Optimization

Lecture 5.1

M.EIC – 2021.2022

Linear Programming

FUNDAMENTAL CONCEPTS ON DUALITY

Duality

Duality is one of the most important findings in Linear Programming. It shows that each linear programming problem is associated to a second linear problem, known as the Dual Problem.

Applications of Duality concepts:

- The relationships between the primal and the dual problems provide important information to sensitivity analysis.
- Resolution of Linear Programming problems:

  Sometimes it is easier to solve the Dual problem instead of solving the original problem (Primal problem). Since the optimal value of both objective functions is the same, and there is a correspondence between the optimal solutions of each problem, it may be easier to solve the dual problem first and then deduce the optimal solution of the primal problem.
- Resolution of Transportation Problems

The standard form and the Dual problem

A maximization problem is in the standard form if all the variables are non-negative and all the constraints are of <= type.

A minimization problem is in the standard form if all the variables are non-negative and all the constraints are of >= type.

Primal

$$\begin{aligned} &\max f = x_1 + 2x_2 \\ &\text{s.to} \\ &x_1 + x_2 \le 4 \\ &x_2 \le 3 \\ &x_1, x_2 \ge 0 \end{aligned}$$

Dual

$$\begin{aligned} &\min g = 4y_1 + 3y_2 \\ &\text{s.to} \\ &y_1 \ge 1 \\ &y_1 + y_2 \ge 2 \\ &y_1, y_2 \ge 0 \end{aligned}$$

Generalization

Primal Problem

$$\begin{aligned} &\max \ c^T.x \\ &\text{s.to} \quad A\,x \le b \\ &\qquad\quad x_j \ge 0 \end{aligned}$$

Dual Problem

$$\begin{aligned} &\min \ b^T.\,y \\ &\text{s.to} \quad A^T\,y \ge c \\ &\qquad\quad y_i \ge 0 \end{aligned}$$

Correspondence between Primal and Dual

max $f(x_j)$	min $g(y_i)$
Constraint i (<=)	Variable y_i (y_i >= 0)
Variable x_j (x_j >= 0)	Constraint j (>=)
f coefficients	Righthand side c
Righthand side b	g coefficients

Example

Primal

$$
\begin{aligned}
\max f &= x_1 + 2x_2 \\
\text{s.a} & \\
x_1 + x_2 &\le 4 \\
x_2 &\le 2 \\
x_1, x_2 &\ge 0
\end{aligned}
$$

Dual

$$
\begin{aligned}
\min g &= 4y_1 + 2y_2 \\
\text{s.a} & \\
y_1 &\ge 1 \\
y_1 + y_2 &\ge 2 \\
y_1, y_2 &\ge 0
\end{aligned}
$$

Canonical Form (Phase 2)

$$
\begin{aligned}
\max f &= x_1 + 2x_2 \\
\text{s.a} & \\
x_1 + x_2 + s_1 &= 4 \\
x_2 + s_2 &= 2 \\
x_1, x_2, s_1, s_2, &\ge 0
\end{aligned}
$$

Canonical Form (Phase 1)

$$
\begin{aligned}
\min G &= a_1 + a_2 = 3 - 2y_1 - y_2 + t_1 + t_2 \\
\text{s.a} & \\
y_1 - t_1 + a_1 &= 1 \\
y_1 + y_2 - t_2 + a_2 &= 2 \\
y_1, y_2, t_1, t_2, a_1, a_2 &\ge 0
\end{aligned}
$$

So far, we have analyzed a maximization primal LP problem and the corresponding dual. What happens if the primal problem is not in the standard form?

standard form

$$\max\ c^T.x$$
$$\text{s.a}\quad A\,x \le b$$
$$x_i \ge 0$$

Converting to the standard form (maximization)

Non-standard form	Standard form
Min f	Max (-f)
$\sum_{j=1}^{n} a_{ij}x_j \ge b_i$	$-\sum_{j=1}^{n} a_{ij}x_j \le -b_i$
$\sum_{j=1}^{n} a_{ij}x_j = b_i$	$\sum_{j=1}^{n} a_{ij}x_j \le b_i \wedge -\sum_{j=1}^{n} a_{ij}x_j \le -b_i$
xj unrestricted in sign	$x_j^+ - x_j^-, x_j^+ \ge 0, x_j^- \ge 0$

