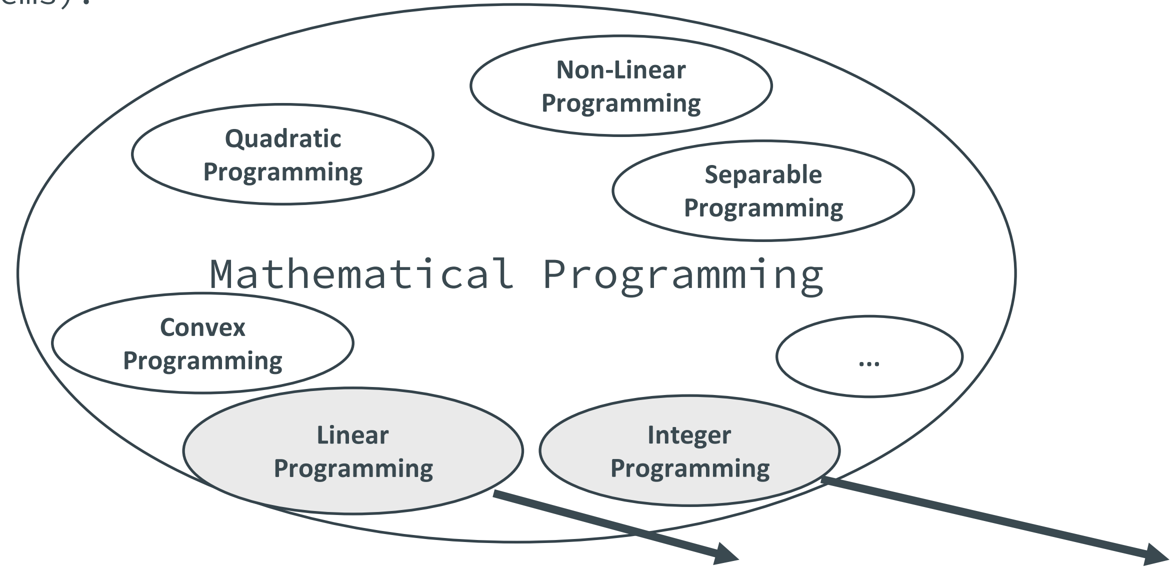
Optimization

Lecture 2.1

Introduction to Linear Programming

Some models we are going to learn in this course

Linear Programming Problems belong to a group of problems known as Mathematical Programming Problems, which are characterized by having a single objective and are subject to a set of constraints (which features are different for each class of problems).



In this course we will only address Linear Programming and Integer Programming

Components of a Mathematical Programming Model

Decision variables (unknowns of the problem)- For example, the decision variables can be the quantities of the resources to be allocated, the number of units to be produced, or both. The decision maker searches for the value set of these unknown variables that will provide an optimal solution to the problem.

The objective function represents the goal/objective of the problem in terms of decision variables. The decision maker intends to either maximize or minimize this function, such as to either maximize the total profit or minimize the total cost of production when producing or selling certain products.

Components of a Mathematical Programming Model (cont.)

Data such as profit (for maximization) or cost (for minimization) per unit product are parameters required in association with the decision variables to form the objective function. These parameters are known as coefficients (profit or cost) of the objective function.

The constraints are known as restrictions or limitations of the problem.

A constraint has two components, usually a function and a constant related by either an equality or inequality sign. For a resource constraint, the function represents the total resource required in terms of the decision variables and the constant specifies the total resource availability. Data such as the resources required per unit product are required to form the constraint functions. These data are known as coefficients associated with the constraints or technological coefficients.

Note that most optimization software products adhere to the convention of having the variable expression on the left-hand side (LHS) of the constraint equation and a constant on the equation's right-hand side (RHS).

