Improved Maximin Guarantees for Subaddertive and Fractionally Subadditive Fair Allocation Problem

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Abstract

In this work, we study the maximin share fairness notion for allocation of indivisible goods in the subadditive and fractionally subadditive settings. While previous work refutes the possibility of obtaining an allocation which is better than $1/2$-MMS, the only positive result for the subadditive setting states that when the number of items is equal to m, there always exists an $\Omega(1/\log m)$-MMS allocation. Since the number of items may be larger than the number of agents (n), such a bound can only imply a weak bound of $\Omega(\frac{1}{n \log n})$-MMS allocation in general.

In this work, we improve this gap exponentially to an $\Omega(\frac{1}{\log n \log \log n})$-MMS guarantee. In addition to this, we prove that when the valuation functions are fractionally subadditive, a $1/4.6$-MMS allocation is guaranteed to exist. This also improves upon the previous bound of $1/5$-MMS guarantee for the fractionally subadditive setting.

1   Introduction

Fair division is a fundamental problem which has received significant attention in economics, political science, mathematics, and more recently in computer science (Brams and Taylor 1995; Budish 2011; Dubins and Spanier 1961; Bezáková and Dani 2005; Kurokawa, Procaccia, and Wang 2018a; Lipton et al. 2004). In this problem the goal is to divide a resource among a set of agents in a fair manner. Both divisible and indivisible settings have been subject to several studies (Brams and Taylor 1995, 1996; Dubins and Spanier 1961; Lipton et al. 2004; Kurokawa, Procaccia, and Wang 2018a,b; Aziz et al. 2017; Barman, Krishnamurthy, and Vaish 2018) though recent years have seen a plethora of developments in the indivisible setting (Kurokawa, Procaccia, and Wang 2018a; Plaut and Roughgarden 2020; Caragiannis et al. 2016; Caragiannis, Gravin, and Huang 2019; Chaudhury, Garg, and Mehlhorn 2020; Chaudhury et al. 2020; Ghodsi et al. 2018; Garg and Taki 2020; Garg, McGlaughlin, and Taki 2019; Aziz, Chan, and Li 2019; Gourvès and Monnot 2019; Amanatidis et al. 2017; Barman and Krishna Murthy 2017) which is the focus of this work.

Unfortunately, most of the guarantees that hold in the divisible setting do not carry over to the indivisible setting. For

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example, well-known fairness criteria such as *envy-freeness*[1] and proportionality[2] that are known to exist in the divisible setting may be violated in the indivisible setting. This led the community to develop more relaxed fairness notions that are better suited for the indivisible setting.

In this paper, we investigate on the maximin-share (MMS) notion which is one of the central measures of fairness in the indivisible setting. This notion is introduced by Budish (Budish 2011) as a relaxation of proportionality for the case of indivisible goods. Let $\mathcal{N}$ be a set of size n that contains the agents. For a set $\mathcal{M}$ of m indivisible goods and an agent a_i, $\mathsf{MMS}_i^n(\mathcal{M})$ is defined as

$$\mathsf{MMS}_i^n(\mathcal{M}) = \max_{\pi_1, \pi_2, \ldots, \pi_n \in \Pi} \min_j U_i(\pi_j),$$

where Π is the set of all partitionings of $\mathcal{M}$ into n bundles and $U_i(\pi_j)$ is the valuation of agent a_i for a bundle π_j. In other words, among all n partitionings of the items, the one that maximizes the minimum value of the partitions for agent a_i gives the MMS value of that agent. When the goal is to allocate the items to n agents, maximin-share of agent a_i is defined to be equal to $\mathsf{MMS}_i^n(\mathcal{M})$. For brevity, we denote this value by MMS_i. An allocation is then said to be MMS, if it guarantees each agent a_i a bundle with utility at least MMS_i to agent a_i.

MMS-allocations have received significant attention both in the additive and non-additive settings. While it may seem that in the additive setting, an MMS-allocation always exists, an elegant counter-example by Kurocawa *et al.* (Kurokawa, Procaccia, and Wang 2018a) reveals that some additive instances admit no MMS allocation. On the positive side, it is been shown that a $2/3$-MMS allocation (an allocation that guarantees each agent a_i a bundle with utility at least $2\mathsf{MMS}_i/3$) always exists (Kurokawa, Procaccia, and Wang 2018a). This bound is improved by Ghodsi *et al.* (2018) to a $3/4$-MMS guarantee. No counter-example refutes the possibility of obtaining a better bound and therefore whether or not a more efficient algorithm can guarantee a better bound remains an open question.

The importance of fair allocation problems goes well beyond the additive setting. For instance, it is quite natural to

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[1]An allocation is called envy-free, if no agent prefers to exchange her bundle with another.

[2]An allocation is called proportional, if each agent receives a bundle which is worth at least $1/n$ of the entire resource to her.

expect that an agent prefers to receive two items of value 400, rather than receiving 1000 items of value 1. Such a constraint cannot be imposed in the additive setting. However, subadditive and fractionally subadditive functions are strong tools for modelling such constraints. Previous work have already made some progress in generalizing the Maxmin fair allocation problem to non-additive settings. Barman and Krishna Murthy (2017) prove that when the valuation functions are submodular, a $1/10$-MMS allocation can be guaranteed for the fair allocation problem. The bound was later improved by Ghodsi *et al.* (2018) to a $1/3$-MMS guarantee for submodular functions. They also prove a $1/5$-MMS guarantee for the fractionally subadditive setting and an $\Omega(1/\log m)$-MMS guarantee for the subadditive setting.

In this paper, we improve the previous results on subadditive and fractionally subadditive settings. Our proof gives an improved guarantee of $\Omega(\frac{1}{\log n \log \log n})$-MMS in the subadditive setting which exponentially improves the prior work of Ghodsi *et al.* (2018). In addition to this, we also improve the $1/5$-MMS guarantee of Ghodsi *et al.* (2018) for the fractionally subadditive setting to $1/4.6$-MMS.

1.1 Related Work

Maximin-share has been initially studied for the additive setting (Budish 2011; Kurokawa, Procaccia, and Wang 2018a; Ghodsi et al. 2018; Kurokawa, Procaccia, and Wang 2016; Amanatidis et al. 2017). The MMS notion is first introduced by Budish (2011), and later used in computer science by the work of Kurokawa *et al.* (2018a). In their paper, Kurokawa *et al.* show that for some instances, no MMS allocation can be guaranteed even in the additive setting. They also show that there always exists an allocation that guarantees a $2/3$ fraction of the MMS value of each agent for her. This ratio is improved in subsequent works to $3/4$ by Ghodsi *et al.* (2018) and $3/4 + o(1)$ by Grag and Taki (2020).

