Curso MIEIC Data / /

Disciplina CMAT Ano Semestre

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AULA 6 : Ex°s Tratados : ficha 2 : 33, 29, 28, 30, 31

Ex°s Propostos : 25, 26, 27, 32

33) $xz^2 - yz^2 + xy^2z - 5 = 0$ e $z = f(x,y)$

Derivando a expressão em ordem a x:

$$z^2 + x\left(2z\frac{\partial z}{\partial x}\right) - y\left(2z\frac{\partial z}{\partial x}\right) + y^2z + xy^2\frac{\partial z}{\partial x} = 0 \Leftrightarrow$$

$$\Leftrightarrow \left(2xz - 2yz + xy^2\right)\frac{\partial z}{\partial x} = -z^2 - y^2z \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial z}{\partial x} = \frac{-z(z+y^2)}{2xz - 2yz + xy^2}$$

Derivando a expressão em ordem a y:

$$x\left(2z\frac{\partial z}{\partial y}\right) - z^2 - y\left(2z\frac{\partial z}{\partial y}\right) + 2xyz + xy^2\frac{\partial z}{\partial y} = 0 \Leftrightarrow$$

$$\Leftrightarrow \left(2xz - 2yz + xy^2\right)\frac{\partial z}{\partial y} = z^2 - 2xyz \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial z}{\partial y} = \frac{z(z-2xy)}{2xz - 2yz + xy^2}$$

Determinemos as coordenadas z (cotas) dos pontos da superfície tais que $x=3$ e $y=1$; substituindo estes valores na expressão que define a superfície obtém-se

$$3z^2 - z^2 + 3z - 5 = 0 \Leftrightarrow 2z^2 + 3z - 5 = 0 \Leftrightarrow$$

$\Leftrightarrow \quad z = \dfrac{-3 \pm \sqrt{49}}{4} \quad \Leftrightarrow \quad z = 1 \ \vee \ z = -\dfrac{5}{2}$, pelo que

existem dois pontos : $P = (3,1,1)$ e $P_1 = (3,1,-\frac{5}{2})$.

Assim, obtém-se para o ponto P

$$\frac{\partial z}{\partial x}(3,1,1) = \frac{-2}{7} \quad e \quad \frac{\partial z}{\partial y}(3,1,1) = -\frac{5}{7}$$

e para o ponto P_1

$$\frac{\partial z}{\partial x}\left(3,1,-\frac{5}{2}\right) = \frac{-15/4}{-7} = \frac{15}{28} \quad e \quad \frac{\partial z}{\partial y}\left(3,1,-\frac{5}{2}\right) = \frac{85/4}{-7} = -\frac{85}{28}$$

29) $e^{\cos(z)} \ln(z+1) = \operatorname{arctg}(2x+y)$ e $z = f(x,y)$

Derivando a expressão em ordem a x :

$$\frac{\partial}{\partial x}\left[e^{\cos(z)}\right] \ln(z+1) + e^{\cos(z)} \frac{\partial}{\partial x}\left[\ln(z+1)\right] = \frac{\partial}{\partial x}\left[\operatorname{arctg}(2x+y)\right] \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial}{\partial x}\left[\cos(z)\right] e^{\cos(z)} \ln(z+1) + e^{\cos(z)} \frac{\frac{\partial}{\partial x}(z+1)}{z+1} = \frac{\frac{\partial}{\partial x}(2x+y)}{1+(2x+y)^2} \Leftrightarrow$$

$$\Leftrightarrow -\frac{\partial z}{\partial x} \operatorname{sen}(z)\, e^{\cos(z)} \ln(z+1) + e^{\cos(z)} \frac{\partial z/\partial x}{z+1} = \frac{2}{1+(2x+y)^2} \Leftrightarrow$$

$$\Leftrightarrow e^{\cos(z)} \left(-\operatorname{sen}(z) \ln(z+1) + \frac{1}{z+1}\right) \frac{\partial z}{\partial x} = \frac{2}{1+(2x+y)^2} \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial z}{\partial x} = \frac{2}{e^{\cos(z)}\left(-\operatorname{sen}(z)\ln(z+1) + \frac{1}{z+1}\right)\left[1+(2x+y)^2\right]}$$

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Derivando a expressão em ordem a y:

$$\frac{\partial}{\partial y}\left[e^{\cos(z)}\right]\ln(z+1)+e^{\cos(z)}\frac{\partial}{\partial y}\left[\ln(z+1)\right]=\frac{\partial}{\partial y}\left[\operatorname{arctg}(2x+y)\right] \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial}{\partial y}\left[\cos(z)\right]e^{\cos(z)}\ln(z+1)+e^{\cos(z)}\frac{\frac{\partial}{\partial y}(z+1)}{z+1}=\frac{\frac{\partial}{\partial y}(2x+y)}{1+(2x+y)^2} \Leftrightarrow$$

$$\Leftrightarrow -\frac{\partial z}{\partial y}\operatorname{sen}(z)\,e^{\cos(z)}\ln(z+1)+e^{\cos(z)}\frac{\partial z/\partial y}{z+1}=\frac{1}{1+(2x+y)^2} \Leftrightarrow$$

$$\Leftrightarrow e^{\cos(z)}\left(-\operatorname{sen}(z)\ln(z+1)+\frac{1}{z+1}\right)\frac{\partial z}{\partial y}=\frac{1}{1+(2x+y)^2} \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial z}{\partial y}=\frac{1}{e^{\cos(z)}\left(-\operatorname{sen}(z)\ln(z+1)+\frac{1}{z+1}\right)\left[1+(2x+y)^2\right]}$$

