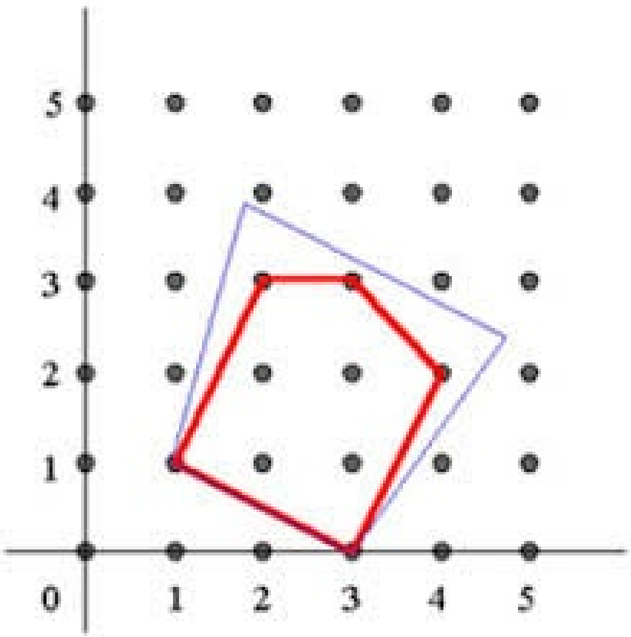


OPTIMIZATION

Faculdade de Engenharia da Universidade do Porto

INTEGER PROGRAMMING

Mestrado em Engenharia Informática e Computação

CONTENTS

EXERCISE 1

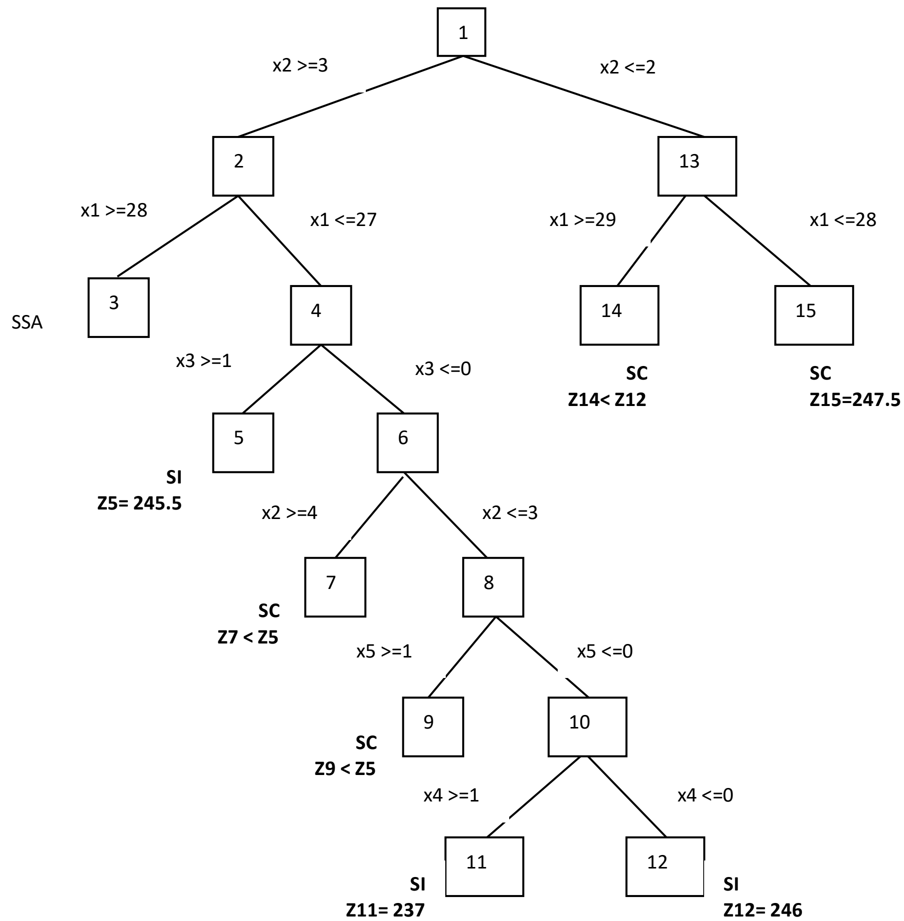
A computational application that implements a branch and bound algorithm provided the following list of solutions (in this order, before being interrupted) for a pure integer programming problem. The criterion of choice of the branch variable is the increasing sequence of the index of the variables.

a) Indicate if the integer problem is a maximization or minimization. Justify.
b) Construct the binary tree of sub-problems associated with this list of solutions (indicate the nodes number), including the restrictions of each branch. What was the node selection strategy of this branch-and-bound tree?
c) Which nodes can be despised ("probed or pruned")? Justify.
d) Is the optimal solution known in the current situation? If not, what is the next branch to be analyzed?

Solution (node)	Z	X1	X2	X3	X4	X5
1	250,667	28	2,667	0	0	0
2	248,667	27,333	3	0	0	0
3	Infeasible	28	3	0	0	0
4	247,875	27	3	0,25	0	0
5	245,5	26	3	1	0	0
6	247,667	27	3,167	0	0	0
7	242,667	25,333	4	0	0	0
8	247,5	27	3	0	0	0,25
9	242	25,75	3	0	0	1
10	247	27	3	0	0,143	0
11	237	25	3	0	1	0
12	246	27	3	0	0	0
13	248	28,5	2	0	0	0
14	245,333	29	1,333	0	0	0
15	247,5	28	2	0	0,5	0

SOLUTION

a) Maximization problem
b) A FIFO (First in-First out) strategy was used



c) The nodes that don't need to be further explored are the following:
Node 3: no feasible solutions;
Node 5: integer solution (first lower bound)
Node 7: the LP solution is inferior to the lower bound given by node 5;
Node 9: the LP solution is inferior to the lower bound given by node 5;
Node 11: integer solution, but inferior to the one given by node 5, so the initial lower bound is kept.
Node 12: integer solution, higher than the value given by node 5. The lower bound is updated being a candidate to the optimal solution.
Node 14: the LP solution is inferior to the lower bound given by node 12;

d) At this moment we know that the optimal solution lies in the interval [246, 247,5].

We know that we reached the optimal solution when the upper and lower bound are equal.

Next we should explore node 15 (non-integer solution) branching into x4 >= 1 e x4 <=0.

EXERCISE 2

The figure below shows the tree of sub-problems at a certain stage of the resolution of a pure integer programming problem by the branch and bound algorithm. The symbol ϕ indicates a region with no permissible solutions for the sub-problem corresponding to that node. SC indicates a continuous solution (there is at least one non-integer variable) and SI indicates an integer solution (all variables have integer values

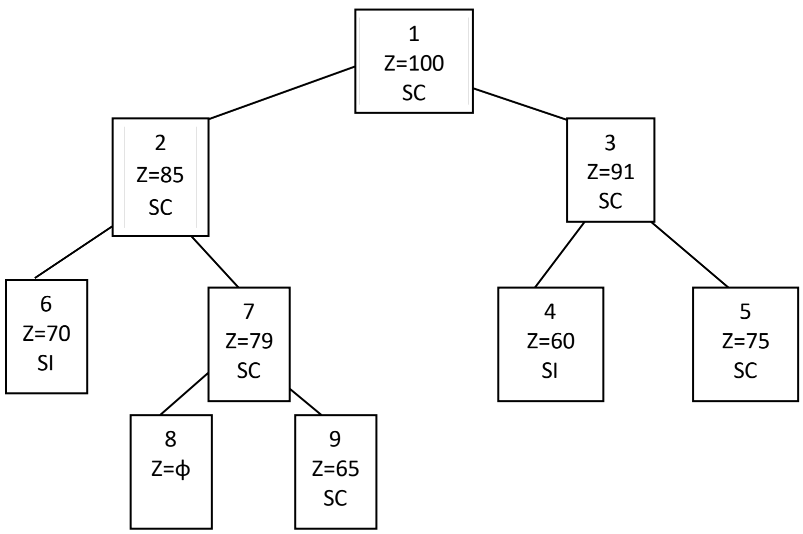
a) Which is the best upper bound for the maximum value of z for the IP?

b) Which is the best lower bound for the maximum value of z?

c) What are the nodes where there is no more branching? Justify.

d) What are the nodes where it is still possible to branch. Justify.

e) Was an optimal solution for the IP found already? If not, what is the quality of the current solution (UB-LI)/LI and which node should be branched?



