Prova sem consulta. Duração: 2h. 2ª Prova de Avaliação

* Todas as folhas devem ser identificadas com o nome completo. Justifique adequadamente todos os cálculos que efetuar;
* A entrega da prova e a desistência só serão possíveis após 1 hora do início da prova;
* Não se pode utilizar telemóveis, máquinas de calcular gráficas e microcomputadores;
* Resolva cada um dos dois grupos utilizando folhas de capa distintas.

GRUPO I

1. **[3,0]** Considere a curva, C, intersecção das superfícies $x^2+y^2+z^2=2$ e $z=1$, percorrida no sentido direto. Calcule $\int_C -xzdx+ydy+ydz$.

2. **[4,5]** Considere o campo vetorial $\vec{f}(x,y)=(2y^3+\beta yx^2+2, \alpha xy^2+x^3+1)$, em que α e β são constantes reais. Seja a curva, C, fronteira da região limitada por $y=1$, $y=x^3$ e $0\leq x\leq 1$, percorrida no sentido direto.

a) Seja $\alpha=\beta=0$. Esboce a curva, C, e calcule $\int_C \vec{f}\cdot d\vec{r}$ usando, se possível, o teorema de Green.

b) Determine os valores de α e β de modo que o campo $\vec{f}(x,y)$ seja gradiente.

c) Para os valores de α e β obtidos em **b)**, obtenha o campo escalar, $\varphi(x,y)$, tal que $\vec{f}=\nabla\varphi$ e calcule $\int_C \vec{f}\cdot d\vec{r}$ entre os pontos $O=(0,0)$ e $P=(1,1)$.

3. **[3,0]** Seja a superfície $z=\sqrt{x^2+y^2}$, $1\leq z\leq 4$. Faça o seu esboço e calcule a sua área.

GRUPO II

4. **[3,0]** Considere o campo vetorial $\vec{f}(x,y,z)=(y,x,z)$ e a superfície $z=xy$, definida em $D: x^2+y^2\leq 1$.

a) Obtenha uma parametrização, $\vec{r}(u,v)$, para a superfície e indique um versor, $\vec{n}(u,v)$, do vetor fundamental.

b) Determine $\iint_S (\vec{f}\cdot\vec{n})dS$.

.... (*continua no verso*)

5. [**4,5**] Considere o integral triplo $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{x^2+y^2-4}^{4-x^2-y^2} dzdydx$.

a) Esboce o domínio de integração.

b) Calcule o valor do integral usando uma mudança de coordenadas apropriada.

c) Reescreva-o de modo que a primeira integração se faça em ordem a y.

6. [**2,0**] Seja $\vec{r}(u,v)$ uma representação paramétrica regular de uma superfície, S, em $\mathbb{R}^3$. Mostre que o vetor fundamental associado a essa representação é, em qualquer ponto de S, um vetor normal à superfície.

(1)

$$C : \begin{cases} x^2+y^2+z^2 = 2 \\ z = 1 \end{cases} \qquad \begin{cases} x^2+y^2 = 1 \\ z = 1 \end{cases}$$

$$\bar{r}(\theta) = (\cos\theta, \operatorname{sen}\theta, 1), \quad \theta \in [0, 2\pi]$$

$$\bar{r}'(\theta) = (-\operatorname{sen}\theta, \cos\theta, 0)$$

$$\bar{f}(x,y,z) = (P, Q, R) = (-xz, y, y)$$

$$\bar{f}[\bar{r}(\theta)] = (-\cos\theta, \operatorname{sen}\theta, \operatorname{sen}\theta)$$

$$\bar{f}[\bar{r}(\theta)] \cdot \bar{r}'(\theta) = \operatorname{sen}\theta\cos\theta + \operatorname{sen}\theta\cos\theta = 2\operatorname{sen}\theta\cos\theta$$

$$\int_C -P\,dx + Q\,dy + R\,dz = \int_0^{2\pi} 2\operatorname{sen}\theta\cos\theta\,d\theta = 0$$

