

# Reduction of Transcendental Decision Problems over the Reals

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# Authors



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Chern”)

I am Rizeng Chen, the people on the left and I am currently a third-year PhD candidate under the supervision of Prof. Xia.



**Author 2:** Bican Xia  
(pronounced as “BeeChan  
Shia”)

## ① Introduction

## ② History

## ③ Reduction

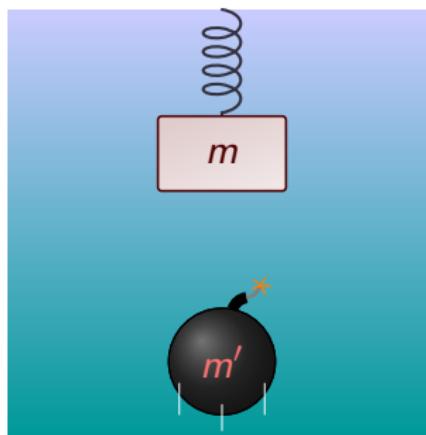
## ④ Proof Sketch

## ⑤ Multivariate Case

## ⑥ Implementation

# Problems with Transcendental Constraints

A **mass** is attached to a **spring** in the water. A **bomb** is thrown beneath the mass. The bomb will **explode** when it hits the mass!



**Question:** Will the mass and the bomb collide at some time  $t > 0$ ?

# Problems with Transcendental Constraints

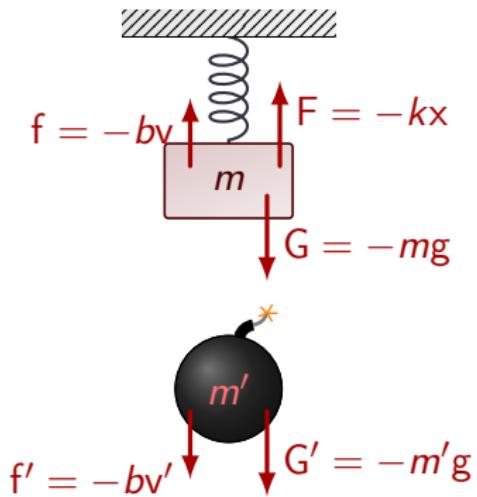


Figure 1: Free Body Diagram

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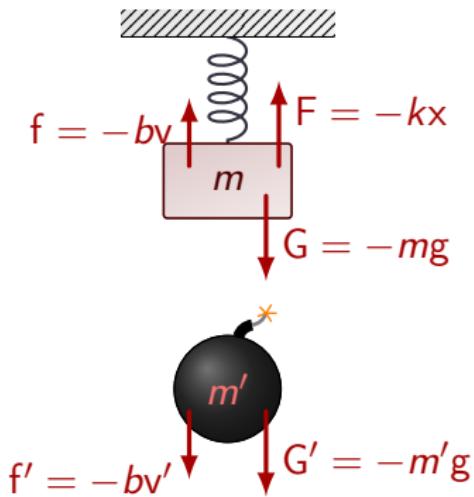


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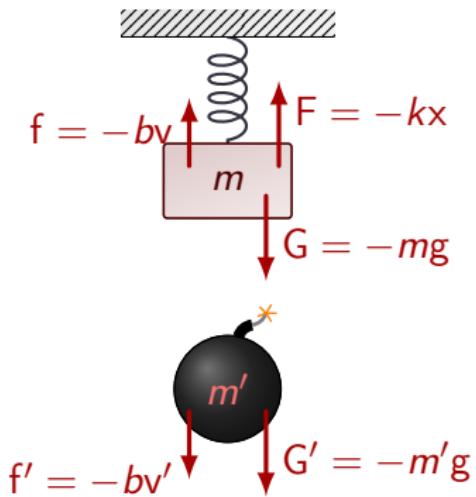
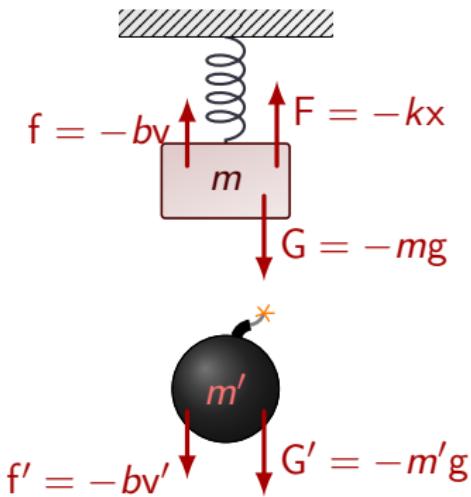