In order to find the dual of any LP problem, we must express it (the primal) in one of the standard forms:

maximize constraints <=	or	minimize constraints >=

Primal

$$\begin{array}{l} \min f = 2x_1 + 4x_2 \\ \text{s.a} \\ \begin{array}{rrcr} 6x_1 & +10x_2 & \leq & 60 \\ 2x_1 & +x_2 & \geq & 10 \\ 2x_1 & +3x_2 & = & 18 \\ & x_1, x_2 & \geq & 0 \end{array} \end{array}$$

Primal

$$\begin{array}{l} \min f = 2x_1 + 4x_2 \\ \text{s.a} \\ \begin{array}{rrcr} -6x_1 & -10x_2 & \geq & -60 \\ 2x_1 & +x_2 & \geq & 10 \\ 2x_1 & +3x_2 & \geq & 18 \\ -2x_1 & -3x_2 & \geq & -18 \\ & x_1, x_2 & \geq & 0 \end{array} \end{array}$$

Dual

$$\begin{array}{l} \max f = -60y_1 + 10y_2 + 18(y_3 - y_4) \\ \text{s.a} \\ \begin{array}{rrrcr} -6y_1 & +2y_2 & +2(y_3 - y_4) & \leq & 2 \\ -10y_1 & +y_2 & +3(y_3 - y_4) & \leq & 4 \\ & y_1, y_2 & y_3, y_4 & \geq & 0 \end{array} \end{array}$$

If we have a variable $x \in \Re$, we replace it by two new variables such as $x = x^+ - x^-$, com $x^+, x^- >= 0$.

To each equality constraint in the primal problem there is a dual variable unrestricted in sign.

...coming back to our example...

Primal problem resolution

1st tableaux (2nd phase)

Basis	x1	x2	s1	s2	Value
s1	1	1	1	0	4
s2	0	1	0	1	2
max f	1	2	0	0	0

2nd tableaux

Basis	x1	x2	s1	s2	Value
s1	1	0	1	-1	2
x2	0	1	0	1	2
f	1	0	0	-2	-4

3rd tableaux

Optimal solution

Basis	x1	x2	s1	s2	Value
x1	1	0	1	-1	2
x2	0	1	0	1	2
f	0	0	-1	-1	-6

Dual problem resolution

1st tableaux (1st phase)

Basis	y1	y2	t1	t2	a1	a2	Value
a1	1	0	-1	0	1	0	1
a2	1	1	0	-1	0	1	2
min G	-2	-1	1	1	0	0	-3
g	4	2	0	0	0	0	0

2nd tableaux

Basis	y1	y2	t1	t2	a1	a2	Value
y1	1	0	-1	0	1	0	1
a2	0	1	1	-1	-1	1	1
G	0	-1	-1	1	2	0	-1
g	0	2	4	0	-4	0	-4

3rd tableaux
optimal solution
1st phase and
optimal solution of the
original problem

Basis	y1	y2	t1	t2	a1	a2	Value
y1	1	0	-1	0	1	0	1
y2	0	1	1	-1	-1	1	1
G	0	0	0	0	1	1	0
g	0	0	2	2	-2	-2	-6

Some Fundamental Duality Properties

Weak Duality Theorem:

If x is a feasible solution for the primal minimization problem and y is a feasible solution for the dual maximization problem, then weak duality implies $g(y) <= f(x)$ where f and g are the objective functions for the primal and dual problems, respectively .

Strong Duality Theorem:

If the primal (dual) problem has a finite optimal solution, then so does the dual (primal) problem and these two values are equal. That is, $g^* = f^*$

These properties describe the key relationships between any pair of primal and dual solutions.
One useful application is for evaluating a proposed solution for the primal problem.

For example, suppose that x is a primal feasible solution proposed for implementation and that a feasible dual solution y was found, such that $cx = yb$. Hence, we can deduce that x must be optimal without solving the problem by the Simplex method.

Even if $cx<yb$, then yb still provides an upper bound on f, so if $yb-cx$ is very small, we can, sometimes, use x as a good approximation of the optimal solution.

Some Fundamental Duality Properties

Complementary Slackeness Property

If, in an optimal solution of a linear program, the value of the dual variable associated with a constraint is nonzero, then that constraint must be satisfied with equality. Further, if a constraint is satisfied with strict inequality, then its corresponding dual variable must be zero.

