Algorithmic Fairness Verification with Graphical Models[*]

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Abstract

In recent years, machine learning (ML) algorithms have been deployed in safety-critical and high-stake decision-making, where the *fairness* of algorithms is of paramount importance. Fairness in ML centers on detecting bias towards certain demographic populations induced by an ML classifier and proposes algorithmic solutions to mitigate the bias with respect to different fairness definitions. To this end, several *fairness verifiers* have been proposed that compute the bias in the prediction of an ML classifier—essentially beyond a finite dataset—given the probability distribution of input features. In the context of verifying *linear classifiers*, existing fairness verifiers are limited by *accuracy* due to imprecise modeling of correlations among features and *scalability* due to restrictive formulations of the classifiers as SSAT/SMT formulas or by sampling.

In this paper, we propose an efficient fairness verifier, called FVGM, that encodes the correlations among features as a Bayesian network. In contrast to existing verifiers, FVGM proposes a *stochastic subset-sum* based approach for verifying linear classifiers. Experimentally, we show that FVGM leads to an *accurate* and *scalable* assessment for more diverse families of fairness-enhancing algorithms, fairness attacks, and group/causal fairness metrics than the state-of-the-art. We also demonstrate that FVGM facilitates the computation of fairness influence functions as a stepping stone to detect the source of bias induced by subsets of features.

1 Introduction

The significant improvement of machine learning (ML) over the decades has led to a host of applications of ML in high-stake decision-making such as college admission (Martinez Neda, Zeng, and Gago-Masague 2021), hiring of employees (Ajunwa et al. 2016), and recidivism prediction (Tollenaar and Van der Heijden 2013; Dressel and Farid 2018). ML algorithms often have an accuracy-centric learning objective, which may cause them to be biased towards certain part of the dataset belonging to a certain economically or socially sensitive groups (Landy, Barnes, and Murphy 1978; Zliobaite 2015; Berk 2019). The following example illustrates a plausible case of unfairness induced by ML algorithms.

Example 1.1. *Following (Ghosh, Basu, and Meel 2021, Example 1.), let us consider an ML problem where the classifier decides the eligibility of an individual for health insurance given their income and fitness (Figure 1). Here, the sensitive feature 'age' (A) follows a Bernoulli distribution, and income (I) and fitness (F) follow Gaussian distributions. We generate 1000 samples from these distributions and use them to train a Support Vector Machine (SVM) classifier. The decision boundary of this classifier is* $9.37I + 9.75F - 0.34A \geq 9.4$, *where* $A = 1$ *denotes the sensitive group 'age* $\geq$ *40'. This classifier selects an individual above and below* 40 *years of age with probabilities* 0.24 *and* 0.86, *respectively. This illustrates a disparate treatment of individuals of two age groups by the SVM classifier.*

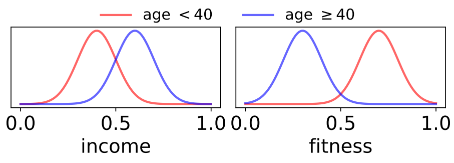


Figure 1: Age-dependent distributions of income and fitness in Example 1.1.

In order to identify and mitigate the bias of ML classifiers, different fairness definitions and fairness algorithms have been proposed (Hardt, Price, and Srebro 2016; Kusner et al. 2017; Mehrabi et al. 2019). In this paper, we focus on two families of fairness definitions: *group* and *causal* fairness. Group fairness metrics, such as disparate impact and equalized odds constrain the probability of the positive prediction of the classifier to be (almost) equal among different sensitive groups (Dwork et al. 2012; Feldman et al. 2015). On the other hand, causal fairness metrics assess the difference in positive predictions if every feature in the causal relation remains identical except the sensitive feature (Nabi and Shpitser 2018; Zhang and Bareinboim 2018). The early works on *fairness verification* focused on measuring fairness metrics of a classifier for a given dataset (Bellamy et al. 2018). Naturally, such techniques were limited in enhancing confidence of users for wide deployment. Consequently, recent verifiers seek to achieve verification beyond finite dataset and in turn focus on the probability distribution of features (Albarghouthi et al. 2017; Bastani, Zhang, and Solar-Lezama 2019; Ghosh, Basu, and Meel 2021). More specifically, the input to the verifier is a classifier and the probability distribution of features, and

[*]Source code: https://github.com/meelgroup/justicia

the output is an estimate of fairness metrics that the classifier obtains given the distribution. For Example 1.1, a fairness verifier takes the SVM classifier and the distribution of features I, F, A as an input and outputs the probability of positive prediction of the classifier for different sensitive groups.

In order to solve the fairness-verification problem, existing works have proposed two principled approaches. Firstly, Ghosh, Basu, and Meel (2021) and Albarghouthi et al. (2017) propose formal methods that reduce the problem into a solution of an SSAT or an SMT formula respectively. Secondly, Bastani, Zhang, and Solar-Lezama (2019) propose a sampling-based approach that relies on extensively enumerating the conditional probabilities of prediction given different sensitive features and thus, incurs high computational cost. Additionally, existing works assume feature independence of non-sensitive features and consider correlated features within a limited scope, such as conditional probabilities of non-sensitive features w.r.t. sensitive features. As a result, the *scalability* and *accuracy* of existing verifiers remains a major challenge.

In this work, we seek to remedy the aforementioned situation. As a first step, we focus on *linear classifiers*, which has attracted significant attention from researchers in the context of fair algorithms (Pleiss et al. 2017; Zafar et al. 2017; Dressel and Farid 2018; John, Vijaykeerthy, and Saha 2020). At this point, it is worth highlighting that our empirical evaluation demonstrates that the existing techniques fail to scale beyond small examples or provide highly inaccurate estimates for comparatively *small* linear classifiers.

Our Contributions. In this paper, we propose a fairness verification framework, namely FVGM, for accurately and efficiently verifying linear classifiers. FVGM proposes a novel *stochastic subset-sum* encoding for linear classifiers with an efficient pseudo-polynomial solution using dynamic programming. To address feature-correlations, FVGM considers a graphical model, particularly a Bayesian Network that represents conditional dependence (and independence) among features in the form of a Directed Acyclic Graph (DAG). Experimentally, FVGM is more accurate and scalable than existing fairness verifiers; FVGM can verify group and causal fairness metrics for multiple fairness algorithms. We also demonstrate two novel applications of FVGM as a fairness verifier: (a) detecting fairness attacks, and (b) computing Fairness Influence Functions (FIF) of features as a mean of identifying (un)fairness contribution of a subset of features.