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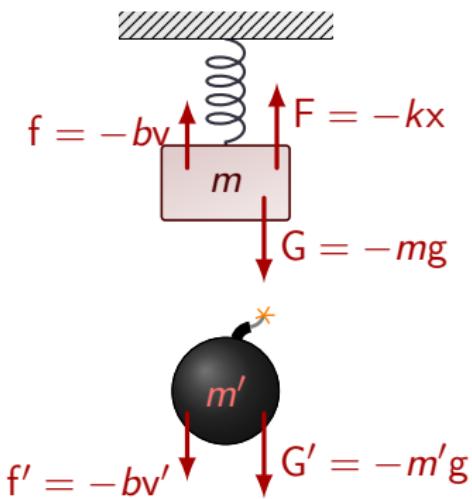
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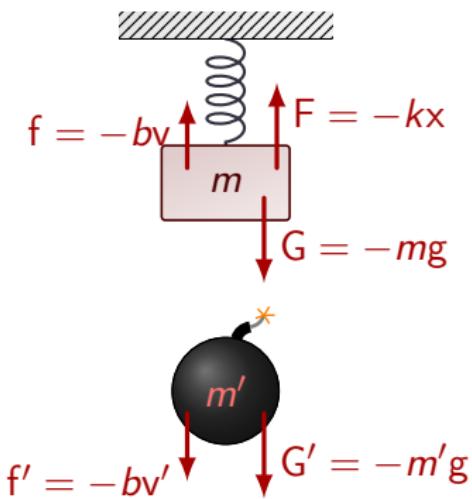


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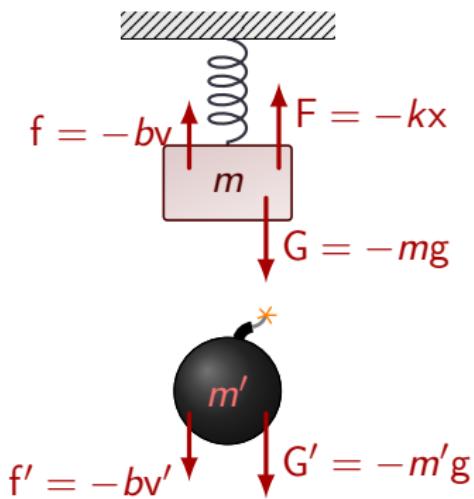
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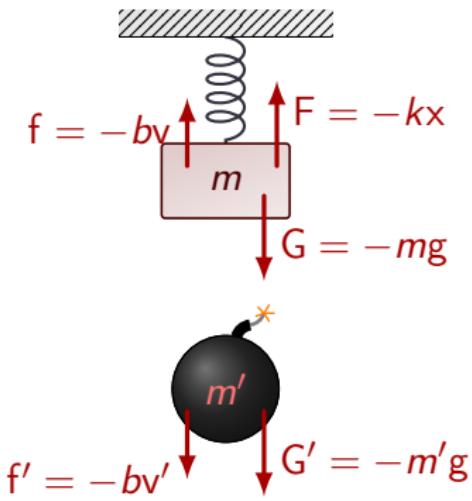


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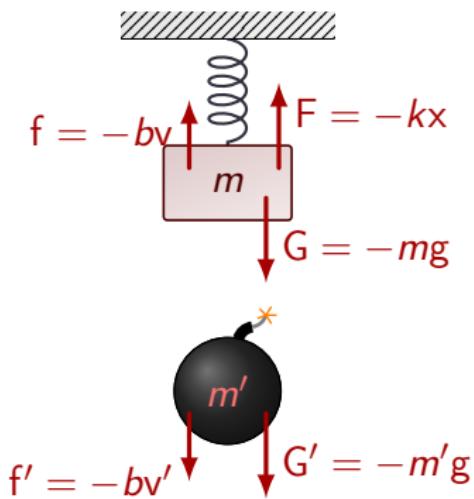


