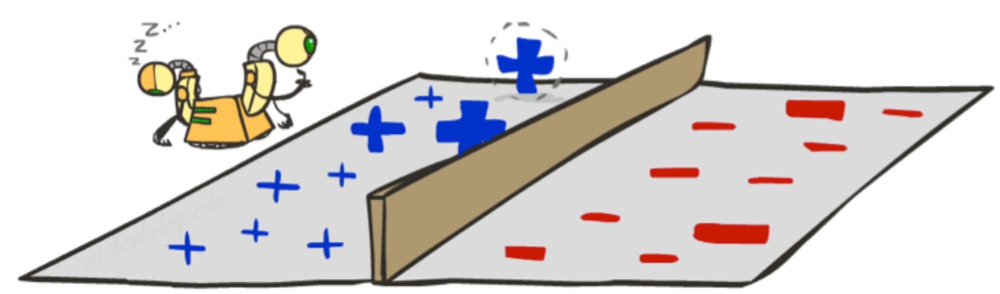
Weight Updates

Learning: Binary Perceptron

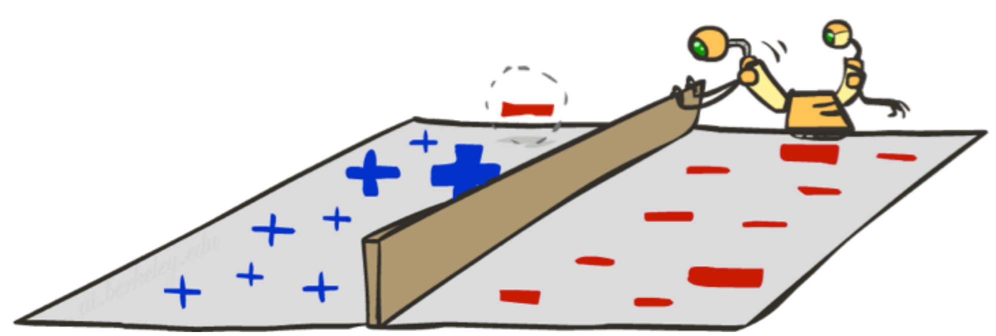
- Start with weights = 0
- For each training instance (**x**, y*):
  - Classify with current weights



  - If correct (i.e., $\hat{y} = y^*$), no change!



  - If wrong: adjust the weight vector



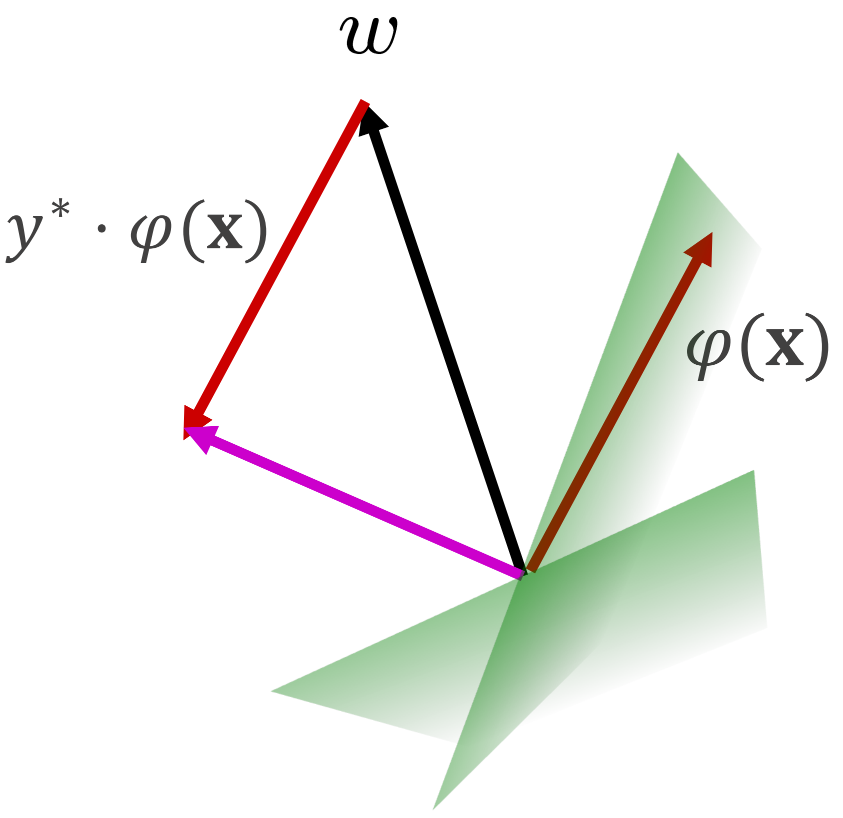
Learning: Binary Perceptron

- Start with weights = 0
- For each training instance (**x**, y*):
  - Classify with current weights

$$\hat{y} = \begin{cases} +1 & \text{if } \mathbf{w} \cdot \varphi(\mathbf{x}) \geq 0 \\ -1 & \text{if } \mathbf{w} \cdot \varphi(\mathbf{x}) < 0 \end{cases}$$

