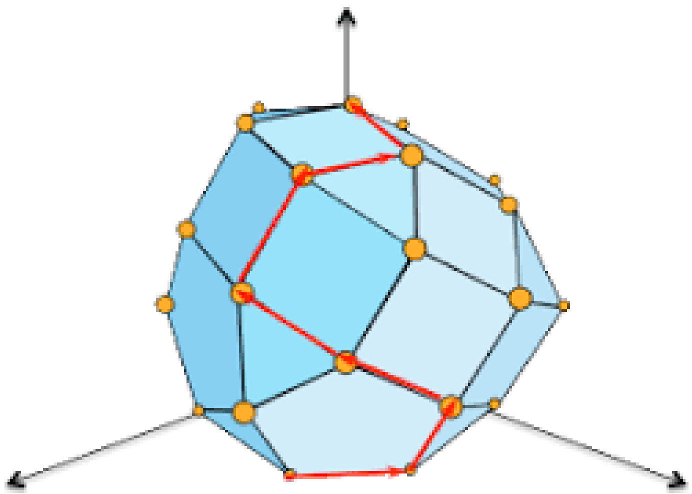


SIMPLEX METHOD

Mestrado em Engenharia Informática e Computação

OPTIMIZATION

Faculdade de Engenharia da Universidade do Porto

CONTENTS

EXERCISE 1

Solve the following LP model applying the Simplex method.

Maximize $f(x) = 80x_1 + 60x_2$

Subject to $5x_1 + 3x_2 \le 30$

$2x_1 + 3x_2 \le 24$

$x_1 + 3x_2 \le 18$

$x_1, x_2 \ge 0$

SOLUTION:

Canonical form:

Maximize $f(x) = 80x_1 + 60x_2$

Subject to $5x_1 + 3x_2 + S_1 = 30$

$2x_1 + 3x_2 + S_2 = 24$

$x_1 + 3x_2 + S_3 = 18$

$x_1, x_2, S_1, S_2, S_3 \ge 0$

Base	x1	x2	S1	S2	S3	Valor
S1	5	3	1	0	0	30
S2	2	3	0	1	0	24
S3	1	3	0	0	1	18
f	80	60	0	0	0	0

Base	x1	x2	S1	S2	S3	Valor
x1	1	3/5	1/5	0	0	6
S2	0	9/5	-2/5	1	0	12
S3	0	12/5	-1/5	0	1	12
f	0	12	-16	0	0	-480

Base	x1	x2	S1	S2	S3	Valor
x1	1	0	1/4	0	-1/4	3
S2	0	0	-1/4	1	-3/4	3
x2	0	1	-1/12	0	5/12	5
f	0	0	-15	0	-5	-540

Optimal solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$ Max $f(x) = 540$

EXERCISE 2

Solve the following LP model applying the Simplex method.

Maximize $f(x) = 20x_1 + 12x_2$

Subject to $x_1 \leq 4$

$x_2 \leq 5$

$3x_1 + x_2 \leq 16$

$x_1, x_2 \geq 0$

SOLUTION:

Canonical form:

Maximize $f(x) = 20x_1 + 12x_2$

Subject to $x_1 + S_1 = 4$

$x_2 + S_2 = 5$

$3x_1 + x_2 + S_3 = 16$

$x_1, x_2, S_1, S_2, S_3 \geq 0$

Base	x1	x2	S1	S2	S3	Valor
S1	1	0	1	0	0	4
S2	0	1	0	1	0	5
S3	3	1	0	0	1	16
f	20	12	0	0	0	0

Base	x1	x2	S1	S2	S3	Valor
x1	1	0	1	0	0	4
S2	0	1	0	1	0	5
S3	0	1	-3	0	1	4
f	0	12	-20	0	0	-80

Base	x1	x2	S1	S2	S3	Valor
x1	1	0	1	0	0	4
S2	0	0	3	1	-1	1
x2	0	1	-3	0	1	4
f	0	0	16	0	-12	-128

Base	x1	x2	S1	S2	S3	Valor
x1	1	0	0	-1/3	1/3	11/3
S1	0	0	1	1/3	-1/3	1/3
x2	0	1	0	1	0	5
f	0	0	0	-16/3	-20/3	-400/3

Optimal solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} 11/3 \\ 5 \\ 1/3 \\ 0 \\ 0 \end{bmatrix} \quad \text{Max } f(x) = 400/3$$

EXERCISE 3

Solve the following LP model applying the Simplex method.

Maximize $f(x) = 8x_1 + 6x_2$

Subject to $x_1 + x_2 \leq 10$

$3x_1 + 3x_2 \leq 24$

$-3x_1 + x_2 \geq -12$

$x_1, x_2 \geq 0$

SOLUTION:

Canonical form:

Maximize $f(x) = 8x_1 + 6x_2$

Subject to $x_1 + x_2 + S_1 = 10$

$3x_1 + 3x_2 + S_2 = 24$

$3x_1 - x_2 + S_3 = 12$

$x_1, x_2, S_1, S_2, S_3 \geq 0$

Base	x1	x2	S1	S2	S3	Valor
S1	1	1	1	0	0	10
S2	3	3	0	1	0	24
S3	3	-1	0	0	1	12
f	8	6	0	0	0	0

Base	x1	x2	S1	S2	S3	Valor
S1	0	4/3	1	0	-1/3	6
S2	0	4	0	1	-1	12
x1	1	-1/3	0	0	1/3	4
f	0	26/3	0	0	-8/3	-32

Base	x1	x2	S1	S2	S3	Valor
S1	0	0	1	-1/3	0	2
x2	0	1	0	1/4	-1/4	3
x1	1	0	0	1/12	1/4	5
f	0	0	0	-13/6	-1/2	-58

Optimal solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} \qquad \text{Max } f(x) = 58$$

EXERCISE 4

Solve the following LP model applying the Simplex method.

Maximize $f(x) = x_1 + x_2 + x_3$

Subject to $x_1 + x_3 \leq 1$

$2x_1 + x_2 + x_3 \leq 8$

$x_1 + x_2 - x_3 \leq 2$

$x_1, x_2, x_3 \geq 0$

SOLUTION

Canonical form:

Maximize $f(x) = x_1 + x_2 + x_3$

Subject to $x_1 + x_3 + S_1 = 1$

$2x_1 + x_2 + x_3 + S_2 = 8$

$x_1 + x_2 - x_3 + S_3 = 2$

$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$

Base	x1	x2	x3	S1	S2	S3	Valor
S1	1	0	1	1	0	0	1
S2	2	1	1	0	1	0	8
S3	1	1	-1	0	0	1	2
f	1	1	1	0	0	0	0

Base	x1	x2	x3	S1	S2	S3	Valor
x3	1	0	1	1	0	0	1
S2	1	1	0	-1	1	0	7
S3	2	1	0	1	0	1	3
f	0	1	0	-1	0	0	-1

Base	x1	x2	x3	S1	S2	S3	Valor
x3	1	0	1	1	0	0	1
S2	-1	0	0	-2	1	-1	4
x2	2	1	0	1	0	1	3
f	-2	0	0	-2	0	-1	-4

Optimal solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 4 \\ 0 \end{bmatrix} \qquad \text{Max } f(x) = 4$$

EXERCISE 5

Consider the following LP problem:

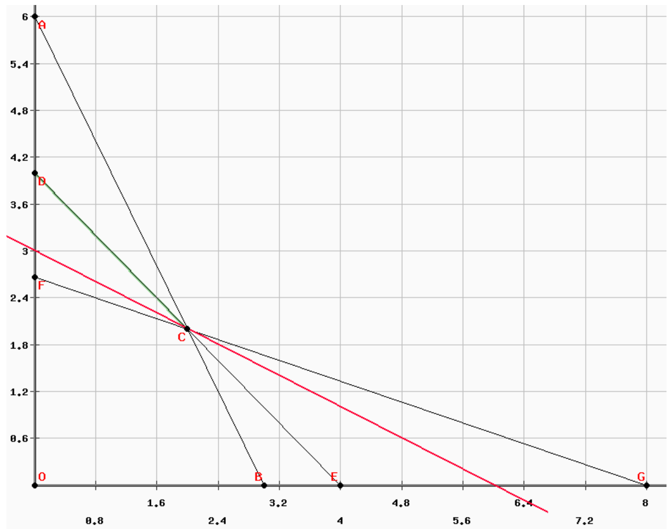
Minimize $f(x) = x_1 + 2x_2$
Subject to $2x_1 + x_2 \le 6$
$x_1 + x_2 = 4$
$x_1 + 3x_2 \ge 8$
$x_1, x_2 \ge 0$

a) Solve the problem graphically.
b) Solve the problem using the Simplex method.
c) If the problem was a maximization problem which would be the optimal solution?

SOLUTION

a) The set of feasible solutions is represented in green, in the figure below. In this particular case, the feasible solutions are a line segment because one of the constraints was n "=" type constraint. The objective function at the optimal solution is represented in red.

