THE GLOBAL WEIGHTED AND UNWEIGHTED BOUNDEDNESS THEORY FOR MULTILINEAR FOURIER INTEGRAL OPERATORS

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Abstract. For the ℓ-linear Fourier integral operators $T_{\sigma,\Psi}$ bounded from $L^{r_1} \times \cdots \times L^{r_\ell}$ to $L^{r,\infty}$, with amplitude function σ and phase function Ψ, we establish criteria for multilinear sparse domination, sharp quantitative weighted bounds (including those for commutators), and new unweighted boundedness results. These findings, when combined with known unweighted bounds, lead to novel conclusions in quantitative weighted estimates. Our work extends beyond the weighted and unweighted results for multilinear pseudodifferential operators of Cao, Xue, and Yabuta (J. Funct. Anal., 2020) [8], and Park and Tomita (J. Funct. Anal., 2024) [36]. Furthermore, by employing amplitude functions $\sigma \in \nabla S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$ and the new phase function classes $L^\infty\phi^1(\mathbb{R}^n \times \mathbb{R}^{n\ell})$ and $\nabla\phi^0(\mathbb{R}^n \times \mathbb{R}^{n\ell})$ —distinct from those used in the works of Rodríguez-López, Rule, and Staubach (2013–2023) [41–45]—we establish sharp estimates. These are sharp in the sense of the exponents of multiple weights for the weighted estimates, and in the sense of the decay order m for the unweighted estimates. As a consequence, we obtain a largely complete global weighted and unweighted boundedness theory for multilinear Fourier integral operators.

1. Introduction and main results

1.1. Background.

In this paper, we investigate the global weighted and unweighted Lebesgue boundedness theory for ℓ-linear Fourier integral operators **(ℓ-FIOs)** $T_{\sigma,\Psi}$ of the form

$$T_{\sigma,\Psi}(\vec{f})(x) = \int_{(\mathbb{R}^n)^\ell} \sigma(x,\vec{\xi})\widehat{f}_1(\xi_1)\cdots\widehat{f}_\ell(\xi_\ell)e^{i\Psi(x,\vec{\xi})}d\vec{\xi},$$

with amplitude function $\sigma \in \nabla S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$ (Defintion 2.1) and phase error function $\overline{\Psi}(x,\vec{\xi}) := \Psi\left(x,\vec{\xi}\right) - \vec{x}\cdot\vec{\xi} \in L^\infty\phi^1(\mathbb{R}^n \times \mathbb{R}^{n\ell})$ (or $\overline{\Psi} \in L^\infty\phi^1 \cap \nabla\phi^0(\mathbb{R}^n \times \mathbb{R}^{n\ell})$, cf. Defintions 2.4, 2.5), where $x \in \mathbb{R}^n$, $\vec{x} := (x,\cdots,x)$, $\vec{\xi} = (\xi_1,\ldots,\xi_\ell)$, and the Fourier transform $\widehat{f}$ of the function f is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}f(x)dx.$$

To achieve the objective of global weighted and unweighted boundedness, it is necessary to employ ideas from the quantitative weighted theory of modern multilinear singular integrals. Specifically, by establishing a sparse bounds criterion for ℓ-FIOs and combining it with the now well-developed multilinear quantitative weighted extrapolation theory introduced by Nieraeth [35], we derive the desired criterion for sharp quantitative weighted bounds. To this end, building upon the known principle of multilinear sparse domination, if we assume that an operator T is bounded from $L^{r_1} \times \cdots \times L^{r_\ell}$ to L^r, it suffices to prove that its associated grand

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maximal truncated operator $\mathcal{M}_T$ is also bounded from $L^{r_1}\times\cdots\times L^{r_\ell}$ to L^r (a result we refer to in this paper as the $\mathcal{M}_T$ theorem). We observe that an ℓ-FIO $T_{\sigma,\Psi}$ can be induced as an ℓ-linear pseudodifferential operator **(ℓ-PDO)** $\mathscr{T}_a$ as follows:

$$T_{\sigma,\Psi}(\vec{f})(x)=\mathscr{T}_a(\vec{f})(x):=\int_{(\mathbb{R}^n)^\ell} a(x,\vec{\xi})\widehat{f}_1(\xi_1)\cdots\widehat{f}_\ell(\xi_\ell)e^{i(x,\cdots,x)\cdot\vec{\xi}}d\vec{\xi}, \tag{1.1}$$

where the induced symbol $a(x,\vec{\xi}):=\sigma(x,\vec{\xi})e^{i\overline{\Psi}(x,\vec{\xi})}$ and the phase error function $\overline{\Psi}(x,\vec{\xi}):=\Psi\left(x,\vec{\xi}\right)-(x,\cdots,x)\cdot\vec{\xi}$. We term a the induced symbol generated by the amplitude σ and the phase Ψ.

To establish the $\mathcal{M}_T$ theorem for $T_{\sigma,\Psi}$ (or $\mathscr{T}_a$), our selection of the phase function class therefore differs from the choices made in previous works by Rodríguez-López, Rule, and Staubach (2013–2023) [41–45]. We instead utilize the new classes $L^\infty\phi^k(\mathbb{R}^n\times\mathbb{R}^d)$, $\nabla\phi^k(\mathbb{R}^n\times\mathbb{R}^d)$ of phase functions (Definition 2.4 and Definition 2.5). The introduction of this class ensures the following crucial facts:

(1) It guarantees that the induced symbol a possesses satisfactory global smoothness.
(2) Through standard integration by parts techniques, it preserves the generalized decay estimates for the Littlewood-Paley projections $a\cdot\widehat{\psi_j}$ of the induced symbol a, cf. Proposition 2.7. Furthermore, for certain variant functions of the induced symbol a, such as $(\xi_i)_v a(x,\vec{\xi})$ and $\partial_{x_v}a(x,\vec{\xi})$, corresponding generalized decay estimates can also be obtained, cf. Proposition 2.8.

Ultimately, these generalized decay estimates, combined with the **dilated cube technique** introduced in this paper within the framework of high-low frequency analysis, allow us to obtain the full-range $\mathcal{M}_T$ theorem for the order $m\in(-\infty,0]$, cf. Theorem 3.2. This, consequently, yields sharp quantitative weighted estimates and quantitative weighted vector-valued inequalities for ℓ-FIOs, cf. Theorem 1.8. Moreover, by applying the multilinear extrapolation theorem for commutators, introduced by Bényi et. al [3], we also derive quantitative weighted estimates and quantitative weighted vector-valued inequalities for generalized commutators of ℓ-FIOs, cf. Theorem 1.9.

The contributions of our main results are summarized as follows. Compared to the previous quantitative estimates by Cao, Xue, and Yabuta (2020) [8], and Park and Tomita (2024) [36], our conclusions surpass theirs in the following aspects:

(1) The ℓ-PDOs they treated can be viewed as a special case of the aforementioned ℓ-FIOs, if $\overline{\Psi}=0\in(L^\infty\phi^1\cap\nabla\phi^0)(\mathbb{R}^n\times\mathbb{R}^{n\ell})$. Indeed, through the class $(L^\infty\phi^1\cap\nabla\phi^0)(\mathbb{R}^n\times\mathbb{R}^{n\ell})$ of phase error function that we introduce, our class of l-FIOs enjoys broader applications (cf. Remark 2.9), particularly in the study of certain PDEs. Examples include nonlinear waterwave and capillary wave equations, nonlinear wave and Klein–Gordon equations, the nonlinear Schrödinger equations, the Korteweg–de Vries-type equations, and higher order nonlinear dispersive equations, see [45].
(2) The effective range for m in our criterion of sharp quantitative weighted estimates is $(-\infty,0]$, cf. Theorems 1.8, 1.9, 1.15, and 1.16, which significantly extends the decay scale permitted for the amplitude function σ in the previous result, cf. Theorems G, K, and L, which yields the desired quantitative sharp weighted L^p improvements.

These improvements are achieved relative to the works of Rodríguez-López, Rule, and Staubach (2013–2023) [41–45]:

(1) Our weighted estimates are not only sharp, but the amplitude functions σ we consider belong to more general Hörmander classes, often of the type $\nabla S^m_{\rho,\delta}$ or $L^\infty S^m_\rho$. In contrast, a substantial body of prior research was confined to considering unweighted boundedness (cf. Theorems A, B, C, and D) only within the classical $S^m_{1,0}$ framework.
(2) We also obtain the new unweighted boundedness, cf. Theorems 1.12, 1.13, 1.14 and Corollaries 1.18, 1.19. Compared with Theorems A, B, C, and D, Theorems 1.12, 1.13, and 1.14 hold under a significantly broader class $L^\infty\phi^1(\mathbb{R}^n \times \mathbb{R}^{n\ell})$ of phase error functions, endowing our conclusions with a high degree of generality, and Corollaries 1.18 and 1.19 realize the sharp range of m under the setting of $S^m_{1,\delta}$.

In the subsequent, we will provide a detailed exposition of the background relevant to the discussions above.

In 2010, Grafakos and Peloso [15, Theorem 2.1] firstly gave that 2-linear FIO is bounded from $L^{r_1} \times L^{r_2}$ to L^r under certain conditions.

In 2013, Rodriguez-Lopez, Rule, and Staubach firstly considered that

Theorem A ([42], Theorem 2.7). *If $\sigma \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^{2n})$ and $\varphi_1, \varphi_2 \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ are non-degenerate phase functions and satisfy that for any α and β with $1 \le |\alpha|$ and $1 \le |\beta|$,*

$$\left|\partial^\beta_{\vec{\xi}} \partial^\alpha_x \varphi_j(x, \vec{\xi})\right| \lesssim_{\alpha,\beta} 1, \text{ for } j = 1, 2,$$

then $T^{\varphi_1,\varphi_2}_\sigma$ is bounded from $L^2 \times L^2$ to L^1.

Moreover, the unweighted boundedness of 2-linear FIO was also estalished by Rodriguez-Lopez, Rule, and Staubach in 2014 as follows.

Theorem B ([43], Theorem 2.7). *Let $r_1, r_2 \in (1, \infty)$ and $\frac{1}{r} := \frac{1}{r_1} + \frac{1}{r_2}$. Set*

$$m \le (1 - n) \left(\left| \frac{1}{2} - \frac{1}{r_1} \right| + \left| \frac{1}{2} - \frac{1}{r_2} \right| \right).$$

If $\sigma \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^{2n})$ is compactly supported in the spatial variable x, and φ_1, φ_2 are non-degenerate phase functions, then $T^{\varphi_1,\varphi_2}_\sigma$ is bounded from $L^{r_1} \times L^{r_2}$ to L^r.

In 2021, they also established the following important result, which improves the bilinear to ℓ-linear.

Theorem C ([44], Theorem 1.4). *Let $n, \ell \ge 2$ and $r_1, \cdots, r_\ell \in (1, \infty)$ with $\frac{1}{r} := \sum_{j=1}^{\ell} \frac{1}{r_j}$, Set*

$$m \le (1 - n) \left(\sum_{j=1}^{\ell} \left| \frac{1}{r_j} - \frac{1}{2} \right| + \left| \frac{1}{r} - \frac{1}{2} \right| \right).$$

If $\sigma \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$ and $\overline{\Psi}(x, \vec{\xi}) := \varphi_0(\xi_1 + \cdots + \xi_\ell) + \sum_{j=1}^{\ell} \varphi_j(\xi_j)$ with $\varphi_0, \cdots, \varphi_\ell \in C^\infty(\mathbb{R}^n \backslash \{0\})$ and is positively homogeneous of degree 1, then $T_{\sigma,\Psi}$ is bounded from $L^{r_1} \times \cdots \times L^{r_\ell}$ to L^r.

Based on Theorem C, in 2023, they also studied that

Theorem D ([45], Theorems 1.4). *Let $n, \ell \in \mathbb{N}$ and $r_1, \cdots, r_\ell \in (1, \infty)$ with $\frac{1}{r} := \sum_{j=1}^{\ell} \frac{1}{r_j}$, Set*

$$m \le -sn \left(\sum_{j=1}^{\ell} \left| \frac{1}{r_j} - \frac{1}{2} \right| + \left| \frac{1}{r} - \frac{1}{2} \right| \right).$$

If $\sigma \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$ *and* $\overline{\Psi}(x,\vec{\xi}) := \varphi_0(\xi_1 + \cdots + \xi_\ell) + \sum_{j=1}^{\ell} \varphi_j(\xi_j)$ *with* $\varphi_0, \cdots, \varphi_\ell \in C^\infty(\mathbb{R}^n)$*, then* $T_{\sigma,\Psi}$ *is bounded from* $L^{r_1} \times \cdots \times L^{r_\ell}$ *to* L^r*.*

Note that when the phase error function $\overline{\Psi}(x,\vec{\xi}) = 0$, since $\sigma = a$, the ℓ-linear FIO $T_{\sigma,\Psi}$ is back to the ℓ-linear PDO $\mathscr{T}_\sigma$.

Our main results also contain the generalization of the PDOs, so we now introduce the background of the PDOs.

It is well known that PDOs play pivotal roles in harmonic analysis and PDEs, (see, among others, Hörmander [23, 24], Coifman-Meyer [11], Fefferman-Kohn [12], and Guan [17].). For $0 < \rho \le 1$ and $0 \le \delta < 1$, Hörmander [21, Theorem 1] proved that the L^2 boundedness of $\mathscr{T}_a$ with $a \in S^m_{\rho,\delta}$ implies $m \le \min\{0, n(\rho-\delta)/2\}$. Alvarez and Hounie [1, Theorems 3.4 and 3.2] obtained the following extremely crucial results.

Theorem E ([1], Theorems 3.4 and 3.2)**.** *Let* $r \in [1,\infty)$*,* $\rho \in (0,1]$*, and* $\delta \in [0,1)$*. If* $a \in S^m_{\rho,\delta}$ *for*

$$m \le n(\rho-1)\left|\frac{1}{r} - \frac{1}{2}\right| + \min\left\{0, \frac{n(\rho-\delta)}{2}\right\}, \tag{1.2}$$

then $\mathscr{T}_a$ *is bounded on* L^r *for* $r \in (1,\infty)$*, and it is bounded from* L^r *to* $L^{r,\infty}$ *for* $r \in [1,\infty)$*. Moreover,* $\mathscr{T}_a$ *is bounded on* L^2 *if and only if* $m \le \min\{0, n(\rho-\delta)/2\}$*.*

Next, we recall some work of the multilinear PDOs. Bilinear pseudo-differential operators were first investigated by Coifman and Meyer [11] at least in the case $m = 0, \rho = 1$ and $\delta = 0$; they utilized them to model the Calderón commutator. It is now well understood that the operators with symbols in $S^0_{1,0}$ are examples of certain singular integrals and fit within the general multilinear Calderón–Zygmund theory developed in [16].

Consider the $\ell \ge 1$. For an ℓ-PDO $\mathscr{T}_a$, let $r_1, \cdots, r_\ell \in [1,\infty)$, and $\frac{1}{r} := \sum_{i=1}^{\ell} \frac{1}{r_i}$. Suppose that $\rho, \delta \in [0,1]$, and $I \subseteq \mathbb{R}^n$ with $\mathcal{I}_0(\vec{r}) := \sup I \le 0$. If $m \in I$ and $a \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$, then we divide the range of m into the following Figure 1.

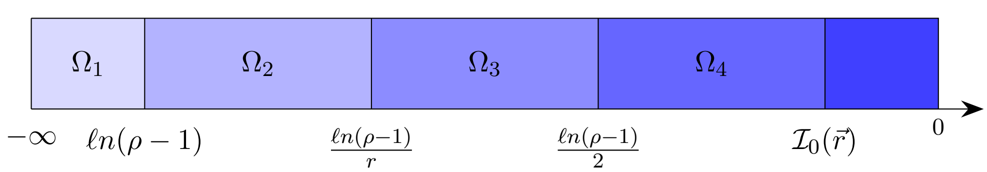

Figure 1. the range of decay order

Remark 1.1. *We need to note that*

$$\frac{\ell n(\rho-1)}{2} \le \mathcal{I}_0(\vec{r}) := n(\rho-1)\left[\sum_{j=1}^{\ell} \max\left(\frac{1}{r_j}, \frac{1}{2}\right) - \min\left(\frac{1}{r}, \frac{1}{2}\right)\right] \le 0.$$

We now present some important recent work. In 2020, Cao, Xue, and Yabuta showed in [8, Proposition 3.7] that ℓ-linear pseudo-differential operators with symbols in $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$, $\rho, \delta \in [0,1]$ and $m + \delta < n\ell(\rho-1)$ fall within the framework of multilinear Calderon-Zygmund theory, and hence have the $L^{p_1} \times \cdots \times L^{p_\ell} \to L^p$ boundedness property for all $1 < p_1, \ldots, p_\ell < \infty$

with $1/p = 1/p_1 + \ldots + 1/p_\ell$. They also considered a slightly larger class $\nabla S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$ of symbols (which requires (2.1) only for $|\alpha| = 0$ or 1), and showed that the strong-type $L^{p_1} \times \cdots \times L^{p_\ell} \to L^p$ bounds hold for $p \in [1,2]$ if $m < n\ell(\rho-1)/p$, and that the weak-type $L^1 \times \ldots \times L^1 \to L^{1/\ell,\infty}$ remain true if $m < n\ell(\rho-1)$ (see [8, Theorem 3.3 and Proposition 3.7]). This extends Kenig–Staubach's results in [26] to the multilinear setting.

Cao, Xue, and Yabuta [8] also established the standard sparse domination for multilinear pseudo-differential operators. In fact, their main contribution is to solve the problem of sharp weighted estimates and sparse domination of multilinear pseudo-differential operators in the first region Ω_1 in the Figure 1 by using multilinear dyadic analysis techniques.

Theorem F ([8], Proposition 4.1). *Let $\ell \in \mathbb{N}$. Suppose that $0 \le \rho, \delta \le 1$. If $m < \ell n(\rho-1)$ and $a \in \nabla S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$, then for any $f_1, \cdots, f_\ell \in L^\infty_c$, there exists a sparse family $\mathcal{S}$ such that for a.e. $x \in \mathbb{R}^n$.*

$$|\mathscr{T}_a(\vec{f})(x)| \lesssim \mathcal{A}_{\mathcal{S},\vec{1}}(|\vec{f}|)(x).$$

Theorem G ([8], Theorem 1.5). *Let $p_1, \cdots, p_\ell \in (1,\infty)$ and $\frac{1}{p} := \sum_{i=1}^{\ell} \frac{1}{p_i}$. Suppose that $\rho, \delta \in [0,1]$, $m < \ell n(\rho-1)$, and $a \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$. If $\vec{\omega} \in A_{\vec{p}}$ with $\omega := \prod_{i=1}^{\ell} \omega_i^{\frac{p}{p_i}}$, then*

$$\|\mathscr{T}_a\|_{L^{p_1}(\omega_1)\times\cdots\times L^{p_\ell}(\omega_\ell)\to L^p(\omega)} \lesssim [\vec{\omega}]_{A_{\vec{p}}}^{\max\limits_{1\le i\le \ell}\left\{1,\frac{p_i'}{p}\right\}}.$$

Theorem H ([8],Proposition 3.1). *Let $r_1, \ldots, r_\ell \in (1,\infty)$ and $\frac{1}{r} := \frac{1}{r_1} + \cdots + \frac{1}{r_\ell}$. Set $\delta, \rho \in [0,1]$ and $m \in \mathbb{R}$. Suppose that $a \in L^\infty S^m_\rho(\mathbb{R}^n \times \mathbb{R}^{n\ell}) \supsetneq \nabla S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell}) \supsetneq S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$. If $m < \ell n(\rho-1)/r$, then $\mathscr{T}_a$ is bounded from $L^{r_1} \times \cdots \times L^{r_\ell}$ to L^r.*

Theorem I ([8], Theorem 3.3). *Let $\delta, \rho \in [0,1]$ and $m \in \mathbb{R}$. Suppose that $a \in \nabla S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell}) \supsetneq S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$. If $m < \ell n(\rho-1)$, then $\mathscr{T}_a$ is bounded from $L^1 \times \cdots \times L^1$ to $L^{\frac{1}{\ell},\infty}$.*

Consequently, research now focuses primarily on regions Ω_2, Ω_3, and Ω_4. In 2024, Park and Tomita [36] gave an affirmative response as follows.

