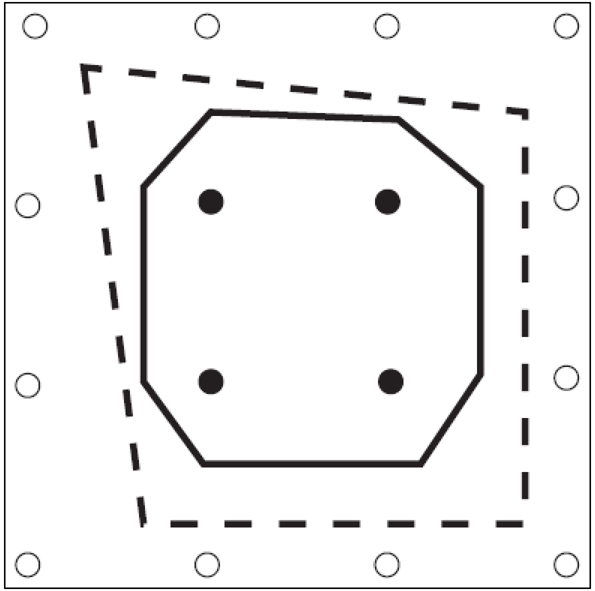
Optimization

Lecture 7.1

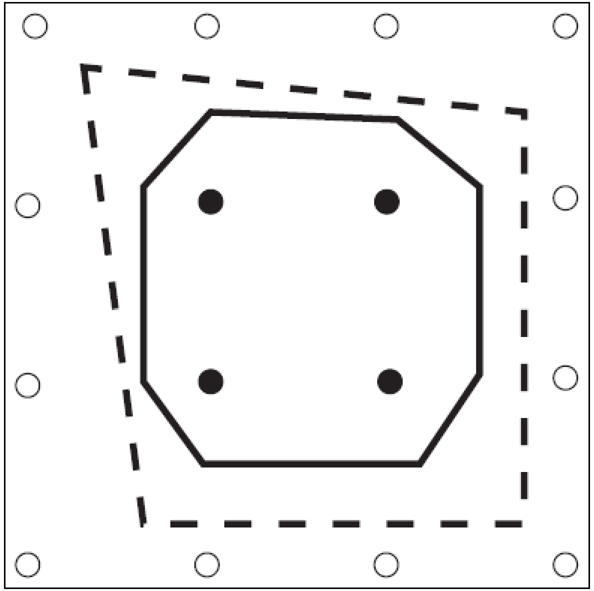
M.EIC – 2021.2022



Linear Programming

INTEGER AND MIXED INTEGER LINEAR PROBLEMS

Integer Programming (IP)



(IP)

$$\max\ f = c^{\mathrm{T}}x$$
$$s.a.$$
$$Ax \leq \mathrm{b}$$
$$\mathrm{x} \geq 0$$
$$x \in Z^n$$

Usually, integer programming problems are very complex, and it is difficult to obtain the optimal solution in useful time.

So, it is very important to obtain lower and upper bounds for the optimal solution through **Linear Relaxation** and **Branch&Bound** method.

History of Integer Programming (IP)

- Introduced in 1951 (Dantzig)
- TSP as special case in 1954 (Dantzig)
- First convergent algorithm in 1958 (Gomory)
- General branch-and-bound technique 1960 (Land and Doig)
- Frequently used to prove bounds on approximation algorithms (late 90s)

Decision Variables in IP Models

- An IP is a mixed integer program (MIP) when some but not all the decision variables are integer.
- If all decision variables are integer, we have a pure IP.
- A binary decision variable must be 0 or 1 (a yes-no decision variable).
- If all decision variables are binary, then the IP is a binary IP (BIP).
- Decision variables that are not required to be integer-valued are continuous variables.

	Cont vars	Int vars	Cont + Int vars
Linear constr	**LP**	**ILP**	**MILP**
Linear + nonlinear constr	**NLP**	**INLP**	**MINLP**

Example of IP: Facility Location

A company is thinking about building new facilities in PO and LX.

	capital needed	expected profit
1. factory in PO	€6M	€9M
2. factory in LX	€3M	€5M
3. warehouse in PO	€5M	€6M
4. warehouse in LX	€2M	€4M

Total capital available for investment: €10M

Question: Which facilities should be built to maximize the total profit?

