Optimization

Lecture 3.1

Introduction to Linear Programming

GRAPHICAL SOLUTION

Cereals, Ltd

Cereals, Ltd is a company specialized in preparing and packing wheat and corn for several retailing stores. There are no limits concerning the supply of these cereals and demand can be considered unlimited (in other words, the whole production is sold).
The production is processed in three phases: Pre-Processing (I), Processing (II) and Packing (III).

	I	II	III	
	Pre-Processing	Processing	Packing	
Weekly production capacity (h)	120	100	150	
Time (h) needed to prepare 1 ton				Profit(€/ton)
Wheat	6 h	1 h	5 h	4
Corn	2 h	4 h	5 h	3

Currently, the company produces 18 tons of wheat and 6 tons of corn per week.
Is this the best option?

Cereals, Ltd - Formulation

Decision variables

x = tons of wheat to produce weekly

y = tons of corn to produce weekly

Constraints

$$6x + 2y \le 120$$
$$x + 4y \le 100$$
$$5x + 5y \le 150$$
$$x \ge 0, y \ge 0$$

Objective function: to maximize the profit $\max 4x + 3y$

Cereals, Ltd | Graphical method

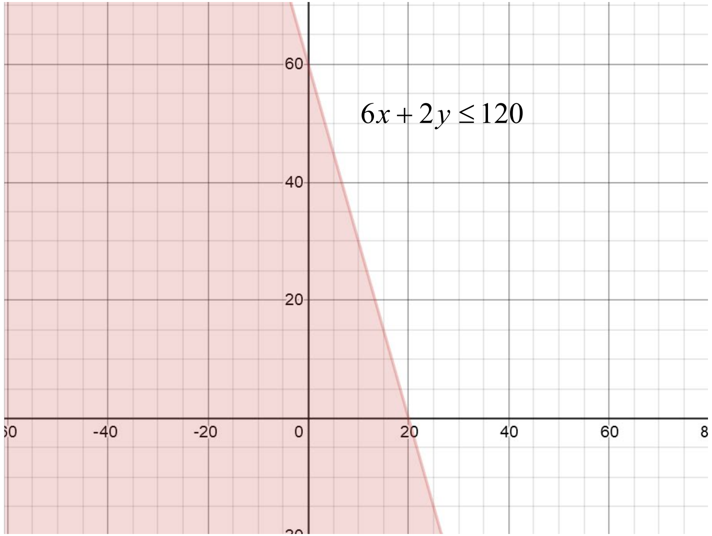
$\max 4x + 3y$

s.to $6x + 2y \leq 120$

$x + 4y \leq 100$

$5x + 5y \leq 150$

$x \geq 0, y \geq 0$



The straight line $6x + 2y = 120$ divides the plane in two half-planes

Which one of them satisfies the inequality $6x + 2y \leq 120$?

Consider, for example, the point (x,y)=(0,0);

Replacing it in the inequality we have $6 \times 0 + 2 \times 0 = 0 \leq 120$

The inequality $6x + 2y \leq 120$ is satisfied by all points in the shaded zone.

Cereals, Ltd | Graphical method

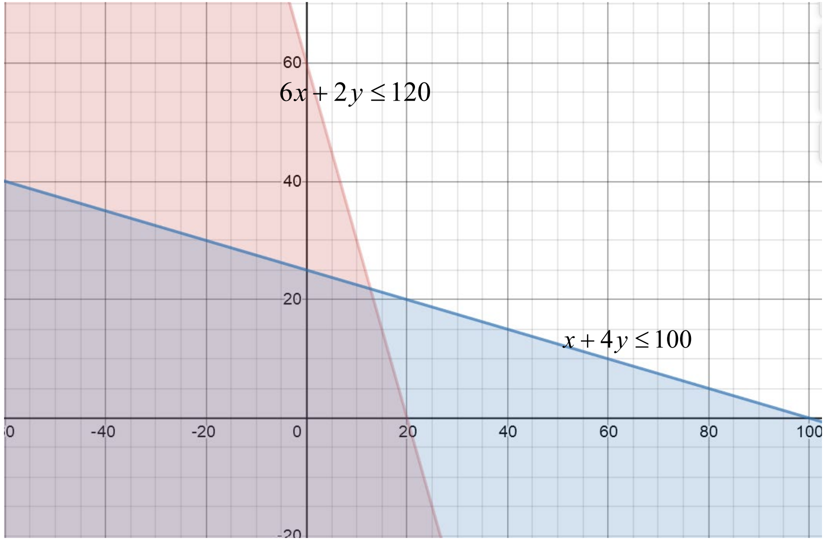
$\max 4x + 3y$

s.to $\quad 6x + 2y \leq 120$

$x + 4y \leq 100$

$5x + 5y \leq 150$

$x \geq 0, y \geq 0$



For the constraint $\quad x + 4y \leq 100$

Consider the straight line $\quad x + 4y = 100$

$x = 0 \Rightarrow y = 25$ $\quad$ Points (0,25) and (100,0) belong to the line

$y = 0 \Rightarrow x = 100$ $\quad$ Point (0,0) satisfies the inequality $x + 4y \leq 100$

