

Learning to Identify Top Elo Ratings with A Dueling Bandits Approach

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Abstract

The Elo rating system is widely adopted to evaluate the skills of (chess) game and sports players. Recently it has been also integrated into machine learning algorithms in evaluating the performance of computerised AI agents. However, an accurate estimation of the Elo rating (for the top players) often requires many rounds of competitions, which can be expensive to carry out. In this paper, to minimize the number of comparisons and to improve the sample efficiency of the Elo evaluation (for top players), we propose an efficient online match scheduling algorithm. Specifically, we identify and match the top players through a dueling bandits framework and tailor the bandit algorithm to the gradient-based update of Elo. We show that it reduces the per-step memory and time complexity to constant, compared to the traditional likelihood maximization approaches requiring $O(t)$ time. Our algorithm has a regret guarantee of $\tilde{O}(\sqrt{T})$, sublinear in the number of competition rounds and has been extended to the multidimensional Elo ratings for handling intransitive games. We empirically demonstrate that our method achieves superior convergence speed and time efficiency on a variety of gaming tasks.

Introduction

In this paper, we investigate the selection of best multi-agent strategies under the Elo rating systems. The evaluation of the competition outcome has received lots of attention, especially in view of the successful usage of reinforcement learning in StarCraft (Vinyals et al. 2019), Game of Go (Silver et al. 2017) and video games (Mnih et al. 2015). The Elo rating system (Elo 1978) is a predominant and valuable algorithm for evaluating and ranking agents. In the widely adopted Bradley-Terry model (Hunter et al. 2004) for Elo, each player is assigned a numerical rating which is updated with competition outcomes via online stochastic gradient descent. Further, for dealing with non-transitive relations between interacting agents such as the game of *Rock-Paper-Scissors*, Balduzzi et al. (2018) proposes multidimensional Elo (mElo), which decomposes a game into transitive and cyclic parts to handle intransitive skills and evaluates different strategies by computing Nash-averaging.

In practical settings when a competition is expensive to conduct, updating Elo rating in a sample efficient way is

highly valuable. To achieve such sample efficiency, we need a way to select the most informative pairs for evaluation. Two popular sampling approaches are Round-robin (Rasmussen and Trick 2008) and Elimination tournament (Groh et al. 2012); The Round-robin (Rasmussen and Trick 2008) is widely used in sport scheduling to balance the total time, venue usage and fairness of tournaments. It would arrange each team to play against all the others in as few as possible days while satisfying some constraints such as each team doesn't play twice in the same day for promoting game fairness. By contrast, the Elimination tournament (Groh et al. 2012) only allows the winners at each round to proceed to the next round, so the stronger team will have the chance to play more times. A recent approach, RG-UCB, Rowland et al. (2019) introduces an adaptive sampling scheme to estimate the accurate ranking among all agents. RG-UCB considers sampling of agent match-ups as a collection of pure exploration bandit problems (Bubeck, Munos, and Stoltz 2011) and requires enough pairwise comparison for estimating each pair of strategies.

However, these tournament matching/sampling methods suffer from two major limitations which prohibit their wide usage in the modern large scale evaluations. Firstly, both the Round-robin and the Elimination tournament organise competitions following a pre-designed schedule, and the Elimination tournament scheduling may need some prior knowledge on the players' skill. Also each pair of players only compete once in both schemes so the results can be highly noisy. Secondly, the main idea behind the matching schemes of the RG-UCB and the Round-robin is random sampling, which fails to pay more attention to more promising players and/or pairs with higher uncertainty in competition outcome. Thus, they are less sample efficient in identifying the best players.

In this work, we propose two sampling algorithms, named MaxIn-Elo and MaxIn-mElo, for the update of Elo and mElo rating systems respectively. Specifically, we maintain a candidate set with promising players using UCB-based (Upper Confidence Bound) dueling bandits and then select the pair with the highest uncertainty in competition outcome at each round. On the one hand, our algorithms are adapted to the gradient-based update of Elo rating systems, thus more memory and time efficient compared to a prior work Saha and Gopalan (2020) which relies on maximum likelihood es-

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timization (MLE). On the other hand, we extend our method to update mElo (multidimensional Elo) ratings to handle intransitive games, while Saha and Gopalan (2020) is based on a generalized linear model and can only fit to the transitive games. To the best of our knowledge, this is the first work that enables online gradient-based update for dueling bandits, and a theoretical guarantee on the cumulative regret is provided. Also compared to a previous dueling bandit method descending through randomly sampled gradients at each time step (Yue and Joachims 2009), our method, by selecting the pairs with higher information gains from a set of top candidates, is more sample efficient and are guaranteed to converge at $\tilde{O}(\sqrt{T})$ ¹.

In summary, our contributions are three-fold: Firstly, we are the first to propose two online active sampling algorithms MaxIn-Elo and MaxIn-mElo that select maximum informative pairs with dueling bandits to update Elo and mElo ratings. Secondly, we give the regret analysis of our proposed algorithms and show that the algorithms converge at $\tilde{O}(\sqrt{T})$. Thirdly, we demonstrate empirically on synthetic and real-world games that our algorithms achieve significantly lower cumulative regret than all baselines. Notably, our methods outperform MaxInP (Saha, Koren, and Mansour 2021) which uses maximum likelihood for more accurate estimation while with lower time and memory complexity. Code of this project is available at <https://github.com/yanxue7/MaxIn-Elo.git>.

