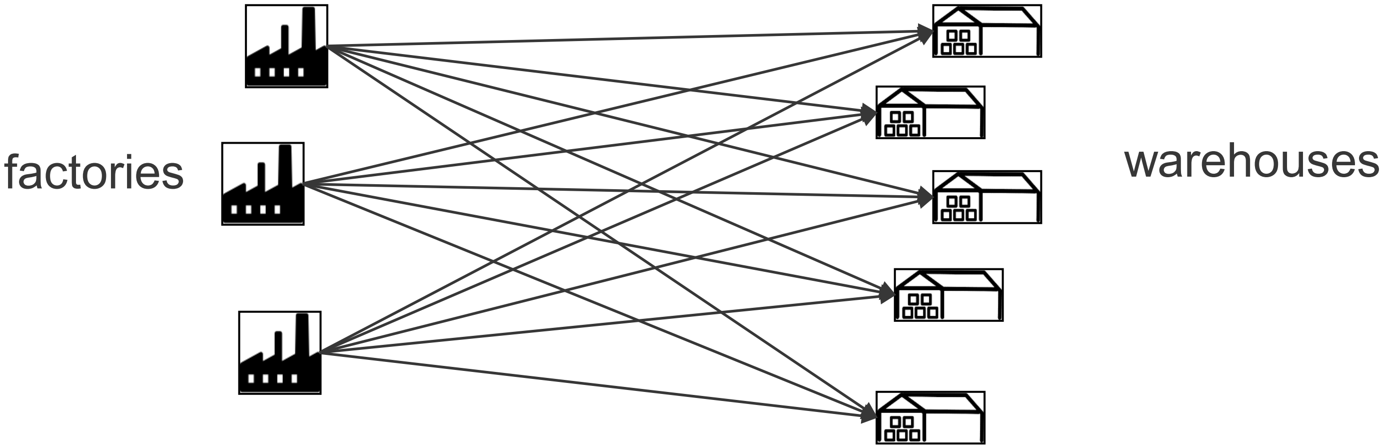
Optimization

Lecture 5.2

M.EIC – 2021.2022

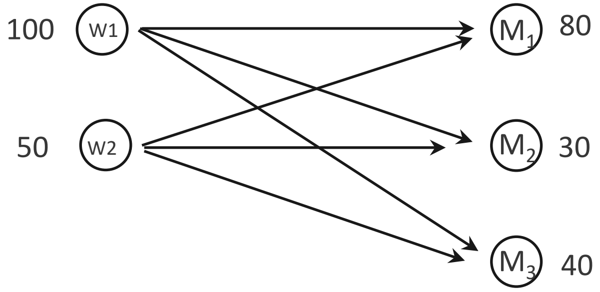


Linear Programming

THE TRANSPORTATION PROBLEM

Transportation Problems

A company has two warehouses W1 and W2 that store 100 and 80 units, respectively, of a given product. From these two warehouses the company supplies three markets $\mathbf{M_1}$, $\mathbf{M_2}$ and $\mathbf{M_3}$ consuming 80, 30 and 40 units of the product, respectively.



Transportation Costs

	M1	**M2**	**M3**
W1	5	3	2
W1	2	2	1

$$
\begin{array}{lllllllll}
\min z = 5x_{11} + 3x_{12} + 2x_{13} + 2x_{21} + 2x_{22} + x_{23} \\
\text{s.a} \\
x_{11} & +x_{12} & +x_{13} & & & & = & 100 \\
 & & & x_{21} & +x_{22} & +x_{23} & = & 50 \\
x_{11} & & & +x_{21} & & & = & 80 \\
 & x_{12} & & & +x_{22} & & = & 30 \\
 & & x_{13} & & & +x_{23} & = & 40 \\
x_{11}, & x_{12}, & x_{13}, & x_{21}, & x_{22}, & x_{23} & \geq & 0
\end{array}
$$

