Multi-Unit Auction in Social Networks with Budgets

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Abstract

We study multi-unit auctions in social networks, where each buyer has a fixed budget and can spread the sale information to the network neighbors. We design a mechanism encouraging buyers to report their valuations truthfully and spread the sale information. Our design uses the idea of the clinching mechanism to decide the transaction price and can be viewed as a network version of the mechanism. Most of the previous clinching mechanisms search for the transaction prices by increasing the current price. Our mechanism directly computes the transaction prices in polynomial time. Furthermore, the mechanism applies a technique to iteratively activate new buyers in the network. This ensures utility preservations of the buyers and benefits the seller. We prove key properties of our mechanism, such as no-positive-transfers, individual rationality, incentive compatibility, non-wastefulness and social welfare preservation.

Introduction

Mechanism design in social networks has become an active research area [Li et al. 2017, 2018; Zhao et al. 2018; Kawasaki et al. 2020; Xu and He 2020]. The classical auctions only sell items to buyers directly connected to the seller and do not consider spreading the commodity information through the network links between buyers to get possibly higher utilities [Nisan et al. 2007; Bajari and Hortacsu 2003]. Auctions in social networks that attract more participants seek mechanisms that encourage buyers to diffuse the commodity information. This model is more economically efficient in attracting audiences than advertising on TV, in newspapers, and search engines [Emek et al. 2011; Leskovec, Adamic, and Huberman 2007].

There is a series of interesting work on spreading influence through social networks [Kempe, Kleinberg, and Tardos 2003; Emek et al. 2011; Borgatti et al. 2009]. However, they do not consider auctions. Li et al. [2017] considered auctions in social networks. They proposed a new auction model under the structure of social networks ensuring that the participants in the social network have an incentive to invite their neighbors into the auction. For the case of selling one single item to one buyer, they gave a mechanism IDM that can encourage participants to report both the bid

and the neighborhood truthfully. Followed by this work, several different auction models that diffuse commodity information through social networks have been studied. Zhao et al. [2018] generalized IDM to the setting of selling multi identical items to single-minded buyers. Takanashi et al. [2019] noted that the generalized IDM is not strategy-proof and proposed a strategy-proof design. In their mechanism, however, the revenue of the seller can be negative. This may discourage sellers' participation in auctions. Recently, Kawasaki et al. [2020] gave a strategy-proof mechanism for this model that guarantees a non-negative revenue.

Most of the previous single-unit and multi-unit auctions in social networks assume that each buyer has unit demand. It is natural to study scenarios where buyers demand more than one item. Under this setting, the following question arises: how many units can each buyer get? Frequently considered scenario is to set a budget for each buyer. For example, budget constraints are often observed in public asset privatization auctions in Eastern Europe [Maskin 2000]. Classical auctions without networking structures under the budget setting have widely been studied [Dobzinski, Lavi, and Nisan 2012; Borgs et al. 2005; Abrams 2006; Dobzinski and Leme 2014; Lu and Xiao 2015]. Addressing budgets properly breaks down the usual quasi-linear setting on the utility of the buyers, and because of this the well-known VCG mechanism [Vickrey 1961; Clarke 1971; Groves 1973] loses incentive compatibility property. The design of incentive compatible mechanisms becomes significantly more involved [Dobzinski, Lavi, and Nisan 2012].

There are two widely studied models with the budgets. In *private budget models* [Borgs et al. 2005; Feldman et al. 2008; Dobzinski, Lavi, and Nisan 2012], the budget is reported by each buyer and incentive compatible mechanisms encourage buyers to report their budgets truthfully. In *public budget models* [Dobzinski, Lavi, and Nisan 2012; Dobzinski and Leme 2014], budgets of the buyers are not considered in the incentive compatibility, and they are regarded as known information. Both models are natural. Here are some scenarios for public budget models. Deposits [Mehta et al. 2007] and bonds [Laffont and Robert 1996] are known forms of public budgets. Before the start of an auction, all buyers should deposit money or buy bonds (the public budget) in the system. Also, some budgets reported by buyers can be regarded as public budgets. For example, the budgets in the

form of fixed assets can be checked by a system and used in auctions. One original motivation for public budgets came from ad-auctions [Dobzinski, Lavi, and Nisan 2008]. Ad-auctions used by search engines, including Google's Ad-Words, are essentially large auctions where businesses place bids for individual keywords, together with limits specifying their maximum daily budget (that is regarded as the public budget) [Mehta et al. 2007]. Auctions for spectrum licences and for pollution permits [Dobzinski, Lavi, and Nisan 2008] are natural applications in this case as well.

The two models are quite different in designing incentive compatible mechanisms. For public budget models, the adaptive clinching auction [Dobzinski, Lavi, and Nisan 2012] based on the idea of locking mechanism [Ausubel 2004] is well recognized. For private budget models, at present no suitable deterministic mechanism guarantees incentive compatibility or other desired properties. Most known mechanisms [Lu and Xiao 2015; Abrams 2006; Borgs et al. 2005] are based on the random sampling techniques [Goldberg et al. 2006]. Private budget models need to consider one more information - the budget, which makes the auction design more difficult. However, in the auction design of practical problems [Mehta et al. 2007; Colini-Baldeschi et al. 2011; Goel, Mirrokni, and Leme 2015], there is no definite conclusion which is better or worse between private budget models and public budget models, even when the budgets are reported by buyers.