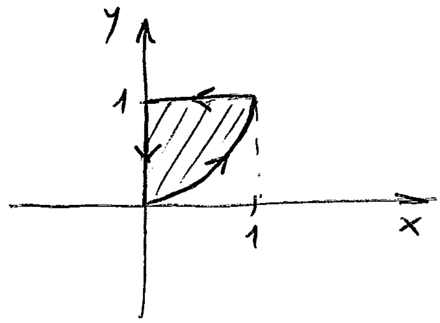
---

(2)

$$\bar{f}(x,y) = (P, Q) = (2y^3 + \beta y x^2 + 2, \alpha x y^2 + x^3 + 1)$$

$$C : y = 1, \; y = x^3, \; 0 \leq x \leq 1 \quad (\text{directo})$$

a) $\alpha = \beta = 0$



$$\bar{f}(x,y) = (2y^3 + 2, x^3 + 1) = (P, Q)$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3x^2 - 6y^2$$

$$\oint_C \bar{f} \cdot d\bar{r} = 3\int_0^1 \int_{x^3}^1 x^2 - 2y^2 \,dy\,dx =$$

$$= 3\int_0^1 \left[x^2 y - \frac{2}{3}y^3\right]_{x^3}^1 dx = 3\int_0^1 x^2 - \frac{2}{3} - x^5 + \frac{2}{3}x^9 \,dx =$$

$$= 3\left[\frac{x^3}{3} - \frac{2}{3}x - \frac{x^6}{6} + \frac{1}{15}x^{10}\right]_0^1 =$$

$$= 3\left(\frac{1}{3} - \frac{2}{3} - \frac{1}{6} + \frac{1}{15}\right) = -1 - \frac{1}{2} + \frac{1}{5} = -\frac{3}{2} + \frac{1}{5} = -\frac{13}{10}$$

b) $\frac{\partial P}{\partial y} = 6y^2 + \beta x^2 \qquad \frac{\partial Q}{\partial x} = \alpha y^2 + 3x^2$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Leftrightarrow \alpha = 6 \wedge \beta = 3$

c) $\frac{\partial \varphi}{\partial x} = P = 2y^3 + 3yx^2 + 2 \Rightarrow \varphi(x,y) = 2xy^3 + yx^3 + 2x + \phi_1(y) + k_1$

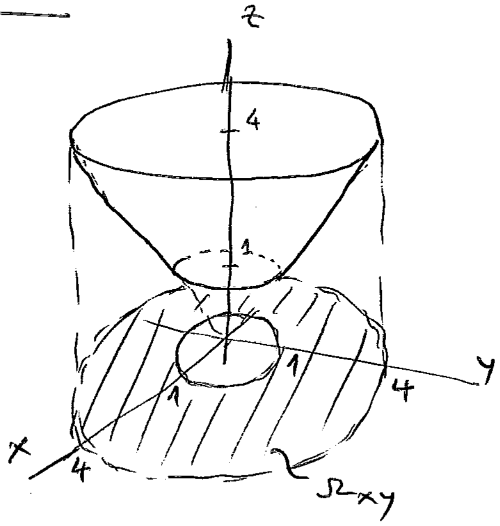
$\frac{\partial \varphi}{\partial y} = Q = 6xy^2 + x^3 + 1 \Rightarrow \varphi(x,y) = 2xy^3 + yx^3 + y + \phi_2(x) + k_2$

$\varphi(x,y) = 2xy^3 + yx^3 + 2x + y + k$

$\int_C \bar{f} \cdot d\bar{r} = \varphi(1,1) - \varphi(0,0) = 2 + 1 + 2 + 1 + k - k = 6$

4

(3) $z = \sqrt{x^2+y^2}, \quad 1 \le z \le 4$



$\Omega_{xy}: 1 \le x^2+y^2 \le 4^2$

$\bar{r}(x,y) = \left(x, y, \sqrt{x^2+y^2}\right)$

$\frac{\partial \bar{r}}{\partial x} = \left(1, 0, \frac{x}{\sqrt{x^2+y^2}}\right)$

