VE477 Introduction to Algorithms
Homework 2

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2021/10/01

Ex. 1 — Basic complexity

1. a) To prove this statement, it is equivalent to prove that $c_1 n^3 \leq n^3 - 3n^2 - n + 1 \leq c_2 n^3$ holds when $n \to \infty$, where c_1 and c_2 are two positive constant number.

   Firstly, we can find that $n^3 - 3n^2 - n + 1 \leq n^3$ when $n \geq 1$. Therefore, $c_2 = 1$ and $n^3 - 3n^2 - n + 1 = \mathcal{O}(n^3)$.

   Then, let $f(n) = (1 - c_1)n^3 - 3n^2 - n + 1$, then take its derivative $f'(n) = 3(1 - c_1)n^2 - 6n - 1$. We can see that $f(n)$ is strictly increasing on $[\frac{1}{1-c_1}, +\infty)$. Take c_1 as $\frac{1}{4}$. After some calculation, we can have $f(n) = (1 - c_1)n^3 - 3n^2 - n + 1 > 0$ for $n \geq 5$. Therefore, $\frac{1}{4}n^3 \leq n^3 - 3^n - n + 1$ for $n \geq 5$, indicating that $n^3 - 3n^2 - n + 1 = \Omega(n^3)$.

   Combining the two results, we can finally get to the conclusion that

   $$n^3 - 3n^2 - n + 1 = \mathbf{\Theta}(n^3)$$

   b) To prove the statement, it is equivalent to prove that $n^2 \leq c2^n$ for some constant c and $n \to \infty$. Take the logarithm of both sides, thus we need to prove that $2\log(n) \leq cn \log 2$.

   It is common knowledge that $\log(n) \leq n$ when $n \geq 1$. Therefore, we just need by choosing $c = \frac{2}{\log 2}$, we can have $2\log(n) \leq cn\log 2$ for all $n \geq 1$. Therefore, $n^2 = \mathcal{O}(2^n)$. Proof done.

   c) Firstly, if $a = 0$, it is obviously true, since $n^b \leq n^b \leq n^b$.

   Then, when $a > 0$, we see that

   $$f(n) = (n+a)^b = \sum_{i=0}^{b} \begin{pmatrix} b \\ i \end{pmatrix} a^i n^{b-i}$$

   $$f(n) \leq (\sum_{i=0}^{b} \begin{pmatrix} b \\ i \end{pmatrix} a^i) n^b$$

   So for $c_1 = (\sum_{i=0}^{b} \begin{pmatrix} b \\ i \end{pmatrix} a^i)$, $(n+a)^b \leq c_1 n^b$. Thus, $(n+a)^b = \mathcal{O}(n^b)$. And since $a > 0$, $(n+a)^b \geq n^b$, so $(n+a)^b = \Omega(n^b)$. Therefore, combining the two results, we can conclude that $(n+a)^b = \mathbf{\Theta}(n^b)$.

When $a < 0$, firstly, same as the above condition, for $c_1 = \sum_{i=0}^{b} \begin{pmatrix} b \\ i \end{pmatrix} a^i$, $(n+a)^b \leq c_1 n^b$, indicating that $(n + a)^b = \mathcal{O}(n^b)$. Also, similar to the question a), for any finite N, we can find a c_2 such that when $n \geq N$, $(n + a)^b \geq c_2 n^b$. It needs attention that $0 < c_2 < 1$. Thus, $(n + a)^b = \mathbf{\Theta}(n^b)$.

Since the above three conditions are all met, we can conclude that $(n + a)^b = \mathbf{\Theta}(n^b)$. Proof done.

2. (a) $f(n) = \mathcal{O}(g(n))$. Construct $h(n) = n^2 - 1 - n\sqrt{n}$, then take its derivative, we can get:

$$h'(n) = 2n - (\sqrt{n} + \frac{\sqrt{n}}{2}) = 2n - \frac{3}{2}\sqrt{n}$$

It is obvious that for $n \geq 1$, $h(n)$ is strictly increasing. Since when $n \geq 2 \Rightarrow h(n) \geq 0$, thus $n\sqrt{n} < n^2 - 1$, we can have $f(n) < g(n)$. Therefore, $f(n) = \mathcal{O}(g(n))$. Proof done.