Decision variables

$\mathbf{x_{ij}}$: amount of product to send from origin i to destination j

LP Formulation

$$\min z = \sum_i \sum_j c_{ij}\, x_{ij}$$

$$\text{s.a} \quad \sum_j x_{ij} = a_i \qquad (i = 1,\dots,m) \qquad \text{supply constraints}$$

$$\sum_i x_{ij} = b_j \qquad (j = 1,\dots,n) \qquad \text{demand constraints}$$

$$x_{ij} \geq 0, \qquad (i = 1,\dots,m;\ j = 1,\dots,n)$$

The particular structure of the coefficient matrix is characterized by the following:

- it only has 1's and 0's
- each xij variable appears only in two constraints (one is a supply constraint, and the other is a demand constraint)
- $\sum_i a_i = \sum_j b_j \Rightarrow \sum_i \sum_j x_{ij} = \sum_j \sum_i x_{ij}$: one of the constraints is redundant because it is the linear combination of the others
- m x n variables
- m+n-1 basic variables (= number of independent constraints)

-(m x n) - (m+n-1) non-basic variables

$$\min z = 5x_{11} + 3x_{12} + 2x_{13} + 2x_{21} + 2x_{22} + x_{23}$$

s.a

$$\left.\begin{array}{lllllllr}
x_{11} & +x_{12} & +x_{13} & & & & = & 100 \\
 & & & x_{21} & +x_{22} & +x_{23} & = & 50
\end{array}\right\} \text{Supply constraints}$$

$$\left.\begin{array}{lllllllr}
x_{11} & & & +x_{21} & & & = & 80 \\
 & x_{12} & & & +x_{22} & & = & 30 \\
 & & x_{13} & & & +x_{23} & = & 40 \\
x_{11}, & x_{12}, & x_{13}, & x_{21}, & x_{22}, & x_{23} & \geq & 0
\end{array}\right\} \text{Demand constraints}$$

Formulation

$$\boxed{\min\ \ c_{11}x_{11} + \dots + c_{1n}x_{1n} + \dots + c_{m1}x_{m1} + \dots + c_{mn}x_{mn}}$$

$$\left.\begin{array}{ll} x_{11} + x_{12} + \dots + x_{1n} & = a_1 \\ x_{21} + x_{22} + \dots + x_{2n} & = a_2 \\ \dots & \dots \\ x_{m1} + x_{m2} + \dots + x_{mn} & = a_m \end{array}\right\} \text{Supply constraints}$$

$$\left.\begin{array}{ll} x_{11} + x_{21} + \dots + x_{m1} & = b_1 \\ x_{12} + x_{22} + \dots + x_{m2} & = b_2 \\ \dots & \dots \\ x_{1n} + x_{2n} + \dots + x_{mn} & = b_n \end{array}\right\} \text{Demand constraints}$$

Example LP formulation in standard form

$$\min z = 5x_{11} + 3x_{12} + 2x_{13} + 2x_{21} + 2x_{22} + x_{23}$$

s.a

$$\begin{array}{l|rrrrrrcr}
u'_1 & x_{11} & +x_{12} & +x_{13} & & & & \geq & 100 \\
u''_1 & -x_{11} & -x_{12} & -x_{13} & & & & \geq & -100 \\
u'_2 & & & & x_{21} & +x_{22} & +x_{23} & \geq & 50 \\
u''_2 & & & & -x_{21} & -x_{22} & -x_{23} & \geq & -50 \\
v'_1 & x_{11} & & & x_{21} & & & \geq & 80 \\
v''_1 & -x_{11} & & & -x_{21} & & & \geq & -80 \\
v'_2 & & x_{12} & & & +x_{22} & & \geq & 30 \\
v''_2 & & -x_{12} & & & -x_{22} & & \geq & -30 \\
v'_3 & & & x_{13} & & & +x_{23} & \geq & 40 \\
v'_3 & & & -x_{13} & & & -x_{23} & \geq & -40 \\
 & x_{11}, & x_{12}, & x_{13}, & x_{21}, & x_{22}, & x_{23} & \geq & 0
\end{array}$$