1st EXAMPLE

A PRODUCT-MIX PROBLEM

A product-mix problem

- A furniture manufacturer produces tables and chairs. The process involves machining, sanding, and assembling the pieces to make the tables and chairs.
- It takes 5h to machine the pieces for a table, 4h to sand the pieces, and 3h to assemble a table.
- A chair requires 2h to machine the pieces, 3h to sand the pieces, and 4h to assemble a chair.
- There are 270h available for machining the pieces, 250h for sanding the pieces, and 200h for assembling.
- If the profit for a table is €100 and for a chair €60, how many tables and chairs should the manufacturer produce in order to maximize the overall profit?
- What if there is an additional requirement to produce four chairs for each table?

Adapted from Optimization modelling : a practical introduction / Ruhul A. Sarker and Charles S. Newton.

Summary of data

Resource/Item	Per Unit Product: Table	Per Unit Product: Chair	Resource Availability (h)
Machining	5	2	270
Sanding	4	3	250
Assembly	3	4	200
Profit per unit ($)	100	60	

Exploring the data

Let T and C be the number of tables and chairs to produce, respectively.

Total machining hours required = 5T + 2C
Total sanding hours required = 4T + 3C
Total assembly hours required = 3T + 4C

Alternative Resource Usage

Alternative Number	Number of Tables (x_T)	Number of Chairs (x_C)	Machining (Limit: 270)	Sanding (Limit: 250)	Assembling (Limit: 200)
1	1	1	7	7	7
2	1	2	9	10	11
3	2	2	14	14	14
:	:	:	:	:	:
:	10	10	70	70	70
:	11	10	75	74	93
:	:	:	:	:	:
:	50	10	270	230	190
k	50	11	272	233	194
:	:	:	:	:	:

Mathematical Model

$$\text{Maximize } Z = 100T + 60C$$

Subject to

$$
\begin{aligned}
5T + 2C &\le 270 && \text{Machining} \\
4T + 3C &\le 250 && \text{Sanding} \\
3T + 4C &\le 200 && \text{Assembly} \\
T \ge 0 \text{ and } C &\ge 0 && \text{Nonnegativity}
\end{aligned}
$$

Revisiting our Model

Remember the last requirement:

What if there is an additional requirement to produce four chairs for each table?

We add the constraint: 4T - C = 0

Maximize $Z = 100T + 60C$

Subject to

$5T + 2C \leq 270$	Machining
$4T + 3C \leq 250$	Sanding
$3T + 4C \leq 200$	Assembly
$4T - C = 0$	Additional
$T \geq 0$ and $C \geq 0$	Nonnegativity

Integer Programming

Tables and chairs are integer by nature! How to consider this in the model?

$$
\begin{aligned}
&\text{Maximize } Z = 100T + 60C \\
&\text{Subject to} \\
&\qquad 5T + 2C \le 270 \qquad \text{Machining} \\
&\qquad 4T + 3C \le 250 \qquad \text{Sanding} \\
&\qquad 3T + 4C \le 200 \qquad \text{Assembly} \\
&\qquad 4T - C = 0 \qquad \text{Additional} \\
&\qquad \boxed{T \ge 0 \text{ and } C \ge 0 \ \text{ and integer}}
\end{aligned}
$$

Integer Programming Models

Pure Integer: Where all the decision variables are integers.

Binary integer: Where all the decision variable values are binary (either zero or one) only.

Mixed integer or mixed integer linear: Linear programs with some integer and some real decision variables.

2ND EXAMPLE

A FINANCIAL MANAGEMENT PROBLEM

A financial management problem

The NNB (NewNew Bank) offers five types of loans. The titles of these loans, with their respective yearly interest rates charged to customers, are presented in the table below:

Loan ID	Type of Loan	Charged (%)
1	Industrial/commercial loans	9%
2	Home extensions	8%
3	First time home loans	6.5%
4	Home loans–supplemented	7.5%
5	Personal loans	10%

The NNB has €50 million available for these loans. NNB's objective is to maximize the yield on investment in loans. The credit union maintains the following policies for their loan investments:

1. Home extension loan investments cannot be greater than 25% of first time home loan investments.

2. Industrial loan investments must be less than or equal to home loan-supplemented investments.

3. The credit union invests at least 70% of the funds in home loans (first time and supplemented).

4. For technical reasons, there must be at least €3 invested in first time home loans for every euro invested in home loans-supplemented.