In contrast to other famous notions such as social welfare, or egalitarian welfare, MMS has received less attention in non-additive settings. For the submodular setting, Barman and Krishna Murthy (2017) prove the existence of a $1/10$-MMS allocation. This factor was later improved by Ghodsi *et al.* (2018) to $1/3$. They also prove an upper-bound of $2/3$ for the submodular setting.

For subadditive and fractionally subadditive settings, which are the focus of this paper, the best known approximation results are $1/5$-MMS for fractionally subadditive and $\Omega(1/\log m)$ for subaddtive settings. For a special case of fractionally subadditive settings where the items form a hereditary set system, Li and Vetta (2018) prove a 0.3667-MMS guarantee.

It is worth mentioning that subadditive and fractionally subadditive settings have been studied for various allocation scenarios and objectives, including maximizing social welfare (Feige 2009), maximizing Nash social welfare (Barman and Sundaram 2020; Barman et al. 2020; Chaudhury, Garg, and Mehta 2020), combinatorial auctions (Dobzinski, Nisan, and Schapira 2010; Bhawalkar and Roughgarden 2011), and envy-freeness up to any item (Plaut and Roughgarden 2020).

2 Preliminaries

Throughout this paper, we assume the set of agents is denoted by $\mathcal{N}$ and the set of items is referred to by $\mathcal{M}$. Let $|\mathcal{N}| = n$ and $|\mathcal{M}| = m$. We refer to the i'th agent by a_i and to the i'th item by b_i, i.e., $\mathcal{N} = \{a_1, a_2, \ldots, a_n\}$ and $\mathcal{M} = \{b_1, b_2, \ldots, b_m\}$. We denote the valuation of an agent a_i for a set S of items by $U_i(S)$. Our interest is in valuation functions that are monotone and non-negative. More precisely, we assume $U_i(S) \geq 0$ for every agent a_i and set $S \subseteq \mathcal{M}$, and for every two sets S_1 and S_2 and every agent a_i we have $U_i(S_1 \cup S_2) \geq \max\{U_i(S_1), U_i(S_2)\}$.

We restrict our attention to two classes of set functions:

- **Fractionally subadditive (XOS)**: A fractionally subadditive set function $V(.)$ can be shown via a finite set of additive functions $\{V_1, V_2, \ldots, V_\alpha\}$ where $V(S) = \max_{i=1}^{\alpha} V_i(S)$ for any subset S of the ground set.

- **Subadditive**: A set function $V(.)$ is subadditive if $V(S_1) + V(S_2) \geq V(S_1 \cup S_2)$ for every two subsets S_1, S_2 of the ground set.

Let Π_r be the set of all partitionings of $\mathcal{M}$ into r disjoint subsets. For every r-partitioning $P^* \in \Pi_r$, we denote the partitions by $P_1^*, P_2^*, \ldots, P_r^*$. For a set function $V(.)$, we define $\mathsf{MMS}_V^r(\mathcal{M})$ as follows:

$$\mathsf{MMS}_V^r(\mathcal{M}) = \max_{P^* \in \Pi_r} \min_{1 \leq j \leq r} V(P_j^*).$$

For brevity we refer to $\mathsf{MMS}_{U_i}^n(\mathcal{M})$ by MMS_i. Since scaling the valuation functions does not affect the optimality of an allocation, we assume without loss of generality that $\mathsf{MMS}_i = 1$ holds for all agents.

An allocation of items to the agents is a vector $A = \langle A_1, A_2, \ldots, A_n \rangle$ where $\bigcup A_i = \mathcal{M}$ and $A_i \cap A_j = \emptyset$ for every two agents $a_i \neq a_j \in \mathcal{N}$. An allocation A is α-MMS, if every agent a_i receives a subset of the items whose value to that agent is at least α times MMS_i. More precisely, A is α-MMS if and only if $U_i(A_i) \geq \alpha \mathsf{MMS}_i$ for every $a_i \in \mathcal{N}$.

We may sometimes give an item to several agents in which case we call it a multiallocation. A multiallocation of items to the agents is $\langle \mathsf{MMS}, k \rangle$ if each agent receives a bundle which is worth at least her MMS value and each item is allocated to at most k agents. Similarly, a multiallocation is $\langle \alpha\text{-MMS}, k \rangle$ if each agent receives an α fraction of her MMS value and no item is allocated to more than k agents.

A well-known technique in finding approximate MMS allocations is reducibility (2018; 2018a; 2017). Here we bring a consequence of this technique, stated in Lemma 2.1.

Lemma 2.1 ((Amanatidis et al. 2017)). *Given that an α-MMS allocation exists under the assumption that the value of each item for each agent is bounded by α, the same guarantee carries over to the general setting without any bounds on the valuations.*

For a threshold $0 < t \leq 1$ and a set function V, Ghodsi *et al.* (2018) define the *bounded welfare function V^t* as:

$$\forall S \subseteq M \quad V^t(S) = \min\{t, V(S)\}. \tag{1}$$

At a high-level, bounded welfare valuations can in fact be seen as a trade-off between efficiency and fairness. Ghodsi *et al.* (2018) prove that V^t is structurally similar to V. More precisely, they prove Proposition 2.1:

Proposition 2.1 (Ghodsi *et al.* (2018)). *For a valuation function V and any $0 < t < 1$,*

- *If V is submodular, then so is V^t.*
- *If V is fractionally subadditive, then so is V^t.*
- *If V is subadditive, then so is V^t.*

We use the notion of bounded welfare functions in both subaddibitive and fractionally subaddibitive settings.

3   Our Contribution

Our main contribution is an improved MMS guarantee for the fair allocation problem under subaddibitive valuations. The previous work of Ghodsi *et al.* (2018) provides a guarantee of $\Omega(1/\log m)$ which we improve in this work.