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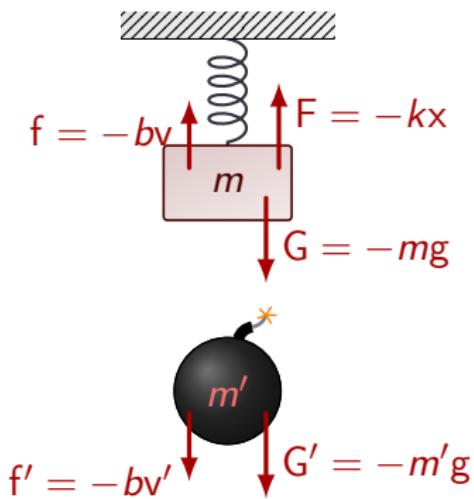


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# Problems with Transcendental Constraints

- Suppose  $m = m' = 1\text{kg}$ ,  $k = 10\text{N/m}$ ,  $b = 2\text{N} \cdot \text{s/m}$  and  $g = 10\text{m/s}^2$ .
- The initial positions are  $x(0) = 0\text{m}$ ,  $y(0) = -5\text{m}$ , and the initial velocities are  $\dot{x}(0) = -12\text{m/s}$  and  $\dot{y}(0) = 9\text{m/s}$ .

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- Solving the ODE:

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- **Transcendental constraints** naturally arise in the real world.
- How do we handle them?

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## 2 History

## 3 Reduction

## 4 Proof Sketch

## 5 Multivariate Case

## 6 Implementation

# Polynomials are easy...

- Tarski (1951) showed that the first-order theory over a real closed field is decidable (polynomial equations and inequalities).

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- Schanuel's Conjecture (SC)

Suppose  $z_1, \dots, z_n \in \mathbb{C}$  are linear independent over  $\mathbb{Q}$ , then

$$\text{tr.deg } \mathbb{Q}(z_1, \dots, z_n, e^{z_1}, \dots, e^{z_n}) \geq n$$

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- At the same time, Strzeboński (2008) studied the real root isolation of exp-log functions (then extended to tame elementary functions and exp-log-arctan functions).

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- It is shown in Chen and Xia (2023) that the theory of univariate mixed trigonometric-polynomials (MTP) is **surprisingly decidable**.

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- Hence, a **reduction** from the unbounded case to the bounded case should be **favorable**.

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- Theorem (ISSAC '24, Chen & Xia Thm. 3.1)

Let  $f_i \in S[\sin x, \cos x]$  for  $i = 1, \dots, s$ , then there are effective bounds  $N, M \in \mathbb{R}$  such that any quantifier-free formula  $\varphi(x)$  whose atoms are of the form  $f_i \triangleright 0$  is true for all  $x \in \mathbb{R}$  if and only if  $\varphi(x)$  is true for all  $x \in [N, M]$ .

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- Our ISSAC'23 result can be directly derived from this by setting  $S = \mathbb{Q}[x]$ .

# Applications of the Reduction Theorem

The Reduction Theorem provides a **general framework** for designing decision procedures. As a consequence, the following rings are **decidable** (SC may be needed).

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As a corollary, the **reachability problem** of a linear differential system  $\dot{x}(t) = Ax(t)$  is decidable, if the imaginary part of **eigenvalues** of  $A$  spans a **1-dimensional space** over  $\mathbb{Q}$ .

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# Unifying $\sin x$ and $\cos x$ by substitution

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- This is the **canonical form** of  $f \in S[\sin x, \cos x]$ :

$$\begin{aligned}\sigma : S[\sin x, \cos x] &\rightarrow S[\tan \frac{x}{2}]_{1+\tan^2 \frac{x}{2}} \\ \sin x &\mapsto \frac{2 \tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}}, \quad \cos x \mapsto \frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}.\end{aligned}$$

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- Notice that

$$f \triangleright 0 \Leftrightarrow \sigma(f) \triangleright 0 \Leftrightarrow (1 + \tan^2 \frac{x}{2})^\ell \sigma(f) \triangleright 0.$$

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- The **singularity** of  $\tan \frac{x}{2}$  at  $x = (2k+1)\pi$  ( $k \in \mathbb{Z}$ ) can be treated separately.

