Optimization

Lecture 4.1

M.EIC – 2021.2022

SIMPLEX METHOD

Linear Programming

- First stated in this form by **George B. Dantzig**, it is an amazing fact that literally thousands of decision (programming) problems from business, industry, government and the military can be stated (or approximated) as linear programming problems.
- Although there were some precursor attempts at stating such problems in mathematical terms, notably by the Russian mathematician **Leonid V. Kantorovich** in 1939, Dantzig's general formulation, combined with his method of solution, the Simplex Method, revolutionized decision making.
- The name "linear programming" was suggested to Dantzig by the economist **Tjalling C. Koopmans**.
- Both Kantorovich and Koopmans were awarded the 1975 Nobel prize in economics for their contributions to the theory of optimum allocation of resources.

The untold story

- Most people familiar with the origins and development of linear programming were amazed and disappointed that Dantzig did not receive the Nobel prize along with Koopmans and Kantorovich (a Nobel prize can be shared by up to three recipients).
- Shortly after the award, Koopmans talked about his displeasure with the Nobel selection and told he had earlier written to Kantorovich suggesting that they both refuse the prize, certainly a most difficult decision for both, but especially so for Kantorovich who was not recognized in URSS….

Kantorovich said :

- “*In the spring of 1939 I gave some more reports – at the Polytechnic Institute and the House of Scientists, but several times met with the objection that the work used mathematical methods, and in the West the mathematical school in economics was an anti-Marxist school and mathematics in economics was a means for apologists of capitalism.*”

Linear Programming and the Simplex method were explained by George Dantzig in 1948 at a meeting held at the University of Wisconsin.

In the discussion after his lecture, someone from the audience said:

"Yes, but... we all know the world is nonlinear..."

John von Neumann, who was also there, stood up and said:

"Mr. Chairman, Mr. Chairman,
if the speaker does not mind, I would like to reply for him.
The speaker titled his talk 'linear programming' and carefully stated his axioms.
If you have an application that satisfies the axioms, well use it.
If it does not, then don't."

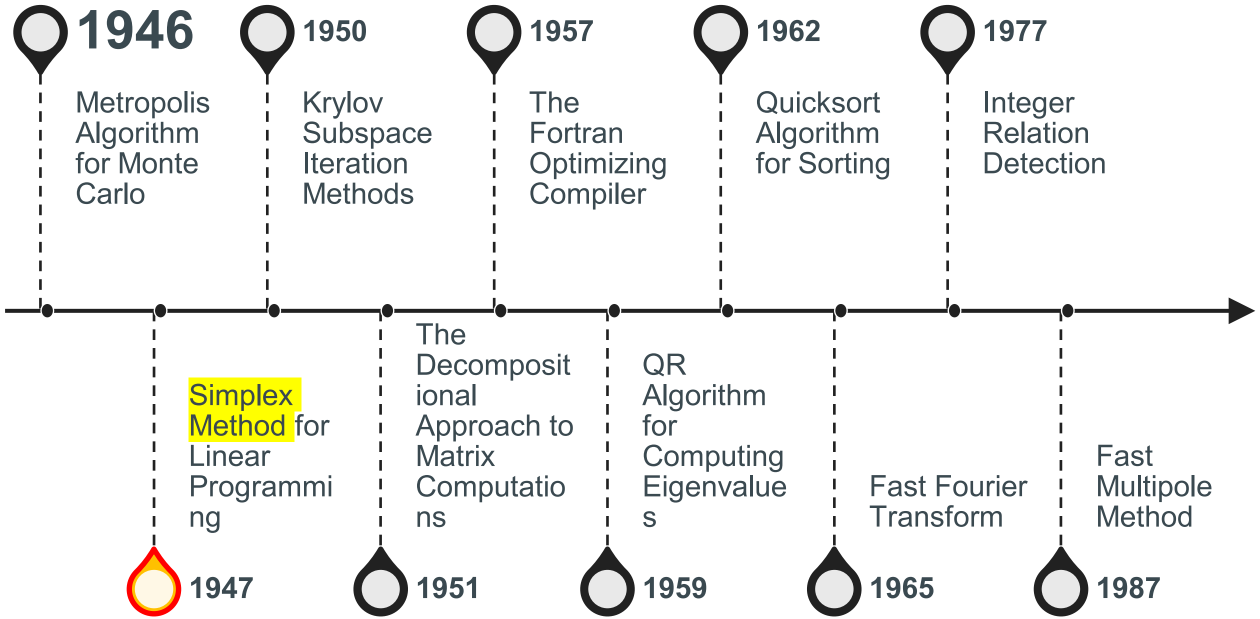
John von Neumann *(1903-1957) was a Hungarian-American mathematician, physicist, inventor, computer scientist. He was a pioneer of quantum mechanics and of concepts of cellular automata, the universal constructor and the digital computer.*

After this episode, Dantzig's colleagues decided to hang this cartoon outside his office…

Top Ten Algorithms of the XXth Century

Computing in Science& Engineering, a joint publication of the American Institute of Physics and the IEEE Computer Society

January/February 2000



Simplex Method

Motivation:

The graphical method cannot be applied to problems with more than 2 variables.

Basic Idea:

The Simplex method is based in the fact that any LP optimal solution lies on a vertex of the feasible region.

Basic Method:

- Start by calculating the objective function value for any vertex of the domain.
- Jump to an adjacent vertex corresponding to a better objective function value.
- Continue with this process until it is no possible to improve the objective function.

Cereals, Ltd

Cereals, Ltd is a company specialized in preparing and packing wheat and corn for several retailing stores. There are no limits concerning the supply of these cereals and demand can be considered unlimited (in other words, the whole production is sold).
The production is processed in three phases: Pre-Processing (I), Processing (II) and Packing (III).