Então, obtém-se para o ponto $P=\left(-\frac{1}{2},1,0\right)$

$$\frac{\partial z}{\partial x}\left(-1/2,1,0\right)=\frac{2}{e(0+1)(1)}=\frac{2}{e}$$

$$\frac{\partial z}{\partial y}\left(-1/2,1,0\right)=\frac{1}{e(0+1)(1)}=\frac{1}{e}$$

28) $x + z + (y+z)^2 = 6$ e $z = f(x,y)$

Derivando a expressão em ordem a x:

$$1 + \frac{\partial z}{\partial x} + 2(y+z)\frac{\partial z}{\partial x} = 0 \Leftrightarrow \frac{\partial z}{\partial x} = \frac{-1}{1+2(y+z)} \quad (1)$$

Derivando a expressão em ordem a y:

$$\frac{\partial z}{\partial y} + \frac{\partial}{\partial y}(y+z)^2 = 0 \Leftrightarrow \frac{\partial z}{\partial y} + 2(y+z)\frac{\partial}{\partial y}(y+z) = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial z}{\partial y} + 2(y+z)\left(1 + \frac{\partial z}{\partial y}\right) = 0 \Leftrightarrow \frac{\partial z}{\partial y}\left(1 + 2(y+z)\right) = -2(y+z) \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial z}{\partial y} = \frac{-2(y+z)}{1+2(y+z)} \quad (2)$$

Derivando a expressão (1) em ordem a y:

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left[\frac{-1}{1+2(y+z)}\right] \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial^2 z}{\partial y\,\partial x} = \frac{-(-1)\frac{\partial}{\partial y}(1+2y+2z)}{[1+2(y+z)]^2} \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial^2 z}{\partial y\,\partial x} = \frac{2 + 2\frac{\partial z}{\partial y}}{[1+2(y+z)]^2}$$, substituindo a expressão (2),

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{2\left[1+\cancel{2(y+z)}\right] + 2\left[\cancel{-2(y+z)}\right]}{[1+2(y+z)]^3} \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial^2 z}{\partial y\,\partial x} = \frac{2}{[1+2(y+z)]^3}$$

Derivando a expressão (2) em ordem a x :

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{-2(y+z)}{1+2(y+z)}\right] \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial^2 z}{\partial x \partial y} = \frac{-2\frac{\partial}{\partial x}(y+z)\left[1+2(y+z)\right] + 2(y+z)\frac{\partial}{\partial x}(1+2y+2z)}{\left[1+2(y+z)\right]^2} \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial^2 z}{\partial x \partial y} = \frac{-2\frac{\partial z}{\partial x}\left[1+\cancel{2(y+z)}\right] + \cancel{2(y+z)}(2)\frac{\partial z}{\partial x}}{\left[1+2(y+z)\right]^2} \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial^2 z}{\partial x \partial y} = \frac{-2\frac{\partial z}{\partial x}}{\left[1+2(y+z)\right]^2}$$, substituindo a expressão (1),

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{2}{\left[1+2(y+z)\right]^3}$$

Convém notar que, neste caso, se verifica que $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial z}$ uma vez que as derivadas parciais $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial x \partial y}$, $\frac{\partial z}{\partial y \partial z}$ são funções contínuas em todos os pontos onde estão definidas, isto é, em todos os pontos de $\mathbb{R}^3$ excepto nos pontos onde $1+2(y+z)=0$

30) $x\ln(y)+y^2z+z^2=6$ e $z=f(x,y)$

Derivando a expressão em ordem a x:

$$\ln(y)+y^2\frac{\partial z}{\partial x}+2z\frac{\partial z}{\partial x}=0 \Leftrightarrow \frac{\partial z}{\partial x}=\frac{-\ln(y)}{y^2+2z} \quad (1)$$

Derivando a expressão em ordem a y:

$$x\left(\frac{1}{y}\right)+2yz+y^2\frac{\partial z}{\partial y}+2z\frac{\partial z}{\partial y}=0 \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial z}{\partial y}=\frac{-x/y-2yz}{y^2+2z}=\frac{-x-2y^2z}{y^3+2yz} \quad (2)$$

Derivando a expressão (1) em ordem a y:

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial y}\left(\frac{-\ln(y)}{y^2+2z}\right) \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial^2 z}{\partial y\partial x}=\frac{-\frac{1}{y}(y^2+2z)+\ln(y)\frac{\partial}{\partial y}(y^2+2z)}{(y^2+2z)^2} \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial^2 z}{\partial y\partial \cancel{x}}=\frac{-(y^2+2z)+y\ln(y)\left(2y+2\frac{\partial z}{\partial y}\right)}{y(y^2+2z)^2} \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial^2 z}{\partial y\partial x}=\frac{-(y^2+2z)+2y^2\ln(y)+2y\ln(y)\frac{\partial z}{\partial y}}{y(y^2+2z)^2} \quad (3)$$

