U.PORTO
FEUP FACULDADE DE ENGENHARIA
UNIVERSIDADE DO PORTO

Curso: MIEIC Data: ___/___/___

Disciplina: CMAT Ano: ___ Semestre: ___

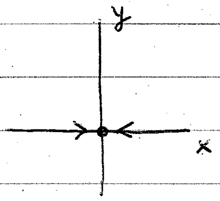
Nome: ___

Espaço reservado para o avaliador: AULA 4 : Ex°s Tratados - Ficha 2 : 4, 6, 8, 10, 9

Ex°s Propostos - ficha 2 : 1, 2, 5g), 7

4) $f(x,y) = \dfrac{xy}{x^2+y^2}$, $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

a)

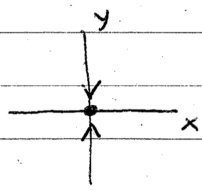


Eixo dos xx : $y = 0$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} f(x,y) = \lim_{x\to 0} f(x,0) =$$

$$= \lim_{x\to 0} \frac{0}{x^2} = 0$$

b)

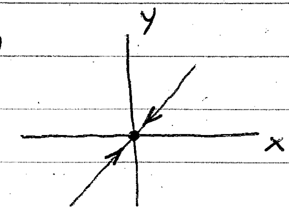


Eixo dos yy : $x = 0$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} f(x,y) = \lim_{y\to 0} f(0,y) =$$

$$= \lim_{y\to 0} \frac{0}{y^2} = 0$$

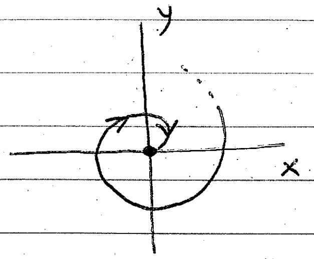
c)



Recta : $y = mx$, $m \neq 0$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} f(x,y) = \lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{mx^2}{(1+m^2)x^2} = \frac{m}{1+m^2}$$

d)



Espiral : $r = \theta$, $\theta > 0$

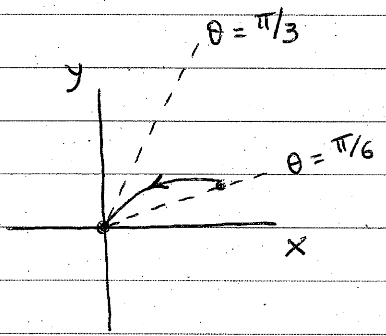
$x = \theta \cos\theta$

$y = \theta \operatorname{sen}\theta$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=\theta\cos\theta\\ y=\theta\operatorname{sen}\theta}} f(x,y) = \lim_{\theta\to 0^+} f(\theta\cos\theta, \theta\operatorname{sen}\theta) =$$

$$= \lim_{\theta\to 0^+} \frac{\theta^2 \operatorname{sen}\theta\cos\theta}{\theta^2(\operatorname{sen}^2\theta+\cos^2\theta)} = \lim_{\theta\to 0^+} \operatorname{sen}\theta\cos\theta = 0$$

e)



Arco : $r = \operatorname{sen}(3\theta)$, $\theta \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right)$

$\theta = \pi/6 \Rightarrow r = 1$

$\theta = \pi/3 \Rightarrow r = 0$

$x = \operatorname{sen}(3\theta)\cos\theta$

$y = \operatorname{sen}(3\theta)\operatorname{sen}\theta$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=\operatorname{sen}(3\theta)\cos\theta\\ y=\operatorname{sen}(3\theta)\operatorname{sen}\theta}} f(x,y) = \lim_{\theta\to \frac{\pi}{3}^-} \frac{\operatorname{sen}^2(3\theta)\operatorname{sen}\theta\cos\theta}{\operatorname{sen}^2(3\theta)(\cos^2\theta+\operatorname{sen}^2\theta)} =$$

$$= \lim_{\theta\to \frac{\pi}{3}^-} \operatorname{sen}\theta\cos\theta = \frac{\sqrt{3}}{4}$$

Curso .... Data ..../..../....