Theorem J ([36], Proposition 1.3). *Let $\ell \in \mathbb{N}$ and $1 < r_1, \ldots, r_\ell < \infty$ with $1/r := 1/r_1 + \cdots + 1/r_\ell$. Suppose that $\rho \in (0,1)$. If*

$$m < \mathcal{I}_0(\vec{r}) := n(\rho-1)\left[\sum_{j=1}^{\ell} \max\left(\frac{1}{r_j}, \frac{1}{2}\right) - \min\left(\frac{1}{r}, \frac{1}{2}\right)\right]$$

and $a \in S^m_{\rho,\rho}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$, then $\mathscr{T}_a$ is bounded from $L^{r_1} \times \cdots \times L^{r_\ell}$ to L^r.

Moreover, Park and Tomita also proved the following conclusion in the second region Ω_2 and third region Ω_3.

Theorem K ([36,37]). *Let $\ell \in \mathbb{N}$, $r \in (1,2]$, and $p_1, \cdots, p_\ell \in (r,\infty)$ with $\frac{1}{p} := \sum_{i=1}^{\ell} \frac{1}{p_i}$. Suppose that $\rho \in [0,1)$ and $a \in S^{\frac{\ell n(\rho-1)}{r}}_{\rho,\rho}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$. If $\vec{\omega} \in A_{\left(\frac{p_1}{r},\cdots,\frac{p_\ell}{r}\right)}$ with $\omega := \prod_{i=1}^{\ell} \omega_i^{\frac{p}{p_i}}$, then $\mathscr{T}_a$ is bounded from $L^{p_1}(\omega_1) \times \cdots \times L^{p_\ell}(\omega_\ell)$ to $L^p(\omega)$.*

For $\ell = 1$, they also give that

Theorem L ([36], Corollary 1.2). *Let $r \in (1,2]$, and $p \in [r,\infty)$. Suppose that $0 \le \delta \le \rho < 1$ and $a \in S^m_{\rho,\delta}$. If $m \le \frac{n(\rho-1)}{r}$ and $\omega \in A_{\frac{p}{r}}$, then $\mathscr{T}_a$ is bounded on $L^p(\omega)$.*

In Theorem L, the authors showed us that the result of the case: $p = r$ can be deduced by combining the result of the case: $p \in (r,\infty)$, the A_1 argument, and interpolation. This makes us only consider the case $p \in (r,\infty)$.

We next introduce the generalized commutators as follows.

Let $\ell \in \mathbb{N}$ and $\mathbf{b} \in \left(L^1_{loc}\right)^\ell$. Given an ℓ-linear operator T and a multi-index $\mathbf{k} = (k_1,\dots,k_\ell) \in (\mathbb{N}_0)^\ell$, Bényi et al. [3] introduced the following generalized commutator defined by

$$\begin{aligned}[T,\mathbf{b}]_{\mathbf{k}}(\vec{f})(x) :=& T\left((b_1(x)-b_1)^{k_1} f_1,\dots,(b_\ell(x)-b_\ell)^{k_\ell} f_\ell\right)(x),\\ :=& \Big[\cdots[T,\mathbf{b}]_{k_1e_1}\cdots\mathbf{b}\Big]_{k_le_l}(\vec{f})(x),\end{aligned}$$

where

$$[T,\mathbf{b}]_{k_ie_i}(\vec{f}) := \Big[\cdots[T,\mathbf{b}]_{e_i}\cdots\mathbf{b}\Big]_{e_i}(\vec{f}),$$

and

$$[T,\mathbf{b}]_{e_i}(\vec{f}) := T(f_1,\cdots,b_if_i,\cdots,f_\ell) - b_iT(f_1,\cdots,f_\ell)$$

All of our main results pertain to the study of the generalized commutators $[T,\mathbf{b}]_{\mathbf{k}}$ of order $|\mathbf{k}|$. Here, we clarify the following cases:

- When $|\mathbf{k}| = 0$, $[T,\mathbf{b}]_{\mathbf{k}} = T$ called a"0-order commutator".
- When $|\mathbf{k}| = 1$ with $\mathbf{k} = e_i := (0,\cdots,1,\cdots,0)$, $[T,\mathbf{b}]_{\mathbf{k}} = [T,\mathbf{b}]_{e_i}$ for $i = 1,\cdots,\ell$, called a “1-order commutator”. The classcial multilinear commutator $T_{\sum\mathbf{b}} := \sum\limits_{i=1}^{\ell}[T,\mathbf{b}]_{e_i}$ was initially introduced by Lerner et al. (2008) in [33] for multilinear Calderón–Zygmund operators and multilinear maximal operators.
- When $|\mathbf{k}| \ge 2$, $[T,\mathbf{b}]_{\mathbf{k}}$ is called a “ higher-order commutator ” studied in [48] for multilinear Calderón–Zygmund operators and multilinear Littlewood-Paley square operators.
- When $\mathbf{k} = \vec{1} := (1,\cdots,1)$, $[T,\mathbf{b}]_{\mathbf{k}}$ is called the “multilinear iterated commutator” initially introduced in [38,47] for multilinear Calderón–Zygmund operators, multilinear fractional maximal operators, and multilinear fractional integral operators.

Now, we recall the following gerneralized multilinear weights.

Definition 1.2 ([3,32,35]). *Let $\vec{p} := (p_1,\dots,p_\ell) \in [1,\infty)^\ell$, $\frac{1}{p} = \sum\limits_{i=1}^{\ell}\frac{1}{p_i}$, and $\vec{\mathbf{r}} := (\vec{r},r_{\ell+1}) = (r_1,\dots,r_\ell,r_{\ell+1})$ with $1 \le r_1,\dots,r_{\ell+1} < \infty$. We say that $\vec{\omega} := (\omega_1,\dots,\omega_\ell) \in A_{\vec{p},\vec{\mathbf{r}}}$, if for $0 < \omega_i < \infty$ a.e. for every $i = 1,\dots,\ell$ and*

$$[\vec{\omega}]_{A_{\vec{p},\vec{\mathbf{r}}}} = \sup_Q\left(\frac{1}{|Q|}\int_Q \omega^{\frac{r'_{\ell+1}}{r'_{\ell+1}-p}}dx\right)^{p\left(\frac{1}{p}-\frac{1}{r'_{\ell+1}}\right)}\prod_{i=1}^{\ell}\left(\frac{1}{|Q|}\int_Q \omega_i^{\frac{r_i}{r_i-p_i}}dx\right)^{p\left(\frac{1}{r_i}-\frac{1}{p_i}\right)} < \infty, \tag{1.3}$$

where $\omega = \prod_{i=1}^{\ell}\omega_i^{\frac{p}{p_i}}$.

Remark 1.3. *Nieraeth in [35] denoted $s' := r_{\ell+1}$ and denoted this class by $[\vec{\omega}]_{\vec{p},(\vec{r},s)}$. In fact, the essence of these two classes of weights is the same. We have that*

$$\left[\omega_1^{\frac{1}{p_1}},\cdots,\omega_\ell^{\frac{1}{p_\ell}}\right]^p_{\vec{p},(\vec{r},r'_{\ell+1})} = [\vec{\omega}]_{A_{\vec{p},\vec{\mathbf{r}}}}.$$

The following classical multilinear weights is introduced by Lerner et al. [33].

Definition 1.4 ([33, 34]). *Let $\vec{p} := (p_1,\ldots,p_\ell) \in [1,\infty)^\ell$ and $\frac{1}{p} = \sum\limits_{i=1}^{\ell}\frac{1}{p_i}$. We say that $\vec{\omega} := (\omega_1,\ldots,\omega_\ell) \in A_{\vec{p}}$, if for $0<\omega_i<\infty$ a.e. for every $i = 1,\ldots,\ell$ and*

$$[\vec{\omega}]_{A_{\vec{p}}} = \sup_Q \left(\frac{1}{|Q|}\int_Q \omega dx\right)\prod_{i=1}^{\ell}\left(\frac{1}{|Q|}\int_Q \omega_i^{1-p_i{}'}dx\right)^{\frac{p}{p_i'}} < \infty, \tag{1.4}$$

where $\omega = \prod_{i=1}^{\ell}\omega_i^{\frac{p}{p_i}}$. If we further assume that $\ell = 1$, it is back to A_p.

One may notice that setting $\vec{\mathbf{r}} = (R,\ldots,R,1)$, we have that

$$[\vec{\omega}]_{A_{\vec{p},(R,\cdots,R,1)}} = [\vec{\omega}]_{A_{\frac{p_1}{R},\cdots,\frac{p_\ell}{R}}}. \tag{1.5}$$

We also need to recall the sparse dyadic cubes.

Definition 1.5. *Let $\delta\in(0,1)$ and we say $\mathcal{S}\subseteq\mathcal{D}$ is a δ-sparse family, if for each $Q\in\mathcal{S}$, there exists a measurable subset $E_Q\subseteq Q$ with $|E_Q|\geq\delta|Q|$, and $\{E_Q\}_{Q\in\mathcal{S}}$ is a pairwise disjoint family.*

We introduce the following operator for realizing sparse domination.

Definition 1.6. *Let $\ell\in\mathbb{N}$ and $r_1,\ldots,r_\ell\in[1,\infty)$. For a sparse family $\mathcal{S}\subseteq\mathcal{D}$, the **$\ell$-linear sparse operator** $\mathcal{A}_{\mathcal{S},\vec{r}}$ is defined by*

$$\mathcal{A}_{\mathcal{S},\vec{r}}(\vec{f})(x) = \sum_{Q\in\mathcal{S}}\langle|f_i|\rangle_{r_i,Q}\cdot 1_Q(x).$$

1.2. Motivations.

Theorems A, B, C, and D establish the unweighted boundedness of ℓ-linear FIO $T_{\sigma,\Psi}$ when the amplitude $\sigma\in S^m_{1,0}$. In parallel, for any ℓ-linear PDOs, Theorems F, G, K, and L develops a corresponding theory of sparse domination and weighted bounds over certain symbol regions, cf. the Ω_1 and Ω_2 in the Figure 1. However, these results exhibit several limitations:

- an insufficiently broad range for the decay order m;
- a lack of generality in the class of amplitudes (often restricted to $S^m_{1,0}$);
- highly restrictive conditions on the phase error functions $\overline{\Psi}$, which are typically required to be independent of x and possess homogeneous structure (cf. Theorems A, B, C, and D).

To overcome these limitations, we propose the following conjectures:

Conjecture 1. *When $\sigma\in\nabla S^m_{\rho,\delta}$(or $S^m_{\rho,\delta}$) with $\rho\in(0,1]$, $\delta\in[0,1]$, and $m\leq 0$, there exists a class of phase error functions $\overline{\Psi}$, which contains at least some phase error function associated with spatial variable x, such that the boundedness $L^{r_1}\times\cdots\times L^{r_\ell}\to L^{r,\infty}$ of ℓ-linear FIO $T_{\sigma,\Psi}$ implies sparse domination. This sparse form would then combine with multilinear extrapolation theory to yield a criterion for sharp quantitative weighted bounds of ℓ-linear FIOs.*

To build a robust theory for multilinear Fourier integral operators, a systematic study of unweighted boundedness—under the assumptions of Conjecture 1—is essential. Not only is this of independent interest, but it also provides a necessary foundation for achieving sharp weighted estimates via the criterion arising from Conjecture 1.

Conjecture 2. *When $\sigma \in \nabla S^m_{\rho,\delta}$(or $S^m_{\rho,\delta}$) with $\rho \in (0,1]$, $\delta \in [0,1]$, and $m \leq 0$, there exists a class of phase error functions $\overline{\Psi}$, which is at least larger than the class in Conjecture 1, such that the ℓ-linear FIO $T_{\sigma,\Psi}$ is bounded from $L^{r_1} \times \cdots \times L^{r_\ell}$ to $L^{r,\infty}$ (or L^r), for some $r_1, \cdots, r_\ell \in [1,\infty]$ with $\frac{1}{r} := \sum_{i=1}^{\ell} \frac{1}{r_i}$.*

Under the framework of Conjectures 1 and 2, we aim to maximize the range of the decay order m of the amplitude σ, going beyond the parameter scope of $S^m_{1,0}$ established in Theorems A, B, C, D, and developing the setting of ℓ-linear PDOs to ℓ-linear FIOs with surpassing the range of decay order m of Theorems F, G, H, I, K, and L. A central challenge is to provide either an explicit description of the permissible phase error functions $\overline{\Psi}$ that make such extended regularity possible.

Conjectures 1 and 2 form the core theoretical motivation of this paper. In the remainder of the work, we develop a new global **dilated cube technique** to address these questions in full generality.

Next, the main results are introduced in the following subsections.

1.3. the criteria for sparse bounds and sharp weighted estimates.

To confirm Conjecture 1, we establish the multilinear sparse bounds criterion for ℓ-FIOs as follows.

Theorem 1.7. *Let $r_1, \cdots, r_\ell \in [1,\infty)$ with $\vec{r} := (r_1, \cdots, r_\ell)$, and $\frac{1}{r} := \sum_{i=1}^{\ell} \frac{1}{r_i}$. Set $\rho \in (0,1]$, $\delta \in [0,1]$, and $m \in (-\infty, 0]$. Suppose that $\sigma \in \nabla S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$ and $\overline{\Psi} \in (L^\infty\phi^1 \cap \nabla\phi^0)(\mathbb{R}^n \times \mathbb{R}^{n\ell})$. Assume that $T_{\sigma,\Psi}$ is bounded from $L^{r_1} \times \cdots \times L^{r_\ell}$ to $L^{r,\infty}$. Then for any $f_i \in L^{r_i}_c$, $i = 1, \cdots, \ell$, there exists a sparse family $\mathcal{S}$ such that for a.e. $x \in \mathbb{R}^n$,*

$$\left|T_{\sigma,\Psi}(\vec{f})(x)\right| \lesssim \mathcal{A}_{\mathcal{S},\vec{r}}(\vec{f})(x).$$

Combining Theorem 1.7 and the multilinear extrapolation theory established by Li, Nieraeth, Bényi et al. [3,32,35] (2018–2020) , the following criterion of sharp weighted bounds and vector-valued weighted estimates for ℓ-FIOs and their generalized commutators can be built immediately as our second main result.

Theorem 1.8. *Let $r_1, \cdots, r_\ell \in [1,\infty)$, $\mathbf{r} := (\vec{r}, 1)$, and $\frac{1}{r} := \sum_{i=1}^{\ell} \frac{1}{r_i}$. Set $\rho \in (0,1]$, $\delta \in [0,1]$, and $m \in (-\infty, 0]$. Suppose that $\sigma \in \nabla S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$ and $\overline{\Psi} \in (L^\infty\phi^1 \cap \nabla\phi^0)(\mathbb{R}^n \times \mathbb{R}^{n\ell})$. Assume that $T_{\sigma,\Psi}$ is bounded from $L^{r_1} \times \cdots \times L^{r_\ell}$ to $L^{r,\infty}$. If $\vec{\omega} \in A_{\vec{p},\mathbf{r}}$ with $p_i \in (r_i, \infty)$ for $i = 1, \cdots, \ell$, and $\frac{1}{p} := \sum_{i=1}^{\ell} \frac{1}{p_i}$, then*

$$\|T_{\sigma,\Psi}\|_{L^{p_1}(\omega_1)\times\cdots\times L^{p_\ell}(\omega_\ell)\to L^p(\omega)} \lesssim [\vec{\omega}]^{\Theta(\vec{r})}_{A_{\vec{p},\mathbf{r}}},$$

where $\Theta(\vec{r}) := \frac{1}{p} \cdot \max\left\{\frac{p_1}{p_1 - r_1}, \cdots, \frac{p_\ell}{p_\ell - r_\ell}, p\right\}$, which is sharp, i.e. it cannot be reduced.

Moreover, for all exponents $\vec{q}$ with $\frac{1}{q} := \sum_{i=1}^{\ell} \frac{1}{q_i}$ and $q_i \in (r_i, \infty)$ for $i = 1, \cdots, \ell$,

$$\|T_{\sigma,\Psi}\|_{L^{p_1}(\omega_1, l^{q_1}) \times \cdots \times L^{p_\ell}(\omega_\ell, l^{q_\ell}) \to L^p(\omega, l^q)} \lesssim [\vec{\omega}]_{A_{\vec{p},\vec{\mathbf{r}}}}^{\Theta(\vec{r}) \cdot \theta(\vec{r})},$$

where $\theta(\vec{r}) := \max\left(\frac{p_1(q_1-r_1)}{q_1(p_1-r_1)}, \cdots, \frac{p_\ell(q_\ell-r_\ell)}{q_\ell(p_\ell-r_\ell)}, \frac{p'(q'-1)}{q'(p'-1)}\right)$.