2 Background

In this section, we define different fairness metrics proposed for classification. Following that, we state basics of stochastic subset sum and Bayesian networks that are the main components of our methodology.

Fairness in ML. We consider[1] a dataset $\mathbf{D}$ as a collection of triples $(\mathbf{X}, \mathbf{A}, Y)$ generated from an underlying distribution $\mathcal{D}$. $\mathbf{X} \triangleq \{X_1, \dots, X_{m_1}\}$ are non-sensitive features whereas $\mathbf{A} \triangleq \{A_1, \dots, A_{m_2}\}$ are categorical sensitive features. $Y \in \{0, 1\}$ is the binary label (or class) of $(\mathbf{X}, \mathbf{A})$. Each non-sensitive feature X_i is sampled from a continuous probability distribution $\mathcal{X}_i$, and each sensitive feature $A_j \in \{0, \dots, N_j\}$ is sampled from a discrete probability distribution $\mathcal{A}_j$. We use $(\mathbf{x}, \mathbf{a})$ to denote the feature-values of $(\mathbf{X}, \mathbf{A})$. For sensitive features, a valuation vector $\mathbf{a} = [a_1, .., a_{m_2}]$ is called a *compound sensitive group*. For example, consider $\mathbf{A} = \{\text{race, sex}\}$ where race $\in$ {Asian, Color, White} and sex $\in$ {female, male}. Thus $\mathbf{a}$ = [Asian, female] is a compound sensitive group. We represent a binary classifier trained on the dataset $\mathbf{D}$ as $\mathcal{M} : (\mathbf{X}, \mathbf{A}) \rightarrow \hat{Y}$. Here, $\hat{Y} \in \{0, 1\}$ is the predicted class of $(\mathbf{X}, \mathbf{A})$.

We now discuss different fairness metrics in the literature based on the prediction of a classifier (Feldman et al. 2015; Hardt, Price, and Srebro 2016; Nabi and Shpitser 2018). In this paper, FVGM verifies two families of fairness metrics: group fairness (first three in the following) and path-specific causal fairness.

1. *Disparate impact* (DI): A classifier satisfies $(1 - \epsilon)$-disparate impact if for $\epsilon \in [0, 1]$, $\min_{\mathbf{a}} \Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}] \geq (1 - \epsilon) \max_{\mathbf{a}'} \Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}']$.
2. *Statistical Parity* (SP): A classifier satisfies ϵ-statistical parity if for $\epsilon \in [0, 1]$, $\max_{\mathbf{a}, \mathbf{a}'} |\Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}] - \Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}']| \leq \epsilon$.
3. *Equalized Odds* (EO): A classifier satisfies ϵ-equalized odds if for $\epsilon \in [0, 1]$, $\max_{\mathbf{a}, \mathbf{a}'} |\Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}, Y = 0] - \Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}', Y = 0]| \leq \epsilon$, and $\max_{\mathbf{a}, \mathbf{a}'} |\Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}, Y = 1] - \Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}', Y = 1]| \leq \epsilon$.
4. *Path-specific Causal Fairness* (PCF): Let $\mathbf{a}_{\max} \triangleq \arg\max_{\mathbf{a}} \Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}]$. We consider mediator features $\mathbf{Z} \subseteq \mathbf{X}$ sampled from the conditional distribution $\mathcal{Z}_{|\mathbf{A} = \mathbf{a}_{\max}}$. This emulates the fact that mediator variables have the same sensitive features $\mathbf{a}_{\max}$. For $\epsilon \in [0, 1]$, path-specific causal fairness is defined as $\max_{\mathbf{a}, \mathbf{a}'} |\Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}, \mathbf{Z}] - \Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}', \mathbf{Z}]| \leq \epsilon$.

For all of the above metrics, lower value of ϵ indicates higher fairness demonstrated by the classifier $\mathcal{M}$. Following the observation of (Ghosh, Basu, and Meel 2021), computing all of the aforementioned fairness metrics is equivalent to computing the maximum and minimum of the *Positive Predictive Value* (PPV) of the classifier, denoted as $\Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}]$, for all compound sensitive groups $\mathbf{a}$ from $\mathbf{A}$. Thus, in Section 3, we focus on computing the maximum and minimum PPVs and then extend it to assess corresponding fairness metrics. We call the group for which PPV is maximum (minimum) as the *most (least) favored group*.

Stochastic Subset Sum Problem (S3P). Let $\mathbf{B} \triangleq \{B_i\}_{i=1}^{|\mathbf{B}|}$ be a set of Boolean variables and $w_i \in \mathbb{Z}$ be the weight of B_i. Given a constraint of the form $\sum_{i=1}^{|\mathbf{B}|} w_i B_i = \tau$, for a constant $\tau \in \mathbb{Z}$, the subset-sum problem seeks to compute $\mathbf{b} \in \{0, 1\}^{|\mathbf{B}|}$ such that the constraint evaluates to true when $\mathbf{B}$ is substituted with $\mathbf{b}$. Subset sum problem is known to be a NP-complete problem and well-studied in theoretical

[1]We represent sets/vectors by bold letters, and the corresponding distributions by calligraphic letters. We express random variables in uppercase, and an assignment of a random variable in lowercase.

computer science (Kleinberg and Tardos 2006). The *counting* version of the subset-sum problem counts all $\mathbf{b}$'s for which the above constraint holds; and this problem belongs to the complexity class #P. In this paper, we consider the counting problem for the constraint $\sum_{i=1}^{|\mathbf{B}|} w_i B_i \geq \tau$ where variables B_i's are stochastic. More precisely, we define a *stochastic subset-sum* problem, namely S3P, that computes $\Pr[\sum_{i=1}^{|\mathbf{B}|} w_i B_i \geq \tau]$. Details of S3P are in Section 3.1.