In this paper, we study auctions with public budgets in social networks. The public budget models in networks can be used in many situations. Often the ad-auction scenarios [Colini-Baldeschi et al. 2011; Goel, Mirrokni, and Leme 2015; Mehta et al. 2007] with public budgets also occur over networks.

Our mechanism runs several rounds. In each round, some units of a product are sold to a buyer with a transaction price. An important step in the mechanism is to decide the transaction price. Our rule to decide the transaction price is inspired by the 'clinching' method [Dobzinski, Lavi, and Nisan 2012; Roughgarden 2016; Ausubel 2004]. Clinching methods form a powerful tool and have been extended to various scenarios, e.g., online auctions [Goel, Mirrokni, and Leme 2020], matching markets [Fiat et al. 2011], auctions under sponsored search setting [Colini-Baldeschi et al. 2011], and combinatorial and double auctions [Fiat et al. 2011; Freeman, Pennock, and Wortman Vaughan 2017]. Almost all clinching mechanisms search for critical prices by continuously increasing the current price. In our mechanism, we compute the critical prices in polynomial time. Furthermore, under the networking structure, our mechanism adopts methods to explore the network by including new activated buyers to the current auction. In this way our approach provides deeper insights into designing mechanisms for multi-unit auctions in a network with budgets. Our contributions advance the-state-of-the-art as follows:

- We propose the first mechanism for multi-unit auctions in a social network with budgets, where buyers can only get the information from their network neighbors.
- In contrast to the previous methods that find transaction

prices, which may skip over the solution or lead to exponential running time, we provide a polynomial-time algorithm to compute the transaction prices directly.

- We analyze the properties of our method and prove that our mechanism is individually rational, incentive compatible, non-wasteful, and has no-positive-transfers.
- We introduce and analyze the social welfare of the mechanism.

The Model and Standard Definitions

We consider the multi-unit auction in social networks with budgets on buyers. A description of our model is the following. There is a social network containing a seller s and n agents (buyers) $N = \{1, 2, ..., n\}$. The seller s wants to sell a set of m identical items (also called units) to buyers. We postulate that the buyers can only get the sale information from their neighbours in the network. This implies that two non-adjacent agents in the network can not communicate directly. Each buyer i has a budget b_i and a valuation v_i for each unit. The seller s also has a valuation v_s for the unit. We assume that the valuations are additive, i.e., each buyer values a collection of units as the sum of the unit values from the collection. In an auction, we require that buyer i's total payment p_i does not exceed the budget b_i. The buyer's budget is public and not defined as part of the reported information by the buyer. But the seller observes each buyer's budget only when the buyer becomes feasible to the seller. In this setting described, our goal is to establish an incentive and self-enforcing system to encourage agents to spread the auction information to their neighbors and report their private valuation truthfully.

For each buyer $i \in N$, the set of her neighbours is denoted by r_i. The seller's neighbor set is denoted by r_s. In an auction, the agent i reports the valuation v_i' and informs a set of neighbors r_i' of the sale, where the valuation v_i' can be different from the truthful private valuation v_i but we require that $v_i' \leq b_i$ and r_i' is a subset of the truthful neighbor set r_i. The reason is that for a valuation $v_i' > b_i$ we can simply let $v_i' = b_i$ and every buyer has no information about other buyers she does not know.

With notations above, the agent i's action profile is a pair $\theta_i' = (v_i', r_i')$. If $\theta_i' = \theta_i$, where $\theta_i = (v_i, r_i)$, we say that agent i is truthful. The profile $\theta_i' = \emptyset$ indicates that agent i does not attend to or is not aware of the auction. The global action profile is $\theta' = (\theta_i', i \in N)$. The global action profile without the agent i is denoted by $\theta_{-i}' = (\theta_j', j \in N \setminus \{i\})$.

Given a global action profile θ', we build the following directed graph on the set of all agents and the seller. There is a directed edge from one agent i to another agent j if agent i reports agent j as one of her neighbors. A pair of opposite edges between two agents are allowed. An agent is *available* if there is a directed path from the seller s to the agent in the graph and *unavailable* otherwise. Only available agents in an auction can get the commodity information.

A global action profile θ' is *feasible* if $\theta_i' = \emptyset$ for each unavailable agent i. Given an arbitrary global action profile, we can obtain a corresponding feasible global action profile

by letting $\theta_i' = \emptyset$ for all unavailable agents i in the social network. In what follows all global action profiles are feasible.

Next, we give a formal description of (direct revelation) mechanisms. The rules of the mechanism will be open for all the participants, including the seller. A mechanism takes a global action profile θ' as the input and decides for each agent how many units to acquire and how much to pay.

Definition 1. *A mechanism* $\mathcal{M}=(\pi, p)$ *takes global action profiles* θ' *as inputs and consists of two components, the allocation rule* $\pi(\theta') = (\pi_i)_{i \in N}$ *and the payment rule* $p(\theta') = (p_i)_{i \in N}$, *where* π_i *is the number of items allocated to agent* i, *and* p_i *is the total payment of agent* i.