Example of IP: Facility Location

Define decision variables (i = 1, 2, 3, 4):

$$x_i = \begin{cases} 1 & \text{if facility i is built} \\ 0 & \text{if not} \end{cases}$$

Then the total expected benefit: $9x_1+5x_2+6x_3+4x_4$

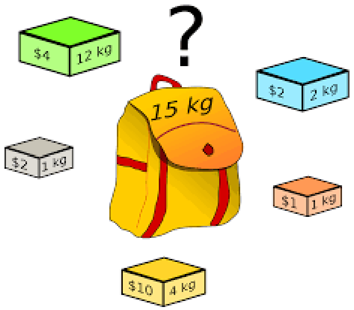
the total capital needed: $6x_1+3x_2+5x_3+2x_4$

Summarizing, the IP model is:

$$\max\ 9x_1+5x_2+6x_3+4x_4$$
$$\text{s.t. } 6x_1+3x_2+5x_3+2x_4 \leq 10$$
$$x_i \in \{0,1\}$$

The Knapsack Problem

- Knapsack with volume 15
- What should you take with you to maximize utility?



Items	1	2	3	4	5
	Paper	Book	Bread	Smartphone	Water
Utility	8	12	7	15	12
Volume	4	8	5	2	6

The Knapsack Problem

- The *knapsack problem* is inspired in the situation where a hiker must decide which objects to include inside the knapsack for his trip.

- c_j is the "value" or utility of including object j that weighs $a_j > 0$ kilos;
- n items to be packed in a knapsack.
- The knapsack can hold up to b kg of items.
- Goal: Pack the knapsack such that the total benefit is maximized.

- Any IP, which has only one constraint, is referred to as a knapsack problem .
- The knapsack problem is NP hard.

IP model for Knapsack problem

- Define decision variables (j = 1, …, n): $x_j = \begin{cases} 1 & \text{if item j is packed} \\ 0 & \text{if not} \end{cases}$
- Total benefit: $\sum_{j=1}^{n} c_i x_j$
- Total weight: $\sum_{j=1}^{n} a_j x_j$
- IP model:

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ s.t. \quad & \sum_{j=1}^{n} a_j x_j \le b \\ & x_j \text{ binary } (j = 1, \ldots, n) \end{aligned}$$

Connection between problems

This version of the facility location problem is a special case of the knapsack problem.

Important modeling skill:

- Suppose you know how to model Problem $A_1,\ldots,A_p$;
- You need to solve Problem B;
- Notice the similarities between Problems A_i and B;
- Build a model for Problem B, using the model for Problem A_i as a prototype.

The Facility Location Problem: Restrictions on the number of options

x_1: factory PO;	*x_2: factory LX*
x_3 : warehouse PO;	*x_4: warehouse LX*

Extra requirement: build *at most one* of the two warehouses.

$$x_3 + x_4 \leq 1$$

Extra requirement: build *at least one* of the two factories.

$$x_1 + x_2 \geq 1$$

Modeling Technique: Restrictions on the number of options

Suppose in a certain problem, n different options are considered. For i=1,…,n

$$x_i = \begin{cases} 1 & \text{if option i is chosen} \\ 0 & \text{if not} \end{cases}$$

Restrictions: At least *p* and at most *q* of the options can be chosen.

The corresponding constraints are:

$$\sum_{i=1}^{n} x_i \geq p \qquad\qquad \sum_{i=1}^{n} x_i \leq q$$

Modeling Technique: Contingent Decisions

x_1: factory PO; x_2: factory LX
x_3 : warehouse PO; x_4: warehouse LX

Back to the facility location problem:

Requirement: Can't build a warehouse *unless* there is a factory in the same city.