The optimal solution is the point x1 = x2 = 2 (point C). C is also a degenerated solution because one of the basic variables is zero (s1). Note that graphically we can also see that more than two constraints are intersected in point C.



b) 1st phase of the 2-phases method

	x1	x2	s1	s2	a1	a2	
s1	2	1	1	0	0	0	6
a1	1	1	0	0	1	0	4
a2	1	3	0	-1	0	1	8
f	1	2	0	0	0	0	0
F	-2	-4	0	1	0	0	-12

	x1	x2	s1	s2	a1	a2	
s1	5/3	0	1	1/3	0	-1/3	10/3
a1	2/3	0	0	1/3	1	-1/3	4/3
x2	1/3	1	0	- 1/3	0	1/3	8/3
min f	1/3	0	0	2/3	0	-2/3	-16/3
min F	-2/3	0	0	-1/3	0	4/3	-4/3

	x1	x2	s1	s2	a1	a2	
s1	0	0	1	-1/2	-5/2	1/2	0
x1	1	0	0	1/2	3/2	-1/2	2
x2	0	1	0	- 1/2	-1/2	1/2	2
f	0	0	0	1/2	-1/2	-1/2	-6
F	0	0	0	0	1	1	0

End of 1st phase and also of the 2nd phase. The solution is optimal (do not forget that we are minimising f)

	x1	x2	s1	s2	
s1	0	0	1	-1/2	0
x1	1	0	0	1/2	2
x2	0	1	0	- 1/2	2
min f	0	0	0	1/2	-6

b) If the problem was a maximisation problem, the optimal solution would be point B (0,4) Solving by the simplex method:

	x1	x2	s1	s2	
s1	0	0	1	-1/2	0
x1	1	0	0	1/2	2
x2	0	1	0	- 1/2	2
max f	0	0	0	1/2	-6

	x1	x2	s1	s2	
s1	1	0	1	0	2
s2	2	0	0	1	4
x2	1	1	0	0	4
max f	-1	0	0	0	-8

EXERCISE 6

Solve the following LP model applying the Simplex method.

Maximize $f(x) = 2x_1 + x_2 + 2x_3$

Subject to $2x_1 + x_2 + 2x_3 = 1$

$-x_1 - 3x_2 - x_3 \leq -3$

$4x_1 + 5x_2 + x_3 \geq 5$

$x_1, x_2, x_3 \geq 0$

SOLUTION:

Using the two-phases method:

Canonical form:

Maximize $f(x) = 2x_1 + x_2 + 2x_3$

Subject to $2x_1 + x_2 + 2x_3 = 1$

$x_1 + 3x_2 + x_3 - S_1 = 3$

Auxiliary problem:

Minimize $F = a_1 + a_2 + a_3 = 9 - 7x_1 - 9x_2 - 4x_3 + S_1 + S_2$

Subject to $2x_1 + x_2 + 2x_3 + a_1 = 1$

$x_1 + 3x_2 + x_3 - S_1 + a_2 = 3$

Base	x1	x2	x3	S1	S2	a1	a2	a3	Valor
a1	2	1	2	0	0	1	0	0	1
a2	1	3	1	-1	0	0	1	0	3
a3	4	5	1	0	-1	0	0	1	5
F	0	0	0	0	0	1	1	1	0
f	2	1	2	0	0	0	0	0	0

Base	x1	x2	x3	S1	S2	a1	a2	a3	Valor
a1	2	1	2	0	0	1	0	0	1
a2	1	3	1	-1	0	0	1	0	3
a3	4	5	1	0	-1	0	0	1	5
F	-7	-9	-4	1	1	0	0	0	-9
f	2	1	2	0	0	0	0	0	0

Base	x1	x2	x3	S1	S2	a1	a2	a3	Valor
x2	2	1	2	0	0	1	0	0	1
a2	-5	0	-5	-1	0	-3	1	0	0
a3	-6	0	-9	0	-1	-5	0	1	0
F	11	0	14	1	1	9	0	0	0
f	0	0	0	0	0	-1	0	0	-1

The value Min F= 0 was achieved for a basis containing artificial variables, so this solution is infeasible (artificial variables are not present in the original canonical form).

Hence, we have to switch (arbitrarily) the artificial variable with non-basic variables from the original canonical form. Only then we can say that the 1st phase of the 2-phases method is complete. We obtained a feasible initial solution (degenerated) for the initial problem and we can proceed to the 2nd phase.

We chose to swap NB variable x1 with a2 and NB variable x3 with a3.

Base	x1	x2	x3	S1	S2	a1	a2	a3	Valor
x2	0	1	0	-2/5	0	-1/5	2/5	0	1
x1	1	0	1	1/5	0	3/5	-1/5	0	0
a3	0	0	-3	6/5	-1	-7/5	-6/5	1	0
F	0	0	3	-6/5	1	-12/5	-11/5	0	0
f	0	0	0	0	0	-1	0	0	-1

Base	x1	x2	x3	S1	S2	a1	a2	a3	Valor
x2	0	1	0	-2/5	0	-1/5	2/5	0	1
x1	1	0	0	3/5	-1/3	2/15	-3/5	1/3	0
x3	0	0	1	-2/5	1/3	7/15	2/5	-1/3	0
F	0	0	0	0	0	1	1	1	0
f	0	0	0	0	0	-1	0	0	-1

Base	x1	x2	x3	S1	S2	Valor
x2	0	1	0	-2/5	0	1
x1	1	0	0	3/5	-1/3	0
x3	0	0	1	-2/5	1/3	0
f	0	0	0	0	0	-1

Optimal solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{Max } f(x) = 1$$

Note: Solve the problem below and verify that, in this case, the minimum value of the auxiliary objective function is not zero (F is minimum, but some artificial variable did not leave the basis, so the problem has no feasible solution.

Minimizar $f(x) = 15x_1 + 20x_2$
s.a.

$5x_1 + 3x_2 \leq 15$
$2x_1 + 3x_2 \geq 20$
$x_1, x_2 \geq 0$

EXERCISE 7

The following simplex tableaux for a maximization problem have some errors. Identify them.

6.1

Base	x1	x2	S1	S2	S3	Valor
x1	1	1/3	5	2	0	12
S3	-1	2/3	4	5	1	10
F(x)	0	0	3	-2	2	24

6.2

Base	x1	x2	S1	S2	Valor
x1	1	1/2	1	3	5
S1	0	4	0	12	-4
F(x)	0	-2/5	3	7	13

SOLUTION:

6.1

Base	x1	x2	S1	S2	S3	Valor
x1	1	1/3	5	2	0	12
S3	**-1**	2/3	4	5	1	10
F(x)	0	0	3	-2	**2**	24

Since we have 3 slack variables (S1, S2, S3) we should also have 3 constraints.
Considering that we corrected this mistake (for example, replacing S3 by x2), we would still have the following erros:
Variable x1 is basic in the first equation, so the coefficients of x1 in the remaining equations must be 0.
Variable S3 is basic in the second equation, so the coefficients of S3 in the remaining equations must be 0.

6.2

Base	x1	x2	S1	S2	Valor
x1	1	1/2	**1**	3	5
S1	0	4	**0**	12	**-4**
F(x)	0	-2/5	**3**	7	13

Variable S1 is basic in the second equation, so S1 must have coefficient 1 in the second equation and 0 in the remaining.
The value of basic variable S1 cannot be negative, sine it breaks the rule $S_1 \geq 0$.

EXERCISE 8

A factory can produce 4 different types of solvents. In order to maximize the profit, the amount of each product x_i (i=1,2,3,4) to be produced monthly has to be defined. We know that the unitary profit of each product is 300, 725, 200 e 450 monetary units (m.u.), respectively.

The amount of each product to be produced is limited by constraints regarding raw material consumption, labor and storage space availability for each product. The following table presents the needs of each production line as well as the monthly availability of each resource.

	Product 1	**Product 2**	**Product 3**	**Product 4**	**Availability**
Raw material **(Kg/month)**	1	3	1	2	60
Labor **(man-hours /mês)**	2	8	2	3	140
Storage space **(m³/month)**	3	5	6	3	100

Based on the information provided it was possible to formulate the problem in the following Linear Programming model:

$$\max Z = 300x_1 + 725x_2 + 200x_3 + 450x_4$$
$$s.a.: x_1 + 3x_2 + x_3 + 2x_4 \leq 60$$
$$2x_1 + 8x_2 + 2x_3 + 3x_4 \leq 140$$
$$3x_1 + 5x_2 + 6x_3 + 3x_4 \leq 100$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Applying the simplex method to this problem, we obtained the following optimal tableaux (Si is the slack variable associated to constraint i):

Base	x1	x2	x3	x4	S1	S2	S3	Valor
x4	0	0	-3/5	1	7/5	2/5	-1/5	8
x2	0	1	-3/10	0	-3/10	3/10	-1/10	14
x1	1	0	31/10	0	-9/10	-1/10	7/10	2
f(x)	0	0	-242,5	0	-142,5	-7,5	-47.5	-14350

a) Describe the obtained solution by identifying what must be produced and how much. Also, identify the resources consumption and indicate if there exists any advantage in increasing the availability of any of the resources.

b) Consider that the unitary profit of solvent 3 changes to 400 m.u. Verify if the optimal solution is still the same.

SOLUTION:

a) The optimal solution indicates that the factory must produce solvents 1, 2 and 4 in the following quantities: 2Kl, 14 Kl e 8 Kl, respectively. The resources are totally consumed (no slack variables in the basis) and we have advantage in increasing their availability unless the cost of this increase in not higher than 142.5 m.u. by unit of raw material, 7.5 m.u. by unit of labor and 47.5 m.u by unit of storage space.

y_i represents the shadow price of resource i (measuring the marginal value of resource i). For example, if we increase 1 unit to the resource 1, the profit will increase 142.5 m.u. (and we would obtain a new optimal solution (x_1, x_2, x_3, x_4)=(1.1; 13.7; 0; 9.4).

b) The solution remains optimal (reduced cost = 242.5). As the increase was only 200 m.u., there is no change in the optimal.