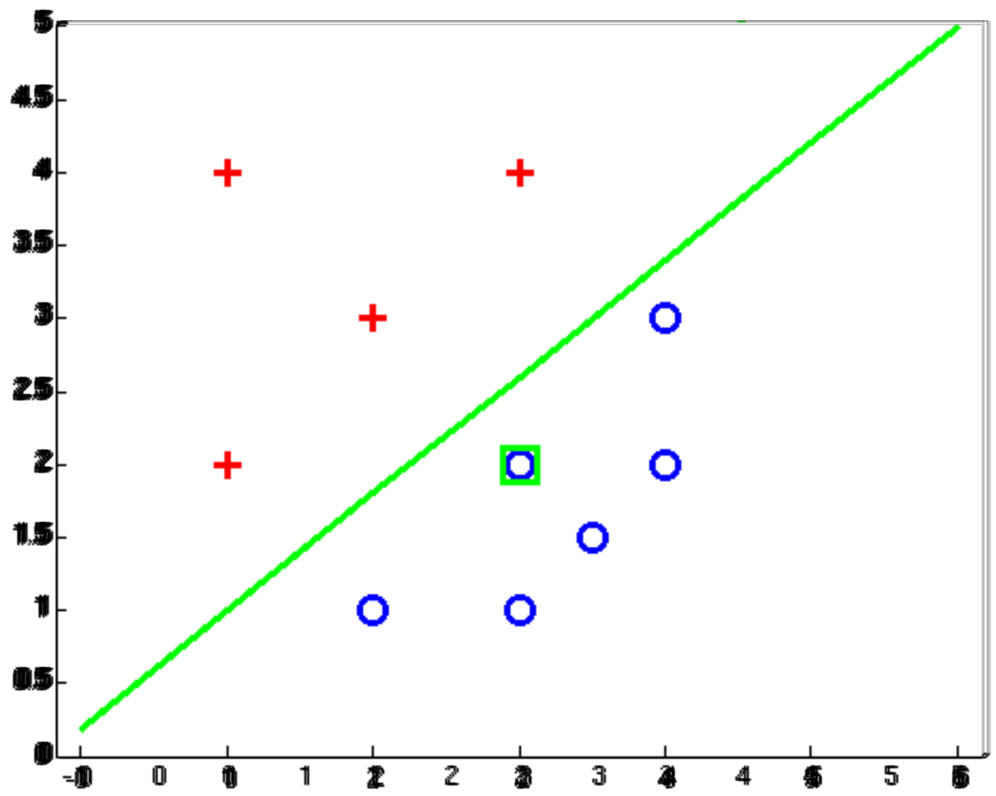
  - If correct (i.e., $\hat{y} = y^*$), no change!
  - If wrong: adjust the weight vector by adding or subtracting the feature vector. Subtract if y^* is -1.

$$\mathbf{w} = \mathbf{w} + y^* \cdot \varphi(\mathbf{x})$$



Examples: Perceptron

- Separable Case



Multiclass Decision Rule

- If we have multiple classes:
  - A weight vector for each class:

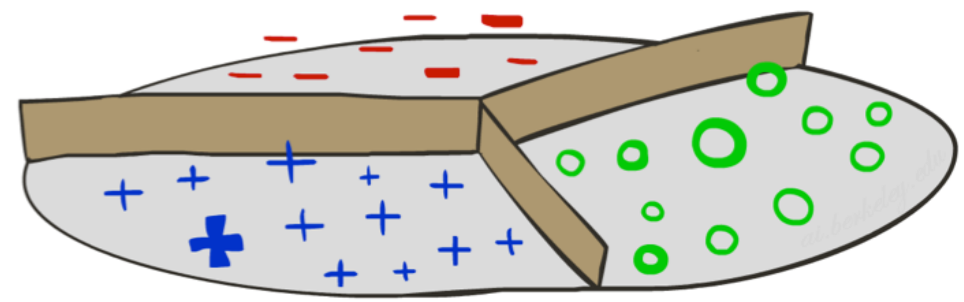
    $$\mathbf{w}_y$$

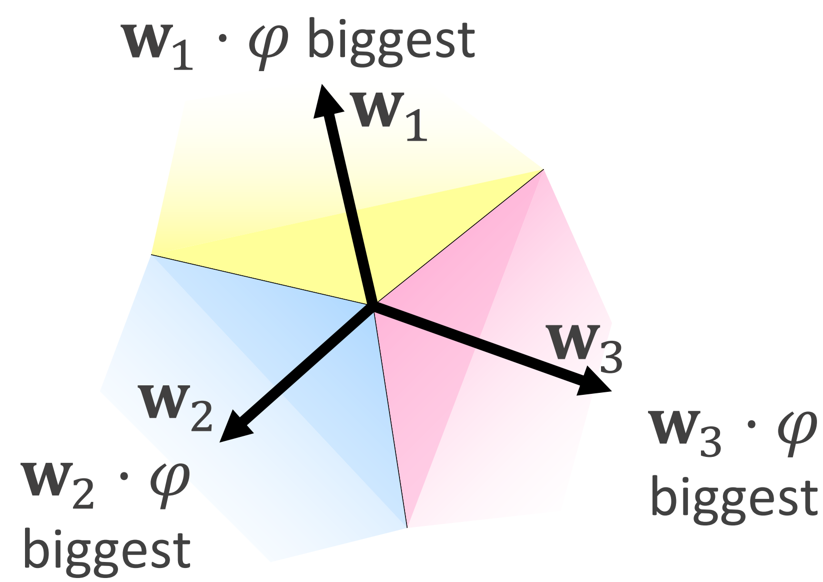
  - Score (activation) of a class y:

    $$\mathbf{w}_y \cdot \varphi(\mathbf{x})$$

  - Prediction highest score wins

    $$\hat{y} = \text{argmax}_y \mathbf{w}_y \cdot \varphi(\mathbf{x})$$





Quiz: Binary Classif. As Multiclass Decision Rule

- Multiclass decision rule

$$\hat{y} = \text{argmax}_y \mathbf{w}_y \cdot \varphi(\mathbf{x})$$

- Denote $\mathbf{w}$ the weight vector of the positive class.
- What could be the weight vector of the negative class?

-w

Learning: Multiclass Perceptron

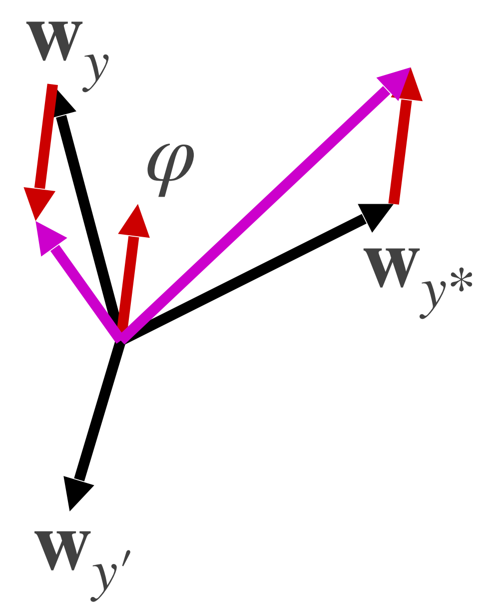
- Start with all weights = 0
- Pick up training examples one by one
- Predict with current weights

$$\hat{y} = \mathrm{argmax}_y \mathbf{w}_y \cdot \varphi(\mathbf{x})$$

- If correct, no change!
- If wrong: lower score of wrong answer, raise score of right answer

$$\mathbf{w}_{\hat{y}} = \mathbf{w}_{\hat{y}} - \varphi(\mathbf{x})$$

$$\mathbf{w}_{y*} = \mathbf{w}_{y*} + \varphi(\mathbf{x})$$



Example: Multiclass Perceptron

"win the vote"

"win the election"

"win the game"

w_{SPORTS}

BIAS	:	1
win	:	0
game	:	0
vote	:	0
the	:	0
...		