Bayesian Network. In general, a Probabilistic Graphical Model (Koller and Friedman 2009), and specifically a *Bayesian network* (Pearl 1985; Chavira and Darwiche 2008), encodes the dependencies and conditional independence between a set of random variables. In this paper, we leverage an access to a Bayesian network on $(\mathbf{X}, \mathbf{A})$ that represents the joint distribution on them. A Bayesian network is denoted by a pair (G, θ), where $G \triangleq (\mathbf{V}, \mathbf{E})$ is a Directed Acyclic Graph (DAG), and θ is a set of parameters that encodes the conditional probabilities induced by the joint distribution under investigation. Each vertex $V_i \in \mathbf{V}$ corresponds to a random variable. Edges $\mathbf{E} \in \mathbf{V} \times \mathbf{V}$ imply conditional dependencies among variables. For each variable $V_i \in \mathbf{V}$, let $\mathrm{Pa}(V_i) \subseteq \mathbf{V} \setminus \{V_i\}$ denote the set of parents of V_i. Given $\mathrm{Pa}(V_i)$ and parameters θ, V_i is independent of its other non-descendant variables in G. Thus, for the assignment v_i of V_i and $\mathbf{u}$ of $\mathrm{Pa}(V_i)$, the aforementioned semantics of a Bayesian network encodes the joint distribution of $\mathbf{V}$ as:

$$\Pr[V_1 = v_1, \dots, V_{|\mathbf{V}|} = v_{|\mathbf{V}|}] = \prod_{i=1}^{|\mathbf{V}|} \Pr[V_i = v_i | \mathrm{Pa}(V_i) = \mathbf{u}; \theta]. \tag{1}$$

3 FVGM: Fairness Verification with Graphical Models

In this section, we present FVGM, a fairness verification framework for linear classifiers that accounts for correlated features represented as a graphical model. The core idea of verifying fairness of a classifier is to compute the PPV of the classifier with respect to all compound sensitive groups. To this end, FVGM solves a stochastic subset sum problem, S3P, that is equivalent to computing the PPV of the classifier for the most (and the least) favored sensitive group. In this section, we first define S3P and present an efficient dynamic programming solution for S3P. We then extend S3P to consider correlated features as input. We conclude by discussing fairness verification based on the solution of S3P in practice.

Problem Formulation. Given a linear classifier $\mathcal{M} : (\mathbf{X}, \mathbf{A}) \rightarrow \hat{Y}$ and a probability distribution $\mathcal{D}$ of $\mathbf{X} \cup \mathbf{A}$, our objective is to compute $\max_{\mathbf{a}} \Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}]$ and $\min_{\mathbf{a}} \Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}]$ with respect to $\mathcal{D}$. In this study, we express a linear classifier $\mathcal{M}$ as

$$\hat{Y} = \mathbb{1}\Big[\sum_i w_{X_i} X_i + \sum_j w_{A_j} A_j \geq \tau\Big].$$

Here, w denotes the weight (or coefficients) of a feature, τ denotes the bias of the classifier, and $\mathbb{1}$ is an indicator function. Hence, the prediction $\hat{Y} = 1$ if and only if the inner inequality holds. Thus, computing the maximum (resp. minimum) PPV is equivalent to finding out the assignment of A_j's for which the probability of satisfying the inner inequality is highest (resp. lowest). We reduce this computation into an instance of S3P. To perform this reduction, we assume weights w and bias τ as integers, and features $\mathbf{X} \cup \mathbf{A}$ as Boolean. In Sec. 3.5, we relax these assumptions and extend to the practical settings.

3.1 S3P: Stochastic Subset Sum Problem

Now, we formally describe the specification and semantics of S3P. S3P operates on a set of Boolean variables $\mathbf{B} = \{B_i\}_{i=1}^n \in \{0,1\}^n$, where $w_i \in \mathbb{Z}$ is the weight of B_i, and $n \triangleq |\mathbf{B}|$. Given a constant threshold $\tau \in \mathbb{Z}$, S3P computes the *probability* of a subset of $\mathbf{B}$ with sum of weights of non-zero variables to be at least τ. Formally,

$$S(\mathbf{B}, \tau) \triangleq \Pr\Big[\sum_i w_i B_i \geq \tau\Big] \in [0,1].$$

Aligning with terminologies in stochastic satisfiability (SSAT) (Littman, Majercik, and Pitassi 2001), we categorize the variables $\mathbf{B}$ into two types: (i) *chance variables* that are stochastic and have an associated probability of being assigned to 1 and (ii) *choice variables* that we optimize while computing $S(\mathbf{B}, \tau)$. To specify the category of variables, we consider a *quantifier* $q_i \in \{Я^{p_i}, \exists, \forall\}$ for each B_i. Elaborately, $Я^p$ is a *random quantifier* corresponding to a chance variable B, where $p \triangleq \Pr[B = 1]$. In contrast, $\exists$ is an *existential quantifier* corresponding to a choice variable B such that a Boolean assignment of B *maximizes* $S(\mathbf{B}, \tau)$. Finally, $\forall$ is an *universal quantifier* for a choice variable B that fixes an assignment to B that *minimizes* $S(\mathbf{B}, \tau)$.

Now, we formally present the semantics of $S(\mathbf{B}, \tau)$ provided that each variable B_i has weight w_i and quantifier q_i. Let $\mathbf{B}[2:n] \triangleq \{B_j\}_{j=2}^n$ be the subset of $\mathbf{B}$ without the first variable B_1. Then $S(\mathbf{B}, \tau)$ is recursively defined as:

$$S(\mathbf{B}, \tau) = \begin{cases} \mathbb{1}[\tau \leq 0], \text{ if } \mathbf{B} = \emptyset \\ S(\mathbf{B}[2:n], \tau - \max\{w_1, 0\}), \text{ if } q_1 = \exists \\ S(\mathbf{B}[2:n], \tau - \min\{w_1, 0\}), \text{ if } q_1 = \forall \\ p_1 \times S(\mathbf{B}[2:n], \tau - w_1) + \\ (1 - p_1) \times S(\mathbf{B}[2:n], \tau), \text{ if } q_1 = Я^{p_1} \end{cases} \tag{2}$$

Observe that when $\mathbf{B}$ is empty, S is computed as 1 if $\tau \leq 0$, and $S = 0$ otherwise. For existential and universal quantifiers, we compute S based on the weight. Specifically, if $q_1 = \exists$, we decrement the threshold τ by the maximum between w_1 and 0. For example, if $w_1 > 0$, B_1 is assigned 1, and assigned 0 otherwise. Therefore, by solving for an existential variable, we maximize S. In contrast, when if $q_1 = \forall$, we fix an assignment of B_1 that minimizes S by choosing between the minimum of w_1 and 0. Finally, for random quantifiers, we decompose the computation of S into two sub-problems: one sub-problem where $B_1 = 1$ and the updated threshold becomes $\tau - w_1$ and another sub-problem where $B_1 = 0$ and the updated threshold remains the same. Herein, we compute S as the expected output of both sub-problems.