(b) $f(n) = \Omega(g(n))$. Construct $h(n) = f(n) - g(n) = 2^n - n^2 - n^4 - n^2$, then take its logarithm, we can get:

$$s(n) = \log h(n) = n \log 2 - 2 \log n - 4 \log n - 2 \log n = n \log 2 - 8 \log n$$

Then take the derivative of $s(n)$, we can get, $s'(n) = \log 2 - 8/n$. For $n \geq 14$, $s'(n) > 0$, therefore, $h(n)$ is strictly increasing on $[14, +\infty)$.

Therefore, when $n \geq 17$, $h(n) > 0$, which indicates that $f(n) > g(n)$ when $n \to +\infty$. In conclusion, $f(n) = \Omega(g(n))$. Proof done.

3. a) Since the requirement for $f(n) = \mathbf{\Theta}(g(n))$ is $f(n) = \mathcal{O}(n)$ and $f(n) = \Omega(g(n))$ at the same time, the statement is false, so we cannot find such two functions.

b) From problem 2(b), $f(n) = 2^n - n^2$ and $g(n) = n^4 + n^2$ can meet the requirement for $f(n) = \Omega(g(n))$ and $f(n) \neq \mathcal{O}(g(n))$.

4. $f_2(n) < f_3(n) < f_1(n) < f_4(n)$.

First we construct $g(n) = f_2(n) - f_3(n) = \sqrt{n} \log n - n\sqrt{\log n}$. Then take its derivative, we can get:

$$
\begin{aligned}
g'(n) &= \frac{\log n}{2\sqrt{n}} + \frac{\sqrt{n}}{n} - (\sqrt{\log n} + n \cdot \frac{1}{2\sqrt{\log n}} \cdot \frac{1}{n}) \\
&= \frac{\log n + 2}{2\sqrt{n}} - \frac{2\log n + 1}{2\sqrt{\log n}} \\
&= \frac{\log n\sqrt{\log n} + 2\sqrt{\log n} - 2\sqrt{n}\log n - \sqrt{n}}{2\sqrt{n\log n}} \\
&= \frac{\log n(\sqrt{\log n} - 2\sqrt{n}) + (2\sqrt{\log n} - \sqrt{n})}{2\sqrt{n\log n}}
\end{aligned}
$$

From the slides we know that $\log n < \sqrt{n} < n$, which indicates that $\sqrt{\log n} < \sqrt{n}$. Thus, $g'(n) < 0$ for $n \geq 1$. In this sense, we can know that for $n \to +\infty$, $f_2(n) < f_3(n)$, which means that $f_2(n) = \mathcal{O}(f_3(n))$.

For $f_1(n) = \sum_{i=1}^{n} \sqrt{i}$, we can take its square, and get:

$$
\begin{aligned}
f_1^2(n) = (\sum_{i=1}^{n} \sqrt{i})^2 &= \sum_{i=1}^{n} i + 2 \sum_{1\leq i<j\leq n} \sqrt{ij} \\
&\geq \frac{n^2+n}{2} + 2\sum_{i=1}^{n-1} \sqrt{i^2}(n-i) \\
&= \frac{n^2+n}{2} + 2\sum_{i=1}^{n-1} (ni - i^2) \\
&= \frac{n^2+n}{2} + 2(n \cdot \frac{n^2-n}{2} - \frac{n(n+1)(2n+1)}{6}) \\
&= \mathbf{\Theta}(n^3)
\end{aligned}
$$

Then we take the square of $f_3(n)$, which gives us $f_3^2(n) = n^2 \log n < n^2\sqrt{n} < n^3$. Therefore, we can conclude that $f_3(n) < f_1(n)$.

For $f_1(n)$, we know that:

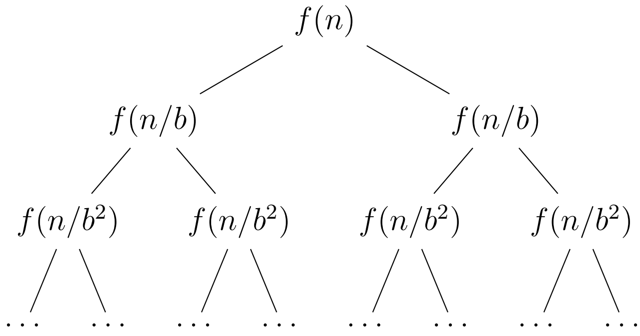
$$\sum_{i=1}^{n} \sqrt{i} \leq \frac{2}{3}(n + \frac{1}{2})^{\frac{3}{2}}$$

Since $f_4(n) = 24n^{\frac{3}{2}} + 4n + 6$, we can simply find that $f_1(n) < f_4(n)$.