Clearly, we should have $\pi_i \in \{0, 1, \ldots, m\}$ and $p_i \in \mathbf{R}$.

Definition 2. *For a global action profile* θ', *an allocation* $\pi(\theta') = (\pi_i)_{i \in N}$ *is* feasible *if* $\pi_j = 0$ *for all unavailable agents* j *and* $\sum_{i \in N} \pi_i \leq m$; *a payment* $p(\theta') = (p_i)_{i \in N}$ *is* feasible *if* $p_i \leq b_i$ *for all agents; and a mechanism is* feasible *if it always maps a global action profile* θ' *to a feasible allocation and a feasible payment.*

Definition 2 constraints the total payment p_i of an agent. We may also put a constraint on the payment for each item:

Definition 3. *A mechanism is* strongly feasible *if it is feasible and the payment p of each item sold to agent i is such that* $v_s \leq p \leq v_i'$.

Now we define utilities and social welfare.

Definition 4. *Given a global action profile* θ' *and a mechanism* $\mathcal{M}$, *the* utility *of buyer i is* $u_i(\theta', \mathcal{M}) := \pi_i v_i - p_i$. *The* utility $u_s(\theta', \mathcal{M})$ *of the seller is* $\sum_{i \in N}(p_i - \pi_i v_s)$.

Definition 5. *The sum of the utilities of the seller and of the agents is called the* social welfare: $sw = \sum_{i \in N} \pi_i(v_i - v_s)$.

The next are standard concepts in evaluation of mechanisms.

Definition 6. *A mechanism* $\mathcal{M}$ *has* no-positive-transfers *if for any feasible global action profile the utility of the seller is non-negative, i.e.,* $u_s(\theta', \mathcal{M}) \geq 0$ *for any feasible* θ'.

Definition 7. *A mechanism* $\mathcal{M}$ *is* individually rational *if for each agent i, the agent's utility is non-negative as long as the agent reports the truthful action profile θ_i, i.e.,* $u_i((\theta_i, \theta_{-i}'), \mathcal{M}) \geq 0$ *holds for any feasible* θ_{-i}'.

With the above two properties, the seller and agents will have the willingness to attend the auction. For auctions with budgets, the following incentive compatibility property is more difficult to achieve [Dobzinski, Lavi, and Nisan 2012].

Definition 8. *A mechanism* $\mathcal{M}$ *is* incentive compatible *if for each agent i, reporting the truthful profile θ_i is a dominant strategy, i.e.,* $u_i((\theta_i, \theta_{-i}'), \mathcal{M}) \geq u_i((\theta_i', \theta_{-i}'), \mathcal{M})$ *holds for any feasible* θ_i' *and* θ_{-i}'.

We also study the concept of non-wastefulness requiring the mechanism to sell as many units as possible. In traditional auction scenarios, there is no network structure and all buyers are aware of the sale. Hence, the concept of non-wastefulness is not interesting. In contrast, under a network structure, to protect the benefit of the buyers, we may hide the sale information from their descendants. Kawasaki et al. [2020] first considered non-wastefulness in social networks. We will also define non-wastefulness in our setting.

Definition 9. *Given a global action profile* θ', *let N^* be the set of all available agents i in the social network with the reported value* $v_i \geq v_s$. *The* basic demand *is defined by*

$$bd(\theta') := \sum_{i \in N^*} \lfloor b_i/v_s \rfloor.$$

The basic demand is an upper bound to the number of items any feasible mechanism can sell to buyers.

Definition 10. *A mechanism* $\mathcal{M}$ *is* non-wasteful *if for any* θ', *it holds that* $\sum_{i \in N} \pi_i(\theta') \geq \min\{m, bd(\theta')\}$.

The Mechanism

We present the SNCA (Social Network Clinching Auction) mechanism for multi-unit auctions in social networks with budgets. The method is inspired by works on clinching mechanisms in the literature. Even for the initial clinching mechanisms, there are several different versions, see the versions in [Roughgarden 2016; Dobzinski, Lavi, and Nisan 2012] that use various methods that handle critical cases. Our design is unique even when there is no network structure. We define two types of critical prices, analyze their properties, and give an algorithm to compute them. Moreover, our method explores the graph and adds new buyers from the network to the auction.

Our mechanism allocates items to agents in rounds. In each round, the auction has a set of fixed participants and a fixed price of each unit. The price in one round can differ from the price in the other round. Each round can be regarded as a normal auction without social networks. We define the concept of *circumstance* (N', p') to characterize the state of the auction in each round, where N' is the set of current participating buyers and p' is the current price of each unit. The circumstance (N', p') means that only agents in N' participate in the auction and the price of each unit is fixed to p' now.

Initially, the circumstance (N', p') is such that $N' = r_s$ (all neighbors of the seller s) and $p' = v_s$. After allocating items to agents (it is allowed to allocate nothing), the mechanism will change the circumstance by either extending N' (inviting more buyers to the current auction) or increasing the current price p' for each unit. The price p' will never decrease.

The rules that extend the participants set N' and increase the price p' are important ingredients of the mechanism. The extension rule is this: once an agent i in N' is 'exhausted', add all neighbors of the agent to N'. Here 'exhausted' means that the current price is higher than the agent's budget b_i or reported private valuation v_i'. This rule encourages agents to report their neighborhood truthfully.