	I	II	III	
	Pre-Processing	Processing	Packing	
Weekly production capacity (h)	120	100	150	
Time (h) needed to prepare 1 ton				Profit(€/ton)
Wheat	6 h	1 h	5 h	4
Corn	2 h	4 h	5 h	3

Cereals, Ltd - Formulation

Decision variables

x = tons of wheat to produce weekly

y = tons of corn to produce weekly

Constraints

$$6x + 2y \leq 120$$

$$x + 4y \leq 100$$

$$5x + 5y \leq 150$$

$$x \geq 0, y \geq 0$$

Objective function: to maximize the profit $\max 4x + 3y$

Simplex Method

Step 1: Writing the problem in the canonical form

Consider that all the inequalities (constraints) are of <= type with positive values in the right side. The LP is in canonical form when the inequalities are changed to equalities by adding a **slack variable** to each constraint.

Maximizar (ou Minimizar) $f = c_1 \cdot x_1 + c_2 \cdot x_2 + \dots + c_n \cdot x_n$

sujeito a

$$a_{11} \cdot x_1 + a_{12} \cdot x_2 + \dots + a_{1n} \cdot x_n = b_1$$
$$a_{21} \cdot x_1 + a_{22} \cdot x_2 + \dots + a_{2n} \cdot x_n = b_2$$
$$\dots\dots$$
$$a_{m1} \cdot x_1 + a_{m2} \cdot x_2 + \dots + a_{mn} \cdot x_n = b_m$$

Com: $m < n$ restrições

$b_1, b_2, \dots, b_m \geq 0$

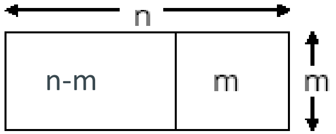
$x_1, x_2, \dots, x_n \geq 0$

$$\max f = 4x_1 + 3x_2$$

$$6x_1 + 2x_2 + s_1 = 120$$
$$x_1 + 4x_2 + s_2 = 100$$
$$5x_1 + 5x_2 + s_3 = 150$$
$$x_1 \geq 0, x_2 \geq 0$$

Simplex Method

Step 2: Find an initial feasible basic solution



Consider a LP in the canonical form with
- n variables
- m constraints (equalities), with m<n

A basic solution is obtained by assigning n−m variables to zero and solving the constraints (equations) for the remaining variables (m).

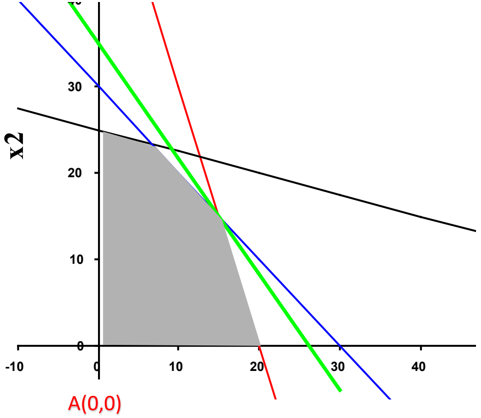
The n−m null variables are referred to as non-basic variables.
The others are the basic variables.

The basic solutions can be:

- feasible basic solutions: when all the basic variables are non-negative.
- infeasible basic solutions: when at least one basic variable is negative.

$$\max f = 4x_1 + 3x_2$$

$$6x_1 + 2x_2 + s_1 = 120$$
$$x_1 + 4x_2 + s_2 = 100$$
$$5x_1 + 5x_2 + s_3 = 150$$
$$x_1 \geq 0, x_2 \geq 0$$



Tabular form

basis	X1	X2	S1	S2	S3	value
S1	6	2	1	0	0	120
S2	1	4	0	1	0	100
S3	5	5	0	0	1	150
f	4	3	0	0	0	0

Basic variables: S1 = 120, S2 = 100, S3 = 150

Non-basic variables: X1 = 0, X2 = 0

Point A

- The coefficient matrix of the basic variables is an identity matrix (or is convertible into one by row or column swaps)
- The coefficients of the basic variables in the objective function are null.

Simplex Method

Step 3: Verify if the basic solution found is optimal:

For a maximization problem, if all the coefficients in the objective function are non-positive (<=0), then the problem is solved.
For a minimization problem, if all the coefficients in the objective function are non-negative (>=0), then the problem is solved.

basis	X1	X2	S1	S2	S3	value
S1	6	2	1	0	0	120
S2	1	4	0	1	0	100
S3	5	5	0	0	1	150
f	4	3	0	0	0	0

- At this moment, the value of x1 and x2 is zero (and also the o.f. value).
- As both coefficients in the objective function (o.f.) are positive numbers, if any of these variables becomes positive, the value of the o.f. would increase (f = 4 x1 + 3 x2).

Which of these two variables should be chosen to enter the basis (become positive)?

Simplex Method

Step 4: Find a new basic solution that improves the objective function

To find a new basic solution, we will choose a **non-basic variable to enter the basis** and a **basic variable to leave the basis.**

Step 4.1: Choose a non-basic variable to enter the basis . The column corresponding to this variable is called *pivot column*.
In a maximization problem choose, amongst the variables with positive coefficients, the one with the highest positive value. In a minimization problem choose, amongst the variables with negative coefficients, the one with the highest negative value

Step 4.2 Choose a basic variable to leave the basis, The row corresponding to this variable is called *pivot row*.
Calculate the ratio between the right-side members of the equations and the corresponding members in the pivot column. From the set of non-negative ratios, the row with the lowest ratio will be the pivot row.

	basis	X1	X2	S1	S2	S3	value		
L1	S1	6	2	1	0	0	120	120/6 =	20
L2	S2	1	4	0	1	0	100	100/1 =	100
L3	S3	5	5	0	0	1	150	150/5 =	30
f	f	4	3	0	0	0	0		

Pivot row

Choose to leave the basis the variable with the lowest non-negative ratio.