Sensitivity Analysis (WinQSB):

Decision Variable	Solution Value	Unit Cost or Profit c(j)	Total Contribution	Reduced Cost	Basis Status	Allowable Min. c(j)	Allowable Max. c(j)
X1	2.0000	300.0000	600.0000	0	basic	232.1429	375.0000
X2	14.0000	725.0000	10.150.0000	0	basic	700.0000	1.200.0000
X3	0	200.0000	0	-242.5000	at bound	-M	442.5000
X4	8.0000	450.0000	3.600.0000	0	basic	348.2143	468.7500
Objective	Function	(Max.) =	14.350.0000				
Constraint	Left Hand Side	Direction	Right Hand Side	Slack or Surplus	Shadow Price	Allowable Min. RHS	Allowable Max. RHS
C1	60.0000	<=	60.0000	0	142.5000	54.2857	62.2222
C2	140.0000	<=	140.0000	0	7.5000	93.3333	160.0000
C3	100.0000	<=	100.0000	0	47.5000	97.1429	140.0000

EXERCISE 9

Consider the following linear problem:

$$\max 3x_1 + 2x_2$$

$$s.a \;\; x_1 - x_2$$

$$2x_1 - 4x_2 \geq 0$$

$$x_1 + x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

a) Solve graphically the problem, identifying clearly the feasible solutions region, the optimal solution and the objective function slope at the optimal solution.

b) Consider the following simplex tableaux. Identify in the graphical solution the point presented in the tableaux.

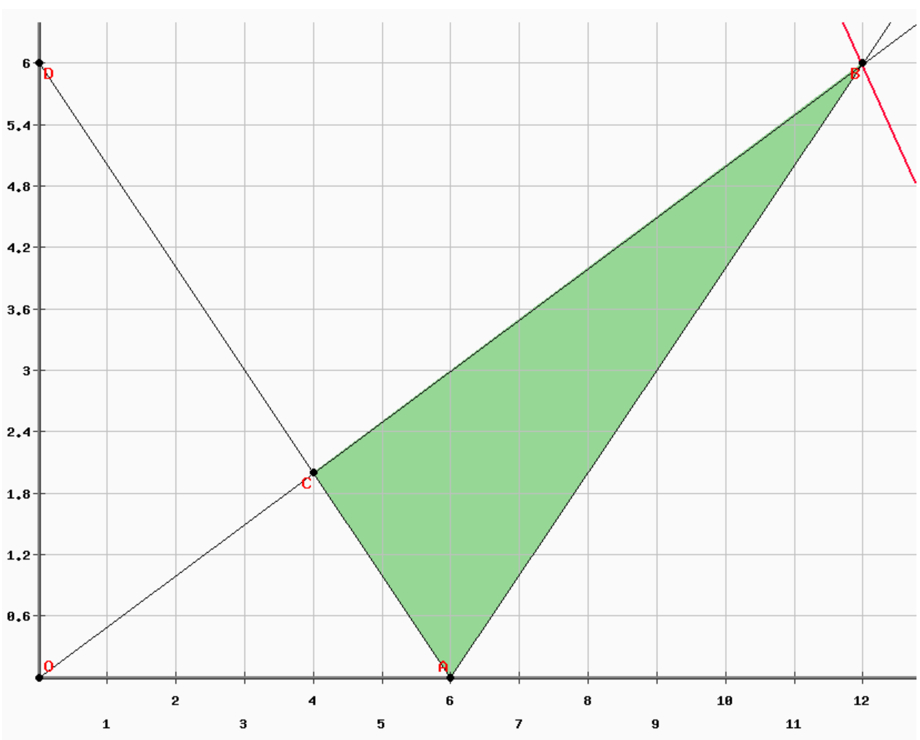
Base	x1	x2	S1	S2	S3	RHS
S3	0	-10	5	0	1	0
S2	0	2	2	1	0	12
x1	1	-1	1	0	0	6
f	0	5	-3	0	0	-18

Perform one iteration of the simplex method. Identify in the graphical solution the new solution obtained. Is the solution optimal? Justify your answer.

c) Consider now that we pretend to minimize the objective function (with the same set of constraints). Which is the optimal solution for the new problem? Is there a decision variable with a non-zero reduced cost? Justify your answer.

SOLUTION:

a)



1st phase of the Método Simplex (point 0)

	x1	x2	s1	s2	s3	a1	
s1	1	-1	1	0	0	0	6
s2	-2	4	0	1	0	0	0
a1	1	1	0	0	-1	1	6
f	3	2	0	0	0	0	0
F	-1	-1	0	0	1	0	-6

Point A – feasible, degenerated solution (end of phase 1)

	x1	x2	s1	s2	s3	a1	
s1	0	-2	1	0	1	-1	0
s2	0	6	0	1	-2	2	12
x1	1	1	0	0	-1	1	6
f	0	-1	0	0	3	-3	-18
F	0	0	0	0	0	1	0

Point A -

	x1	x2	s1	s2	s3	
s3	0	-2	1	0	1	0
s2	0	2	2	1	0	12
x1	1	-1	1	0	0	6
f	0	5	-3	0	0	-18

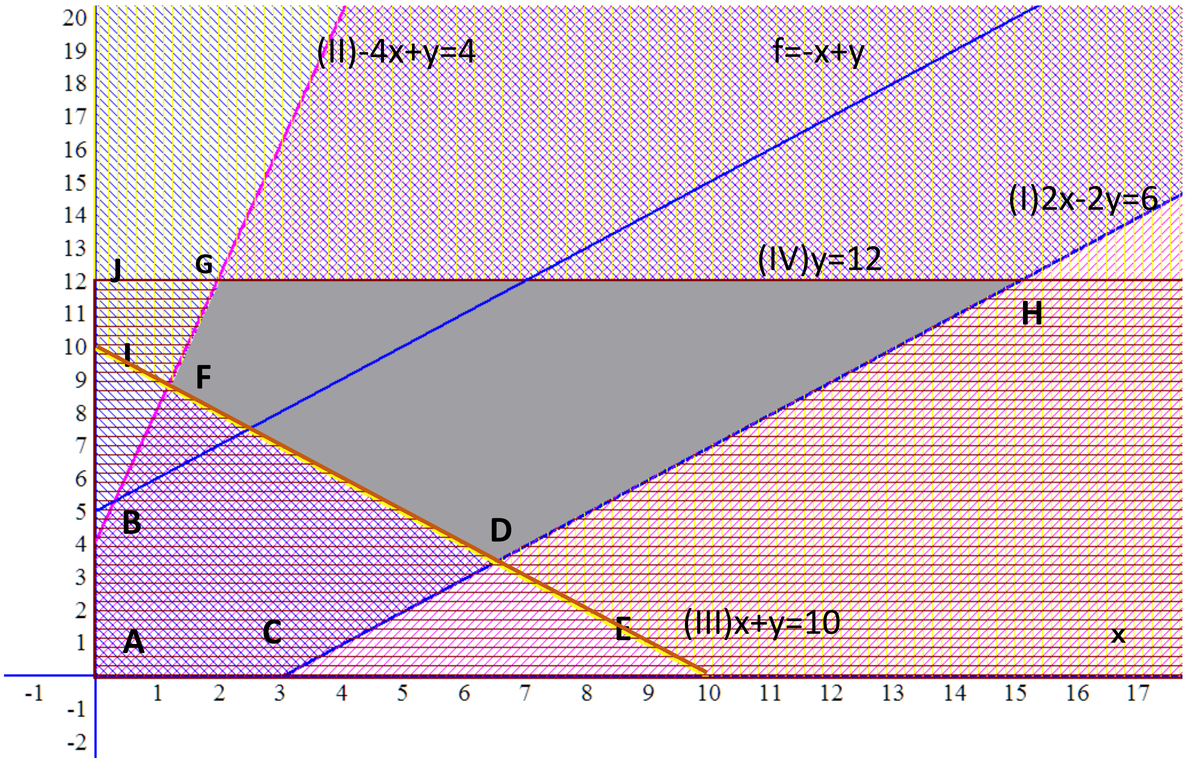
Point B – Optimal solution

	x1	x2	s1	s2	s3	
s3	0	0	3	1	1	12
x2	0	1	1	0,5	0	6
x1	1	0	2	0,5	0	12
f	0	0	-8	-2,5	0	-48

EXERCISE 10

Consider the following linear problem which graphical solution is presented in the figure below.

$$\begin{aligned} &\text{Max } f = -x + y \\ &\text{s.a} \quad 2x - 2y \le 6 \\ &\qquad -4x + y \le 4 \\ &\qquad x + y \ge 10 \\ &\qquad y \le 12 \\ &\qquad x, y \ge 0 \end{aligned}$$



The figure presents the problem's constraints as well as the objective function f when f= 5. The shading represents the feasible solutions area and the letters correspond to points of intersection of the constraints (vertices).

The following tables were obtained by application the Simplex method, but they are not sorted in the order they were obtained. In some of these tables, the coefficient values were omitted and replaced by unknown factors (k1, k2 and k3).

a) What would be the correct order to obtain the tableaux S1 to S5 by applying the Simplex method? To what stage of Figure 1 does each tableau correspond? Fill in the table below, indicating the values of x, y and f for each one.

b)

Iteration	Tableau	Point	x	y	f
1ª iteration (original)		A			0
2ª iteration					-3
3ª iteration	S3				
4ª iteration					
5ª iteration (Final)					10

c) What are the values of k1, k2 and k3? If you are not able to calculate the values, specify the variation intervals (for example, ki >= 0).

d) If this were a minimization problem, which would be the optimal solution? Specify the tableau and the corresponding vertex. Is the solution unique? Justify your answer.