The rule that increases the price p' is more complex and it is based on the concept of critical price that we define later. The critical price is related to the market supply and demand under the given circumstance (N', p'):

Definition 11. *The* demand *of agent i is*

$$D_i(N', p') = \begin{cases} 0, & \text{if } i \notin N' \text{ or } v_i' < p', \\ \min\{\lfloor \frac{b_i}{p'} \rfloor, m\}, & \text{otherwise,} \end{cases}$$

which is the maximum amount of the items the agent can get. The total demand *is* $D(N', p') = \sum_{i \in N} D_i(N', p')$.

Definition 12. *We will call the market is* undersupplying *if* $m < D(N', p')$ *and* oversupplying *if* $m \geq D(N', p')$.

Next, assume that in the circumstance (N', p'), the set N' can not be extended. We consider two cases:

Case 1: The market is oversupplying: $m \geq D(N', p')$. In this case, we do not increase the price and allocate some items to buyers as follows. Find an agent $i \in N'$ with the highest 'priority γ_1' (see Definition 13) and assign her $D_i(N', p')$ items with price p'. After this, agent i is exhausted and the mechanism has the new circumstance $(N' \cup r_i', p')$. The advantage of this operation is that it encourages agents to report their neighborhoods truthfully.

Definition 13. *Under the* priority γ_1, *the priority of agent i is higher than agent j if $b_i > b_j$, or $b_i = b_j$ and $|r_i'| > |r_j'|$; and for two agents i and j with $b_i = b_j$ and $|r_i'| = |r_j'|$, ties are broken arbitrarily.*

Case 2: The market is undersupplying: $m < D(N', p')$. In this case, we may increase p' to a critical point and assign some items with price p'. We have two critical prices. The first is based on the idea of increasing the price p' to the highest price by still keeping the market undersupplying.

Definition 14. *The circumstance (N', p^*) is* I-critical, *if*

$$D(N', p^*) > m \geq D(N', p^* + \epsilon), \ \text{ for any } \epsilon > 0,$$

and the price p^* is called the I-critical price *of N'.*

Interestingly, directly increasing p' to the I-critical price might fail incentive compatibility property. Indeed, consider the seller with 2 items, the reserve price $v_s = 2$, and two buyers 1 and 2 with $(v_1 = 3, b_1 = 3)$ and $(v_2 = 3, b_2 = 6)$. Assume that buyer 1 reports the truthful value 3 and buyer 2 reports v_2' with $3 \geq v_2' \geq 2$. Then v_2' is I-critical price. If the mechanism increases the price from 2 to v_2', then buyer 2 gets the two items with price v_2' and gets more utility by reporting $v_2' = 2$ instead of truthful $v_2' = 3$. Thus, to ensure incentive compatibility, we need to consider another kind of critical price using the idea of important agent under (N', p'):

Definition 15. *Agent i is* important *if we have*

$$D(N', p') - D_i(N', p') < m \leq D(N', p'), \quad (1)$$

and the importance number *of agent i is defined as*

$$IMP_i = m - (D(N', p') - D_i(N', p')).$$

Definition 16. *Given a participant set N', the price p^* is called the* II-critical price *of N' if circumstance (N', p^*) has no important agents and circumstance $(N', p^* + \epsilon)$ has some important agents for any $\epsilon > 0$. The corresponding circumstance (N', p^*) is called* II-critical.

II-critical prices may not exist under some circumstances but I-critical prices always exists. We will prove this when we discuss how to compute I-critical and II-critical prices in the next section. Here we give a simple example where there is no II-critical price in this. Initially, $N' = \{A, B, C\}$ where $v_A = 2$ and $b_A = 200$, $v_B = 3$ and $b_B = 198$,

and $v_C = 2$ and $v_C = 200$, and $m = 100$. Specially, $D(N', p') > 100$ for any $p' < 2$, and the price 2 is I-critical price. It is not too hard to see that there is no II-critical price.

Thus, in our mechanism, we consider the II-critical prices first, and then move to the I-critical price. Based on this, here is our description of the allocation rules.

Find an II-critical price p^* not less than p' (which might not exist). Increase the current price from p' to p^* to get the II-critical circumstance (N', p^*). Given the circumstance (N', p^*), find an important agent $i \in N'$ with the highest priority γ_1 and assign her IMP_i number of items under circumstance $(N', p^* + \epsilon)$ with the price p^*. Update the running variables m and b_i, and execute the next round.

If no II-critical price $\geq p'$ exists, find the I-critical price (which always exist as we will prove this later). Increase the current price from p' to the I-critical price p^* to get an I-critical circumstance (N', p^*). Given the I-critical circumstance (N', p^*), select an agent $i \in N'$ with the highest 'priority γ_2' (see Definition 17) and assign her $D_i(N', p^*)$ items with the price p^*. After this, agent i is exhausted.

Definition 17. *Under the* priority γ_2, *agents i with $D_i(N', p' + \epsilon) > 0$ have a higher priority than agents j with $D_j(N', p' + \epsilon) = 0$; for two agents i and j such that either $\min\{D_i(N', p' + \epsilon), D_j(N', p' + \epsilon)\} > 0$ or $D_i(N', p' + \epsilon) = D_j(N', p' + \epsilon) = 0$, the agent having the bigger budget or the same budget but more neighbors has a higher priority; ties are broken arbitrarily.*

Algorithm 1 implements the ideas described above.