$\frac{\partial \bar{r}}{\partial y} = \left(0, 1, \frac{y}{\sqrt{x^2+y^2}}\right)$

$\bar{N}(x,y) = \left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1\right)$

$\| N(x,y) \| = \sqrt{\frac{x^2+y^2}{x^2+y^2} + 1} = \sqrt{2}$

$A(S) = \iint_\Omega \|N(x,y)\| \, dx\, dy = \sqrt{2} \iint_\Omega dx\, dy = \sqrt{2}\, A(\Omega) = 15\sqrt{2}\,\pi$

(4) $S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq 1 \wedge z = xy\}$

$\bar{f}(x,y,z) = (P,Q,R) = (y,x,z)$

a) $\bar{r}(x,y) = (x, y, xy) \longrightarrow R_1 : 0 \leq x^2+y^2 \leq 1$

$\bar{r}(u,v) = (u\cos v, u \operatorname{sen} v, u^2 \operatorname{sen} v \cos v)$ (30)

$(u,v) \in R \qquad R: 0 \leq v \leq 2\pi \wedge 0 \leq u \leq 1$

$\frac{\partial \bar{r}}{\partial x} = (1,0,y)$

$\frac{\partial \bar{r}}{\partial u} = (\cos v, \operatorname{sen} v, 2u \operatorname{sen} v \cos v)$

$\frac{\partial r}{\partial y} = (0,1,x)$

$\frac{\partial \bar{r}}{\partial v} = (-u\operatorname{sen} v, u\cos v, u^2\cos^2 v - u^2 \operatorname{sen}^2 v) =$

$= u(-\operatorname{sen} v, \cos v, u\cos^2 v - u\operatorname{sen}^2 v)$

$\bar{N}(x,y) = (-y,-x,1)$

$\bar{N}(u,v) = u(u \operatorname{sen} v \cos^2 v - u\operatorname{sen}^3 v - 2u \operatorname{sen} v \cos^2 v, -2u\operatorname{sen}^2 v \cos v - u\cos^3 v +$
$+ u\cos v \operatorname{sen}^2 v, 1) =$

$= u(-u\operatorname{sen}^3 v - u \operatorname{sen} v \cos^2 v, -u\cos^3 v - u\cos v \operatorname{sen}^2 v, 1)$

$= u(-u\operatorname{sen} v(\operatorname{sen}^2 v + \cos^2 v), -u\cos v(\cos^2 v + \operatorname{sen}^2 v), 1) =$ (40)

$= u(-u\operatorname{sen} v, -u\cos v, 1)$

$\|\bar{N}(x,y)\| = \sqrt{1+(x^2+y^2)}$

$\|\bar{N}(u,v)\| = u\sqrt{u^2+1}$

$\bar{n}(u,v) = \frac{\bar{N}(u,v)}{\|\bar{N}(u,v)\|} = \frac{1}{\sqrt{u^2+1}}(-u\operatorname{sen} v, -u\cos v, 1)$

$\bar{n}(x,y) = \frac{1}{\sqrt{1+(x^2+y^2)}}(-y,-x,1)$ (30)

b) $\bar{f}[\bar{r}(u,v)] = (u\operatorname{sen} v, u\cos v, u^2\operatorname{sen} v\cos v)$

$\bar{f}[\bar{r}(u,v)] \cdot \bar{N}(u,v) = u[-u^2\operatorname{sen}^2 v - u^2\cos^2 v + u^2\operatorname{sen} v\cos v] =$

$= u^3(-1 + \operatorname{sen} v\cos v)$

$\bar{f}[\bar{r}(x,y)] = (y,x,xy)$

$\iint_S (\bar{f}\cdot\bar{n})\, dS = \int_0^{2\pi}\int_0^1 u^3(-1+\operatorname{sen} v\cos v)\, du\, dv =$

$[\bar{f}[\bar{r}(x,y)] \cdot \bar{N}(x,y) = -y^2 - x^2 + xy$

$= \frac{1}{4}\int_0^{2\pi} -1 + \operatorname{sen} v\cos v \, dv = -\frac{\pi}{2}$