The values of the optimal dual solution are the shadow prices in the primal problem

Symmetry Property

For any primal and corresponding dual problems, all the relations are symmetrical because the dual of the dual problem is the primal.

Duality Theorem

The following are the only possible relationships between the primal and dual problems:

1. If one of the problems (primal or dual) has feasible solutions and a bounded objective function (and so has an optimal solution), then so does the other problem.
2. If one of the problems has feasible solutions and an unbounded objective function (and so no optimal solutions), then the other problem has no feasible solutions.
3. If one of the problems has no feasible solutions, then the other problem has either no feasible solutions or an unbounded objective function.

When do we solve the dual problem?

Instead of solving the primal, solving the dual LP problem is sometimes easier in following cases:

a) The dual has fewer constraints than primal. The time required for solving LP problems is directly affected by the number of constraints, i.e., number of iterations necessary to converge to an optimum solution, which in Simplex method usually ranges from 1.5 to 3 times the number of structural constraints in the problem

b) The dual involves the maximization of an objective function. It may be possible to avoid artificial variables that otherwise would be used in a primal minimization problem.

Economic interpretation

Usual economic interpretation of a primal problem is the standard form: :

Quantity	Interpretation
x_j	Level of activity j
c_j	Unit profit from activity j
f	Total profit from all activities
b_i	Amount of resource i available
a_{ij}	Amount of resource i consumed per unit of activity j

Economic interpretation of the dual problem

The economic interpretation of the dual problem is directly based on the corresponding interpretation of the primal in its standard form.
Since the dual optimal value is equal to the optimal primal one (profit), we can say that:

$$\boxed{\begin{array}{ll} \min & g = b^T . y \\ s.a & A^T y \geq c \\ & y_i \geq 0 \end{array}}$$

yi is the value of resource i
yi >= 0 says that the contribution to profit per unit of resource *i* (*i* 1, 2, . . . , *m*) must be nonnegative: otherwise, it would be better not to use this resource at all.

In other words, y_i are the shadow prices.

$$g = \sum_{i=1}^{m} b_i y_i$$

corresponds to minimizing the total implicit value of the resources consumed by the activities.

Economic interpretation of the dual problem

$$\boxed{\begin{array}{ll} \min & g = b^T . y \\ \text{s.a} & A^T y \geq c \\ & y_i \geq 0 \end{array}}$$

Since each unit of activity j in the primal problem uses aij units of resource i,

$$\sum_{i=1}^{m} a_{ij} y_i$$

is interpreted as the current contribution to profit of the mix of resources that would be consumed if 1 unit of activity *j* were used (*j* = 1, 2, . . . , *n*)..

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j$$

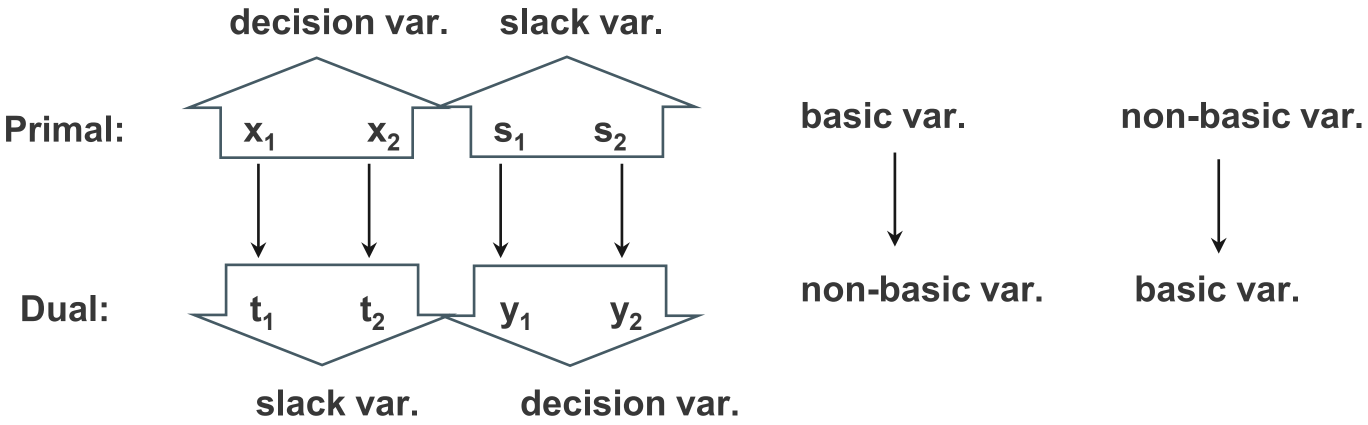
says that the actual contribution to profit of the above mix of resources must be at least as much as if they were used by 1 unit of activity *j;* otherwise, we would not be making the best possible use of these resources.