SOLUTION

a) UB = 75 (node 5)

b) LB = 70 (node 6)

c) Node 6: because it is an integer solution (and also a LB)

Node 8: no feasible solutions

Node 9: because the LP solution is inferior to the solution of node 6

Node 4: because it is an integer solution (inferior to the LB)

d) Only node 5 because the LP value is still superior to the LB

e) Not yet. Relative error = (UB-LB)/LB) = 5/70 =7%

EXERCISE 3

The problem below was solved using the *branch and bound* algorithm, by applying the LIFO rule (*last in, first out* – the last node to enter the tree is the first to leave the tree). Figure 1 shows the tree generated by the algorithm.

$$\max z = 5x_1 + 3x_2 + 2x_3$$
$$\text{s. a} \quad 7x_1 + 4x_2 + 2x_3 \leq 15$$
$$4x_1 + 3x_2 + 4x_3 \leq 12$$
$$x_1, x_2, x_3 \in \mathbb{Z}^+$$

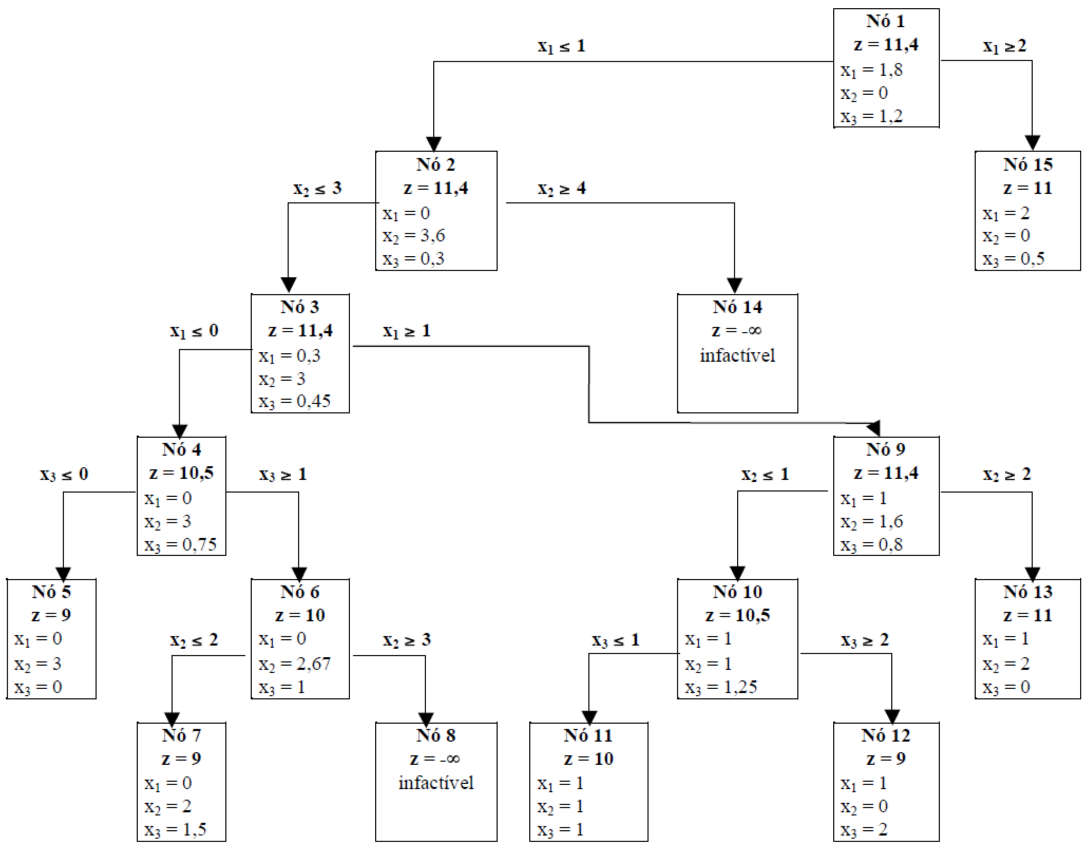


Figura 1. Árvore gerada pelo método *branch- and- bound*

a) Suppose the branch- and- bound algorithm was initiated and at this point the tree only has the nodes 1, 2, 3 e 4. What is the range [upper bound, lower bound] in which the optimal solution is found? Why can we say that the optimal solution is necessarily in this range?

b) Now consider that node 4 was branched generating nodes 5 and 6 (tree contains nodes 1, 2, 3, 4, 5, and 6). What is the range [lower limit, upper limit] in which the optimal solution is found? Why can we say that the optimal solution is necessarily in this range?

c) Describe the update sequence for the lower bound of the problem, from node 1 to node 15. Indicate each node where the limit was updated. Explain the optimal solution.

d) Why was the branching of node 5 not done?

e) Why was the branching of nodes 7 and 15 not done?

f) Why was node 10 branched?

SOLUTION

a) [0,11]
b) [9,11]
c) …
d) Because the solution is already integer.
e) Node 7 was not branched because the LP solution is not better than the current LB (which value is 9 in node 5).
   Node 15 was not branched because the LP solution is equal to the current LB (which value is 11 in node 13).
   Hence, the optimal solution is in node 13 with z=11.
f) When node 10 was branched, LB (node 5) = UB (node7) = 9. The LP value in node 10 is 10.5. Since we are maximizing, it is worth to explore this node. Later we verified that this node did not lead to the optimal solution.

EXERCISE 4

The problem below was solved by the Branch&Bound method, applying the LIFO rule (last in, first out - the last node to enter the tree is the first to leave the tree). Figure 2 shows the tree generated by the method.

$$\max z = 5x_1 + 3x_2 + 7x_3$$
$$\text{s. a} \quad 4x_1 + 5x_2 + 7x_3 \leq 35$$
$$6x_1 + 3x_2 + 5x_3 \leq 29$$
$$x_1, x_2, x_3 \in \mathbb{Z}^+$$

a) Suppose the branch- and- bound algorithm was initiated and at this point the tree only has the nodes 1, 2 and 3. What is the range [upper bound, lower bound] in which the optimal solution is found? Why can we say that the optimal solution is necessarily in this range?

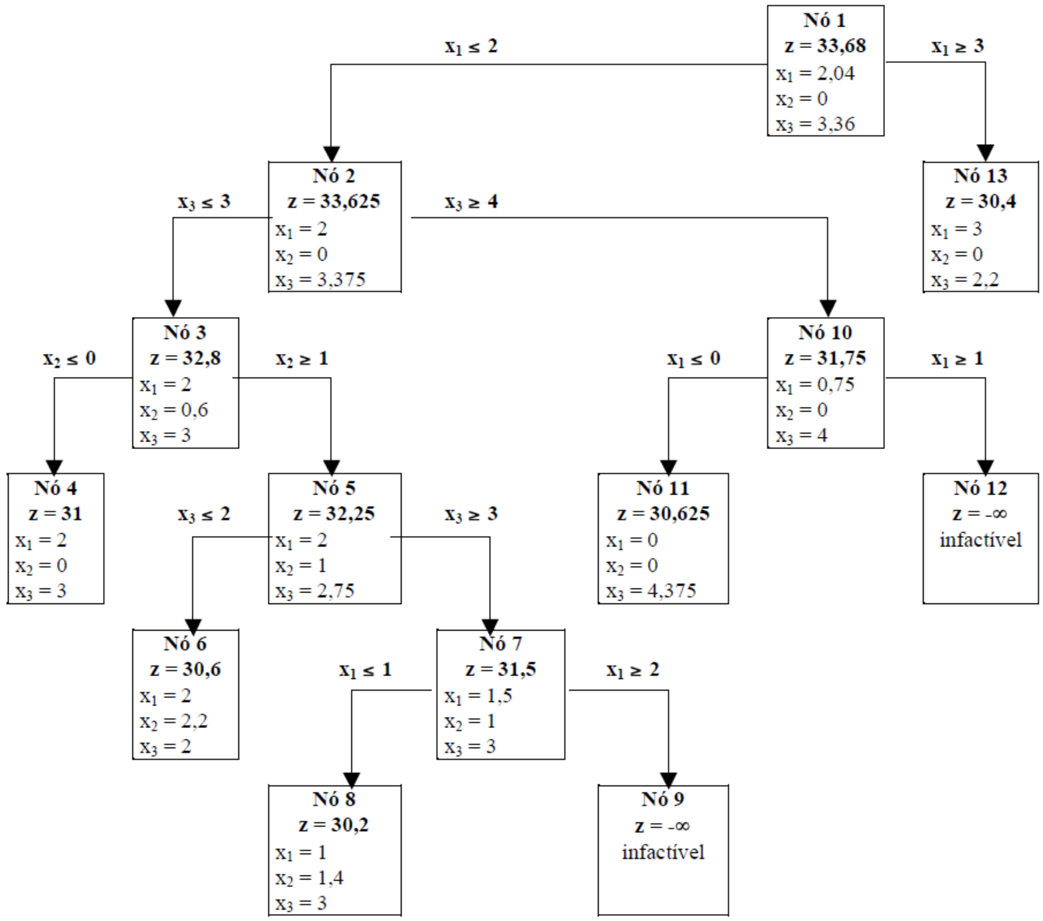
b) Now consider that node 3 was branched generating nodes 4 and 5 (tree contains nodes 1, 2, 3, 4 and 5). What is the range [lower limit, upper limit] in which the optimal solution is found? Why can we say that the optimal solution is necessarily in this range?

c) Describe the update sequence for the lower bound of the problem, from node 1 to node 13. Indicate each node where the limit was updated. Explain the optimal solution.

d) Why was the branching of node 4 not done?

e) Why was the branching of nodes 6 and 13 not done?

f) Why was node 7 branched?