Related Work

Multi-agent evaluation has attracted wide attention in ranking of players (Silver et al. 2017; Lai 2015; Arneson, Hayward, and Henderson 2010; Gruslys et al. 2018) and in selecting stronger strategies in meta games (Muller et al. 2020; Czarnecki et al. 2020). There are many methods used for multi-agent evaluation problem. The Elo rating system is widely used for two-player games such as chess and tennis. It increases (decreases) player’s rating according to player wins (loss) a competition, and updates ratings by online stochastic gradient descent (SGD), which is computationally efficient and simple to implement. While the Elo rating system cannot handle intransitive games such as rock-paper-scissors, multidimensional Elo (Melo) (Balduzzi et al. 2018) was introduced. It decomposes the win-loss matrix of an intransitive game into the transitive component and cyclic component baked in Hodge decomposition theory (Jiang et al. 2011). α -rank (Omidshafiei et al. 2019) is also a popular counterpart in tackling intransitive games, however, it requires the evaluation of all pairs before computing the stationary distribution and is not updated in an online manner. α -IG (Rashid, Zhang, and Ciosek 2021) selects the agent pairs with the largest information gain about the α -rank belief. Despite various evaluation algorithms discussed, how to sample agent pairs at each round is of high value to realize these algorithms in large-scale evaluation tasks. We adopt dueling bandits as a promising sampling scheme.

Similar to the binary competition results in evaluation, the dueling bandits also receive a binary preference between two

selected arms at each round. The concept of dueling bandits was firstly proposed in (Yue and Joachims 2009). Compared to traditional bandits algorithms which pull one arm at each round and receive the reward of this arm directly, dueling bandits pull arm-pair at each round and only get a binary comparison result. DBGD (Yue and Joachims 2009) models a convex optimization problem as the dueling bandits problem which aims to find the best point in a convex space, and DBGD uses a random gradient as the direction of exploration for selecting the next arm-pair. Yue et al. (2012) formulates the best player identification as a dueling bandits problem with noisy comparison results and an underlying winning probability matrix, and proposes two algorithms as well as their corresponding regret bounds. These algorithms identify best arms based on the observed binary feedback however do not learn the player’s skills (ratings), which is helpful in predicting future competition outcomes. Szörényi et al. (2015) regards the ranking of M alternatives (e.g. human players or agents) as a dueling bandits problem. They introduce the confidence interval of rating or winning probability into the dueling bandits problem, and design algorithms to identify the close-to-optimal item or to obtain the close-to-optimal whole ranking respectively. Saha, Koren, and Mansour (2021) studies the adversarial setting, in which the winning probability is non-stationary because players’ skill may change over time. And they measure arms’ abilities by estimating Borda score, however Borda score does not possess predictive power of future competition results and the algorithm of (Saha, Koren, and Mansour 2021) estimates Borda score only by simply calculating the frequency of wins. Heckel et al. (2019) gives an active ranking algorithm that can solve the top- k player identification problem and find the entire sequential ranking among all players.

While existing algorithms could rely on Borda score to update the rankings or design specific sort algorithms to obtain the ranking of items, they are not suitable for the Elo rating systems that adopt stochastic gradient descent to update ratings. The earliest attempt Ding, Hsieh, and Sharpnack (2021) proposes SGD-TS for contextual bandit problem, which learns parameters in the generalized linear model through online SGD instead and employs Thompson Sampling (TS) (Thompson 1933; Agrawal and Goyal 2012, 2013) to encourage exploration in arm-pulling. Compared to UCB-GLM (Li, Lu, and Zhou 2017) that adopt maximum likelihood estimators, SGD-TS achieves a similar theoretical cumulative regret bound, but lower time and memory complexity. However, no prior work has studied the SGD update in dueling bandits setting.

In this work, we will tame dueling bandits for the online match scheduling in the Elo rating system that adopts stochastic gradient descent. Saha and Gopalan (2020) proposes the algorithm that selects Maximum-Informative-Pair (MaxInP) for K -armed contextual dueling bandits. This algorithm utilizes the MLE to estimate parameter $\hat{\theta}$ and uses UCB (Auer, Cesa-Bianchi, and Fischer 2002) estimator to narrow down the set of candidate pairs from which the pair of arms with the maximum uncertainty is selected at each round. Our algorithms adopt a similar design as MaxInP Saha and Gopalan (2020) in calculating the uncertainty of a

¹ \tilde{O} ignores poly-logarithmic factors

pair. However, we use an online batch SGD instead of MLE to update Elo rating, which is more time and memory efficient.