Example. Figure 1 shows how SNCA works. The vector (v_i', b_i) beside agent i denotes agent i's reported value v_i' and the budget b_i, and the information of r_i' is given by the graph structure (a direction from i_1 to i_2 means $i_2 \in r_{i_1}'$).
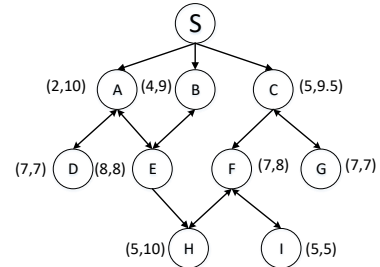

Figure 1: An example

In Figure 1, $m = 8$ and $v_s = 1$. The initial circumstance is $(N' = \{A, B, C\}, p' = 1)$. The set N' has no exhausted agents and Steps 2-7 will not execute. Under the current circumstance, the market is undersupplying as $D(N', p') = 8 + 8 + 8 > 8 = m$. The mechanism executes Step 13. Now we compute critical prices (this is shown in the next section). There is an II-critical price 2: there is no important agent under circumstance $(N', 2)$, and C and B are important agents under circumstance $(N', 2 + \epsilon)$. Agent C has a higher priority γ_1 than agent B. In Step 16, the mechanism assigns $IMP_C = 8 - (8 - 4) = 4$ items with price 2 to agent C. So, $m = 4$. The budget of agent C

Algorithm 1: The Social Network Clinching Auction Mechanism (SNCA)
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1: Initialize the circumstance (N', p') by letting $N' = r_s$ and $p' = v_s$.
2: **while** there is an unmarked exhausted agent $i \in N'$ **do**
3:     Update N' by letting $N' = N' \cup r'_i$ and mark the agent.
4: **end while**
5: **if** $m = 0$ or all agents in N' are marked and exhausted **then**
6:     Terminate.
7: **end if**
8: **if** the market is oversupplying (i.e., $m \geq D(N', p')$) **then**
9:     (1) Assign the demand number $D_i(N', p')$ of items with price p' to agent i who is the agent of the highest priority γ_1 in N'.
10:     (2) Update the running variables ($m \leftarrow m - D_i(N', p')$ and $b_i \leftarrow b_i - D_i(N', p')p'$).
11:     (3) Go to Step 2.
12: **end if**
13: **if** the market is undersupplying (i.e., $m < D(N', p')$) **then**
14:     **if** there exists an II-critical price $p^* \geq p'$ **then**
15:         (1) $p' \leftarrow p^*$.
16:         (2) Assign the important number IMP_i of items with price p' to agent i who is an important agent of the highest priority γ_1 in N'.
17:         (3) Update the running variables m and b_i.
18:         (4) Go to Step 2.
19:     **end if**
20:     **if** there exists an I-critical price $p^* \geq p'$ **then**
21:         (1) $p' \leftarrow p^*$.
22:         (2) Assign the demand number $D_i(N', p')$ of items with price p' to agent i who is an agent of the highest priority γ_2 in N'.
23:         (2) Update the running variables m and b_i.
24:         (4) Go to Step 2.
25:     **end if**
26: **end if**

becomes 1.5 and the agent is now exhausted. The mechanism goes to Step 2. Two agents F and G are put into N'. The new circumstance is ($N' = \{A, B, C, F, G\}$, $p' = 2$). The market is still undersupplying. We find an I-critical price 4. The mechanism executes Step 22. Agent F has the highest priority γ_2. Allocate $D_F(N', 4) = 2$ items to agent F with price 4. So, $m = 2$, and agents A and F are now exhausted. In the next round, agents $\{D, E, H, I\}$ are put into N' in Steps 2-3. The current circumstance becomes ($N' = \{A, B, C, D, E, F, G, H, I\}, p' = 4$). The market is still undersupplying and the I-critical price is 7. Agent E has the highest priority γ_2, and $D_E(N', 7) = 1$ item with price 7 is assigned to agent E. After this, the circumstance will not change and Step 20 is executed. We still have the same I-critical price 7. Agents D and G have the highest priority γ_2 and we break tie arbitrarily, say we select G. Assign $D_G(N', 7) = 1$ item with price 7 to agent G. All eight items are allocated and the algorithm terminates. Finally, agent C gets four items with payment 2, agent F gets two items with payment 4, and agents E and G get one item with payment 7. Table 1 shows how the mechanism works. The numbers in the agent columns are demands of the agents under the corresponding circumstance. The empty units mean that the corresponding agent is not activated under the circumstance.

m	price p'	A	B	C	F	G	D	E	H	I	
8	2	2: 5	4	4							II-critical
		2+ϵ: 0	4	4							
4	4	4: 0	2	0	2	1					I-critical
		4+ϵ: 0	0	0	1	1					
2	7	7: 0	0	0	0	1	1	1	0	0	I-critical
		7+ϵ: 0	0	0	0	0	1	0	0	0	
1	7	7: 0	0	0	0	1	1	0	0	0	I-critical
		7+ϵ: 0	0	0	0	0	0	0	0	0	

Table 1: At II-critical price 2, four items are allocated to agent C with unit price 2; at I-critical price 4, two items are allocated to agent F with price 4; at I-critical price 7 (twice), one item is allocated to agent E (or agent G) with price 7.

