

## PROOF BY INDUCTION EXERCISES

- 1 Prove that, for every natural number,  $1+3+5+ \dots+(2n+1) = (n+1)^2$ . ➡
- 2 Consider the following definition of a tree. ➡

Definition of tree:

1. A simple node is a tree.
2. If  $T_1, T_2, \dots, T_k$  are trees, the structure that results from taking a new node  $N$  and connecting with an edge each of the trees  $T_1, T_2, \dots, T_k$  to  $N$  is a tree.
3. Nothing else is a tree other than that obtained from 1. and 2.

Prove that a tree has a number of nodes  $N$  larger than the number of edge  $E$  by one unit.

- 3 A palindrome is a string that can be read in the same way both from left-to-right and from right-to-left. Consider the following inductive definition of the related concept **pal**: ➡
  1. Each character of the alphabet is a **pal**.
  2. If  $\alpha$  is a **pal** so is the result of concatenating any character before and after  $\alpha$ .
  3. Nothing is a **pal** except if obtained from 1. and 2.

Prove by induction that every **pal** is a palindrome. Is the converse true? If so, prove it. Otherwise, correct the definition so that it becomes true.

- 4 An ATM has only 20€ and 50€ bills. Prove by induction that this machine can supply any amount that is a multiple of 10€, equal or larger than 40€.

Define the structure over which you'll make your proof. Show the different steps of the proof. ➡

- 5 Prove that, for every natural number  $n \geq 2$ ,  $(1-1/2)(1-1/3)\dots(1-1/n) = 1/n$ .
- 6 Prove that, for every natural number  $n \geq 1$ , 2 is a factor of  $n^2+n$ .
- 7 Prove that the sum of the first  $n$  perfect cubes is a perfect square. [Note: you need to prove a stronger proposition (Inventor's Paradox).]
- 8 A binary tree can be recursively defined in the following manner: "*a binary tree is constituted by one node and 0, 1 or 2 sub-trees connected by edges to that node. A full node is a node that has two sub-trees. A leaf is a node that has no sub-trees.*"

We want to prove, using mathematical induction, that "*for every binary tree the number of leafs equals the number of full nodes plus one*".

Present the structure over which you will apply the proof by induction and explain its inductive definition.