Methodology

Background

Suppose there are n players, the Elo rating system (Elo 1978) assigns a rating $r_x, x \in [n]$ to each player representing its skill. Let r^* denote the true ratings of n players. Our aim is to identify the best player among all n players:

$$x^* = \arg \max_{x \in [n]} r_x^* \quad (1)$$

Denote P as the true winning probability matrix, p_{xy} as the underlying groundtruth probability of x beating y . Based on the Bradley-Terry model (Hunter et al. 2004), the predicted probability of player x winning y is

$$\hat{p}_{xy} = \sigma(r_x - r_y). \quad (2)$$

$\sigma(x)$ is a sigmoid function with $\sigma(x) = \frac{1}{1+e^{-x}}$. Elo ratings are updated by maximizing the likelihood of win-loss predictions which corresponds to minimizing the loss:

$$\ell_{\text{Elo}}(p_{xy}, \hat{p}_{xy}) = -p_{xy} \log \hat{p}_{xy} - (1 - p_{xy}) \log (1 - \hat{p}_{xy}). \quad (3)$$

At time t player x compete with player y with outcome o_{xy}^t : $o_{xy}^t = 1$ if x wins and $o_{xy}^t = 0$ otherwise. We can use o_{xy}^t to compute the gradient of Eq. (3) and update Elo by gradient descent:

$$r_x^{t+1} \leftarrow r_x^t - \eta \cdot \nabla_{r_x} \ell_{\text{Elo}}(o_{xy}^t, \hat{p}_{xy}^t) = r_x^t + \eta \cdot (o_{xy}^t - \hat{p}_{xy}^t). \quad (4)$$

Let T denote the total number of rounds, at each round $t \in [T]$, we adopt a system that will pull a pair of players $(x_t, y_t) \in [n] \times [n]$ and get comparison result $o_t(x_t, y_t) \sim \text{Bern}(p_{xy})$. The cumulative regret of T rounds is defined as

$$R(T) = \sum_{t=1}^T [r_{x^*}^* - \frac{1}{2}(r_{x_t}^* + r_{y_t}^*)]. \quad (5)$$

The definition is consistent with (Saha and Gopalan 2020). It measures the reward difference between the best arm and the two selected arms at each round.

MaxIn-Elo Algorithm

Notations: We use the notations below in the followed presentations.

- n : the number of players.
- τ : the batch size.
- r : a vector of n players' ratings.
- r^* : the true ratings of n players.
- \hat{r}_t : ratings estimated by MLE on the t round comparisons.
- \tilde{r}_j : the SGD estimator at batch j .
- $\bar{r} = \sum_{q=1}^j \tilde{r}_q$: the average of previous SGD iterations.
- $\|x\| = \sqrt{x^\top x}$: the standard ℓ_2 norm.
- e_i : the i -th unit base vector, i.e., the i -dimension equals 1 and all other components equal 0.

Algorithm 1: MaxIn-Elo: Dueling bandits with online SGD for top player identification.

Input: batch size τ , maximum number of rounds T , N players' strategies, parameters α, γ .

Output: output r

- 1: Randomly choose a pair to compare and record as x_t, y_t, o_t for $t \in [\tau]$
- 2: $V_{\tau+1} = \sum_{t=1}^{\tau} (e_{x_t} - e_{y_t})(e_{x_t} - e_{y_t})^\top$
- 3: Calculate the maximum-likelihood estimator \hat{r}_τ by solving $\nabla_r \sum_{t=1}^{\tau} \ell_{\text{Elo}}(o_t, \hat{p}(x_t, y_t)) = 0$
- 4: Maintain convex set $\mathcal{C} = \{r : \|r - \hat{r}_\tau\| \leq 2\}$
- 5: **for** $t = \tau + 1, \tau + 2, \dots, T$ **do**
- 6: **if** $t \% \tau = 1$ **then**
- 7: $j \leftarrow \lfloor (t-1)/\tau \rfloor$ and $\eta_j = \frac{1}{\alpha_j}$
- 8: Calculate gradient $\nabla_r l_{j,\tau}(\tilde{r}_{j-1})$ through Eq. (4)
- 9: Update ratings \tilde{r}_j through Eq. (8)
- 10: Compute $\bar{r} = \frac{1}{j} \sum_{q=1}^j \tilde{r}_q$
- 11: **end if**
- 12: Define a candidate optimal set $\mathcal{S} = \{x \mid \bar{r}_x - \bar{r}_y + \gamma \|e_x - e_y\|_{V_t^{-1}} > 0, \forall y \in [n] \setminus \{x\}\}$
- 13: Select a pair as:
 $(x_t, y_t) = \arg \max_{(x,y) \in \mathcal{S}} \|e_x - e_y\|_{V_t^{-1}}$.
- 14: Let players (x_t, y_t) compete and observe $o_t(x_t, y_t)$
- 15: Compute $V_{t+1} = V_t + (e_{x_t} - e_{y_t})(e_{x_t} - e_{y_t})^\top$
- 16: **end for**

- V_t : the history matrix recording previous t pulling information defined by $V_t = \sum_{i=1}^{t-1} (e_{x_i} - e_{y_i})(e_{x_i} - e_{y_i})^\top$.
- $\|x\|_V$: a special ℓ_2 -norm associated with matrix V defined by $\|x\|_V = \sqrt{x^\top V x}$.
- \mathcal{B} a neighborhood of r^* with $\mathcal{B} = \{r \mid \|r - r^*\| \leq 3\}$.
- \mathcal{C} : a neighborhood of \hat{r}_τ with $\mathcal{C} = \{r \mid \|r - \hat{r}_\tau\| \leq 2\}$.
- $\prod_{\mathcal{C}}(\cdot)$: the projection operation defined by:

$$\prod_{\mathcal{C}}(r) = \hat{r}_\tau + \frac{2 * (r - \hat{r}_\tau)}{\min\{2, \|r - \hat{r}_\tau\|\}} \quad (6)$$

