Aplicação em transformações lineares

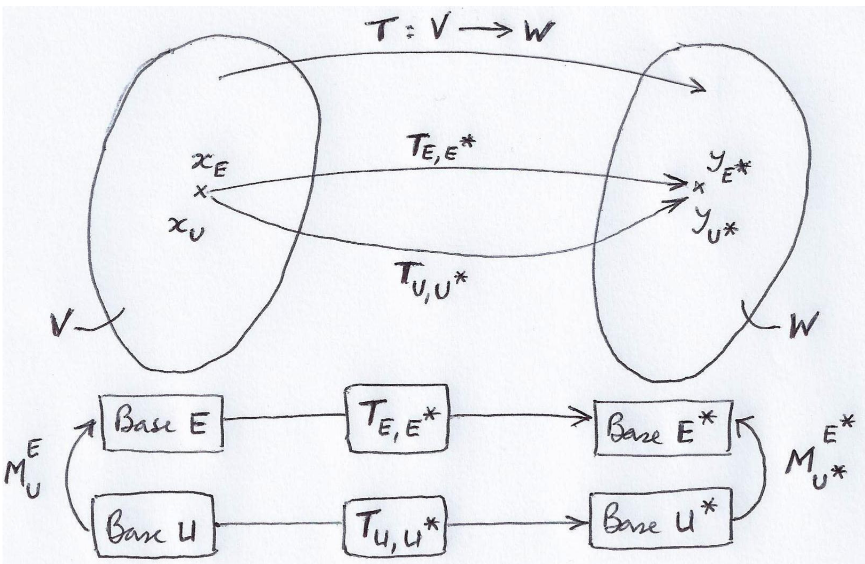
- Sejam V e W espaços lineares sobre um corpo Ω, tais que $\dim \mathrm{V} = n$ e $\dim \mathrm{W} = m$.

Teorema [4.2]: Seja a transformação linear $T : \mathrm{V} \to \mathrm{W}$. Admita que $\boldsymbol{T}_{\mathrm{E},\mathrm{E}^*} = m(T)_{\mathrm{E},\mathrm{E}^*}$ é a *representação matricial* de T *em relação às bases ordenadas* $\mathrm{E} = \{e_1, e_2, \ldots, e_n\} \subset \mathrm{V}$ e $\mathrm{E}^* = \{e_1^*, e_2^*, \ldots, e_m^*\} \subset \mathrm{W}$ e, por outro lado, que $\boldsymbol{T}_{\mathrm{U},\mathrm{U}^*} = m(T)_{\mathrm{U},\mathrm{U}^*}$ é a sua *representação matricial em relação às bases ordenadas* $\mathrm{U} = \{u_1, u_2, \ldots, u_n\} \subset \mathrm{V}$ e $\mathrm{U}^* = \{u_1^*, u_2^*, \ldots, u_m^*\} \subset \mathrm{W}$.

Se $\boldsymbol{M}_{\mathrm{U}}^{\mathrm{E}} = \boldsymbol{M}_{\mathrm{U}\to\mathrm{E}}$ e $\boldsymbol{M}_{\mathrm{U}^*}^{\mathrm{E}^*} = \boldsymbol{M}_{\mathrm{U}^*\to\mathrm{E}^*}$ são, respectivamente, as *matrizes mudança de base de* U *para* E e *de* U^* *para* E^*, então as representações matriciais referidas verificam a relação matricial

$$m(T)_{\mathrm{U},\mathrm{U}^*} = \left(\boldsymbol{M}_{\mathrm{U}^*}^{\mathrm{E}^*}\right)^{-1} m(T)_{\mathrm{E},\mathrm{E}^*}\ \boldsymbol{M}_{\mathrm{U}}^{\mathrm{E}} = \left(\boldsymbol{M}_{\mathrm{U}^*\to\mathrm{E}^*}\right)^{-1} m(T)_{\mathrm{E},\mathrm{E}^*}\ \boldsymbol{M}_{\mathrm{U}\to\mathrm{E}}$$

$$\boldsymbol{T}_{\mathrm{U},\mathrm{U}^*} = \left(\boldsymbol{M}_{\mathrm{U}^*}^{\mathrm{E}^*}\right)^{-1} \boldsymbol{T}_{\mathrm{E},\mathrm{E}^*}\ \boldsymbol{M}_{\mathrm{U}}^{\mathrm{E}} = \left(\boldsymbol{M}_{\mathrm{U}^*\to\mathrm{E}^*}\right)^{-1} \boldsymbol{T}_{\mathrm{E},\mathrm{E}^*}\ \boldsymbol{M}_{\mathrm{U}\to\mathrm{E}}$$