Proof. Theorem 1.7 implies that

$$\left\langle \left| T_{\sigma,\Psi}(\vec{f}) \right|, g \right\rangle \lesssim \sup_{\mathcal{S} \subseteq \mathcal{D}} \left\langle \mathcal{A}_{\mathcal{S},\vec{r}}(\vec{f}), g \right\rangle =: \sup_{\mathcal{S} \subseteq \mathcal{D}} \Lambda_{(\vec{r},1)}(\vec{f}, g). \tag{1.6}$$

The desired follows from combining Theorem 1.7 with (1.6), and [35, Corollary 4.2]. □

Theorem 1.9. *Let $r_1, \cdots, r_\ell \in [1, \infty)$, $\vec{\mathbf{r}} := (\vec{r}, 1)$, and $\frac{1}{r} := \sum_{i=1}^{\ell} \frac{1}{r_i}$. Set $\mathbf{k} = (k_1, \cdots, k_\ell) \in \mathbb{N}_0^\ell$ and $\mathbf{b} = (b_1, \ldots, b_\ell) \in (\mathrm{BMO})^\ell$. Set $\rho \in (0, 1]$, $\delta \in [0, 1]$, and $m \in (-\infty, 0]$. Suppose that $\sigma \in \nabla S_{\rho,\delta}^m(\mathbb{R}^n \times \mathbb{R}^{n\ell})$, and $\overline{\Psi} \in (L^\infty\phi^1 \cap \nabla\phi^0)(\mathbb{R}^n \times \mathbb{R}^{n\ell})$. Assume that $T_{\sigma,\Psi}$ is bounded from $L^{r_1} \times \cdots \times L^{r_\ell}$ to $L^{r,\infty}$. If $\vec{\omega} \in A_{\vec{p},\vec{\mathbf{r}}}$ with $p_i \in (r_i, \infty)$ for $i = 1, \cdots, \ell$, and $\frac{1}{p} := \sum_{i=1}^{\ell} \frac{1}{p_i}$, then*

$$\left\| [T_{\sigma,\Psi}, \mathbf{b}]_{\mathbf{k}} \right\|_{L^{p_1}(\omega_1) \times \cdots \times L^{p_\ell}(\omega_\ell) \to L^p(\omega)} \lesssim \prod_{i=1}^{\ell} \|b_i\|_{\mathrm{BMO}}^{k_i} \cdot [\vec{\omega}]_{A_{\vec{p},\vec{\mathbf{r}}}}^{\Theta(\vec{r}) + |\mathbf{k}|\Xi},$$

where $|\mathbf{k}| = \sum_{i=1}^{\ell} k_i$, $\Theta(\vec{r}) := \frac{1}{p} \cdot \max\left\{\frac{p_1}{p_1-r_1}, \cdots, \frac{p_\ell}{p_\ell-r_\ell}, p\right\}$, and $\Xi = \frac{1}{p}\max\left\{\frac{p_1 r_1}{p_1-r_1}, \cdots, \frac{p_\ell r_\ell}{p_\ell-r_\ell}, p\right\}$.

Furthermore, for all exponents $\vec{q}$ with $\frac{1}{q} := \sum_{i=1}^{\ell} \frac{1}{q_i}$ and $q_i \in (r_i, \infty)$ for $i = 1, \cdots, \ell$,

$$\|[T_{\sigma,\Psi}, \mathbf{b}]_{\mathbf{k}}\|_{L^{p_1}(\omega_1, l^{q_1}) \times \cdots \times L^{p_\ell}(\omega_\ell, l^{q_\ell}) \to L^p(\omega, l^q)} \lesssim \prod_{i=1}^{\ell} \|b_i\|_{\mathrm{BMO}}^{k_i} \cdot [\vec{\omega}]_{A_{\vec{p},\vec{\mathbf{r}}}}^{(\Theta(\vec{r}) + |\mathbf{k}|\Xi) \cdot \theta(\vec{r})},$$

where $\theta(\vec{r}) := \max\left(\frac{p_1(q_1-r_1)}{q_1(p_1-r_1)}, \cdots, \frac{p_\ell(q_\ell-r_\ell)}{q_\ell(p_\ell-r_\ell)}, \frac{p'(q'-1)}{q'(p'-1)}\right)$.

Proof. The first result derives from Theorem 1.8 and [3, Theorem A.2]. The second result can be deduced from combining [35, the proof of Corollary 4.6] and the first result. □

Remark 1.10. *The sharpness of the $\Theta(\vec{r})$ and $\Theta(\vec{1})$ is proved in [34, 35] for multilinear Calderón–Zygmund operators and multilinear maximal operators. When $m < \ell n(\rho - 1)$, the sharp weighted estimate is proved in Theorem G. Our proof of sharpness is similar to theirs, and we omit it here.*

Remark 1.11. *Building upon Remark 2.9 below, we observe that our class $(L^\infty\phi^1 \cap \nabla\phi^0)(\mathbb{R}^n \times \mathbb{R}^{n\ell})$ of admissible phase error functions $\overline{\Psi}$ in fact encompasses a wide range of functions that depend on the spatial variable x and are not necessarily homogeneous. This provides further evidence for the conditions imposed in Conjecture 1.*

1.4. Unweighted results.

In order to verify Conjecture 2, we establish the unweighted strong and weak type boundedness of the ℓ-linear FIOs. The first result improves the results of Kenig and Staubach [26, Proposition 2.3], and Cao, Xue, and Yabuta (Theorem H) as follows.

Theorem 1.12. *Let $1 < r_1, \ldots, r_\ell < \infty$ and $\frac{1}{r} := \sum_{i=1}^{\ell} \frac{1}{r_i}$ with $1 \leq r \leq 2$. Suppose that $\rho \in [0, 1]$ and $m < \ell n(\rho - 1)/r$. If $\sigma \in L^\infty S_\rho^m\left(\mathbb{R}^n \times \mathbb{R}^{n\ell}\right)$ and $\overline{\Psi} \in L^\infty\phi^1\left(\mathbb{R}^n \times \mathbb{R}^{n\ell}\right)$, then $T_{\sigma,\Psi}$ is bounded from $L^{r_1}(\mathbb{R}^n) \times \cdots \times L^{r_\ell}(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$.*

The following also gives the L^{∞}-estimate for ℓ-linear FIOs.

Theorem 1.13. *Suppose that $\rho \in [0,1]$ and $m < \ell n(\rho-1)/2$. If $\sigma \in L^{\infty}S^m_{\rho}\left(\mathbb{R}^n \times \mathbb{R}^{n\ell}\right)$ and $\overline{\Psi} \in L^{\infty}\phi^1\left(\mathbb{R}^n \times \mathbb{R}^{n\ell}\right)$, then $T_{\sigma,\Psi}$ is bounded from $L^{\infty}(\mathbb{R}^n) \times \cdots \times L^{\infty}(\mathbb{R}^n)$ to $L^{\infty}(\mathbb{R}^n)$.*

For the weak type result of Cao, Xue, and Yabuta (Theorem I), we improve the range of m and extend ℓ-linear PDOs to ℓ-linear FIOs by our **dilated cube technique** as follows.

Theorem 1.14. *Let $\sigma \in L^{\infty}S^m_{\rho}\left(\mathbb{R}^n \times \mathbb{R}^{n\ell}\right)$ with $\rho \in (0,1]$ and $m \in \mathbb{R}$. If $m < \ell n(\rho-1)/2$ and $\overline{\Psi} \in L^{\infty}\phi^1\left(\mathbb{R}^n \times \mathbb{R}^{n\ell}\right)$, then $T_{\sigma,\Psi}$ is bounded from $L^1(\mathbb{R}^n) \times \cdots \times L^1(\mathbb{R}^n)$ to $L^{\frac{1}{\ell},\infty}(\mathbb{R}^n)$. Moreover, the result still holds when $\rho = 0$ and $m < -\ell n$.*

1.5. **FIOs: the important results beyond the previous.**

Combining Theorems 1.8, 1.9, 1.14 and (1.5), by taking $\vec{r} = \vec{1}$, we can obtain the sharp weighted bounds for ℓ-linear FIOs and their generalized commutators.

Theorem 1.15. *Let $p_1, \cdots, p_\ell \in (1,\infty)$ and $\frac{1}{p} := \sum_{i=1}^{\ell} \frac{1}{p_i}$. Set $\rho \in (0,1]$, $\delta \in [0,1]$, and $m \in (-\infty, \ell n(\rho-1)/2)$. Suppose that $\sigma \in \nabla S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$, and $\overline{\Psi} \in (L^{\infty}\phi^1 \cap \nabla\phi^0)(\mathbb{R}^n \times \mathbb{R}^{n\ell})$. If $\vec{\omega} \in A_{\vec{p}}$, then*

$$\|T_{\sigma,\Psi}\|_{L^{p_1}(\omega_1)\times\cdots\times L^{p_\ell}(\omega_\ell)\to L^p(\omega)} \lesssim [\vec{\omega}]_{A_{\vec{p}}}^{\max\limits_{1\le i\le \ell}\left\{1,\frac{p_i'}{p}\right\}},$$

where the exponent is sharp, i.e. it cannot be reduced.

Moreover, for all exponents $\vec{q}$ with $\frac{1}{q} := \sum_{i=1}^{\ell} \frac{1}{q_i}$ and $q_i \in (1,\infty)$ for $i = 1, \cdots, \ell$,

$$\|T_{\sigma,\Psi}\|_{L^{p_1}(\omega_1,l^{q_1})\times\cdots\times L^{p_\ell}(\omega_\ell,l^{q_\ell})\to L^p(\omega,l^q)} \lesssim [\vec{\omega}]_{A_{\vec{p}}}^{\max\limits_{1\le i\le \ell}\left\{1,\frac{p_i'}{p}\right\}\cdot\max\left(\frac{p_1'}{q_1'},\cdots,\frac{p_\ell'}{q_\ell'},\frac{p}{q}\right)}.$$

Theorem 1.16. *Let $p_1, \cdots, p_\ell \in (1,\infty)$ and $\frac{1}{p} := \sum_{i=1}^{\ell} \frac{1}{p_i}$. Set $\mathbf{k} = (k_1, \cdots, k_\ell) \in \mathbb{N}_0^\ell$ and $\mathbf{b} = (b_1, \ldots, b_\ell) \in (\mathrm{BMO})^\ell$. Set $\rho \in (0,1]$, $\delta \in [0,1]$, and $m \in (-\infty, \ell n(\rho-1)/2)$. Suppose that $\sigma \in \nabla S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$, and $\overline{\Psi} \in (L^{\infty}\phi^1 \cap \nabla\phi^0)(\mathbb{R}^n \times \mathbb{R}^{n\ell})$. If $\vec{\omega} \in A_{\vec{p}}$, then*

$$\left\|[T_{\sigma,\Psi},\mathbf{b}]_{\mathbf{k}}\right\|_{L^{p_1}(\omega_1)\times\cdots\times L^{p_\ell}(\omega_\ell)\to L^p(\omega)} \lesssim \prod_{i=1}^{\ell}\|b_i\|_{\mathrm{BMO}}^{k_i} \cdot [\vec{\omega}]_{A_{\vec{p}}}^{(|\mathbf{k}|+1)\max\limits_{1\le i\le \ell}\left\{1,\frac{p_i'}{p}\right\}},$$

where $|\mathbf{k}| = \sum_{i=1}^{\ell} k_i$.

Furthermore, for all exponents $\vec{q}$ with $\frac{1}{q} := \sum_{i=1}^{\ell} \frac{1}{q_i}$ and $q_i \in (1,\infty)$ for $i = 1, \cdots, \ell$,

$$\|[T_{\sigma,\Psi},\mathbf{b}]_{\mathbf{k}}\|_{L^{p_1}(\omega_1,l^{q_1})\times\cdots\times L^{p_\ell}(\omega_\ell,l^{q_\ell})\to L^p(\omega,l^q)} \lesssim \prod_{i=1}^{\ell}\|b_i\|_{\mathrm{BMO}}^{k_i} \cdot [\vec{\omega}]_{A_{\vec{p}}}^{(|\mathbf{k}|+1)\max\limits_{1\le i\le \ell}\left\{1,\frac{p_i'}{p}\right\}\cdot\max\left(\frac{p_1'}{q_1'},\cdots,\frac{p_\ell'}{q_\ell'},\frac{p}{q}\right)}.$$

Remark 1.17. *Not only do Theorems 1.15 and 1.16 extend the previous results–Theorems G, J, and K on ℓ-PDOs by Cao et al. [8] and Park et al. [36] to the setting of ℓ-linear FIOs, but also improve and surpass their range of order m.*

If $\rho = 1$ and $\vec{\omega} = \vec{1}$, we can observe that the range of m in Corollaries 1.15 and 1.16 will become $(-\infty, 0)$, which is clearly sharp, since the results generally do not hold for $m > 0$. Thus, the following unweighted results can be regarded as the improvements in the work of Rodríguez-López, Rule, and Staubach (2013–2023) [41–45].

Corollary 1.18. *Let $p_1, \cdots, p_\ell \in (1,\infty)$ and $\frac{1}{p} := \sum_{i=1}^{\ell} \frac{1}{p_i}$. Set $\delta \in [0,1]$, and $m \in (-\infty, 0)$. Suppose that $\sigma \in \nabla S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell}) \supseteq S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$, and $\overline{\Psi} \in (L^\infty\phi^1 \cap \nabla\phi^0)(\mathbb{R}^n \times \mathbb{R}^{n\ell})$. Then $T_{\sigma,\Psi}$ is bounded from $L^{p_1} \times \cdots \times L^{p_\ell}$ to L^p. The range of m is sharp.*

Moreover, for all exponents $\vec{q}$ with $\frac{1}{q} := \sum_{i=1}^{\ell} \frac{1}{q_i}$ and $q_i \in (1,\infty)$ for $i = 1, \cdots, \ell$, $T_{\sigma,\Psi}$ is bounded from $L^{p_1}(l^{q_1}) \times \cdots \times L^{p_\ell}(l^{q_\ell})$ to $L^p(l^q)$.

Corollary 1.19. *Let $p_1, \cdots, p_\ell \in (1,\infty)$ and $\frac{1}{p} := \sum_{i=1}^{\ell} \frac{1}{p_i}$. Set $\mathbf{k} = (k_1, \cdots, k_\ell) \in \mathbb{N}_0^\ell$ and $\mathbf{b} = (b_1, \ldots, b_\ell) \in (\mathrm{BMO})^\ell$. Set $\delta \in [0,1]$ and $m \in (-\infty, 0)$. Suppose that $\sigma \in \nabla S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell}) \supseteq S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$, and $\overline{\Psi} \in (L^\infty\phi^1 \cap \nabla\phi^0)(\mathbb{R}^n \times \mathbb{R}^{n\ell})$. Then*

$$\left\|[T_{\sigma,\Psi}, \mathbf{b}]_{\mathbf{k}}\right\|_{L^{p_1}\times\cdots\times L^{p_\ell}\to L^p} \lesssim \prod_{i=1}^{\ell} \|b_i\|_{\mathrm{BMO}}^{k_i},$$

where $|\mathbf{k}| = \sum_{i=1}^{\ell} k_i$. The range of m is sharp.

Furthermore, for all exponents $\vec{q}$ with $\frac{1}{q} := \sum_{i=1}^{\ell} \frac{1}{q_i}$ and $q_i \in (1,\infty)$ for $i = 1, \cdots, \ell$,

$$\|[T_{\sigma,\Psi}, \mathbf{b}]_{\mathbf{k}}\|_{L^{p_1}(l^{q_1})\times\cdots\times L^{p_\ell}(l^{q_\ell})\to L^p(l^q)} \lesssim \prod_{i=1}^{\ell} \|b_i\|_{\mathrm{BMO}}^{k_i},$$

where $|\mathbf{k}| = \sum_{i=1}^{\ell} k_i$.

Remark 1.20. *In contrast to Theorems A, B, C, and D, Corollaries 1.18 and 1.19 give the sharp range of m for which the multilinear Fourier integral operator admits unweighted boundedness. Moreover, Theorems 1.12, 1.13, and 1.14 hold under a significantly broader class $L^\infty\phi^1(\mathbb{R}^n \times \mathbb{R}^{n\ell})$ of phase error functions, endowing our conclusions with a high degree of generality.*

1.6. **PDOs: the important results beyond the previous.**

Now, let $\overline{\Psi} = 0$ in (1.1), then we can obtain the following improved conclusions for ℓ-linear PDOs $\mathscr{T}_a$. Let $a \in S^m_{\rho,\rho}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$ with $\rho \in (0,1]$. Combining Theorems J, 1.8, and 1.9, we can deduce that

Corollary 1.21. *Let $r_1, \cdots, r_\ell \in (1,\infty)$, $\frac{1}{r} := \sum_{i=1}^{\ell} \frac{1}{r_i}$, and $\vec{\mathbf{r}} := (r_1, \ldots, r_\ell, 1)$. Suppose that $\rho \in (0,1]$, $m \in (-\infty, \mathcal{I}_0(\vec{r}))$ with $\mathcal{I}_0(\vec{r}) := n(\rho-1)\left[\sum_{j=1}^{\ell} \max\left(\frac{1}{r_j}, \frac{1}{2}\right) - \min\left(\frac{1}{r}, \frac{1}{2}\right)\right]$, and $a \in S^m_{\rho,\rho}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$. If*

1. *when $m \in (-\infty, \ell n(\rho-1)/2)$, $\vec{\omega} \in A_{\vec{p}}$ with $p_1, \ldots, p_\ell \in (1,\infty)$, and $\frac{1}{p} := \sum_{i=1}^{\ell} \frac{1}{p_i}$;*
2. *when $m \in [\ell n(\rho-1)/2, \mathcal{I}_0(\vec{r}))$, $\vec{\omega} \in A_{\vec{p},\vec{\mathbf{r}}}$ with $p_i \in (r_i, \infty)$ for $i = 1, \cdots, \ell$, and $\frac{1}{p} := \sum_{i=1}^{\ell} \frac{1}{p_i}$,*

then

$$\|\mathscr{T}_a\|_{\prod_{i=1}^{\ell} L^{p_i}(\omega_i)\to L^p(\omega)} \lesssim \begin{cases} [\vec{\omega}]_{A_{\vec{p}}}^{\Theta(\vec{1})}, & m \in (-\infty, \ell n(\rho-1)/2); \\ [\vec{\omega}]_{A_{\vec{p},\vec{\mathbf{r}}}}^{\Theta(\vec{r})}, & m \in [\ell n(\rho-1)/2, \mathcal{I}_0(\vec{r})), \end{cases}$$

where $|\mathbf{k}|=\sum_{i=1}^{\ell} k_i$, $\Theta(\vec{r}):=\frac{1}{p}\cdot\max\left\{\frac{p_1}{p_1-r_1},\cdots,\frac{p_\ell}{p_\ell-r_\ell},p\right\}$, *and* $\Theta(\vec{r})$ *and* $\Theta(\vec{1})$ *are both sharp, i.e. they cannot be reduced.*

Corollary 1.22. *Let* $r_1,\cdots,r_\ell\in(1,\infty)$, $\frac{1}{r}:=\sum_{i=1}^{\ell}\frac{1}{r_i}$, *and* $\vec{\mathbf{r}}:=(r_1,\ldots,r_\ell,1)$. *Set* $\mathbf{k}=(k_1,\cdots,k_\ell)\in\mathbb{N}_0^\ell$ *and* $\mathbf{b}=(b_1,\ldots,b_\ell)\in(\mathrm{BMO})^\ell$. *Suppose that* $\rho\in(0,1]$, $m\in(-\infty,\mathcal{I}_0(\vec{r}))$ *with* $\mathcal{I}_0(\vec{r}):=n(\rho-1)\left[\sum_{j=1}^{\ell}\max\left(\frac{1}{r_j},\frac{1}{2}\right)-\min\left(\frac{1}{r},\frac{1}{2}\right)\right]$, *and* $a\in S^m_{\rho,\rho}(\mathbb{R}^n\times\mathbb{R}^{n\ell})$. *If*

(1) *when* $m\in(-\infty,\ell n(\rho-1)/2)$, $\vec{\omega}\in A_{\vec{p}}$ *with* $p_1,\ldots,p_\ell\in(1,\infty)$, *and* $\frac{1}{p}:=\sum_{i=1}^{\ell}\frac{1}{p_i}$;

(2) *when* $m\in[\ell n(\rho-1)/2,\mathcal{I}_0(\vec{r}))$, $\vec{\omega}\in A_{\vec{p},\vec{\mathbf{r}}}$ *with* $p_i\in(r_i,\infty)$ *for* $i=1,\cdots,\ell$, *and* $\frac{1}{p}:=\sum_{i=1}^{\ell}\frac{1}{p_i}$,

then

$$\|[\mathscr{T}_a,\mathbf{b}]_{\mathbf{k}}\|_{\prod_{i=1}^{\ell}L^{p_i}(\omega_i)\to L^p(\omega)}\lesssim\begin{cases}\prod_{i=1}^{\ell}\|b_i\|_{\mathrm{BMO}}^{k_i}\cdot[\vec{\omega}]_{A_{\vec{p}}}^{\Theta(\vec{1})+|\mathbf{k}|\Xi}, & m\in(-\infty,\ell n(\rho-1)/2);\\ \prod_{i=1}^{\ell}\|b_i\|_{\mathrm{BMO}}^{k_i}\cdot[\vec{\omega}]_{A_{\vec{p},\vec{\mathbf{r}}}}^{\Theta(\vec{r})+|\mathbf{k}|\Xi}, & m\in[\ell n(\rho-1)/2,\mathcal{I}_0(\vec{r})),\end{cases}$$

where $|\mathbf{k}|=\sum_{i=1}^{\ell}k_i$, $\Theta(\vec{r}):=\frac{1}{p}\cdot\max\left\{\frac{p_1}{p_1-r_1},\cdots,\frac{p_\ell}{p_\ell-r_\ell},p\right\}$, *and* $\Xi=\frac{1}{p}\max\left\{\frac{p_1r_1}{p_1-r_1},\cdots,\frac{p_\ell r_\ell}{p_\ell-r_\ell},p\right\}$.