Cereals, Ltd | Graphical method

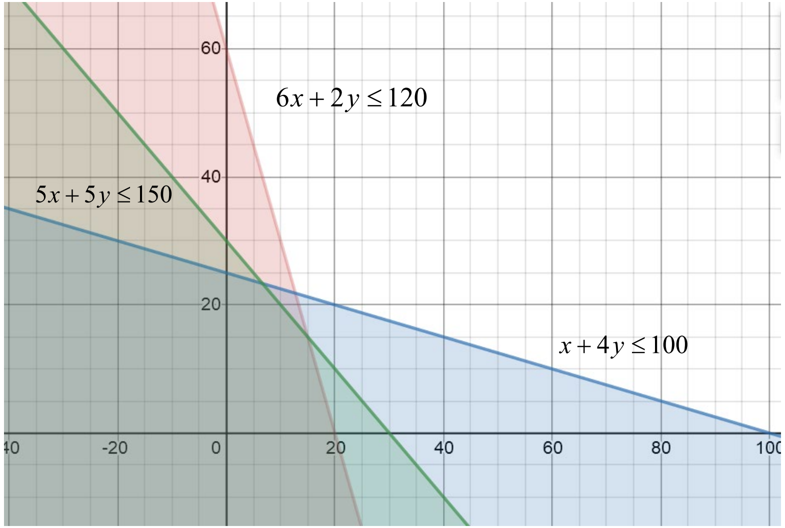
$$\max\ 4x+3y$$

s.to $6x+2y \le 120$

$x+4y \le 100$

$5x+5y \le 150$

$x \ge 0, y \ge 0$



For the constraint $5x+5y \le 150$

Consider the straight line $5x+5y = 150$

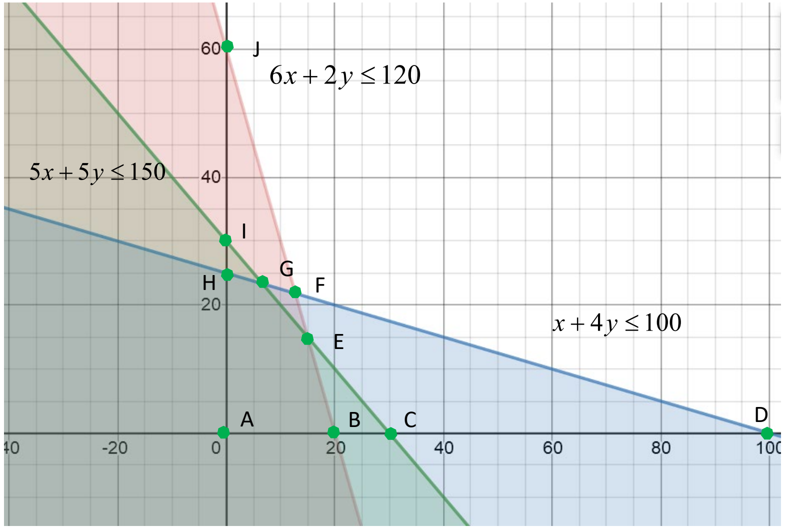
$x=0 \Rightarrow y=30$ Points (0,30) and (30,0) belong to the straight line

$y=0 \Rightarrow x=30$ Point (0,0) satisfies the inequality $5x+5y \le 150$

Cereals, Ltd | Graphical method

$$\max\ 4x + 3y$$

s.a
$$6x + 2y \le 120$$
$$x + 4y \le 100$$
$$5x + 5y \le 150$$
$$x \ge 0, y \ge 0$$



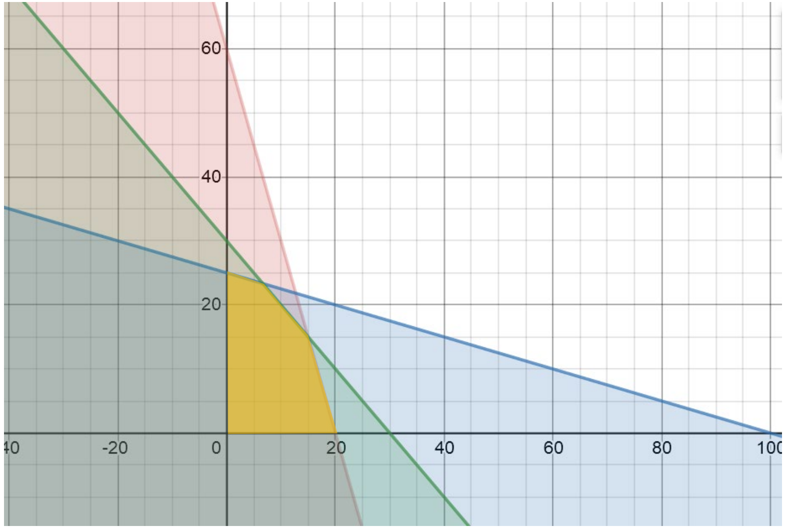
Which points are feasible solutions for the represented problem?
Where is the optimal solution?

Cereals, Ltd | Graphical method

$$\max 4x + 3y$$

s.a

$$6x + 2y \leq 120$$
$$x + 4y \leq 100$$
$$5x + 5y \leq 150$$
$$x \geq 0, y \geq 0$$



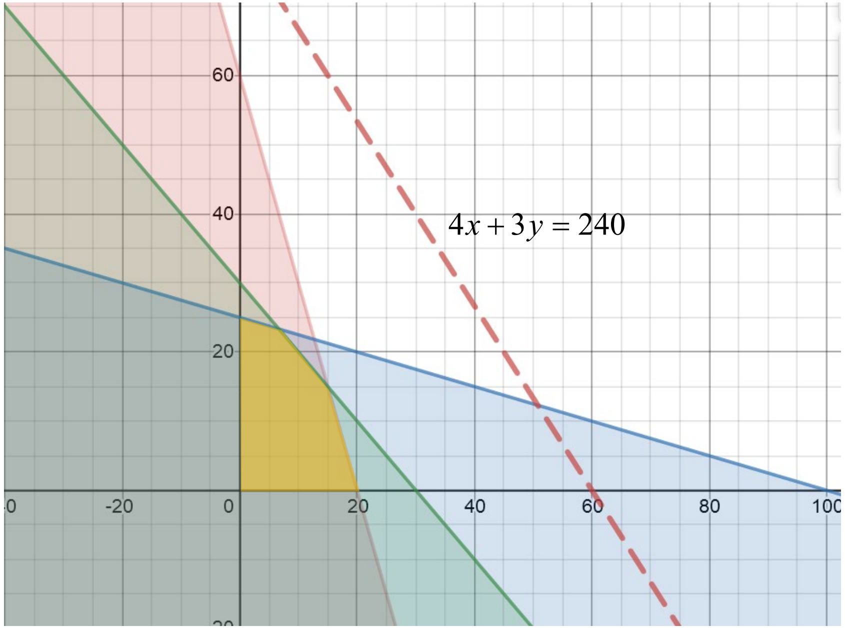
Considering also the non-negativity constraints $x \geq 0, y \geq 0$

The shaded area represents the feasible solutions region

Cereals, Ltd | Graphical method

$$\max\ \boxed{4x+3y}$$

$$\text{s.a}\quad \begin{aligned} 6x+2y &\le 120 \\ x+4y &\le 100 \\ 5x+5y &\le 150 \\ x\ge 0, y &\ge 0 \end{aligned}$$



How can we represent the objective function?

Assign an arbitrary value to the objective function.

