Analytical solution to the Age of Earth problem

Let y(x,t) be the temperature. The partial differential equation that we want to solve is:

$$\frac{\partial y}{\partial t} = D\frac{\partial^2 y}{\partial x^2}$$

with the boundary condition:

$$y(0,t) = y_{s,} \quad for \quad t > 0 \quad ,$$

and the initial condition:

$$y(x,0) = y_{0,} \quad for \quad x > 0$$

D is the diffusivity.

Now, instead of the half plane, consider a full plane with

$$y(-x,t) - y_s = -\left[y(x,t) - y_s\right]$$

Because the equation is linear, $u = \frac{\partial y}{\partial x}$ also satisfies the equation, i.e.:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

with the boundary conditions

$$u(-\infty,t) = u(\infty,t) = 0$$

and initial condition:

$$u(x,0) = 2(y_0 - y_s)\delta(x)$$

where $\delta(x)$ is the delta function.

Diffusion equation with this set of boundary and initial conditions has this solution (from the central limit theorem):

$$u(x,t) = \frac{y_0 - y_s}{\sqrt{\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right)$$

i.e.:

$$\frac{\partial y}{\partial x}(x,t) = \frac{y_0 - y_s}{\sqrt{\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right)$$