- A expressão anterior pode, ainda, ser reescrita sob as formas

$$\boldsymbol{T}_{\mathrm{E,E^*}} = \boldsymbol{M}_{\mathrm{U^*}}^{\mathrm{E^*}}\ \boldsymbol{T}_{\mathrm{U,U^*}} \left(\boldsymbol{M}_{\mathrm{U}}^{\mathrm{E}}\right)^{-1} = \boldsymbol{M}_{\mathrm{U^*\to E^*}}\ \boldsymbol{T}_{\mathrm{U,U^*}} \left(\boldsymbol{M}_{\mathrm{U\to E}}\right)^{-1}$$

$$\boldsymbol{T}_{\mathrm{U,U^*}} = \boldsymbol{M}_{\mathrm{E^*}}^{\mathrm{U^*}}\ \boldsymbol{T}_{\mathrm{E,E^*}} \left(\boldsymbol{M}_{\mathrm{E}}^{\mathrm{U}}\right)^{-1} = \boldsymbol{M}_{\mathrm{E^*\to U^*}}\ \boldsymbol{T}_{\mathrm{E,E^*}} \left(\boldsymbol{M}_{\mathrm{E\to U}}\right)^{-1}$$

$$\boldsymbol{T}_{\mathrm{E,E^*}} = \left(\boldsymbol{M}_{\mathrm{E^*}}^{\mathrm{U^*}}\right)^{-1} \boldsymbol{T}_{\mathrm{U,U^*}}\ \boldsymbol{M}_{\mathrm{E}}^{\mathrm{U}} = \left(\boldsymbol{M}_{\mathrm{E^*\to U^*}}\right)^{-1} \boldsymbol{T}_{\mathrm{U,U^*}}\ \boldsymbol{M}_{\mathrm{E\to U}}$$

- Tenhamos em atenção os seguintes casos particulares:

  i) As *bases ordenadas* U e E são a mesma base para V

$$\boldsymbol{U} = \boldsymbol{E}\ ,\ \ \boldsymbol{M}_{\mathrm{U}}^{\mathrm{E}} = \boldsymbol{I}_n \ \text{ e }\ \boldsymbol{X}_{\mathrm{E}} = \boldsymbol{X}_{\mathrm{U}}$$

$$\boldsymbol{T}_{\mathrm{E,U^*}} = \left(\boldsymbol{M}_{\mathrm{U^*}}^{\mathrm{E^*}}\right)^{-1} \boldsymbol{T}_{\mathrm{E,E^*}} = \boldsymbol{M}_{\mathrm{E^*}}^{\mathrm{U^*}}\ \boldsymbol{T}_{\mathrm{E,E^*}}$$

$$\boldsymbol{T}_{\mathrm{E,E^*}} = \boldsymbol{M}_{\mathrm{U^*}}^{\mathrm{E^*}}\ \boldsymbol{T}_{\mathrm{E,U^*}} = \left(\boldsymbol{M}_{\mathrm{E^*}}^{\mathrm{U^*}}\right)^{-1} \boldsymbol{T}_{\mathrm{E,U^*}}$$

  ii) As *bases ordenadas* U^* e E^* são a mesma base para W

$$\boldsymbol{U}^* = \boldsymbol{E}^*\ ,\ \ \boldsymbol{M}_{\mathrm{U^*}}^{\mathrm{E^*}} = \boldsymbol{I}_m \ \text{ e }\ \boldsymbol{Y}_{\mathrm{E^*}} = \boldsymbol{Y}_{\mathrm{U^*}}$$

$$\boldsymbol{T}_{\mathrm{U,E^*}} = \boldsymbol{T}_{\mathrm{E,E^*}}\ \boldsymbol{M}_{\mathrm{U}}^{\mathrm{E}} = \boldsymbol{T}_{\mathrm{E,E^*}} \left(\boldsymbol{M}_{\mathrm{E}}^{\mathrm{U}}\right)^{-1}$$

$$\boldsymbol{T}_{\mathrm{E,E^*}} = \boldsymbol{T}_{\mathrm{U,E^*}} \left(\boldsymbol{M}_{\mathrm{U}}^{\mathrm{E}}\right)^{-1} = \boldsymbol{T}_{\mathrm{U,E^*}}\ \boldsymbol{M}_{\mathrm{E}}^{\mathrm{U}}$$