For example, to obtain a profit of 240 €:

$$4x+3y=240$$

Cereals, Ltd | Graphical method

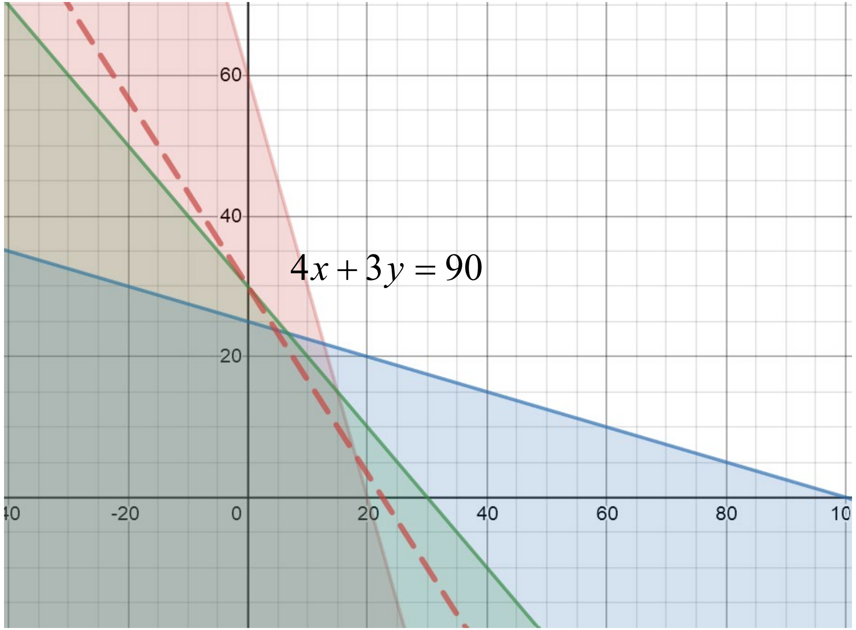
$$\max \; 4x + 3y$$

s.a $\quad 6x + 2y \leq 120$

$x + 4y \leq 100$

$5x + 5y \leq 150$

$x \geq 0, y \geq 0$



Currently, the company produces 18 tons of wheat and 6 tons of corn each week.

Hence the week profit is 90 €.

$$4x + 3y = 90$$

Is this the optimal solution?

Cereals, Ltd | Graphical method

$\max 4x + 3y$

s.a $\quad 6x + 2y \leq 120$

$x + 4y \leq 100$

$5x + 5y \leq 150$

$x \geq 0, y \geq 0$

To obtain the optimal solution, we calculate the coordinates of (A), the intersection point of the straight lines:

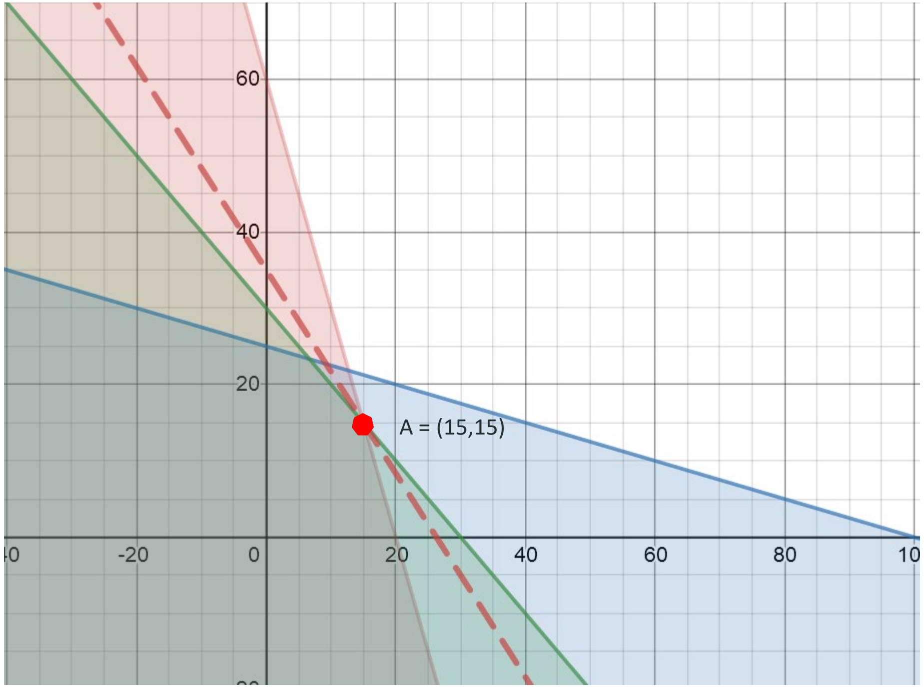
$$6x + 2y = 120$$

$$5x + 5y = 150$$

which is A = (15,15)