$x_3 \le x_1$ (PO) $\qquad$ $x_4 \le x_2$ (LX)

Requirement: Can't select option 3 (x_3) *unless* at least one of options 1 and 2 (x_1 or x_2) is selected.

$$x_3 \le x_1 + x_2$$

Requirement: Can't select option 4 (x_4) *unless* at least two of options 1, 2 and 3 are selected ($x_1 + x_2 + x_3$)

$$2x_4 \le x_1 + x_2 + x_3$$

Modeling Technique: Variables with k possible values

Suppose that variable y can only take one of the values d_1, d_2, …, d_k .

How to achieve that in the model?

Introduce new decision variables. For i=1,…,k,

$$x_i = \begin{cases} 1 & \text{if y takes value } d_i \\ 0 & \text{otherwise} \end{cases}$$

Then we need the following constraints:

$$\sum_{i=1}^{k} x_i = 1 \quad \text{(y can take only one value)}$$

$$y = \sum_{i=1}^{k} d_i x_i \quad (\text{y should take value } d_i \text{ if } x_i = 1)$$

Logical constraints: Alternative Constraints

Consider a situation with the alternative constraints:

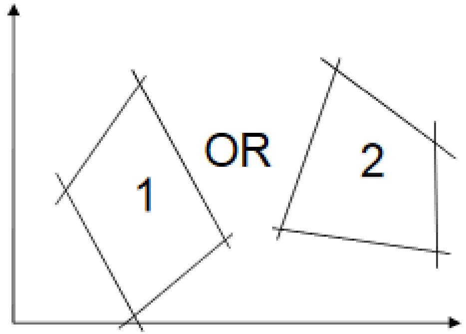
$$f_1(x_1, x_2, \ldots, x_n) \leq b_1,$$
$$f_2(x_1, x_2, \ldots, x_n) \leq b_2.$$

At least one, but not necessarily both, of these constraints must be satisfied.

This restriction can be modeled by combining the technique just introduced with a multiple-choice constraint as follows:

$$f_1(x_1, x_2, \ldots, x_n) \leq b_1 + M\, y_1$$
$$f_2(x_1, x_2, \ldots, x_n) \leq b_2 + M\, y_2$$
$$y_1 + y_2 \leq 1,$$
$$y_1, y_2 \text{ binary}$$

and M is chosen to be large enough



Logical constraints: K out of N constraints must hold

N constraints

$$f_1(x_1,\dots,x_n) \le b_1$$
$$\vdots$$
$$f_N(x_1,\dots,x_n) \le b_N$$

At least K out of N must be satisfied.

$$f_1(x_1,\dots,x_n) \le b_1 + M(1 - y_1)$$
$$\vdots$$
$$f_N(x_1,\dots,x_n) \le b_N + M(1 - y_N)$$

$$\sum_{i=1}^{N} y_i = K$$

$$y_i \in \{0, 1\}, \quad i = 1,\dots,N$$

M is chosen to be large enough

Example: A production system has N potential quality control inspection strategies. The management has decided that K of these strategies should be adopted.

Logical constraints: Conditional Constraints

Nota:
$a \Rightarrow b \Leftrightarrow \sim a \lor b$

These constraints have the form:

$$f_1(x_1, x_2, \ldots, x_n) > b_1 \text{ \underline{implies} that } f_2(x_1, x_2, \ldots, x_n) \leq b_2.$$

The conditional constraint is logically equivalent to the alternative constraints:

$$f_1(x_1, x_2, \ldots, x_n) \leq b_1 \text{ or } f_2(x_1, x_2, \ldots, x_n) \leq b_2,$$

where at least one must be satisfied.

Hence, this situation can be modeled by alternative constraints as indicated before.

Modelling minimum constraints

Consider that our integer variable X must satisfy $X = \min(A,B)$.
Then, we may write the following equivalent constraints:

1. $X \leq A$
2. $X \leq B$
3. $X = A$ OR $X = B$

Then we have:

1. $X \leq A$
2. $X \leq B$
3. Create two new decision binary variables Y1 and Y2, M is chosen to be large enough
   - 3.1 $X-A-M*Y1 = 0$
   - 3.2 $X-B-M*Y2 = 0$
   - 3.3 $Y1+Y2 = 1$

Uncapacitated Facility Location Problem

Problem: Open a set of facilities and assign each customer to one facility such that cost is minimized.