Remark. *$S(\mathbf{B}, \tau)$ does not depend on the order of $\mathbf{B}$.*

Computing Minimum and Maximum PPVs of Linear Classifiers Using S3P**.** For computing $\max_{\mathbf{a}} \Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}]$ of a linear classifier, we set existential quantifiers $\exists$ to sensitive features A_j, randomized quantifiers R to non-sensitive features X_i and construct a set $\mathbf{B} = \mathbf{A} \cup \mathbf{X}$. The coefficients w_{A_j} and w_{X_i} of the classifier become weights of $\mathbf{B}$. Also, we get $n = m_1 + m_2$. For non-sensitive variables X_i, which are chance variables, we derive their marginal probability $p_i = \Pr[X_i = 1]$ from the distribution $\mathcal{D}$. According to semantic of S3P, setting $\exists$ quantifiers on $\mathbf{A}$ computes the maximum value of $S(\mathbf{B}, \tau)$ that equalizes the maximum PPV of the classifier. In this case, the *inferred* assignment of $\mathbf{A}$ implies the most favored group $\mathbf{a}_{\max} = \arg\max_{\mathbf{a}} \Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}]$. In contrast, to compute the minimum PPV, we instead assign each variable A_j a universal quantifier while keeping random quantifiers over X_i, and infer the least favored group $\mathbf{a}_{\min} = \arg\min_{\mathbf{a}} \Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}]$.

3.2 A Dynamic Programming Solution for S3P

We propose a dynamic programming approach (Pisinger 1999; Woeginger and Yu 1992) to solve S3P as the problem has overlapping sub-problem properties. For example, $S(\mathbf{B}, \tau)$ can be solved by solving $S(\mathbf{B}[2:n], \tau')$, where the updated threshold τ', called the *residual threshold*, depends on the original threshold τ and the assignment of B_1 as shown in Eq. (2). Building on this observation, we propose a recursion and terminating condition leading to our dynamic programming algorithm.

Recursion. We consider a function $\mathsf{dp}(i, \tau)$ that solves the sub-problem $S(\mathbf{B}[i:n], \tau)$, for $i \in \{1, \ldots, n\}$. The semantics of $S(\mathbf{B}, \tau)$ in Eq. (2) induces the recursive definition of $\mathsf{dp}(i, \tau)$ as:

$$\mathsf{dp}(i, \tau) = \begin{cases} \mathsf{dp}(i+1, \tau - \max\{w_i, 0\}), & \text{if } q_i = \exists \\ \mathsf{dp}(i+1, \tau - \min\{w_i, 0\}), & \text{if } q_i = \forall \\ p_i \times \mathsf{dp}(i+1, \tau - w_i) + \\ (1 - p_i) \times \mathsf{dp}(i+1, \tau), & \text{if } q_i = \text{R}^{p_i} \end{cases} \quad (3)$$

Eq. (3) shows that $S(\mathbf{B}, \tau)$ can be solved by instantiating $\mathbf{dp}(1, \tau)$, which includes all the variables in $\mathbf{B}$.

Terminating Condition. Let w_{neg}, w_{pos}, and w_{all} be the sum of negative, positive, and all weights of $\mathbf{B}$, respectively. We observe that $w_{neg} \leq w_{all} \leq w_{pos}$. Thus, for any i, if the residual threshold $\tau \leq w_{neg}$, there is always a subset of $\mathbf{B}[i:n]$ with sum of weights at least τ. Conversely, when $\tau > w_{pos}$, there is no subset of $\mathbf{B}[i:n]$ with sum of weights at least τ. We leverage this bound and tighten the terminating conditions of $\mathsf{dp}(i, \tau)$ in Eq. (4).

$$\mathsf{dp}(i, \tau) = \begin{cases} 1 \text{ if } \tau \leq w_{neg} \\ 0 \text{ if } \tau > w_{pos} \\ \mathbb{1}[\tau \leq 0] \text{ if } i = n+1 \end{cases} \quad (4)$$

Eq. (3) and (4) together define our dynamic programming algorithm. While deploying the algorithm, we store $\mathsf{dp}(i, \tau)$ in memory to avoid repetitive computations. This allows us to achieve a pseudo-polynomial algorithm (Lemma 1) instead of a naïve exponential algorithm enumerating all possible assignments.

Lemma 1. *Let n' be the number of existential and universal variables in $\mathbf{B}$. Let $w_\exists = \sum_{B_i \in \mathbf{B} | q_i = \exists} \max\{w_i, 0\}$ and $w_\forall = \sum_{B_i \in \mathbf{B} | q_i = \forall} \min\{w_i, 0\}$ be the considered sum of weights of existential and universal variables, respectively. We can exactly solve* S3P *using dynamic programming with time complexity $\mathcal{O}((n - n')(\tau + |w_{neg}| - w_\exists - w_\forall) + n')$. The space complexity is $\mathcal{O}((n - n')(\tau + |w_{neg}| - w_\exists - w_\forall))$.*

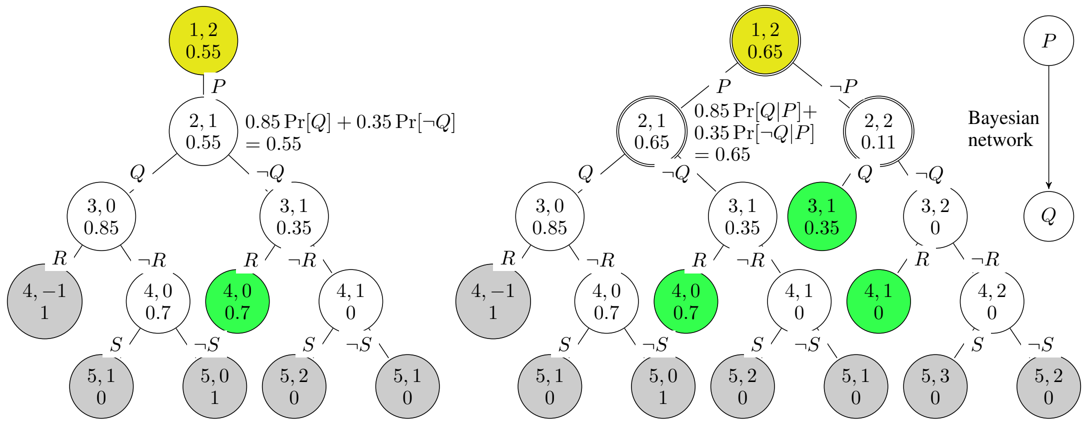
A Heuristic for Faster Computation. We propose two improvements for a faster computation of the dynamic programming solution. Firstly, we observe that in Eq. (3), existential/universal variables are deterministically assigned based on their weights. Hence, we reorder $\mathbf{B}$ such that existential/universal variables appear earlier in $\mathbf{B}$ than random variables. This allows us to avoid unnecessary repeated exploration of existential/universal variables in dp. Moreover, according to the remark in Section 3.1, reordering $\mathbf{B}$ still produces the same exact solution of S3P. Secondly, to reach the terminating condition of $\mathsf{dp}(i, \tau)$ more frequently, we sort $\mathbf{B}$ based on their weights—more specifically, within each cluster of random, existential, and universal variables. In particular, if $\tau \leq 0.5(w_{pos} - w_{neg})$, τ is closer to w_{pos} than w_{neg}. Hence, we sort each cluster in descending order of weights. Otherwise, we sort in ascending order. We illustrate our dynamic programming approach in Example 3.1.