$$4x + 3y = 105$$

This production plan yields a profit of 105 €



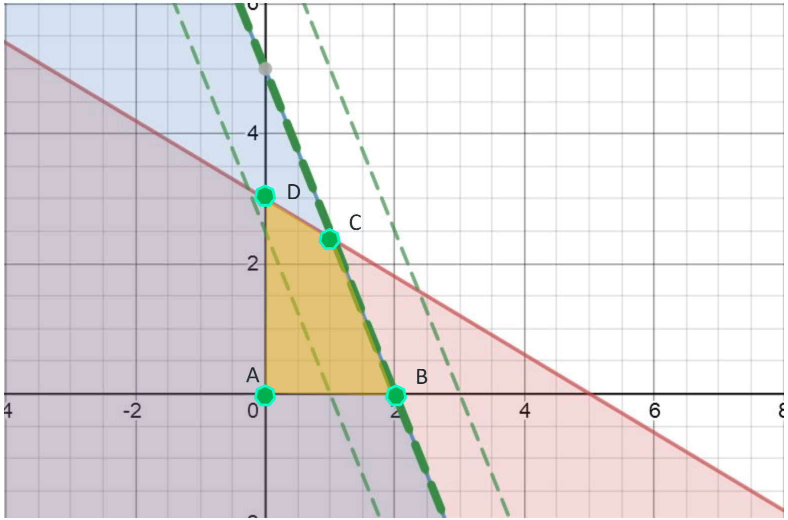
Particular cases of Linear Programming

Infinite optimal solutions

$$\max 10x + 4y$$

s.a

$$3x + 5y \leq 15$$
$$5x + 2y \leq 10$$
$$x \geq 0, y \geq 0$$



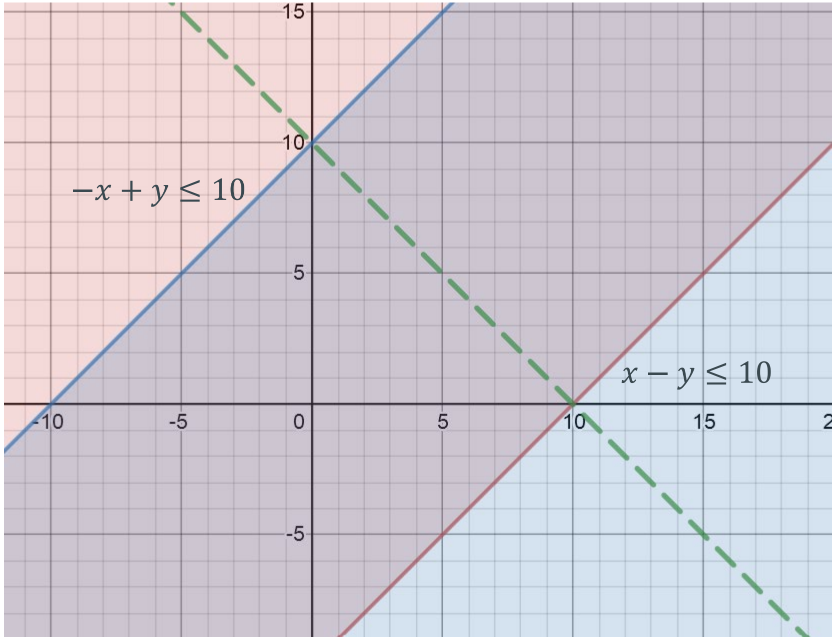
Particular cases of Linear Programming

Unlimited optimal solution

$$\max\ x + y$$

s.a

$$x - y \leq 10$$
$$-x + y \leq 10$$
$$x \geq 0, y \geq 0$$



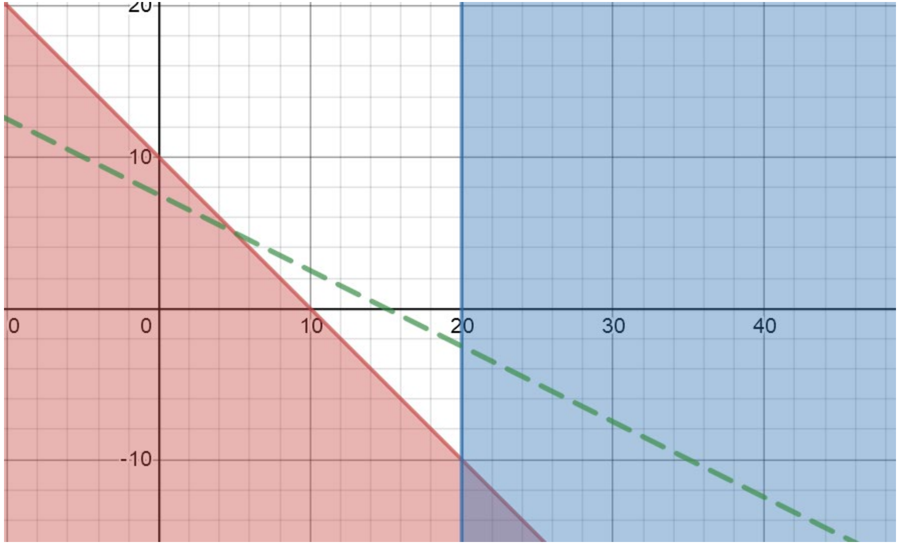
Particular cases of Linear Programming

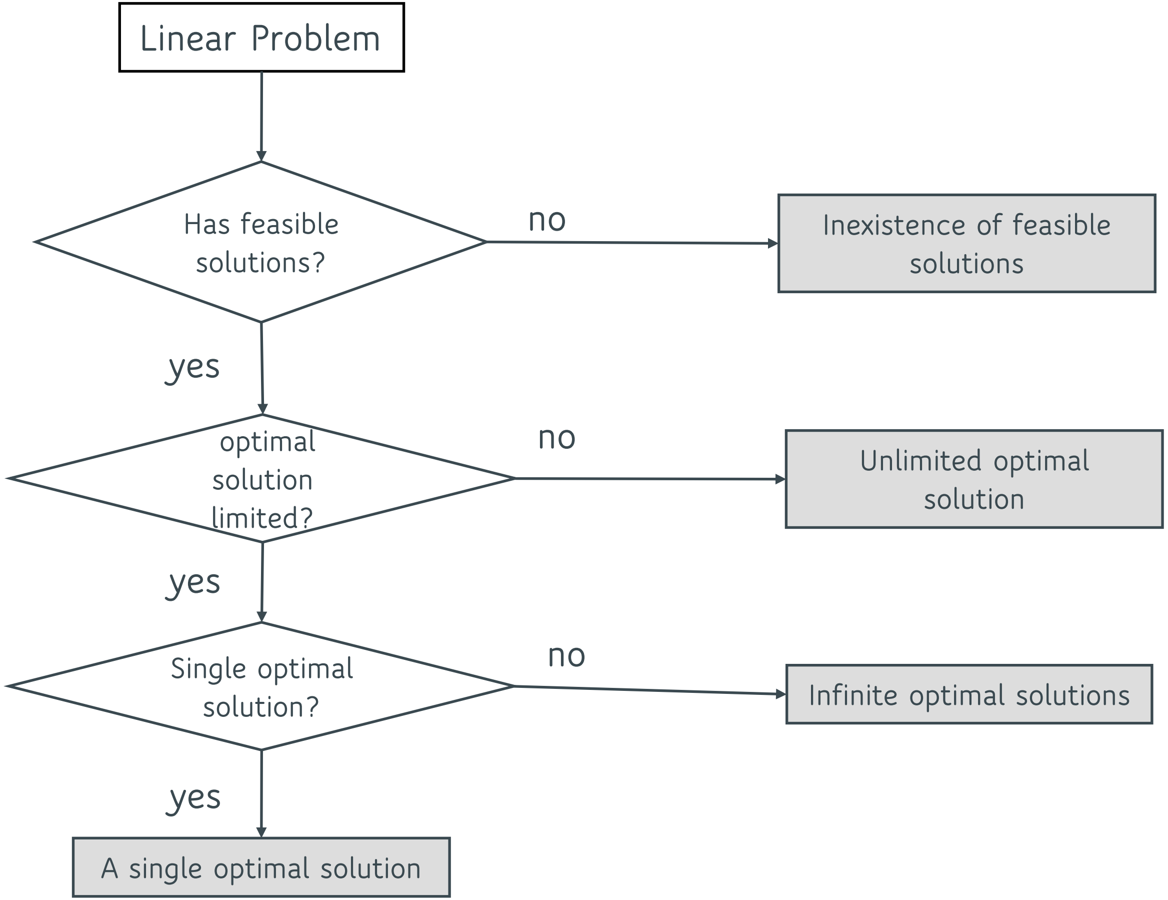
Inexistence of a feasible solution

$$\max\ x + 2y$$

s.a

$$x + y \leq 10$$
$$x \geq 20$$
$$x \geq 0, y \geq 0$$





Linear Problem
Has feasible solutions?
no
Inexistence of feasible solutions
yes
optimal solution limited?
no
Unlimited optimal solution
yes
Single optimal solution?
no
Infinite optimal solutions
yes
A single optimal solution

Cereals, Ltd

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Currently, the company produces 18 tons of wheat and 6 tons of corn per week.
Is this the best option?

Cereals, Ltd Revisited

$$\max 4x + 3y$$

s.a

$6x + 2y \le 120$ Section 1

$x + 4y \le 100$ Section 2

$5x + 5y \le 150$ Section 3

$x \ge 0, y \ge 0$

Optimal solution A = (15,15)
Profit: 105€

What if the profit of each ton of wheat (x) increases to 4,35 €?

What if the production capacity in section 3 (packing) is reduced to 125 h/week?