Primal-dual relationships

- The optimal solutions in both problems are equal:

$$\max f = \min g$$

- there is a correspondence between primal and dual variables



Comparison between the final Primal and Dual simplex tableaux

Final Primal Tableau

Basis	x1	x2	s1	s2	Value
x1	1	0	1	-1	2
x2	0	1	0	1	2
max f	0	0	-1	-1	-6

Final Dual tableau

Basis	y1	y2	t1	t2	Value
y1	1	0	-1	0	1
y2	0	1	1	-1	1
min g	0	0	2	2	-6

Primal-dual relationships

the value of a primal (dual) variable is the absolute value of the o,f. coefficient of the corresponding dual (primal) variable.

$x_1 = 2$ $\quad = |$ coeff. of t_1 in $g = 2| = 2$

$x_2 = 2$ $\quad = |$ coeff. of t_2 in $g = 2| = 2$

$y_1 = 1$ $\quad = |$ coeff. of s_1 in $f = -1| = 1$

$y_2 = 1$ $\quad = |$ coeff. of s_2 in $f = -1| = 1$

the coefficient that in the primal tableau is at the intersection of a row (basic var.) and a non-basic column is the symmetrical of the coefficient that, in the dual tableau is at the intersection of the corresponding column and row dual variables.

Primal – Intersection of x_1 (basic) and s_2 (non-basic) = -1

Dual – Intersection of t_1 (non-basic) and y_2 (basic) = 1

Wood Company

A company producing two products A and B intends to optimize its week production plans to maximize the profit. The unit profits of A and B are of €8 and €5.
The production of A and B implies the consumption of other resources, namely, timber and labour hours.
The following table shows the number of resources used per unit of A and B produced.

Product	Timber(m)	Labour hours
A	30	5
B	20	10

The company has 300 m of timber/week and 110 hours of week hours available.
(a) Which is the optimal production plan?
(b) Market studies indicate that the profit of B will increase of € 1/6 per week, while the profit of A will remain constant. Taking this tendency into account, for how many weeks will the previous optimal solution remain optimal?
(c) At what price will the company pay for extra labour hours?