Pivot column

In a maximization problem, choose to enter the basis the variable with the highest positive value in the o.f.

-f = 0

This value is the symmetric of the o.f. value for the current solution.

x1 was chosen to enter the basis, keeping x2 = 0 (non-basic).

x1 should take the highest possible value while satisfying the constraints.

$$s_1 = 120 - 6x_1 \quad \text{Since } s_1 \geq 0, 120 - 6x_1 \geq 0 \Leftrightarrow \boxed{x_1 \leq \frac{120}{6} = 20}$$

$$s_2 = 100 - x_1 \quad \text{Since } s_2 \geq 0, 100 - x_1 \geq 0 \Leftrightarrow x_1 \leq 100$$

$$s_3 = 150 - 5x_1 \quad \text{Since } s_3 \geq 0, 150 - 5x_1 \geq 0 \Leftrightarrow x_1 \leq \frac{150}{5} = 30$$

Choose the most restrictive condition

When x1 = 20, S1 is null (leaves the basis)

	basis	X1	X2	S1	S2	S3	value		
L1	S1	6	2	1	0	0	120	120/6 =	20
L2	S2	1	4	0	1	0	100	100/1 =	100
L3	S3	5	5	0	0	1	150	150/5 =	30
f	f	4	3	0	0	0	0		

Pivot row

Choose to leave the basis the variable with the lowest non-negative ratio.

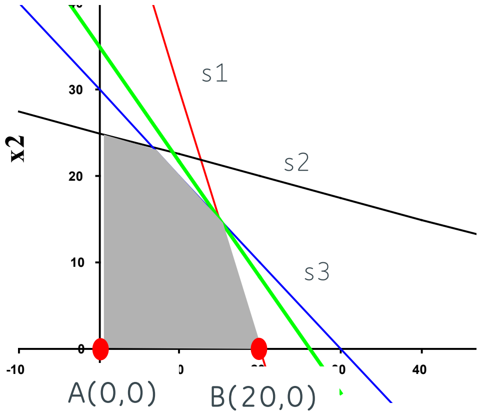
Pivot column

-f = 0

In a maximization problem, choose to enter the basis the variable with the highest positive value in the o.f.

X1 enters the basis, taking a non-negative value
S1 leaves the basis , taking the null value

Going from point A to Point B (20,0)



Simplex Method

Step 5: Update Simplex tableaux to identify the new basic solution

The procedure is based in algebraic operations performed on the rows of the Simplex tableaux in order to build a new identity matrix with the rows and columns of the basic variables.

We perform algebraic operations in order to set the value 1 to the intersection of the pivot row and the pivot column and zero values in all the other coefficients of the pivot column (including the o.f.).

After identifying the new basic solution, go to step 3 to verify if the new solution is optimal.

Iteration 0 (point A)

	basis	X1	X2	S1	S2	S3	value		
L1	S1	6	2	1	0	0	120	120/6 =	20
L2	S2	1	4	0	1	0	100	100/1 =	100
L3	S3	5	5	0	0	1	150	150/5 =	30
f	f	4	3	0	0	0	0		

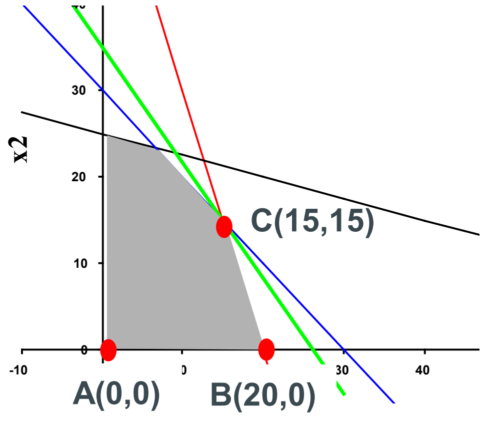
Divide by 6 all the values in this line

Iteration 1 (point B)

	basis	X1	X2	S1	S2	S3	value		
L'1 = L1/6	X1	1	0.3333	0.1667	0	0	20	20/0,33=	60
L'2 = L2 - L'1	S2	0	3.6667	-0.167	1	0	80	80/3,66=	21.82
L'3 = L3 - 5.L'1	S3	0	3.3333	-0.833	0	1	50	50/3,33=	15
f' = f' - 4.L'1	f	0	1.6667	-0.667	0	0	-80		

X2 enters the basis with a non-negative value
S3 leaves the basis, with null value

Going from point B to Point C (15,15)




BRACE YOURSELVES
(ANOTHER) ITERATION IS COMING
memegenerator.net

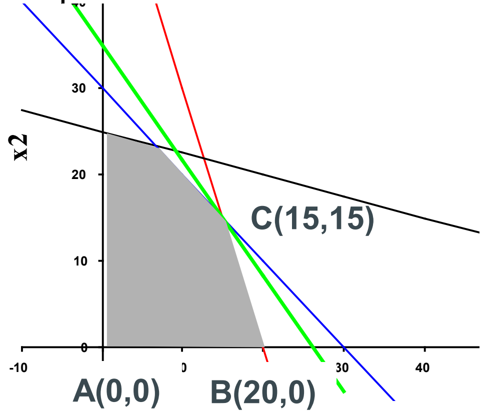
Iteration 1 (point B)

basis	X1	X2	S1	S2	S3	value
X1	1	0,3333	0,1667	0	0	20
S2	0	3,6667	-0,167	1	0	80
S3	0	3,3333	-0,833	0	1	50
f	0	1,6667	-0,667	0	0	-80