Tableau S1								
	x	y	s1	s2	s3	s4	a1	RHS
x	1	-1	0,5	0	0	0	0	3
s2	0	-3	2	1	0	0	0	16
a1	0	k1	-0,5	0	-1	0	1	7
s4	0	1	0	0	0	1	0	12
f	0	0	0,5	0	0	0	0	3
F	0	-2	0,5	0	1	0	0	-7

Tableau S2							
	x	y	s1	s2	s3	s4	RHS
x	1	0	0	-0,25	0	0,25	2
s1	0	0	1	0,5	0	1,5	26
y	0	1	0	0	0	1	12
s3	0	0	0	-0,25	1	1,25	4
f	0	0	0	k2	0	-0,8	-10

Tableau S3								
	x	y	s1	s2	s3	s4	a1	RHS
x	1	0	0,25	0	-0,5	0	0,5	6,5
s2	0	0	1,25	1	-1,5	0	1,5	26,5
y	0	1	-0,25	0	-0,5	0	0,5	3,5
s4	0	0	0,25	0	0,5	1	-0,5	8,5
f	0	0	0,5	0	0	0	0	3
F	0	0	0	0	0	0	1	0

Tableau S4								
	x	y	s1	s2	s3	s4	a1	RHS
s1	2	-2	1	0	0	0	0	6
s2	-4	1	0	1	0	0	0	4
a1	1	1	0	0	-1	0	1	10
s4	0	1	0	0	0	1	0	12
f	-1	1	0	0	0	0	0	0
F	-1	-1	0	0	1	0	0	-10

Tableau S5							
	x	y	s1	s2	s3	s4	RHS
x	1	0	0	-0,2	-0,2	0	1,2
s1	0	0	1	0,8	-1,2	0	21,2
y	0	1	0	0,2	-0,8	0	8,8
s4	0	0	0	-0,2	K3	1	3,2
f	0	0	0	-0,4	0,6	0	-7,6

SOLUTION:

a)

Iteration	Tableau	Point	x	y	f
1ª iteration (original)	**S4**	A	**0**	**0**	0
2ª iteration	**S1**	**C**	**3**	**0**	-3
3ª iteration	S3	**D**	**6,5**	**3,5**	**-3**
4ª iteration	**S5**	**F**	**1,2**	**8,8**	**7,6**
5ª iteration (Final)	**S2**	**G**	**2**	**12**	10

b)

Coefficient	Value
k1	2 (ou >0)
k2	-0,25 (ou <0)
k3	0,8 (ou >0)

e) In this case, the optimal solution would be tableau S3 (vertex D). The solution is not unique, since the variable S3 is non-basic and has null coefficient in the objective function. Besides, we also can see in the graphical solution that the objective function is parallel to constraint (I). Vertex H is an alternative optimal solution.

EXERCISE 11

Consider a particular LP problem that was solved by the Simplex method. The tables below show the initial tableau and the optimal tableau obtained during the resolution, in which:

• x1, x2 and x3 are the three decision variables of the problem;

• s1 and s2 represent slack variables;

• a1 and a2 represent auxiliary variables;

• α, β, σ, ϒ, θ ≥ 0.

	x_1	x_2	x_3	s_1	s_2	a_1	a_2	
	4	1	5	0	1	0	0	3,2
	3,5	4	3	-1	0	1	0	3,5
	1	1	1	0	0	0	1	1
(-z)	28	30	20	0	0	0	0	0

	x_1	x_2	x_3	s_1	s_2	
	1	0	0	1	-4	0,2
	0,5	1	0	0	-1	0,5
	0,5	0	1	0	1	0,5
(-z)	α	β	σ	ϒ	θ	- 25

a) Formulate the LP problem that originated the initial table. Indicate, justifying, whether it is a problem of minimization or maximization.

b) Indicate the optimal solution of the problem and its objective function.

c) Assign a value to the parameters α, β, σ, Y and θ such that:

   I. Increasing a unit to the right side of the 2nd restriction will increase the objective function to 35.

   II. It is necessary to reduce the coefficient of the objective function of x1 to a value lower than 25 to enter the basis.

SOLUTION

a) Min z = 28 x_1 + 30 x_2 + 20 x_3
   Subject to
   4 x_1 + x_2 + 5 x_3 ≤ 3.2
   3.5 x_1 + 4 x_2 + 2 x_3 ≥ 3.5
   x_1 + x_2 + x_3 = 1
   x_1 ≥ 0 , x_2 ≥ 0, x_3 ≥ 0
   It is a minimization problem (all the coefficients in the objective function are non-negative).

b) x1 = 0, x2 = 0,5, x3 = 0,5 (z = 25)

c) α = 3, β = 0, σ = 0, ϒ = 0 e θ = 10

EXERCISE 12

An automobile producer company can produce four products denoted P1,P2,P3,P4. Each product may be processed in each of two factories. Nevertheless, due to human resources and machinery differences, the production lead times differ, as shown in Table 2:

Table 2 - Factory Lead Times, in hours of labor per unit

	P_1	P_2	P_3	P_4
Factory 1	3	4	8	6
Factory 2	6	2	5	8

Each factory has 400 hours of labor available. The profit margins are 4,6,10 and 9 thousand euros per each unit of P_1, P_2, P_3 e P_4 produced, respectively. Everything that is produced may be sold. The problem formulation is the following:

Max $4P_1 + 6P_2 + 10P_3 + 6P_4$

s.a.

$3P_1 + 4P_2 + 8P_3 + 6P_4 <= 400$

$6P_1 + 2P_2 + 5P_3 + 8P_4 <= 400$

$P_1, P_2, P_3, P_4 >= 0$

Introducing slack variables F_1 and F_2 we applied the simplex method and obtained the final tableau presented in Table 3.

Table 3 – Final Simplex tableau

Base	P_1	P_2	P_3	P_4	F_1	F_2	Valor
?	3/4	1	2	3/2	1/4	0	100
?	9/2	0	-9	-11	-1/2	1	200
F	-1/2	0	-2	-3	-3/2	0	-600

a) Perform one single iteration of Simplex method (start on the initial Simplex tableau)

b) Based on the final tableau presented on Table 3, please find the optimal production plan.

c) There was an error in the production module of the ERP and 20 units of P_3 were produced. The company's board demands compensation from the ERP provider. What would be an adequate compensation value?

d) The company's board wants to assess if it is worth duplicating the production capacity of Factory 1. The optimal production plan was made for a monthly period and that this investment will cost 20 million euros. How many months would be necessary to recover the investment?

SOLUTION

a) Initial tableau

Base	P_1	P_2	P_3	P_4	F_1	F_2	Valor
F1	3	4	8	6	1	0	400
F2	6	2	5	8	0	1	400
F	4	6	10	6	0	0	0

1st iteration

Base	P_1	P_2	P_3	P_4	F_1	F_2	Valor
P3	3/8	1/2	1	3/4	1/8	0	50
F2	33/8	-1/2	0	17/4	-5/8	1	150
F	1/4	1	0	-3/2	-5/4	0	-500

b) max f = 600 k euros; $P_1 = 0$, $P_2 = 100$, $P_3 = 0$, $P_4 = 0$. P_2 and F_2 are in the basis.

c) The reduced cost of P_3 is -2 (f row in the final tableau). Hence, the effect on the profit of producing units of P_3 is $-2P_3$. If 20 units were produced by mistake, then the profit would decrease by 2*20 k euros than the optimal value found previously. The compensation from the ERP provider should be at least this value.

d) The shadow price corresponding to the capacity increase in Factory 1 is 1.5 k euros. Doubling the capacity implies having an extra monthly profit of 400*1.5 = 600 k euros (in total, 1.2 millions euros, twice the current profit). Therefore, we would need around 20/1.2 = 16.66 = 17 months.

EXERCISE 13

The tableau given below corresponds to a maximization problem with decision variables

$x_j \geq 0 \ (j = 1,2, \dots, 5)$:

Basis	x_1	x_2	x_3	x_4	x_5	Values
x_3	-1	a_1	1	0	0	4
x_4	a_2	-4	0	1	0	1
x_5	a_3	3	0	0	1	b
$-f$	c	-2	0	0	0	-10

Identify conditions on all five unknowns a_1, a_2, a_3, b, and c, such that the following statements are true.

a) The current solution is optimal and unique.
b) The current solution is optimal and has multiple optimal solutions.
c) The current solution is optimal and unbounded.
d) The current solution is optimal and degenerated.
e) The current solution is not optimal (assume that $b \geq 0$). Indicate the variable that enters the basis and the variable that leaves the basis.