Example 3.1. *We consider a linear classifier $P + Q + R - S \geq 2$. Herein, P is a Boolean sensitive feature, and Q, R, S are Boolean non-sensitive features with $\Pr[Q] = 0.4, \Pr[R] = 0.5$, and $\Pr[S] = 0.3$. To compute the maximum PPV of the classifier, we impose an existential quantifier on P and randomized quantifiers on others. This leads us to the observation that $P = 1$ is the optimal assignment as $w_P = 1 > 0$. We now require to compute $\Pr[Q + R - S \geq 1]$, which by dynamic programming, is computed as 0.55. The solution is visualized as a search tree in Figure 2a, where we observe that storing solution of sub-problems in the memory avoids repetitive computation, such as exploring the node $(4, 0)$. Similarly, the minimum PPV of the classifier is 0.14 (not shown in Figure 2a) where we impose a universal quantifier on P to obtain $P = 0$ as the optimal assignment.*

3.3 S3P with Correlated Variables

In S3P presented in Section 3.1, we consider all Boolean variables to be probabilistically independent. This independence assumption often leads to an *inaccurate estimate* of PPVs because both sensitive and non-sensitive features can be correlated in practical fairness problems. Therefore, we extend S3P to include correlations among variables.

We consider a Bayesian network $\mathsf{BN} = (G, \theta)$ to represent correlated variables, where $G \triangleq (\mathbf{V}, \mathbf{E})$, $\mathbf{V} \subseteq \mathbf{B}$, $\mathbf{E} \subseteq \mathbf{V} \times \mathbf{V}$, and θ is the parameter of the network. In BN, we constrain that there is no conditional probability of choice (i.e., existential and universal) variables as we optimize their assignment in S3P. Choice variables, however, can impose conditions on chance (i.e., random) variables. In practice,



(a) Known marginal probabilities. (b) Probabilities computed with Bayesian network.

Figure 2: Search tree representation of S3P for computing the maximum PPV of the classifier on variables $\mathbf{B} = \{P, Q, R, S\}$ with weights $\{1, 1, 1, -1\}$ and threshold $\tau = 2$. Each node is labeled by (i, τ'), where i is the index of $\mathbf{B}$ and τ' is the residual threshold. The tree is explored using Depth-First Search (DFS) starting with left child. Within a node, the value in the bottom denotes $\mathsf{dp}(i, \tau')$ that is solved recursively based on sub-problems $\mathsf{dp}(i+1, \cdot)$ in child nodes. Yellow nodes denote *existential* variables and all other nodes are *random* variables. Additionally, a green node denotes a collision, in which case a previously computed dp solution is returned. Leaf nodes (gray) are computed based on terminating conditions in Eq. 4. In Figure 2b, nodes with double circles, such as $\{(1, 2), (2, 1), (2, 2)\}$, are enumerated exponentially to compute conditional probabilities from the Bayesian network.

we achieve this by allowing no incoming edge on choice variables while learning BN (ref. Section 4).

For a chance variable $B_i \in \mathbf{V}$, let $\mathrm{Pa}(B_i)$ denote its parents. According to Eq. (1), for an assignment $\mathbf{u}$ of $\mathrm{Pa}(B_i)$, BN ensures B_i to be independent of other non-descendant variables in $\mathbf{V}$. Hence, in the recursion of Eq. (3), we substitute p_i with $\Pr[B_i = 1|\mathrm{Pa}(B_i) = \mathbf{u}]$. In order to explicate the dependence on $\mathbf{u}$, we denote the expected solution of $S(\mathbf{B}[i : n], \tau)$ as $\mathsf{dp}(i, \tau, \mathbf{u})$, which for $B_i \in \mathbf{V}$ is modified as:

$$\mathsf{dp}(i,\tau, \mathbf{u}) = \Pr[B_i = 0|\mathrm{Pa}(B_i) = \mathbf{u}]\mathsf{dp}(i + 1, \tau, \mathbf{u} \cup \{0\}) \\ + \Pr[B_i = 1|\mathrm{Pa}(B_i) = \mathbf{u}]\mathsf{dp}(i + 1, \tau - w_i, \mathbf{u} \cup \{1\}).$$

Since $\mathsf{dp}(i, \tau, \mathbf{u})$ involves $\mathbf{u}$, we initially perform a topological sort of $\mathbf{V}$ to know the assignment of parents before computing dp on the child. Moreover, there are $2^{|\mathrm{Pa}(B_i)|}$ assignments of $\mathrm{Pa}(B_i)$, and we compute $\mathsf{dp}(i, \tau, \mathbf{u}')$ for all $\mathbf{u}' \in \{0, 1\}^{|\mathrm{Pa}(B_i)|}$ to incorporate all conditional probabilities $\Pr[B_i = 1|\mathrm{Pa}(B_i) = \mathbf{u}']$ into S3P. For this enumeration, we do not store $\mathsf{dp}(i, \tau, \mathbf{u})$ in memory. However, for $B_i \notin \mathbf{V}$ that does not appear in the network, we instead compute $\mathsf{dp}(i, \tau)$ and store it in memory as in Section 3.2, because B_i is not correlated with other variables. Lemma 2 presents the complexity of solving S3P with correlated variables, wherein unlike Lemma 1, the complexity differentiates based on variables in $\mathbf{V}$ (exponential) and $\mathbf{B} \setminus \mathbf{V}$ (pseudo-polynomial).