$$\max L = 8X_1 + 5X_2$$

$$30X_1 + 20X_2 \leq 300$$

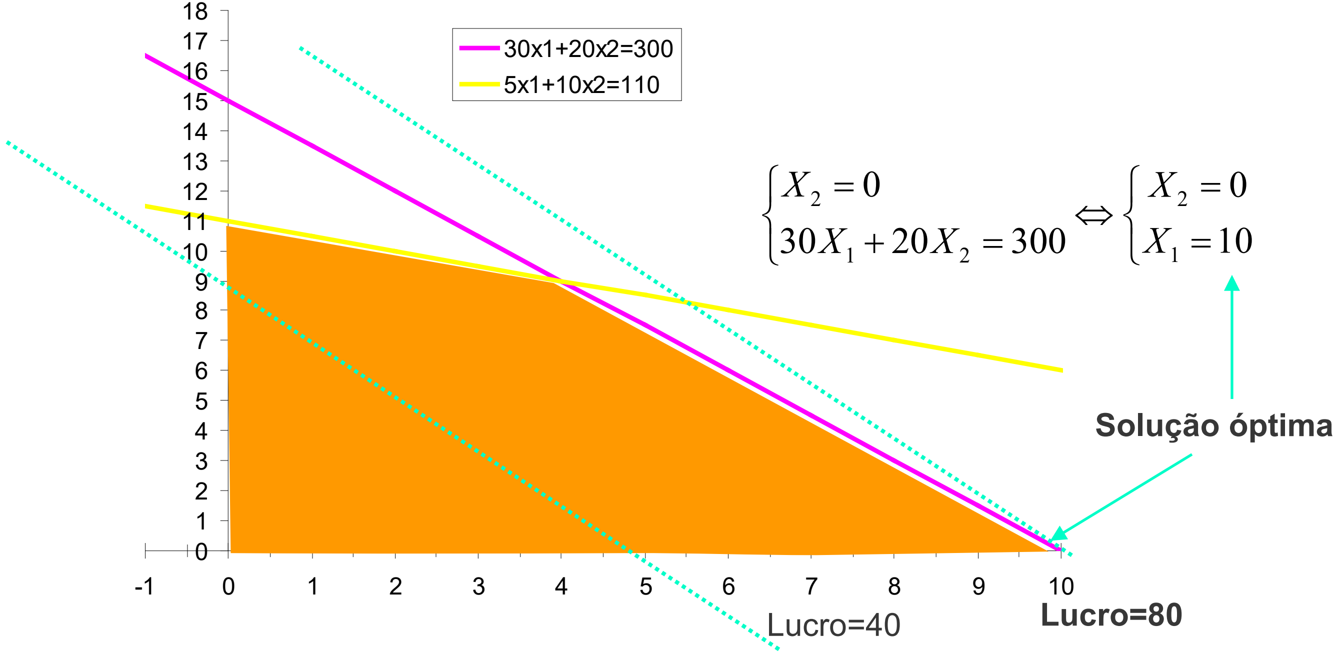
$$5X_1 + 10X_2 \leq 110$$

$$X_1, X_2 \geq 0$$

Notes:

- **Reduced cost of X2**: if the profit of X2 increases to 5.33, it becomes profitable to produce X2 (X2 >0)
- **Shadow price of timber constraint**: if we one exta unit of timber the profit increases from €80 to €80,2667

$$\begin{cases} X_2 = 0 \\ 30X_1 + 20X_2 = 300 \end{cases} \Leftrightarrow \begin{cases} X_2 = 0 \\ X_1 = 10 \end{cases}$$



Primal

Variable -->	X1	X2	Direction	R. H. S.
Maximize	8	5		
C1	30	20	<=	300
C2	5	10	<=	110
LowerBound	0	0		
UpperBound	M	M		
VariableType	Continuous	Continuous		

Iter. 1

		X1	X2	Slack_C1	Slack_C2		
Basis	C(j)	8,0000	5,0000	0	0	R. H. S.	Ratio
Slack_C1	0	30,0000	20,0000	1,0000	0	300,0000	10,0000
Slack_C2	0	5,0000	10,0000	0	1,0000	110,0000	22,0000
	C(j)-Z(j)	8,0000	5,0000	0	0	0	

Iter. 2

		X1	X2	Slack_C1	Slack_C2		
Basis	C(j)	8,0000	5,0000	0	0	R. H. S.	Ratio
X1	8,0000	1,0000	0,6667	0,0333	0	10,0000	
Slack_C2	0	0	6,6667	-0,1667	1,0000	60,0000	
	C(j)-Z(j)	0	-0,3333	-0,2667	0	80,0000	

Dual

Variable -->	C1	C2	Direction	R. H. S.
Minimize	300	110		
X1	30	5	>=	8
X2	20	10	>=	5
LowerBound	0	0		
UpperBound	M	M		
VariableType	Continuous	Continuous		

Iter. 1

		C1	C2	Surplus_X1	Surplus_X2	Artificial_X1	Artificial_X2		
Basis	C(j)	300,0000	110,0000	0	0	0	0	R. H. S.	Ratio
Artificial_X1	M	30,0000	5,0000	-1,0000	0	1,0000	0	8,0000	0,2667
Artificial_X2	M	20,0000	10,0000	0	-1,0000	0	1,0000	5,0000	0,2500
	C(j)-Z(j)	300,0000	110,0000	0	0	0	0	0	
	* Big M	-50,0000	-15,0000	1,0000	1,0000	0	0	0	

Iter. 2

		C1	C2	Surplus_X1	Surplus_X2	Artificial_X1	Artificial_X2		
Basis	C(j)	300,0000	110,0000	0	0	0	0	R. H. S.	Ratio
Artificial_X1	M	0	-10,0000	-1,0000	1,5000	1,0000	-1,5000	0,5000	0,3333
C1	300,0000	1,0000	0,5000	0	-0,0500	0	0,0500	0,2500	M
	C(j)-Z(j)	0	-40,0000	0	15,0000	0	-15,0000	75,0000	
	* Big M	0	10,0000	1,0000	-1,5000	0	2,5000	0	