SOLUTION

a) c < 0, b >= 0, a1, a2, a3 ∈ ℝ

b) c = 0, b >= 0, a1∈ ℝ, 1/a2 >= 0 ∨ b/a3 >= 0

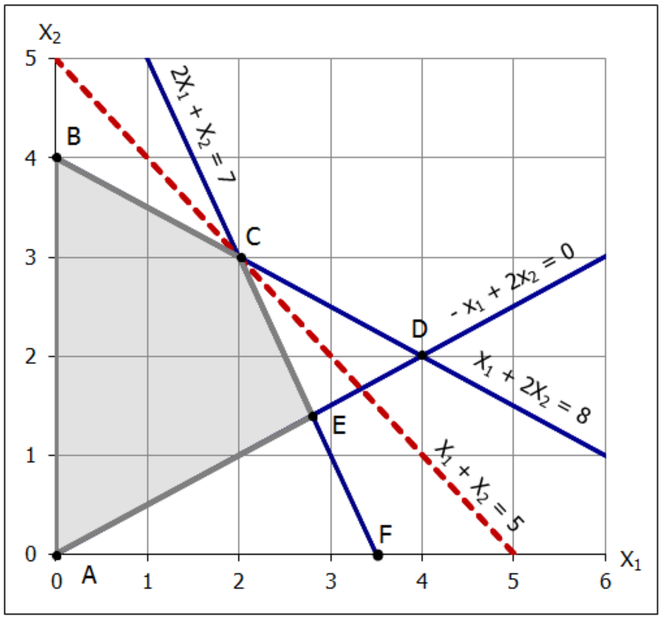
c) c > 0, b >= 0, a2, a3 <= 0, a1 ∈ ℝ

d) b = 0, c < 0, a1, a2, a3 ∈ ℝ

e) c > 0, x1 enters.
X4 exits if 1/a2 >0 ∧ 1/a2< b/a3, if b/a3 >0.
X5 exits if b/a3>0 ∧ b/a3 < 1/a2, 1/a2 >0

EXERCÍCIO 14

Consider the LP represented graphically in the following figure, in which the shadowed region depicts the feasible solutions space, and the dotted line depicts the objective function at the optimal solution. Points A-F represent the intersection of the constraints in the graphic.



a) Formulate the linear programming problem represented in the figure, identifying clearly:
   - Decision variables;
   - Objective function;
   - Constraints.

b) Identify the lines inside which the slope of the objective function can vary (by changing the coefficients of the objective function) without changing the optimal solution. Justify your answer.

c) Consider the Simplex tableau presented in below, that was obtained through the resolution of the same LP problem. Identify in Figure 1 which is the point corresponding to the solution presented in the tableau.

	x_1	x_2	s_1	s_2	s_3	Valor
x_2	0	1	0.2	0	-0.4	1.4
s_2	0	0	-0.8	1	0.6	2.4
x_1	1	0	0.4	0	0.2	2.8
-Z	0	0	-0.6	0	0.2	4.2

d) Identify in the graphic that is the solution that would be obtained after the tableau presented in Figure 2. Justify your answer without performing an iteration of the Simplex method.

SOLUTION

a) Decision variables: x1, x2
   Objective function: Maximize $X_1 + X_2$
   Constraints:
   $2X_1 + X_2 \leq 7$

$X_1 + 2X_2 \leq 8$

$-X_1 + 2X_2 \geq 0$

$X_1, X_2 \geq 0$

b) The objective function (with slope=-1) can vary between the slope of lines $2x_1 + x_2 = 7$ (slope = -2) and $x_1 + 2x_2 = 8$ (slope = -0.5). If the slope is less than -2, the optimal solution would be E and if the slope is higher than -0.5, the optimal solution would be B.

c) The solution represented in Figure 2 is vertex E.

d) The solution of the next Simplex tableau is vertex C. There are two possible justifications:

   i. S3 is entering the basis and S2 is leaving the basis. Therefore, $-X_1 + 2X_2 = 0$ will become a non-active constraint (not binding) and $X_1 + 2X_2 = 8$ will become active (binding), resulting in point C.

   ii. C in the only vertex in the feasible region with a objective function value higher than E. As the problem is convex and we are maximizing, C is the only possible solution.

EXERCISE 16

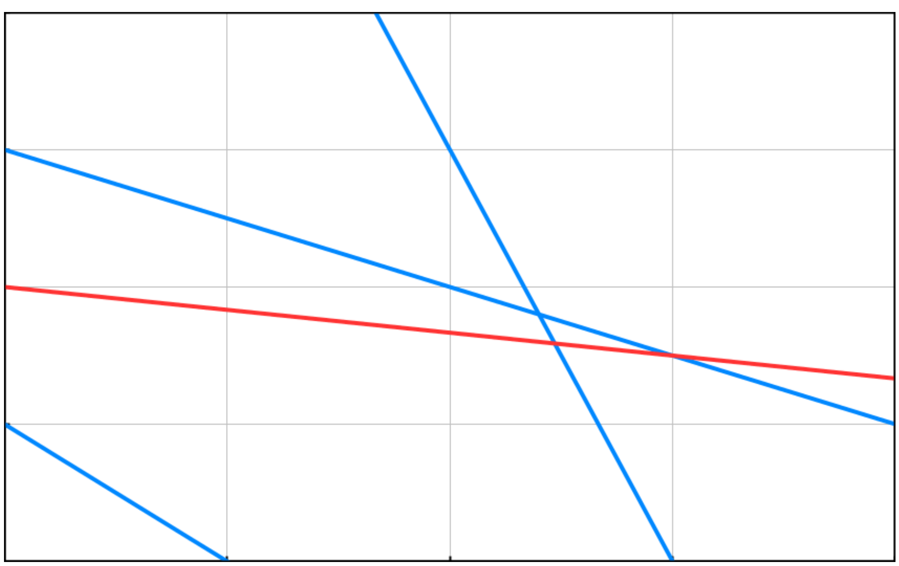
Consider the following linear programming problems and the graphical solution presented below.

Max $Z = x_2 + \frac{1}{6}x_1$

$x_2 + x_1 \geq 1$ (1)

$x_2 + \frac{1}{2}x_1 \leq 3$ (2)

$x_2 + 3x_1 \leq 9$ (3)



Classify the following sentences as True or False. Justify your answer.

a) The optimal solution of this problem is: $x_2 = 1{,}8;\ x_1 = 2{,}4$.
b) It will be necessary to user the two-phases method in order to apply the Simplex method to this problem.
c) The reduced cost of x_2 in the optimal solution is 0.
d) The shadow price of constraint (3) in the optimal solution is 3.
e) If the problem were a minimization problem, we would have a degenerated optimal solution.
f) If constraint (2) was replaced by $x_2 \leq 2$, the optimal solution would change.

SOLUTION:

a) F. $x_2 = 3;\ x_1 = 0$.
b) T. The point (0,0) does not belong to the feasible solution region.
c) T. x_2 é basic.
d) F. The shadow price is null.
e) F. Single solution. $x_2 = 0;\ x_1 = 1$.
f) T. $x_2 = 2;\ x_1 = 2$.

EXERCISE 17

Consider a linear programming problem (LP), in which the decision variables X and Y represent the number of wafer packages of type X and Y to be produced.

To produce these wafers, 50 kilograms of cream, 120 kilograms of flour and an unlimited stock of sugar are available for the producer. Each package of wafers of type X requires a total of 2 kilograms of sugar and 3 kilograms of flour. Each package of type Y needs only 1 kilogram of cream and 1 kilogram of flour. While producing each package of wafers of type X, 2 kilograms of cream are generated with the sugar used.

To promote a new type of wafers, the producer decided to offer his customers at least 90 samples of the new wafers. He has decided to offer 6 wafers with each package of type X and 1 wafer with each package of type Y. Due to business issues the producer does not want to produce more than 20 packages of wafers of type X.

The constraints of the problem and the corresponding lines are shown below.

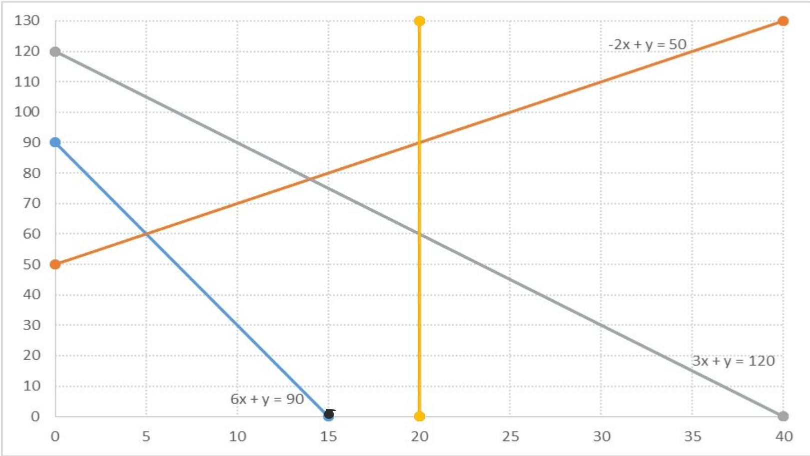
-2X + Y <= 50 (A)

3X + Y <= 120 (B)

6X + Y >= 90 (C)

X <= 20 (D)

X, Y >= 0



a) Indicate on the graph the area of feasible solutions. (Note: You will have to copy the graph to your exam sheet)

b) Assuming that each package of wafers of type Y has a cost of 2 cents, define an objective function that results in point P as the single production plan with the lowest cost.

c) Now consider that the aim of the producer is to maximize his profit regarding the sales of wafers of type X and Y with the following objective function: Max 1.5X + Y.

   i. Identify in the graph the optimal solution of this maximization problem and indicate its value (without making iterations in Simplex tableaux).

   ii. Write the Simplex initial tableau and identify in the graph the corresponding point.