$w_{POLITICS}$

BIAS	:	0
win	:	0
game	:	0
vote	:	0
the	:	0
...		

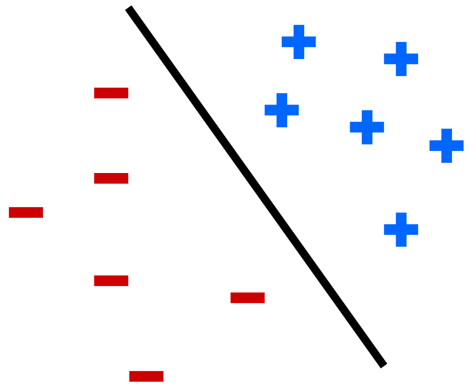
w_{TECH}

BIAS	:	0
win	:	0
game	:	0
vote	:	0
the	:	0
...		

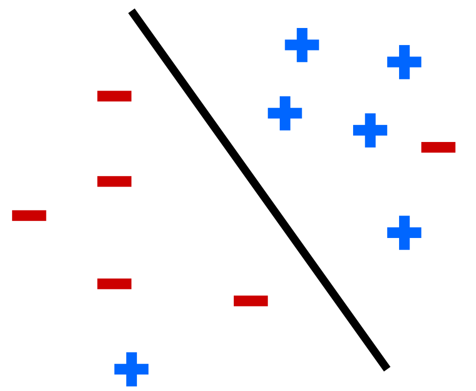
Properties of Perceptrons

- Separability: true if some parameters get the training set perfectly correct
- Convergence: if the training set is separable, perceptron will eventually converge (binary case)
- Mistake Bound: the maximum number of mistakes (binary case) related to the *margin* or degree of separability

Separable



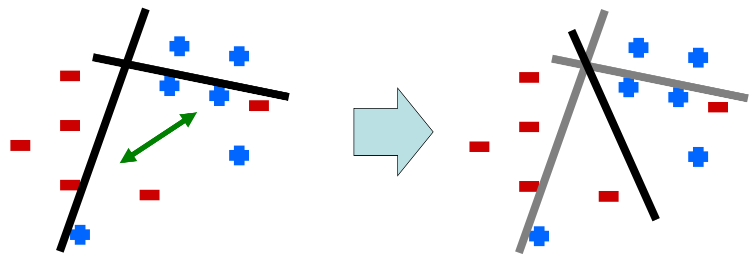
Non-Separable

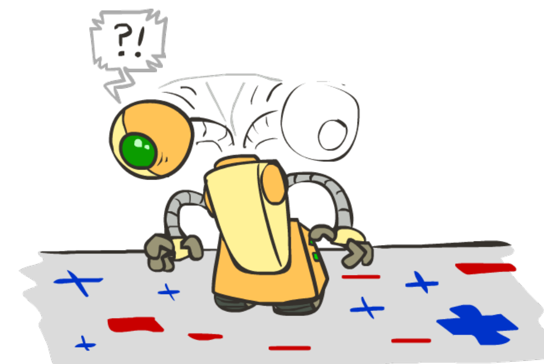


$$\text{mistakes} < \frac{k}{\delta^2}$$

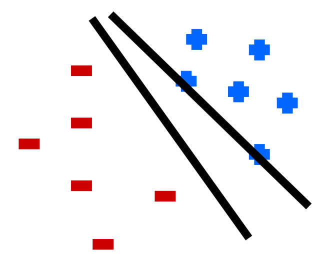
Problems with the Perceptron

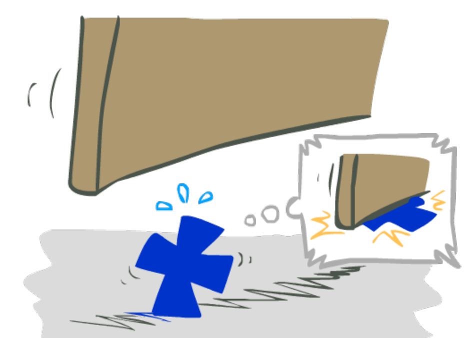
- Noise: if the data isn't separable, weights might thrash
  - Averaging weight vectors over time can help (averaged perceptron)



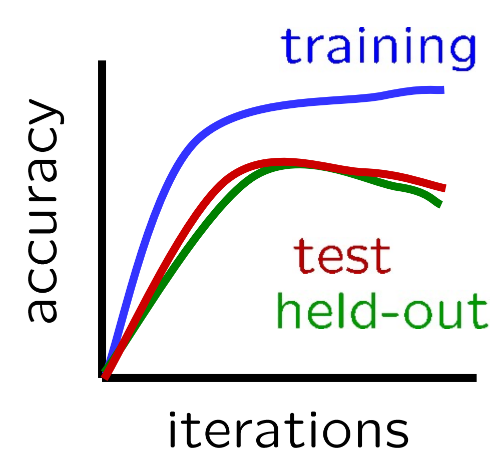


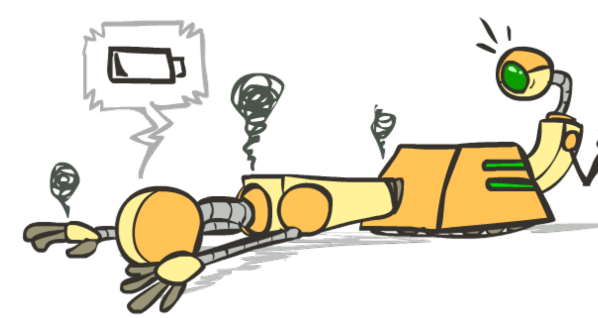
- Mediocre generalization: finds a "barely" separating solution