Remark 1.23. *Corollaries 1.21 and 1.22 realize the quantitative weighted improvements of Theorems J, in term of the range of* m, *the* ℓ*-linear PDOs, and the multilinear weighted theory.*

Although we have improved the important results of ℓ-linear PDOs, one might wonder whether the quantitative weighted improvements of PDOs and generalized commutators when $\ell=1$ would be better than the linear result of Park and Tomita (Theorem L). Combining Theorems E, 1.8, 1.9, 1.14 (for $m<n(\rho-1)/2$), and (1.5) can give the following positive answer for $\ell=1$.

Corollary 1.24. *Let* $r\in[1,\infty)$ *and* $p\in(r,\infty)$. *Set* $\rho\in(0,1]$, $\delta\in[0,1]$, *and* $m\in(-\infty,n(\rho-1)\left|\frac{1}{r}-\frac{1}{2}\right|+\min\left\{0,\frac{n(\rho-\delta)}{2}\right\})$. *Suppose that* $a\in\nabla S^m_{\rho,\delta}(\mathbb{R}^n\times\mathbb{R}^n)$. *If* $\omega\in A_{p/r}$ *for* $m\geq\ell n(\rho-1)/2$, *and* $\omega\in A_p$ *for* $m<\ell n(\rho-1)/2$, *then*

$$\|\mathscr{T}_a\|_{L^p(\omega)\times\to L^p(\omega)}\lesssim\begin{cases}[\vec{\omega}]_{A_p}^{\max\left\{1,\frac{1}{p-1}\right\}}, & m\in(-\infty,\ell n(\rho-1)/2);\\ [\vec{\omega}]_{A_{p/r}}^{\max\left\{1,\frac{1}{p-r}\right\}}, & m\in[\ell n(\rho-1)/2,n(\rho-1)\left|\frac{1}{r}-\frac{1}{2}\right|+\min\left\{0,\frac{n(\rho-\delta)}{2}\right\}).\end{cases}$$

where the exponents are both sharp, i.e. it cannot be reduced.

Moreover, for all $q\in(r,\infty)$,

$$\|\mathscr{T}_a\|_{L^p(\omega,l^q)\to L^p(\omega,l^q)}\lesssim\begin{cases}[\vec{\omega}]_{A_p}^{\max\left\{1,\frac{1}{p-1}\right\}\cdot\max\left\{\frac{p(q-1)}{q(p-1)},\frac{p'(q'-1)}{q'(p'-1)}\right\}}, & m\in(-\infty,n(\rho-1)/2);\\ [\vec{\omega}]_{A_{p/r}}^{\max\left\{1,\frac{1}{p-r}\right\}\cdot\max\left\{\frac{p(q-r)}{q(p-r)},\frac{p'(q'-1)}{q'(p'-1)}\right\}}, & m\in[n(\rho-1)/2,n(\rho-1)\left|\frac{1}{r}-\frac{1}{2}\right|+\min\left\{0,\frac{n(\rho-\delta)}{2}\right\}).\end{cases}$$

Corollary 1.25. *Let* $r\in[1,\infty)$ *and* $p\in(r,\infty)$. *Set* $\rho\in(0,1]$, $\delta\in[0,1]$, *and* $m\in(-\infty,n(\rho-1)\left|\frac{1}{r}-\frac{1}{2}\right|+\min\left\{0,\frac{n(\rho-\delta)}{2}\right\})$. *Suppose that* $a\in\nabla S^m_{\rho,\delta}(\mathbb{R}^n\times\mathbb{R}^n)$. *If* $\omega\in A_{p/r}$

for $m \geq \ell n(\rho-1)/2$, and $\omega \in A_p$ for $m < \ell n(\rho-1)/2$, then

$$\|[\mathscr{T}_a, \mathbf{b}]_k\|_{L^p(\omega)\to L^p(\omega)} \lesssim \|b_i\|_{\mathrm{BMO}}^k \cdot \begin{cases} [\vec{\omega}]_{A_p}^{\max\left\{1,\frac{1}{p-1}\right\}}, & m \in (-\infty, n(\rho-1)/2); \\ [\vec{\omega}]_{A_{p/r}}^{\max\left\{1,\frac{1}{p-r}\right\}}, & m \in [n(\rho-1)/2, n(\rho-1)\left|\frac{1}{r}-\frac{1}{2}\right| + \min\left\{0, \frac{n(\rho-\delta)}{2}\right\}). \end{cases}$$

where the exponents are both sharp, i.e. it cannot be reduced.

Moreover, for all $q \in (r,\infty)$,

$$\|[\mathscr{T}_a, \mathbf{b}]_k\|_{L^p(\omega,l^q)\to L^p(\omega,l^q)} \lesssim \|b_i\|_{\mathrm{BMO}}^k \cdot \begin{cases} [\vec{\omega}]_{A_p}^{\max\left\{1,\frac{1}{p-1}\right\}\cdot\max\left\{\frac{p(q-1)}{q(p-1)},\frac{p'(q'-1)}{q'(p'-1)}\right\}}, & m \in (-\infty, n(\rho-1)/2); \\ [\vec{\omega}]_{A_{p/r}}^{\max\left\{1,\frac{1}{p-r}\right\}\cdot\max\left\{\frac{p(q-r)}{q(p-r)},\frac{p'(q'-1)}{q'(p'-1)}\right\}}, & m \in [n(\rho-1)/2, n(\rho-1)\left|\frac{1}{r}-\frac{1}{2}\right| + \min\left\{0, \frac{n(\rho-\delta)}{2}\right\}). \end{cases}$$

Remark 1.26. *Corollaries 1.24 and 1.25 yield the weighted improvements of Theorem E. Furthermore, the range of order m is better than Theorem L, since when $\rho \geq \delta$ and $r \in [1,4]$,*

$$0 \geq n(\rho-1)\left|\frac{1}{r}-\frac{1}{2}\right| + \min\left\{0, \frac{n(\rho-\delta)}{2}\right\} \geq \mathcal{I}_0(r) := n(\rho-1)\left|\frac{1}{r}-\frac{1}{2}\right| \geq \frac{n(\rho-1)}{r}.$$

1.7. Notation and Structure.

- For some positive constant C independent of the main parameters, we denote $A \lesssim B$ to mean that $A \leq CB$, and $A \approx B$ to indicate that $A \lesssim B$ and $B \lesssim A$. Additionally, $A \lesssim_{\alpha,\beta} B$ means that $A \leq C_{\alpha,\beta}B$, where $C_{\alpha,\beta}$ depends on α and β.
- We say $\omega : \mathbb{R}^n \to [0,\infty]$ is a weight, if ω is a locally integrable function satisfying $0 < \omega(x) < \infty$ almost everywhere. We define the measure $d\omega(x) = \omega(x)dx$ and $\omega(E) = \int_E \omega\, dx$ for measurable $E \subseteq \mathbb{R}^n$.
- For any $E \subseteq \mathbb{R}^n$, the average bump of f is defined by

  $$\langle f\rangle_{r,E} := \left(\frac{1}{|E|}\int_E f^r\right)^{\frac{1}{r}}.$$

  We always denote $\langle f\rangle_E := \langle f\rangle_{1,E}$.
- For $\mathbf{k} = (k_i)_{i=1}^{\ell}$ and $\mathbf{t} = (t_i)_{i=1}^{\ell}$, we use $\mathbf{t} \leq \mathbf{k}$ to denote that $t_i \leq k_i$ for all i; $\mathbf{t} < \mathbf{k}$ denote that $t_i < k_i$ for all i.
- Throughout this paper, we always assume $\ell \in \mathbb{N}$.

The remainder of this paper is organized as follows. In Sect. 2, we present the preparatory material, which includes the definitions of the classes of amplitude and phase functions, the corresponding decay estimates, and several operators used in the dyadic analysis. In Sect. 3, we prove Theorem 1.7 by combining the techniques of dyadic analysis and the dilated cube method. Finally, in Sect. 4, we establish the unweighted estimates, including Theorems 1.12, 1.13, and 1.14.

2. Preliminaries

2.1. Amplitudes and Phases.

In our investigation of the regularity properties of Fourier integral operators, we will be concerned with both smooth and non-smooth amplitudes and phase functions. Below, we shall recall some basic definitions and fix some notations which will be used throughout the paper.

Definition 2.1. *Let $p \in (0,\infty]$, $m \in \mathbb{R}$, $\rho, \delta \in [0,1]$, and $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^{\ell n})$.*

- *We say $a \in \nabla S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$ if for each tuple $(\alpha, \beta_1, \ldots, \beta_\ell)$ of multi-indices there exists a constant $C_{\alpha,\beta}$ such that*
$$\left|\partial_x^\alpha \partial_{\xi_1}^{\beta_1} \cdots \partial_{\xi_\ell}^{\beta_\ell} a(x, \vec{\xi})\right| \le C_{\alpha,\beta}(1+|\vec{\xi}|)^{m-\rho\sum_{j=1}^{\ell}|\beta_j|+\delta|\alpha|}. \tag{2.1}$$
- *We say $a \in \nabla S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$ if for each tuple $(\alpha, \beta_1, \ldots, \beta_\ell)$ of multi-indices with $|\alpha| \in \{0,1\}$, there exists a constant $C_{\alpha,\beta}$ such that (2.1) holds.*
- *We say $a \in L^p S^m_\rho(\mathbb{R}^n \times \mathbb{R}^{n\ell})$ if for each tuple $(\beta_1, \ldots, \beta_\ell)$ of multi-indices there exists a constant C_β such that*
$$\left\|\partial_{\xi_1}^{\beta_1} \cdots \partial_{\xi_\ell}^{\beta_\ell} a(\cdot, \vec{\xi})\right\|_{L^p(\mathbb{R}^n)} \le C_\beta(1+|\vec{\xi}|)^{m-\rho\sum_{j=1}^{\ell}|\beta_j|}.$$

Clearly, $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell}) \subsetneq \nabla S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell}) \subsetneq L^\infty S^m_\rho(\mathbb{R}^n \times \mathbb{R}^{n\ell})$.

We also need to describe the type of phase functions that we will deal with.

Definition 2.2. *A real valued function φ belongs to the class $\Phi^k(\mathbb{R}^n \times \mathbb{R}^d)$, if $\varphi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^d \setminus 0)$, is positively homogeneous of degree 1 in the frequency variable ξ, and satisfies that for any pair of multi-indices α and β, satisfying $|\alpha| + |\beta| \ge k$,*
$$\sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^d\setminus 0} |\xi|^{-1+|\beta|}|\partial_\xi^\beta \partial_x^\alpha \varphi(x,\xi)| < +\infty. \tag{2.2}$$

We also introduce the non-smooth version of the class Φ^k which is used in the classical work, cf. [13].

Definition 2.3. *A real valued function φ belongs to the phase class $L^\infty\Phi^k(\mathbb{R}^n \times \mathbb{R}^d)$, if it is positively homogeneous of degree 1 and smooth on $\mathbb{R}^d \setminus 0$ in the frequency variable ξ, bounded measurable in the spatial variable x, and if for all multi-indices $|\beta| \ge k$ it satisfies*
$$\sup_{\xi\in\mathbb{R}^d\setminus 0} |\xi|^{-1+|\beta|}\|\partial_\xi^\beta \varphi(\cdot,\xi)\|_{L^\infty(\mathbb{R}^n)} < +\infty. \tag{2.3}$$

We introduce the following new the classes of the phase functions.

Definition 2.4. *A real valued function φ belongs to the class $L^\infty\phi^k(\mathbb{R}^n \times \mathbb{R}^d)$, if $\varphi(x,\cdot) \in C^\infty(\mathbb{R}^d)$, satisfying that for any multi-indice β with $|\beta| \ge k$,*
$$\sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^d} \left|\partial_\xi^\beta \varphi(x,\xi)\right| < +\infty.$$

Definition 2.5. *A real valued function φ belongs to the class $\nabla\phi^k(\mathbb{R}^n \times \mathbb{R}^d)$, if $\varphi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^d)$, satisfying that for any multi-indices β and α with $|\alpha| = 1$ and $|\beta| \ge k$,*
$$\sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^d} \left|\partial_\xi^\beta \partial_x^\alpha \varphi(x,\xi)\right| < +\infty.$$

We next introduce the family of functions of Littlewood–Paley partition of unity.

Definition 2.6. *Let $\psi_k \in \mathscr{S}$ for any $k \in \mathbb{N}_0$. We say $\{\psi_k\}_{k\in\mathbb{N}_0}$ is a Littlewood–Paley partition of unity, if*

- *for any $k \in \mathbb{N}$, $\operatorname{supp}(\widehat{\psi_k}) \subseteq \{\xi : 2^{k-1} \le |\xi| \le 2^{k+1}\}$;*
- *$\widehat{\psi_0} \in C_c^\infty(B(0,2))$ such that $\widehat{\psi_0}(\vec{\xi}) = 1$ for any $\vec{\xi} \in B(0,1)$, and $\widehat{\psi_0}(\vec{\xi}) \in [0,1]$ for any $\vec{\xi} \in B(0,2)$;*

- $\sum_{k=0}^{\infty} \widehat{\psi_k}(\xi) = 1, \quad \forall \xi \in \mathbb{R}^n.$

We present the main technique of our proof, called generalized decay estimates for the rough amplitudes.

Proposition 2.7. *Suppose that $\rho \in [0,1]$ and $m \in \mathbb{R}$. If $\sigma \in L^\infty S^m_\rho(\mathbb{R}^n \times \mathbb{R}^{n\ell})$ and $\overline{\Psi} \in L^\infty \phi^1(\mathbb{R}^n \times \mathbb{R}^{n\ell})$, then for any $\mathcal{N} \geq 0$, any $j \in \mathbb{N}_0$, any $i = 1, \cdots, \ell$, and any $v = 1, \cdots, n$,*

$$\sup_{x,y \in \mathbb{R}^n} |\vec{y}|^{\mathcal{N}} \left| \int_{\mathbb{R}^{n\ell}} a(x, \vec{\xi}) \widehat{\psi_j}(\vec{\xi}) e^{i\vec{y}\cdot\vec{\xi}} d\vec{\xi} \right| \lesssim_{\mathcal{N}} 2^{j(n\ell + m - \rho\mathcal{N})}, \tag{2.4}$$

$$\sup_{x,y \in \mathbb{R}^n} |\vec{y}|^{\mathcal{N}} \left| \int_{\mathbb{R}^{n\ell}} (\xi_i)_v a(x, \vec{\xi}) \widehat{\psi_j}(\vec{\xi}) e^{i\vec{y}\cdot\vec{\xi}} d\vec{\xi} \right| \lesssim_{\mathcal{N}} 2^{j(n\ell + m + 1 - \rho\mathcal{N})}, \tag{2.5}$$

where $\vec{y} := (y, \cdots, y)$, $a(x, \vec{\xi}) := \sigma(x, \vec{\xi}) e^{i\overline{\Psi}(x, \vec{\xi})}$, and $\{\psi_j\}_{j \in \mathbb{N}_0}$ is a Littlewood–Paley partition of unity, cf. Definition 2.6.

Proof. We firstly claim that for any $j \in \mathbb{N}_0$ and any $\beta \in \mathbb{N}_0^{n\ell}$,

$$\left| \partial^\beta_{\vec{\xi}} \left(a(x, \vec{\xi}) \widehat{\psi_j}(\vec{\xi}) \right) \right| \lesssim \left\langle \vec{\xi} \right\rangle^{m - \rho|\beta|}. \tag{2.6}$$

If $\mathcal{N} \in \mathbb{N}_0$, integration by parts and (2.6) give that

$$\begin{aligned} &\sup_{x,y \in \mathbb{R}^n} |\vec{y}|^{\mathcal{N}} \left| \int_{\mathbb{R}^{n\ell}} a(x, \vec{\xi}) \widehat{\psi_j}(\vec{\xi}) e^{i\vec{y}\cdot\vec{\xi}} d\vec{\xi} \right| \\ \lesssim_N &\sup_{x,y \in \mathbb{R}^n} \sum_{|\alpha| = \mathcal{N}} |\vec{y}^\alpha| \left| \int_{\mathbb{R}^{n\ell}} a(x, \xi) \widehat{\psi_j}(\vec{\xi}) \partial^\alpha_{\vec{\xi}} \left(e^{i\vec{y}\cdot\vec{\xi}} \right) d\xi \right| \\ = &\sup_{x,y \in \mathbb{R}^n} \sum_{|\alpha| = \mathcal{N}} \left| \int_{\mathbb{R}^{n\ell}} \partial^\alpha_{\vec{\xi}} \left(a(x, \vec{\xi}) \widehat{\psi_j}(\vec{\xi}) \right) e^{i\vec{y}\cdot\vec{\xi}} d\vec{\xi} \right| \lesssim 2^{j(n\ell + m - \rho\mathcal{N})}. \end{aligned} \tag{2.7}$$

For any $\mathcal{N} \geq 0$, there exists a $\theta \in [0,1]$ and $k \in \mathbb{N}_0$, such that $\mathcal{N} = \theta k + (1-\theta)(k+1)$. Using (2.7) for k and $k+1$, we have

$$\begin{aligned} \sup_{x,y \in \mathbb{R}^n} |\vec{y}|^{\mathcal{N}} |K_j(x, \vec{y})| &= \sup_{x,y \in \mathbb{R}^n} \left(|\vec{y}|^k |K_j(x, \vec{y})| \right)^\theta \left(|\vec{y}|^{k+1} |K_j(x, \vec{y})| \right)^{1-\theta} \\ &\lesssim \left(2^{j(n\ell + m - \rho k)} \right)^\theta \left(2^{j(n\ell + m - \rho(k+1))} \right)^{1-\theta} \\ &= 2^{j(n\ell + m)} \cdot 2^{-j\rho[\theta k + (1-\theta)(k+1)]} = 2^{j(n\ell + m - \rho\mathcal{N})}, \end{aligned}$$

where $K_j(x, \vec{y}) = \int_{\mathbb{R}^{n\ell}} a(x, \vec{\xi}) \widehat{\psi_j}(\vec{\xi}) e^{i\vec{y}\cdot\vec{\xi}} d\vec{\xi}$.