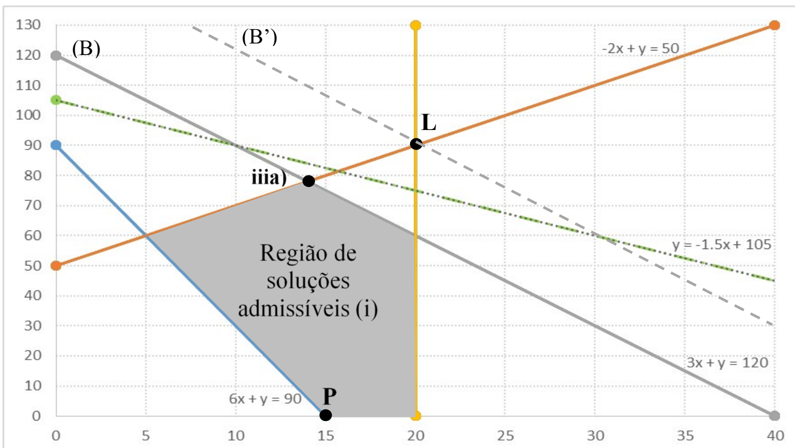
   iii. Starting from the tableau given below, perform a single iteration of the Simplex method and indicate its new point.

Basis	X	Y	S1	S2	S3	S4	RHS
S1	0	4/3	1	0	-1/3	0	80
S2	0	1/2	0	1	1/2	0	75
X	1	1/6	0	0	-1/6	0	15
S4	0	-1/6	0	0	1/6	1	5
f	0	3/4	0	0	1/4	0	-22.5

iv. Indicate what would be the optimal solution if the producer withdrew the maximum number of packages of wafers of type X to be produced? Justify.

v. The flour supplier informs the producer that he is doing a promotion campaign and the producer can buy more flour (in addition to the 120 kilograms of flour that he already has) for a cheaper price. In this promotion, each 40 kilogram bag of flour costs 10 cents. Should the producer take advantage of the promotion? If yes, how many bags should he buy?

SOLUTION



ii) Considering this problem aims at minimizing the production costs, the optimal solution will be point P if the slope of the objective function lies between the slope of the lines representing constraints C (blue) and Y >=0. Being C1 and C2 the coefficients of X and Y in the objective function, respectively:

Slope (6X+Y = 90) <= Slope (Z=C1X+C2Y) <= Slope (Y=0)

$$-6 \leq -C1/C2 \leq 0$$

Setting C2=2, 12>=C1>=0.

Hence, we can define the objective function as “Min 1X+2Y”, for example.

iii) Max 1.5X + Y.

a) Drawing in the graphic a line representing the objective function (green), we verify that the optimal solution of this maximization problem is the intersection of constraints (A) and (B) (Traçando no gráfico a reta da função objetivo (verde), verificamos que a solução ótima deste problema de maximização é o ponto de interseção entre as retas das restrições (A) e (B). The optimal solution is the point (14.78) and the optimal value of the objective function is 99.

$$\begin{cases} -2X+Y=50 \\ 3X+Y=120 \end{cases} \quad \begin{cases} X=14 \\ Y=78 \end{cases} \qquad FO=1.5x14+1x78=99$$

b) When we write the original problem in the canonical form, we can see that the point (0,0) is not a feasible solution, since S3=-90. Hence, we have to solve an auxiliary problem to get a feasible solution for the original problem.

Original problem in canonical form

Max f = 1.5X + Y

-2X + Y + S1 = 50

3X + Y + S2 = 120

6X + Y - S3 = 90

X + S4 = 20

X, Y, S1, S2, S3, S4 >= 0

Auxiliary problem in canonical form

Min F= 90 - 6X - Y + S3

-2X + Y +S1 = 50

3X + Y + S2 = 120

6X + Y - S3 + A1= 90

X + S4 = 20

X, Y, S1, S2, S3, S4, A1 >= 0

The initial Simplex tableau is:

Base	X	Y	S1	S2	S3	S4	A1	RHS
S1	-2	1	1	0	0	0	0	50
S2	3	1	0	1	0	0	0	120
A1	6	1	0	0	-1	0	1	90
S4	1	0	0	0	0	1	0	20
f	1.5	6	0	0	0	0	0	0
F	-6	-1	0	0	1	0	0	-90

c) The constraint representing the maximum number of packages of type X wafers is (D), represented n yellow. In this maximization problem, this constraint is not binding (active). Therefore, if we remove this constraint, the solution will remain the intersection of constraints (A) and (B) with the same optimal solution.

d) The constraint regarding the flour (B) is active, so it does not have slack. If we increase the amount of flour available, the value of the objective function in the optimal solution could increase. In order to understand if it would be advantageous to buy the bags of flour by 10 cents, we have to calculate the shadow price of constraint (B).

$$\begin{cases} -2X+Y=50 \\ 3X+Y=121 \end{cases} \quad \begin{cases} X=14.2 \\ Y=78.4 \end{cases} \quad \begin{array}{l} FO=1.5x14.2+1x78.4=99.7 \\ \text{Shadow price (B)} = 99.7-99=0.7 \end{array}$$

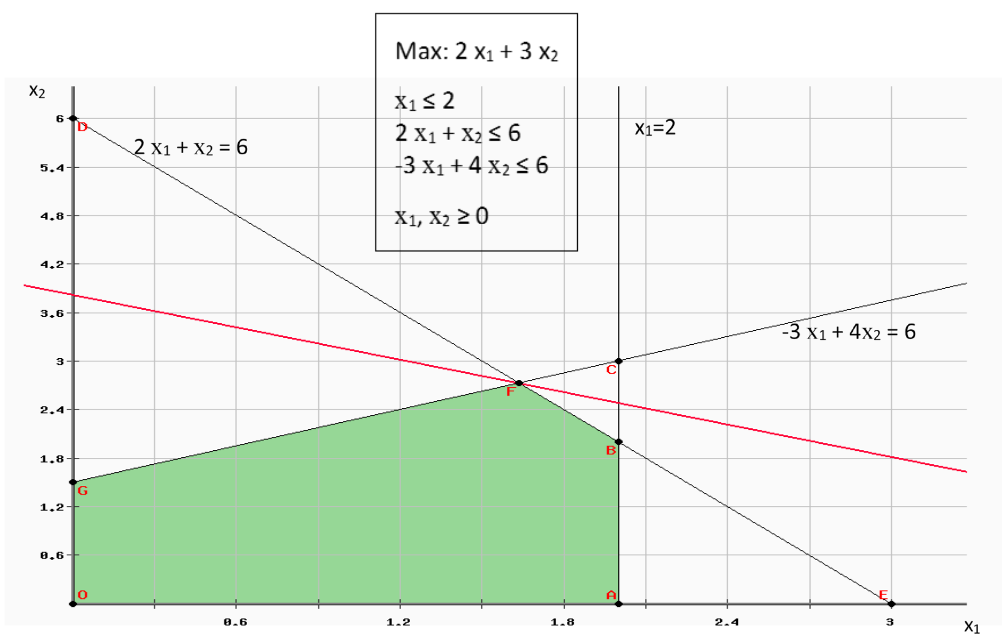
For each additional kg of flour, the objective function will increase 0.7 cents. However, this shadow price is only valid if the optimal solution is bound by the same set of constraints. This means that the amount of flour that is used can increase while this constraint is below line (B'). In the figure we can see that the intersection point of (B') with (A) is point L (20,90). Using this point, we can determine that the amount of flour can increase until 3*20 + 90 =150 kg.

Therefore, it would be advantageous for the producer to buy 150-120 = 30 kg of flour if the price per kg is inferior to 0.7 cents, in a total of 30*0.7=21 cents. If the producer buys more than 30 kg of flour, the constraint becomes inactive and the shadow price would be zero.

If the producer buys a bag of flour with 40 kg by 10 cents he would have an additional profit of 11 cents. In fact, the profit would be given by the value of f in point L and would be f* = 1.5*20+ 90 = 120. We have to discount the cost of the flour bag (10 cents) so we would have 120-10 = 110. Comparing with the original problem (f=99), the additional profit of buying 40kg by 10 cents would be 110-99 = 11 cents.

EXERCISE 18

Consider the linear problem represented below.



Consider the following set of statements. Choose the alternative that you consider the most correct. Each wrong answer will have a penalty of 25%.

1. In the Simplex method, when converting a "≤" constraint into a "=":
   a) We must add an artificial variable
   b) We must add a slack variable
   c) We must subtract a slack variable
   d) We must add a slack variable and an artificial variable
   e) It depends if the problem is a maximization or a minimization problem.