Lemma 2. *Let $\mathbf{V} \subseteq \mathbf{B}$ be the set of vertices in the Bayesian network and n'' be the number of existential and universal variables in $\mathbf{B} \setminus \mathbf{V}$. Let $w'_\exists = \sum_{B_i \in \mathbf{B}\setminus\mathbf{V}|q_i=\exists} \max\{w_i, 0\}$ and $w'_\forall = \sum_{B_i \in \mathbf{B}\setminus\mathbf{V}|q_i=\forall} \min\{w_i, 0\}$ be the sum of considered weights of existential and universal variables, respectively that only appear in $\mathbf{B} \setminus \mathbf{V}$. To exactly compute S3P with correlated variables in dynamic programming approach, time complexity is $\mathcal{O}(2^{|\mathbf{V}|} + (n - n'' - |\mathbf{V}|)(\tau + |w_{neg}| - w'_\exists - w'_\forall) + n'')$ and space complexity is $\mathcal{O}((n - n'' - |\mathbf{V}|)(\tau + |w_{neg}| - w'_\exists - w'_\forall))$.*

A Heuristic for Faster Computation. We observe that to encode conditional probabilities, we enumerate all assignments of variables in $\mathbf{V}$. For computing PPVs of a linear classifier with correlated features, we consider a heuristic to sort variables in $\mathbf{B} = \mathbf{A} \cup \mathbf{X}$. Let $\mathbf{V} \subseteq \mathbf{B}$ be the set of vertices in the network and $\mathbf{V}^c = \mathbf{B} \setminus \mathbf{V}$. In this heuristic, we sort sensitive variables $\mathbf{A}$ by positioning $\mathbf{A} \cap \mathbf{V}$ in the beginning followed by $\mathbf{A} \cap \mathbf{V}^c$. Then we order the variables $\mathbf{B}$ such that variables in $\mathbf{A}$ precedes those in $\mathbf{X} \cap \mathbf{V}$, and the variables in $\mathbf{X} \cap \mathbf{V}^c$ follows the ones in $\mathbf{X} \cap \mathbf{V}$. This sorting allows us to avoid repetitive enumeration of variables in $\mathbf{V} \subseteq \mathbf{B}$ as they are placed earlier in $\mathbf{B}$.

Example 3.2. *We extend Example 3.1 with a Bayesian Network (G, θ) with $\mathbf{V} = \{P, Q\}$ and $\mathbf{E} = \{(P, Q)\}$. Parameters θ imply conditional probabilities $\Pr[Q|P] = 0.6$ and $\Pr[Q|\neg P] = 0.3$. In Figure 2b, we enumerate all assignment of P and Q to incorporate all conditional probabilities of Q given P. We, however, observe that the dynamic programming solution in Section 3.2 still prunes search space for variables that do not appear in $\mathbf{V}$, such as $\{R, S\}$. Hence following the calculation in Figure 2b, we obtain the maximum PPV as 0.65 for $P = 1$. The minimum PPV (not shown) is similarly calculated as 0.11 for $P = 0$.*

3.4 Fairness Verification with Computed PPVs

Given a classifier $\mathcal{M}$, a distribution $\mathcal{D}$, and a fairness metric f, verifying whether a classifier is ϵ-fair ($\epsilon \in [0, 1]$) is equivalent to computing $\mathbb{1}[f(\mathcal{M}|\mathcal{D}) \le \epsilon]$. We now compute $f(\mathcal{M}|\mathcal{D})$ based on the maximum PPV $\max_{\mathbf{a}} \Pr[\hat{Y} = 1|\mathbf{A} = \mathbf{a}]$ and the minimum PPV $\min_{\mathbf{a}} \Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}]$ of a classifier.

For measuring fairness metric SP, we compute the difference $\max_{\mathbf{a}} \Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}] - \min_{\mathbf{a}} \Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}]$. We, however, deploy FVGM twice while measuring EO: one for the distribution $\mathcal{D}$ conditioned on $Y = 1$ and another for $Y = 0$. In each case, we compute $\max_{\mathbf{a}} \Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}, Y = y] - \min_{\mathbf{a}} \Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}, Y = y]$ for $y \in \{0, 1\}$ and take the maximum difference as the value of EO. For measuring causal metric PCF, we compute $\max_{\mathbf{a}} \Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}, \mathbf{Z}]$ and $\min_{\mathbf{a}} \Pr[\hat{Y} = 1, \mathbf{Z} | \mathbf{A} = \mathbf{a}, \mathbf{Z}]$ conditioned on mediator features $\mathbf{Z}$ and take their difference. To measure DI, we compute the ratio $\max_{\mathbf{a}} \Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}] / \min_{\mathbf{a}} \Pr[\hat{Y} = 1 | \mathbf{A} = \mathbf{a}]$. In contrast to other fairness metrics, DI closer to 1 indicates higher fairness level. Thus, we verify whether a classifier achieves $(1-\epsilon)$-DI by checking $\mathbb{1}[DI(\mathcal{M}|\mathcal{D}) \geq 1 - \epsilon]$.

3.5 Extension to Practical Settings

For verifying linear classifiers with real-valued features and coefficients, we preprocess them so that FVGM can be invoked. Let $X_c \in \mathbb{R}$ be a continuous real-valued feature with coefficient $w_c \in \mathbb{R}$ in the classifier. We discretize X_c to a set of k Boolean variables $\mathbf{B}_c$ based on the binning-based discretization and assign a Boolean variable to each bin. Hence, $B_i \in \mathbf{B}_c$ becomes 1, when X_c belongs to the i^{th} bin. Let μ_i denote the mean of feature-values within i^{th} bin. We then set the coefficient of B_i as $w_c \mu_i$. By law of large numbers, $X_c \approx \sum_i \mu_i B_i$ for infinitely many bins (Grimmett and Stirzaker 2020). Finally, we multiply the coefficients of discretized variables by $l \in \mathbb{N} \setminus \{0\}$ and round to an integer. Accuracy of the preprocessing step relies on the number of bins k and the multiplier l. Therefore, we empirically fine-tune both k and l by comparing the processed classifier with the initial classifier on a validation dataset.

4 Empirical Performance Analysis

In this section, we empirically evaluate the performance of FVGM. We first present the experimental setup and the objective of our experiments, followed by experimental results.[2]

Experimental Setup. We implement a prototype of FVGM in Python (version 3.8). We deploy the Scikit-learn library for learning linear classifiers such as Logistic Regression (LR) and Support Vector Machine (SVM) with linear kernels. We perform five-fold cross-validation. While the classifier is trained on continuous features, we discretize them to Boolean features to be invoked by FVGM. While discretizing, we apply a gird-search to estimate the best bin-size within a maximum bin of 10. To convert the coefficients of features into integers, we employ another grid-search to choose the best multiplier within $\{1, 2, \dots, 100\}$. For learning a Bayesian network on the converted Boolean data, we deploy the PGMPY library (Ankan and Panda 2015). For network learning, we apply a Hill-climbing search algorithm that learns a DAG structure by optimizing K2 score (Koller and Friedman 2009). For estimating parameters of the network, we use Maximum Likelihood Estimation (MLE) algorithm.