Algorithm Overview The main idea of our MaxIn-Elo algorithm is to maintain a candidate set of promising items via UCB and select the most informative pairs out of the set to evaluate at each round. Firstly, the ratings are initialized by maximizing likelihood of Eq. (3) on a batch of randomly sampled pairs with batch size τ . The solution is denoted as \hat{r}_τ and $\tilde{r}_0 = \hat{r}$. Then starting from round $t = \tau + 1$, we update \tilde{r}_j every τ rounds by solving the following objective function

$$l_{j,\tau}(r) = \sum_{t=(j-1)\tau+1}^{j\tau} \ell_{\text{Elo}}(o_t, \hat{p}(x_t, y_t)). \quad (7)$$

The stochastic gradient update of \tilde{r}_j reads

$$\tilde{r}_j \leftarrow \prod_{\mathcal{C}}(\tilde{r}_{j-1} - \eta_j \nabla_r l_{j,\tau}(\tilde{r}_{j-1})). \quad (8)$$

First, the strong convexity of the objective function is required for fast convergence, and if we select a suitable τ through Eq. (14), the aggregated objective function $l_{j,\tau}(r)$ is a α -strong convex function when $r \in \mathcal{B}$. Second, to ensure $\tilde{r}_j \in \mathcal{B}$, \tilde{r}_j is projected into the convex set \mathcal{C} (also discussed in the proof of Lemma 2).

For each update, a batch of pairs are selected that lead to maximal information gain. The UCB score of a pair is defined by:

$$h(x_t, y_t) = \bar{r}_{x_t} - \bar{r}_{y_t} + \gamma \|e_{x_t} - e_{y_t}\|_{V_t^{-1}}, \quad (9)$$

with the balance parameter γ . The specific V_t^{-1} norm $\gamma \|e_{x_t} - e_{y_t}\|_{V_t^{-1}}$ measures the uncertainty between two arms. The UCB estimator balances the exploitation and exploration through combining ratings estimation \bar{r} and the uncertainty term.

At each round t , we obtain a set of optimal player candidates \mathcal{S} with positive UCB scores:

$$\mathcal{S} = \{x | h(x, y) > 0, \forall y \in [n] \setminus \{x\}\}. \quad (10)$$

From the candidate set \mathcal{S} , we then pull a pair of arms with highest uncertainty by

$$(x_t, y_t) = \arg \max_{(x,y) \in \mathcal{S} \times \mathcal{S}} \|e_x - e_y\|_{V_t^{-1}} \quad (11)$$

to induce sufficient exploration. A detailed algorithm of our MaxIn-Elo is shown in Algorithm 1.

Compared to (Saha and Gopalan 2020) which uses MLE at each iteration, MaxIn-Elo uses SGD to update the Elo rating r as traditional Elo does. Thus, our method is more efficient in both computation and time, and simple to implement. Because the objective function $l_{j,\tau}$ is the aggregation of a batch with τ steps for the α -strong convexity, the online SGD on this objective function is actually the mini-batch gradient descent on τ single steps. The mini-batch update provides diversity and makes our algorithm more smooth and steady. Thus, the selection of the batch size τ is important to achieve the fast convergence speed and smoothness during learning. While RG-UCB randomly selects a pair to evaluate, our MaxIn-Elo selects the maximum informative pair and trade-off exploration and exploitation. To our best knowledge, this is the first algorithm that allows stochastic gradient descent update in dueling bandits settings. See Table 1 for a comparison on time and memory.

MaxIn-mElo Algorithm

To enable the rating system to handle the intransitive skills, we extend the online sampling algorithm to multidimensional Elo ratings (mElo) (Balduzzi et al. 2018). Baking in the Hodge decomposition theory (Jiang et al. 2011), mElo proposed to decompose the antisymmetric logits matrix of win-loss probabilities into a transitive component, i.e. gradient flow of rating vector, and a cyclic component to capture the intransitive relations. By learning a $2k$ -dimensional vector c_x and a rating r_x per player, the win-loss prediction for mElo $_{2k}$ is defined as:

$$\hat{p}_{xy} = \sigma(r_x - r_y + c_x^\top \cdot \Omega_{2k \times 2k} \cdot c_y). \quad (12)$$

Table 1: Comparison of regret, time complexity and memory with other algorithms. Our MaxIn-Elo and the MaxInP achieve the lowest regret bound $\tilde{O}(\sqrt{T})$, but our MaxIn-Elo has lower time and memory complexity than the MaxInP.

Algorithms	Regret	Time Complexity	Memory
DBGD	$O(T^{2/3})$	$O(T)$	$O(n)$
RG-UCB	No	$O(T)$	$O(n)$
Random	No	$O(T)$	$O(n)$
MaxInP	$\tilde{O}(\sqrt{T})$	$O(nT^2 + n^2T)$	$O(nT)$
MaxIn-Elo	$\tilde{O}(\sqrt{T})$	$O(n^2T)$	$O(n^2)$

where $\Omega_{2k \times 2k} = \sum_{i=1}^k (e_{2i-1}e_{2i}^\top - e_{2i}e_{2i-1}^\top)$.

The UCB estimate of a pair (x_t, y_t) for mElo then becomes:

$$h(x_t, y_t) = \bar{r}_{x_t} - \bar{r}_{y_t} + \bar{c}_x^\top \Omega \bar{c}_y + \gamma \|e_{x_t} - e_{y_t}\|_{V_t^{-1}}. \quad (13)$$

Notice that compared to Elo ratings with $k = 0$ (Eq. (2)), mElo ratings assign a feature vector per player to approximate intransitive interactions. We present the details for the mElo ratings and Algorithm 2 for MaxIn-mElo in Appendix.