SOLUTION

a) [0,33]

b) [31, 33]

c) …

d) Because it is already na integer solution (at this moment, the LB)

e) Nodes 6 and 13 were not branched because the LP solution is worse than the current LB (given by node 4)

f) Node 7 was branched because the LP solution was better than the best solution obtained so far (given by node 4).

EXERCISE 5

Consider the following integer programming problem (note that it is nor referred if this is a maximization or minimization problem):

$$Z = 10x1 + 7x2 + 4x3$$

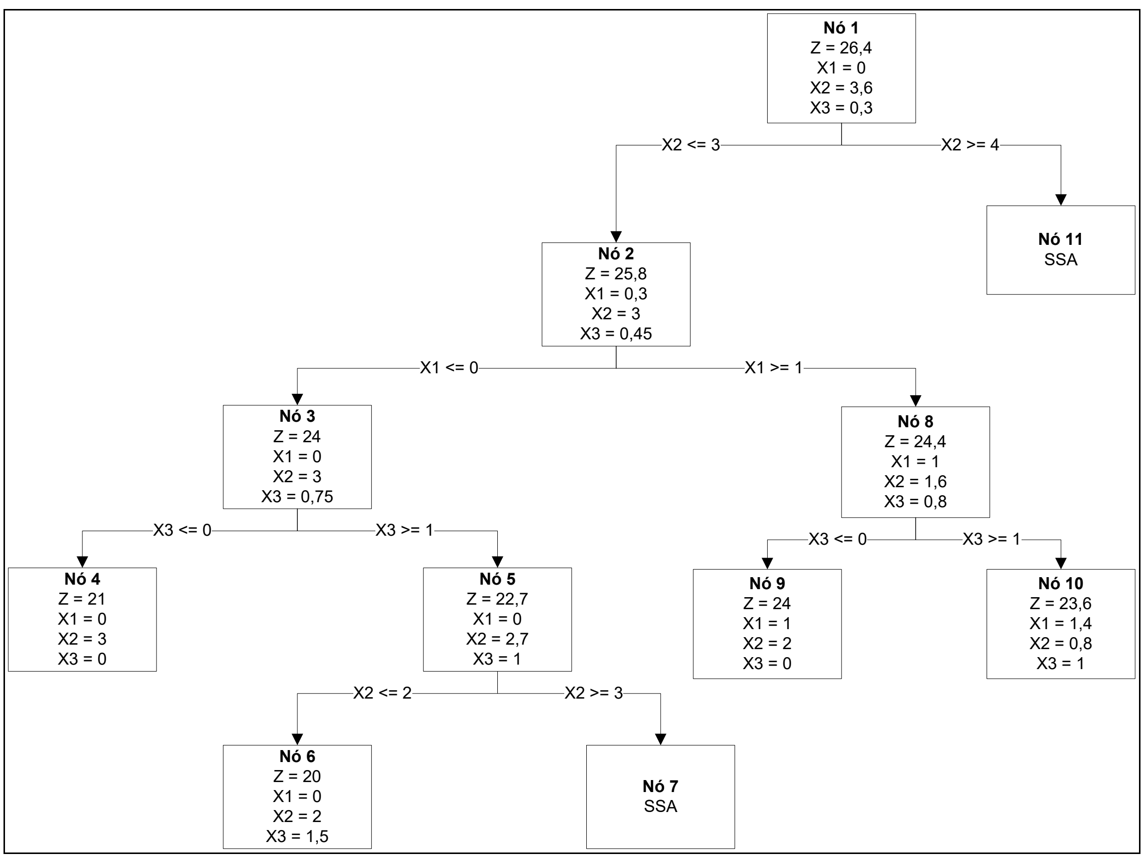
$$\text{s.a} \quad 14x_1 + 8x_2 + 4x_3 \leq 30$$

$$8x_1 + 6x_2 + 8x_3 \leq 24$$

$$x_1,\ x_2,\ x_3 \ \in Z^+$$

Figure 2 shows the sub-problem tree generated in a given phase applying the branch-and-bound method. The order of generation of the nodes is represented by the identification numbers. For each node is also represented the value of the decision variables and the objective function. The nodes with "SSA" correspond to sub-problems without admissible solutions.

a) Formulate the sub-problem correspondent to node 5, referring if this is a maximization or minimization problem.
b) For each unbranched node specify if it has been exploded or not. Justify your answer.
c) Indicate which is the optimal solution for the problem. In the case it has not been determined yet, indicate the interval (lower and upper limits) in which the optimal solution is. Justify your answer.
d) The problem considered was solved applying a depth-first search strategy. If instead of this strategy a breadth-first search strategy was chosen, would it be necessary the resolution of a bigger or smaller number of sub-problems? Justify your answer



SOLUTION

a)

$$Max\ Z = 10x1 + 7x2 + 4x3$$

Subject to

$$14x_1 + 8x_2 + 4x_3 \leq 30$$
$$8x_1 + 6x_2 + 8x_3 \leq 24$$
$$x_2 \leq 3$$
$$x_1 \leq 0$$
$$x_3 \geq 1$$
$$x_1, x_2, x_3 \geq 0$$

Maximization because the solution gets worse as we go exploring - the best solution is always the root solution (linear problem with fewer constraints). Additional constraints are the cuts that result from branching.

b) All nodes are already exploited. The justification is as follows:

- Nodes 4 and 9 - Already exploited because they correspond to integer solutions.
- Nodes 6 and 10 - Already explored because they can be pruned. They correspond to sub-problems with a non-integer optimal solution and a value for the objective function lower than the value of an already existing intger solution (Node 4 for subproblem 6 and Node 9 for subproblem 10).
- Nodes 7 and 11 - Already exploited because they correspond to problems without a feasible solution.

c) All nodes have already been explored and therefore the optimal solution has already been determined, corresponding to node 9 (Z = 24)

d) It would be necessary to solve a smaller number of sub-problems - it would not be necessary to branch node 5 since it would have an objective function value lower than node 9, already explored at that time.

EXERCISE 6

Figure 1 presents a branch-and-bound tree used to solve a minimization problem. For each node there are indicated the values of the decision variables (x and y) and the objective function (Z).

a) Specify which is the interval (upper and lower limits) in which the optimal solution is and what is the maximum error if the method "Branch-and-Bound" was finished in this phase.
b) Which of the unbranched nodes can lead to a better solution for the problem? Justify your answer.
c) Complete the tree with the nodes presented below, which were obtained after continuing solving the problem.