$\iint_{R_1} \underbrace{\cancel{}}_{xy-x^2-y^2} dR_1 = \int_0^{2\pi}\int_0^1 \underbrace{\cancel{}}_{r^3 \operatorname{sen}\theta\cos\theta - r^3} dr\, d\theta = 2\pi\left[\frac{r^2}{2}+\frac{r^4}{4}\right]_0^1 = -\frac{\pi}{2}$

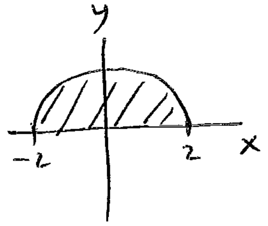
75 (60)

25

(5)

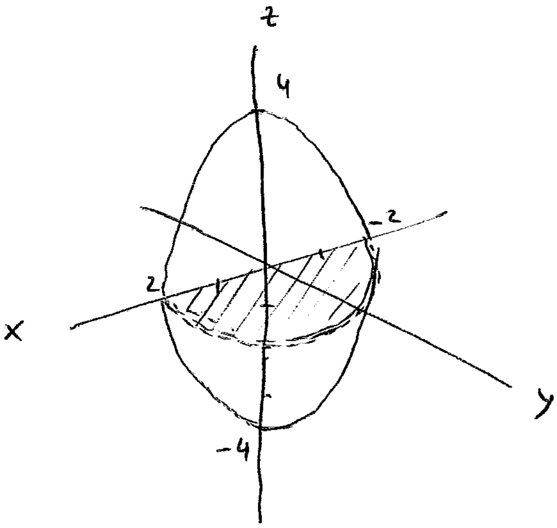
$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{x^2+y^2-4}^{4-x^2-y^2} dz\, dy\, dx$$

a)



$z = x^2+y^2-4$

$z = 4-(x^2+y^2)$



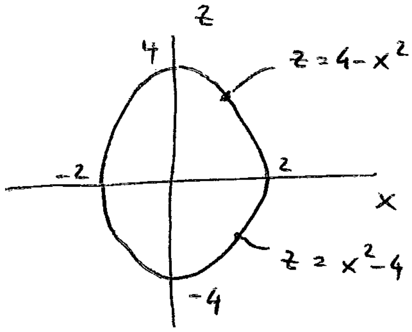
b) $0 \leq \theta \leq \pi \quad \wedge \quad 0 \leq r \leq 2$

$r^2-4 \leq z \leq 4-r^2$

$$\int_0^{\pi}\int_0^{2}\int_{r^2-4}^{4-r^2} r\, dz\, dr\, d\theta = \pi \int_0^2 r\left[4-r^2-r^2+4\right] dr =$$

$$= \pi\, 2\int_0^2 4r - r^3\, dr = 2\pi\left[2r^2-\frac{1}{4}r^4\right]_0^2 = 8\pi$$

c) $\Omega_{xz}: \; -2 \leq x \leq 2 \quad \wedge \quad x^2-4 \leq z \leq 4-x^2$



$\Omega_1: \; -2 \leq x \leq 2 \quad \wedge \quad 0 \leq z \leq 4-x^2$

$0 \leq y \leq \sqrt{4-x^2-z}$

$\Omega_2: \; -2 \leq x \leq 2 \quad \wedge \quad x^2-4 \leq z \leq 0$

$0 \leq y \leq \sqrt{4+z-x^2}$

$$\int_{-2}^{2}\int_0^{4-x^2}\int_0^{\sqrt{4-x^2-z}} dy\, dz\, dx + \int_{-2}^{2}\int_{x^2-4}^{0}\int_0^{\sqrt{4+z-x^2}} dy\, dz\, dx$$

$4-x^2-y^2 = z$

$y^2 = 4-x^2-z$

$y = \pm\sqrt{4-x^2-z}$

$x^2+y^2-4 = z$

$y^2 = 4+z-x^2$

$y = \pm\sqrt{4+z-x^2}$