Thus, (2.4) is valid.

We now prove (2.6) as follows. Let $\overline{\Psi}(x, \vec{\xi}) := \Psi\left(x, \vec{\xi}\right) - (x, \cdots, x) \cdot \vec{\xi}$. For any $\alpha \in \mathbb{N}_0^{n\ell}$, Note that

$$\partial^\alpha_{\vec{\xi}} a(x, \vec{\xi}) = \sum_{\gamma + \gamma' = \alpha} C_{\gamma, \gamma'} \partial^\gamma_{\vec{\xi}} \sigma(x, \vec{\xi}) \cdot \partial^{\gamma'}_{\vec{\xi}} e^{i\overline{\Psi}(x, \vec{\xi})},$$

and $\partial_{\vec{\xi}}^{\gamma'} e^{i\overline{\Psi}(x,\vec{\xi})} = e^{i\overline{\Psi}(x,\vec{\xi})} Q_{\gamma'}(x,\vec{\xi})$ with

$$Q_{\gamma'}(x,\vec{\xi}) := \sum_{k=1}^{|\gamma'|} \sum_{\substack{\gamma_1+\cdots+\gamma_k=\gamma' \\ |\gamma_1|,\cdots,|\gamma_k|\geq 1}} C_{k,\gamma_1,\ldots,\gamma_k} \prod_{j=1}^{k} \partial_{\vec{\xi}}^{\gamma_j} \overline{\Psi}\left(x,\vec{\xi}\right).$$

Due to $|\gamma_1|,\cdots,|\gamma_k| \geq 1$ and the definition of $L^\infty \phi^1(\mathbb{R}^n \times \mathbb{R}^{n\ell})$, we have that

$$\left|\partial_{\vec{\xi}}^{\gamma'} e^{i\overline{\Psi}(x,\vec{\xi})}\right| \lesssim 1. \tag{2.8}$$

On the one hand, (2.8) implies the fact that

$$\left|\partial_{\vec{\xi}}^{\alpha} a(x,\vec{\xi})\right| \lesssim \left|\sum_{\gamma\leq\alpha} C_\gamma \partial_{\vec{\xi}}^{\gamma} \sigma(x,\vec{\xi})\right| \lesssim \left\langle\vec{\xi}\right\rangle^m.$$

On the other hand, since $\widehat{\psi_j} \in \mathscr{S}$, for any $\alpha', \beta \in \mathbb{N}_0^{n\ell}$, we have that $\left|\partial_{\vec{\xi}}^{\alpha'} \widehat{\psi_j}(\vec{\xi})\right| \lesssim \left\langle\vec{\xi}\right\rangle^{-\rho|\beta|}$.

Thus, these two aspects can give that for any $\beta \in \mathbb{N}_0^{n\ell}$,

$$\left|\partial_{\vec{\xi}}^{\beta}\left(a(x,\vec{\xi})\widehat{\psi_j}(\vec{\xi})\right)\right| = \left|\sum_{\alpha+\alpha'=\beta} C_{\alpha,\alpha'} \partial_{\vec{\xi}}^{\alpha} a(x,\vec{\xi}) \cdot \partial_{\vec{\xi}}^{\alpha'} \widehat{\psi_j}(\vec{\xi})\right| \lesssim \left\langle\vec{\xi}\right\rangle^{m-\rho|\beta|}, \tag{2.9}$$

which means that (2.6) holds.

Finally, similar to before, (2.5) follows from the fact that $(\xi_i)_v a_j(z,\vec{\xi}) \in L^\infty S_\rho^{m+1}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$. □

Based on Proposition 2.7, we also give the following decay estimate.

Proposition 2.8. *Suppose that $\rho, \delta \in [0,1]$ and $m \in \mathbb{R}$. If $\sigma \in \nabla S_{\rho,\delta}^m(\mathbb{R}^n \times \mathbb{R}^{n\ell})$ and $\overline{\Psi} \in (L^\infty\phi^1 \cap \nabla\phi^0)(\mathbb{R}^n \times \mathbb{R}^{n\ell})$, then for any $\mathcal{N} \geq 0$ and any $v = 1,\cdots,n$,*

$$\sup_{x,y\in\mathbb{R}^n} |\vec{y}|^{\mathcal{N}} \left|\int_{\mathbb{R}^{n\ell}} \partial_{x_v} a(x,\vec{\xi})\widehat{\psi_j}(\vec{\xi}) e^{i\vec{y}\cdot\vec{\xi}} d\vec{\xi}\right| \lesssim_{\mathcal{N}} 2^{j(n\ell+m+1-\rho\mathcal{N})}, \tag{2.10}$$

where $\vec{y} := (y,\cdots,y)$, $a(x,\vec{\xi}) := \sigma(x,\vec{\xi})e^{i\overline{\Psi}(x,\vec{\xi})}$, and $\{\psi_j\}_{j\in\mathbb{N}_0}$ is a Littlewood–Paley partition of unity, cf. Definition 2.6.

Proof. Similar to (2.6), we only need to show that for any $\beta \in \mathbb{N}_0^{n\ell}$,

$$\left|\partial_{\vec{\xi}}^{\beta}\left(\partial_{x_v} a(x,\vec{\xi})\widehat{\psi_j}(\vec{\xi})\right)\right| \lesssim \left\langle\vec{\xi}\right\rangle^{m+\delta-\rho|\beta|}. \tag{2.11}$$

We write

$$\partial_{x_v} a_j(x,\vec{\xi}) := \left(\partial_{x_v}\sigma(x,\vec{\xi})\right) e^{i\overline{\Psi}(x,\vec{\xi})} + i\left(\sigma(x,\vec{\xi})\partial_{x_v}\overline{\Psi}(x,\vec{\xi})\right) e^{i\overline{\Psi}(x,\vec{\xi})} =: b_1(x,\vec{\xi}) + ib_2(x,\vec{\xi}).$$

Note that $\partial_{x_v}\sigma(x,\vec{\xi}) \in \nabla S_{\rho,\delta}^{m+\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$, and

$$\partial_{\vec{\xi}}^{\alpha} b_1(x,\vec{\xi}) = \sum_{\gamma+\gamma'=\alpha} C_{\gamma,\gamma'} \partial_{\vec{\xi}}^{\gamma} \partial_{x_v}\sigma(x,\vec{\xi}) \cdot \partial_{\vec{\xi}}^{\gamma'} e^{i\overline{\Psi}(x,\vec{\xi})}.$$

Similar to the method of getting (2.9), we have for any $\beta \in \mathbb{N}_0^{n\ell}$,

$$\left|\partial_{\vec{\xi}}^{\beta}\left(b_1(x,\vec{\xi})\widehat{\psi_j}(\vec{\xi})\right)\right| \lesssim \left\langle\vec{\xi}\right\rangle^{m+\delta-\rho|\beta|}. \tag{2.12}$$

Next, it suffices to prove that for any $\beta \in \mathbb{N}_0^{n\ell}$,

$$\left|\partial_{\vec{\xi}}^{\beta}\left(b_2(x,\vec{\xi})\widehat{\psi_j}(\vec{\xi})\right)\right| \lesssim \left\langle\vec{\xi}\right\rangle^{m+1-\rho|\beta|}. \tag{2.13}$$

On the one hand, the Leibniz's rule and (2.8) give that for any $\beta_1 \in \mathbb{N}_0^{n\ell}$,

$$\left|\partial_{\vec{\xi}}^{\beta_1} b_2(x,\vec{\xi})\right| = \left|\sum_{\gamma_1+\gamma_2+\gamma_3=\beta_1} C_{\gamma_1,\gamma_2,\gamma_3}\partial_{\vec{\xi}}^{\gamma_1}\sigma(x,\vec{\xi})\cdot\partial_{\vec{\xi}}^{\gamma_2}\partial_{x_v}\overline{\Psi}(x,\vec{\xi})\cdot\partial_{\vec{\xi}}^{\gamma_3}e^{i\overline{\Psi}(x,\vec{\xi})}\right| \lesssim \left\langle\vec{\xi}\right\rangle^{m+1},$$

where we used the fact that for any $\gamma_2 \in \mathbb{N}_0^{n\ell}$ and any $x \in \mathbb{R}^n$, $\vec{\xi} \in \mathbb{R}^{n\ell}$,

$$\left|\partial_{\vec{\xi}}^{\gamma_2}\partial_{x_v}\bar{\Psi}(x,\vec{\xi})\right| \lesssim 1. \tag{2.14}$$

On the other hand, since $\widehat{\psi_j} \in \mathscr{S}$, for any $\beta_2, \beta \in \mathbb{N}_0^{n\ell}$, we have that $\left|\partial_{\vec{\xi}}^{\beta_2}\widehat{\psi_j}(\vec{\xi})\right| \lesssim \left\langle\vec{\xi}\right\rangle^{-\rho|\beta|}$. Therefore, (2.13) derives from the Leibniz's rule and the two aspects as above. □

Remark 2.9. *By simple calculation, we note that*

$$\left\{\sum_{j=1}^{v} a_j(x)(\vec{\lambda}_j\cdot\vec{\xi}) : \vec{\lambda}_j \in \mathbb{R}^{n\ell}, a_j \in C^\infty \cap L^\infty(\mathbb{R}^n), \nabla a_j \in L^\infty(\mathbb{R}^n)\right\}$$

$$\subsetneq \left\{\sum_{j=1}^{v} a_j(x)h_j(\vec{\lambda}_j\cdot\vec{\xi}) : \vec{\lambda}_j \in \mathbb{R}^{n\ell}, a_j \in C^\infty \cap L^\infty(\mathbb{R}^n), \nabla a_j \in L^\infty(\mathbb{R}^n), {h_j}^{(k)} \in L^\infty(\mathbb{R}) \text{ for any } k \in \mathbb{N}\right\}$$

$$\subsetneq (L^\infty\phi^1 \cap \nabla\phi^0)(\mathbb{R}^n \times \mathbb{R}^{n\ell}).$$

From this, we can see that the class $L^\infty\phi^1 \cap \nabla\phi^0$ contains a large number of nonlinear functions for the frequency variable $\vec{\xi}$.

2.2. **Dyadic analysis.**

We call $\mathcal{D}(Q)$ the dyadic grid obtained by repeatedly subdividing Q and its descendants into 2^n cubes.

Definition 2.10. *A dyadic lattice $\mathcal{D}$ in $\mathbb{R}^n$ is a family of cubes that satisfies the following properties:*

(1) If $Q \in \mathcal{D}$ then each descendant of Q is in $\mathcal{D}$ as well.
(2) For every two cubes $Q_1, Q_2 \in \mathcal{D}$, we can find a common ancestor, that is, a cube $Q \in \mathcal{D}$ such that $Q_1, Q_2 \in \mathcal{D}(Q)$.
(3) $\mathcal{D}$ has the regularity property, i.e. for every compact set $K \subset \mathbb{R}^n$, there exists a cube $Q \in \mathcal{D}$ such that $K \subseteq Q$.

Define the grand maximal truncated operator as

$$\mathcal{M}_T(\vec{f})(x) := \sup_{Q\in\mathcal{D}} \operatorname*{ess\,sup}_{\xi\in Q}\left|T(\vec{f})(\xi) - T(\vec{f}1_{3Q})(\xi)\right| \cdot 1_Q(x).$$

Given a cube Q_0 and $x \in Q_0$, the localized grand maximal truncated operator is defined as

$$\mathcal{M}_{T,Q_0}(\vec{f})(x) := \sup_{Q\in\mathcal{D},Q\subseteq Q_0} \operatorname*{ess\,sup}_{\xi\in Q}\left|T(\vec{f}1_{3Q_0})(\xi) - T(\vec{f}1_{3Q})(\xi)\right| \cdot 1_Q(x).$$

For a dyadic lattice $\mathcal{D}$ and $r_1, \cdots, r_\ell \in [1,\infty)$, define the dyadic multi-sublinear maximal operator $\mathcal{M}_{\vec{r}}$ by

$$\mathcal{M}_{\vec{r}}(\vec{f})(x) := \sup_{Q\in\mathcal{D}} \prod_{i=1}^{\ell} \langle |f_i| \rangle_{r_i,Q} \cdot 1_Q(x). \tag{2.15}$$

When $r_1 = \cdots = r_\ell = r$, we denote it by $\mathcal{M}_r$. Note that when $r = 1$, $\mathcal{M}_1$ is the multi-sublinear maximal operator $\mathcal{M}$ introduced in [33].

3. Proof of Sparse bounds criterion

This section is dedicated to proving Theorem 1.7. We firstly need the following sparse domination principle.

Theorem 3.1. *Let $q_1, \cdots, q_\ell \in [1,\infty)$ with $\frac{1}{q} := \sum_{i=1}^{\ell} \frac{1}{q_i}$, $r_1, \cdots, r_\ell \in [1,\infty)$ with $\frac{1}{r} := \sum_{i=1}^{\ell} \frac{1}{r_i}$, and $q_i \ge r_i$ for every $i = 1, \ldots, \ell$. If T is bounded from $L^{r_1} \times \cdots \times L^{r_\ell}$ to $L^{r,\infty}$ and $\mathcal{M}_T$ is bounded from $L^{q_1} \times \cdots \times L^{q_\ell}$ to $L^{q,\infty}$, then for any $f_i \in L_c^{q_i}(\mathbb{R}^n)$, $i = 1, \cdots, \ell$, there exists a sparse family $\mathcal{S}$ such that for almost every $x \in \mathbb{R}^n$,*

$$\left|T(\vec{f})(x)\right| \lesssim \mathcal{A}_{\mathcal{S},\vec{r}}(\vec{f})(x) \tag{3.1}$$

where

$$C_T = c_{n,\vec{p},\vec{r}} \left(\|T\|_{L^{r_1}\times\cdots\times L^{r_\ell}\to L^{r,\infty}} + \|\mathcal{M}_T\|_{L^{q_1}\times\cdots\times L^{q_\ell}\to L^{q,\infty}} \right).$$

The proof of Theorem 3.1 is standard, and we can get it simply by making a slight modification for [27, Theorem 3.1] and [31, Theorem 1.1].

Based on Remark 2.9, we next establish the $\mathcal{M}_T$ theorem for $T_{\sigma,\Psi}$ as follows, which is our key result in this section.

Theorem 3.2. *Let $r_1, \cdots, r_\ell \in [1,\infty)$ with $\vec{r} := (r_1, \cdots, r_\ell)$, and $\frac{1}{r} := \sum_{i=1}^{\ell} \frac{1}{r_i}$. Suppose that $\rho \in (0,1]$, $\delta \in [0,1]$, $m \in (-\infty, 0]$, $\sigma \in \nabla S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^{n\ell})$, and $\overline{\Psi} \in (L^\infty\phi^1 \cap \nabla\phi^0)(\mathbb{R}^n \times \mathbb{R}^{n\ell})$. If $T_{\sigma,\Psi}$ is bounded from $L^{r_1} \times \cdots \times L^{r_\ell}$ to $L^{r,\infty}$, then for any $x \in \mathbb{R}^n$ and $s \in (0,r)$,*

$$\mathcal{M}_{T_{\sigma,\Psi}}(\vec{f})(x) \lesssim \left(\|T_{\sigma,\Psi}\|_{\prod_{i=1}^{\ell} L^{r_i}\to L^{r,\infty}} + 1 \right) \mathcal{M}_{\vec{r}}(\vec{f})(x) + M_s(T_{\sigma,\Psi}(\vec{f}))(x). \tag{3.2}$$

Furthermore, (3.2) and [14, Exercise 2.1.13] yield that

$$\left\|\mathcal{M}_{T_{\sigma,\Psi}}\right\|_{\prod_{i=1}^{\ell} L^{r_i}\to L^{r,\infty}} \lesssim \|T_{\sigma,\Psi}\|_{\prod_{i=1}^{\ell} L^{r_i}\to L^{r,\infty}} + 1.$$

Proof of Theorem 1.7. Theorem 1.7 follows from combining Theorems 3.1 and 3.2. □

Proof of Theorem 3.2.

It suffices to prove for the PDO $\mathscr{T}_a$ as in (1.1). We present the **dilated cubes trick** as follows. Given a cube Q, we define the concentric dilated cube Q_β satisfying $l(Q_\beta) = l(Q)^\beta$, where $\beta \in [1,\infty)$ is a parameter to be chosen later. Let $x \in Q$, define $E := E_1 \cap E_2$, where

$$E_1 := \left\{ y \in Q : |\mathscr{T}_a(\vec{f} 1_{2Q})(y)| \le 2^{\frac{n+2}{r}} \|\mathscr{T}_a\| \prod_{i=1}^{\ell} \langle |f_i| \rangle_{r_i,2Q} \right\};$$

$$E_2 := \left\{ y \in Q : |\mathscr{T}_a(\vec{f})(y)| \leq 4^{1/s} M_s(\mathscr{T}_a(\vec{f}))(x) \right\}.$$

The weak-type boundedness and the definition of maximal operator M_s yield that

$$\begin{aligned}|Q\backslash E| \leq& |Q\backslash E_1| + |Q\backslash E_2| \\ \leq& \frac{\left\|\mathscr{T}_a(\vec{f}\cdot 1_{2Q})\right\|^r_{L^{r,\infty}(Q)}}{\left(2^{\frac{n+2}{r}}\|\mathscr{T}_a\| \prod_{i=1}^{\ell}\langle |f_i|\rangle_{r_i,2Q}\right)^r} + \frac{\left\|\mathscr{T}_a(\vec{f})\right\|^s_{L^{s,\infty}(Q)}}{\left(4^{1/s}M_s(\mathscr{T}_a(\vec{f}))(x)\right)^s} \leq \frac{1}{2}|Q|.\end{aligned}$$

This gives that $|E| > \frac{1}{2}|Q| > 0$.