[2]Due to page limit, we present additional experimental results, proof of lemmas, and miscellaneous details in Appendix.

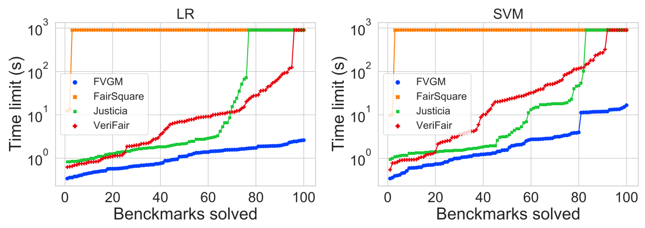


Figure 3: A cactus plot to present the scalability of different fairness verifiers. The number of solved benchmarks are on the X-axis and the time taken on the Y-axis; a point (x, y) implies that a verifier takes less than or equal to y seconds to compute fairness metrics of x many benchmarks. We consider 100 benchmarks generated from 5 real-world datasets using 5-fold cross-validation. In each fold, we consider $\{25, 50, 75, 100\}$ percent of non-sensitive features.

We compare FVGM with three existing fairness verifiers: Justicia (Ghosh, Basu, and Meel 2021), FairSquare (Albarghouthi et al. 2017), and VeriFair (Bastani, Zhang, and Solar-Lezama 2019).

4.1 Scalability Analysis

Benchmarks. We perform the scalability analysis on five real-world datasets studied in fair ML literature: UCI Adult, German-credit (Doshi-Velez and Kim 2017), COMPAS (Angwin et al. 2016), Ricci (McGinley 2010), and Titanic (https://www.kaggle.com/c/titanic). We consider 100 benchmarks generated from 5 real-world datasets and report the computation times (for DI and SP) of different verifiers.

Results. In Figure 3, we present the scalability results of different verifiers. First, we observe that FairSquare often times out ($= 900$ seconds) and can solve ≤ 5 benchmarks. This indicates that SMT-based reduction for linear classifiers cannot scale. Similarly, SSAT-based verifier Justicia that performs pseudo-Boolean to CNF translation for linear classifiers, times out for around 20 out of 100 benchmarks. Sampling-based framework, VeriFair, has comparatively better scalability than SMT/SSAT based frameworks and can solve more than 90 benchmarks. Finally, FVGM achieves impressive scalability by solving all 100 benchmarks with 1 to 2 orders of magnitude runtime improvements than existing verifiers. Therefore, S3P-*based framework* FVGM *proves to be highly scalable in verifying fairness properties of linear classifiers than the state-of-the-art.*

4.2 Accuracy Analysis

Benchmark Generation. To perform accuracy analysis, we require the ground truth, which is not available for real-world instances. Therefore, we focus on generating synthetic benchmarks for analytically computing the ground truth of different fairness metrics, such as DI, from the known distribution of features. In each benchmark, we consider $n \in \{2, 3, 4, 5\}$ features including one Boolean sensitive feature, say A, generated from a Bernoulli distribution with mean 0.5. We generate non-sensitive features X_i from Gaussian distributions such that $\Pr[X_i | A = 1] \sim \mathcal{N}(\mu_i, \sigma^2)$ and $\Pr[X_i | A = 0] \sim$

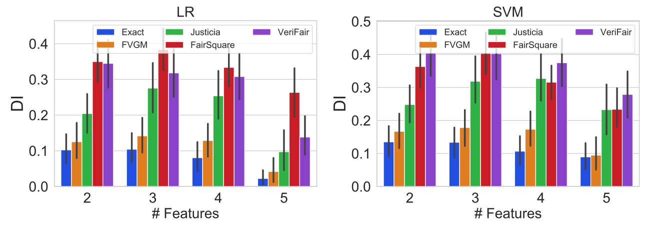


Figure 4: Comparing the average accuracy of different verifiers over 100 synthetic benchmarks while varying the number of features. FVGM yields the closest estimation of the analytically calculated *Exact* values of DI for LR and SVM classifiers.

D	A	Algo.	ΔDI	ΔPCF	ΔSP	ΔEO
Adult	race	RW	**0.53**	**-0.06**	**-0.06**	**-0.02**
		OP	**0.57**	**-0.07**	**-0.07**	**-0.02**
	sex	RW	**0.96**	**-0.16**	**-0.15**	**-0.08**
		OP	**0.43**	**-0.08**	**-0.08**	0.03

Table 1: Verification of fairness algorithms using FVGM. **D** and **A** denote dataset and sensitive features. RW and OP refer to reweighing and optimized-preprocessing algorithms. Numbers in bold refer to fairness improvement.

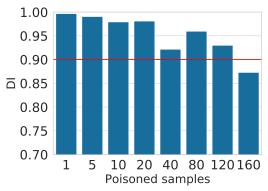


Figure 5: Verifying poisoning attack against fairness using FVGM. The red line denotes the safety margin of the ML model against the attack.

$\mathcal{N}(\mu_i', \sigma^2)$, where $\mu_i, \mu_i' \in [0,1]$, $\sigma = 0.1$, and μ_i, μ_i' are chosen from a uniform distribution in $[0,1]$. Finally, we create label $Y = \mathbb{1}[\sum_{i=1}^{n-1} X_i \geq 0.5 \sum_{i=1}^{n-1}(\mu_i + \mu_i')]$ such that Y does not directly depend on the sensitive feature. For each n, we generate 100 random benchmarks, learn LR and SVM classifiers on them, and compute DI[3] using different verifiers.

Results. We assess the accuracy of the competing verifiers in estimating fairness metrics, specifically DI with LR and SVM classifiers. In Figure 4, FVGM computes DI closest to the *Exact* value for different number of features and both type of classifiers. In contrast, Justicia, FairSquare, and VeriFair measure DI far from the *Exact* because of ignoring correlations among the features. For example, for SVM classifier with $n = 5$ (right plot in Figure 4), *Exact* DI is 0.089 (average over 100 random benchmarks). Here, FVGM computes DI as 0.094, while all other verifiers compute DI as at least 0.233. Therefore, FVGM *is more accurate than existing verifiers as it explicitly considers correlations among features*.