Derivando a expressão (2) em ordem a x:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{-x-2y^2z}{y^3+2yz}\right) \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial^2 z}{\partial x\partial y}=\frac{\frac{\partial}{\partial x}(-x-2y^2z)(y^3+2yz)+(x+2y^2z)\frac{\partial}{\partial x}(y^3+2yz)}{(y^3+2yz)^2} \Leftrightarrow$$

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$$\Leftrightarrow \frac{\partial^2 z}{\partial x \partial y} = \frac{\left(-1-2y^2\frac{\partial z}{\partial x}\right)(y^3+2yz)+(x+2y^2z)\left(2y\frac{\partial z}{\partial x}\right)}{(y^3+2yz)^2} \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial^2 z}{\partial x \partial y} = \frac{-y^3-2yz-2y^5\frac{\partial z}{\partial x}+2xy\frac{\partial z}{\partial x}}{(y^3+2yz)^2} \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial^2 z}{\partial x \partial y} = \frac{-y^3-2yz-2y(y^4-x)\frac{\partial z}{\partial x}}{(y^3+2yz)^2} \qquad (4)$$

Assim, as derivadas parciais de primeira ordem em $P=(1,1,2)$ são:

$$\frac{\partial z}{\partial x}(1,1,2) = \frac{0}{5} = 0 \qquad (5)$$

$$\frac{\partial z}{\partial y}(1,1,2) = \frac{-1-4}{5} = -1 \qquad (6)$$

Recorrendo a (6) e à expressão (3) obtém-se:

$$\frac{\partial^2 z}{\partial y \partial x}(1,1,2) = \frac{-5+0+0}{25} = -\frac{1}{5}$$

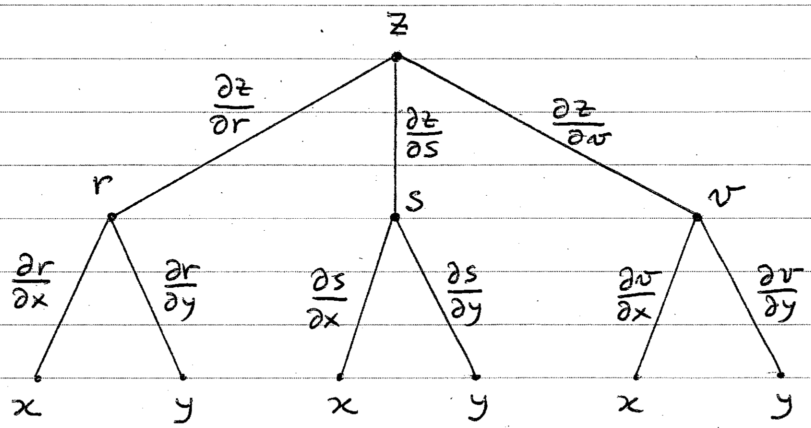
Recorrendo a (5) e à expressão (4) obtém-se:

$$\frac{\partial^2 z}{\partial x \partial y}(1,1,2) = \frac{-1-4-2(0)}{25} = -\frac{1}{5}$$

31) $z(r,s,v) = \dfrac{r+s}{v}$ em que

$r(x,y) = x\cos(y)$, $s(x,y) = y\,\mathrm{sen}(x)$, $v(x,y) = 2x-y$

Considere-se o diagrama de árvore



Sabe-se que:

$$\frac{\partial z}{\partial r} = \frac{1}{v} \quad , \quad \frac{\partial z}{\partial s} = \frac{1}{v} \quad , \quad \frac{\partial z}{\partial v} = -\frac{r+s}{v^2}$$

$$\frac{\partial r}{\partial x} = \cos(y) \quad , \quad \frac{\partial r}{\partial y} = -x\,\mathrm{sen}(y)$$

$$\frac{\partial s}{\partial x} = y\cos(x) \quad , \quad \frac{\partial s}{\partial y} = \mathrm{sen}(x)$$

$$\frac{\partial v}{\partial x} = 2 \quad , \quad \frac{\partial v}{\partial y} = -1$$

Então:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial z}{\partial x} = \frac{\cos(y)}{v} + \frac{y\cos(x)}{v} - 2\,\frac{r+s}{v^2} \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial z}{\partial x} = \frac{\cos(y) + y\cos(x)}{2x-y} - 2\,\frac{x\cos(y) + y\,\mathrm{sen}(x)}{(2x-y)^2}$$

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$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial z}{\partial y} = \frac{-x\,\text{sen}(y)}{v} + \frac{\text{sen}(x)}{v} + \frac{r+s}{v^2} \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial z}{\partial y} = \frac{-x\,\text{sen}(y) + \text{sen}(x)}{2x-y} + \frac{x\cos(y) + y\,\text{sen}(x)}{(2x-y)^2}$$