# Essential Ingredient From Real Algebraic Geometry

Collins (1975) proposed Cylindrical Algebraic Decomposition (CAD). The key theorem in the paper is:

## Theorem (Col75, Thm. 1)

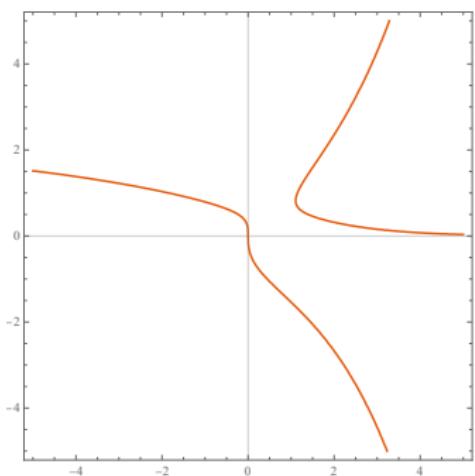
Let  $f(\vec{x}; y)$  be a parametric univariate polynomial and let  $C$  be a connected parameter region.

Suppose  $\text{LC}_y(f) \neq 0$  for all  $\vec{x} \in C$  and the number of distinct complex roots of  $f(\vec{x}; y)$  is invariant for all  $\vec{x} \in C$ .

Then  $f$  is delineable over  $C$ , i.e. the real roots of  $f$  are continuous functions in the parameters.

He counted the complex roots by his subresultant theory.

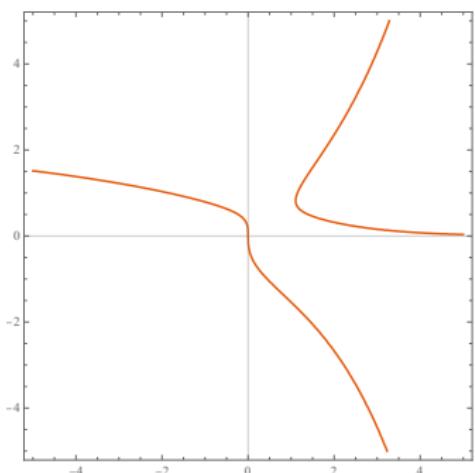
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**Figure 2:** The locus of  
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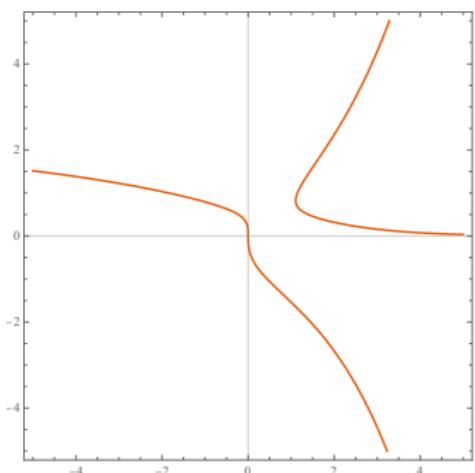


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①  $\Delta_0 = 27x^2 - 12e^x + 12e^{2x} - 4e^{3x} + 4;$

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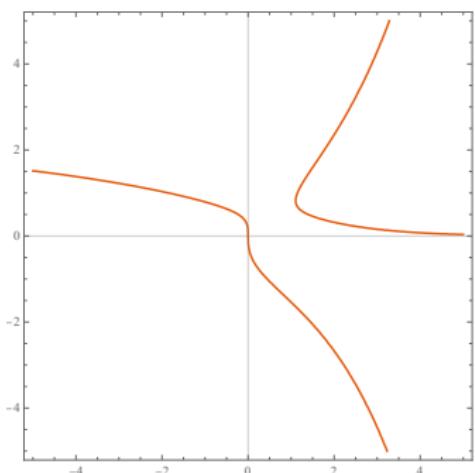


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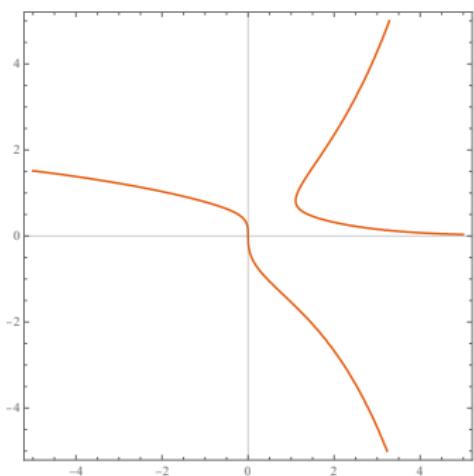