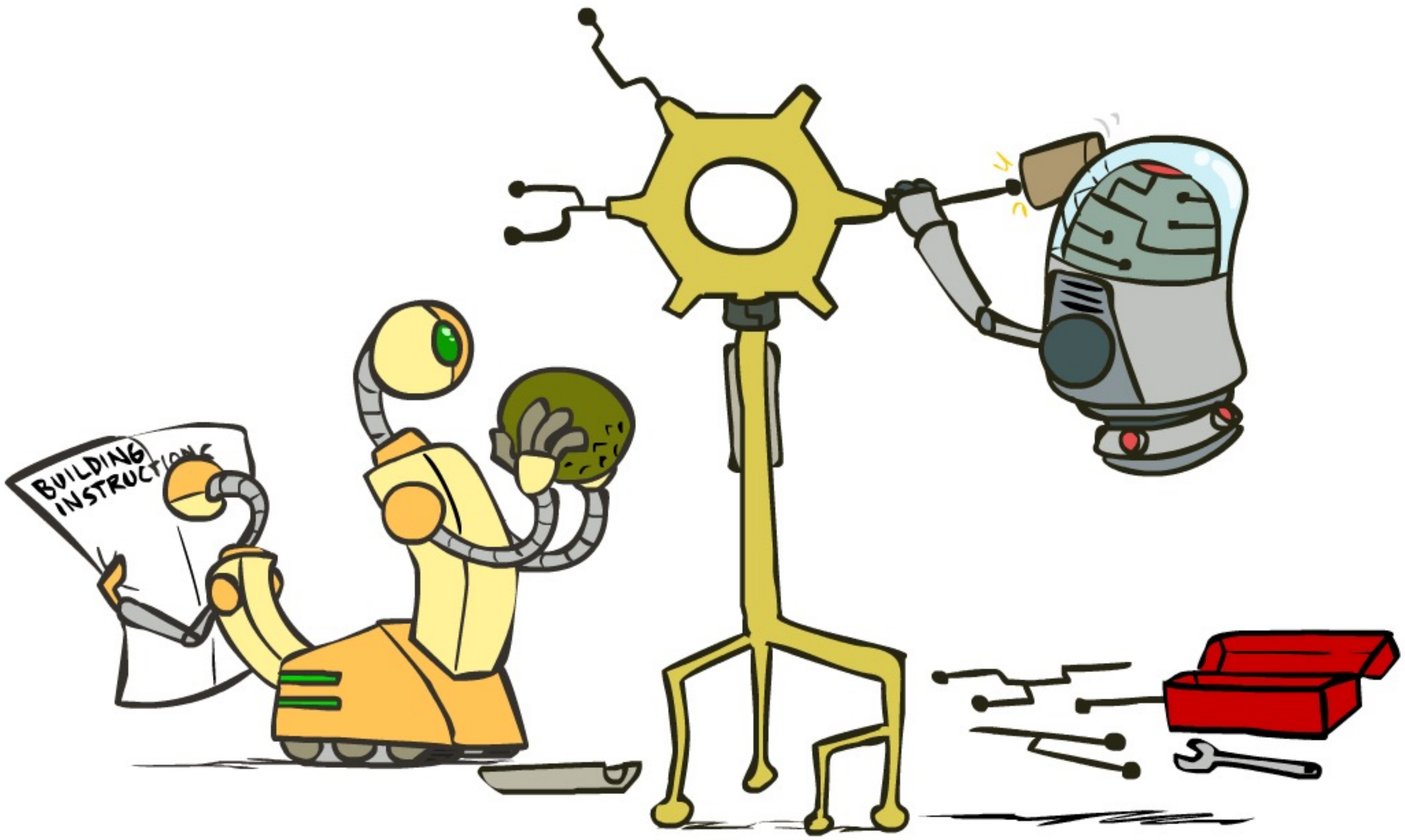


- Overtraining: test/validation accuracy usually rises, then falls
  - Overtraining is a kind of overfitting