Iteration 2 (point C)

	basis	X1	X2	S1	S2	S3	value
L"1 = L'1 - 0,33.L"3	X1	1	0	0,25	0	-0,1	15
L"2 = L'2 - 3,67.L"3	S2	0	0	0,75	1	-1,1	25
L"3 = L'3 / 3,33	X2	0	1	-0,25	0	0,3	15
f" = f' - 1,67.L"3	f	0	0	-0,25	0	-0,5	-105

***Point C is the optimal solution**, since none of the coefficients in the o.f is positive (we are solving a maximization problem)*



f = 4*X1+3*X2 = 4*15+3*15 = 60+45=**105**

Exercise 1

$$\max \text{f} = -x_1 + 2x_2 + x_3$$
$$2x_1 + x_2 - x_3 \leq 2$$
$$2x_1 - x_2 + 5x_3 \leq 6$$
$$4x_1 + x_2 + x_3 \leq 6$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Iteration 0 : Point (0,0,0)

	basis	X1	X2	X3	S1	S2	S3	value		
L1	S1	2	1	-1	1	0	0	2	2/1 =	2
L2	S2	2	-1	5	0	1	0	6		
L3	S3	4	1	1	0	0	1	6	6/1 =	6
f	f	-1	2	1	0	0	0	0		

Variable X2 enters the basis and variable S1 leaves the basis

Iteration 1 : Point (0,2,0)

	basis	X1	X2	X3	S1	S2	S3	value		
L'1 = L1	X2	2	1	-1	1	0	0	2		
L'2 = L2+L'1	S2	4	0	4	1	1	0	8	8/4 =	2
L'3 =L3-L'1	S3	2	0	2	-1	0	1	4	4/2 =	2
f' = f-2*L'1	f	-5	0	3	-2	0	0	-4		

Variable X3 enters the basis and variable S2 leaves the basis

Iteration 1 : Point (0,2,0)

	basis	X1	X2	X3	S1	S2	S3	value		
L'1 = L1	X2	2	1	-1	1	0	0	2		
L'2 = L2+L'1	S2	4	0	4	1	1	0	8	8/4 =	2
L'3 =L3-L'1	S3	2	0	2	-1	0	1	4	4/2 =	2
f' = f-2*L'1	f	-5	0	3	-2	0	0	-4		

Variable X3 enters the basis and variable S2 leaves the basis

Iteration 2 : Point (0,4,2)

	basis	X1	X2	X3	S1	S2	S3	value
L"1 =L'1+L"2	X2	3,00	1	0	1,25	0,25	0	4,00
L"2 = L'2/4	X3	1,00	0	1,00	0,25	0,25	0	2,00
L"3 =L'3-2L"2	S3	0	0	0	-1,5	-0,5	1	0
f" = f' - 3*L"2	f	-8	0	0	-2,75	-0,75	0	-10

Optimal solution:

X1 = 0
X2 = 4
X3 = 2

Optimal value of o.f. = 10

Linear Programming

THE TWO-PHASE SIMPLEX METHOD

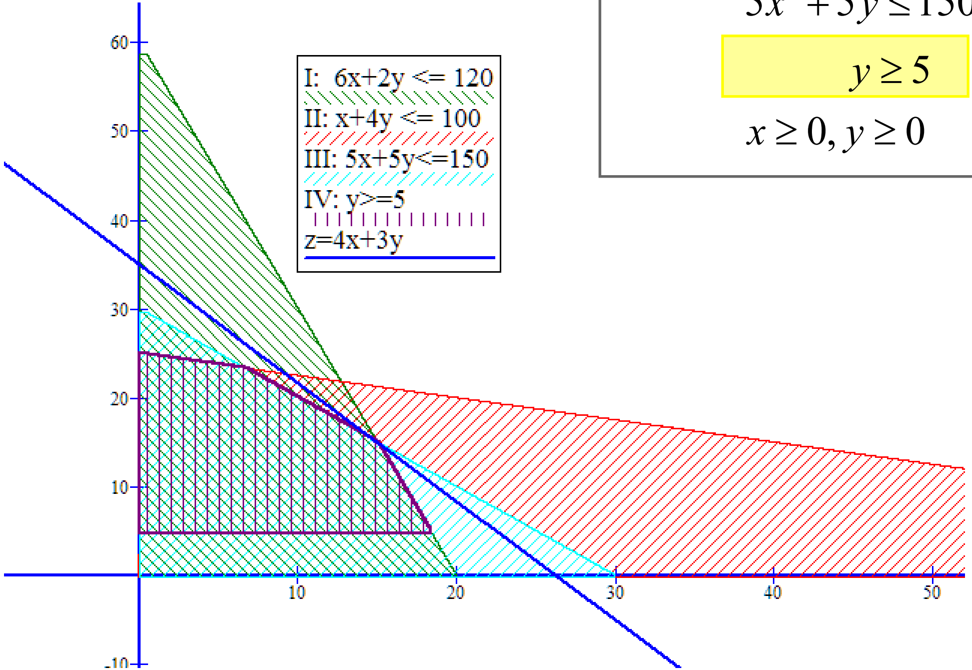
The Two-phase Simplex Method

Cereals Ltd revisited:

Consider that a new constraint is added to the original problem, imposing that the minimum amount of corn (y) to produce is 5 tons.

Graphically, we can see that the optimal solution does not change.