	A	**B**	**C**	**D**	**E**	**F**
Z	36,4	38	SSA*	35,7	39,8	36
X	3,3	5		4	0	3
Y	11	8		11,6	4,3	12

*SSA – no feasible solutions

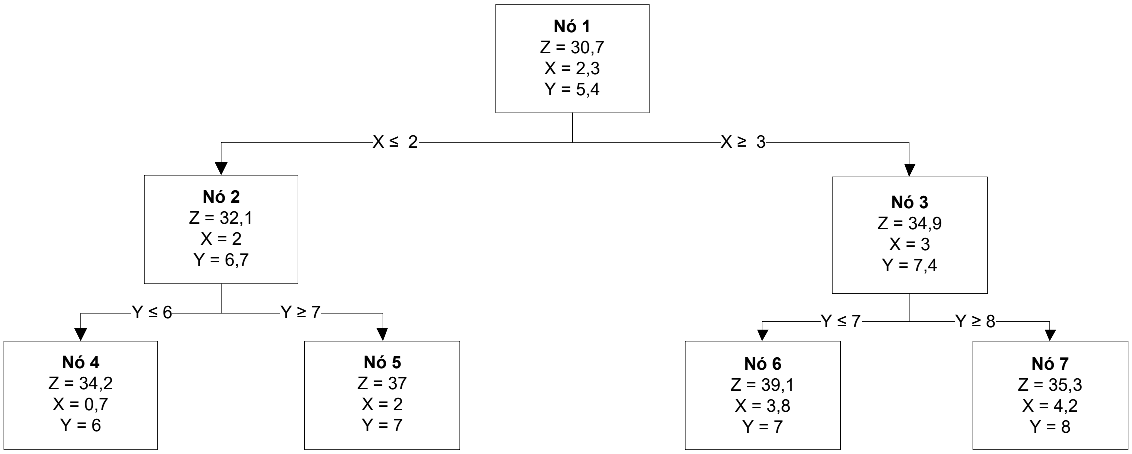
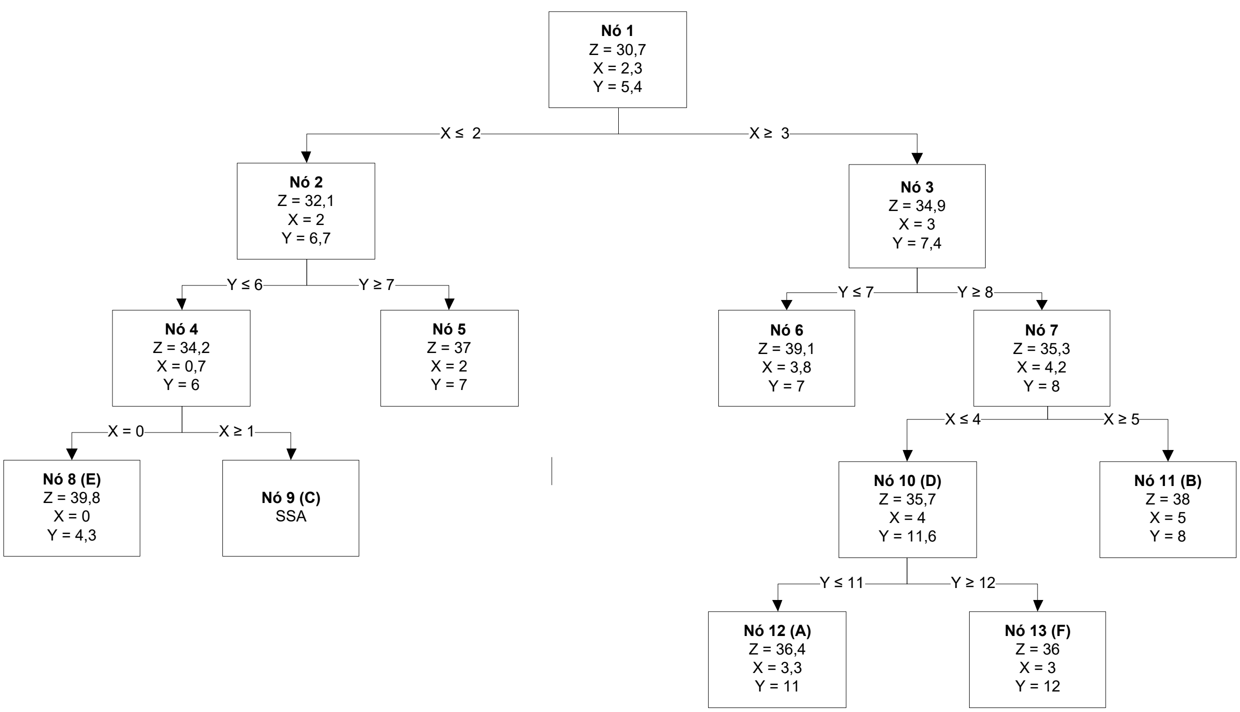


Figure 1

SOLUTION

a) Lower bound – 34,2
Upper bound – 37
Max absolute Error = Upper bound - Lower bound = 2

b) b) The non-branched nodes that can provide a better solution to the problem are nodes 4 and 7. In these nodes the objective function is inferior to the best integer solution so far (Node 5), thus there is potential to obtain a better solution to the problem.



EXERCISE 7

Consider the following integer programming problem, presented in Figure 1:

$$\begin{aligned} max\ Z = \ & -x_1 + 4x_2 \\ & -10x_1 + 20x_2 \le 22 \\ & 5x_1 + 10x_2 \le 49 \\ & x_1 \le 5 \\ & x_1, x_2 \ge 0, integer \end{aligned}$$

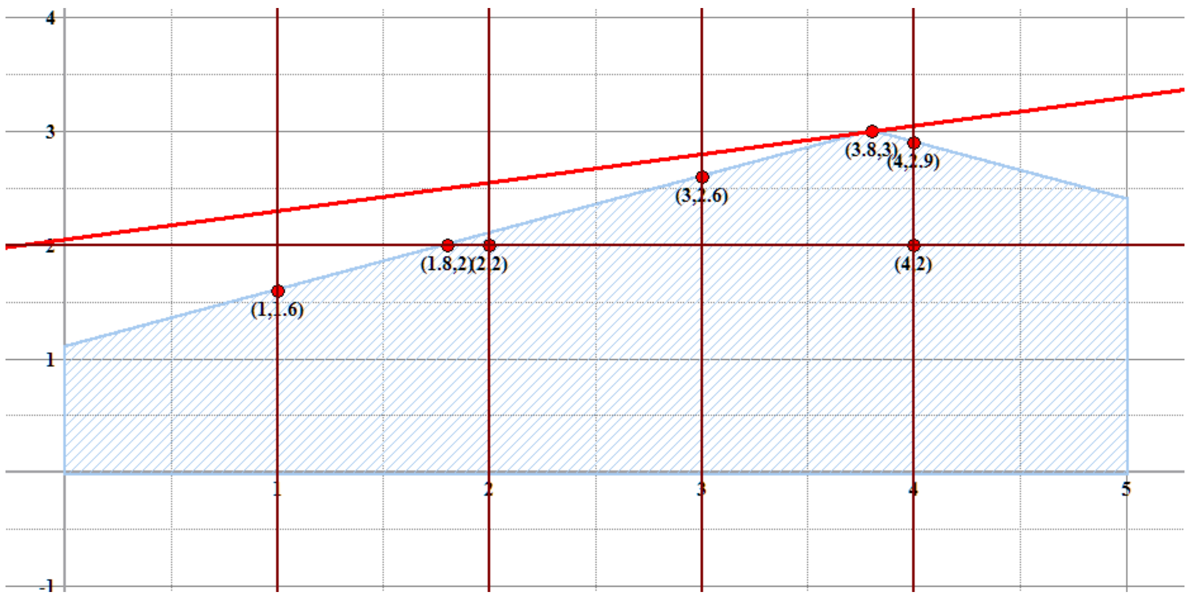


Figure 1

Using the Branch&Bound method the following solutions were obtained (Table 1). Analyzing the graphical resolution presented, indicate the branch& bound tree for this problem. The optimal solution was obtained? Justify your answerosta.

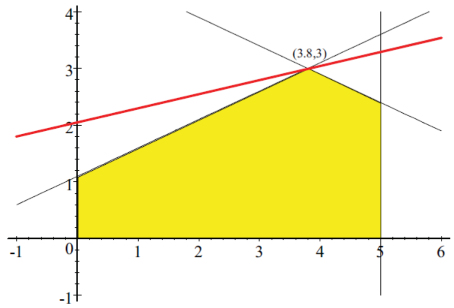
X	Y
1	1.6
1.8	2
2	2
3	2.6
3.8	3
4	2
4	2.9

SOLUTION

$$Max\ Z = -x_1 + 4x_2$$

$$\begin{aligned} -10x_1 + 20x_2 &\leq 22 \\ 5x_1 + 10x_2 &\leq 49 \\ x_1 &\leq 5 \end{aligned}$$

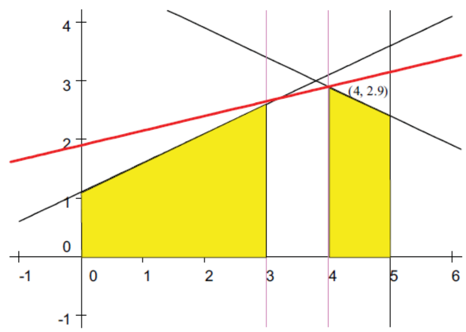
$$x_i \geq 0,\ x_i\text{'s are integers}$$



Solution: x1 = 3.8; x2= 3, z= 8.2

$$\text{Max } Z = -x_1 + 4x_2$$

$$\begin{aligned} \text{s.t.} \quad -10x_1 + 20x_2 &\leq 22 \\ 5x_1 + 10x_2 &\leq 49 \\ x_1 &\leq 5 \\ x_1 &\geq 4 \\ x_2 &\geq 0 \end{aligned}$$