Iter. 3

		C1	C2	Surplus_X1	Surplus_X2	Artificial_X1	Artificial_X2		
Basis	C(j)	300,0000	110,0000	0	0	0	0	R. H. S.	Ratio
Surplus_X2	0	0	-6,6667	-0,6667	1,0000	0,6667	-1,0000	0,3333	
C1	300,0000	1,0000	0,1667	-0,0333	0	0,0333	0	0,2667	
	C(j)-Z(j)	0	60,0000	10,0000	0	-10,0000	0	80,0000	
	* Big M	0	0	0	0	1,0000	1,0000	0	

Final Primal tableau

		X1	X2	Slack_C1	Slack_C2		
Basis	C(j)	8,0000	5,0000	0	0	R. H. S.	Ratio
X1	8,0000	1,0000	0,6667	0,0333	0	10,0000	
Slack_C2	0	0	6,6667	-0,1667	1,0000	60,0000	
	C(j)-Z(j)	0	-0,3333	-0,2667	0	80,0000	

Reduced cost of X2: 0,33
Shadow price of Slack_C1: 0,2667

	basic	non-basic	non-basic	basic
Primal	**X1**	**X2**	**Slack_C1**	**Slack_C2**
Dual	**Surplus_X1**	**Surplus_X2**	**C1**	**C2**
	non-basic	basic	basic	non-basic

Final Dual tableau

		C1	C2	Surplus_X1	Surplus_X2	Artificial_X1	Artificial_X2		
Basis	C(j)	300,0000	110,0000	0	0	0	0	R. H. S.	Ratio
Surplus_X2	0	0	-6,6667	-0,6667	1,0000	0,6667	-1,0000	0,3333	
C1	300,0000	1,0000	0,1667	-0,0333	0	0,0333	0	0,2667	
	C(j)-Z(j)	0	60,0000	10,0000	0	-10,0000	0	80,0000	
	* Big M	0	0	0	0	1,0000	1,0000	0	

Exercise A

a) Define the dual problem corresponding to the following linear program

$$\max f = 2x_1 + 3x_2$$

s.a

$$\begin{array}{rrcr} 0.25x_1 & +0.50x_2 & \leq & 40 \\ 0.40x_1 & +0.20x_2 & \leq & 40 \\ & 0.80x_2 & \leq & 40 \\ & x_1, x_2 & \geq & 0 \end{array}$$

b) Knowing that the following tableau corresponds to the optimal primal solution, write the final dual tableau.

Basis	x1	x2	s1	s2	s3	Value
x1	1	0	-1,33	3,33	0	80
s3	0	0	-2,13	1,33	1	8
x2	0	1	2,67	-1,67	0	40
max f	0	0	-5,33	-1,67	0	-280

Resolution of Exercise A

a) Dual problem

$$\min g = 40y_1 + 40y_2 + 40y_3$$

s.a

$$\begin{array}{llll} 0.25y_1 & +0.40y_2 & & \geq 2 \\ 0.50y_1 & +0.20y_2 & +0.8y_3 & \geq 3 \\ & y_1, y_2 & & \geq 0 \end{array}$$

	b	**b**	nb	nb	**b**
Primal	**x1**	**x2**	s1	s2	**s3**
Dual	s4	s5	**y1**	**y2**	y3
	nb	nb	**b**	**b**	nb

b) Final dual Simplex tableau.

Primal

Base	x1	x2	s1	s2	s3	Valor
x1	1	0	-1,33	3,33	0	80
s3	0	0	-2,13	1,33	1	8
x2	0	1	2,67	-1,67	0	40
max f	0	0	-5,33	-1,67	-1	-280

Dual

Base	y1	y2	y3	s4	s5	Valor
y1	1	0	2,13	1,33	-2,67	5,33
y2	0	1	-1,33	-3,33	1,67	1,67
min g	0	0	8	80	40	-280

Linear Programming

THE DUAL SIMPLEX METHOD

Simplex Method vs. Dual Simplex Method

- Simplex method starts with a nonoptimal but feasible solution where as dual simplex method starts with an optimal but infeasible solution.