For any $z \in Q$, we define the good set $E_\beta(z) := E \cap B(z, 2\sqrt{n}l(Q_\beta))$. We claim that $E_\beta(z)$ is non-empty for a.e. $z \in Q$. In fact, if $l(Q) \geq 1$, then $|E_\beta(z)| = |E| > 0$. If $l(Q) < 1$, it derives from the Lebesgue density theorem that there is a $r \in (0, l(Q_\beta)) \subsetneq (0,1)$, for a.e. $z \in Q$,

$$|E_\beta(z)| \geq |E \cap B(z,r)| \geq \frac{1}{2}|B(z,r)| \approx r^n > 0.$$

For any $z \in Q$ and any $z' \in E_\beta(z)$,

$$|\mathscr{T}_a(\vec{f})(z) - \mathscr{T}_a(\vec{f}1_{2Q})(z)| \leq |\mathscr{T}_a(\vec{f})(z')| + |\mathscr{T}_a(\vec{f}1_{2Q})(z')| + J_a(z,z') \tag{3.3}$$

where

$$J_a(z,z') := \left|(\mathscr{T}_a(\vec{f})(z) - \mathscr{T}_a(\vec{f}1_{2Q})(z)) - \left(\mathscr{T}_a(\vec{f})(z') - \mathscr{T}_a(\vec{f}1_{2Q})(z')\right)\right|. \tag{3.4}$$

If we can prove that any $z \in Q$ and any $z' \in E_\beta(z)$,

$$J_a(z,z') \lesssim \mathcal{M}_r(f)(x), \tag{3.5}$$

then it follows from the definition of $E_\beta(z)$ and Hölder's inequality that for any $x \in Q$, a.e. $z \in Q$, and any $z' \in E_\beta(z)$,

$$\begin{aligned}&|\mathscr{T}_a(\vec{f})(z) - \mathscr{T}_a(\vec{f}1_{2Q})(z)| \\ \leq& |\mathscr{T}_a(f)(z')| + |\mathscr{T}_a(f1_{2Q})(z')| + J_a(z,z') \\ \lesssim& M_s(\mathscr{T}_a(\vec{f}))(x) + \|\mathscr{T}_a\|\langle f\rangle_{r,2Q} + M_r(\vec{f})(x) \\ \lesssim& M_s(\mathscr{T}_a(\vec{f}))(x) + (\|\mathscr{T}_a\|+1)M_r(\vec{f})(x),\end{aligned}$$

which proves that (3.2) is valid by replace Q with $\frac{3}{2}Q$..

To prove the fact (3.5), we first make the following observation.

$$\begin{aligned}J_a(z,z') \leq& \sum_{j=0}^{\infty}\left|\left(T_{a_j}(\vec{f})(z) - T_{a_j}(\vec{f}1_{2Q})(z)\right) - \left(T_{a_j}(\vec{f})(z') - T_{a_j}(\vec{f}1_{2Q})(z')\right)\right| \\ =:& \left(\sum_{j:1\leq 2^j l(Q_\beta)} + \sum_{j:2^j l(Q_\beta)<1}\right) J_{a_j}(z,z'), \\ =:& J_{high}(z,z') + J_{low}(z,z'),\end{aligned} \tag{3.6}$$

where the decomposition is based on a Littlewood–Paley partition of unity $\{\psi_j\}_{j\in\mathbb{N}_0}$ (Definition 2.6), with $a_j(x,\vec{\xi}) := a(x,\vec{\xi})\widehat{\psi_j}(\vec{\xi})$ so that $a(x,\vec{\xi}) = \sum_{j=0}^{\infty} a_j(x,\vec{\xi})$.

Estimate for Low frequency part:

We find that

$$J_{a_j}(z,z') = \left| \int_{(\mathbb{R}^n\setminus 2Q)^\ell} \int_{\mathbb{R}^{\ell n}} \left(a_j(z,\vec{\xi}) e^{i\sum_{k=1}^{\ell}(z-y_k)\cdot\xi_k} - a_j(z',\vec{\xi}) e^{i\sum_{k=1}^{\ell}(z'-y_k)\cdot\xi_k} \right) d\vec{\xi} \prod_{i=1}^{\ell} f_i(y_i) d\vec{y} \right|.$$

By the mean value theorem, the Leibniz's rule, and Propositions 2.7, 2.8, we have

$$\begin{aligned}
&J_{a_j}(z,z')\\
&\le \sum_{k=1}^{\infty} \int_{(2^{k+1}Q)^\ell\setminus(2^kQ)^\ell} \left| \int_{\mathbb{R}^{\ell n}} \left(a_j(z,\vec{\xi}) e^{i\sum_{i=1}^{\ell}(z-y_i)\cdot\xi_i} - a_j(z',\vec{\xi}) e^{i\sum_{i=1}^{\ell}(z'-y_i)\cdot\xi_i} \right) d\vec{\xi} \right| \prod_{i=1}^{\ell} |f_i(y_i)| d\vec{y}\\
&= \sum_{k=1}^{\infty} \int_{(2^{k+1}Q)^\ell\setminus(2^kQ)^\ell} \left| \int_{\mathbb{R}^{\ell n}} \int_0^1 \nabla_z \left(a_j(z(t),\vec{\xi}) e^{i\sum_{i=1}^{\ell}(z(t)-y_i)\cdot\xi_i} \right) \cdot (z-z') dt d\vec{\xi} \right| \prod_{i=1}^{\ell} |f_i(y_i)| d\vec{y}\\
&= \sum_{k=1}^{\infty} \int_{(2^{k+1}Q)^\ell\setminus(2^kQ)^\ell}\\
&\left| \int_{\mathbb{R}^{\ell n}} \sum_{v=1}^{n} (z_v - z'_v) \int_0^1 t \left(\partial_{z_v} a_j(z(t),\vec{\xi}) + i \left(\sum_{i=1}^{\ell} \xi_i \right)_v a_j(z(t),\vec{\xi}) \right) e^{i\sum_{i=1}^{\ell}(z(t)-y_i)\cdot\xi_i} dt d\vec{\xi} \right| \prod_{i=1}^{\ell} |f_i(y_i)| d\vec{y}\\
&\lesssim \sum_{k=1}^{\infty} \int_{(2^{k+1}Q)^\ell\setminus(2^kQ)^\ell}\\
&|z-z'| \int_0^1 t \sum_{v=1}^{n} \left| \int_{\mathbb{R}^{\ell n}} \left(\partial_{z_v} a_j(z(t),\vec{\xi}) + i \sum_{i=1}^{\ell} (\xi_i)_v a_j(z(t),\vec{\xi}) \right) e^{i\sum_{i=1}^{\ell}(z(t)-y_i)\cdot\xi_i} d\vec{\xi} \right| dt \prod_{i=1}^{\ell} |f_i(y_i)| d\vec{y}\\
&\lesssim \sum_{k=1}^{\infty} \int_{(2^{k+1}Q)^\ell\setminus(2^kQ)^\ell} |z-z'| \int_0^1 \frac{t \cdot 2^{j(\ell n+m-\rho\mathcal{N}+1)}}{\left(\sum_{i=1}^{\ell} |y_i - z(t)|\right)^{\mathcal{N}}} dt \prod_{i=1}^{\ell} |f_i(y_i)| d\vec{y}\\
&\lesssim \sum_{k=1}^{\infty} \int_{(2^{k+1}Q)^\ell\setminus(2^kQ)^\ell} l(Q_\beta) \frac{2^{j(\ell n+m-\rho\mathcal{N}+1)}}{(2^k l(Q))^{\mathcal{N}}} \prod_{i=1}^{\ell} |f_i(y_i)| d\vec{y}\\
&\lesssim \left(2^{j(\ell n+m-\rho\mathcal{N}+1)} l(Q)^{\ell n-\mathcal{N}+\beta} \right) \left(\sum_{k=1}^{\infty} 2^{k(\ell n-\mathcal{N})} \right) \prod_{i=1}^{\ell} \langle |f_i| \rangle_{1,2^{k+1}Q}\\
&\lesssim \left(2^{j(\ell n+m-\rho\mathcal{N}+1)} l(Q)^{\ell n-\mathcal{N}+\beta} \right) \left(\sum_{k=1}^{\infty} 2^{k(\ell n-\mathcal{N})} \right) \mathcal{M}_r(\vec{f})(x),
\end{aligned}$$

where $z(t) = (1-t)z' + tz$ with $t \in [0,1]$, and we used the fact $\sum_{i=1}^{\ell} |y_i - z(t)| \approx |y_j - z(t)| \approx 2^k l(Q)$ since $z(t) \in Q$ and $y_j \in 2^{k+1}Q \setminus 2^kQ$ for some $j \in \{1, \cdots, \ell\}$.

Therefore, we have

$$\begin{aligned}
J_{low}(z,z') &= \sum_{j:2^j l(Q_\beta)<1} J_{a_j}(z,z')\\
&\lesssim \left(\sum_{j:2^j l(Q_\beta)<1} 2^{j(\ell n+m-\rho\mathcal{N}+1)} l(Q)^{\ell n-\mathcal{N}+\beta} \right) \left(\sum_{k=1}^{\infty} 2^{k(\ell n-\mathcal{N})} \right) \mathcal{M}_r(\vec{f})(x), \qquad (3.7)
\end{aligned}$$

For any $j \in \mathbb{N}_0$ with $1 \leq 2^j < l(Q_\beta)^{-1}$ and any $\beta > 0$, we only need to consider $l(Q_\beta), l(Q) < 1$. We now proceed by considering two situations:

- $m \leq \ell n(\rho - 1) - 1$;
- $\ell n(\rho - 1) - 1 < m \leq 0$.

Situation 1: $\ell n(\rho - 1) - 1 < m \leq 0$**.**

Now, (3.7) requires that we find a $\mathcal{N} \geq 0$ such that $\mathcal{N} > \ell n$ and

$$\sum_{j:2^j l(Q_\beta)<1} 2^{j(\ell n+m-\rho\mathcal{N}+1)} l(Q)^{\ell n-\mathcal{N}+\beta} \lesssim_N l(Q)^{\ell n-\mathcal{N}-\beta(\ell n+m-\rho\mathcal{N})} \lesssim_N 1. \tag{3.8}$$

In this situation, a sufficient condition for (3.8) is the existence of a $\mathcal{N} \geq 0$, satisfying the following three conditions.

$$(1)\, \ell n + m - \rho\mathcal{N} + 1 \geq 0; \quad (2)\, \ell n - \mathcal{N} - \beta(\ell n + m - \rho\mathcal{N}) \geq 0; \quad (3)\, \mathcal{N} > \ell n. \tag{3.9}$$

The conditions (3.9) are implied by the stronger condition:

$$\max\left\{\ell n, \frac{\ell n\beta + \beta m}{\rho\beta - 1}\right\} < \mathcal{N} \leq \frac{m + \ell n + 1}{\rho}, \tag{3.10}$$

where we chose $\beta > 1/\rho$.

Since $\ell n(\rho - 1) - 1 < m$, it is easy to see that

$$\frac{m + \ell n + 1}{\rho} - \ell n > 0.$$

Furthermore, since $\rho \in (0, 1]$, $m \in (-\infty, 0]$, and $\eta \in [0, \ell)$, choosing

$$\beta > \frac{1 + n\ell}{\rho} > \frac{1}{\rho}, \tag{3.11}$$

ensures that

$$\frac{m + \ell n + 1}{\rho} > \frac{\ell n\beta + \beta m}{\rho\beta - 1}.$$

Consequently, there exists a $\mathcal{N} \geq 0$ such that both conditions (3.10) and (3.9).

Situation 2: $m \leq \ell n(\rho - 1) - 1$**.**

Now, (3.7) requires that we find a $\mathcal{N} \geq 0$ such that $\mathcal{N} > \ell n$ and

$$\sum_{j:2^j l(Q_\beta)<1} 2^{j(\ell n+m-\rho\mathcal{N}+1)} l(Q)^{\ell n-\mathcal{N}+\beta} \lesssim_N l(Q)^{\ell n-\mathcal{N}+\beta} \leq 1. \tag{3.12}$$

Therefore, (3.12) holds if there exists a $\mathcal{N} \geq 0$ that fulfills the following three conditions.

$$(4)\, \ell n + m - \rho\mathcal{N} + 1 \leq 0; \quad (5)\, \ell n - \mathcal{N} + \beta \geq 0; \quad (6)\, \mathcal{N} > \ell n. \tag{3.13}$$

Note that (3.13) can be yielded by the following condition:

$$\max\left\{\ell n, \frac{m + \ell n + 1}{\rho}\right\} < \mathcal{N} \leq \ell n + \beta. \tag{3.14}$$

Futhermore, (3.14) is valid because of the condition (3.11).

Thus, there exists a $\mathcal{N} \geq 0$ satisfying both conditions (3.14) and (3.13).

Estimate for high frequency part:

For any $z \in Q$, Proposition 2.7 gives that

$$
\begin{aligned}
J_{high}(z,z') &\leq 2 \cdot \sum_{j:1\leq 2^j l(Q_\beta)} \sup_{z\in Q} \left| \mathscr{T}_a(\vec{f})(z) - \mathscr{T}_a(\vec{f}1_{2Q})(z) \right| \\
&\lesssim \sum_{j:1\leq 2^j l(Q_\beta)} \sum_{k=1}^{\infty} \int_{(2^{k+1}Q)^\ell \setminus (2^kQ)^\ell} \sup_{z\in Q} \left| \int_{\mathbb{R}^{\ell n}} a_j(z,\vec{\xi}) e^{i\sum_{i=1}^{\ell}(z-y_i)\cdot\xi_i} d\vec{\xi} \right| \prod_{i=1}^{\ell} |f_i(y_i)| d\vec{y} \\
&\lesssim \sum_{j:1\leq 2^j l(Q_\beta)} \sum_{k=1}^{\infty} \int_{(2^{k+1}Q)^\ell \setminus (2^kQ)^\ell} \sup_{z\in Q} \frac{2^{j(\ell n+m-\rho\mathcal{N})}}{\left(\sum_{i=1}^{\ell} |y_i - z|\right)^{\mathcal{N}}} \prod_{i=1}^{\ell} |f_i(y_i)| d\vec{y} \\
&\lesssim \sum_{j:1\leq 2^j l(Q_\beta)} \sum_{k=1}^{\infty} \frac{2^{j(\ell n+m-\rho\mathcal{N})}}{(2^k l(Q))^{\mathcal{N}-\ell n}} \left(2^k l(Q)\right)^{n\eta} \prod_{i=1}^{\ell} \langle |f_i| \rangle_{1,2^{k+1}Q} \\
&\lesssim \left(\sum_{j:1\leq 2^j l(Q_\beta)} 2^{j(\ell n+m-\rho\mathcal{N})} l(Q)^{\ell n-\mathcal{N}} \right) \left(\sum_{k=1}^{\infty} 2^{k(\ell n-\mathcal{N})} \right) \mathcal{M}_r(\vec{f})(x).
\end{aligned} \tag{3.15}
$$

Now we require

$$
\sum_{j:1\leq 2^j l(Q_\beta)} 2^{j(\ell n+m-\rho\mathcal{N})} l(Q)^{\ell n-\mathcal{N}} \lesssim_N l(Q)^{\ell n-\mathcal{N}-\beta(\ell n+m-\rho\mathcal{N})} \leq 1. \tag{3.16}
$$

Next, we consider several cases.

Case 1: $l(Q) \leq 1$.

To guarantee this, we claim that there exists a $\mathcal{N} \geq 0$ satisfying

$$
(7)\, \ell n + m - \rho\mathcal{N} < 0; \quad (8)\, \ell n - \mathcal{N} - \beta(\ell n + m - \rho\mathcal{N}) \geq 0; \quad (9)\, \mathcal{N} > \ell n. \tag{3.17}
$$

Indeed, by taking (3.11), a sufficient condition for the above is

$$
\mathcal{N} > \max\left\{ \ell n, \frac{\ell n + m}{\rho}, \frac{\ell n(\beta-1)+\beta m}{\rho\beta - 1} \right\}. \tag{3.18}
$$

Case 2: $l(Q) > 1$.

This case is handled similarly to Case 1 and we just need to select $\mathcal{N} > \max\{\frac{n\ell+m}{\rho}, n\ell\}$ so that those corresponding terms in (3.15) have negative exponents. This choice is far simpler than in the Case 1, and hence we can obviously obtain the desired conclusion.

Consequently, we obtain the desired estimates for $J_{high}(z,z')$ and $J_{low}(z,z')$ and prove the fact (3.5) by taking β satisfies (3.11). $\square$

4. Proof of the unweighted results

In this section, we shall prove Theorems 1.12, 1.13, and 1.14 as follows. **Due to (1.1), we just need to provide the proof for $\mathscr{T}_a$ in the following.**

Proof of Theorem 1.12. Firstly, we have

$$
\mathscr{T}_a(\vec{f})(x) := \sum_{j=0}^{\infty} \mathscr{T}_{a_j}(f_1,\ldots,f_\ell)(x), \tag{4.1}
$$

where the decomposition is based on a Littlewood–Paley partition of unity $\{\psi_j\}_{j\in\mathbb{N}_0}$ (Definition 2.6), with

$$a_j(x,\vec{\xi}) := a(x,\vec{\xi})\widehat{\psi_j}(\vec{\xi}) = \sigma(x,\vec{\xi})e^{i\overline{\Psi}(x,\vec{\xi})}\widehat{\psi_j}(\vec{\xi})$$

so that $a(x,\vec{\xi}) = \sum_{j=0}^{\infty} a_j(x,\vec{\xi})$.