Regret analysis

We give the cumulative regret bound of MaxIn-Elo, as for as we know, this is the first work that combines the online gradient update with dueling bandits and gives the cumulative regret of dueling bandits while being updated with SGD. We make a mild assumption on the link function σ .

Assumption 1. Define $c_\eta = \inf_{\{\|r-r^*\| \leq \eta\}} \sigma'(r_x - r_y)$, where $(x, y) \in [n] \times [n]$, and we assume $c_3 > 0$.

This assumption is similar to that in (Ding, Hsieh, and Sharpnack 2021). Our main results rely on the following concentration events and the proofs of which are deferred to Appendix.

Lemma 1. Suppose we sample a sequence of arm pairs $\{(x_1, y_1), (x_2, y_2), \dots, (x_t, y_t)\}$ through Algorithm 1 up to round t , and assume the selected batch size τ satisfy that $\lambda_{\min}(V_{\tau+1}) \geq 1$, where $\lambda_{\min}(V_{\tau+1})$ means the minimum eigenvalue of the matrix $(V_{\tau+1})$, Then $\forall t > \tau$,

$$\sum_{i=\tau+1}^t \|(e_{x_i} - e_{y_i})\|_{V_i^{-1}} < \sqrt{2nt \log \left(\frac{2\tau + t}{n} \right)}.$$

Lemma 1 gives the bound of the sum of selected pair's uncertainty from round $\tau + 1$ to t . And this lemma will be adopted to derive the cumulative regret bound. In the following Lemma 2, we show that when the batch size τ is chosen as Eq. (14), we have the concentration property of the averaged SGD estimator \bar{r} .

Lemma 2. Assume that there exists a positive constant λ_f such that $\lambda_{\min}(\mathbb{E}[(e_{x_t} - e_{y_t})(e_{x_t} - e_{y_t})^\top]) \geq \lambda_f$ holds at each round $t > \tau$, where (x_t, y_t) is sampled through Algo-

Algorithm 1. Let the batch size τ satisfies

$$\begin{aligned}\tau_1 &= 2 \left(\frac{C_1 \sqrt{n} + C_2 \sqrt{2 \log T}}{\lambda_{\min}(B)} \right)^2 + \frac{16(n + 2 \log T)}{c_1^2 \lambda_{\min}(B)}, \\ \tau_2 &= 2 \left(\frac{C_1 \sqrt{n} + C_2 \sqrt{2 \log T}}{\lambda_f} \right)^2 + \frac{4\alpha}{c_3 \lambda_f}, \\ \tau &= \lceil \max \{ \tau_1, \tau_2 \} \rceil,\end{aligned}\quad (14)$$

where $B = \mathbb{E}_{(x,y) \sim \text{iid}_{[n] \times [n]}} [(e_x - e_y)(e_x - e_y)^T]$. Define $g_1(t)$ and $g_2(j)$,

$$g_1(t) = \frac{1}{2c_1} \sqrt{\frac{n}{2} \log \left(1 + \frac{2t}{n} \right) + 2 \log T}, \quad (15)$$

$$g_2(j) = \frac{\tau}{\alpha} \sqrt{1 + \log j}. \quad (16)$$

For a constant $\alpha \geq c_3$, there exists two positive constants C_1, C_2 such that if the batch size τ is chosen as Eq. (14), then we have that at each round $t > \tau$ corresponding to batch $j = \lfloor \frac{t}{\tau} \rfloor$, event $E_1(t)$ holds with probability at least $1 - \frac{5}{T^2}$, where $E_1(t) = \{ \forall (x, y) : |(e_x - e_y)^T (\bar{r}_j - r^*)| \leq g_1(t) \|e_x - e_y\|_{V_t^{-1}} + g_2(j) \frac{\sqrt{2}}{\sqrt{j}} \}$.

The following Lemma 3 shows how to select a suitable balanced parameter γ of UCB score that ensures the best player is always in the candidate set.

Lemma 3. At each round $t > \tau$, if we let UCB balanced parameter $\gamma = g_1(t)$ and α satisfies that $\alpha \geq \frac{\sqrt{2\tau} \sqrt{1 + \log j}}{\Delta \sqrt{j}}$, then we have $x^* \in \mathcal{S}$ holds with probability at least $1 - \frac{5}{T^2}$, where $j = \lfloor \frac{t}{\tau} \rfloor$, Δ is the difference between ratings of optimal player x^* and sub-optimal player x' . Recall $x^* = \arg \max_{x \in [n]} r_x^*$, and define $x' = \arg \max_{x \in [n]/x^*} r_x^*$, $\Delta = r_{x^*}^* - r_{x'}^*$.

Lemma 3 shows that if we properly select UCB balanced parameter γ and parameter α which describes objective function $l_{j,\tau}$ as a α -strongly convex, then it is promised that the best player x^* is in candidates set \mathcal{S} with high probability. This property is helpful for the top-1 identification because the candidate set \mathcal{S} will become tighter with the time, and x^* always in \mathcal{S} , thus candidate set \mathcal{S} only contains x^* eventually. Together we are ready to present our main results in Theorem 1.