- Simplex method maintains the feasibility during successive iterations where as dual simplex method maintains the optimality.

Steps involved in the dual simplex method

1. All the constraints (except those with equality (=) sign) are modified to 'less-than-equal-to' sign. Constraints with 'greater-than-equal-to sign' are multiplied by -1 through out so that inequality sign gets reversed. Finally, all these constraints are transformed to equality sign by introducing required slack variables.

2. Modified problem, as in step one, is expressed in the form of a simplex tableau. If all the cost coefficients are positive (i.e., optimality condition is satisfied) and one or more basic variables have negative values (i.e., non-feasible solution), then dual simplex method is applicable.

Steps involved in the dual simplex method

3. Selection of exiting variable: The basic variable with the highest negative value is the exiting variable. If there are two candidates for exiting variable, any one is selected. The row of the selected exiting variable is marked as pivotal row.

4. Selection of entering variable: Cost coefficients, corresponding to all the negative elements of the pivotal row, are identified. Their ratios are calculated after changing the sign of the elements of pivotal row, i.e.,

$$ratio = \left(\frac{Cost\ Coefficients}{-1 \times Elements\ of\ \ pivotal\ row} \right)$$

The column corresponding to minimum ratio is identified as the pivotal column and associated decision variable is the entering variable.

Steps involved in the dual simplex method

5. Pivotal operation: Pivotal operation is exactly same as in the case of simplex method, considering the pivotal element as the element at the intersection of pivotal row and pivotal column.

6. Check for optimality: If all the basic variables have nonnegative values then the optimum solution is reached. Otherwise, Steps 3 to 5 are repeated until the optimum is reached.

Dual Simplex Method: An Example

Primal: max $3x_1+6x_2+3x_3$

s.t. $3x_1+4x_2+x_3 \leq 3$ y1

$x_1+3x_2+x_3 \leq 2$ y2

$x_1, x_2 \geq 0$

$x_3 \leq 0$

Dual: min $3y_1+2y_2$

s.t. $3y_1+ y_2 \geq 3$

$4y_1+3y_2 \geq 6$

$y_1+ y_2 \leq 3$

$y_1, y_2 \geq 0$

Solving the primal problem

We use the normal Simplex method.
There is no need to use the dual Simplex method since (0, 0, 0) is a feasible solution.

$$
\begin{aligned}
\text{Primal:max} \quad & 3x_1+6x_2+3x_3 \\
\text{s.t.} \quad & 3x_1+4x_2+x_3 \leq 3 \\
& x_1+3x_2+x_3 \leq 2 \\
& x_1, x_2 \geq 0 \\
& x_3 \leq 0
\end{aligned}
$$

Note that x3 <=0, so a change of variable was made, replacing x3 by –x3 and imposing that the new variable –x3 >=0

PRIMAL

Iter 0	basis	X1	X2	X3	S1	S2	RHS	*ratio*
	S1	3.00	4.00	-1.00	1.00	-	3.00	*3/4*
	S2	1.00	3.00	-1.00	-	1.00	2.00	*2/3*
max	f	3.00	6.00	-3.00	-	-	-	

iter 1	basis	X1	X2	X3	S1	S2	RHS	*ratio*
	S1	1.67	-	0.33	1.00	-1.33	0.33	*1/5*
	X2	0.33	1.00	-0.33	-	0.33	0.67	*2*
	f	1.00	-	-1.00	-	-2.00	-4.00	

iter 2	basis	X1	X2	X3	S1	S2	RHS
	X1	1.00	-	0.20	0.60	-0.80	0.20
	X2	-	1.00	-0.40	-0.20	0.60	0.60
	f	-	-	-1.20	-0.60	-1.20	-4.20

optimal

Solving the dual using the Dual Simplex method

In this case the point (0,0,0) is not a feasible solution.
We may use the Two-phases method or the Dual Simplex method. In this case we will use the Dual Simplex method.

$$
\begin{aligned}
\text{Dual:}\quad \min \quad & 3y_1+2y_2 \\
\text{s.t.} \quad & 3y_1+\ y_2 \geq 3 \\
& 4y_1+3y_2 \geq 6 \\
& y_1+\ y_2 \leq 3 \\
& y_1, y_2 \geq 0
\end{aligned}
$$