2. A simplex tableau for a maximization LP corresponds to the optimal solution when the last row (the coefficients of the objective funciton):
   a) Does not have null values
   b) Does not have positive values
   c) Does not have negative values
   d) All the values are null
   e) None of the previous alternatives

3. If, when applying the Simplex method to a maximization problem, all the calculated ratios for choosing the leaving variable are negative, then we can conclude that:
   a) The problem does not have feasible solutions
   b) The problem has a degenerated solution
   c) The problem has infinite optimal solutions.
   d) The problem is unbound
   e) The optimal solution has been found

4. For the problem represented above, the vertex sequence of the application of the Simplex method is:
   a) O → A → B → C → F
   b) A → B → F
   c) O → G → F
   d) O → A → B → F
   e) O → G → D → F
5. The optimal solution of the problem represented above is point F. In this point, the active constraints (no slack) are:
   a) "$x_1 \leq 2$" and "$2\ x_1 + x_2 \leq 6$"
   b) "$x_1 \leq 2$" and "$-3\ x_1 + 4\ x_2 \leq 6$"
   c) "$2\ x_1 + x_2 \leq 6$" and "$-3\ x_1 + 4\ x_2 \leq 6$"
   d) All the constraints are active
   e) None of the constraints is active

6. The following Simplex tableau was obtained for the problem represented above.

Base	X1	X2	X3	X4	X5	b
X3	1	0	1	0	0	2
X4	2,75	0	0	1	-0,25	4,5

X2	-0,75	1	0	0	0,25	1,5
-Z	4,25	0	0	0	-0,75	-4,5

a) The tableau corresponds to point A
b) The tableau corresponds to point B
c) The tableau corresponds to point C
d) The tableau corresponds to point F
e) The tableau corresponds to point G

7. If the objective function changes to max $-x_1+3x_2$, i.e., if the x_1 coefficient in the objective function becomes -1,
   a) The optimal solution is the same
   b) The optimal solution is now G
   c) The optimal solution is now C
   d) The problem does not have feasible solutions
   e) The problem has infinite optimal solutions

8. If constraint "$x_1 \leq 2$" is replaced by constraint "$x_1 \geq 1$":
   a) The basic variables in the optimal solution remain the same
   b) The basic variables in the optimal solution are different
   c) The optimal solution is degenerated
   d) The problem does not have feasible solutions
   e) The optimal solution is now B

SOLUTION

1. b)
2. b)
3. d)
4. c)
5. c)
6. e)
7. a)
8. a)

EXERCISE 19

Meireles Company manufactures two different types of cars (Cabriolet and Standard), with a unitary net profit of e €160 e €200, respectively. The factory has 20 m² available for storage. The cabriolet version needs 1 m² while the standard version needs 2 m². Additionally, both cars need 18 units of steel and the amount of steel available for production 216 units. The work in the factory is limited to 240 h/month. To produce a Cabriolet car, 24 hours of labor are needed and, in order to produce a standard car, 16 hours are required. Finally, the factory needs to produce at least 3 Standard cars in order to accomplish the contract.

The manager intends to calculate the number of units of each type of car to produce in a month in order to maximize the profit.

He proposes the following formulation for the problem.

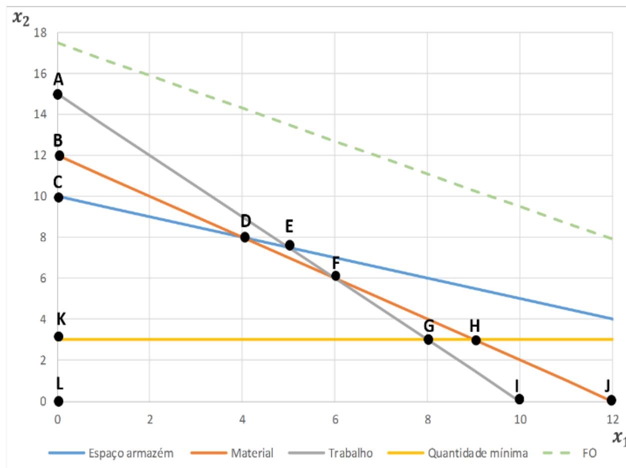
$$maximize\ Z = 160x_1 + 200x_2$$

$$1x_1 + 2x_2 \leq 20 \text{ (storage)}$$

$$18x_1 + 18x_2 \leq 216 \text{ (steel)}$$

$$24x_1 + 16x_2 \leq 240 \text{ (worl)}$$

$$x_2 \geq 3$$



a) Find the points in the figure that belong to the feasible solutions area (from A to L).

b) Present the first Simplex tableau.

c) Consider the following tableau. Perform an iteration (indicate all calculations) and identify the corresponding point in the figure. Is that the optimal solution??

	X1	X2	S1	S2	S3	S4	
S1	1	0	1	0	0	2	14
S2	18	0	0	1	0	18	162
S3	24	0	0	0	1	16	192
X2	0	1	0	0	0	-1	3
Z	160	0	0	0	0	200	-600

d) Consider the following tableau:

	X1	X2	S1	S2	S3	S4	
S4	0	0	1	-1/18	0	1	5

X1	1	0	-1	1/9	0	0	4
S3	0	0	8	-16/9	1	0	16
X2	0	1	1	-1/18	0	0	8
Z	0	0	-40	-20/3	0	0	-2240

d1. Indicate the point corresponding to this solution. Is the solution of this tableau optimal? Justify your answer.
d2. Are all the resources being totally used in this scenario? Justify your answer.
d3. There is the possibility of renting 1 m² for storage to someone for a value of 30€, reducing the space available for the cars. Do you think that the manager should take this opportunity? Justify your answer.

e) Between which lines can the slope of the objective function vary, maitaing the same optimal solution? Justify your answer.

SOLUTION

a) C, D, F, G, K

b) First tableau:

	X1	X2	S1	S2	S3	S4	A1	
S1	1	2	1	0	0	0	0	20
S2	18	18	0	1	0	0	0	216
S3	24	16	0	0	1	0	0	240
A1	0	1	0	0	0	-1	1	3
F	0	-1	0	0	0	1	0	-3
Z	160	200	0	0	0	0	0	0

c)

	X1	X2	S1	S2	S3	S4	
S4	1/2	0	1/2	0	0	1	7
S2	9	0	-9	1	0	0	36
S3	16	0	-8	0	1	0	80
X2	1/2	1	1/2	0	0	0	10
Z	60	0	-100	0	0	0	-2000

d)

d1. Point D. Yes, all the objective function coefficients are negative.
d2. No, resource 3 (labor) is not being totally used.
d3. No, he would loose 10€ (30€-40€).

e) Between the slopes of lines orange and blue (steel and storage).

EXERCISE 20

Consider the following linear problem:

$\min 5x1 + 3x2 - 2x3$

$s.a \quad x1 + x2 + x3 \geq 4$

$2x1 + 3x2 - x3 \geq 9$

$x1 + x2 \geq 4$
$x1, x2 \geq 0$

The following tableau corresponds to a particular iteration of the Simplex method.

	X1	X2	X3	S1	S2	S3	A1	A2	
X3	0,25	0	1	-0,75	0,25	0	0,75	-0,25	0,75
X2	0,75	1	0	-0,25	-0,25	0	0,25	0,25	3,25
S3	0,25	0	0	0,25	0,25	1	-0,25	-0,25	1,75
f	3,25	0	0	-0,75	1,25	0	0,75	0,75	-8,25
F	0	0	0	0	0	0	1	1	0

The following tableau corresponds to the subsequent iteration and also the optimal solution.

	X1	X2	X3	S1	S2	S3	
X3	1	0	1	0	1	A	6
X2	1	B	0	0	0	1	5
S1	C	0	0	1	1	4	D
f	4	0	0	0	2	3	-3

1. Find the values of A, B, C and D presented in the above tableau.
2. If the coefficient of x1 in the objective function changes from 5 to 8, which will be the new value of the objective function in the optimal solution? Justify your answer and present an economic interpretation.
3. If the independent term of the second constraint changes from 9 to 12, which will be the new value of the objective function in the optimal solution? Justify your answer and present an economic interpretation.

SOLUTION

	X1	X2	X3	S1	S2	S3	A1	A2	
A1	1	1	1	-1	0	0	1	0	4
A2	2	3	-1	0	-1	0	0	1	9
S3	1	1	0	0	0	1	0	0	5
f	5	3	-2	0	0	0	0	0	0
F	-3	-4	0	1	1	0	0	0	-13

	X1	X2	X3	S1	S2	S3	A1	A2	
A1	1/3	0	4/3	-1	1/3	0	1	-1/3	1
X2	2/3	1	-1/3	0	-1/3	0	0	1/3	3
S3	1/3	0	1/3	0	1/3	1	0	-1/3	2
f	3	0	-1	0	1	0	0	1	-9
F	-1/3	0	-4/3	1	-1/3	0	0	4/3	-1

	X1	X2	X3	S1	S2	S3	A1	A2	
X3	0.25	0	1	-0.75	0.25	0	0.75	-0.25	0.75
X2	0.75	1	0	-0.25	-0.25	0	0.25	0.25	3.25
S3	0.25	0	0	0.25	0.25	1	-0.25	-0.25	1.75
f	3.25	0	0	-0.75	1.25	0	0.75	0.75	-8.25
F	0	0	0	0	0	0	1	1	0

	X1	X2	X3	S1	S2	S3	
X3	1	0	1	0	1	**A=3**	6
X2	1	**B=1**	0	0	0	1	5
S1	**C=1**	0	0	1	1	4	**D=7**
f	4	0	0	0	2	3	-3

1. A=3, B=1, C=1, D=7
2. The reduced cost of X1 is 4 (X1=0). The reduced cost is the decrement (it is a minimization problem) of the X1 coefficient such that X1 could have a positive value in the optimal solution.
   In this case, we have an increment (from 5 to 8) hence, the objective function value does not change, and X1 remains 0.

2) Changing the RHS of the second constraint from 9 to 12, we have an increase of 3 units. The shadow price of the 2nd constraint (s2) is 2, which means that the objective function will worse (increase) from 3 to 3 + 3*2 = 9.