We would like to compare our result to previous work before proceeding to the techniques and results. First, m denotes the number of items and can be exponentially large in terms of the number of agents. Thus, the $\Omega(1/\log m)$ guarantee of Ghodsi *et al.* (2018) does not explicitly give any bound in terms of the number of agents n. We show in Appendix D that any guarantee that holds for $m = n^n$ items also carries over to $m > n^n$ items. Unfortunately, this only gives us a weak bound of $\Omega(1/(n \log n))$-MMS when plugging the reduction into the bound of Ghodsi *et al.* (2018).

We improve this bound exponentially and obtain an $\Omega(\frac{1}{\log n \log \log n})$-MMS guarantee in the subaddibitive setting. In addition, we improve the analysis of Ghodsi *et al.* (2018) for fractionally subaddibitive valuations, yielding a $1/4.6$-MMS guarantee for the fractionally subaddibitive setting.

setting	previous guarantee	our improvement
fractionally subaddibitive	$1/5$ (2018)	$1/4.6$ Theorem 4
subaddibitive	$\Omega(\frac{1}{n \log n})$ (2018)	$\Omega(\frac{1}{\log n \log \log n})$ Theorem 1

Table 1: A summary of the results of this paper.

3.1   Subaddibitive Setting

Let us first point out to the main difficulty of the subaddibitive setting. Unlike the previously studied settings such as additive, submodular, and fractionally subaddibitive settings, the subaddibitive setting seems to be particularly challenging to tackle when it comes to randomized and probabilistic methods. Let us show this with an example: Let V be a monotone subaddibitive set function and S be a subset of the ground elements. It follows from the subaddibitivity of V that if we put each element of S in a set S' uniformly at random with probability α then $\mathbb{E}[V(S')] \geq \alpha\mathbb{E}[V(S)]$ holds (in expectation). This is a strong bound that has been used in previous studies (Feige 2009) when the goal is to bound the expected value of the outcome. For our problem, the goal is to bound the MMS guarantee in the worst case and therefore instead of a bound on the expected utilities of the agents, we need a bound on the utilities of the agents in the worst case. Thus, a question that becomes relevant to our analysis is how well is the value of $V(S')$ concentrated around its expectation?

While the answer to the above question is positive for additive, submodular, and fractionally subaddibitive functions, there are several counter examples that show the value of $V(S')$ may well deviate from its expectation. That is, with a considerable probability, $V(S')$ may be smaller than $(1 - \epsilon)\mathbb{E}[V(S')]$ which is a highly undesirable situation in our analysis. Moreover, lower tail bounds on subaddibitive functions of i.i.d chosen sets are not well-understood. Indeed, the authors are not aware of any bound that guarantees for some **constant** values $c_1, c_2 > 1$, $\Pr[V(S') \geq \frac{\mathbb{E}[V(S')]}{c_1}] \geq 1/c_2$.

For reasons that will become clear later in the section, our analysis needs such a bound in the subaddibitive setting. As part of our analysis, we show a weaker lower tail bound for subaddibitive functions which is of independent interest.

Lemma 3.1. *Let V be a monotone subaddibitive function with non-negative values such that for a set S we have $V(S) = 1$. In addition, assume that for some value $0 < t < 1$, for every element $e_i \in S$ we have $V(\{e_i\}) \leq \frac{t}{\log 1/t}$. Let S' be a set made randomly from S such that each element of S appears in S' independently with probability at least t. Then we have $\Pr[V(S') \leq t/3] \leq 0.77$.*

Notice that Lemma 3.1 gives us a tail-bound on the valuation of a randomly chosen subset of items but this bound only holds with constant probability. Therefore, another challenge that we have in our analysis is to improve the guarantee of the bound down to $1 - 1/n - \epsilon$ such that by taking the union bound on the undesirable possibilities we can prove that a desired scenario exists for all agents at once. In what follows, we show how we prove such a guarantee.

Our algorithm consists of two steps. In the first step, we find a multiallocation of the items to the agents such that each item is given to at most $O(\log n)$ agents and moreover, each agent can divide her items into $\Omega(\log n)$ bundles such that each bundle is worth at least $1/8$ to her (recall that we assume all the MMS values are equal to 1). In the second step, we make an allocation out of our multiallocation by giving each item to one of the agents that receives the item in the multiallocation uniformly at random. The bound of Lemma 3.1 then implies that the bundle given to each agent is worth at least $\Omega(1/\log n)$ to her with probability more than $1 - 1/n$. Intuitively, this follows since in a bad event, each of the independent $\Omega(\log n)$ high-value bundles of an agent in the multiallocation should provide a small utility to that agent and thus the probability that none of the bundles provides such a utility decreases exponentially.

Therefore, the main algorithmic difficulty is to show that there exists a multiallocation of items to the agents with the desired properties. To this end, we leverage two techniques: First we define a modified utility function for each agent in a way that for an integer $c \geq 1$, the value of a set S is at least c if and only if items of S can be divided into $\Omega(c)$ disjoint subsets each having a large value for the corresponding agent. We then write a configuration LP that fractionally allocates the items to the agents in a way that meets our conditions. We then leverage the proof of Fuige (2009) that shows the integrality gap of the LP is bounded by 2. This implies that there is an integer solution for the LP in which for a considerable portion of the agents, the allocated bundle

maintains our property. Finally, we show that by repeating the same procedure $O(\log n)$ times we can obtain the desired multiallocation.

Theorem 1. *Any maximin fair allocation problem with subadditive agents admits an* $\Omega(\frac{1}{\log n \log \log n})$*-MMS allocation.*

The additional $\log \log n$ term in the denominator of the guarantee in Theorem 1 comes from the reducibility argument. Since the bound of Lemma 3.1 holds only if each item is worth no more than $O(\frac{1}{\log n \log \log n})$ to each agent, then we lose an additional $\log \log n$ factor in the guarantee.

3.2 Fractionally Subadditive Setting

Fractionally subadditive functions are special cases of subadditive functions. Ghodsi *et al.* (2018) show that when the valuation functions are fractionally subadditive, there always exists a $1/5$-MMS allocation. We improve this result to 0.2192235-MMS. Our method is based on the notion of bounded welfare, introduced by Ghodsi *et al.* (2018).

The structure of our proof is similar to that of (Ghodsi et al. 2018): we assume without loss of generality that the MMS values of the agents are equal to 1. For a certain threshold $0 < t$, we prove that an allocation A that maximizes $\sum_i U_i^t(A_i)$ is $t/2$-MMS. Ghodsi *et al.*(2018) prove this claim for $t = 2/5$ and thus imply that a $1/5$-MMS allocation always exists. Via a more in-depth analysis, we prove that this holds for a slightly larger $t > 2/5$ but the analysis involves a more intricate process and a deeper analysis of the valuation functions.