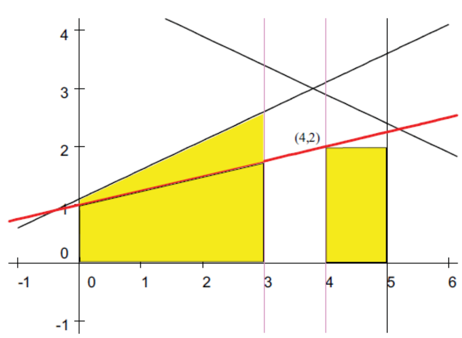
Solution: x1 = 4; x2= 2.9, z= 7.6

$$\text{Max } Z = -x_1 + 4x_2$$

$$\begin{aligned} \text{s.t.} \quad -10x_1 + 20x_2 &\leq 22 \\ 5x_1 + 10x_2 &\leq 49 \\ 4 &\leq x_1 \leq 5 \\ x_2 &\geq 3 \end{aligned}$$

$$\text{Max } Z = -x_1 + 4x_2$$

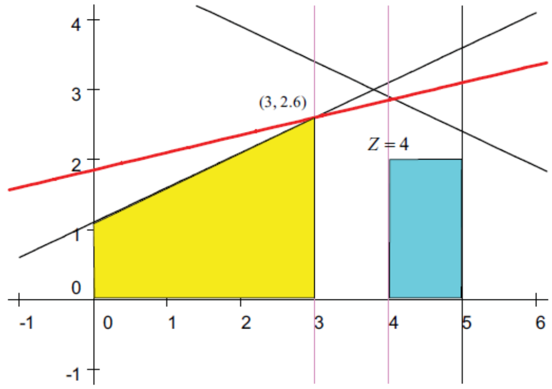
$$\begin{aligned} \text{s.t.} \quad -10x_1 + 20x_2 &\leq 22 \\ 5x_1 + 10x_2 &\leq 49 \\ 4 &\leq x_1 \leq 5 \\ 0 &\leq x_2 \leq 2 \end{aligned}$$



Solution: x1 = 4; x2= 2, z= 4

$$\text{Max } Z = -x_1 + 4x_2$$

$$\begin{array}{rrcl} \text{s.t.} & -10x_1 + 20x_2 & \leq & 22 \\ & 5x_1 + 10x_2 & \leq & 49 \\ & x_1 & \leq & 3 \\ & 0 & \leq & x_i \end{array}$$



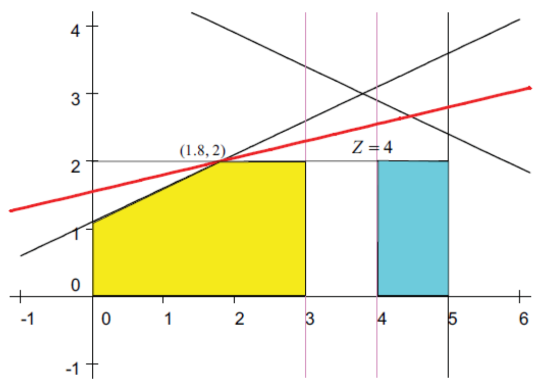
Solution: x1 = 3; x2= 2.6, z= 7.4

$$\text{Max } Z = -x_1 + 4x_2$$

$$\begin{array}{rrcl} \text{s.t.} & -10x_1 + 20x_2 & \leq & 22 \\ & 5x_1 + 10x_2 & \leq & 49 \\ & x_1 & \leq & 3 \\ & x_2 & \geq & 3 \end{array}$$

$$\text{Max } Z = -x_1 + 4x_2$$

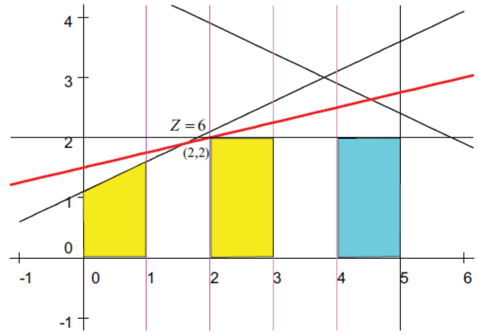
$$\begin{array}{rrcl} \text{s.t.} & -10x_1 + 20x_2 & \leq & 22 \\ & 5x_1 + 10x_2 & \leq & 49 \\ & 0 & \leq & x_1 \leq 3 \\ & 0 & \leq & x_2 \leq 2 \end{array}$$



Solution: x1 = 1.8; x2= 2, z= 6.2

$$\text{Max } Z = -x_1 + 4x_2$$

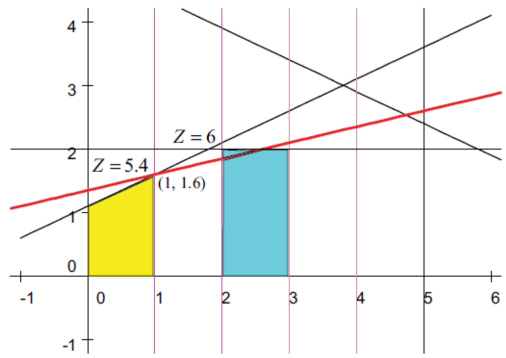
$$\begin{aligned} \text{s.t.} \quad -10x_1 + 20x_2 &\leq 22 \\ 5x_1 + 10x_2 &\leq 49 \\ 2 &\leq x_1 \leq 3 \\ 0 &\leq x_2 \leq 2 \end{aligned}$$



Solution: x1 = 2; x2= 2, z= 6

$$\text{Max } Z = -x_1 + 4x_2$$

$$\begin{aligned} \text{s.t.} \quad -10x_1 + 20x_2 &\leq 22 \\ 5x_1 + 10x_2 &\leq 49 \\ 0 &\leq x_1 \leq 1 \\ 0 &\leq x_2 \leq 2 \end{aligned}$$



Solution: x1 = 1; x2= 1.6, z= 5.4

EXERCISE 8

Consider a maximization Integer Programming problem with three binary variables. (y1, y2, y3) ∊ {0,1}. The following table presents the enumeration of all the candidate solutions obtained through linear relaxation. Symbol "#" in the "Partial" column indicates that there is not still any restriction imposed by the branch-and-bound method in that variable.

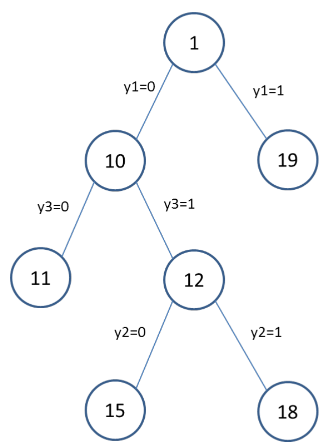
Solve the Branch & Bound method adopting a depth-first search strategy. Draw the Branch & Bound tree and identify the optimal solution.

Obs: The table presents all the possible solutions and not the iterations for the Branch & Bound method.

	Partial	(y1;y2;y3)	z
1	(#,#,#)	(0.2; 1; 0)	82.80
2	(#,#,0)	(0.2; 1; 0)	82.80
3	(#,#,1)	(0; 0.8; 1)	79.40
4	(#,0,#)	(0.7; 0; 0)	81.80
5	(#,0,0)	(0.7; 0; 0)	81.80
6	(#,0,1)	(0.4; 0; 1)	78.60
7	(#,1,#)	(0.2; 1; 0)	82.80
8	(#,1,0)	(0.2; 1; 0)	82.80
9	(#,1,1)	(0; 1; 1)	77.00
10	(0,#,#)	(0;1; 0.67)	80.67
11	(0,#,0)	(0; 1; 0)	28.00
12	(0,#,1)	(0; 0.8; 1)	79.40
13	(0,0,#)	Infeasible	
14	(0,0,0)	Infeasible	

	Partial	(y1,y2,y3)	z
15	(0; 0; 1)	Infeasible	
16	(0; 1; #)	(0; 1; 0.67)	80.67
17	(0; 1; 0)	(0; 1; 0)	28.00
18	(0; 1; 1)	(0; 1; 1)	77.00
19	(1; #; #)	(1; 0; 0)	74.00
20	(1; #; 0)	(1; 0; 0)	74.00
21	(1; #; 1)	(1; 0; 1)	63.00
22	(1; 0; #)	(1; 0; 0)	74.00
23	(1; 0; 0)	(1; 0; 0)	74.00
24	(1; 0; 1)	(1; 0; 1)	63.00
25	(1; 1; #)	(1; 1; 0)	62.00
26	(1; 1; 0)	(1; 1; 0)	62.00
27	(1; 1; 1)	(1; 1; 1)	51.00