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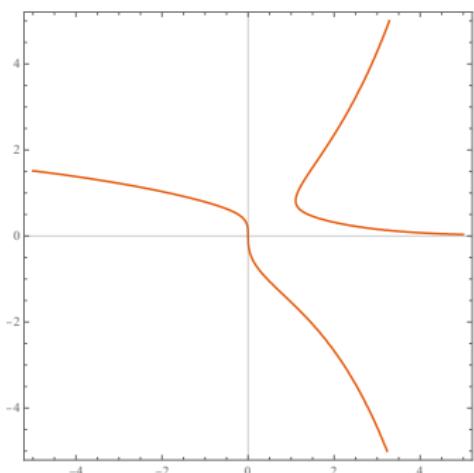
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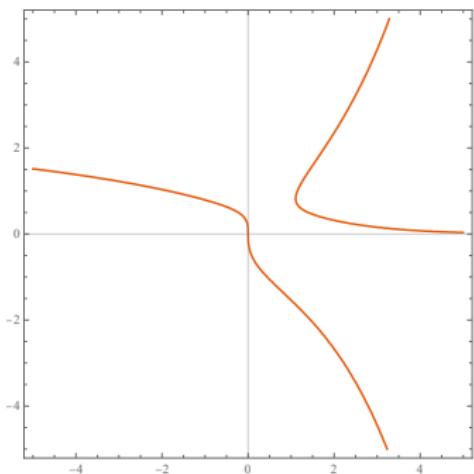
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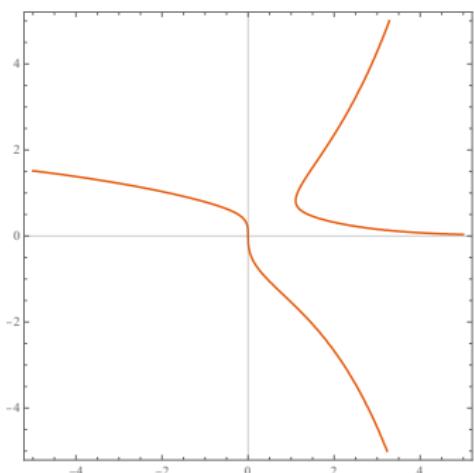
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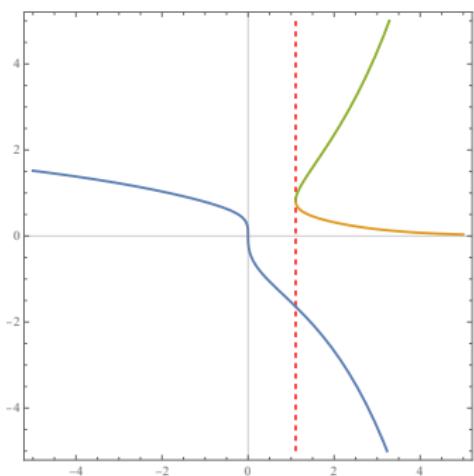


Figure 2:  $f(x, y)$  is delineable when  $x > 1.105 \dots$

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- For all  $x_0 > 1.105 \dots$ ,  $\Delta_0 \neq 0$ , thus the real roots are continuous functions  $y_1(x)$ ,  $y_2(x)$  and  $y_3(x)$ .

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- Let  $\varphi$  be a quantifier-free formula whose atoms are  $g_i \triangleright 0$  ( $g_i \in S[\tan \frac{x}{2}]$ ).

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- e.g.  $\tilde{\varphi}(x, y) = "y^3 - (e^x - 1)y + x < 0"$ .
- Note that the subdiscriminants  $\Delta_0, \Delta_1, \dots$  of  $h \in S[y]$  is always in  $S$ , so they have finitely many real roots if they are not identically zero.

# The Proof for the Reduction Theorem (cont'd)

- Then there is an interval  $I = (M, +\infty)$  such that  $h$  is delineable over  $I$ .