Theorem 2. *For any instance of the fair allocation problem with fractionally subadditive agents a* 0.219225*-MMS allocation always exists.*

4 Subadditive Valuations

In this section, we prove that an $\Omega(\frac{1}{\log n \log \log n})$-MMS allocation is guaranteed to exist when the valuations are subadditive. The high-level ideas of our algorithms is explained in Section 3.1. Here we discuss the proof in detail. Recall that we assume without loss of generality that the MMS value for each agent is equal to 1. From a technical point of view, our proof relies on two combinatorial and probabilistic techniques which we bring in the following.

At a high-level, the first observation implies that no matter what the MMS values are, we can always allocate the items to the agents in a way that a constant fraction of the agents receive a bundle whose value to them is at least a constant fraction of their MMS values.

Lemma 4.1. *For any instance of the fair allocation problem with subadditive valuations there always exists an allocation that guarantees* $1/4$*-MMS to at least* $1/3$ *of the agents.*

We use Lemma 4.1 in an indirect way. As explained in Section 4.1, the first step of our algorithm is to find a multiallocation in a way that each item is given to at most $O(\log n)$ agents and that each agent can divide her items into $\Omega(\log n)$ bundles such that the value of each bundle to her is at least a constant fraction of her MMS value. In order to prove such a multiallocation exists, we first introduce a modified valuation function U_i' for each agent a_i such that (i) for each

subset of size $O(n/\log n)$ of agents, the MMS values of the agents with respect to the modfied valuation functions are $O(\log n)$ times larger than their original MMS values. (ii) if an agent receives a bundle of items whose value to her is a constant fraction of her new MMS value, then she can divide her bundle into $O(\log n)$ parts such that her original valuation for each part is at least some constant value. Via using Lemma 4.1 in an iterative manner, we prove that such a multiallocation exists. We then leverage Lemma 3.1 to turn our multiallocation into a desired allocation.

Lemma 3.1. *Let V be a monotone subadditive function with non-negative values such that for a set S we have $V(S) = 1$. In addition, assume that for some value $0 < t < 1$, for every element $e_i \in S$ we have $V(\{e_i\}) \leq \frac{t}{\log 1/t}$. Let S' be a set made randomly from S such that each element of S appears in S' independently with probability at least t. Then we have* $\Pr[V(S') \leq t/3] \leq 0.77$.

We defer the proofs of Lemmas 3.1 and 4.1 to Section 4.3 and Appendix B, respectively. Our allocation algorithm consists of two parts. In the first part, based on Lemma 4.1, we find a multiallocation with desired properties and in the second part based on Lemma 3.1 we use a randomized procedure to convert this multiallocation into an $\Omega(\frac{1}{\log n \log \log n})$-MMS allocation. These two steps are explained in Sections 4.1 and 4.2.

4.1 Constructing the Multiallocation

In the first part of our algorithm, we construct a multiallocation A with the following properties:

- Each agent a_i can partition her items into $6 \log n$ bundles each with value at least $1/8$ to her.
- No item is allocated to more than $168 \log n$ agents.

Let us begin our discussion in this section with a corollary of Lemma 4.1. Let $r < n$ be a parameter, and let $\mathcal{N}'$ be an arbitrary subset of $\mathcal{N}$ with size n/r. For each agent $a_i \in \mathcal{N}'$, let $P_{i,1}, P_{i,2}, \ldots, P_{i,n}$ be the optimal MMS-partitioning of agent a_i, that is $U_i(P_{i,j}) \geq 1$ for all $1 \leq j \leq n$. We define a new valuation function U_i' for agent a_i as follows: for each subset S of goods,

$$U_i'(S) = \max_{0 \leq j < n/r} \left(\sum_{1 \leq l \leq r} U_i(S \cap P_{i,jr+l}) \right).$$

See Figure 2 in Appendix E for a representation of U_i'. We show in Lemma 4.2 that the new valuation is subadditive.

Lemma 4.2. *For each agent $a_i \in \mathcal{N}'$, U_i' is subadditive.*

Now, consider an instance of the fair allocation problem with agents in $\mathcal{N}'$, valuation U_i' for each agent a_i, and all the items. Also, let MMS_i' be the maximin-share value of agent a_i in this instance. By the way we define the valuations for this instance, we know that for each agent a_i, we have $\mathsf{MMS}_i' \geq r$. By Lemma 4.1, we can allocate to $|\mathcal{N}'|/3$ of the agents in $\mathcal{N}'$, a subset of items with value at least $r/4$ to them. Let a_i be one of these agents. For agent a_i, define set Q_i as set of bundles in the original MMS partitioning of a_i, that contribute a subset with value at least $1/8$ to A_i, that is $Q_i = \{P_{i,j} | U_i(P_{i,j} \cap A_i) \geq 1/8\}$. We claim $|Q_i| \geq r/7$.

Algorithm 1: Finding a multiallocation.

Procedure `Allocate` ($\mathcal{N}$: *set of remaining agents, M: set of goods*) :

> **if** $\mathcal{N} = \emptyset$ **then**
> > | return
>
> **end**
> $\mathcal{N}' =$ a subset of size $\min(n/r, |a|)$ of $\mathcal{N}$
> **foreach** $a_i \in \mathcal{N}'$ **do**
> > | Construct U_i'.
>
> **end**
> A = Allocation defined in Corollary 4.1
> $\mathcal{N}'' =$ agents that receive a bundle in A.
> **foreach** $a_i \in \mathcal{N}''$ **do**
> > Allocate A_i to a_i
> > Remove a_i from $\mathcal{N}$
>
> **end**
> `Allocate` ($\mathcal{N}, M$);

Lemma 4.3. *Let $a_i \in \mathcal{N}'$ be an agent that has received a bundle A_i with $U_i'(A_i) \geq r/4$. We have $|Q_i| \geq r/7$.*

Corollary 4.1 (Lemmas 4.1 and 4.3)**.** *Given a set of n agents with subadditive valuations. For any arbitrary subset $\mathcal{N}'$ of the agents with size at most n/r, it is possible to select a subset of at least $\frac{|\mathcal{N}'|}{3}$ of the agents in $\mathcal{N}'$, and allocate each agent a_i a bundle A_i of the items such that $|Q_i| \geq r/7$.*

Based on Corollary 4.1, we perform the first stage of our allocation algorithm by choosing $r = 42 \log n$ and iteratively running the following steps until no agent remains:

- Select a set $\mathcal{N}'$ of the remaining agents with size n/r. If the total number of the remaining agents is less than n/r, select all the remaining agents.
- Using corollary 4.1, find a subset of size at least $|\mathcal{N}'|/3$ of the agents in $\mathcal{N}'$ and allocate to each agent a_i in this subset a bundle A_i of items such that $|Q_i| \geq r/7$.
- Remove the agents that receive a bundle in the previous step and repeat these steps for the remaining agents and **all the items**. Note that, the goal is to find a multiallocation, so an item might be allocated in multiple rounds.