To bound $\mathscr{T}_a$, we rewrite

$$\mathscr{T}_{a_j}(f_1,\ldots,f_\ell)(x) = \int_{\mathbb{R}^{\ell n}} K_j(x,\vec{y})\prod_{j=1}^{\ell} f_j(x-y_j)d\vec{y},$$

where

$$K_j(x,\vec{y}) = \int_{\mathbb{R}^{\ell n}} a_j(x,\vec{\xi})e^{i\vec{y}\cdot\vec{\xi}}d\vec{\xi} = \mathcal{F}^{-1}(a_j(x,\cdot))(\vec{y}).$$

For $j\in\mathbb{N}$, we consider the regions where $|\vec{y}|\le 2^{-j\rho}$ and $|\vec{y}|>2^{-j\rho}$. Then we have

$$\int_{|\vec{y}|\le 2^{-j\rho}} 2^{j\ell n\rho}d\vec{y} + \int_{|\vec{y}|>2^{-j\rho}} 2^{j\rho(\ell n-p\kappa)}\frac{d\vec{y}}{|\vec{y}|^{p\kappa}} \lesssim 1,$$

where κ is a fixed constant satisfying $p\kappa>\ell n$. Let the multi-index $\alpha\in\mathbb{N}^n\times\cdots\times\mathbb{N}^n$ with $|\alpha|=\kappa$. It follows from Hausdorff-Young's theorem and (2.6) that

$$\begin{aligned}
&\left\|\mathscr{T}_{a_j}(f_1,\ldots,f_\ell)\right\|^r_{L^r(\mathbb{R}^n)}\\
&= \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^{\ell n}} K_j(x,\vec{y})\prod_{j=1}^{\ell} f_j(x-y_j)\,d\vec{y}\right|^r dx\\
&\lesssim \int_{\mathbb{R}^n}\left[2^{-j\ell n\rho/r}\left(\int_{\mathbb{R}^{\ell n}}|K_j(x,\vec{y})|^{p'}d\vec{y}\right)^{1/r'} + 2^{-j\rho(\ell n/r-\kappa)}\left(\int_{\mathbb{R}^{\ell n}}|\vec{y}^{\kappa}K_j(x,\vec{y})|^{p'}d\vec{y}\right)^{1/r'}\right]^r\\
&\quad\times\left(\int_{|\vec{y}|\le 2^{-j\rho}}\prod_{j=1}^{\ell}|f_j(x-y_j)|^r\,2^{j\ell n\rho}d\vec{y} + \int_{|\vec{y}|>2^{-j\rho}}\prod_{j=1}^{\ell}|f_j(x-y_j)|^r\,2^{j\rho(\ell n-p\kappa)}|\vec{y}|^{-p\kappa}d\vec{y}\right)dx\\
&\lesssim \int_{\mathbb{R}^n}\left[2^{-j\ell n\rho/r}\left(\int_{\mathbb{R}^{\ell n}}|a_j(x,\vec{\xi})|^p d\vec{\xi}\right)^{1/r} + 2^{-j\rho(\ell n/r-\kappa)}\left(\int_{\mathbb{R}^{\ell n}}|\partial_{\xi}^{\kappa}a_j(x,\vec{\xi})|^p d\vec{\xi}\right)^{1/r}\right]^r \times\cdots dx\\
&\lesssim \int_{\mathbb{R}^n}\left[2^{-j\ell n\rho/r}\left(\int_{|\vec{\xi}|\approx 2^j} 2^{jrp}d\vec{\xi}\right)^{1/r} + 2^{-j\rho(\ell n/r-\kappa)}\left(\int_{|\vec{\xi}|\approx 2^j} 2^{jp(m-\rho\kappa)}d\vec{\xi}\right)^{1/r}\right]^r\times\cdots dx\\
&\lesssim 2^{j(mp-\ell n(\rho-1))}\prod_{j=1}^{\ell}\|f_j\|^r_{L^{r_j}(\mathbb{R}^n)}\left(\int_{|\vec{y}|\le 2^{-j\rho}} 2^{j\ell n\rho}d\vec{y} + \int_{|\vec{y}|>2^{-j\rho}} 2^{j\rho(\ell n-p\kappa)}|\vec{y}|^{-p\kappa}d\vec{y}\right)\\
&\lesssim 2^{j(mp-\ell n(\rho-1))}\prod_{j=1}^{\ell}\|f_j\|^r_{L^{r_j}(\mathbb{R}^n)}.
\end{aligned}$$

For $j=0$, it can be deduced from (2.7) (taking $\mathcal{N}=0$ and $\mathcal{N}>0$) that for any $\mathcal{N}\in\mathbb{N}_0$,

$$|K_0(x,\vec{y})|\lesssim (1+|\vec{y}|)^{-\mathcal{N}}\approx\min\{1,|\vec{y}|^{-\mathcal{N}}\}.$$

Thus, it yields that for $\mathcal{N}>\ell n$

$$|\mathscr{T}_{a_0}(f_1,\ldots,f_\ell)(x)|\lesssim\int_{\mathbb{R}^{\ell n}}\frac{\prod_{j=1}^{\ell}|f_j(x-y_j)|}{(1+|\vec{y}|)^{\mathcal{N}}}d\vec{y}$$

$$
\begin{aligned}
&= \left(\int_{B(0,1)^\ell} + \sum_{k=0}^{\infty} \int_{B(0,2^{k+1})^\ell \setminus B(0,2^k)^\ell} \right) \frac{\prod_{j=1}^{\ell} |f_j(x-y_j)|}{(1+|\vec{y}|)^{\mathcal{N}}} d\vec{y} \\
&\lesssim \mathcal{M}(f_1,\ldots,f_\ell)(x), \qquad (4.2)
\end{aligned}
$$

which implies that

$$\|\mathscr{T}_{a_0}(f_1,\ldots,f_\ell)\|_{L^r(\mathbb{R}^n)} \lesssim \|\mathcal{M}(\vec{f})\|_{L^r(\mathbb{R}^n)} \lesssim \prod_{j=1}^{\ell} \|f_j\|_{L^{r_j}(\mathbb{R}^n)}.$$

Therefore, since $m < \ell n(\rho-1)/r$,

$$
\begin{aligned}
\|\mathscr{T}_a(f_1,\ldots,f_\ell)\|_{L^r(\mathbb{R}^n)} &\le \sum_{j=0}^{\infty} \|\mathscr{T}_{a_j}(f_1,\ldots,f_\ell)\|_{L^r(\mathbb{R}^n)} \\
&\lesssim \left(1+\sum_{j=1}^{\infty} 2^{j(m-\ell n(\rho-1)/r)}\right) \prod_{j=1}^{\ell} \|f_j\|_{L^{r_j}(\mathbb{R}^n)} \lesssim \prod_{j=1}^{\ell} \|f_j\|_{L^{r_j}(\mathbb{R}^n)}.
\end{aligned}
$$

□

Proof of Theorem 1.13. Due to homogeneity and multilinearity, assume that $\|f_j\|_{L^\infty(\mathbb{R}^n)} = 1$ for $j = 1,\ldots,\ell$. On the one hand, it follows from the Cauchy-Schwarz inequality, Hausdorff-Young's theorem, and (2.6) that

$$
\begin{aligned}
\left\|T_{a_j}(f_1,\ldots,f_\ell)\right\|_{L^\infty(\mathbb{R}^n)} &\le \prod_{j=1}^{\ell} \|f_j\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^{\ell n}} |K_j(x,\vec{y})|\, d\vec{y} \\
&= \int_{|\vec{y}|\le 2^{-j\rho}} + \int_{|\vec{y}|>2^{-j\rho}} |K_j(x,\vec{y})|\, d\vec{y} \\
&\le \left(\int_{|\vec{y}|\le 2^{-j\rho}} d\vec{y}\right)^{1/2} \left(\int_{\mathbb{R}^{\ell n}} |K_j(x,\vec{y})|^2\, d\vec{y}\right)^{1/2} \\
&\quad + \left(\int_{|\vec{y}|>2^{-j\rho}} |\vec{y}|^{-2\kappa} d\vec{y}\right)^{1/2} \left(\int_{\mathbb{R}^{\ell n}} |\vec{y}|^{2\kappa} |K_j(x,\vec{y})|^2\, d\vec{y}\right)^{1/2} \\
&\lesssim 2^{-j\rho\ell n/2} \cdot 2^{j(m+\ell n/2)} + 2^{-j\rho(\ell n/2-\kappa)} \cdot 2^{j(\ell n/2+m-\rho\kappa)} \\
&\approx 2^{j(m-\ell n(\rho-1)/2)}.
\end{aligned}
$$

Obviously, a sufficient condition for the convergence of the series is that $m < \ell n(\rho-1)/2$.

On the other hand, the inequality (4.2) implies that

$$\|T_{\sigma_0}(f_1,\ldots,f_\ell)\|_{L^\infty(\mathbb{R}^n)} \lesssim \prod_{j=1}^{\ell} \|f_j\|_{L^\infty(\mathbb{R}^n)}.$$ □

Proof of Theorem 1.14. Applying the Calderón-Zygmund decomposition to each f_j at height $\lambda^{1/\ell}$ yields $f_j = g_j + b_j$, where $b_j = \sum_{k_j} b_j^{(k_j)}$ over a collection of disjoint cubes $\left\{Q_j^{(k_j)}\right\}$. The following properties hold:

- For each k_j, $\operatorname{supp} b_j^{(k_j)} \subseteq Q_j^{(k_j)}$, $\int_{\mathbb{R}^n} b_j^{(k_j)} dx = 0$, and $\left\|b_j^{(k_j)}\right\|_{L^1} \lesssim \lambda^{1/m} \left|Q_j^{(k_j)}\right|$.
- The cubes satisfy $\left|\bigcup_{k_j} Q_j^{(k_j)}\right| \lesssim \lambda^{-1/\ell}$.

- The good part satisfies $\|g_j\|_{L^s} \lesssim \lambda^{\frac{1}{\ell s'}}$, for $1 \leq s \leq \infty$.

Due to homogeneity and multilinearity, assume that each $\|f_j\|_{L^1(\mathbb{R}^n)} = 1$. It is suffices to prove

$$|\{x \in \mathbb{R}^n; T_{\sigma,\Psi}(f_1, \ldots, f_\ell)(x) > \lambda\}| \leq C\lambda^{1/\ell}.$$

By symmetry, it suffices to show

(1): $\left|\left\{x \in \mathbb{R}^n; \mathscr{T}_a(g_1, \ldots, g_\ell)(x) > \lambda/2^\ell\right\}\right| \lesssim \lambda^{1/\ell}$.

(2): $\left|\left\{x \in \mathbb{R}^n; \mathscr{T}_a(b_1, g_2, \ldots, g_\ell)(x) > \lambda/2^\ell\right\}\right| \lesssim \lambda^{1/\ell}$.

(3): $\left|\left\{x \in \mathbb{R}^n; \mathscr{T}_a(b_1, \ldots, b_\kappa, g_{\kappa+1}, \ldots, g_\ell)(x) > \lambda/2^\ell\right\}\right| \lesssim \lambda^{1/\ell}$.

For **(1)**, it follows from Proposition 1.12 that

$$\begin{aligned}
&\left|\left\{x \in \mathbb{R}^n; T_{\sigma,\Psi}(g_1, \ldots, g_\ell)(x) > \lambda/2^\ell\right\}\right| \\
&\lesssim \lambda^{-2}\|T_{\sigma,\Psi}(g_1, \ldots, g_\ell)\|^2_{L^2(\mathbb{R}^n)} \lesssim \lambda^{-2}\prod_{j=1}^{\ell}\|g_j\|^2_{L^{q_j}(\mathbb{R}^n)} \\
&\lesssim \lambda^{-2}\prod_{j=1}^{\ell}\lambda^{2/\ell q_j'} = \lambda^{-1/\ell},
\end{aligned}$$

where we chose $q_1, \cdots, q_\ell$ such that $\frac{1}{2} = \sum_{j=1}^{\ell}\frac{1}{q_j}$.

For **(2)** and **(3)**, since $\left|\bigcup_{j=1}^{\ell}\bigcup_{k_j} 2Q_j^{(k_j)}\right| \lesssim \lambda^{-1/\ell}$, it suffices to show

$$\left|\left\{x \notin \bigcup_{k_1} 2Q_1^{(k_1)}; T_{\sigma,\Psi}(b_1, g_2, \ldots, g_\ell)(x) > \lambda/2^\ell\right\}\right| \lesssim \lambda^{-1/\ell},$$

and

$$\left|\left\{x \notin \bigcup_{j=1}^{\ell}\bigcup_{k_j} 2Q_j^{(k_j)}; T_{\sigma,\Psi}(b_1, \ldots, b_\kappa, g_{\kappa+1}, \ldots, g_\ell)(x) > \lambda/2^\ell\right\}\right| \lesssim \lambda^{-1/\ell}.$$

By considering each term in the sum $b_j = \sum_{k_j} b_j^{(k_j)}$, we only need to prove that

$$\int_{\mathbb{R}^n \setminus 2Q_1^{(k_1)}} \left|T_{\sigma,\Psi}\left(b_1^{(k_1)}, g_2, \ldots, g_\ell\right)(x)\right| dx \lesssim \lambda \left|Q_1^{(k_1)}\right| \tag{4.3}$$

and

$$\int_{\mathbb{R}^n \setminus \bigcup_{j,k_j} 2Q_j^{(k_j)}} |T_{\sigma,\Psi}(b_1, \ldots, b_\kappa, g_{\kappa+1}, \ldots, g_\ell)(x)|^{1/\kappa} dx \lesssim \lambda^{1/\kappa - 1/\ell}, \tag{4.4}$$

where the implied constants are independent of j and k_j.

We find that inequality (4.3) is a direct consequence of Lemma 4.1. Moreover, we use the subsequent Lemma 4.2 to obtain

$$\int_{\mathbb{R}^n \setminus \bigcup_{j,k_j} 2Q_j^{(k_j)}} |T_{\sigma,\Psi}(b_1, \ldots, b_\kappa, g_{\kappa+1}, \ldots, g_\ell)(x)|^{1/\kappa} dx$$

$$
\begin{aligned}
&\leq \int_{\mathbb{R}^n \setminus \bigcup_{j,k_j} 2Q_j^{(k_j)}} \left(\sum_{k_1,\ldots,k_\kappa} \left| T_{\sigma,\Psi}\left(b_1,\ldots,b_\kappa,g_{\kappa+1},\ldots,g_\ell\right)(x) \right| \right)^{1/\kappa} dx \\
&\lesssim \int_{\mathbb{R}^n \setminus \bigcup_{j,k_j} 2Q_j^{(k_j)}} \left(\prod_{j=1}^{\kappa} \sum_{k_j} \lambda^{1/\ell} \frac{l\left(Q_j^{(k_j)}\right)^{n+\epsilon}}{\left| x - c_{Q_j^{(k_j)}} \right|^{n+\epsilon}} \right)^{1/\kappa} \left(\prod_{j=\kappa+1}^{\ell} \lambda^{1/\ell} \right)^{1/\kappa} dx \\
&\leq \lambda^{1/\kappa} \prod_{j=1}^{\kappa} \left(\sum_{k_j} \left| Q_j^{(k_j)} \right| \int_{\mathbb{R}^n \setminus 2Q_j^{(k_j)}} \frac{l\left(Q_j^{(k_j)}\right)^{\epsilon}}{\left| x - c_{Q_j^{(k_j)}} \right|^{n+\epsilon}} dx \right)^{1/\kappa} \\
&\leq \lambda^{1/\kappa} \prod_{j=1}^{\kappa} \left(\sum_{k_j} \left| Q_j^{(k_j)} \right| \right)^{1/\kappa} \lesssim \lambda^{1/\kappa - 1/\ell}.
\end{aligned}
$$

□

Lemma 4.1. *Let $\rho \in (0,1]$ and $m \leq 0$. Given a cube Q, assume that for some index $k \in \{1,\ldots,\ell\}$, the function f_k satisfies*

$$
\operatorname{supp}(f_k) \subseteq Q \quad \textit{and} \quad \int_{\mathbb{R}^n} f_k(x)\, dx = 0,
$$

and $f_j \in L^\infty(\mathbb{R}^n)$ for $j \neq k$. If $\sigma \in L^\infty S_\rho^m\left(\mathbb{R}^n \times \mathbb{R}^{n\ell}\right)$ and $\overline{\Psi} \in L^\infty \phi^1\left(\mathbb{R}^n \times \mathbb{R}^{n\ell}\right)$, then

$$
\int_{\mathbb{R}^n \setminus 2Q} \left| T_{\sigma,\Psi}\left(f_1,\ldots,f_\ell\right)(x) \right| \, dx \lesssim \|f_k\|_{L^1(\mathbb{R}^n)} \prod_{\substack{j=1 \\ j \neq k}}^{\ell} \|f_j\|_{L^\infty(\mathbb{R}^n)} .
$$

Moreover, the result still holds if $\rho = 0$ and $m < -n\ell$.

Proof of Lemma 4.1. Without loss of generality, we let $k = \ell$. We employ the same dilated cube technique as we do in the proof of Theorem 3.2. For an arbitrary cube Q and each point $z \in Q$, we use the same symbol Q_β and $E_\beta(z)$ as there, where the parameter $\beta \in [1,\infty)$ will be chosen later. In light of (4.1), we need to prove that

$$
\begin{aligned}
&\int_{\mathbb{R}^n \setminus 2Q} \left| \mathscr{T}_a\left(f_1,\ldots,f_\ell\right)(x) \right| dx \leq \sum_{j=0}^{\infty} \int_{\mathbb{R}^n \setminus 2Q} \left| \mathscr{T}_{a_j}(f_1,\ldots,f_\ell)(x) dx \right| \\
&\lesssim \underbrace{\sum_{2^j l(Q_\beta) \geq 1} \int_{\mathbb{R}^n \setminus 2Q} \left| \mathscr{T}_{a_j}(f_1,\ldots,f_\ell)(x) dx \right|}_{H_{high}} + \underbrace{\sum_{2^j l(Q_\beta) < 1} \cdots}_{H_{low}} \\
&\approx \|f_\ell\|_{L^\ell(\mathbb{R}^n)} \prod_{j=1}^{\ell-1} \|f_j\|_{L^\infty(\mathbb{R}^n)} .
\end{aligned} \tag{4.5}
$$

We establish the estimation in two different scenarios.

Estimate for high frequency part:

Define $p_j(x,\vec{y},\vec{\xi}) := a_j(x,\vec{\xi})e^{i(x-\vec{y})\cdot\vec{\xi}}$. Using Proposition 2.7 we demonstrate that

$$
\begin{aligned}
&\int_{\mathbb{R}^n\setminus 2Q} \left|\mathscr{T}_{a_j}(f_1,\dots,f_\ell)(x)\right| dx\\
&\leq \int_Q |f_\ell(y_\ell)| \int_{\mathbb{R}^n\setminus 2Q} \left(\int_{\mathbb{R}^{n(\ell-1)}} \prod_{j=1}^{\ell-1} |f_j(y_j)| \left| \int_{\mathbb{R}^{\ell n}} p_j(x,\vec{y},\vec{\xi}) d\vec{\xi} \right| dy_1 \dots dy_{\ell-1} \right) dx dy_\ell\\
&\lesssim \int_Q |f_\ell(y_\ell)| \int_{\mathbb{R}^n\setminus 2Q} \left(\prod_{j=1}^{\ell-1} \|f_j\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^{n(\ell-1)}} \frac{2^{j(\ell n+m-\rho\mathcal{N})}}{\left(\sum_{j=1}^{\ell}|x-y_j|\right)^{\mathcal{N}}} dy_1\dots dy_{\ell-1} \right) dx dy_\ell\\
&\lesssim \int_Q |f_\ell(y_\ell)| \int_{\mathbb{R}^n\setminus 2Q} \left(\frac{2^{j(\ell n+m-\rho\mathcal{N})}\prod_{j=1}^{\ell-1}\|f_j\|_{L^\infty(\mathbb{R}^n)}}{|x-y_\ell|^{\mathcal{N}-n(\ell-1)}} \int_{\mathbb{R}^{n(\ell-1)}} \frac{dy_1\dots dy_{\ell-1}}{\left(1+\sum_{j=1}^{\ell-1}|x-y_j|\right)^{\mathcal{N}}} \right) dx dy_\ell\\
&\lesssim \int_Q |f_\ell(y_\ell)| \int_{\mathbb{R}^n\setminus 2Q} \left(\prod_{j=2}^{\ell} \|f_j\|_{L^\infty(\mathbb{R}^n)} 2^{j(\ell n+m-\rho\mathcal{N})} |x-y_\ell|^{n(\ell-1)-\mathcal{N}} \right) dx dy_\ell\\
&\lesssim \|f_\ell\|_{L^\ell(\mathbb{R}^n)} \prod_{j=1}^{\ell} \|f_j\|_{L^\infty(\mathbb{R}^n)} 2^{j(\ell n+m-\rho\mathcal{N})} \ell(Q)^{\ell n-\mathcal{N}}.
\end{aligned}
\tag{4.6}
$$

where in the last step we used the fact

$$\int_{|y-y_0|>t} \frac{dy}{|y-y_0|^{n+\delta}} \lesssim t^{-\delta}, \quad \forall t>0, \delta>0. \tag{4.7}$$

Now we require that

$$\sum_{2^j l(Q_\beta)\geq 1} 2^{j(\ell n+m-\rho\mathcal{N})} l(Q)^{-(\mathcal{N}-\ell n)} \approx l(Q)^{-(\mathcal{N}-\ell n)-\beta(\ell n+m-\rho\mathcal{N})} \lesssim_N 1.$$

For $l(Q)\leq 1$, we claim that there exist an $\mathcal{N}\geq 0$ satisfying

$$\begin{cases} \mathcal{N} > \ell n;\\ \ell n+m-\rho\mathcal{N} < 0;\\ -(\mathcal{N}-\ell n)-\beta(\ell n+m-\rho\mathcal{N}) \geq 0. \end{cases} \tag{4.8}$$

Thus, we directly obtain

$$H_{high} \lesssim \|f_\ell\|_{L^\ell(\mathbb{R}^n)} \prod_{j=1}^{\ell} \|f_j\|_{L^\infty(\mathbb{R}^n)}. \tag{4.9}$$

Finally, to realize (4.8), we take

$$\mathcal{N} > \max\left\{ \ell n, \frac{m+\ell n}{\rho}, \frac{\left(1-\frac{1}{\beta}\right)\ell n+m}{\rho-\frac{1}{\beta}} \right\}, \tag{4.10}$$

where β is chosen as $\beta > \frac{1}{\rho}$.