However, in order to apply the Simplex Method we need to find a feasible basic solution, since the point (0,0) does no longer belong to the feasible region.



$$\max f = 4x + 3y$$

$$\text{s.to} \quad \begin{aligned} 6x + 2y &\le 120 \\ x + 4y &\le 100 \\ 5x + 5y &\le 150 \\ \boxed{y \ge 5} & \\ x \ge 0, y &\ge 0 \end{aligned}$$

The Two-phase Simplex Method

Canonical form

$$\max \text{ f} = 4x_1 + 3x_2$$
$$\text{s.a} \quad 6x_1 + 2x_2 \le 120$$
$$x_1 + 4x_2 \le 100$$
$$5x_1 + 5x_2 \le 150$$
$$\boxed{x_2 \ge 5}$$
$$x_1 \ge 0, x_2 \ge 0$$

$$\max \text{ f} = 4x_1 + 3x_2$$
$$\begin{aligned} 6x_1 + 2x_2 + s_1 \qquad\qquad &= 120 \\ x_1 + 4x_2 \qquad + s_2 \qquad\quad &= 100 \\ 5x_1 + 5x_2 \qquad\quad + s_3 \quad &= 150 \\ x_2 \qquad\qquad\qquad - s_4 &= 5 \\ x_1, x_2, s_1, s_2, s_3, s_4 \ge 0 & \end{aligned}$$

The point (0,0) is not a feasible solution, hence (0,0,120,100,150,-5) is not a feasible basic solution.

Remember that all the variables must be non-negative and, in this solution, S_4 = -5 !!!

How can we find a feasible basic solution?

The Two-phase Simplex Method

Original problem in canonical form

$$\max \mathrm{f} = 4x_1 + 3x_2$$

$$\begin{aligned}
6x_1 + 2x_2 + s_1 \quad &= 120 \\
x_1 + 4x_2 \quad + s_2 \quad &= 100 \\
5x_1 + 5x_2 \quad + s_3 \quad &= 150 \\
x_2 \quad - s_4 &= 5 \\
x_1, x_2, s_1, s_2, s_3, s_4 &\geq 0
\end{aligned}$$

Auxiliary problem in canonical form

$$\boxed{\min \mathrm{F} = a_1 = 5 - x2 + s4}$$

$$\begin{aligned}
6x_1 + 2x_2 \quad + s_1 \quad &= 120 \\
x_1 + 4x_2 \quad + s_2 \quad &= 100 \\
5x_1 + 5x_2 \quad + s_3 \quad &= 150 \\
x_2 \; - s_4 \quad + \boxed{\mathrm{a}_1} \quad &= 5 \\
x_1, x_2, s_1, s_2, s_3, s_4, a_1 &\geq 0
\end{aligned}$$

artificial variable

To set all the artificial variables to zero, solve a linear programming problem derived from the canonical form of the original problem by replacing the original objective function by the minimization of the sum of all the artificial variables.
If the optimal value of the modified objective function is not equal to zero, then the problem (system of constraints) is not feasible.

The Two-phase Simplex Method

1st phase: Find a feasible basic solution for the original problem

- For each '>=' or '=' constraint, add an artificial variable.
- The artificial variables allow us to have a basic solution for an auxiliary problem for which the objective function is the minimization of the sum of the artificial variables.
- The Simplex Method is then applied to this auxiliary problem.
- The optimal solution of the auxiliary problem will be (if exists) a feasible basic solution for the original problem.

2nd phase: Once obtained a feasible basic solution for the original problem, the Simplex method is applied to solve it.

1^{st} phase: Find a feasible basic solution for the original problem

Solve the auxiliary problem:

$$\min F = a_1 = 5 - x2 + s4$$
$$6x_1 + 2x_2 + s_1 = 120$$
$$x_1 + 4x_2 + s_2 = 100$$
$$5x_1 + 5x_2 + s_3 = 150$$
$$x_2 - s_4 + a_1 = 5$$
$$x_1, x_2, s_1, s_2, s_3, s_4, a_1 \geq 0$$

A(0,0)

Point A is not feasible for the original problem

Iteration 0 : Point A (0, 0)

	basis	X1	X2	S1	S2	S3	S4	a1	value		
L1	S1	6	2	1	0	0	0	0	120	120/2 =	60
L2	S2	1	4	0	1	0	0	0	100	100/4 =	25
L3	S3	5	5	0	0	1	0	0	150	150/5 =	30
L4	a1	0	1	0	0	0	-1	1	5	**5/1 =**	**5**
	f	4	3	0	0	0	0	0	0		
max min	F	0	-1	0	0	0	1	0	-5		

We are minimizing the auxiliary problem F. So, we choose to enter the basis the variable with the most negative coefficient in the objective function F.

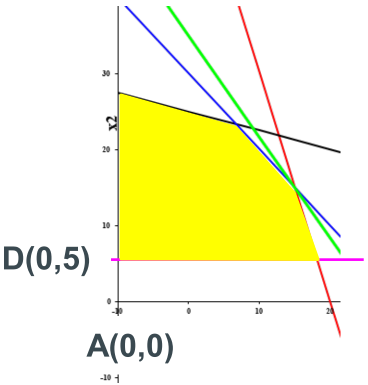
1st phase: Find a feasible basic solution for the original problem

Iteration 0 : Point A (0, 0)

	basis	X1	X2	S1	S2	S3	S4	a1	value		
L1	S1	6	2	1	0	0	0	0	120	120/2 =	60
L2	S2	1	4	0	1	0	0	0	100	100/4 =	25
L3	S3	5	5	0	0	1	0	0	150	150/5 =	30
L4	a1	0	1	0	0	0	-1	1	5	**5/1 =**	**5**
	f	4	3	0	0	0	0	0	0		
	F	0	-1	0	0	0	1	0	-5		