Algorithm 1 shows a pseudocode of our method for this step.

Lemma 4.4. *At the end of Algorithm 1, the following properties hold:*

- *Each agent a_i can partition her items into $6 \log n$ bundles each with value at least $1/8$ to her.*
- *Each item is allocated to at most $168 \log n$ agents.*

4.2 From Multiallocation to Allocation

Recall that in a multiallocation, we might allocate a good to multiple agents. Let A be the multiallocation obtained in Section 4.1. We know by Lemma 4.4 that in A, each item belongs to at most $168 \log n$ agents. In this step, we convert A into an allocation via a simple procedure: for each item that

Algorithm 2: Random Allocation Algorithm.

Input: A multiallocation A obtained in the first part.
Output: An $\Omega(\log n)$-MMS allocation.
foreach $b \in M$ **do**
> | Let $S = \{a_i | b \in A_i\}$ Allocate b to one of the agents in S uniformly at random.

end

is allocated to multiple agents, we select one of them independently and uniformly at random and allocate the item to her. Algorithm 2 shows a pseudocode for this procedure.

In Lemma 4.5 we prove that assuming that the items are small enough, with a non-zero probability, this process guarantees for each agent a bundle with a value at least $\Omega(1/\log n)$ to her.

Lemma 4.5. *Let A be the multiallocation of Algorithm 1, and let A' be the allocation obtained by running Algorithm 2 on A and let $t = \frac{1}{1344 \log n}$. Then, assuming that the value of each item for each agent is less than $\frac{t}{\lceil \log \frac{1}{t} \rceil}$, in A' with probability more than $1 - 1/n$ each agent receives a bundle with a value of $\frac{1}{4032 \log n}$ to her.*

Proof. Consider an arbitrary agent a_i. By Corollary 4.1 we know that a_i can partition her share into $k \geq 6 \log n$ bundles, each with value at least $1/8$ to her. Let $B_1, B_2, \ldots, B_k$ be these bundles. By definition, for every B_j we have $U_i(B_j) \geq 1/8$. Let B_j' be the items in B_j that remain for agent a_i after running Algorithm 2. Since each good belongs to at most $168 \log n$ agents, each item remains for a_i with probability at least $1/(168 \log n)$ and therefore,

$$\forall j \qquad \mathbb{E}[U_i(B_j')] \geq \frac{U_i(B_j)}{168 \log n} \geq \frac{1}{1344 \log n}.$$

By Lemma 4.3, assuming that the value of each item to each agent is smaller than $\frac{1}{1344 \log n \log(1344 \log n)}$, for every $1 \leq j \leq k$ we have $\Pr\left[U_i(B_j') \leq \frac{1}{4032 \log n}\right] \leq 0.77$. Therefore, with probability at least $1 - (0.77)^k$, for at least one bundle $1 \leq j \leq k$ we have

$$U_i(A_i') \geq U_i(B_j') \geq \frac{1}{4032 \log n}. \qquad (2)$$

Since $k \geq 6 \log n$ and,

$$1 - 0.77^k \geq 1 - 0.77^{6 \log n}$$

$$\geq 1 - 0.77^{2 \log_{0.77}(1/2) \log n} \geq 1 - \frac{1}{n^2},$$

using union bound we conclude that with probability at least $1 - n(1/n^2) = 1 - 1/n$, Inequality (2) holds for all the agents. This in turn implies that with non-zero probability, our allocation is $\frac{1}{4032 \log n}$-MMS. Therefore, such an allocation always exists. $\qquad \square$

Finally, note that in order for Lemma 4.5 to hold, we need the value of each item for each agent to be upper bounded by $\frac{1}{1344 \log n \log(1344 \log n)}$. To resolve this, we choose the

objective to find a $\frac{1}{1344 \log n \log(1344 \log n)}$-MMS allocation. By Lemma 2.1, for this objective we can assume that the value of each item for each agent is upper bounded by $\frac{1}{1344 \log n \log(1344 \log n)}$, and hence, the condition of Lemma 4.5 is satisfied. This reduces the final approximation factor to $\Omega(1/(\log n \log \log n))$.

Theorem 1. *Any maximin fair allocation problem with subadditive agents admits an $\Omega(\frac{1}{\log n \log \log n})$-MMS allocation.*

4.3 Satisfying a Fraction of Agents

In this section, we prove Lemma 4.1. This Lemma states that for an instance of the fair allocation problem with subadditive valuations, there always exists an allocation that allocates to at least a fraction $1/3$ of the agents a bundle with value at least $1/4$. Here we bring the statement of Lemma 4.1.