According to all the calculations above, we can finally conclude that $f_2(n) < f_3(n) < f_1(n) < f_4(n)$. Proof done.

Ex. 2 — Master Theorem

1. a) The recurrence tree is shown below:



b) i) Let the depth of tree be d, then $n/b^d = 1$ when it reaches the bottom. Therefore, the depth of the tree is $\log_b n$.

ii) The number of leaves is a to the power of depth, which is $a^{\log_b n}$.

iii) The total cost for all the nodes at each depth is the cost of each node at that depth multiplied by the number of nodes. Suppose that the depth is d, so the total cost is $a^d f(n/b^d)$.

iv) The overall cost is the sum of the costs of all the depths, which is calculated as:

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{\log_b n} a^i f(n/b^i) \\
&= a^{\log_b n} f(n/b^{\log_b n}) + \sum_{i=0}^{\log_b n-1} a^i f(n/b^i) \\
&= n^{\log_b a} f(1) + \sum_{i=0}^{\log_b n-1} a^i f(n/b^i) \\
&= \boldsymbol{\Theta}(n^{\log_b a}) + \sum_{i=0}^{\log_b n-1} a^i f(n/b^i)
\end{aligned}
$$

2. (a) i) Since $f(n) = \boldsymbol{\Theta}(n^{\log_b a})$, we can find some constant c_1, c_2, such that:

$$c_1 n^{\log_b a} \le f(n) \le c_2 n^{\log_b a}$$

Then, by substituting n with n/b^j, for $0 \le j \le \log_b n - 1$, and multiplied by a^j, we can have:

$$c_1(a^j(\frac{n}{b^j})^{\log_b a}) \le a^j f(\frac{n}{b^j}) \le c_2(a^j(\frac{n}{b^j})^{\log_b a})$$

Calculating the sum for j from 0 to $\log_b n - 1$, we can get:

$$c_1 \sum_{j=0}^{\log_b n-1} a^j(\frac{n}{b^j})^{\log_b a} \le \sum_{j=0}^{\log_b n-1} a^j f(\frac{n}{b^j}) \le c_2 \sum_{j=0}^{\log_b n-1} a^j(\frac{n}{b^j})^{\log_b a}$$

Since $g(n) = \sum_{j=0}^{\log_b n-1} a^j f(n/b^j)$, we can finally reach the conclusion that $g(n) = \mathbf{\Theta}(\sum_{j=0}^{\log_b n-1} a^j(\frac{n}{b^j})^{\log_b a})$. Proof done.

ii) Firstly, we can do some formula transformation:

$$a^j(\frac{n}{b^j})^{\log_b a} = a^j \frac{n^{\log_b a}}{(b^j)^{\log_b a}} = a^j \frac{n^{\log_b a}}{(b^{\log_b a})^j} = a^j \frac{n^{\log_b a}}{a^j} = n^{\log_b a}$$

We can find that the deduced expression does not contain j. Therefore, we can simply get to the point:

$$\sum_{j=0}^{\log_b n-1} a^j(\frac{n}{b^j})^{\log_b a} = \sum_{j=0}^{\log_b n-1} n^{\log_b a} = n^{\log_b a} \log_b n$$

Proof done.

iii) From problem 2.a.i, we get:

$$c_1 \sum_{j=0}^{\log_b n-1} a^j(\frac{n}{b^j})^{\log_b a} \le g(n) \le c_2 \sum_{j=0}^{\log_b n-1} a^j(\frac{n}{b^j})^{\log_b a}$$

From problem 2.a.ii, we get:

$$\sum_{j=0}^{\log_b n-1} a^j(\frac{n}{b^j})^{\log_b a} = n^{\log_b a} \log_b n$$

Combining the two formulas, we can finally get:

$$c_1 n^{\log_b a} \log_b n \le g(n) \le c_2 n^{\log_b a} \log_b n$$

Therefore, $g(n) = \mathbf{\Theta}(n^{\log_b a} \log n)$. Proof done.