EXERCISE 21

A manufacturer produces two types of cork flooring. These have the trade names CorkTreePlus (CTP) and GoldenSummer (GS) and are produced in panels. One panel of CTP requires 8 kg of cork, 2.5 kg of a plastic and 2 kg of organic glue. One panel of GS needs 10 kg of cork, 1kg of plastic, and 4 kg of organic glue. The company has in stock 80,000 kg of cork, 20,000 kg of plastic, and 30,000 kg of organic glue. Both products can be produced by alternate parameter settings of the production plant, which is able to produce at the rate of 12 panels per hour. A total of 750 production plant hours are available for the next planning period. The contribution to profit on CTP is €10 per panel and on GS is €20 per panel. The company has a contract to deliver at least 3,000 panels of CTP. What production plan should be implemented to maximize the contribution to the firm's profit from this product division.

Definition of variables:

x = Number of panels of CTP produced

y = Number of panels of GS produced

$$\max \mathrm{Z} = 10x + 20\mathrm{y}$$

subject to

$$\begin{array}{llcll}
(1) & 8x + 10\mathrm{y} & \leq & 80000 & (cork\ ) \\
(2) & 2.5x + y & \leq & 20000 & (plastic) \\
(3) & 2\mathrm{x} + 4\mathrm{y} & \leq & 30000 & (organic\ glue) \\
(4) & x + y & \leq & 9000 & (plant\ capacity) \\
(5) & x & \geq & 3000 & (contract) \\
 & \mathrm{x}, \mathrm{y} & \geq & 0 &
\end{array}$$

After using a solver the following report was obtained:

Answer Report:

Cell	Objective Name	Value
$J$5	Obj. function	142000

Cell	Variable Name	Value
$F$5	Obj. function x (CTP)	3000
$G$5	Obj. function y(GS)	5600
$H$5	Obj. function	0

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$F$5	Obj. function x (CTP)	0	3000	Contin
$G$5	Obj. function y(GS)	0	5600	Contin
$H$5	Obj. function	0	0	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$I$12	contract	3000	$I$12>=$K$12	Binding	0
$I$8	cork	80000	$I$8<=$K$8	Binding	0
$I$9	plastic	13100	$I$9<=$K$9	Not Binding	6900
$I$10	organic glue	28400	$I$10<=$K$10	Not Binding	1600
$I$11	plant capacity	8600	$I$11<=$K$11	Not Binding	400

Sensitivity Report:

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$F$5	Obj. function x (CTP)	3000	0	10	6	1E+30
$G$5	Obj. function y(GS)	5600	0	20	1E+30	7.5
$H$5	Obj. function	0	0	0	0	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$I$12	contract	3000	-6	3000	2000	1333.333333
$I$8	cork	80000	2	80000	4000	56000
$I$9	plastic	13100	0	20000	1E+30	6900
$I$10	organic glue	28400	0	30000	1E+30	1600
$I$11	plant capacity	8600	0	9000	1E+30	400

1. How many panels of GS should be manufactured? ________________________________
2. How much of the available plastic will be used? ________________________________
3. How much of the available glue will be unused? ______________________________________
4. Suppose that the company can purchase 2000 kg of additional cork for €2.50 per kg. Should they make the purchase?______Why?________________________________
5. Regardless of your answer in (4), suppose that they do purchase 2000 kg of additional cork. This is equivalent to
   a) decreasing the slack in row 1 by 2000
   b) increasing the slack in row 1 by 2000
   c) none of the alternatives
   d) no sufficient information
6. 6. If the company purchases 2000 kg of additional cork, what is the total amount of GS that they should deliver? (Choose nearest value?)
   a) 5500 panels
   b) 5600 panels
   c) 5700 panels
   d) 5800 panels
   e) 5900 panels
   f) no sufficient information

7. How will the decision to purchase 2000 kg of additional cork change the quantity of plastic use during the next planning period?
   a. increase by 100 kg
   b. decrease by 200 kg
   c. decrease by 100 kg
   d. increase by 200 kg
   e. no sufficient information
   f. none of the alternatives
8. If the profit contribution from GS were to decrease to €11per panel, will the optimal solution change? _______. Justify your answer.__________________________
9. If the profit contribution from CTP were to increase to €15 per panel, will the optimal solution change?_________. Justify your answer.__________________________
10. Suppose that the company could deliver 1000 panels less than the contracted amount of CTP by paying a penalty of €5 per panel shortage. Should they do so?_________. Justify your answer.__________________________

SOLUTION

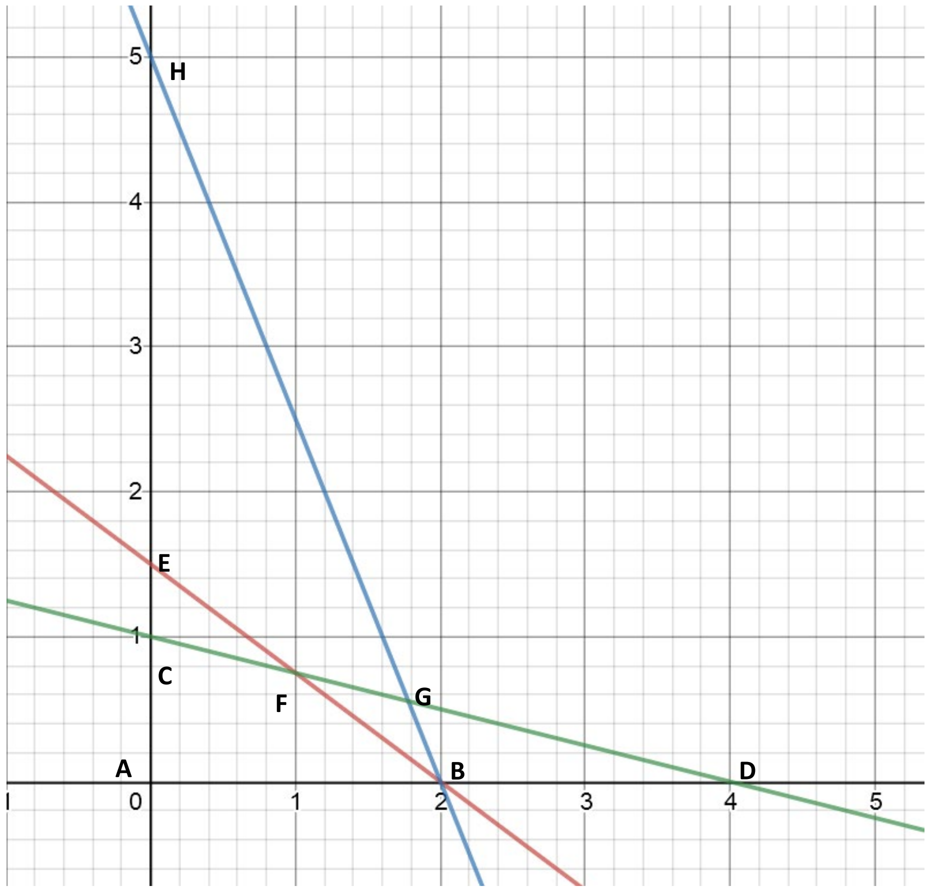
1. 5600
2. 13100
3. 1600
4. No, because the dual price is 2, inferior to the price they would by the cork (2.5)
5. a)
6. d)
7. d)
8. Yes, the allowable decrease is 7.5
9. No, the allowable increase is 6
10. Yes, the shadow price of the contract constraint is -6

EXERCISE 23

Consider the following LP:

$$\min Z = 8x_1 + 4x_2$$

$$\begin{array}{lrcl} \text{subject to} & 3x_1 + 4x_2 & \geq & 6 \\ & 5x_1 + 2x_2 & \leq & 10 \\ & x_1 + 4x_2 & \leq & 4 \\ & x_1, x_2 & \geq & 0 \end{array}$$



1. Which are of the vertices displayed in the graphical solution are feasible?
2. Where is the optimal solution?

EXERCISE 22

1. The following linear problem has:

$$\max Z = 15x_1 + 20x_2$$

$$\text{subject to} \quad \begin{aligned} 12x_1 + 4x_2 &\geq 36 \\ 12x_1 - 6x_2 &\leq 24 \\ x_1, x_2 &\geq 0 \end{aligned}$$

a) Infeasible solution
b) Unbounded solution
c) Multiple optimal solutions
d) Degenerate solution

2. The following linear problem has:

$$\max Z = 3x_1 + 2x_2$$

$$\begin{aligned} \text{subject to } -2x_1 + 3x_2 &\leq 9 \\ x_1 - 5x_2 &\geq -20 \\ x_1, x_2 &\geq 0 \end{aligned}$$

a) No feasible solution
b) Unbounded solution
c) Multiple optimal solutions
d) Degenerate solution

3. The following linear problem has:

$$\max Z = 3x_1 + 7x_2$$

$$\begin{aligned} \text{subject to } 3x_1 + 7x_2 &\leq 10 \\ 4x_1 + 6x_2 &\leq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$

a) Exactly one optimal solution
b) Unbounded solution
c) Multiple optimal solutions
d) No feasible solutions

4. The following linear problem has:

$$\max Z = 3x_1 + 2x_2$$

$$\begin{aligned} \text{subject to } x_1 &\leq 4 \\ x_2 &\leq 6 \\ 3x_1 + 2x_2 &\leq 18 \\ x_1, x_2 &\geq 0 \end{aligned}$$

a) Exactly one optimal solution
b) Unbounded solution
c) Multiple optimal solutions
d) No feasible solutions

SOLUTION

1. b
2. b
3. a
4. c