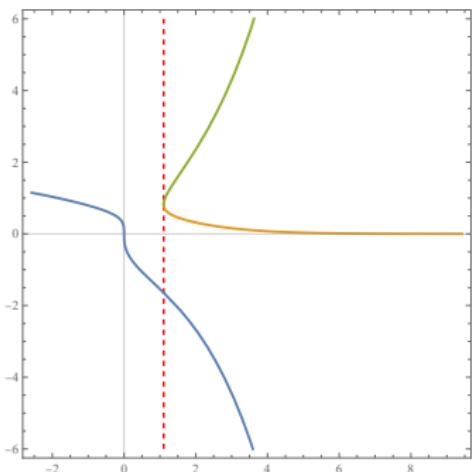


Figure 3:  $h(x, y)$  is delineable over  $I = (1.105 \dots, +\infty)$

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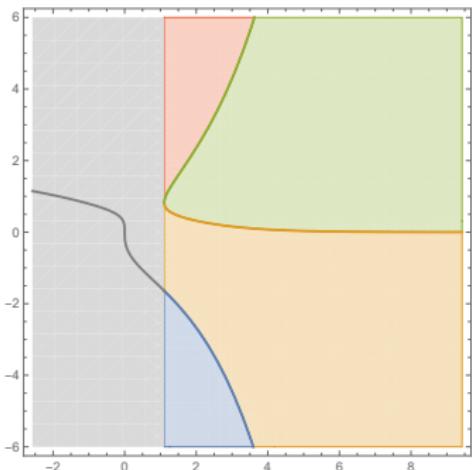
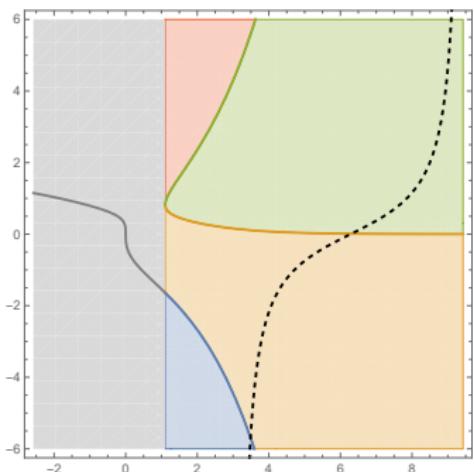


Figure 3:  $I \times \mathbb{R}$  consists of cylindrically stacked cells

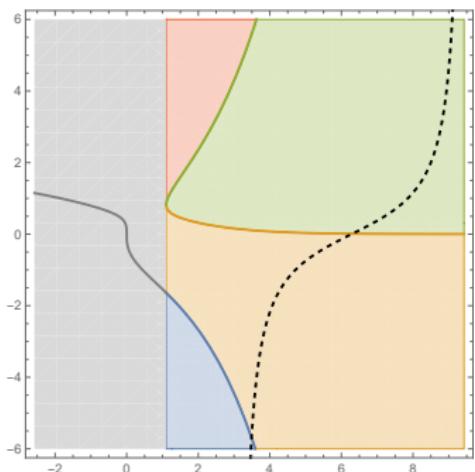
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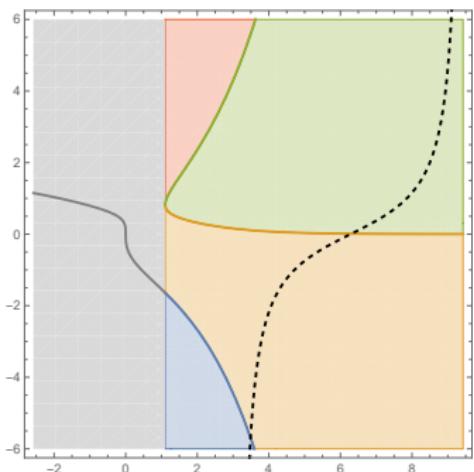


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- It is easy to observe that the graph of  $y = \tan \frac{x}{2}$  intersects with each cell in each period.
- Hence, if  $\tilde{\varphi}(x, y)$  holds in cell  $C$ ,  $\varphi = \tilde{\varphi}(x, \tan \frac{x}{2})$  is also satisfiable in each period  $(2k\pi - \pi, 2k\pi + \pi) \subseteq I$ . It suffices to look at one of them.

## ① Introduction

## ② History

## ③ Reduction

## ④ Proof Sketch

## ⑤ Multivariate Case

## ⑥ Implementation

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- Hence, our result in the univariate case is **not very far from being optimal**.

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# Package TranscendentalProblems

We implement the reduction algorithm with Mathematica 13. Our package `TranscendentalProblems` is available at:

[https://github.com/xiaxueqaq/TranscendentalProblems.](https://github.com/xiaxueqaq/TranscendentalProblems)

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# wasted

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## mission passed!

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# Experiments

We report some experimental data here.

Examples	1	2	3	4	5	6
Time (s)	1.531	0.969	1.797	3.562	1.891	1.438
Examples	7	8	9	10	11	12
Time (s)	0.125	0.031	0.188	0.078	8.406	2.109

Table 1: Running Time

# *Thank you!*

You are more than welcome to give any suggestion!

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