Iteration 1 : Point D (0, 5)

	basis	X1	X2	S1	S2	S3	S4	a1	value		
L'1 = L1-2L'4	S1	6	0	1	0	0	2	-2	110	110/6 =	18,3
L'2 = L2-4L'4	S2	1	0	0	1	0	4	-4	80	80/1 =	80
L'3 =L3-5L'4	S3	5	0	0	0	1	5	-5	125	125/5=	25
L'4 = L4	X2	0	1	0	0	0	-1	1	5		
f' = f-3L'4		4	0	0	0	0	3	-3	-15		
F' = F+L'4		0	0	0	0	0	0	1	0		



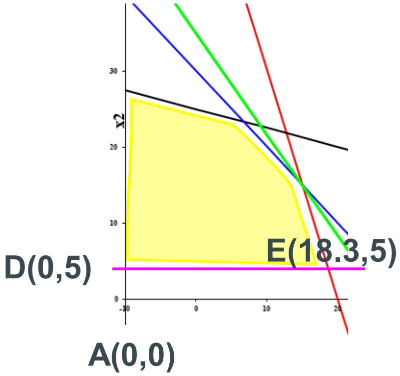
End of 1st phase: we found a feasible basic solution for the original problem: **point D**

2nd phase: apply the Simplex method to the original problem

Iteration 1 : Point D (0, 5)

	basis	X1	X2	S1	S2	S3	S4	a1	value		
L'1 = L1-2L'4	S1	6	0	1	0	0	2	-2	110	110/6 =	18.3
L'2 = L2-4L'4	S2	1	0	0	1	0	4	-4	80	80/1 =	80
L'3 =L3-5L'4	S3	5	0	0	0	1	5	-5	125	125/5=	25
L'4 = L4	X2	0	1	0	0	0	-1	1	5		
Lf' = f-3L'4	f	4	0	0	0	0	3	-3	-15		
LF' = LF+L'4	F	0	0	0	0	0	0	1	0		

Remove the auxiliary problem column (F) and the artificial variable row (a1) from the tableaux.
Solve the original problem (f): the variable that enters the basis is x1



	basis	X1	X2	S1	S2	S3	S4	value	ratio	
L"1 = L'1/6	X1	1	0	0.1667	0	0	0.3333	18.333	18,3/0,33 =	55
L"2 = L'2-L"1	S2	0	0	-0.167	1	0	3.6667	61.667	6,67/2,67 =	17
L"3 = L'3-5L"1	S3	0	0	-0.833	0	1	3.3333	33.333	33,3/3,3 =	10
L"4 = L'4	X2	0	1	0	0	0	-1	5		
f" = f'-4L"1		0	0	-0.667	0	0	1.6667	-88.33		

Iteration 2 : Point E (18.3, 5)

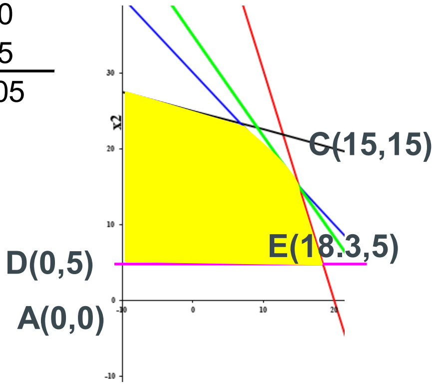
2nd phase: apply the Simplex method to the original problem.

Iteration 2 : Point E (18.3, 5)

	basis	X1	X2	S1	S2	S3	S4	value	ratio	
L"1 = L'1/6	X1	1	0	0,1667	0	0	0,3333	18,333	18,3/0,33 =	55
L"2 = L'2-L"1	S2	0	0	-0,167	1	0	3,6667	61,667	6,67/2,67 =	17
L"3 = L'3-5L"1	S3	0	0	-0,833	0	1	3,3333	33,333	33,3/3,3 =	10
L"4 = L'4	X2	0	1	0	0	0	-1	5		
f" = f-4L"1		0	0	-0,667	0	0	1,6667	-88,33		

Iteration 3 : Point C (15,15)

	basis	X1	X2	S1	S2	S3	S4	value
L"'1=L"1-0,33L"'3	X1	1	0	0,25	0	-0,099	0	15
L"'2=L"2-3,67L"'3	S2	0	0	0,75	1	-1,1	0	25
L"'3=L"3/3,33	S4	0	0	-0,25	0	0,3	1	10
L"'4=L"4+L"'3	X2	0	1	-0,25	0	0,3	0	15
f"'=f"-1,667L"'3		0	0	-0,25	0	-0,5	0	-105



End of 2nd phase

Optimal solution: ***fmax = 105***

Point C : X1 = 15; X2 = 15

The Two-phase Simplex Method: final remarks

At the end of the Phase 1 we can distinguish between two cases, namely:

Case 1: $F_{min} > 0$: The optimal value of F is greater than zero.
 we conclude that the LP problem under consideration has no feasible solutions.
Case 2: $F_{min} = 0$: The optimal value of F is equal to zero.
 implies that the constraints are feasible, hence the problem under consideration has a feasible solution.

It should be noted that although Case 2 implies that all the artificial variables are equal to zero, this does not mean that they are all out of the basis. So, it is necessary to consider Case 2 in more detail, namely:

Case 2.1: All the artificial variables are non-basic.

- we proceed to Phase 2 of the 2-Phase method replacing the objective function of Phase 1 with the original objective function.

Case 2.2: some of the artificial variables are in the basis.