iii) Se E, E^*, U e U^* são *bases ortonormais*, então $\boldsymbol{E}$, $\boldsymbol{E}^*$, $\boldsymbol{U}$ e $\boldsymbol{U}^*$ são matrizes *ortogonais*

$$\boldsymbol{E}^{-1} = \boldsymbol{E}^{\mathrm{T}} \ , \ \left(\boldsymbol{E}^*\right)^{-1} = \left(\boldsymbol{E}^*\right)^{\mathrm{T}} \ , \ \boldsymbol{U}^{-1} = \boldsymbol{U}^{\mathrm{T}} \ \text{e} \ \left(\boldsymbol{U}^*\right)^{-1} = \left(\boldsymbol{U}^*\right)^{\mathrm{T}}$$

pelo que

$$\left(\boldsymbol{M}_{\mathrm{U}}^{\mathrm{E}}\right)^{-1} = \left(\boldsymbol{M}_{\mathrm{U}}^{\mathrm{E}}\right)^{\mathrm{T}} \ \text{e} \ \left(\boldsymbol{M}_{\mathrm{E}}^{\mathrm{U}}\right)^{-1} = \left(\boldsymbol{M}_{\mathrm{E}}^{\mathrm{U}}\right)^{\mathrm{T}}$$

$$\left(\boldsymbol{M}_{\mathrm{U}^*}^{\mathrm{E}^*}\right)^{-1} = \left(\boldsymbol{M}_{\mathrm{U}^*}^{\mathrm{E}^*}\right)^{\mathrm{T}} \ \text{e} \ \left(\boldsymbol{M}_{\mathrm{E}^*}^{\mathrm{U}^*}\right)^{-1} = \left(\boldsymbol{M}_{\mathrm{E}^*}^{\mathrm{U}^*}\right)^{\mathrm{T}}$$

isto é, as *matrizes mudança de base* são matrizes *ortogonais*. Assim,

$$\boldsymbol{T}_{\mathrm{U,U}^*} = \left(\boldsymbol{M}_{\mathrm{U}^*}^{\mathrm{E}^*}\right)^{\mathrm{T}} \boldsymbol{T}_{\mathrm{E,E}^*} \ \boldsymbol{M}_{\mathrm{U}}^{\mathrm{E}} = \boldsymbol{M}_{\mathrm{E}^*}^{\mathrm{U}^*} \ \boldsymbol{T}_{\mathrm{E,E}^*} \left(\boldsymbol{M}_{\mathrm{E}}^{\mathrm{U}}\right)^{\mathrm{T}}$$

$$\boldsymbol{T}_{\mathrm{E,E}^*} = \boldsymbol{M}_{\mathrm{U}^*}^{\mathrm{E}^*} \ \boldsymbol{T}_{\mathrm{U,U}^*} \left(\boldsymbol{M}_{\mathrm{U}}^{\mathrm{E}}\right)^{\mathrm{T}} = \left(\boldsymbol{M}_{\mathrm{E}^*}^{\mathrm{U}^*}\right)^{\mathrm{T}} \boldsymbol{T}_{\mathrm{U,U}^*} \ \boldsymbol{M}_{\mathrm{E}}^{\mathrm{U}}$$

Exemplo 2 [4.4]: Seja a transformação linear $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, definida pela *matriz*

$$\boldsymbol{T} = m(T) = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

em relação às bases canónicas $\mathrm{E}_3 = \{\vec{i}, \vec{j}, \vec{k}\}$ e $\mathrm{E}_2 = \{\vec{i}_1, \vec{j}_1\} = \{(1,0),(0,1)\}$ para os espaços lineares $\mathbb{R}^3$ e $\mathbb{R}^2$, respectivamente. Sejam ainda as bases ordenadas para $\mathbb{R}^3$ e para $\mathbb{R}^2$

$$\mathrm{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \{(1,-1,0),(0,1,1),(1,0,-1)\} \subset \mathbb{R}^3$$

$$\mathrm{W} = \{\vec{w}_1, \vec{w}_2\} = \{(1,1),(1,-1)\} \subset \mathbb{R}^2$$

Determine:

a) A *matriz* $\boldsymbol{T}_{\mathrm{V,E_2}} = m(T)_{\mathrm{V,E_2}}$, que representa T em relação às *bases ordenadas* V e E_2.

b) A *matriz* $\boldsymbol{T}_{\mathrm{E_3,W}} = m(T)_{\mathrm{E_3,W}}$, que representa T em relação às *bases ordenadas* E_3 e W.

c) A *matriz* $\boldsymbol{T}_{\mathrm{V,W}} = m(T)_{\mathrm{V,W}}$, que representa T em relação às *bases ordenadas* V e W.