Theorem 1. We run our Algorithm 1 to get a sequence of arm-pair, and let the learning rate parameter $\alpha \geq \max \{ c_3, \frac{\sqrt{2\tau} \sqrt{1 + \log j}}{\Delta \sqrt{j}} \}$, the balanced parameter $\gamma = g_1(t)$, there exists two positive parameter C_1, C_2 such that if the batch size τ is chosen as Eq. (14), then we have the cumulative regret satisfies that:

$$R(T) \leq \tau * \Delta_{\max} + 2g_1(T) \sqrt{2nT \log \left(\frac{2\tau + T}{n} \right)} + 4g_2(J) \sqrt{\tau T},$$

with probability at least $1 - \frac{10}{T}$, where $J = \lfloor \frac{T}{\tau} \rfloor$, $\Delta_{\max} = \max_i r_i^* - \min_i r_i^*$ and $g_1(T), g_2(J)$ is defined in Eq. (15).

Note that $\tau \sim O(\max \{ n, \log T \})$ (Eq. (14)), $g_1(T) \sim O(\sqrt{n \log T})$, $g_2(J) \sim O(\sqrt{\log T})$. Combining the above

analysis, we have $R(T) \sim O(n \log T \sqrt{T})$ (or $\tilde{O}(\sqrt{T})$). This regret upper bound is equivalent to that in (Saha and Gopalan 2020) which employs MLE estimators. However, our algorithm improves the efficiency in terms of memory and time. The memory cost is constant with respect to T while MaxInP's memory cost is linear in the time horizon T . The time complexity of our MaxIn-Elo is $O(n^2 T)$, while MaxInP's time complexity is $O(nT^2 + n^2 T)$. See Table 1 for a detailed comparison. Detailed proofs are referred to Appendix.

Experiments

We consider the following two batteries of experiments to evaluate the performance of our algorithms in the scenarios of transitive and intransitive real world meta-games. Ablation studies of parameter γ , dimension of mElo and the batch size τ can be found in Appendix.

Baselines

Random: The pairwise matching scheme of the classical Round-robin (Rasmussen and Trick 2008) tournament is based on random sampling. We construct a simple baseline that randomly select a pair from all $n * (n - 1) / 2$ pairs with replacement. After sampling a pair, we use the Elo/mElo model to update the ratings.

RG-UCB (Rowland et al. 2019): This algorithm adopts a pure exploration sampling scheme, which uniformly samples a pair from the set containing pairs that need to be estimated. And the stopping condition $C(\delta)$ controls the total number of comparisons of each pair, where δ is a hyper parameter deciding the confidence level of estimated competitive results.

DBGD (Yue and Joachims 2009): This dueling bandits algorithm is popular in ranking tasks when only pair-wise binary feedback is available. It maintains one winning arm at each round, and randomly synthesizes a gradient to obtain the opponent arm in the contextual bandit setting. In our feature free setting, this is equivalent to randomly selecting a player as the opponent.

α -IG (Rashid, Zhang, and Ciosek 2021): This is an active sampling algorithm used for estimating the α -rank (Omidshafiei et al. 2019). This algorithm selects a pair with largest information gain at each round. In the transitive case, the top player has an α -rank score equal to 1. Due to the high computation cost at each round (computing α -rank for 80000 times in a 4×4 game), we only compare with it in a 4×4 transitive game: the '2 Good, 2Bad' game given by α -IG.

MaxInP (Saha and Gopalan 2020): This algorithm is for the generalized linear contextual dueling bandits problem, in which arms are represented as feature vectors. And it uses MLE to estimate model parameters θ relying on all historical pairwise comparisons at each round t . This algorithm calculates a candidate set containing advanced arms and pulls an arm-pair with the largest uncertainty. In order to fit their model, each player is described as a one-hot vector, and the estimated parameters θ correspond to players' ratings in our setting.

MaxIn-Elo: Our first algorithm adopts dueling bandits to adaptively sample pairs for Elo rating update in Eq. (2). The

aim of our MaxIn-Elo is to identify the advanced players gradually, and to minimize the cumulative regret described in Eq. (5) simultaneously.

MaxIn-mElo: Our second algorithm tames the intransitive scenarios. Different to MaxIn-Elo, there is an extra vector c to capture intransitive relationship in competition outcome prediction. The dimension of c is set to 8 in experiments. For the MaxIn-mElo algorithm, we hope to identify players with superior mElo ratings and to minimize cumulative regret on mElo ratings.

Experiments setting

Real world games We do our experiments on twelve real-games released by Czarnecki et al. (2020), most of which are implemented on the OpenSpiel framework (Lanctot et al. 2019). The six games used for evaluating Elo are Triangular game, Transitive game, Elo game, and three noisy variants of Elo games. The first three are transitive games; the three variants of Elo game are Elo games with additive Gaussian noises. The six intransitive games used for evaluating mElo are Kuhn-poker, AlphaStar, tic_tac_toe, hex, Blotto and 5,3-Blotto game.

The intransitivity of games can be revealed by sink strongly connected components (SSCCs) (Omidshafiei et al. 2019), which is a set of strategies that cannot be defeated by external strategies and all internal strategies become a circle, such as Rock, Paper, Scissors. The statistics of these games is shown in Table 2 in Appendix.

Metrics Except the cumulative regret defined in Eq. (5), we introduce three other metrics for Reciprocal Rank (RR), Normalized Discounted Cumulative Gain (NDCG), and Hit Ratio (HR). RR is used for the results on generating top-1 players. NDCG and HR report discrete performance for top-1 performance and are thus used in top- k results.