- This case represents a degenerate basis, namely a situation where one or more of the basic variables are equal to zero. To take a degenerate artificial variable out of the basis we pivot on any non-artificial variable whose coefficient in the row of the artificial is not equal to zero and enter it into the basis.
- If the coefficients of all the non-artificial variables in that row are zeros, then the conclusion is that the constraint is redundant and thus can be ignored.

Special cases in Simplex Method

1. Infeasibility of the problem (no feasible solutions)
2. Unbound optimal value
3. Multiple optimal solutions
4. Degeneracy

1. Infeasibility of the problem

$$\max f = x_1 + x_2$$

s.to

$$\begin{array}{rrcl} x_1 & + x_2 & \leq & 1 \\ 2x_1 & + 3x_2 & \geq & 6 \\ & x_1, x_2 & \geq & 0 \end{array}$$

Auxiliary Problem (phase 1)

$$\min F = a_1$$

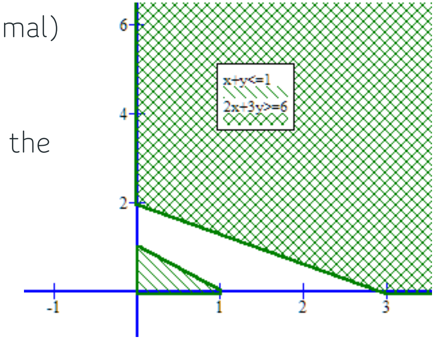
s.to

$$\begin{array}{rrrrrcl} x_1 & + x_2 & + s_1 & & & = & 1 \\ 2x_1 & + 3x_2 & & - s_2 & + a_1 & = & 6 \\ x_1, & x_2, & s_1, & s_2, & a_1 & \geq & 0 \end{array}$$

Basis	x1	x2	s1	s2	a1	Value
s1	1	1	1	0	0	1
a1	2	3	0	-1	1	6
F	-2	-3	0	1	0	-6
f	1	1	0	0	0	0

Basis	x1	x2	s1	s2	a1	Value
x2	1	1	1	0	0	1
a1	-1	0	-3	-1	1	3
F	1	0	3	1	0	-3
f	0	0	-1	0	0	-1

At the end of phase 1, F is optimal (minimal) and positive while a1 is still basic (and positive).
Hence there are no feasible solutions for the original problem (see slide 11).



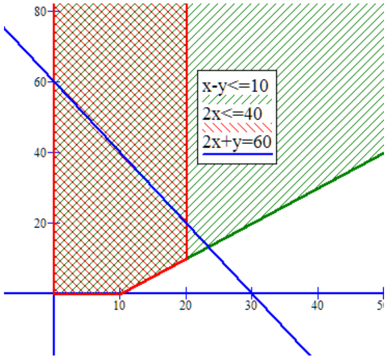
2. Unbound optimal value

$$\max f = 2x_1 + x_2$$
s.a
$$\begin{array}{lllr} x_1 & -x_2 & \leq & 10 \\ 2x_1 & & \leq & 40 \\ & x_1, x_2 & \geq & 0 \end{array}$$

Base	x1	x2	s1	s2	Valor
s1	1	-1	1	0	10
s2	2	0	0	1	40
f	2	1	0	0	0

Base	x1	x2	s1	s2	Valor
x1	1	-1	1	0	10
s2	0	2	-2	1	20
f	0	3	-2	0	-20

Base	x1	x2	s1	s2	Valor
x1	1	0	0	1/2	20
x2	0	1	-1	1/2	10
f	0	0	1	-3/2	-20



- At some iteration of the simplex method, a non-basic variable with positive coefficient can enter the basis without a bound on its value (for a maximization problem)
- This means we can bring that variable in the basis and increase the z-value to $+\infty$ (since the variable can be increased to $+\infty$).

3. Multiple optimal solutions

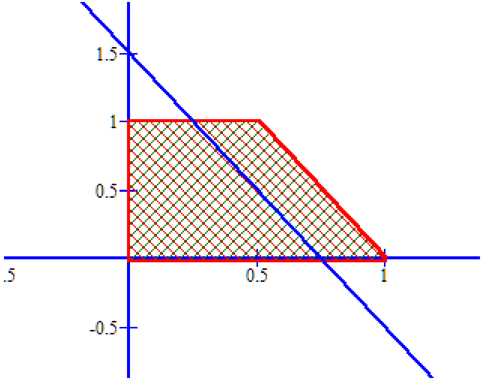
$$\max f = 4x_1 + 2x_2$$
s.a
$$\begin{array}{rrcl} 2x_1 & +x_2 & \leq & 2 \\ & x_2 & \leq & 1 \\ & x_1, x_2 & \geq & 0 \end{array}$$

Basis	x1	x2	s1	s2	Value
s1	2	1	1	0	2
s2	0	1	0	1	1
f	4	2	0	0	0

Basis	x1	x2	s1	s2	Value
x1	1	0.5	0.5	0	1
s2	0	1	0	1	1
f	0	0	-2	0	-4

- There is a non-basic variable (x2) with 0 coefficient in the optimal table.
- This means we can bring that variable in the basis without changing the f-value.

The new solution would also be optimal.



5. Degeneracy

Degeneracy: occurs when a basic feasible solution has one or more basic variables with value 0

In a n-dimensional space, we have degeneracy when at least n+1 constraints intersect at the same vertex of the feasible region.

For example, when n=2, we have a degeneracy if 3 or more constraints intersect in the same vertex of the feasible region.

A degeneracy may occur:

(i) at the initial basic solution, or

(ii) at any iteration, when a tie occurs in the candidate variables to leave the basis (we can choose arbitrarily, although it is common practice to choose the lowest index variable).