(b) i) Since $f(n) = \mathcal{O}(n^{\log_b a - \varepsilon})$, we can find some constant c, such that:

$$f(n) \leq cn^{\log_b a - \varepsilon}$$

Then, by substituting n with n/b^j, for $0 \leq j \leq \log_b n - 1$, and multiplied by a^j, we can have:

$$a^j f(\frac{n}{b^j}) \leq c_2(a^j(\frac{n}{b^j})^{\log_b a - \varepsilon})$$

Calculating the sum for j from 0 to $\log_b n - 1$, we can get:

$$\sum_{j=0}^{\log_b n-1} a^j f(\frac{n}{b^j}) \leq c_2 \sum_{j=0}^{\log_b n-1} a^j(\frac{n}{b^j})^{\log_b a - \varepsilon}$$

Since $g(n) = \sum_{j=0}^{\log_b n-1} a^j f(n/b^j)$, we can finally reach the conclusion that

$$g(n) = \mathcal{O}(\sum_{j=0}^{\log_b n-1} a^j(\frac{n}{b^j})^{\log_b a - \varepsilon})$$

Proof done.

ii) Firstly, we can do some formula transformation:

$$\begin{aligned} a^j(\frac{n}{b^j})^{\log_b a - \varepsilon} &= a^j \frac{n^{\log_b a - \varepsilon}}{(b^j)^{\log_b a - \varepsilon}} \\ &= a^j \frac{n^{\log_b a - \varepsilon}}{(b^{\log_b a - \varepsilon})^j} \\ &= a^j \frac{n^{\log_b a - \varepsilon}}{(\frac{a}{b^\varepsilon})^j} \\ &= n^{\log_b a - \varepsilon} b^{j\varepsilon} \end{aligned}$$

Then we calculate the sum for $0 \le j \le \log_b n - 1$:

$$\begin{aligned}
\sum_{j=0}^{\log_b n-1} a^j (\frac{n}{b^j})^{\log_b a-\varepsilon} &= \sum_{j=0}^{\log_b n-1} n^{\log_b a-\varepsilon} b^{j\varepsilon} \\
&= n^{\log_b a-\varepsilon} \sum_{j=0}^{\log_b n-1} b^{j\varepsilon} \\
&= n^{\log_b a-\varepsilon} \frac{b^0(1-(b^\varepsilon)^{\log_b n})}{1-b^\varepsilon} \\
&= n^{\log_b a-\varepsilon} \frac{(b^{\log_b n})^\varepsilon}{b^\varepsilon - 1} \\
&= \frac{n^\varepsilon - 1}{b^\varepsilon - 1} n^{\log_b a-\varepsilon}
\end{aligned}$$

Proof done.

iii) From problem 2.b.i, we get:

$$g(n) = \mathcal{O}(\sum_{j=0}^{\log_b n-1} a^j (\frac{n}{b^j})^{\log_b a-\varepsilon})$$

From problem 2.b.ii, we get:

$$\sum_{j=0}^{\log_b n-1} a^j (\frac{n}{b^j})^{\log_b a-\varepsilon} = \frac{n^\varepsilon - 1}{b^\varepsilon - 1} n^{\log_b a-\varepsilon}$$

Combine the two formulas, we can get:

$$g(n) = \mathcal{O}(\frac{n^\varepsilon - 1}{b^\varepsilon - 1} n^{\log_b a-\varepsilon})$$

This tells us that there exists some constant c such that:

$$g(n) \le c(\frac{n^\varepsilon - 1}{b^\varepsilon - 1} n^{\log_b a-\varepsilon})$$

Since $\varepsilon > 0$, $n^\varepsilon - 1 > 0$ and $b^\varepsilon - 1 > 0$, thus:

$$\frac{n^\varepsilon - 1}{b^\varepsilon - 1} n^{\log_b a-\varepsilon} < \frac{n^\varepsilon}{b^\varepsilon - 1} n^{\log_b a-\varepsilon} < n^{\log_b a} \cdot \frac{1}{b^\varepsilon - 1}$$

where $1/(b^\varepsilon - 1)$ is a constant. Therefore, we can have the following:

$$g(n) \le c(\frac{n^\varepsilon - 1}{b^\varepsilon - 1} n^{\log_b a-\varepsilon}) < \frac{c}{b^\varepsilon - 1} n^{\log_b a}$$

So we can conclude that $g(n) = \mathcal{O}(n^{\log_b a})$. Proof done.