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f) Curva : $\vec{r}(t) = \left(\frac{1}{t}, \frac{\operatorname{sen} t}{t}\right), t>0$

$x = \frac{1}{t} \wedge y = \frac{\operatorname{sen} t}{t}$

$$\lim_{\substack{(x,y)\to(0,0) \\ x=t^{-1} \\ y=t^{-1}\operatorname{sen} t}} f(x,y) = \lim_{t\to\infty} \frac{t^{-2}\operatorname{sen} t}{t^{-2}(1+\operatorname{sen}^2 t)} = \lim_{t\to\infty} \frac{\operatorname{sen} t}{1+\operatorname{sen}^2 t} \Rightarrow$$

$\Rightarrow$ não existe limite

6) a) $f(x,y,z) = x e^{y} \operatorname{sen}(x+z)$

$$\frac{\partial f}{\partial x}(x,y,z) = e^{y}\operatorname{sen}(x+z) + x e^{y}\cos(x+z)$$

$$\frac{\partial f}{\partial y}(x,y,z) = x e^{y}\operatorname{sen}(x+z)$$

$$\frac{\partial f}{\partial z}(x,y,z) = x e^{y}\cos(x+z)$$

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) =$$

$$= e^{y}\left(\operatorname{sen}(x+z) + x\cos(x+z),\ x\operatorname{sen}(x+z),\ x\cos(x+z)\right)$$

b) $g(x,y,z) = (-x+2y)^5 + 2z^{-1}$

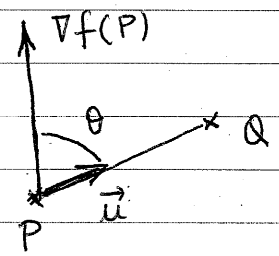
$$\frac{\partial g}{\partial x}(x,y,z) = -5(-x+2y)^4$$

$$\frac{\partial g}{\partial y}(x,y,z) = 10(-x+2y)^4$$

$$\frac{\partial g}{\partial z}(x,y,z) = -2z^{-2}$$

$$\nabla g(x,y,z) = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right) = \left(-5(-x+2y)^4, 10(-x+2y)^4, -2z^{-2}\right)$$

8) $f(x,y,z) = z \ln\left(\frac{x}{y}\right)$ $(y \neq 0)$ $\wedge$ $(x/y > 0)$



$$\frac{\partial f}{\partial x}(x,y,z) = z \frac{1/y}{x/y} = \frac{z}{x}$$

$$\frac{\partial f}{\partial y}(x,y,z) = z \frac{-xy^{-2}}{x/y} = -\frac{z}{y}$$

$P = (1,2,-2)$

$Q = (2,2,1)$

$\overrightarrow{PQ} = (1,0,3)$

$$\frac{\partial f}{\partial z}(x,y,z) = \ln\left(\frac{x}{y}\right)$$

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\frac{z}{x}, -\frac{z}{y}, \ln\left(\frac{x}{y}\right)\right)$$

$$\nabla f(1,2,-2) = \left(-2, 1, -\ln(2)\right) = \nabla f(P)$$

Seja o versor $\vec{u} = \frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = \frac{1}{\sqrt{10}}(1,0,3)$

Então $f'_{\vec{u}}(P) = \nabla f(P) \cdot \vec{u} = \frac{1}{\sqrt{10}}\left(-2-3\ln(2)\right) < 0$,

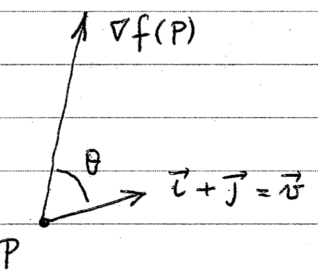
pelo que a função $f(x,y,z)$ tem um comportamento decrescente no ponto P segundo o versor $\vec{u}$ (na figura $\theta \in]\pi/2, \pi[$), ou na direcção do vector $\overrightarrow{PQ}$.