Degeneracy and infinite loops

- Theoretically, on degenerated solutions, the algorithm of the Simplex method can get into an infinite loop without progressing to the optimum solution.
- However, the resolution of real problems has shown that this issue rarely occurs.
- Software implementations of the Simplex algorithm ensure that it does not get into a loop.

Degeneracy at the initial basic solution

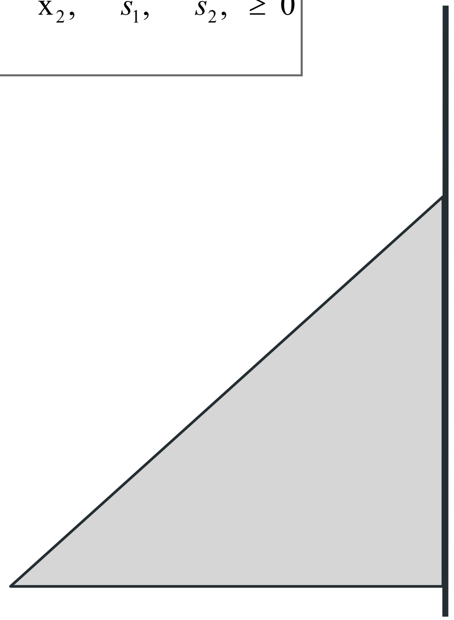
$$
\begin{aligned}
&\max f = x_1 + 2x_2 \\
&\text{s.a} \\
&\begin{array}{rrcl}
x_1 & & \le & 4 \\
-x_1 & +x_2 & \le & 0 \\
 & x_1, x_2 & \ge & 0
\end{array}
\end{aligned}
$$

$$
\begin{aligned}
&\max f = x_1 + 2x_2 \\
&\text{s.a} \\
&\begin{array}{rrrrcl}
x_1 & & +s_1 & & = & 4 \\
-x_1 & +x_2 & & +s_2 & = & 0 \\
x_1, & x_2, & s_1, & s_2, & \ge & 0
\end{array}
\end{aligned}
$$

Base	x1	x2	s1	s2	Valor
s1	1	0	1	0	4
s2	-1	1	0	1	0
f	1	2	0	0	0

Base	x1	x2	s1	s2	Valor
s1	1	0	1	0	4
x2	-1	1	0	1	0
f	3	0	0	-2	0

Base	x1	x2	s1	s2	Valor
x1	1	0	1	0	4
x2	0	1	1	1	4
f	0	0	-3	-2	-12



Degeneracy at any iteration of Simplex

$$\begin{aligned} &\max f = x_1 + 2x_2 \\ &\text{s.a} \\ &\quad\quad\quad x_2 \leq 1 \\ &x_1 + x_2 \leq 1 \\ &x_1, x_2 \geq 0 \end{aligned}$$

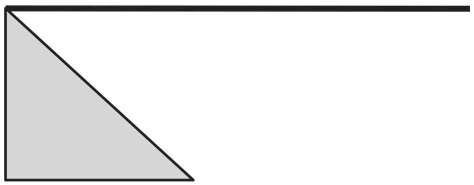
$$\begin{aligned} &\max f = x_1 + 2x_2 \\ &\text{s.a} \\ &\quad\quad x_2 + s_1 = 1 \\ &x_1 + x_2 + s_2 = 1 \\ &x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$

Base	x1	x2	s1	s2	Valor
s1	0	1	1	0	1
s2	1	1	0	1	1
f	1	2	0	0	0

Base	x1	x2	s1	s2	Valor
x2	0	1	1	0	1
s2	1	0	-1	1	0
f	1	0	-2	0	-2

Base	x1	x2	s1	s2	Valor
x2	0	1	1	0	1
x1	1	0	-1	1	0
f	0	0	-1	-1	-2

Optimal solution
x1 = 0
x2 = 1



Exercise

The following Simplex tableaux represents the final solution of a maximization linear problem. Which are the values that constants *a*, *b*, *c* e *d* can take in the following situations:

i. The solution is optimal and unique;
ii. There are several alternative optimal solutions;
iii. The optimal solution is unbounded;
iv. The optimal solution is degenerated.

Base	x_1	x_2	x_3	x_4	x_5	Valor
x_4	-4	$\boldsymbol{b}$	0	1	0	1
x_3	2	-1	1	0	0	4
x_5	3	$\boldsymbol{c}$	0	0	1	$\boldsymbol{a}$
f	-2	$\boldsymbol{d}$	0	0	0	-10

Solution

(i) Para a solução ser óptima $\Rightarrow d \leq 0$. Dado que tem que ser única $\Rightarrow d \neq 0$;
Dada a restrição de não negatividade das variáveis: $a \geq 0$;
Assim: $d < 0 \wedge a \geq 0$.

ii) $d = 0$ por forma a que, se a variável não básica $x2$ entrasse para a base, o valor da função objectivo não se alterasse;
Dada a restrição de não negatividade das variáveis: $a \geq 0$;
Para garantir que, ao entrar para a base a variável não básica $x2$, saia desta uma outra variável, é necessário que $\left(\frac{1}{b} \geq 0 \vee \frac{a}{c} \geq 0\right)$, que é equivalente a $(b > 0 \vee c > 0)$

Assim: $d = 0 \wedge a \geq 0 \wedge (b > 0 \vee c > 0)$

iii) Para a solução ser óptima e não limitada é necessário que ao entrar a variável não básica $x2$ para a base, não saia de lá nenhuma variável básica, ou seja, $x2$ pode entrar para a base com um valor qualquer ilimitado.

Assim, $d > 0 \wedge a \geq 0 \wedge c < 0 \wedge b < 0$

iv) $a = 0 \wedge d < 0$ (b e c quaisquer) .