SOLUTION



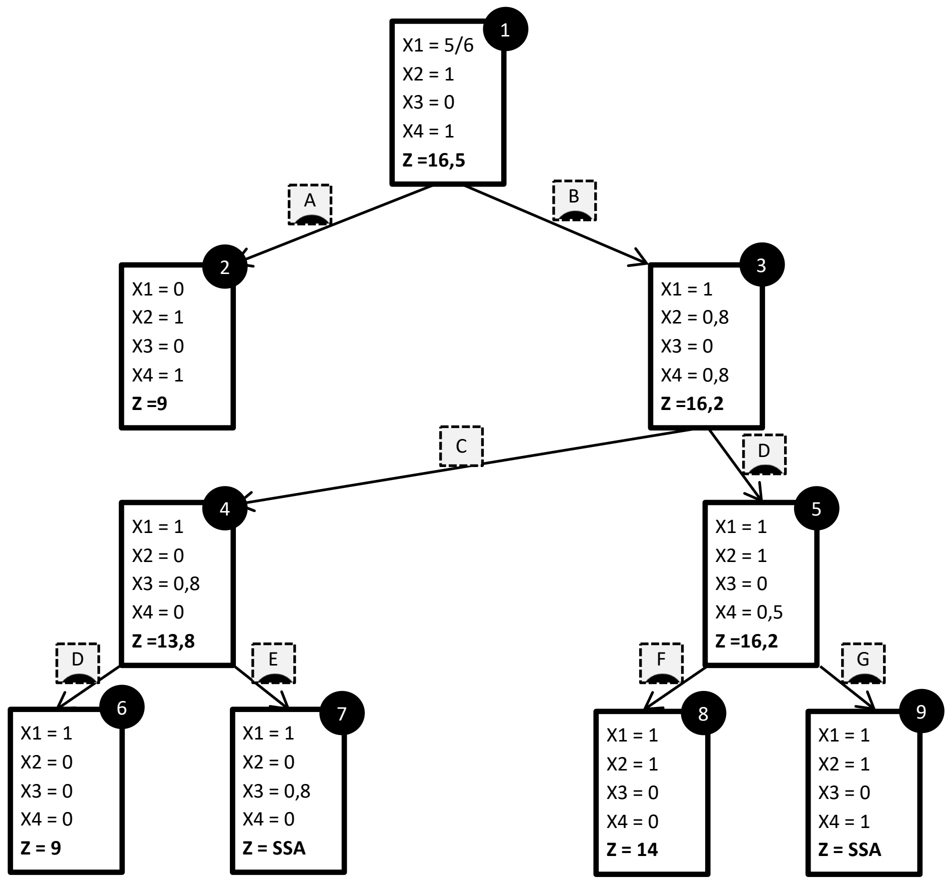
EXERCISE 9

Consider the following Integer Programming Problem:

$$\text{Max } Z = 9X_1 + 5X_2 + 6X_3 + 4X_4$$

$$\begin{aligned} \text{s.a.}\quad & 6X_1+3X_2+5X_3+2X_4 \leq 10 \\ & X_2 + X_4 \leq 10 \\ & X_1+X_3 \leq 0 \\ & -X_2+X_4 \leq 0 \\ & X_i \text{ é binária, } \forall\, i \end{aligned}$$

After running nine iterations on solver software, the following tree was obtained. Each node contains a number that identifies its processing order (the number in the black circle):



a) What was the branching strategy applied? Comment its degree of adequacy to particular problem.
b) Replace each letter on each branch by its corresponding constraint.
c) Enunciate the upper and lower bounds throughout the execution of the algorithm, for each node of the tree.
d) Enunciate the nodes that could be cut, justifying adequately.
e) Has the optimal solution of the problem already been found? Justify your answer.

SOLUTION

a) Breadth-First. In this particular case, we verified that this strategy allowed finding a solution (not optimal) for the problem in the second iteration. However, the optimal solution would only be found in the eighth iteration, so a Depth-First or Promising Node strategy would be more effective.
b) A: $X_1 = 0$, B: $X_1 = 1$, C: $X_2 = 0$, D: $X_2 = 1$, E: $X_3 = 0$, F: $X_3 = 1$, G: $X_4 = 0$, H: $X_4 = 1$

c) Sequence of bounds:

	LI	LS
1	0	16.5
2	9	16.5
3	9	16.2
4	=	=
5	9	16
6	=	=
7	=	=
8	14	16
9	14	14

d) 2 -> SI, 6 -> SI (<= LI known), 7 and 9 -> SSA

e) Yes, because LB = UB.

EXERCISE 10

Represent graphically the following IP problem:

$$Maximize\ z = x_1 + 5x_2,$$

subject to :

$$-4x_1 + 3x_2 \leq 6,$$

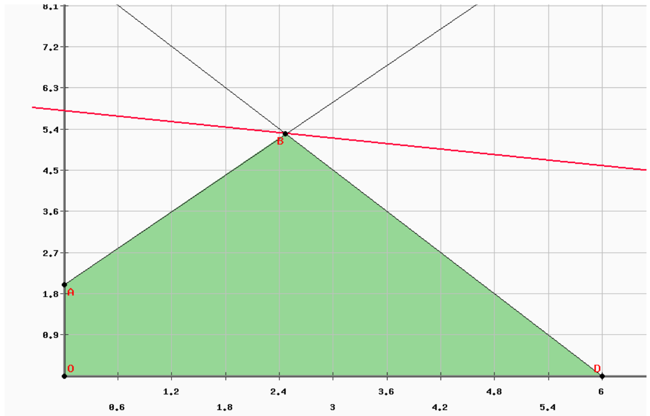
$$3x_1 + 2x_2 \leq 18,$$

x_1, x_2 inteiros.

Apply the branch-and-bound algorithm, solving graphical each linear sub-problem that you find. Draw the branch-and-bound tree.

SOLUÇÃO

Solving the original LP (P1)

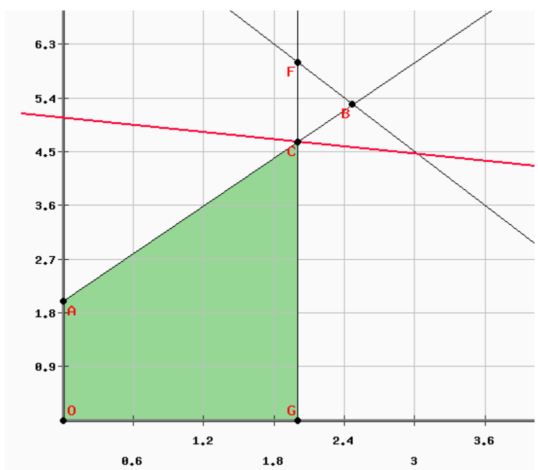


Optimal solution (point B): x1=2,4705882352941; x2=5.2941176470588

Z=28.941176470588

X<=2 or X>=3

P2: adding the constraint X<=2

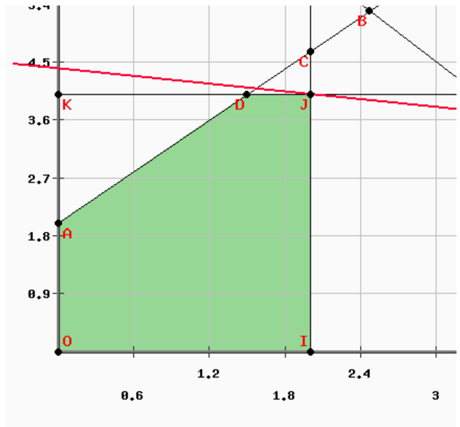


Optimal solution (point c): x1=2; x2=4.6666666666667

Z=25.333333333333

X2 <=4 ou x2 >=5

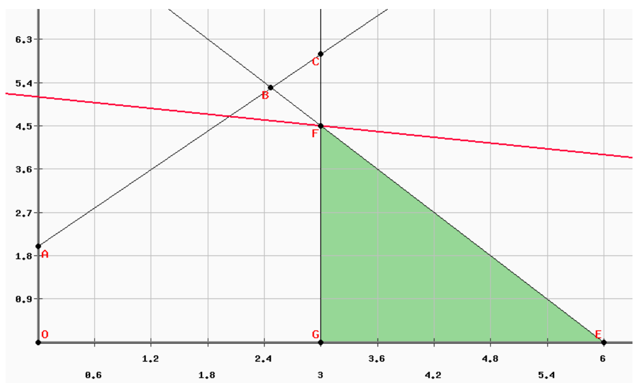
P3: adding the constraint X2 <= 4



Optimal solution of P3 (point J, integer): x1 =2; x2 =4

Problem P4 results from P2 with the constraint X2 >=5 but doesn't have feasible solutions.

Problem P6 results from P1 with the constraint x1 >= 3



Optimal solution of P6 (point F, continuous): x1 =3; x2 =

EXERCISE 11

A team of employees of the ConsultWithUs company developed a mathematical model of integer programming to solve an optimization problem that they found with a customer. To solve the model, the team sees two possible alternatives:

Alternative 1: Solve a linear relaxation of the model, rounding the variables so as to obtain an integer solution;

Alternative 2: Resolve the original problem by the Branch-and-bound algorithm, stopping its execution after a limited period of time.

The team chose the second alternative and obtained the sub-problems' tree presented in figure 1. In this tree, the value "Z" corresponds to the objective Function and the values X1, X2 and X3 correspond to the decision variables.