Lemma 4.1. *For any instance of the fair allocation problem with subadditive valuations there always exists an allocation that guarantees $1/4$-MMS to at least $1/3$ of the agents.*

In our proof, we use the method of Feige (2009) for maximizing welfare when the valuations are subadditive. Assume that the agents' valuations are subadditive and the objective is to maximize social welfare. This problem can be formulated as the following integer program:

$$
\begin{aligned}
\max \quad & \sum_{i,S} x_{i,S} \cdot U_i(S) \\
\text{s.t.} \quad & \sum_{i,S | b_j \in S} x_{i,S} \leq 1, \quad \forall b_j \\
& \sum_{S \subseteq M} x_{i,S} \leq 1, \quad \forall a_i \\
& x_{i,S} \in \{0,1\}, \quad \forall a_i \text{ and } S \subseteq M
\end{aligned} \tag{3}
$$

Roughly speaking, Program (3) allocates the items to the agents in a way that each agent receives at most one subset (first set of constraints) and each item is allocated to at most one agent (second set of constraints). The linear relaxation of Program (3) is a famous linear program, especially in allocation problems. This LP is known as *configuration LP*.

$$
\begin{aligned}
\max \quad & \sum_{i,s} x_{i,S} \cdot U_i(S) \\
\text{s.t.} \quad & \sum_{i,S | b_j \in S} x_{i,S} \leq 1, \quad \forall b_j \\
& \sum_{S \subseteq M} x_{i,S} \leq 1, \quad \forall a_i \\
& x_{i,S} \geq 0, \quad \forall a_i \text{ and } S \subseteq M
\end{aligned} \tag{4}
$$

Note that, despite the exponential number of constraints, assuming demand queries can be answered in polynomial time, it is possible to find a solution to LP (4) in polynomial time. In (Feige 2009) Feige proposes a randomized rounding technique to produce a feasible integer allocation with expected welfare at least half of the value of LP (4). In other words, Feige (2009) proves that the integrality gap of the configuration LP is at most 2 for subadditive valuations. Here, we use this fact to prove that there always exists an allocation that satisfies the conditions of Lemma 4.1.

Recall the definition of bounded welfare. In Proposition 2.1, we state a very useful property of these valuations: for a monotone and subadditive set function V, V^t is also subadditive. According to this fact, consider the following LP:

$$
\begin{aligned}
\max \quad & \sum_{i,s} x_{i,S} \cdot U_i^1(S) \\
\text{s.t.} \quad & \sum_{i,S | j \in S} x_{i,S} \leq 1, \quad \forall b_j \\
& \sum_{S \in P(M)} x_{i,S} \leq 1, \quad \forall a_i \\
& x_{i,S} \geq 0, \quad \forall a_i \text{ and } S \subseteq M
\end{aligned} \tag{5}
$$

Note that LP (5) is similar to LP (4), except that U_i is replaced by U_i^1. Since for any subset S of items, we know that $U_i^1(S) \leq 1$, the objective of LP (5) is upper bounded by n. Also, consider the following fractional solution: for every set S and agent a_i, if S is one of the bundles in the optimal MMS partitioning of agent a_i, set $x_{i,S} = 1/n$ and set $x_{i,S} = 0$ otherwise. One can easily verify that this is a feasible solution to LP (5), with an expected welfare of n. Therefore, the answer of LP (5) is exactly n. Since the integrality gap of the configuration LP is bounded by 2, there exists an integral solution (an allocation) that obtains an objective of at least $n/2$. Denote such an allocation by $A = \langle A_1, A_2, \ldots, A_n \rangle$. We know that the bounded social welfare of this allocation is at least $n/2$, that is $\sum_{1 \leq i \leq n} U_i^1(A_i) \geq n/2$.

Based on the above observation, we prove that allocation A satisfies the conditions of Lemma 4.1.

Proof of Lemma 4.1. Let S be the set of agents that receive a bundle with value at least $1/3$ to them, and assume for contradiction that $|S| < n/3$. The contribution of these agents to the social welfare is at most $|S|$. Also, the contribution of the rest of the agents to the social welfare is less than $(n - |S|)/4$. Therefore, the social welfare is upper bounded by $(n - |S|)/4 + |S| = n/4 + 3|S|/4 < n/4 + n/4 = n/2$. But we already know that the bounded social welfare of A is at least $n/2$, which is a contradiction. $\qquad\square$

5 Fractionally Subadditive Valuations

We improve the result of Ghodsi *et al.* (2018) for fractionally subadditive valuations and show that a 0.2192235-MMS allocation always exists. Our method is based on the notion of bounded welfare, introduced by Ghodsi *et al.* (2018).

The structure of our proof is similar to that of (Ghodsi et al. 2018): we assume without loss of generality that the MMS values of the agents are equal to 1. For a certain threshold $2/5 < t < 1/2$, we prove that an allocation A that maximizes $\sum_i U_i^t(A_i)$ is $t/2$-MMS. Ghodsi *et al.* (2018) prove this claim for $t = 2/5$ and thus imply that a $1/5$-MMS allocation always exists. Via a more in-depth analysis, we prove that this holds for a slightly larger $t > 2/5$ but the analysis involves a more intricate process and a deeper analysis of the valuation functions.

Recall that for a subadditive function V, $V^t(S)$ is defined as $\min(V(S), t)$. Fix a constant t (we later determine the exact value of t) and let A be an allocation that maximizes the bounded social welfare, that is, $w = \sum_j U_j^t(A_j)$. Since

for every agent a_i, the value of $U_i^t(S)$ for any set S of goods is upper bounded by t, a trivial upper bound on the value of w is nt. We show that for a properly chosen threshold $2/5 < t < 1/2$, we can guarantee that every agent receives a bundle in A whose value for the agent is at least $t/2$. We first define the contribution of the items to w.

Definition 3. *For every agent a_j let $\{U_{j,1}^t, U_{j,2}^t, \ldots, U_{j,\alpha_j}^t\}$ be the set of additive functions such that for every subset S of items, $U_j^t = \max_{1 \le l \le \alpha_j} U_{j,l}^t(S)$. Then, for every $S \subseteq M$, we define the contribution of S to w, denoted by $C(S)$ as*

$$C(S) = \sum_{1 \le j \le n} U_{j,l_j}^t(S \cap A_j),$$

where $l_j = \arg\max_{1 \le l \le \alpha_j} U_{j,l}^t(A_j)$.

One can easily observe that function $C(\cdot)$ is additive. Also, since for every agent a_j, U_j^t is fractionally subadditive, we have

$$\forall S \subseteq A_j \qquad U_j^t(A_j \setminus S) \ge U_j^t(A_j) - C(S). \quad (6)$$

Now, assume that there exists an agent a_i such that $U_i^t(A_i) < t/2$. Since $\mathsf{MMS}_i = 1$, agent a_i can partition the goods into n sets with value at least 1 to her. Since $w < nt$[3], the contribution of at least one of these bundles to the value of w is less than t. Let $S = \{b_1, b_2, \ldots, b_k\}$ be the set of goods in this bundle. We assume without loss of generality that the goods in S are sorted according to their *value per contribution* that is,

$$\frac{U_i(\{b_1\})}{C(\{b_1\})} \ge \frac{U_i(\{b_2\})}{C(\{b_2\})} \ge \cdots \ge \frac{U_i(\{b_k\})}{C(\{b_k\})} \quad (7)$$

(see Figure 3 in Appendix E). For a set $T \subseteq S$, we define $\Delta(T) := U_i^t(T) - C(T)$. Since allocation A maximizes the bounded social welfare, there is no way to increases w by modifying A. This yields Observation 5.1.