Sensitivity Analysis

Once the optimal solution of a linear problem is obtained, what happens if changes in the parameters occur?

Sensitivity analysis measures the effect of (small) changes on the parameter values in the optimal solution.

Case 1: changes in the coefficients of the objective function (c_j)

Example: what is the possible variation for unitary profit of wheat (x) and corn (y) without changing the optimal solution (x=15 and y=15)?

Case 2: changes in the right side of constraints (b_i)

Example: what is the effect of changing the production capacity in each section (I, II and III)?

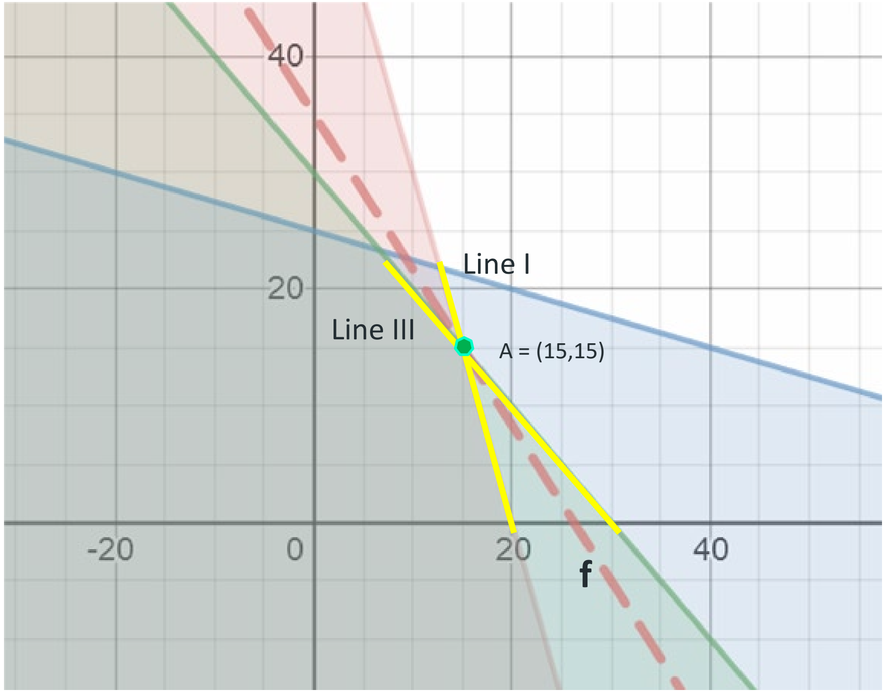
Case 1 - changes in the coefficients of the objective function (c_j)

Observe the slopes of each line

Lines I and III make an angle $\boldsymbol{\alpha}$ at point A.

If **f** rotates inside angle $\boldsymbol{\alpha}$, the optimal solution is maintained.

Rotating **f** inside angle $\boldsymbol{\alpha}$ means that the slope of **f** varies between the slopes of I and III.



Case 1 - changes in the coefficients of the objective function (cj)

Line I: $6x+2y=120 \Leftrightarrow y=-3x+60$

Slope of line I = - 3

Line III: $5x+5y=150 \Leftrightarrow y=-x+30$

Slope of line III = -1

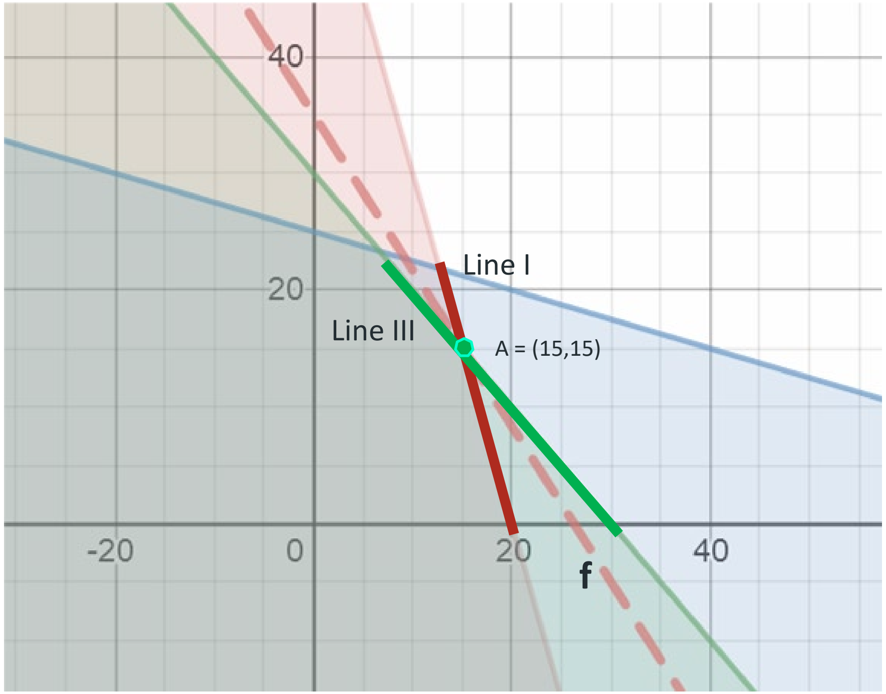
f: $4x+3y=k \Leftrightarrow y=-\frac{4}{3}x+\frac{k}{3}$

Slope of f = - 4 / 3

Just checking…

$$slope(I) \le slope(f) \le slope(III)$$

$$-3 \le -\frac{4}{3} \le -1$$



Case 1 - changes in the coefficients of the objective function (cj)

Let a, b be the coefficients of the objective function $f(x,y) = ax + by = k \Leftrightarrow y = -\frac{a}{b}x + \frac{k}{b}$

$$slope(f) = -\frac{a}{b}$$

The optimal solution (i.e., the values of x and y) remains unchanged if $-3 \leq -\frac{a}{b} \leq -1$

although the **f** value (in this example, the profit) may vary.