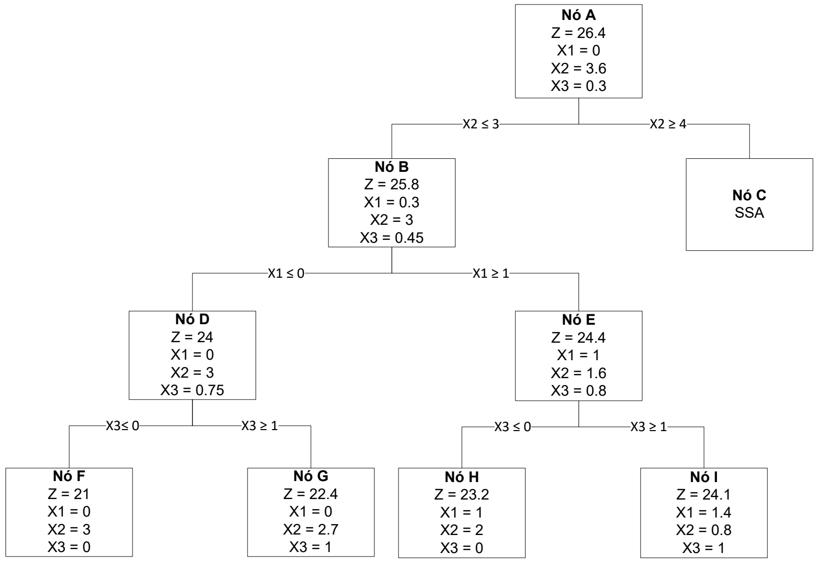


Figure 1 - "Branch-and-Bound" tree (SSA – no feasible solutions)

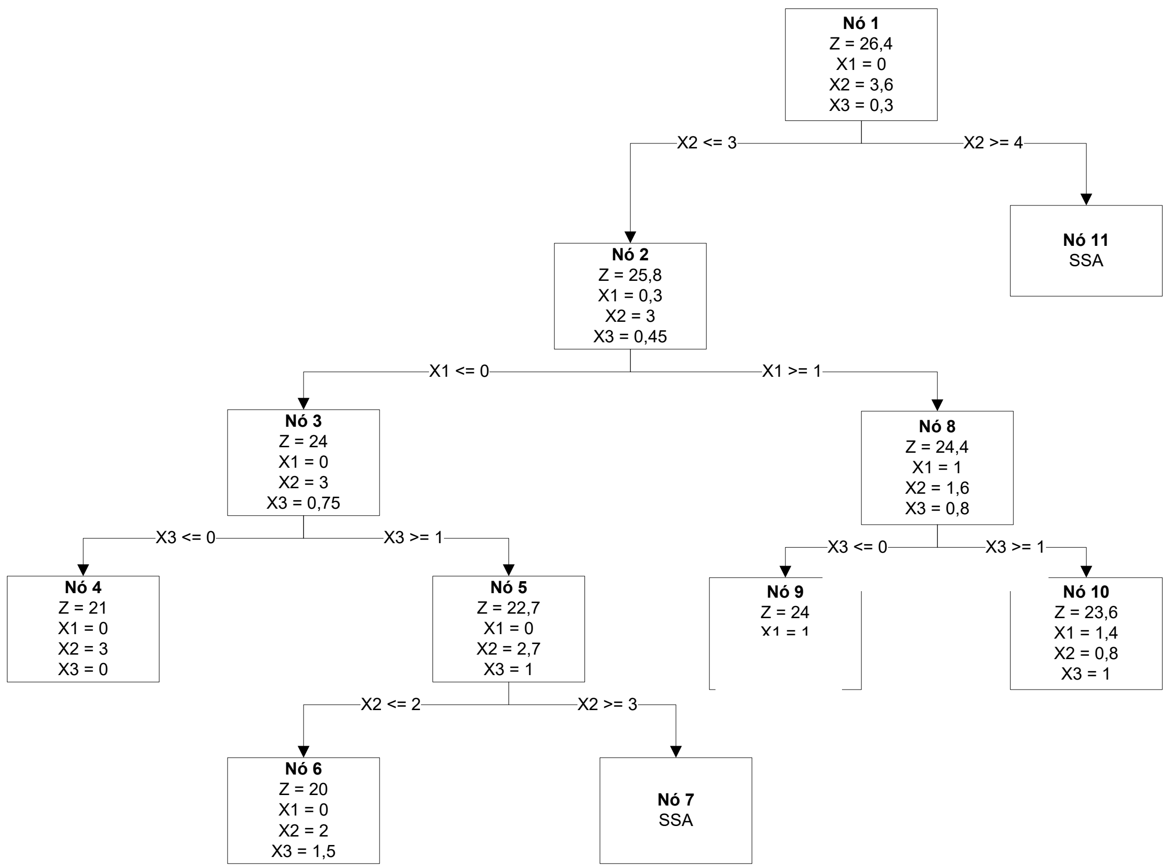
a) Which nodes were already explores and which are yet to explore? Justify your answer.
b) With the deadlines approaching, and with many other problems to solve, every minute count for the team, who does not know if they should continue with the execution of the algorithm, in order to obtain a better solution. What advice would you give the team? Justify by presenting the range [lower limit, upper limit] in which the optimal solution is.
c) Should the team have previously opted for Alternative 1? Present an advantage and a disadvantage of solving the model by Alternative 1 instead of Alternative 2.

SOLUTION

a) Explored – C (no feasible solutions), F (integer solution), G (solution worse than LB) e H (integer solution)
   Unexplored – Node I
b) [23.2; 24.1]
c) Advantage: Speed. Disadvantage: The round solution may not satisfy all constraints.

EXERCISE 12

Consider the following branch-and-bound tree corresponding to the resolution of an entire integer programming problem. The objective function is represented by Z and the decision variables by X1, X2 and X3. SSA means no feasible solution.



a) Indicate what solution you would find for the original problem if the domain of the variables belonged to $\mathbb{R}^+$ instead of belonging to $\mathbb{Z}^+$.

b) Indicate all constraints arising from the execution of the branch-and-bound method that are used in solving the linear programming problem associated with Node 10.

c) Characterize Node 9 (objective function and decision variables) so that the solution of this sub-problem is the optimal solution.

SOLUTION:

a) A solução do Nó 1.

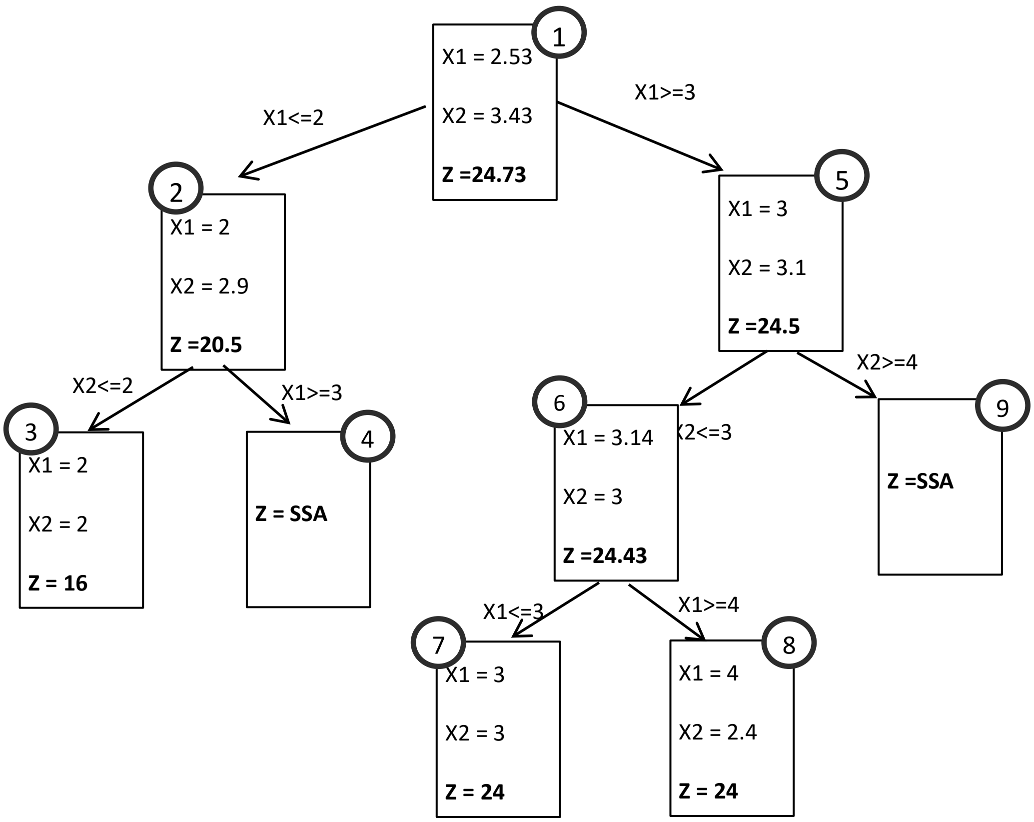
b) $x_2 \leq 3;\ x_1 \geq 1;\ x_3 \geq 1.$

c) Z=24, $x_2 \leq 3;\ x_1 \geq 1;\ x_3 = 0.$

EXERCISE 13

Consider the following integer programming problem:

$$\begin{aligned}
&\text{Max } Z = 3X_1 + 5X_2 \\
&\text{s.a.} \quad -10X_1 + 10X_2 \leq 9 \\
&\qquad 7X_1 + 10X_2 \leq 52 \\
&\qquad X_i \geq 0, \forall i \\
&\qquad X_i \in \mathbb{Z}^+, \forall i
\end{aligned}$$



After a set of iterations in a known software, the following tree was achieved. Each node has a number (identified in the black circle) that indicates the order of the calculations.

a) Which exploration strategy was used?

b) Formulate the correspondent sub-problem of node 8.

c) Indicate the sequence of the limits along the execution of the algorithm, specifying in order, the upper and lower bounds of each tree node.

d) Indicate which nodes can be pruned/cut, justifying it.

e) Was the optimal solution of the problem found? If yes, indicate which is and justify. If not indicate the next step to be made.