Computing Critical Prices

In the SNCA mechanism, we need to compute the I-critical and II-critical prices. This can be done by checking all possible prices if the price domain is discrete; or search by increasing the price in a searching interval each time if the domain of the price is continuous. Most of the previous clinching mechanisms use this type of simple search method to find critical values. However, this method may lead to exponential running time and may even skip over the solution. Here we first analyze the existence of the critical prices and then introduce an algorithm to find our critical prices in polynomial time. The following lemma is easy to prove.

Lemma 1. *There is at most one II-critical price p_1 and exactly one I-critical price p_2 for all participant sets N'. Furthermore, if the II-critical price p_1 exists, then*

$$p_1 \leq p_2. \tag{2}$$

Lemma 2. *I-critical and II-critical prices are computed in polynomial time.*

Proof. To prove this lemma, we give an algorithm that the critical price p^* directly. Assume that $N' = \{1, 2, \ldots, l\}$ and $i \in N'$. Define *change-points* for agent i as values of $\hat{p}$ between 0 and $\min\{v_i, b_i\}$ such that $D_i(N', \hat{p}) \neq D_i(N', \hat{p} + \epsilon)$ for any $\epsilon > 0$. Associate a *change-value* to each change-point defined as $D_i(N', \hat{p}) - D_i(N', \hat{p} + \epsilon)$. Observe that for agent $i \in N'$, the sum of all her change-values equals $D_i(N', p')$.

We claim that any critical price (I-critical or II-critical) is a change-point for at least one agent in N'. Indeed, for any critical price p'', we have $D(N', p'') \neq D(N', p'' + \epsilon)$. Hence, $D_i(N', p'') \neq D_i(N', p'' + \epsilon)$ for at least one agent $i \in N'$.

Thus, to find critical prices we only need to check change-points of the agents. Now note that for an agent $i \in N'$, the demand $D_i(N', p')$ is an integer between 0 and m. There are at most $m + 1$ possible values for $D_i(N', p')$ and thus each agent has at most m change-points. So, we compute all the change-points (together with their change-values) not greater than $\min\{v_i, b_i\}$ for each agent $i \in N'$ by using Algorithm 2.

List all change-points in an increasing order L and combine the same values into a single value with the change-value being the sum of these change-values. These values

Algorithm 2: Computing change-points for agent i
1: Initially let $x = D_i(N', 0)$ and $\hat{p} = 0$.
2: **while** $\hat{p} \leq \min\{v_i, b_i\}$ **do**
3:     $\hat{p} \leftarrow b_i/x$.
4:     **if** $\hat{p} < \min\{v_i, b_i\}$ **then**
5:        Print $\hat{p}$ as a change-point with the change-value being 1 and let $x = x - 1$.
6:     **else**
7:        Print $\min\{v_i, b_i\}$ as a change-point with the change-value being x and halt.
8:     **end if**
9: **end while**

will be used to check if a change-point is a critical price. In total, there are at most $m|N'| \leq mn$ change-points for all agents, and then the ordering and combination can be done in $O(mn \log(mn))$ by using a standard sorting algorithm.

For each change-point in L, check in the increasing order if it satisfies the conditions of I-critical or II-critical prices by using the change-values. Note that the sum of all the change-values of all agents is $D(N', p')$. The first change-point $\bar{p}$ on L such that the sum of the change-values of the agents on the left of $\bar{p}$ (including $\bar{p}$) is at most $D(N', p') - m$ is I-critical.

There are at most mn change-points on L. For each change-point checking if the point is critical takes $O(|N'|)$ time by using the information given by the change-values. Hence, the algorithm runs in $O(mn^2)$ time. $\square$

Remark. Algorithm 2 works for continuous price domains. If the domain is discrete, then we need to round down to the next discrete price. For the current circumstance (N', p'), we will only consider critical prices not less than the current price p' in our algorithm. So in Algorithm 2, we can only consider change-points between p' and $\min\{v_i, b_i\}$.

Example. Consider the profile in Fig. 1, in the first round we have $(N' = \{A, B, C\}, p' = 0)$. Consider Algorithm 2 for agent A. Initially, $D_A(N', p') = 8$, $x = 8$, and $\hat{p} = 0$. By Algorithm 2, we get change-points $L_A = \{\frac{5}{4}(1), \frac{10}{7}(1), \frac{5}{3}(1), 2(5)\}$, where inside the brackets we have corresponding change-values. For agents B and C, Algorithm 2 outputs $L_B = \{\frac{9}{8}(1), \frac{9}{7}(1), \frac{3}{2}(1), \frac{9}{5}(1), \frac{9}{4}(1), 3(1), 4(2)\}$ and $L_C = \{\frac{9.5}{8}(1), \frac{9.5}{7}(1), \frac{9.5}{6}(1), \frac{9.5}{5}(1), \frac{9.5}{4}(1), \frac{9.5}{3}(1), \frac{9.5}{2}(1), 5(1)\}$. By combining L_A, L_B and L_C, we get the order L as in Fig. 2. By checking each change-point on L, we find the II-critical price 2. For the other rounds, the critical prices are computed similarly.