If we modify the value of **a**, keeping **b = 3**:

$$-3 \leq -\frac{a}{3} \leq -1 \Leftrightarrow -9 \leq -a \leq -3 \Leftrightarrow 3 \leq a \leq 9$$

If we change the value of **b** instead, keeping **a = 4**:

$$-3 \leq -\frac{4}{b} \leq -1 \Leftrightarrow 1 \leq -\frac{4}{b} \leq 3 \Leftrightarrow \frac{4}{3} \leq b \leq 4$$

Case 1 - changes in the coefficients of the objective function (c_j)

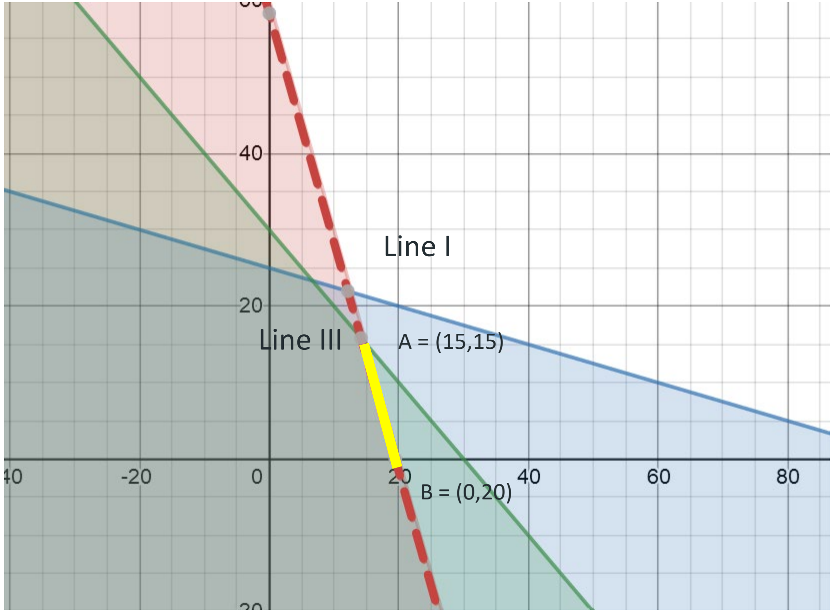
If $-3 < -\frac{a}{b} < -1$, the optimal solution is unique (point A).

But If $\boxed{-\frac{a}{b} = -3}$

($e.g,\ a=9,\ b=3$),

the slope of **f** is equal to the slope of I.

In this case, we will have an infinite number of optimal solutions.



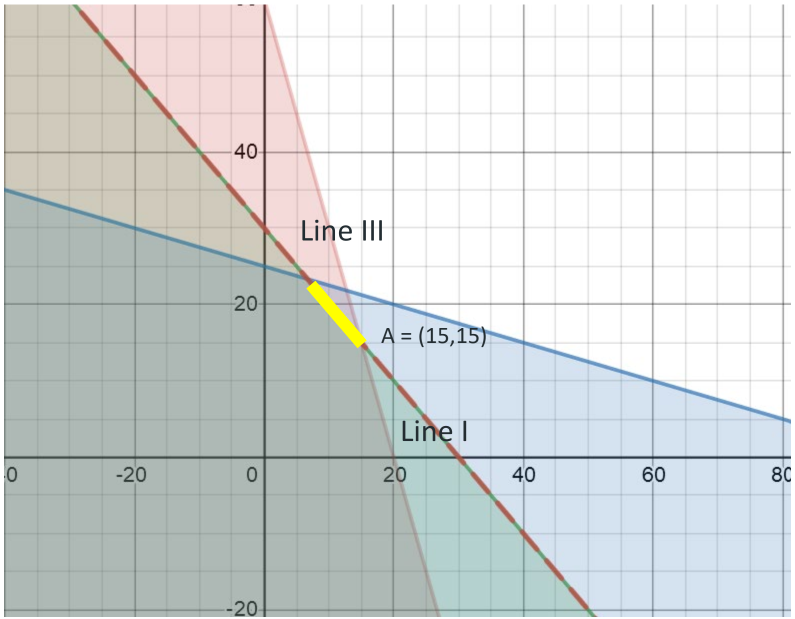
Case 1 - changes in the coefficients of the objective function (c_j)

If $\boxed{-\frac{a}{b} = -1}$

(*e.g, a = 4, b=4 or a=3, b=3. ...*),

The slope of **f** is equal to the slope of III and we will also have an infinite number of optimal solutions.

Note that, for the same production plan (x=15, y=15), the profit is different if we have 4x+4y or 3x+3y.

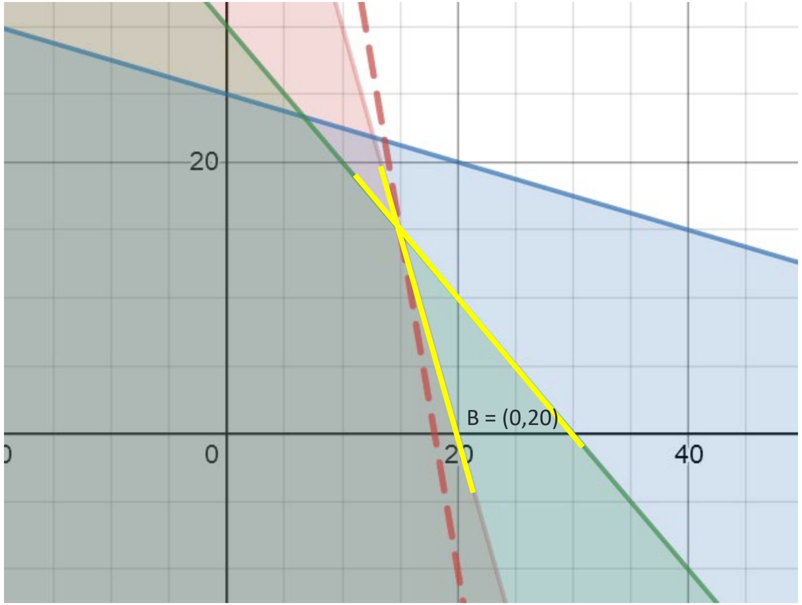


Case 1 - changes in the coefficients of the objective function (c_j)

What happens if **f** rotates beyond angle $\boldsymbol{\alpha}$?

If $-\frac{a}{b} < -3$, we can see graphically that the new optimal solution is B.

But, in the general, we can only say that the optimal solution will change, and it is necessary to solve the new problem.



Case 2: changes in the right side of constraints

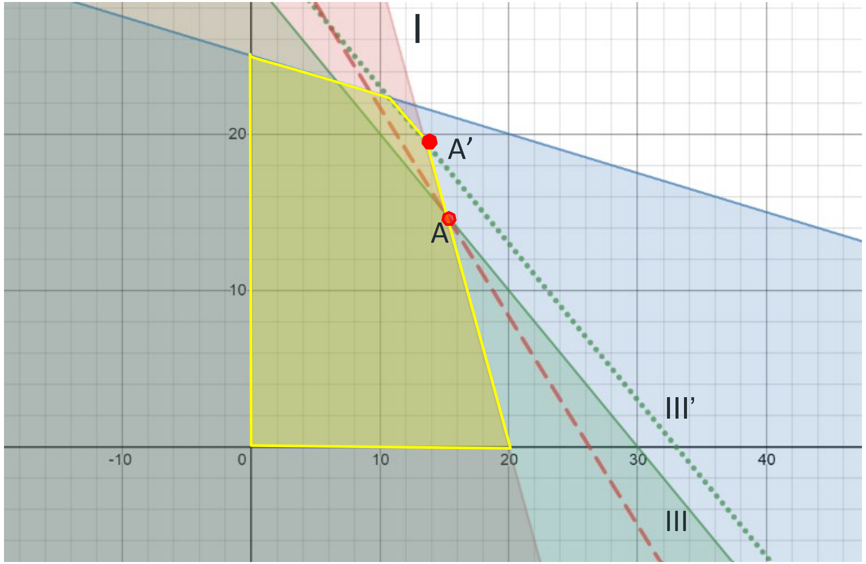
Consider now constraint III:

$$\boxed{5x + 5y \leq 150}$$

What happens if we increase the production capacity (k) in section III?