The cost could be a function of distance from a facility to a customer or could be based on a response time (e.g., locating fire stations).

Data:

f_i : cost of opening a facility at location i

c_{ij} : cost of assigning customer j to facility i

Decision variables:

y_i : open facility at location i (1 = yes, 0 = no)

x_{ij} : assign customer j to location i (1 = yes, 0 = no)

Uncapacitated Facility Location Problem: IP formulation

$$\text{Min} \quad \sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} + \sum_{i\in I} f_i y_i$$

$$\text{s.t.} \quad \sum_{i\in I} x_{ij} = 1, \qquad \forall\, j \in J$$

Each customer is assigned to exactly one facility

$$x_{ij} \le y_i, \qquad \forall\, i \in I,\ j \in J$$

Setup constraint: Can assign customer *j* to facility *i* only if we open facility *i*

$$x_{ij} \in \{0,1\}\ y_i \in \{0,1\}, \quad \forall\, i \in I,\ j \in J$$

Capacitated Facility Location Problem

Problem: A company has m potential warehouse sites and n customers. A manager must decide which warehouses to use to meet the demand of the customers.

Data:

f_i = fixed operating cost for warehouse **i** , if opened (for example, a cost to lease the warehouse),

c_{ij} = per-unit operating cost at warehouse i plus the transportation cost for shipping from warehouse **i** to customer **j**.

d_j : demand for customer j

s_i : capacity (supply) of warehouse i

Decision variables:

y_i : build a warehouse at site i (1 = yes, 0 = no)

x_{ij} : shipments from warehouse i to customer j

Facility Location IP Model

$$\text{Min} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{i=1}^{m} f_i y_i$$

$$\text{s.t.} \quad \sum_{i=1}^{m} x_{ij} = d_j \quad j = 1,\ldots,n$$

Satisfy each customer's demand

$$\sum_{j=1}^{n} x_{ij} \le s_i y_i \quad i = 1,\ldots,m$$

each warehouse can ship no more than its supply if it is built

$$x_{ij} \ge 0, \quad i = 1,\ldots,m, \; j = 1,\ldots,n$$

$$y_i \in \{0,1\}, \quad i = 1,\ldots,m$$

Set Covering Problem

A hospital ER needs to keep doctors on call, so that a qualified individual is available to perform every medical procedure that might be required (there is an official list of such procedures).
For each of several doctors available for on-call duty, the additional salary they need to be paid, and which procedures they can perform, is known. The goal to choose doctors so that each procedure is covered, at a minimum cost.

Example:

	Doctor 1	Doctor 2	Doctor 3	Doctor 4	Doctor 5	Doctor 6
Procedure 1	X			X		
Procedure 2	X				X	
Procedure 3		X	X			
Procedure 4	X					X
Procedure 5		X	X			X
Procedure 6		X				

Set Covering Problem

<u>Data representation</u>: incidence matrix.
With m procedures and n available doctors, the data can be represented as

A (mxn), where $a_{ij} = 1$ *if doctor j can perform procedure i and*
$a_{ij} = 0$ *otherwise.*

Also, let c_j , $j = 1, .., n$ be the additional salary that will need to be paid to doctor j for on-call duty.

Variables: $x_j = 1$ if doctor j is on call, and 0 otherwise.

Set Covering and Partitioning

Given n sets and m items:

n columns = sets
m Rows = items

$$A_{ij} = \begin{cases} 1, & \text{if set j includes item i} \\ 0, & \text{otherwise} \end{cases}$$

$$c_j = \text{cost of set j}$$

$$x_j = \begin{cases} 1, & \text{if set j is included} \\ 0, & \text{otherwise} \end{cases}$$

Set covering:

minimize: $c^T x$

subject to: $Ax \geq 1$, x binary

Set partitioning:

minimize: $c^T x$

subject to: $Ax = 1$, x binary