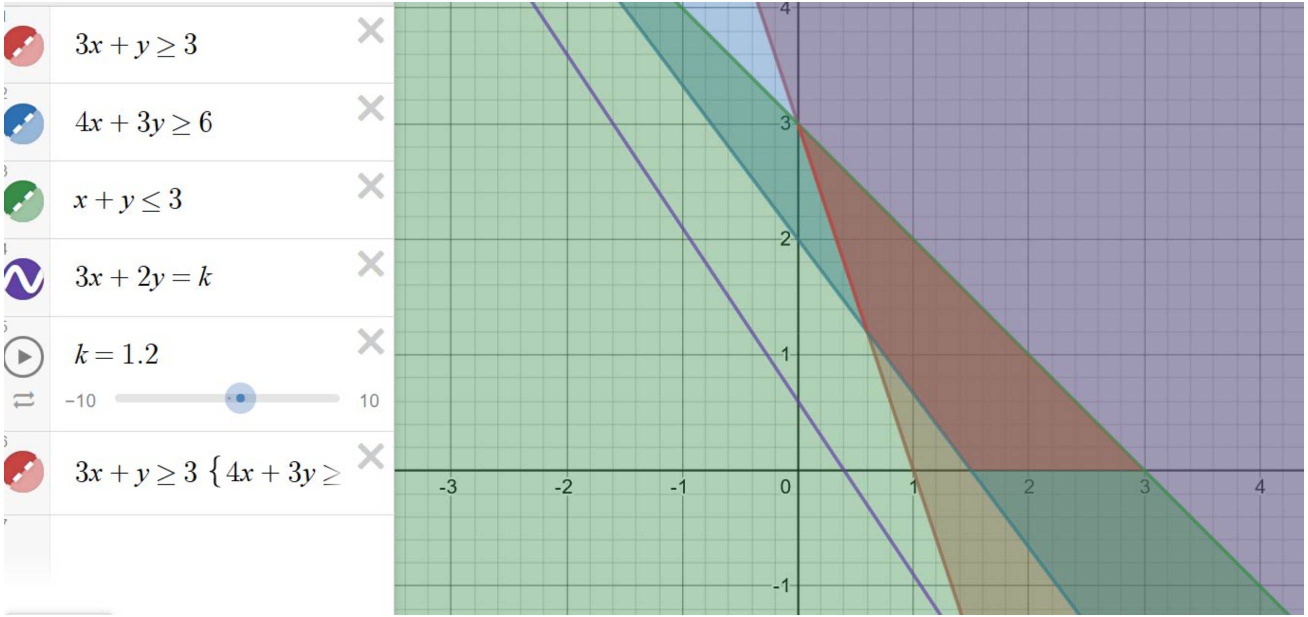
Note that using the dual Simplex, first we choose the row of the leaving variable (t2 in the iteration 0) and then the column of the entering variable using the ratio condition (y2 in the iteration 0)

DUAL

iter 0	basis	y1	y2	t1	t2	t3	RHS
	t1	-3	-1	1	0	0	-3
	t2	-4	-3	0	1	0	-6
	t3	1	1	0	0	1	3
min	g	3	2	0	0	0	0
ratio		0.75	0.67				

iter 1	basis	y1	y2	t1	t2	t3	RHS
	t1	- 1.67	-	1.00	- 0.33	-	- 1.00
	y2	1.33	1.00	-	- 0.33	-	2.00
	t3	- 0.33	-	-	0.33	1.00	1.00
min	g	0.33	-	-	0.67	-	- 4.00
ratio		0.20			2.00		

iter 2	basis	y1	y2	t1	t2	t3	RHS
	y1	1.00	-	- 0.60	0.20	-	0.60
	y2	-	1.00	0.80	- 0.60	-	1.20
	t3	-	-	- 0.20	0.40	1.00	1.20
min	g	-	-	0.20	0.60	-	- 4.20
ratio							optimal


$3x + y \geq 3$
$4x + 3y \geq 6$
$x + y \leq 3$
$3x + 2y = k$
$k = 1.2$
−10
10
$3x + y \geq 3 \{4x + 3y \geq$
-3
-2
-1
0
1
2
3
4
-1
1
2
3
4