[3]To analytically compute DI, let the coefficients of the classifier be w_i for X_i and w_A for A, and bias be τ. Since all non-sensitive features are from Gaussian distributions, we compute the probability of the predicted class $\Pr[\hat{Y}|A=1] \sim \mathcal{N}(\sum_{i=1}^{n-1} w_i\mu_i, \sigma_{\hat{Y}}^2)$ and $\Pr[\hat{Y}|A=0] \sim \mathcal{N}(\sum_{i=1}^{n-1} w_i\mu_i', \sigma_{\hat{Y}}^2)$ with $\sigma_{\hat{Y}}^2 = (\sum_{i=1}^{n-1} w_i^2)\sigma^2$. Hence, the PPV of the classifier is $1 - \mathsf{CDF}_{\hat{Y}|A=1}(\tau - w_A)$ for $A = 1$ and $1 - \mathsf{CDF}_{\hat{Y}|A=0}(\tau)$ for $A = 0$, where CDF is the cumulative distribution function. Finally, we compute DI by taking the ratio of the minimum and the maximum of the two PPVs.

5 Applications of FVGM

In this section, we apply FVGM for verifying fairness-enhancing algorithms and depreciation attacks. We also show that FVGM facilitates computation of fairness influence functions enabling detection of bias due to individual features.

Verifying Fairness-enhancing Algorithms. We deploy FVGM in verifying the effectiveness of fairness-enhancing algorithms designed to ameliorate bias. For example, fairness pre-processing algorithms can be validated by applying FVGM on the unprocessed and the processed data separately and comparing different fairness metrics. In Table 1, we report the effect of fairness algorithms w.r.t. four fairness metrics: disparate impact (DI), path-specific causal fairness (PCF), statistical parity (SP), and equalized odds (EO). Note that, fairness is improved if DI increases and the rest of the metrics decrease. In most instances for Adult dataset, reweighing (RW) (Kamiran and Calders 2012) and optimized pre-processing (OP) (Calmon et al. 2017) algorithms are successful in improving fairness. Except for the unfairness regarding the sensitive feature 'sex', where OP algorithm fails. Thus, FVGM *verifies the enhancement and decrement in fairness by fairness-enhancing algorithms.*

Verifying Fairness Attacks. We apply FVGM in verifying a fairness poisoning-attack algorithm. This algorithm injects a small fraction of poisoned samples into the training data such that the classifier becomes relatively unfair (Solans, Biggio, and Castillo 2020). We apply this attack to add $\{1, 5, \ldots, 160\}$ poisoned samples and measure the corresponding DI's. In Figure 5, FVGM verifies that DI of the classifier decreases, i.e. *the classifier becomes more unfair, as the number of poisoned samples increases*. Therefore, FVGM shows the potential of being deployed in safety-critical applications to detect fairness attacks. For example, if we set 0.9 as an acceptable threshold of DI, FVGM can raise an alarm once 160 poisoned samples are added.

Fairness Influence Function (FIF): Tracing Sources of Unfairness. Another application of FVGM as a fairness verifier is to quantify the effect of a subset of features on fairness. Thus, we define fairness influence function (FIF) that computes the effect of a subset of features $\mathbf{S}$ on the PPV of a classifier given a specific sensitive group $\mathbf{A} = \mathbf{a}$, $\mathsf{FIF}(\mathbf{S}) \triangleq \Pr[\hat{Y} = 1|\mathbf{A} = \mathbf{a}, \mathcal{D}] - \Pr[\hat{Y} = 1|\mathbf{A} = \mathbf{a}, \mathcal{D}_{-\mathbf{S}}]$. FIF allows us to explain the sources of unfairness in the classifier. In practice, we compute FIF of $\mathbf{S}$ by replacing the probability distribution of $\mathbf{S}$ with a uniformly random distribution, referred to as $\mathcal{D}_{-\mathbf{S}}$, and reporting differences in the conditional PPVs. Further details of FIF are in Appendix.

In Figure 6, we compute FIF for all features in COMPAS dataset, separately for two sensitive groups: Female ('sex' $= 1$) and Male. This dataset concerns the likelihood of a person of re-offending crimes within the next two years. We first observe that the base PPV values are different for the two groups (0.46 vs 0.61 for Female and Male), thereby showing Male as more probable to re-offend crimes than Female. Moreover, FIF of the feature 'age' is comparatively higher in magnitude for Male than Female. This implies that while deciding recidivism the algorithm assumes that

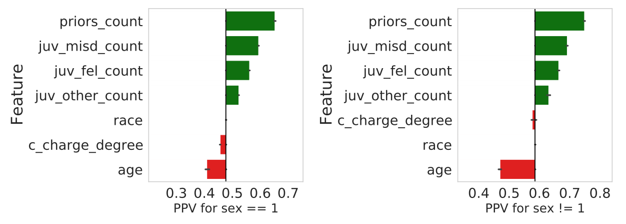


Figure 6: FIF computation for COMPAS dataset. For Female (left plot) and Male, influence of 'age' decreases and the PPV of the classifier increases by different magnitudes.

Female individuals across different ages re-offend crimes with almost the same probability and the probability of re-offending for Male individuals highly depends on age. Thus, applying FVGM to compute FIF provides us insights about sources of bias and thus, indicators to improve fairness.

6 Conclusion

In this paper, we propose FVGM, an efficient fairness verification framework for linear classifiers based on a novel stochastic subset-sum problem. FVGM encodes a graphical model of feature-correlations, represented as a Bayesian Network, and computes multiple group and causal fairness metrics accurately. We experimentally demonstrate that FVGM is not only more accurate and scalable than the existing verifiers but also applicable in practical fairness tasks, such as verifying fairness attacks and enhancing algorithms, and computing the fairness influence functions. As a future work, we aim to design fairness-enhancing algorithms certified by fairness verifiers, such as FVGM. Since FVGM serves as an accurate and scalable fairness verifier for linear classifiers, it will be interesting to design such verifiers for other ML models.

Acknowledgments

This work was supported in part by National Research Foundation, Singapore under its NRF Fellowship Programme [NRF-NRFFAI1-2019-0004] and AI Singapore Programme [AISG-RP-2018-005], and NUS ODPRT Grant [R-252-000-685-13]. The computational work for this article was performed on resources of Max Planck Institute for Software Systems, Germany and the National Supercomputing Centre, Singapore (https://www.nscc.sg).

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