(c) i. From the expression of $g(n)$, we can deduce that:

$$g(n) = \sum_{j=0}^{\log_b n-1} a^j f(n/b^j) = \sum_{j=1}^{\log_b a-1} +f(n) \geq f(n)$$

Therefore, $g(n) = \Omega(f(n))$. Proof done.

ii. Since $af(n/b) \leq cf(n)$, by changing n to n/b^{j-1} and multiplying a^{j-1} to both sides, we can get $af(n/b^i) \leq cf(n/b^{i-1}))$ for $1 \leq i \leq \log_b n - 1$. Therefore, we can do the following recurrence:

$$a^j f(\frac{n}{b^j}) \leq c \cdot a^{j-1} f(\frac{n}{b^{j-1}}) \leq c^2 \cdot a^{j-2} f(\frac{n}{b^{j-2}}) \leq \cdots \leq c^{j-1} \cdot af(\frac{n}{b}) \leq c^j f(n)$$

Proof done.

iii. From the definition of $g(n)$ and the result of problem 2.c.ii, we can have:

$$g(n) = \sum_{j=0}^{\log_b n-1} a^j f(n/b^j) \leq \sum_{j=0}^{\log_b n-1} c^j f(n) = \frac{1 - c^{\log_b n}}{1 - c} f(n)$$

Since $\frac{1-c^{\log_b n}}{1-c}$ is a constant, we can finally say that $g(n) = \mathcal{O}(f(n))$. Proof done.

iv. Combining the conclusions of problem 2.c.i $g(n) = \Omega(f(n))$ and 2.c.iii $g(n) = \mathcal{O}(f(n))$, we can simply say that $g(n) = \mathbf{\Theta}(f(n))$. Proof done.

3. Let $a \geq 1$, $b > 1$ be two constants, $f(n)$ be a function, and $T(n) = aT(n/b) + f(n)$, where $n = b^i$ with i a positive integer, and $T(1) = O(1)$ be a recurrence relation over the positive integers.

Firstly, for $f(n) = \mathbf{\Theta}(n^{\log_b a})$, we will have:

$$\begin{aligned}
T(n) &= \sum_{j=0}^{\log_b n-1} a^j f(\frac{n}{b^j}) + \sum_{j=0}^{i-1} f(n/b^j) \\
&= \mathbf{\Theta}(n^{\log_b a} \log n) + \mathbf{\Theta}(n^{\log_b a}) \\
&= \mathbf{\Theta}(n^{\log_b a} \log n)
\end{aligned}$$

Secondly, for $f(n) = \mathcal{O}(n^{\log_b a-\varepsilon})$, we will have:

$$\begin{aligned}
T(n) &= \sum_{j=0}^{\log_b n-1} a^j f(\frac{n}{b^j}) + \sum_{j=0}^{i-1} f(n/b^j) \\
&= \mathcal{O}(n^{\log_b a}) + \mathcal{O}(n^{\log_b a-\varepsilon}) \\
&= \mathcal{O}(n^{\log_b a})
\end{aligned}$$

Finally, when $af(n/b) \leq cf(n)$ for some $c < 1$ and $f(n) = \Omega(n^{\log_b a+\varepsilon})$, since $g(n) = \mathbf{\Theta}(f(n))$, we will have $T(n) = \mathbf{\Theta}(f(n))$.

In all, we can sum up the Master Theorem as:

$$T(n) = \begin{cases} \mathbf{\Theta}(n^{\log_b a}\log n) & \text{if } f(n) = \mathbf{\Theta}(n^{\log_b a}) \\ \mathbf{\Theta}(n^{\log_b a}) & \text{if } f(n) = \mathcal{O}(n^{\log_b a-\varepsilon}), \varepsilon > 0 \\ \mathbf{\Theta}(f(n)) & \text{if } f(n) = \Omega(n^{\log_b a+\varepsilon}), \varepsilon > 0, \text{ and } af(n/b) \leq cf(n), c < 1 \end{cases}$$

Proof done.