SOLUTION

a) In-depth search

b) Max $Z = 3X_1 + 5X_2$

s.a. $-10X_1+10X_2 <= 9$

$7X_1 +10X_2 <= 52$

$X_1 >= 3$

$X_2 <= 3$

$X_1 >=4$

$X_i >=0, \forall i$

X_i integer, $\forall$ i

c) Node 1: UB=24.73 LB=0

Node 2: UB=24.73 LB=0

Node 3: UB=24.73 LB=16

Node 4: UB=24.73 LB=16

Node 5: UB=24.5 LB=16

Node 6: UB=24.5 LB=16

Node 7: UB=24.5 LB=24

Node 8: UB=24.5 LB=24

Node 9: UB=24 LB=24

d) All nodes could already be pruned because they are integer solutions (nodes 3 and 7), solutions not feasible (nodes 4 and 9) or solution with objective function value less than or equal to the current LB.

e) Yes, the optimal solution has already been found since UB = LB. The optimal solution is Z = 24, corresponding to node 7. However, there may be alternative solutions (node 8 would have to be explored to confirm).

EXERCISE 14

A company will launch a promotional action for customers who purchase a mobile phone of their brand next week. The promotional action aims to offer customers two types of experiences: a romantic experience in a 4 star hotel (X1) and an adventure experience in a theme park of your choice (X2). In order to know the number of romantic experiences and the number of adventure experiences that should hire an event supplier, the company built the integer programming model presented below.

$$\text{Max } Z = 5X_1 + 4X_2$$

suj. a

$X_1 + X_2 \leq 5$

$5X_1 + 3X_2 \leq 22.5$

$X_2 \leq 4.5$

$X_1, X_2 \geq 0$ e inteiras

a) Represent graphically the whole programming problem. Do not forget to mark the feasible solutions of the problem, the constraints as well as the iso-line of the objective function.
b) Represent the "Branch and Bound" tree corresponding to the resolution below. Do not forget to indicate in each of the branches the constraint added in each branch.

Sub-problem	1	2	3	4	5	6	7
X_1	3.75	3	4	4.5	-	4	-
X_2	1.25	2	0.83	0	-	0	-
Z	23.75	23	23.33	22.5	SSA	20	SSA

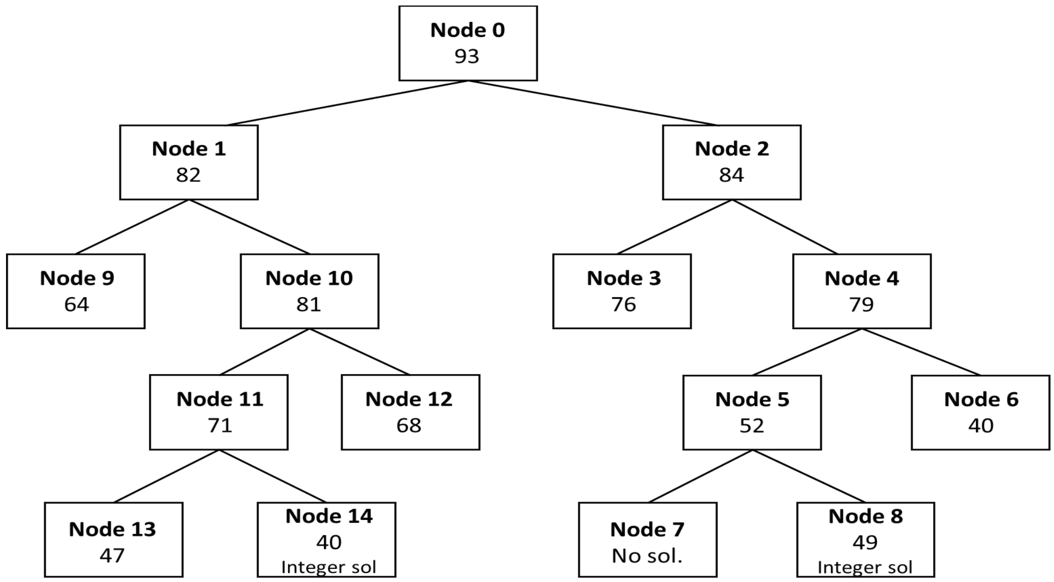
c) Reformulate the "Branch and Bound" tree obtained in (b) to include only the branches necessary to determine the optimum solution.

d) Indicate the optimal solution to the integer programming problem. Comment the solution obtained.

SOLUÇÃO

EXERCISE 15

The company *BestColoring* is responsible for painting one type of cars. The company is now deciding what colors to produce in the following year. The available colors are black, white, red, blue and gray. The decision variables are binary, taking only values of 0 and 1. The figure below shows the branch and bound tree in a given stage of the solution process.



a) Answer the following questions concerning the figure above.
   a1. Indicate if the this is a minimization or a maximization problem. Justify your answer.
   a2. Which are the best upper and lower bounds for the maximum value of z for the IP?
   a3. What are the nodes where it is still possible to branch? Justify.
   a4. Was an optimal solution for the IP found already? If not, what is the quality of the current solution?

b) Consider that the company has additional restrictions in the production of cars. Define the constraints that ensure the following:
   b1. At least two different colors should be produced and at most 4 colors.
   b2. The color "gray" should only be produced if colors "white" and "black" are produced as well.

Note: Consider the following binary decision variables:

$X_j = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$ Produce the color *j*

X_1	Produce back cars
X_2	Produce white cars
X_3	Produce grey cars
X_4	Produce blue cars
X_5	Produce red cars

SOLUTION

a1)Maximization.

a2) LS= 76 and LI=49

a3) Nodes 3, 9, 12.

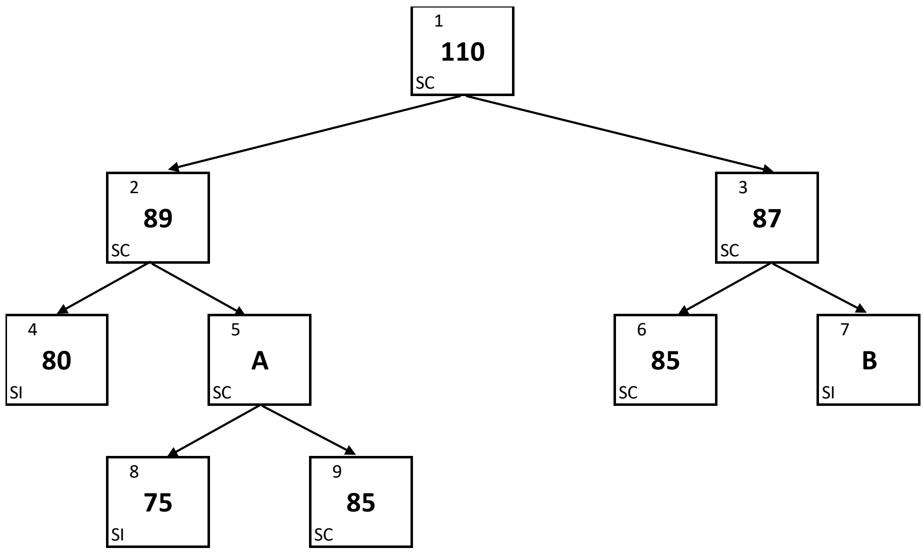
a4) The optimal solution was not found yet. Gap is (76-49)/49.

b1) $\sum_j X_j \geq 2 \;\; e \;\; \sum_j X_j \leq 4$

b2) $2X_3 \leq X_1 + X_2$

EXERCISE 16

Consider the following branch-and-bound tree corresponding to the resolution of an entire scheduling problem. The value in the upper left corner indicates the order in which the nodes were explored. SI represents an integer solution and SC a continuous solution



1) What are the upper and lower bounds when we analyze the tree up to node 4 (inclusive)?
2) Assume that you have covered all nodes presented and that B takes the value 84. Which nodes can be cut / pruned?
3) Assuming that we have already found the optimal solution, among what values can the solution of node 7 be (value of B)?
4) Between what values can the solution of node 5 (value of A) be?

SOLUTION

1) LS=89. LI=80
2) Nós 4, 8, 7
3) [85, 87]
4) [85, 89]