Observation 5.1. *For every subset $T \subseteq S$, $\Delta(T) < t/2$.*

A simple corollary of Observation 5.1 is that agent a_i cannot divide her goods into two subsets T_1 and T_2 ($T_1 \cap T_2 = \emptyset$), such that $U_i(T_1), U_i(T_2) \ge t$. Otherwise, for at least one of these subsets, say T, we have $\Delta(T) \ge t/2$.

Corollary 5.1 (Observation 5.1). *There are no subsets $T_1, T_2 \subseteq S$ such that $T_1 \cap T_2 = \emptyset$, $U_i^t(T_1) \ge t$, and $U_i^t(T_2) \ge t$.*

Let l be the smallest index such that $U_i^t(\{b_1, b_2, \ldots, b_l\}) = t$. By Corollary 5.1, we know that $U_i^t(\{b_{l+1}, b_{l+2}, \ldots, b_k\}) < t$. Let

$$\gamma = t - U_i^t(\{b_1, \ldots, b_{l-1}\}), \gamma' = t - U_i^t(\{b_{l+1}, \ldots, b_k\}). \quad (8)$$

Notice that both γ and γ' are larger than 0. Since the value of S to agent a_i is at least 1, $U_i^t(\{b_l\}) \ge 1 - 2t + \gamma + \gamma'$.

Observation 5.2. *We have $C(\{b_1, \ldots, b_{l-1}\}) < t/2$ and $C(\{b_{l+1}, \ldots, b_k\}) < t/2$.*

---

[3]Note that, at least one agent has received a bundle with value strictly less than t.

Based on Observation 5.2 define $\delta, \delta' > 0$ such that

$$\delta = t/2 - C(\{b_1, \ldots, b_{l-1}\}), \delta' = t/2 - C(\{b_{l+1}, \ldots, b_k\}). \quad (9)$$

Note that by Observation 5.1, $\delta < \gamma$ and $\delta' < \gamma'$. Also, since $C(S) < t$, $C(\{b_j\}) \le \delta + \delta'$, and by Inequality (7), we have

$$\frac{t - \gamma}{t/2 - \delta} \ge \frac{U_i^t(\{b_l\})}{C(\{b_l\})} \ge \frac{t - \gamma'}{t/2 - \delta'}. \quad (10)$$

Finally, assuming that the goal is to find a $t/2$-MMS allocation, by Lemma 2.1, we can restrict our attention to the cases that the value of each good to each agent is less than $t/2$. Therefore,

$$1 - 2t + \gamma + \gamma' < t/2, \qquad \delta + \delta' < t/2. \quad (11)$$

To conclude, if for every subset T of goods $\Delta(T) < t/2$ holds, the following inequalities must be satisfied:

$$\frac{t - \gamma}{t/2 - \delta} \ge \frac{U_i^t(\{b_l\})}{C(\{b_l\})}, \qquad \text{Inequality (10)}$$
$$\frac{U_i^t(\{b_l\})}{C(\{b_l\})} \ge \frac{t - \gamma'}{t/2 - \delta'} \qquad \text{Inequality (10)}$$
$$1 - 2t + \gamma + \gamma' \le U_i^t(\{b_l\})$$
$$C(\{b_l\}) \le \delta + \delta'$$
$$C(\{b_l\}) \ge \delta, \delta' \qquad \text{Observation 5.1}$$
$$U_i^t(\{b_l\}), C(\{b_l\}) < t/2 \qquad \text{Inequality (11)}$$
$$t > \gamma, \quad t > \gamma'$$
$$\gamma > \delta, \quad \gamma' > \delta' \qquad \text{Observation 5.1}$$
$$\gamma, \gamma', t, \delta, \delta' > 0$$

We show in Appendix C that in order for all the above inequalities to hold, the value of t cannot be arbitrarily small. Indeed, we show that the answer of the following program is at least $t \simeq 0.438447$.

$$\min \quad t$$
$$\frac{t - \gamma}{t/2 - \delta} \ge \frac{U_i^t(\{b_l\})}{C(\{b_l\})},$$
$$\frac{U_i^t(\{b_l\})}{C(\{b_l\})} \ge \frac{t - \gamma'}{t/2 - \delta'}$$
$$1 - 2t + \gamma + \gamma' \le U_i^t(\{b_l\}) \qquad (12)$$
$$C(\{b_l\}) \le \delta + \delta'$$
$$C(\{b_l\}) \ge \delta, \delta'$$
$$U_i^t(\{b_l\}), C(\{b_l\}) < t/2$$
$$t > \gamma, \quad t > \gamma'$$
$$\gamma > \delta, \quad \gamma' > \delta'$$
$$\gamma, \gamma', t, \delta, \delta' > 0$$

This means that for any threshold t less than 0.438447, the set of inequalities in Optimization Program (12) cannot be simultaneously met and therefore, there is a subset T with $\Delta(T) \ge t/2$. This contradicts Observation 5.1. Thus, Lemma 5.1 holds for $t = 0.438447$.

Lemma 5.1. *Let $t \le 0.438447$, and let A be an allocation that maximizes $\sum_i U_i^t(A_i)$. Then, every agent i in A receives a bundle with value at least $t/2$ to her.*

Lemma 5.1 states that for any $t \le 0.438447$, there exists a $t/2$-MMS allocation. Therefore, Theorem 4 holds.

Theorem 4. *For any instance of the fair allocation problem with fractionally subadditive agents a 0.219225-MMS allocation always exists.*

References

Amanatidis, G.; Markakis, E.; Nikzad, A.; and Saberi, A. 2017. Approximation algorithms for computing maximin share allocations. *ACM Transactions on Algorithms (TALG)*, 13(4): 52.

Aziz, H.; Chan, H.; and Li, B. 2019. Weighted maxmin fair share allocation of indivisible chores. *arXiv preprint arXiv:1906.07602*.