Let $5x + 5y \leq k$

As k increases, we will have lines (like III') parallel to III, and the optimal solution is in the intersection point of I and III' (A').



Case 2: changes in the right side of constraints

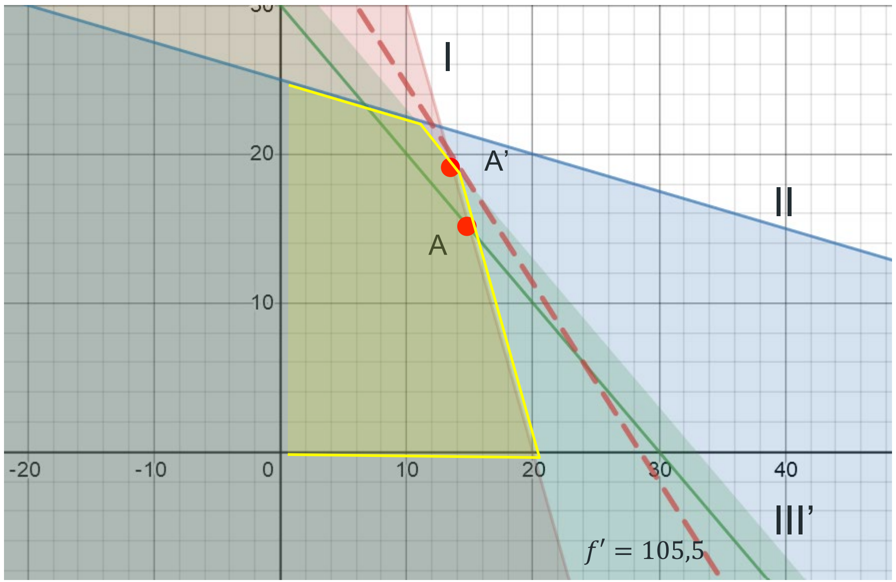
If we increase one unit to resource III:
k = 151.

The new optimal solution is the intersection point of:

$$\begin{cases} 5x+5y=151 \\ 6x+2y=120 \end{cases} \Leftrightarrow \begin{cases} x=14{,}9 \\ y=15{,}3 \end{cases}$$

The optimal profit changed from f* =105 to f'* =105,5

$$4x+3y = 4\times 14{,}9 + 3\times 15{,}3 = 105{,}5$$



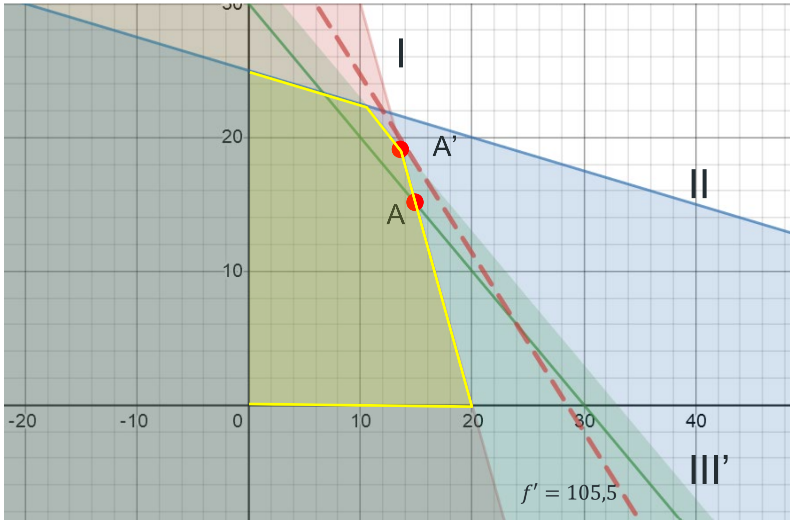
Shadow price

$$\max\ 4x + 3y$$

$$6x + 2y \le 120 \quad \text{I}$$
$$x + 4y \le 100 \quad \text{II}$$
$$5x + 5y \le 151 \quad \text{III'}$$
$$x \ge 0,\ y \ge 0$$

- The amount added to profit (in this case) as a result of the additional unit of resources is seen as the marginal value of the resources and is referred to as the opportunity cost or the shadow price.

- In this example, the shadow price of resource III is the marginal profit obtained when we have an additional hour in the packing section.
- Since the profit has increased from 105 to 105,5, the shadow price of constraint III is 0,5 €.



Case 2: changes in the right side of constraints

Consider again the variation in constraint III,

$$5x + 5y \leq k$$

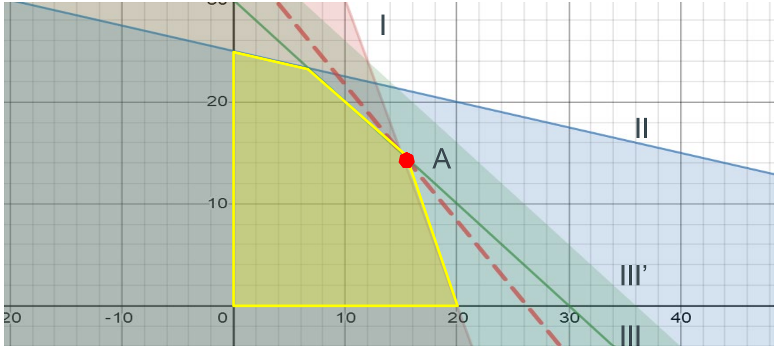
For example, the number of hours in packing section increases to 180

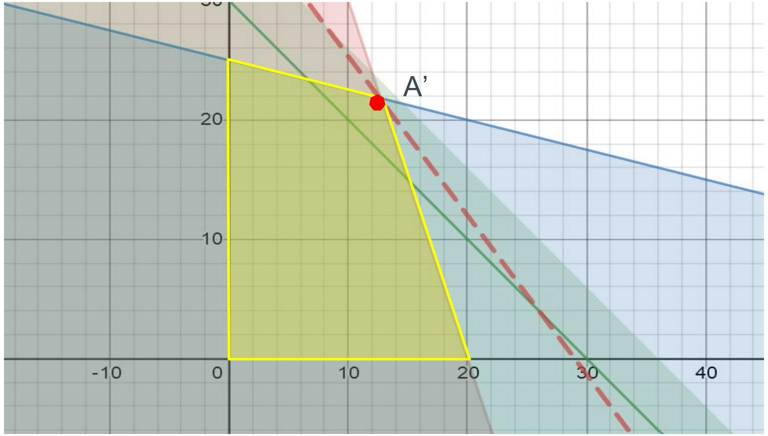
$$5x + 5y \leq 180$$

Now, the optimal solution A’ lies in the intersection point of I and II and constraint III’ becomes redundant.