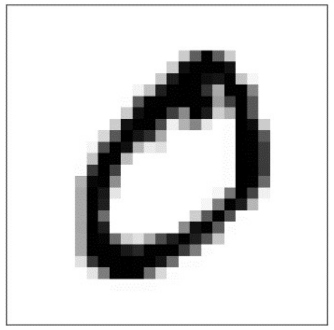
Improving the Perceptron



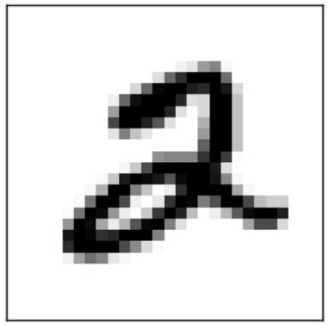
Probabilistic Classification

- Naïve Bayes provides *probabilistic* classification

$$\text{Answers the query: } P(Y = y_i | x_1, \ldots, x_n)$$



1: 0.001
2: 0.001
…
0: 0.991

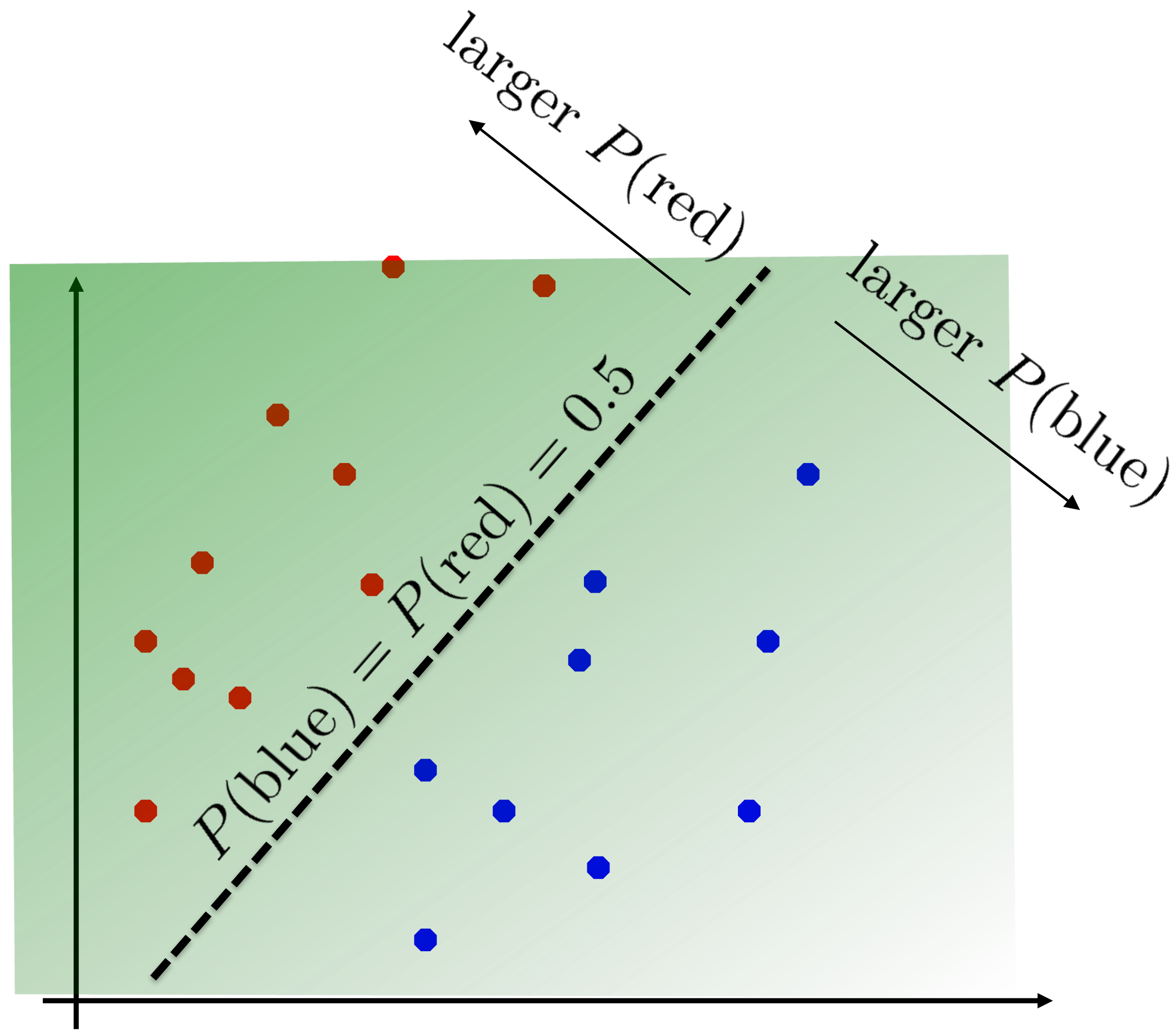


1: 0.001
2: 0.703
…
6: 0.264
…
0: 0.001

- Perceptron just gives us a class prediction
  - Can we get it to give us probabilities?
  - Turns out it also makes it easier to train!

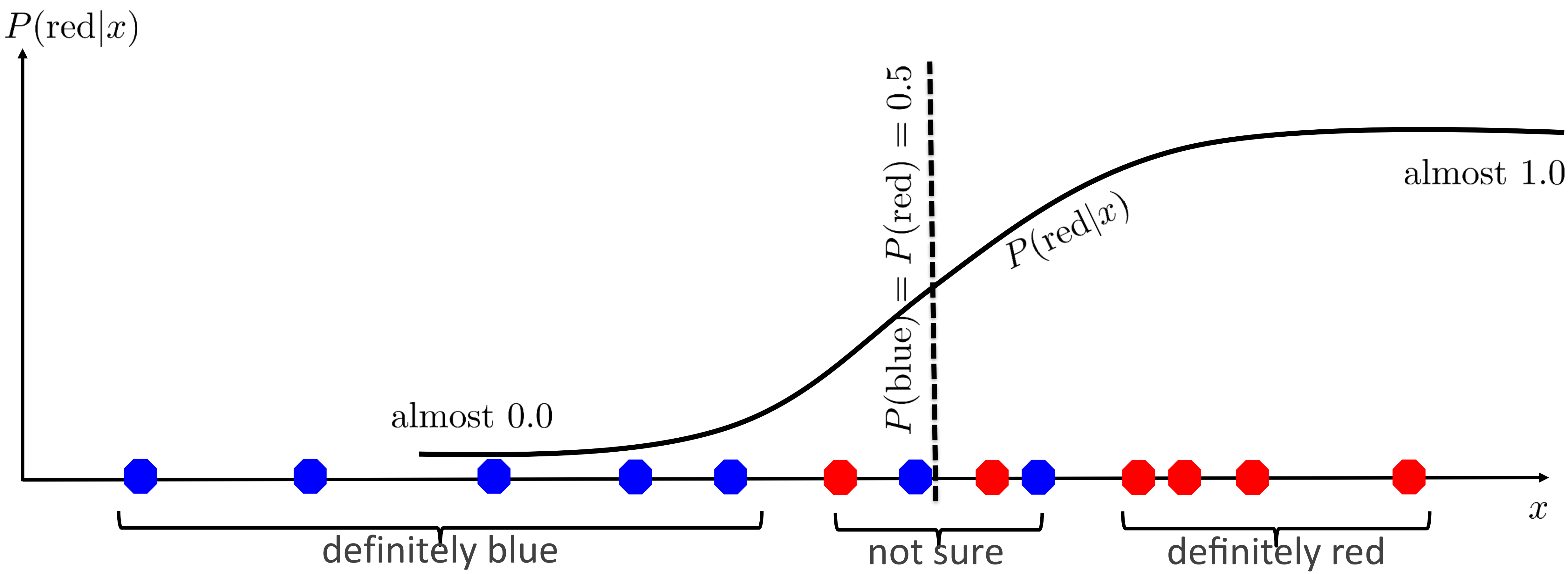
Note: To simplify notations, "x" denotes "$\varphi(\mathbf{x})$" from now on