3RD EXAMPLE

Another Product Mix Problem

Another Product Mix Problem

- BIC Manufacturing Company produces and markets three products P1, P2 and P3.
- The raw material requirements for each product and the availability of the raw materials are given in the following table.

Raw material	Requirements per unit of product (kg)			Total availability (kg)
	P1	P2	P3	
R1	6	4	9	5,000
R2	3	7	6	6,000

- If the whole labour force is engaged in only producing product P1, time will permit 1600 units of P1 to be produced.
- Time to manufacture product P1 is twice that for P2 and thrice that for P3 and the products are to be produced in the ratio 3:4:5.
- There is a demand for at least 185, 250 and 200 units of products P1, P2 and P3 to be produced and the profit earned per unit is €50, €40 and €70 respectively. Find the quantities of P1, P2 and P3 to be produced.

Adapted from Optimization modelling : a practical introduction / Ruhul A. Sarker and Charles S. Newton.

Another Product Mix Problem

Decision Variables: X_1, X_2, X_3 denote the number of units of products P1, P2 and P3 to be manufactured

Raw material availability (R1 and R2)

$$6x_1 + 4x_2 + 9x_3 \leq 5000$$
$$3x_1 + 7x_2 + 6x_3 \leq 6000$$

Production capacity: Assume each product P1 takes t amount of time, so

$$tx_1 + (t/2)x_2 + (t/3)x_3 \leq 1600t$$
$$x_1 + (1/2)x_2 + (1/3)x_3 \leq 1600$$

Market demand constraints

$$x_1 \geq 185$$
$$x_2 \geq 250$$
$$x_3 \geq 200$$

Product ratio constraints: Since the products P1, P2, and P3 are to be produced in the ratio 3:4:5 , $x_1:x_2:x_3 = 3:4:5$ or

$$\frac{x_1}{3} = \frac{x_2}{4} = \frac{x_3}{5}$$
$$4x_1 - 3x_2 = 0$$
$$5x_2 - 4x_3 = 0$$

Another Product Mix Problem

Maximize $50x_1 + 40x_2 + 70x_3$

s. to

$6x_1 + 4x_2 + 9x_3 \leq 5000$

$3x_1 + 7x_2 + 6x_3 \leq 6000$

$x_1 + (1/2)x_2 + (1/3)x_3 \leq 1600$

$x_1 \geq 185$

$x_2 \geq 250$

$x_3 \geq 200$

$4x_1 - 3x_2 = 0$

$5x_2 - 4x_3 = 0$

$x_1, x_2, x_3 \geq 0$

4TH EXAMPLE

A CAPITAL BUDGETING PROBLEM

A capital budgeting problem

Problem

Local councils and organizations frequently face situations where they have to select one or more projects (investment opportunities) from a number of competing projects. Consider the following list of projects.

If €30 million is available, which projects should be selected?

Project Number	Project	Cost (€ million)	Expected Utility
1	After school program	6	18
2	Road security	18	16
3	Crime reduction	10	12
4	Road extension	9	25
5	Child care facility	4	14

A capital budgeting problem

Model

$$\text{Maximize } Z = 18x_1 + 16x_2 + 12x_3 + 25x_4 + 14x_5$$

Subject to

$$6x_1 + 18x_2 + 10x_3 + 9x_4 + 4x_5 \leq 30$$

$$x_1, x_2, x_3, x_4, x_5 \text{ are either 1 or 0}$$

$$\text{where } x_i = \begin{cases} 1 & \text{, if project i is selected} \\ 0 & \text{0 otherwise} \end{cases}$$

These type of integer problems are also known as 0-1 knapsack problems, which are NP-complete problems, despite the slim formulation…