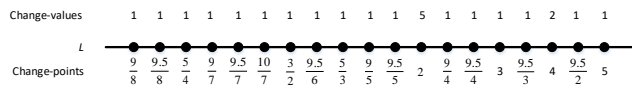| Change-values	1	1	1	1	1	1	1	1	1	1	1	5	1	1	1	1	2	1	1
L																			
Change-points	$\frac{9}{8}$	$\frac{9.5}{8}$	$\frac{5}{4}$	$\frac{9}{7}$	$\frac{9.5}{7}$	$\frac{10}{7}$	$\frac{3}{2}$	$\frac{9.5}{6}$	$\frac{5}{3}$	$\frac{9}{5}$	$\frac{9.5}{5}$	2	$\frac{9}{4}$	$\frac{9.5}{4}$	3	$\frac{9.5}{3}$	4	$\frac{9.5}{2}$	5

Figure 2: Computing change-points/-values

Properties of SNCA Mechanism

We now address key properties of the SNCA mechanism. Due to the space limitation, we may only give the outline of some proofs.

Theorem 1. *Any strongly feasible mechanism is no-positive-transfers and individually rational.*

Proof (Outline). The payment price of each item is at least the reserving price v_s of the seller and then the utility of the seller is not negative. The payment p of an item sold to buyer i is such that $p \leq v'_i$. If the buyer reports truthfully, then $p \leq v_i$. So the buyer has no negative utility. $\square$

Theorem 2. *The SNCA mechanism is strongly feasible.*

Proof (Outline). The first two conditions of feasible mechanisms are easy to observe. At any round, x items are assigned to agent j if $p \leq v'_j$ and $xp \leq b_j$. The total payment of a buyer j does not exceed the initial budget b_j. Hence, SNCA is feasible. Also, the price never decrease. It is also always the case that $v_s \leq p$. We know that $v_s \leq p \leq v'_j$. Therefore, SNCA is strongly feasible. $\square$

Theorem 3. *The SNCA mechanism is non-wasteful.*

Proof (Outline). Assume that the market is undersupplying. Then the market stays undersupplying. So, in this case, all the items will be sold. If the market is never undersupplying, then the price never changes. Hence, items will iteratively be sold to buyers at price v_s until no buyer can by any more. This shows that the SNCA mechanism is non-wasteful. $\square$

Theorem 4. *The SNCA mechanism is incentive compatible.*

Proof. We need to prove the following: (i) Agent i can not get more benefit by reporting a proper subset of r_i; (ii) Agent i can not get more benefit by misreporting her value v_i.

We prove (i). The mechanism extends the participant set by including the neighbors of agent i to N' when the agent cannot buy any item. So the crowd spread by an agent i will not compete with herself. Also, reporting a proper subset of r_i may decrease her priorities. In the auction, both the set of participants and the price will not decrease. So by reporting a small subset of r_i the agent can get neither items under a smaller price nor more benefit.

We prove (ii). For the reported value v'_i, we have two cases.

Case 1: $v'_i > v_i$. No matter agent i reports a value $v'_i > v_i$ or the truthful value v_i, the execution of the mechanism before the price exceeding v_i would be identical. By reporting v'_i or v_i, the allocations before the price exceeding v_i are the same. However, reporting v'_i will stop any further allocations to agent i with a price between v_i and v'_i, which will hurt the benefit of the agent. So reporting v'_i higher than the truthful value is not benificial.

Case 2: $v'_i < v_i$. Let I and I' be two instances where agent i reports the truthful value v_i and a value $v'_i < v_i$, respectively. We compare them. Before the price reaches v'_i, the allocations of the two instances are the same. If v'_i is not a transaction price in I', then I' will miss some further allocations to agent i with some price between v'_i and v_i, compared to I. Then agent i may get less utility in I'. Next, we assume that v'_i is a transaction price in I'. We also assume

that in I' some items are allocated to agent i at the price v'_i, because if there is no allocation to agent i at price v'_i, then agent i also can not get any further utility in I'. Since some items are allocated to agent i at price v'_i and $D_i(N', v'_i + \varepsilon) = 0$, we know that the price v'_i can only be an II-critical price in I'. Furthermore, since agent i is selected to be allocated with some items at the II-critical price v'_i, we know that agent i has a higher priority γ_2 than other agents. Then $D_j(N', v'_i + \varepsilon) = 0$ holds for all other agents j. In I, only agent i holds that $D_i(N', v'_i + \varepsilon) > 0$. So agent i will be allocated with $D_j(N', v'_i)$ items at the price v'_i. Thus, in I agent i still can get at least as the same number of items at the price v'_i.