Solução:

a) A matriz $\boldsymbol{T}_{\mathrm{V,E_2}} = m(T)_{\mathrm{V,E_2}}$ é obtida a partir da matriz $\boldsymbol{T} = m(T)$ através da relação matricial

$$\boldsymbol{T}_{\mathrm{V,E_2}} = \boldsymbol{T}\ \boldsymbol{M}_{\mathrm{V}}^{\mathrm{E_3}}$$

onde $\boldsymbol{M}_{\mathrm{V}}^{\mathrm{E_3}}$ é a matriz *mudança de base de* V *para* E_3, definida por

$$\boldsymbol{M}_{\mathrm{V}}^{\mathrm{E_3}} = \boldsymbol{E}_3^{-1}\ \boldsymbol{V} = \boldsymbol{I}_3\ \boldsymbol{V} = \boldsymbol{V} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Assim,

$$\boldsymbol{T}_{\mathrm{V},\mathrm{E}_2} = \boldsymbol{T}\ \boldsymbol{M}_{\mathrm{V}}^{\mathrm{E}_3} = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 5 \\ -1 & 2 & -1 \end{bmatrix}_{\mathrm{V},\mathrm{E}_2}$$

b) A matriz $\boldsymbol{T}_{\mathrm{E}_3,\mathrm{W}} = m(T)_{\mathrm{E}_3,\mathrm{W}}$ é obtida a partir da matriz $\boldsymbol{T} = m(T)$ através da relação matricial

$$\boldsymbol{T}_{\mathrm{E}_3,\mathrm{W}} = \left(\boldsymbol{M}_{\mathrm{W}}^{\mathrm{E}_2}\right)^{-1} \boldsymbol{T}$$

onde $\boldsymbol{M}_{\mathrm{W}}^{\mathrm{E}_2}$ é a matriz *mudança de base de* W *para* E_2, definida por

$$\boldsymbol{M}_{\mathrm{W}}^{\mathrm{E}_2} = \boldsymbol{E}_2^{-1}\ \boldsymbol{W} = \boldsymbol{I}_2\ \boldsymbol{W} = \boldsymbol{W} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

e tem como matriz inversa

$$\left(\boldsymbol{M}_{\mathrm{W}}^{\mathrm{E}_2}\right)^{-1} = \boldsymbol{W}^{-1} = \frac{1}{|\ \boldsymbol{W}\ |}\left[\boldsymbol{Cof}\ \boldsymbol{W}\right]^{\mathrm{T}} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}^{\mathrm{T}} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Obtém-se, então,

$$\boldsymbol{T}_{\mathrm{E}_3,\mathrm{W}} = \left(\boldsymbol{M}_{\mathrm{W}}^{\mathrm{E}_2}\right)^{-1} \boldsymbol{T} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & 1 & -1 \\ 3 & -1 & -3 \end{bmatrix}_{\mathrm{E}_3,\mathrm{W}}$$

c) A matriz $\boldsymbol{T}_{\mathrm{V},\mathrm{W}} = m(T)_{\mathrm{V},\mathrm{W}}$ é obtida a partir da matriz $\boldsymbol{T} = m(T)$ através da relação matricial

$$\boldsymbol{T}_{\mathrm{V},\mathrm{W}} = \left(\boldsymbol{M}_{\mathrm{W}}^{\mathrm{E}_2}\right)^{-1} \boldsymbol{T}\ \boldsymbol{M}_{\mathrm{V}}^{\mathrm{E}_3}$$

ou seja, por exemplo,

$$\boldsymbol{T}_{\mathrm{V},\mathrm{W}} = \boldsymbol{T}_{\mathrm{E}_3,\mathrm{W}}\ \boldsymbol{M}_{\mathrm{V}}^{\mathrm{E}_3} = \frac{1}{2}\begin{bmatrix} 3 & 1 & -1 \\ 3 & -1 & -3 \end{bmatrix}_{\mathrm{E}_3,\mathrm{W}} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -2 & 3 \end{bmatrix}_{\mathrm{V},\mathrm{W}}$$