Reciprocal Rank (RR) (Donmez, Svore, and Burges 2009) give the reciprocal of predicted ranking of the best player x^* . Define

$$RR = \frac{1}{R(x^*)},$$

where $R(x)$ returns the ranking of player x relying on currently predicted ratings \bar{r} . Larger RR corresponds to better performance on the top-1 player identification.

Hit Ratio@ K (He et al. 2015) is defined as the ratio of the predicted top- k that belongs to the true top- k . For hit ratio doesn't consider the positions of correct predictions, we also adopts NDCG (Donmez, Svore, and Burges 2009) which assigns higher importance to results at top ranks. Normalized Discounted Cumulative Gain (NDCG) is widely used in the evaluation of rankings and the NDCG@ k measures the importance of predicted top- k players. It is given by

$$NDCG@K = \frac{1}{N_K} \sum_{i=1}^K \frac{2^{l(d_i)} - 1}{\log(i + 1)},$$

where N_k is a normalizer to ensure that the perfect ranking would result in $NDCG@K = 1$. d_i denote the index of predicted i -th player, and $l(x) \in \{0, 1\}$ is the relevance level about top- k identification, we set $l(x) = 1$ if player x in true top- k otherwise 0.

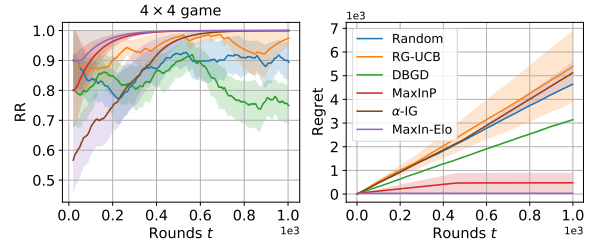


Figure 1: Results on 4 × 4 game (2 Good 2 Bad).

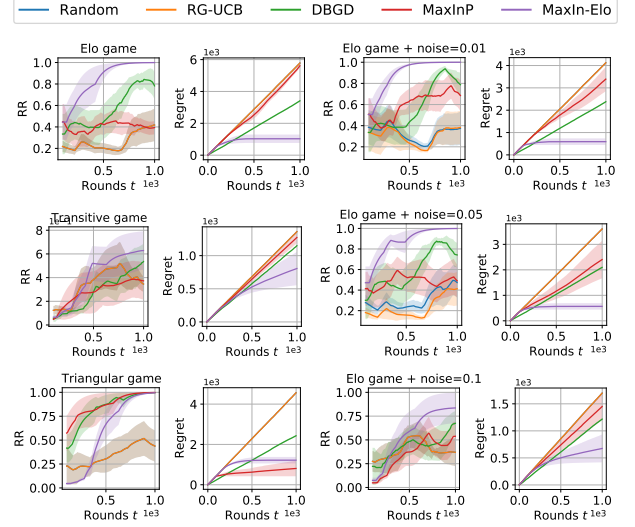


Figure 2: Results of Elo on transitive games.

Parameters setting For Random, DBGD, and RG-UCB baseline, we perform a grid search for the initial step size η in the range $\{0.01, 0.05, 0.1, 0.5, 1, 5, 10\}$. For RG-UCB, stopping confidence $\delta = 0.2$. For MaxInP, we tune the UCB balanced parameter $\gamma \in \{0.2, 0.4, 0.6, \dots, 2.0\}$. For MaxIn-Elo and MaxIn-mElo, we tune the initialized learning rate $\eta \in \{0.01, 0.05, 0.1, 0.5, 1, 5, 10\}$, and the learning rate at batch j is set as $\frac{\eta}{j}$. And the UCB balanced parameter $\gamma \in \{0.2, 0.4, 0.6, \dots, 2.0\}$. The batch size τ of MaxInP, MaxIn-Elo and MaxIn-mElo is set to $0.7 * n$. When baselines uses mElo model to calculate ratings, we set the dimension of the extra vector c as 8. We use the parameters that report the best performance for α -IG. We repeat experiments 5 times with different random seeds and plot the averaged performance with standard deviations.

All experiments were run in a single x86_64 GNU/Linux machine with 256 AMD EPYC 7742 64-Core Processor and 2 A100 PCIe 40GB GPU. We use sklearn(0.24.2) to solve the MLE.

Results

Figure 1, 2, 3 show the results of top-1 identification on 13 games. To ensure a fair comparison between all baselines, we perform a grid search to select parameters with the best RR performance for each random seed. If the winning prob-

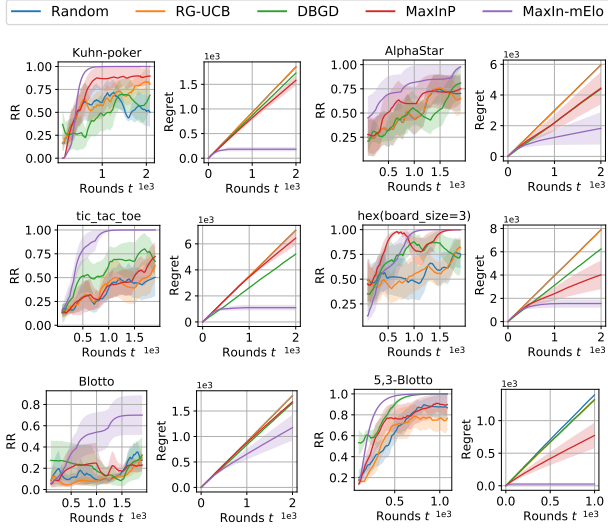


Figure 3: Results of mElo on intransitive games.

ability matrix can be fitted into the Elo model, then we calculate the true ratings through Eq. (2), otherwise we use the mElo ratings as the true ratings through Eq. (12).