Thus, we have proved that no agent can get more utility by misreporting her profile. □

Social Welfare

In this section, we analyze the social welfare. Recall, see Definition 5, that the social welfare is the sum of the utilities of the seller and all buyers: $sw = \sum_{i \in N} \pi_i(v_i - v_s)$. This only relates to the allocations $(\pi_i)_{i \in N}$ but not the payments $(p_i)_{i \in N}$. To obtain a greater social welfare, we need to sell items to buyers with higher values v_i. The *optimal social welfare*, denoted by sw_o, is the maximum social welfare that can be obtained by strongly feasible mechanisms. We can compute sw_o as follows. List the buyers by their values v_i in descending order and sell as many items as possible to buyer i at the lowest price v_s according to this order, i.e., sell $\lfloor b_i/v_s \rfloor$ items to buyer i. Let $\pi^* = (\pi^*_1, \ldots, \pi^*_n)$ denote the assignment obtained by the above algorithm to maximize the social welfare, which is also called the *SW-optimal assignment*. Thus, we have $sw_o = \sum_{i \in N} \pi^*_i(v_i - v_s)$. Recall that r_s is the set of buyers directly adjacent to the seller s in the network. We define the "domination ratio":

$$\beta = \frac{\sum_{i \in r_s} \pi^*_i(v_i - v_s)}{\sum_{i \in N} \pi^*_i(v_i - v_s)}.$$

With no network links, we regard $r_s = N$ and then $\beta = 1$.

The following results show incentive compatible mechanisms cannot guarantee optimality of social welfare. A mechanism without payment to buyers means that in the mechanism, the payment p_i of each buyer i is nonnegative.

Theorem 5. *No incentive compatible mechanism without payment to buyers can guarantee a social welfare at least* $\min\{sw_o, \beta \cdot sw_o + \epsilon\}$ *for any constant $\epsilon > 0$.*

Proof. We give an example to show that no incentive compatible mechanism can get that bound. In this example, there are $m = 3$ items and three buyers $\{1, 2, 3\}$. Buyers 1 and 2 are directly adjacent to the seller, buyer 3 is only adjacent to buyer 2. The reserve price v_s is a positive value. We have that $v_1 = b_1 = 2v_s$, $v_2 = b_2 = v_s + \epsilon/3$ and $v_3 = b_3 = 2v_s - \epsilon/3$.

If all buyers report the truth, then the SW-optimal assignment will assign two items to buyer 1 and one item to buyer 3 at price v_s, and it holds that $sw_o = 2(v_1 - v_s) + (v_3 - v_s) = 3v_s - \epsilon/3$. Note that $\beta sw_o + \epsilon = 2v_s + \epsilon > 2(v_1 - v_s) + (v_2 - v_s)$. Any mechanism with the social welfare at least $\beta sw_o + \epsilon$ must assign two items to buyer 1 and one item to

buyer 3, which means that buyer 2 will not get anything and her utility is 0.

If buyer 2 does not report her neighbor buyer 3, then we can regard it as an instance without a network. The SW-optimal assignment will assign two items to buyer 1 and one item to buyer 2, and the social welfare is $2(v_1 - v_s) + (v_2 - v_s) = 2v_s + \epsilon/3$. Any mechanism with the social welfare at least $\min\{sw_o, \beta \cdot sw_o + \epsilon\} = sw_o$ will also assign one item to buyer 2. Buyer 2 will get one item and her utility is at least $\epsilon/3$.

So buyer 2 can get more utility by misreporting. Then no incentive compatible mechanism can guarantee a social welfare at least $\min\{sw_o, \beta \cdot sw_o + \epsilon\}$. □

In the example in the proof of Theorem 5, we may pay some to buyer 2 to encourage her to report her neighbors if payments to buyers are allowed. So have the following theorem.

Theorem 6. *No incentive compatible and no-positive-transfers mechanism can guarantee a social welfare at least* $\min\{sw_o, \beta \cdot sw_o + \epsilon\}$ *for any constant $\epsilon > 0$.*

We give the outline of the proof, which is to construct an example based on the example in the proof of Theorem 5. We add a long path P between buyer 2 and buyer 3 and the buyers closer to buyer 3 will have a slightly higher value v_i. If all buyers report the truth, the SW-optimal assignment still needs to assign one item to buyer 3. However, any buyer on P (including buyer 2) can get more utility without reporting the neighbour. If we give each buyer in P a payment (at least her utility without reporting her neighbour), the total payment will be too large as the length of P is long and then it will not be no-positive-transfers.

In spite of these theorems, we can still analyze certain bounds for SNCA. The proof the following theorem can be found in the full version of this paper.

Theorem 7. *SNCA can guarantee a social welfare of* $O(\frac{1}{\sqrt{m}} \cdot \beta \cdot sw_o)$.

Discussion

In this paper, we first consider multi-unit auctions in social networks with budgets on buyers. For private budgets, we design a clinching-based mechanism that can encourage the buyers to tell truthfully and also guarantee the benefit of the seller. Different from previous clinching mechanisms, our algorithm computes transaction prices in polynomial time and it also works when there is no network. While we can get a clean mechanism with neat results for the public budget model, the private budget model becomes significantly harder. For instance, it was mentioned in [Dobzinski, Lavi, and Nisan 2012; Dobzinski and Leme 2014; Lu and Xiao 2015] that even if there is no network, the budgets are private and buyers' utility is additive, there is no truthful, incentive compatible, and Pareto optimal mechanism. Furthermore, no deterministic mechanisms have been found yet that can guarantee the incentive compatibility for private budgets.

Acknowledgements

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