10) $f(x,y,z) = \left(x+y^2+z^3\right)^2$, $(x,y,z) \in \mathbb{R}^3$

$\dfrac{\partial f}{\partial x}(x,y,z) = 2\left(x+y^2+z^3\right)$

$\dfrac{\partial f}{\partial y}(x,y,z) = 4y\left(x+y^2+z^3\right)$

$\dfrac{\partial f}{\partial z}(x,y,z) = 6z^2\left(x+y^2+z^3\right)$



$P = (1,-1,1)$

$\vec{v} = \vec{i}+\vec{j} = (1,1,0)$

$\nabla f(x,y,z) = 2\left(x+y^2+z^3\right)\left(1, 2y, 3z^2\right)$

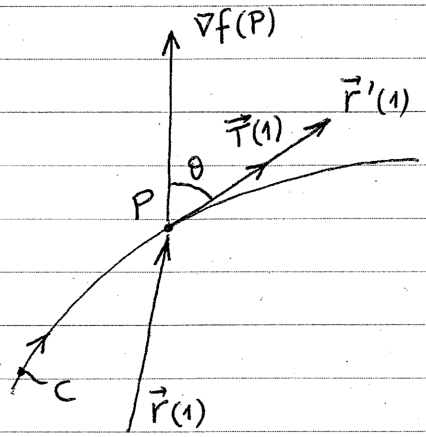
$\nabla f(P) = \nabla f(1,-1,1) = 2(3)(1,-2,3) = 6(1,-2,3)$

Seja o versor $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{1}{\sqrt{2}}(1,1,0)$

Então $f'_{\vec{u}}(P) = \nabla f(P) \cdot \vec{u} = \dfrac{6}{\sqrt{2}}(-1) = -3\sqrt{2} < 0$, pelo que

a função $f(x,y,z)$ tem um comportamento decrescente no ponto P na direcção de $\vec{u}$ (ou do vector $\vec{v}$); na figura acima $\theta \in \left]\pi/2, \pi\right[$ (o produto escalar é negativo).

9) $f(x,y,z) = x\,e^{y^2-z^2}$, $(x,y,z)\in\mathbb{R}^3$



$\frac{\partial f}{\partial x}(x,y,z) = e^{y^2-z^2}$

$\frac{\partial f}{\partial y}(x,y,z) = 2xy\,e^{y^2-z^2}$

$\frac{\partial f}{\partial z}(x,y,z) = -2xz\,e^{y^2-z^2}$

$\nabla f(x,y,z) = e^{y^2-z^2}\,(1, 2xy, -2xz)$

Curva C : $\vec{r}(t) = (t, 2\cos(t-1), -2e^{t-1})$, $t\in\mathbb{R}$

$P = (1,2,-2) = \vec{r}(1)$

$\nabla f(P) = \nabla f(1,2,-2) = (1,4,4)$

Considere-se a linha tangente à curva C no ponto P, definida pelo vector tangente $\vec{r}\,'(1)$:

$$\vec{r}\,'(t) = (1, -2\,\mathrm{sen}(t-1), -2e^{t-1}) \Rightarrow \vec{r}\,'(1) = (1,0,-2)$$

O versor da tangente em P é : $\vec{T}(1) = \frac{\vec{r}\,'(1)}{\|\vec{r}\,'(1)\|} = \frac{1}{\sqrt{5}}(1,0,-2) = \vec{u}$

Então $f'_{\vec{u}}(P) = \nabla f(P)\cdot\vec{u} = \frac{1}{\sqrt{5}}(-7) = -\frac{7\sqrt{5}}{5} < 0$, pelo que a função $f(x,y,z)$ tem um comportamento decrescente no ponto P na direcção de $\vec{u}$ (ou do vector $\vec{r}\,'(1)$) ; na figura acima $\theta \in\,]\pi/2, \pi[$ (o produto escalar é negativo).