A *Probabilistic* Perceptron



As $w_y \cdot x$ gets bigger, $P(y|x)$ gets bigger

A 1D Example

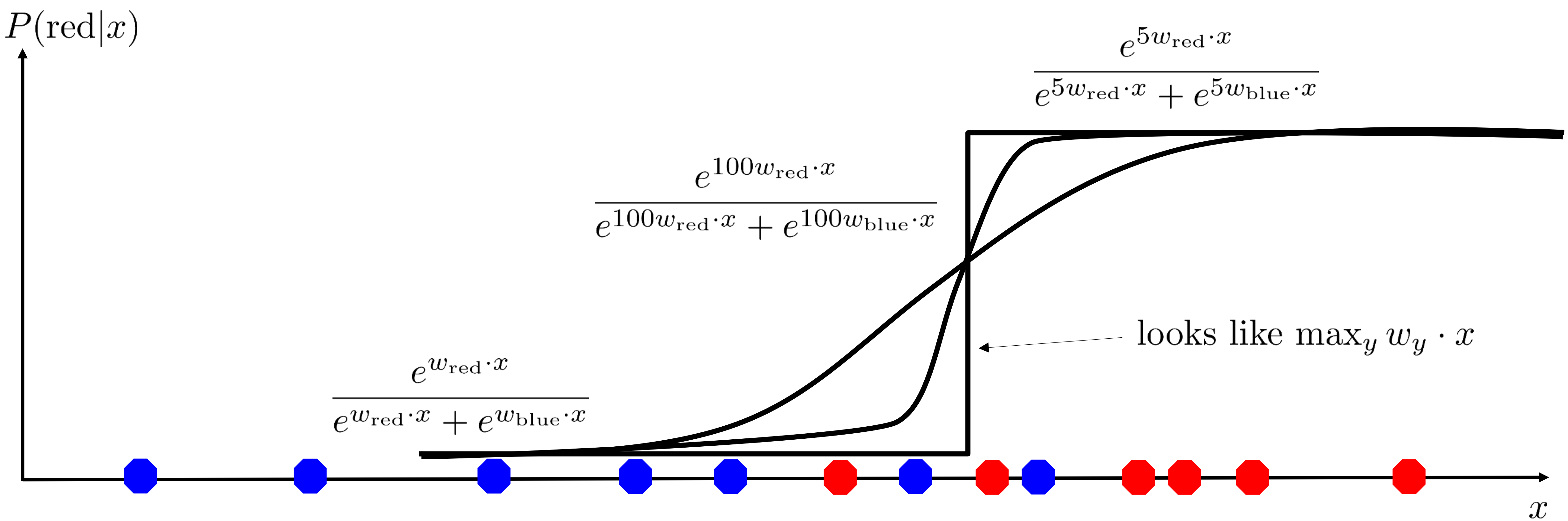


probability increases exponentially as we move away from boundary

$$P(\text{red}|x) = \frac{e^{w_{\text{red}} \cdot x}}{e^{w_{\text{red}} \cdot x} + e^{w_{\text{blue}} \cdot x}}$$

normalizer

The *Soft* Max

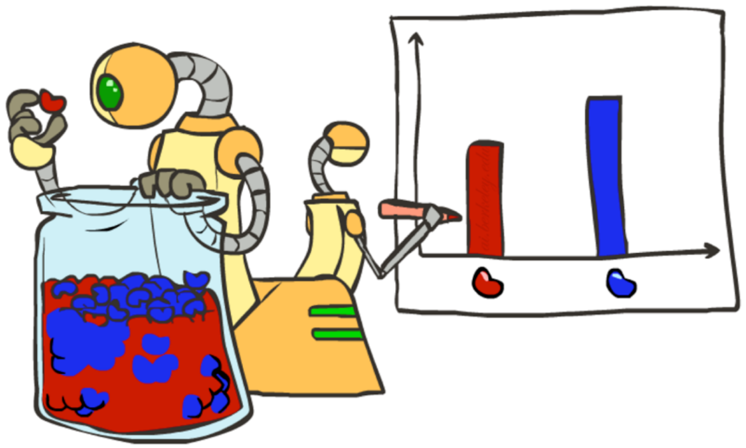


$$P(\text{red}|x) = \frac{e^{w_{\text{red}} \cdot x}}{e^{w_{\text{red}} \cdot x} + e^{w_{\text{blue}} \cdot x}}$$

How to Learn?

- Maximum likelihood estimation

$$\theta_{ML} = \arg\max_{\theta} P(\mathbf{X}|\theta)$$
$$= \arg\max_{\theta} \prod_i P_\theta(X_i)$$



- Maximum *conditional* likelihood estimation

$$\theta^\star = \arg\max_{\theta} P(\mathbf{Y}|\mathbf{X}, \theta)$$
$$= \arg\max_{\theta} \prod_i \underbrace{P_\theta(y_i|x_i)}$$
$$\ell(w) = \prod_i \frac{e^{w_{y_i}\cdot x_i}}{\sum_y e^{w_y \cdot x_i}}$$

$$\ell\ell(w) = \sum_i \log P_w(y_i|x_i)$$
$$= \sum_i w_{y_i} \cdot x_i - \log \sum_y e^{w_y \cdot x_i}$$

Local Search

- Simple, general idea:
  - Start wherever
  - Repeat: move to the best neighboring state
  - If no neighbors better than current, quit
  - Neighbors = small perturbations of w