Aziz, H.; and Mackenzie, S. 2016. A discrete and bounded envy-free cake cutting protocol for any number of agents. In *Foundations of Computer Science (FOCS), 2016 IEEE 57th Annual Symposium on*, 416–427. IEEE.

Aziz, H.; Rauchecker, G.; Schryen, G.; and Walsh, T. 2017. Algorithms for Max-Min Share Fair Allocation of Indivisible Chores. In *AAAI*, volume 17, 335–341.

Barman, S.; Bhaskar, U.; Krishna, A.; and Sundaram, R. G. 2020. Tight approximation algorithms for p-mean welfare under subadditive valuations. *arXiv preprint arXiv:2005.07370*.

Barman, S.; and Krishna Murthy, S. K. 2017. Approximation algorithms for maximin fair division. In *Proceedings of the 2017 ACM Conference on Economics and Computation*, 647–664. ACM.

Barman, S.; Krishnamurthy, S. K.; and Vaish, R. 2018. Finding fair and efficient allocations. In *Proceedings of the 2018 ACM Conference on Economics and Computation*, 557–574.

Barman, S.; and Sundaram, R. G. 2020. Uniform welfare guarantees under identical subadditive valuations. *arXiv preprint arXiv:2005.00504*.

Barman, S.; and Verma, P. 2020. Existence and Computation of Maximin Fair Allocations Under Matroid-Rank Valuations. *arXiv preprint arXiv:2012.12710*.

Bezáková, I.; and Dani, V. 2005. Allocating indivisible goods. *ACM SIGecom Exchanges*, 5(3): 11–18.

Bhawalkar, K.; and Roughgarden, T. 2011. Welfare guarantees for combinatorial auctions with item bidding. In *Proceedings of the twenty-second annual ACM-SIAM symposium on Discrete Algorithms*, 700–709. SIAM.

Brams, S. J.; and Taylor, A. D. 1995. An envy-free cake division protocol. *American Mathematical Monthly*, 9–18.

Brams, S. J.; and Taylor, A. D. 1996. *Fair Division: From cake-cutting to dispute resolution*. Cambridge University Press.

Budish, E. 2011. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy*, 119(6): 1061–1103.

Caragiannis, I.; Gravin, N.; and Huang, X. 2019. Envy-freeness up to any item with high Nash welfare: The virtue of donating items. In *Proceedings of the 2019 ACM Conference on Economics and Computation*, 527–545.

Caragiannis, I.; Kurokawa, D.; Moulin, H.; Procaccia, A. D.; Shah, N.; and Wang, J. 2016. The unreasonable fairness of maximum Nash welfare. In *Proceedings of the 2016 ACM Conference on Economics and Computation*, 305–322. ACM.

Chaudhury, B. R.; Garg, J.; and Mehlhorn, K. 2020. EFX exists for three agents. In *Proceedings of the 21st ACM Conference on Economics and Computation*, 1–19.

Chaudhury, B. R.; Garg, J.; and Mehta, R. 2020. Fair and efficient allocations under subadditive valuations. *arXiv preprint arXiv:2005.06511*.

Chaudhury, B. R.; Kavitha, T.; Mehlhorn, K.; and Sgouritsa, A. 2020. A little charity guarantees almost envy-freeness. In *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, 2658–2672. SIAM.

Dobzinski, S.; Nisan, N.; and Schapira, M. 2010. Approximation algorithms for combinatorial auctions with complement-free bidders. *Mathematics of Operations Research*, 35(1): 1–13.

Dubins, L. E.; and Spanier, E. H. 1961. How to cut a cake fairly. *American mathematical monthly*, 1–17.

Feige, U. 2009. On maximizing welfare when utility functions are subadditive. *SIAM Journal on Computing*, 39(1): 122–142.

Feige, U.; Mirrokni, V. S.; and Vondrak, J. 2007. Maximizing Non-Monotone Submodular Functions. In *Foundations of Computer Science, 2007. FOCS'07. 48th Annual IEEE Symposium on*, 461–471. IEEE.

Foley, D. K. 1967. Resource allocation and the public sector. *YALE ECON ESSAYS, VOL 7, NO 1, PP 45-98, SPRING 1967. 7 FIG, 13 REF.*

Garg, J.; McGlaughlin, P.; and Taki, S. 2019. Approximating maximin share allocations. *Open access series in informatics*, 69.

Garg, J.; and Taki, S. 2020. An improved approximation algorithm for maximin shares. In *Proceedings of the 21st ACM Conference on Economics and Computation*, 379–380.

Ghodsi, M.; HajiAghayi, M.; Seddighin, M.; Seddighin, S.; and Yami, H. 2018. Fair Allocation of Indivisible Goods: Improvements and Generalizations. In *Proceedings of the 2018 ACM Conference on Economics and Computation*, 539–556. ACM.

Gourvès, L.; and Monnot, J. 2019. On maximin share allocations in matroids. *Theoretical Computer Science*, 754: 50–64.

Kurokawa, D.; Procaccia, A. D.; and Wang, J. 2016. When Can the Maximin Share Guarantee Be Guaranteed? In *AAAI*, volume 16, 523–529.

Kurokawa, D.; Procaccia, A. D.; and Wang, J. 2018a. Fair Enough: Guaranteeing Approximate Maximin Shares. *Journal of the ACM (JACM)*, 65(2): 8.

Kurokawa, D.; Procaccia, A. D.; and Wang, J. 2018b. Fair Enough: Guaranteeing Approximate Maximin Shares. *Journal of the ACM (JACM)*, 65(2): 8.

Li, Z.; and Vetta, A. 2018. The fair division of hereditary set systems. In *International Conference on Web and Internet Economics*, 297–311. Springer.

Lipton, R. J.; Markakis, E.; Mossel, E.; and Saberi, A. 2004. On approximately fair allocations of indivisible goods. In *Proceedings of the 5th ACM conference on Electronic commerce*, 125–131. ACM.

Plaut, B.; and Roughgarden, T. 2020. Almost envy-freeness with general valuations. *SIAM Journal on Discrete Mathematics*, 34(2): 1039–1068.

Schechtman, G. 2003. Concentration, results and applications. In *Handbook of the geometry of Banach spaces*, volume 2, 1603–1634. Elsevier.

Steinhaus, H. 1948. The problem of fair division. *Econometrica*, 16(1).

Stromquist, W. 1980. How to cut a cake fairly. *The American Mathematical Monthly*, 87(8): 640–644.