In this case the meaning of the shadow price no longer applies, since the increase of a unit in resource III has no impact in the objective function

We will need to solve the new problem.





Case 2: changes in the right side of constraints

If we decrease the number of hours in section III, for example, if k=90,

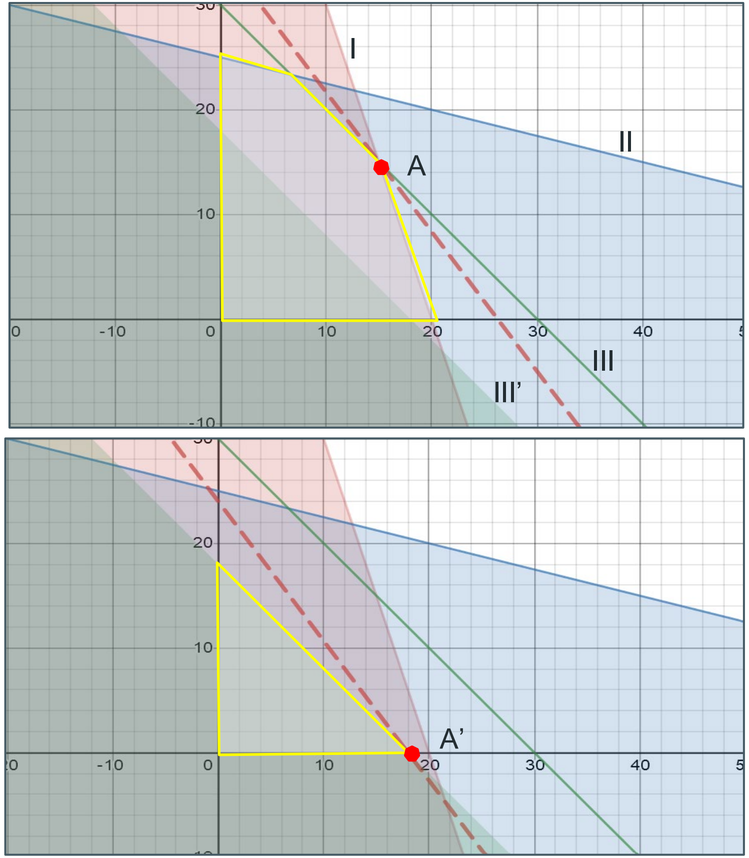
$$5x + 5y \leq 90$$

the optimal solution no longer lies in the intersection of I with III.

Now it lies in the intersection of III’ with y=0 (we say that the basis has changed).

Now we have a slack in constraint I. The value and the meaning of the shadow price associated to constraint III no longer apply.

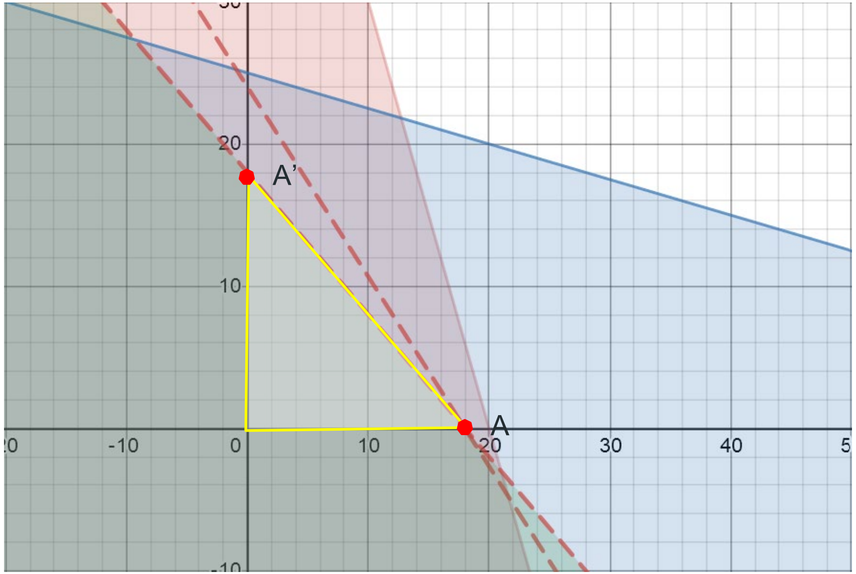
We will need to solve the new problem.



Reduced cost

- In this case, the optimal solution indicates that the optimal production plan should not include the production of corn (y=0).
- The reduced cost of this product shows the increment that the corn (y) unitary profit should have in order to include it in the optimal production plan.
- In this example, if the corn unitary profit is 4 €/ton (f'), it may be considered in the production plan.

  Hence, since the increment is of 1€/ton, the **reduced cost of y is 1**.



In short, for a maximization problem

Case 1 - changes in the coefficients of the objective function

- For a particular optimal solution, and for each coefficient of the objective function (cj), it is possible to determine an interval of variation that will keep the optimal solution unchanged (note that the value of the objective function may change).

- If, in the optimal solution, the value of a decision variable is zero (xi = 0), its reduced cost is the increment that the corresponding coefficient in the objective function should have in order to include that variable in the optimal solution (xi > 0).

Case 2: changes in the right side of constraints

(i) If a constraint is active (there is no slack or surplus) then increasing or decreasing the amount of the resource associated to that constraint could lead to a change in the value of the objective function in the optimal solution.

The shadow price of a resource is the increment in the objective function generated by an additional unit of that resource.

(ii) If there is a slack in a constraint (the constraint is inactive), the value of the objective function in the optimal solution does not alter if we increase the amount of available resource. However, if we decrease the amount of available resource, the value of the objective function in the optimal solution may change.

Solution of Cereals, Ltd obtained by Lindo software (http://www.lindo.com)

LP OPTIMUM FOUND AT STEP 1

OBJECTIVE FUNCTION VALUE

1) 105.0000

VARIABLE	VALUE	REDUCED COST
X	15.000000	0.000000
Y	15.000000	0.000000

ROW	SLACK OR SURPLUS	**DUAL PRICES**
2)	0.000000	0.250000
3)	25.000000	0.000000
4)	0.000000	0.500000

SENSITIVITY ANALYSIS

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VAR	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
X	4.000000	5.000000	1.000000
Y	3.000000	1.000000	1.666667

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	120.000000	60.000000	33.333332
3	100.000000	INFINITY	25.000000
4	150.000000	22.727272	49.999996