Our Status

- Our objective $ll(w)$

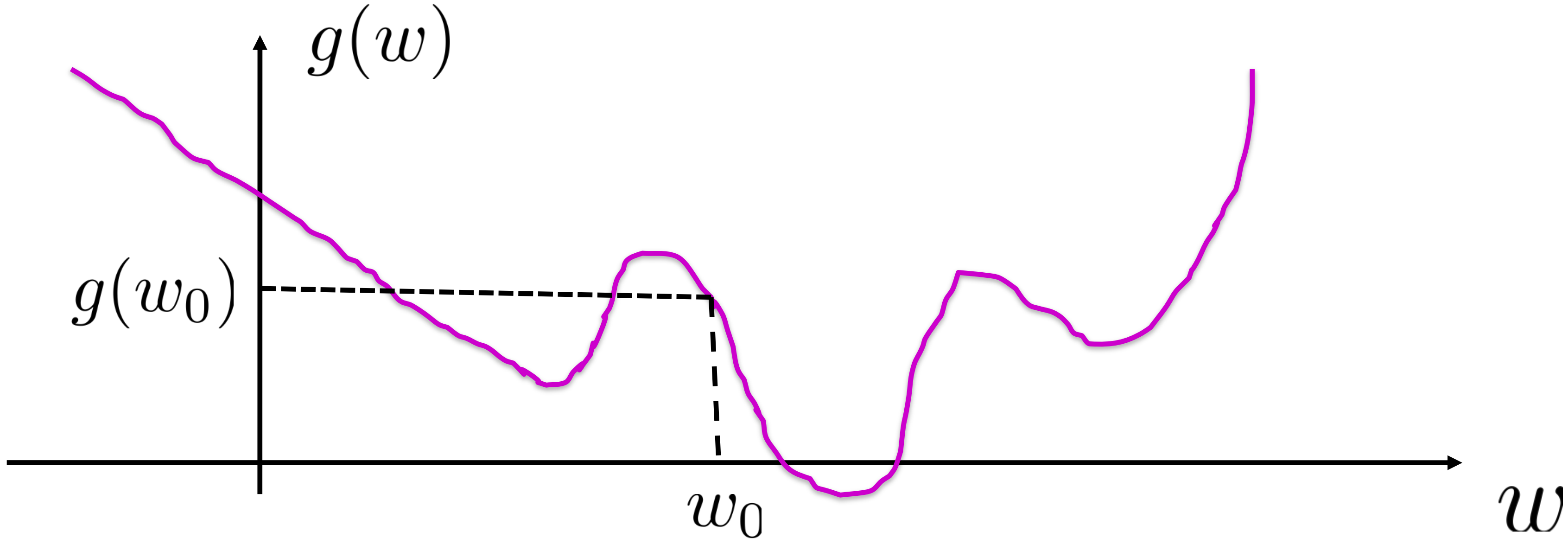
- Challenge: how to find a good w ?

$$\max_w ll(w)$$

- Equivalently:

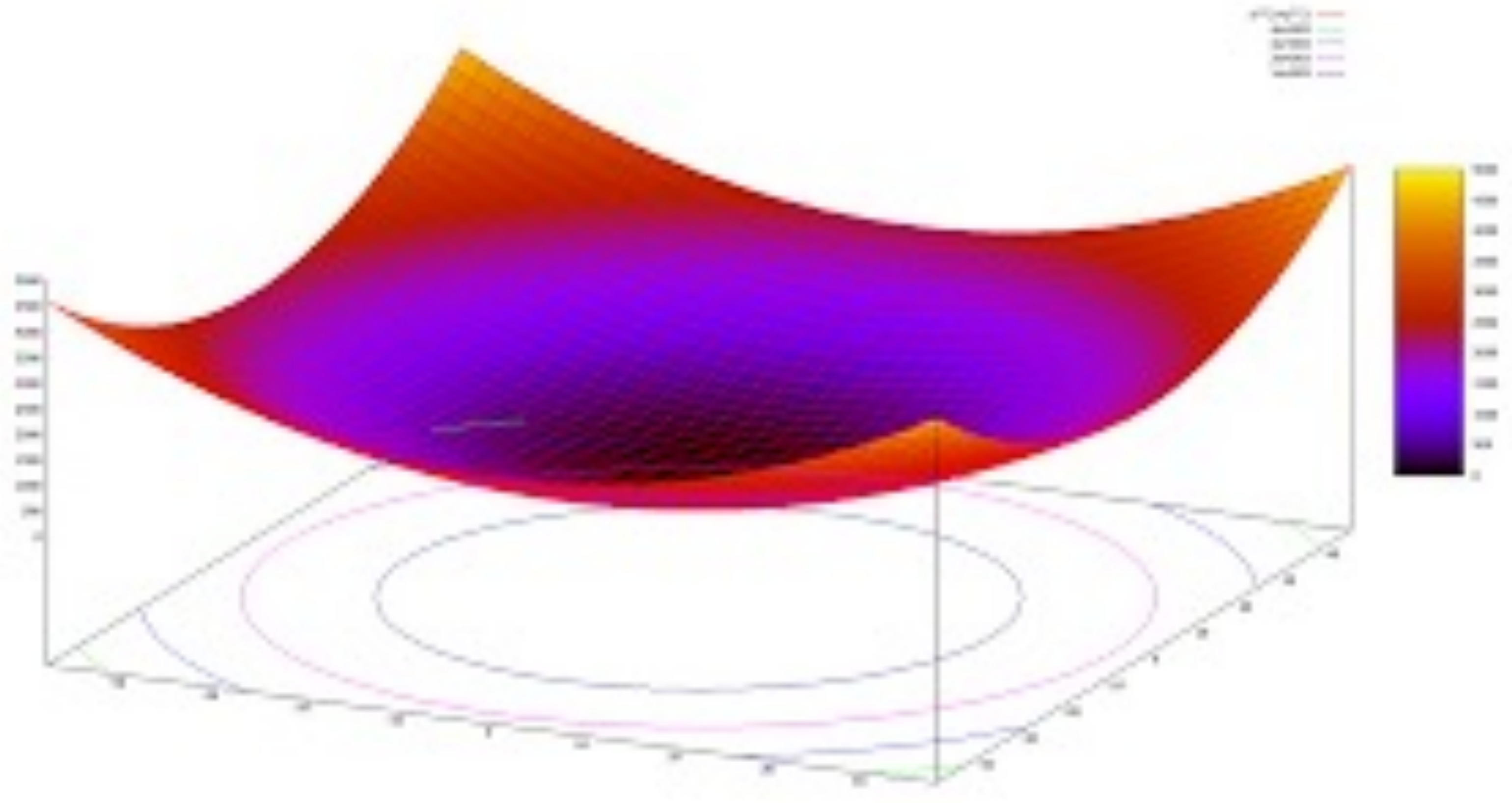
$$\min_w -ll(w)$$

1D optimization



- Could evaluate $g(w_0 + h)$ and $g(w_0 - h)$
- Then step in best direction
- Or, evaluate derivative: $\dfrac{\partial g(w_0)}{\partial w} = \lim_{h \to 0} \dfrac{g(w_0 + h) - g(w_0 - h)}{2h}$
- Which tells which direction to step into

2-D Optimization



Source: Thomas Jungblut's Blog

Steepest Descent

- Idea:
  - Start somewhere
  - Repeat: Take a step in the steepest descent direction