For $l(Q)>1$, using the same discussion as in the proof of Theorem 3.2, we can take $N > \max\{\frac{n\ell+m}{\rho}, n\ell\}$, such that those corresponding terms in (4.6) have negative exponents. Then (4.9) holds.

Estimate for low frequency part:

In view of the cancellation of f_ℓ, for any $y_\ell \in Q$ and $y'_\ell \in E_\beta(y_\ell)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus 2Q} \left|\mathscr{T}_{a_j}(f_1,\dots,f_\ell)(x)\right| dx \\
&= \int_{\mathbb{R}^n\setminus 2Q} \left| \int_{\mathbb{R}^{\ell n}} \int_{\mathbb{R}^{\ell n}} \left(p_j(x,\vec{y},\vec{\xi}) - p_j\left(x,y_1,y_2,\dots,y'_\ell,\vec{\xi}\right)\right) f_\ell(y_\ell) \prod_{j=1}^{\ell-1} f_j(y_j)\, d\vec{y} d\vec{\xi} \right| dx \\
&\le \int_{\mathbb{R}^n} |f_\ell(y_\ell)| \int_{\mathbb{R}^n\setminus 2Q} \int_{\mathbb{R}^{n(\ell-1)}} \prod_{j=1}^{\ell-1} |f_j(y_j)| \\
&\quad \times \left| \int_{\mathbb{R}^{\ell n}} \left(p_j(x,\vec{y},\vec{\xi}) - p_j\left(x,y_1,y_2,\dots,y'_\ell,\vec{\xi}\right)\right) d\vec{\xi} \right| dy_1 \dots dy_{\ell-1} dx dy_\ell.
\end{aligned}$$

For any $0 \le t \le 1$, we set $y_\ell(t) = y_\ell + t\left(y'_\ell - y_\ell\right)$. It follows that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^{\ell n}} \left(p_j(x,\vec{y},\vec{\xi}) - p_j\left(x,y_1,y_2,\dots,y'_\ell,\vec{\xi}\right)\right) d\vec{\xi} \right| \\
&= \left| \int_{\mathbb{R}^{\ell n}} \int_0^1 \left(y_\ell - y'_\ell\right) \cdot \nabla_{y_\ell} \left(p_j\left(x,y_1,y_2,\dots,y_\ell(t),\vec{\xi}\right)\right) dt d\vec{\xi} \right| \\
&\le \sum_{k=1}^n \left| y_{\ell,k} - y'_{\ell,k} \right| \int_0^1 \left| \int_{\mathbb{R}^{\ell n}} \partial_{y_{\ell,k}} \left(p_j\left(x,y_1,y_2,\dots,y_\ell(t),\vec{\xi}\right)\right) d\vec{\xi} \right| dt \\
&\le l(Q_\beta) \sum_{k=1}^n \int_0^1 (1-t) \left| \int_{\mathbb{R}^{\ell n}} \left(\partial_{y_{\ell,k}} p_j\right)\left(x,y_1,y_2,\dots,y_\ell(t),\vec{\xi}\right) d\vec{\xi} \right| dt, \\
&\lesssim l(Q_\beta) \sup_{1\le k\le n} \sup_{0\le t\le 1} \left| \int_{\mathbb{R}^{\ell n}} \left(\partial_{y_{\ell,k}} p_j\right)\left(x,y_1,y_2,\dots,y_\ell(t),\vec{\xi}\right) d\vec{\xi} \right|.
\end{aligned}$$

where we use the fact that $\sum_{k=1}^n \left| y_{\ell,k} - y'_{\ell,k} \right| = |y_\ell - y'_\ell| \approx l(Q_\beta)$.

It follows from Proposition 2.7 that

$$\begin{aligned}
&\int_{\mathbb{R}^{n(\ell-1)}} \prod_{j=1}^{\ell-1} |f_j(y_j)| \left| \int_{\mathbb{R}^{\ell n}} \left(p_j(x,\vec{y},\vec{\xi}) - p_j\left(x,y_1,y_2,\dots,y_\ell(t),\vec{\xi}\right)\right) d\vec{\xi} \right| dy_1 \dots dy_{\ell-1} \\
&\lesssim l(Q_\beta) \int_{\mathbb{R}^{n(\ell-1)}} \prod_{j=1}^{\ell-1} |f_j(y_j)| \\
&\quad \times \sup_{1\le k\le n} \sup_{0\le t\le 1} \left| \int_{\mathbb{R}^{\ell n}} \left(\partial_{y_{\ell,k}} p_j\right)\left(x,y_1,y_2,\dots,y_\ell(t),\vec{\xi}\right) d\vec{\xi} \right| dy_1 \dots dy_{\ell-1} \\
&\lesssim l(Q_\beta) \int_{\mathbb{R}^{n(\ell-1)}} \sup_{0\le t\le 1} \frac{2^{j(\ell n+m+1-\rho\mathcal{N})}}{\left(|x-y_\ell(t)| + \sum_{j=1}^{\ell-1} |x-y_j|\right)^{\mathcal{N}}} \prod_{j=1}^{\ell-1} |f_j(y_j)|\, dy_1 \dots dy_{\ell-1} \\
&\lesssim l(Q_\beta) 2^{j(\ell n+1+m-\rho\mathcal{N})} \sup_{0\le t\le 1} \frac{\prod_{j=1}^{\ell-1} \|f_j\|_{L^\infty(\mathbb{R}^n)}}{|x-y_\ell(t)|^{\mathcal{N}-n(\ell-1)}}
\end{aligned}$$

$$\approx l(Q_\beta)2^{j(\ell n+1+m-\rho\mathcal{N})}\frac{\prod_{j=1}^{\ell-1}\|f_j\|_{L^\infty(\mathbb{R}^n)}}{|x-c_Q|^{\mathcal{N}-n(\ell-1)}}.$$

Then the inequality (4.7) gives that

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus 2Q}\left|\mathscr{T}_{a_j}\left(f_1,\ldots,f_\ell\right)(x)\right|dx\\
&\lesssim \|f_\ell\|_{L^\ell(\mathbb{R}^n)}\prod_{j=1}^{\ell-1}\|f_j\|_{L^\infty(\mathbb{R}^n)}\,l(Q)2^{j(\ell n+1+m-\rho\mathcal{N})}\int_{\mathbb{R}^n\setminus 2Q}\frac{dx}{|x-c_Q|^{\mathcal{N}-n(\ell-1)}}\\
&\lesssim \|f_\ell\|_{L^\ell(\mathbb{R}^n)}\prod_{j=1}^{\ell-1}\|f_j\|_{L^\infty(\mathbb{R}^n)}\,l(Q)^{\ell n+\beta-\mathcal{N}}2^{j(\ell n+1+m-\rho\mathcal{N})}.
\end{aligned}$$

We will prove the case where m is divided into two scenarios: $\ell n(\rho-1)-1\le m\le 0$ and $m\le \ell n(\rho-1)-1$.

Case 1: $\ell n(\rho-1)-1\le m\le 0$**.**

We claim that there exist an $\mathcal{N}\ge 0$ satisfying

$$\begin{cases}\mathcal{N}>(\ell-1)n;\\ \ell n+m+1-\rho\mathcal{N}\ge 0;\\ \ell n+\beta-\mathcal{N}-\beta(\ell n+1+m-\rho\mathcal{N})\ge 0.\end{cases}\tag{4.11}$$

Choose $\beta>\frac{1}{\rho}$, (4.11) is actually can be reduced that

$$(\ell-1)n<\mathcal{N}\le\left\{\frac{\ell n+m+1}{\rho},\ell n-\beta m\right\},$$

and the existence of $\mathcal{N}$ is obvious.

Case 2: $m\le \ell n(\rho-1)-1$**.**

On the one hand, for (3.8) to hold, it is sufficient to find a positive constant $\mathcal{N}=\mathcal{N}(\ell,n,\rho,m,\beta)$ that satisfies the conditions as follows.

$$\begin{cases}\mathcal{N}>(\ell-1)n;\\ \ell n+m+1-\rho\mathcal{N}\le 0;\\ \ell n+\beta-\mathcal{N}-\beta(\ell n+1+m-\rho\mathcal{N})\ge 0.\end{cases}\tag{4.12}$$

The above conditions are equivalent to the following inequality that $\mathcal{N}$ need satisfy:

$$\left\{(\ell-1)n,\frac{\ell n+1+m}{\rho}\right\}\le\mathcal{N}\le\ell n+\beta.\tag{4.13}$$

On the other hand, since $\beta>\frac{1}{\rho}$, the condition: $m\le\ell n(\rho-1)-1$ ensures that the existence of $\mathcal{N}$ in (4.13). □

Lemma 4.2. *Let $\rho\in(0,1]$ and $m\le 0$. Given cubes $Q_1,\ldots,Q_\kappa$ with $2\le\kappa\le\ell$. Suppose that $f_j\in L^1(\mathbb{R}^n)$ with $\operatorname{supp}(f_j)\subseteq Q_j$ and $\int_{\mathbb{R}^n}f_j(x)dx=0$, $j=1,\ldots,\kappa$. If $\sigma\in L^\infty S^m_\rho\left(\mathbb{R}^n\times\mathbb{R}^{n\ell}\right)$ and $\overline{\Psi}\in L^\infty\phi^1\left(\mathbb{R}^n\times\mathbb{R}^{n\ell}\right)$, then there exists a positive constant $\epsilon=\epsilon(n,r,\rho)$ such that*

$$\left|T_{\sigma,\Psi}(\vec{f})(x)\right|\lesssim\prod_{j=1}^{\ell}\frac{l\left(Q_j\right)^\epsilon}{\left|x-c_{Q_j}\right|^{n+\epsilon}}\|f_j\|_{L^1(\mathbb{R}^n)}\prod_{j=\kappa+1}^{\ell}\|f_j\|_{L^\infty(\mathbb{R}^n)},\quad x\notin\bigcup_{j=1}^{l}2Q_j.$$

Moreover, the result still holds when $\rho=0$ and $m<-\ell n$.

Proof of Lemma 4.2. We use the same signs as in the proof of Lemma 4.1. Similar to (4.5), it suffices to prove that for any $x \notin \bigcup_{j=1}^{l} 2Q_j$

$$\begin{aligned}
&\left|\mathscr{T}_a\left(f_1, \ldots, f_\ell\right)(x)\right| dx \leq \sum_{j=0}^{\infty}\left|\mathscr{T}_{a_j}(f_1, \ldots, f_\ell)(x) dx\right| \\
&\lesssim \underbrace{\sum_{j: 2^j l(Q_\beta) \geq 1}\left|\mathscr{T}_{a_j}(f_1, \ldots, f_\ell)(x) dx\right|}_{R_{high}} + \underbrace{\sum_{j: 2^j l(Q_\beta)<1} \cdots}_{R_{low}} \\
&\lesssim \prod_{k=1}^{\ell} \frac{l\left(Q_k\right)^\epsilon}{\left|x-c_{Q_k}\right|^{n+\epsilon}}\left\|f_k\right\|_{L^1(\mathbb{R}^n)} \prod_{k=\kappa+1}^{\ell}\left\|f_k\right\|_{L^{\infty}(\mathbb{R}^n)}, \qquad (4.14)
\end{aligned}$$

where $\epsilon=(\mathcal{N}-\ell n)/\kappa$.

Assume that $l\left(Q_1\right)=\min_{1 \leq j \leq \kappa} l\left(Q_j\right)$ and perform the estimation in two parts.

Estimate for high frequency part:

Since $l\left(Q_j\right) \lesssim\left|x-c_{Q_j}\right| \approx\left|x-z_j\right|$ for any $z_j \in Q_j$ and $j=1, \ldots, \kappa$, arguing as in the proof (Estimate for high frequency part) of Lemma 4.1 yields that

$$\begin{aligned}
&\left|\mathscr{T}_{a_j}\left(f_1, \ldots, f_\ell\right)(x)\right| \\
&\leq \int_{\mathbb{R}^{\ell n}} \prod_{j=1}^{\ell}\left|f_j\left(y_j\right)\right|\left|\int_{\mathbb{R}^{\ell n}} p_j(x, \vec{y}, \vec{\xi}) d\vec{\xi}\right| d\vec{y} \\
&\lesssim \int_{\mathbb{R}^{\kappa n}} \prod_{j=1}^{\kappa}\left|f_j(y_j)\right|\left(\int_{\mathbb{R}^{n(\ell-\kappa)}}\left|\int_{\mathbb{R}^{\ell n}} p_j(x, \vec{y}, \vec{\xi}) d\vec{\xi}\right| dy_{\kappa+1} \ldots dy_\ell\right) dy_1 \ldots dy_\kappa \cdot \prod_{j=\kappa+1}^{\ell}\left\|f_j\right\|_{L^{\infty}(\mathbb{R}^n)} \\
&\lesssim \int_{\mathbb{R}^{\kappa n}} \prod_{j=1}^{\kappa}\left|f_j(y_j)\right| \cdot \frac{2^{j(\ell n+m-\rho \mathcal{N})}}{\left(\sum_{j=1}^{\kappa}\left|x-y_j\right|\right)^{\mathcal{N}-n(\ell-\kappa)}} dy_1 \ldots dy_\kappa \cdot \prod_{j=\kappa+1}^{\ell}\left\|f_j\right\|_{L^{\infty}(\mathbb{R}^n)} \\
&\lesssim 2^{j(\ell n+m-\rho \mathcal{N})} l(Q_1)^{\ell n-\mathcal{N}} \prod_{j=1}^{\kappa} \frac{l\left(Q_1\right)^{(\mathcal{N}-\ell n)/\kappa}}{\left|x-c_{Q_j}\right|^{(\mathcal{N}-n(\ell-\kappa))/\kappa}}\left\|f_j\right\|_{L^1(\mathbb{R}^n)} \prod_{j=\kappa+1}^{\ell}\left\|f_j\right\|_{L^{\infty}(\mathbb{R}^n)} \\
&\leq 2^{j(\ell n+m-\rho \mathcal{N})} l(Q_1)^{\ell n-\mathcal{N}} \prod_{j=1}^{\ell} \frac{l\left(Q_j\right)^\epsilon}{\left|x-c_{Q_j}\right|^{n+\epsilon}}\left\|f_j\right\|_{L^1(\mathbb{R}^n)} \prod_{j=\kappa+1}^{\ell}\left\|f_j\right\|_{L^{\infty}(\mathbb{R}^n)}. \qquad (4.15)
\end{aligned}$$

Estimate for low frequency part:

Similar to the proof (Estimate for low frequency part) of Lemma 4.1, for any $y_1 \in Q_1$ and any $\vec{y}'=(y_1', y_2, \ldots, y_\ell)$ with $y_1' \in E_\beta(y_1)$, the mean value theorem implies

$$\begin{aligned}
&\int_{\mathbb{R}^{n(\ell-\kappa)}}\left|\int_{\mathbb{R}^{\ell n}}\left(p_j(x, \vec{y}, \vec{\xi})-p_j\left(x, \vec{y}', \vec{\xi}\right)\right) d\vec{\xi}\right| dy_{\kappa+1} \ldots dy_\ell \\
&\lesssim \int_{\mathbb{R}^{n(\ell-\kappa)}} \int_0^1 \frac{l\left(Q_{1,\beta}\right) 2^{j(\ell n+1+m-\rho \mathcal{N})}}{\left(\left|x-ty_1-(1-t) y_1'\right|+\sum_{j=2}^{\kappa}\left|x-y_j\right|\right)^{\mathcal{N}}} dt dy_{\kappa+1} \ldots dy_\ell \\
&\lesssim \sup_{0 \leq t \leq 1} \frac{l\left(Q_{1,\beta}\right) 2^{j(\ell n+1+m-\rho \mathcal{N})}}{\left(\left|x-ty_1-(1-t) y_1'\right| \sum_{j=2}^{\kappa}\left|x-y_j\right|\right)^{\mathcal{N}-n(\ell-l)}} \int_{\mathbb{R}^{n(\ell-l)}} \frac{dy_{\kappa+1} \ldots dy_\ell}{\left(1+\left|y_{l+1}\right|+\cdots+\left|y_\ell\right|\right)^{\mathcal{N}}}
\end{aligned}$$

$$\lesssim \frac{l\left(Q_{1,\beta}\right) 2^{j(\ell n+1+m-\rho \mathcal{N})}}{\prod_{j=1}^{l}\left|x-c_{Q_j}\right|^{(\mathcal{N}-n(\ell-\kappa))/\kappa}}.$$

In addition, it follows from the cancelation of f_j for $j=1,\ldots,\ell$ that

$$\begin{aligned}
&\left|\mathscr{T}_{a_j}\left(f_1,\ldots,f_\ell\right)(x)\right| \\
&\leq \int_{\mathbb{R}^{\ell n}} \prod_{j=1}^{\ell}\left|f_j\left(y_j\right)\right| \left|\int_{\mathbb{R}^{\ell n}} \left(p_j(x,\vec{y},\vec{\xi}) - p_j\left(x,y_1',y_2,\vec{\xi}\right)\right) d\vec{\xi}\right| d\vec{y} \\
&\lesssim l(Q_1)^{\ell n+\beta-\mathcal{N}} 2^{j(\ell n+1+m-\rho\mathcal{N})} \prod_{j=1}^{\ell} \frac{l\left(Q_1\right)^{(\mathcal{N}-\ell n)/\kappa}}{\left|x-c_{Q_j}\right|^{(\mathcal{N}-n(\ell-\kappa))/\kappa}} \|f_j\|_{L^1(\mathbb{R}^n)} \cdot \prod_{j=\kappa+1}^{\ell} \|f_j\|_{L^\infty(\mathbb{R}^n)} \\
&\leq l(Q_1)^{\ell n+\beta-\mathcal{N}} 2^{j(\ell n+1+m-\rho\mathcal{N})} \prod_{k=1}^{\ell} \frac{l\left(Q_k\right)^{\epsilon}}{\left|x-c_{Q_k}\right|^{n+\epsilon}} \|f_k\|_{L^1(\mathbb{R}^n)} \cdot \prod_{k=\kappa+1}^{\ell} \|f_j\|_{L^\infty(\mathbb{R}^n)}. \qquad (4.16)
\end{aligned}$$

After the summations $\sum_{j:2^j l(Q_\beta)\geq 1}$ and $\sum_{j:2^j l(Q_\beta)<1}$ over j for the (4.15) and (4.16), respectively, following the same procedure as in the proof (Estimate for low frequency part) of Lemma 4.1, there always exists a positive number $\mathcal{N}=\mathcal{N}(\ell,n,\rho,m,\beta)$ such that (4.14) is valid. Finally, the proof is completed. □

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Declarations

Conflict of Interest The author(s) state that there is no conflict of interest.

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