Dual Formulation in standard form

$$
\begin{aligned}
\max g = {} & 100u_1^{'} - 100u_1^{''} + 50u_2^{'} - 50u_2^{''} + \\
& + 80v_1^{'} - 80v_1^{''} + 30v_2^{'} - 30v_2^{''} + 40v_3^{'} - 40v_3^{''}
\end{aligned}
$$

s.a.

$$u_1^{'} - u_1^{''} + v_1^{'} - v_1^{''} \le 5$$

$$u_1^{'} - u_1^{''} + v_2^{'} - v_2^{''} \le 3$$

$$u_1^{'} - u_1^{''} + v_3^{'} - v_3^{''} \le 2$$

$$u_2^{'} - u_2^{''} + v_1^{'} - v_1^{''} \le 2$$

$$u_2^{'} - u_2^{''} + v_2^{'} - v_2^{''} \le 2$$

$$u_2^{'} - u_2^{''} + v_3^{'} - v_3^{''} \le 1$$

$$u_1^{'}, u_1^{''}, u_2^{'}, u_2^{''}, v_1^{'}, v_1^{''}, v_2^{'}, v_2^{''}, v_3^{'}, v_3^{''} \ge 0$$

Dual Formulation in standard form (simplified)

Let

$$u_i = u_i^{'} - u_i^{''}, i = 1,\ldots,2$$

$$v_j = v_j^{'} - v_j^{''}, j = 1,\ldots,3$$

$$\max g = 100u_1 + 50u_2 +$$
$$+ 80v_1 + 30v_2 + 40v_3$$

s.a.

$$u_1 + v_1 \leq 5$$
$$u_1 + v_2 \leq 3$$
$$u_1 + v_3 \leq 2$$
$$u_2 + v_1 \leq 2$$
$$u_2 + v_2 \leq 2$$
$$u_2 + v_3 \leq 1$$
$$u_1, u_2, v_1, v_2, v_3 \in \Re$$

Generalization

$$\max\ a_1(U'_1 - U''_1) + \ldots + a_m(U'_m - U''_m) + b_1(V'_1 - V''_1) + \ldots + b_n(V'_n - V''_n)$$

$$(U'_1 - U''_1) + (V'_1 - V''_1) \leq c_{11}$$

$$(U'_1 - U''_1) + (V'_2 - V''_2) \leq c_{12}$$

$$(U'_2 - U''_2) + (V'_1 - V''_1) \leq c_{21}$$

$$(U'_m - U''_m) + (V'_n - V''_n) \leq c_{mn}$$

$$U'_i,\ U''_i,\ V'_j,\ V''_j \geq 0$$

Dual of a Transportation Problem

$$\max\ a_1U_1 + \ldots + a_mU_m + b_1V_1 + \ldots + b_nV_n$$

$$\text{s.a} \quad U_i + V_j \leq c_{ij}$$

$$U_i,\ V_j \in \mathbb{R}$$

$$U_i = U_i^{'} - U_i^{''}, i = 1,\ldots,m$$

$$V_j = V_j^{'} - V_j^{''}, j = 1,\ldots,n$$

Primal formulation

$$\min z = \sum_i \sum_j c_{ij}\, x_{ij}$$

$$\text{s.a } \sum_j x_{ij} = a_i \qquad (i = 1,\ldots,m)$$

$$\sum_i x_{ij} = b_j \qquad (j = 1,\ldots,n)$$

$$x_{ij} \geq 0, \qquad (i = 1,\ldots,m;\ j = 1,\ldots,n)$$

Dual formulation

$$\max\ a_1U_1 + \ldots + a_mU_m + b_1V_1 + \ldots + b_nV_n$$

$$\text{s.a} \quad U_i + V_j \leq c_{ij}$$

$$U_i\ ,\ V_j \in \mathbb{R}$$

$$i=1,\ldots m,\ j=1..n$$

Dual in canonic form:

$$\max\ a_1U_1 + \ldots + a_mU_m + b_1V_1 + \ldots + b_nV_n$$

$$\text{s.a} \quad \mathbf{U_i + V_j + S_{ij} = c_{ij}}$$

$$U_i\ ,\ V_j \in \mathbb{R}$$

$$i=1,\ldots m,\ j=1..n$$