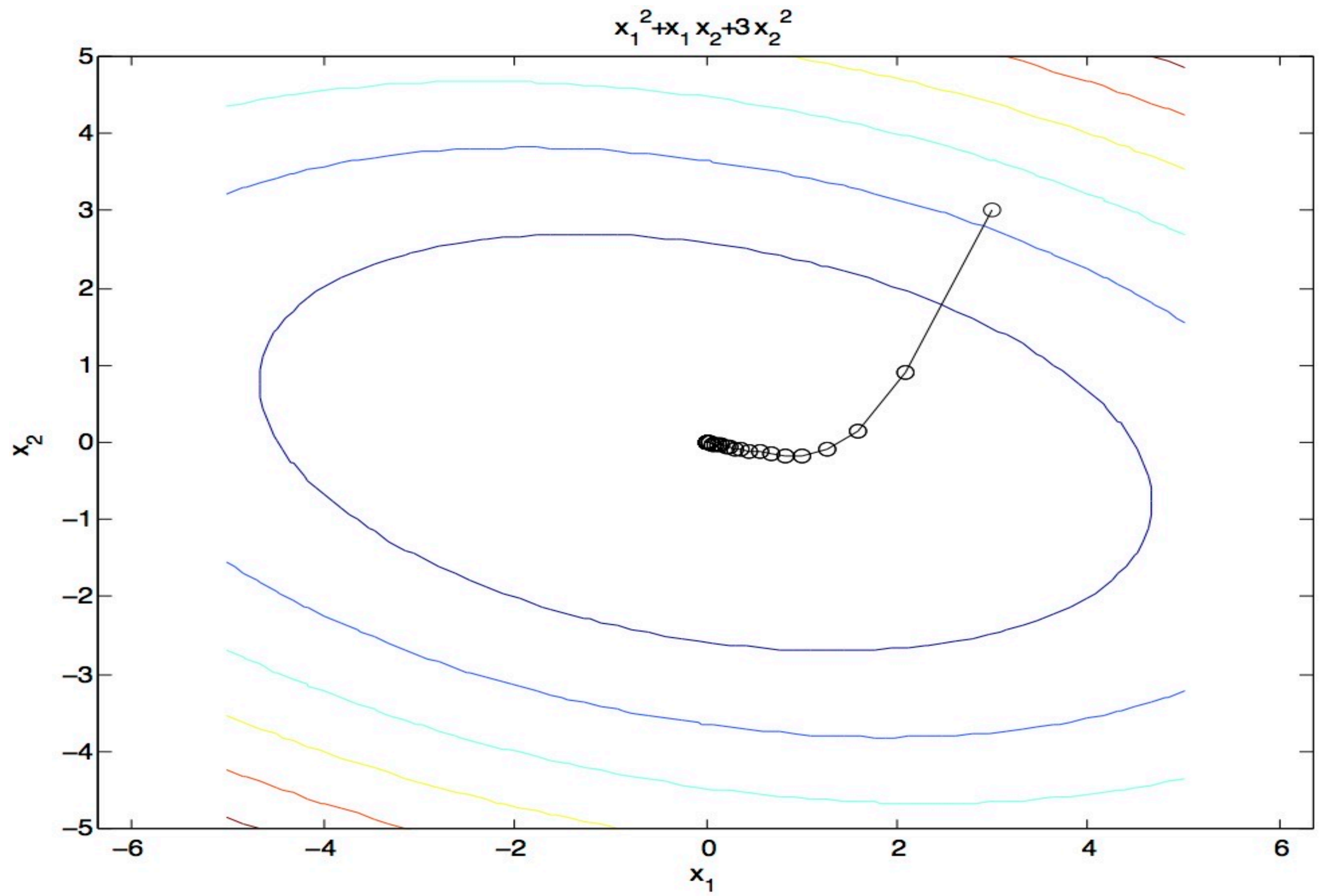


Figure source: Mathworks

Steepest Direction

- Steepest Direction = direction of the gradient

$$\nabla g = \begin{bmatrix} \frac{\partial g}{\partial w_1} \\ \frac{\partial g}{\partial w_2} \\ \cdots \\ \frac{\partial g}{\partial w_n} \end{bmatrix}$$

How to Learn?

$$\ell\ell(w) = \sum_i \log P_w(y_i|x_i)$$

$$= \sum_i w_{y_i} \cdot x_i - \log \sum_y e^{w_y \cdot x_i}$$

$$\frac{d}{dw_y} \log P_w(y_i|x_i) = \begin{cases} x_i - x_i \frac{e^{w_y \cdot x_i}}{\sum_{y'} e^{w_{y'} \cdot x_i}} & \text{if } y = y_i \\ -x_i \frac{e^{w_y \cdot x_i}}{\sum_{y'} e^{w_{y'} \cdot x_i}} & \text{otherwise} \end{cases}$$

$$= x_i(I(y = y_i) - P(y|x_i))$$

Optimization Procedure: Gradient Descent

initialize w (e.g., randomly)

repeat for K iterations:

  for each example (x_i, y_i):

    compute gradient $\Delta_i = -\nabla_w \log P_w(y_i|x_i)$

  compute gradient $\nabla_w \mathcal{L} = \sum_i \Delta_i$

  $w \leftarrow w - \alpha \nabla_w \mathcal{L}$

$$\frac{d}{dw_y} \log P_w(y_i|x_i) = x_i(I(y = y_i) - P(y|x_i))$$

- α: learning rate —- hyperparameter that needs to be chosen carefully
- How? Try multiple choices
  - Crude rule of thumb: update should change w by about 0.1-1%

Stochastic Gradient Descent

initialize w (e.g., randomly)

repeat for K iterations:

for each example (x_i, y_i):

compute gradient $\Delta_i = -\nabla_w \log P_w(y_i|x_i)$

$w \leftarrow w - \alpha \Delta_i$

$$\frac{d}{dw_y} \log P_w(y_i|x_i) = x_i(I(y = y_i) - P(y|x_i))$$

if $y_i = y$, move w_y toward x_i

with weight $\underbrace{1 - P(y_i|x_i)}_{\text{probability of \textit{incorrect} answer}}$

if $y_i \neq y$, move w_y away from x_i

with weight $\underbrace{P(y|x_i)}_{\text{probability of \textit{incorrect} answer}}$

compare this to the multiclass perceptron: probabilistic weighting!

Logistic Regression Demo!

https://playground.tensorflow.org/