Evaluation of MaxIn-Elo Figure 1 shows the results on a 4×4 transitive game. MaxIn-Elo has the highest convergence rate on both RR and cumulative regret metrics, and MaxIn-Elo has the lowest cumulative regret close to 0. As shown in Figure 2, MaxIn-Elo significantly outperforms all other baselines on five games and achieves similar performance on Triangular game. Regarding the RR metric, MaxIn-Elo can converge to 1 on four games. Even on Transitive game and Elo game + noise=0.1, RR scores as up to 0.6 and 0.8 respectively, which indicates that the rank of the top player no more than 2. Thus we think MaxIn-Elo has the ability to effectively identify the top player. On the Elo game, Elo game + noise=0.01, and Elo game + noise=0.01, the cumulative regret are closed to convergence at around 500 rounds. When the cumulative regret meets convergence, the candidate optimal set \mathcal{S} only contains the top player, and no regret increasing.

Different from the other 5 stochastic games, Triangular game is a deterministic game with all winning probabilities are equal to 1 or 0, thus it is easy to evaluate. For DBGD baseline, it maintains the current best player and randomly selects an opponent, so it could find the best player more quickly, but has a large cumulative regret because of randomly selected opponents.

Evaluation of MaxIn-mElo Figure 3 shows the results of baselines on six real-world intransitive games. MaxInP is based on the Elo model for it is a special generalized linear model only with rating parameter r without cyclic vector parameter c , but all other baselines are based on the mElo model. As the Figure 3 shows, MaxIn-mElo has the lowest cumulative regret and the highest RR on all six games. With regard to the RR, MaxIn-mElo can be up to 1 on all games except for Blotto. One possible reason why MaxIn-mElo cannot be up to 1 on Blotto may be that its size of top SSCC is very large. The other reason is that we use the low-rank ap-

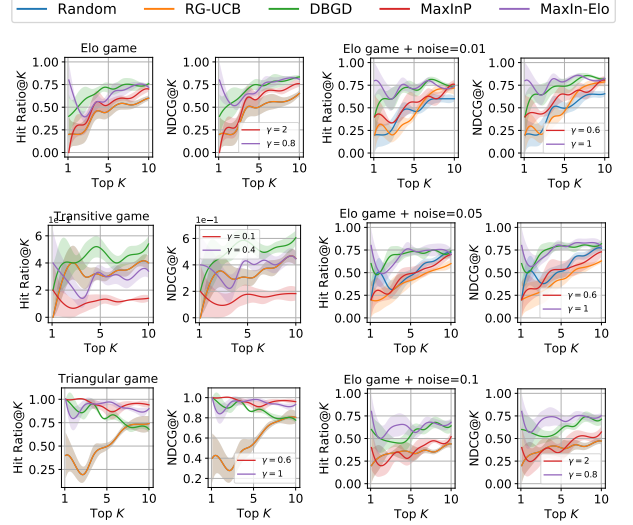


Figure 4: Results of Top- k player identification on transitive games. γ in red and purple indicates that reports best performance for MaxInP and MaxIn-Elo respectively.

proximation of the probability matrix’s rotation on the mElo model. Although we misidentified the top-1 player, we are still better than all other baselines.

Results of Top- k player identification Figure 4 gives the results of top- k predictions on transitive games. MaxIn-Elo and MaxInP both have a parameter γ used to balance exploration and exploitation, larger γ can lead to a larger candidate set then lead to better top- k performance. We keep other parameters fixed and run experiments with different $\gamma \in \{0.2, 0.4, 0.6, \dots, 2.0\}$, and we report the performance of MaxInP and MaxIn-Elo under the best γ . Figure 4 shows that MaxIn-Elo has the best performance of the top-1 identification on all games, and it achieves the comparable performance of top- k identification on most games. Results of different γ can be found in Appendix.

Discussions

This work studied the problem of multi-agent evaluation with Elo ratings. We have adopted an online match scheduling framework to improve the sample efficiency of the Elo rating system and its extension mElo for the intransitive settings. Both empirical and theoretical results justify that our algorithms can achieve higher sample efficiency and lower regret on most of the tasks.

Limitations and Future work We consider two limitations of this work. Firstly, the match outcome prediction in our algorithm is based on only ratings without considering features that describe players. Future work may consider adding features into the match prediction. Secondly, our algorithm focuses more on identifying the best player without being tailored for identifying top- k players. Future work can consider active sampling that achieves better results on both top-1 and top- k cases.

Ethical Statement

This work mainly focuses on identifying top Elo ratings among a set of AI agents or real-world sports games such as football. Our method is based on the assumption that the true skill will not be changed as the increase of the number of comparisons, which is consistent with the competitions of the AI agents. We did not consider the influence of energy and the skill's self-improvement on the evaluation of humans. Nevertheless, we provide a method that can guide sports schedules that is more efficient than the random-based methods such as Round-Robin.

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