

From FAs to Regular Expressions

MIEIC, 2nd Year

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Conversion from FAs to Regular Expressions

- ▶ Given a Finite Automata (FA) how to generate an equivalent regular expression (RE)?
- ▶ Two techniques:
 - ▶ State Elimination
 - ▶ Construction of Paths
- ▶ Both algorithms work with Finite Automata (FA) as input, i.e., DFAs, NFAs, and ε -NFAs

Construction of Paths

⊙ Hopcroft's formula

- $R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)} \cdot (R_{kk}^{(k-1)})^* \cdot R_{kj}^{(k-1)}$
- $R_{ij}^{(n)}$ is the regular expression of all paths from i to j (n is the number of states)
- States are numbered 1 to n
- $R_{ij}^{(k)}$ is regular expression of all paths from i to j passing thru nodes less than k
- Computed for all i,j for k=0, then k=1,...,n
- $R_{sf1}^{(n)} + \dots + R_{sfk}^{(n)}$ is the regular expression of the DFA considering:
 - s is the start state, f_1, \dots, f_k are accepting states, n is the number of states.

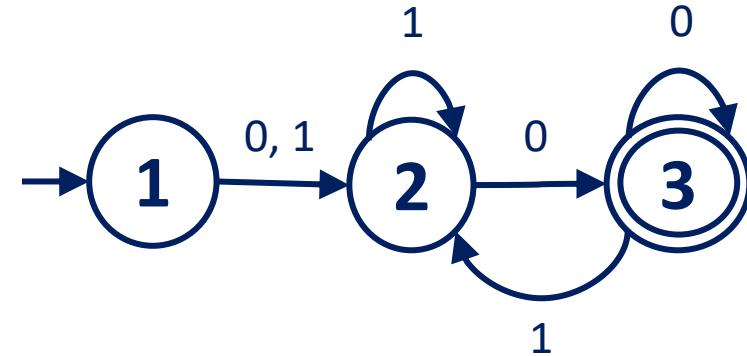
Algorithm based on Path Construction

- ▶ Numerate the nodes (states) from 1 to n
- ▶ $R_{ij}^{(k)}$
 - ▶ Regular expression defining the language consisting of the set of strings w such that w is the label of a path between nodes i and j , without passing in any intermediate node higher than k
- ▶ Induction in the number of nodes (k)
 - ▶ $R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)} \cdot (R_{kk}^{(k-1)})^* \cdot R_{kj}^{(k-1)}$
 - ▶ Computed for all i, j for $k=0$, then $k=1, \dots, n$
- ▶ $R_{s,f1}(n) + \dots + R_{s,fk}(n)$ is the regular expression of the DFA considering:
- ▶ s is the start state, $f1, \dots, fk$ are accepting states, n is the number of states.

Algorithm based on Path Construction (example)

- ▶ DFA below
 - ▶ 3 states
 - ▶ 1 final state
 - ▶ DFA states already labeled from 1 to 3
- ▶ Regular expression representing the language of the DFA:
 - ▶ $RE = RE_{13}^{(3)}$

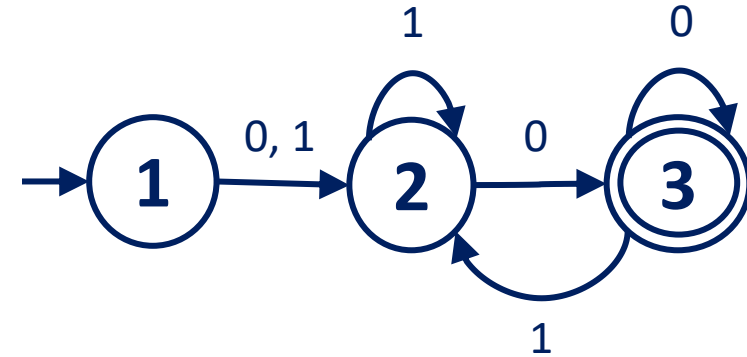
DFA:



Algorithm based on Path Construction

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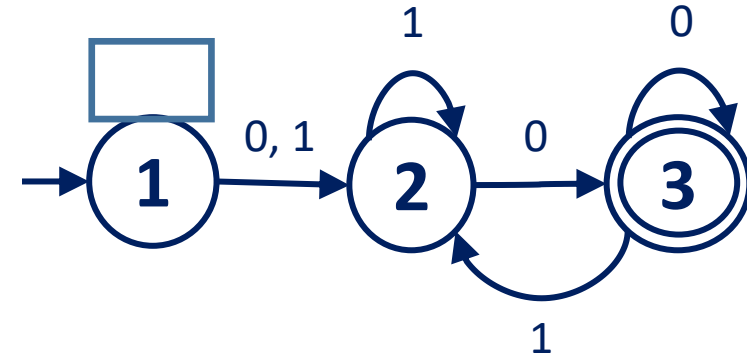
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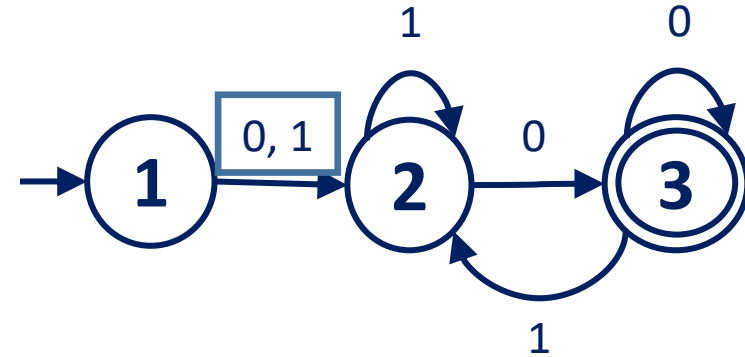
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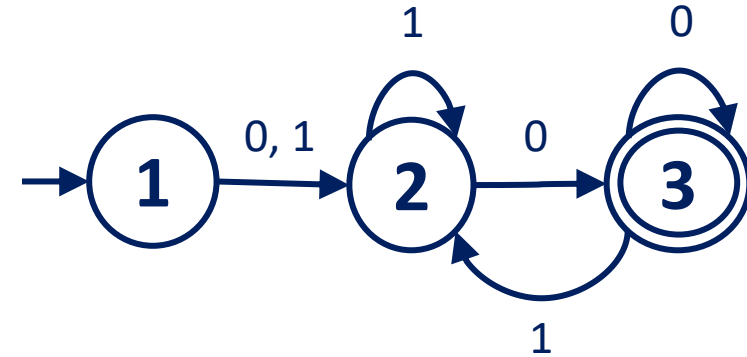
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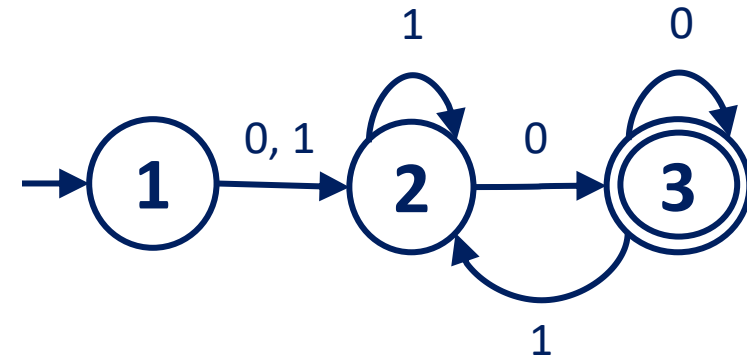
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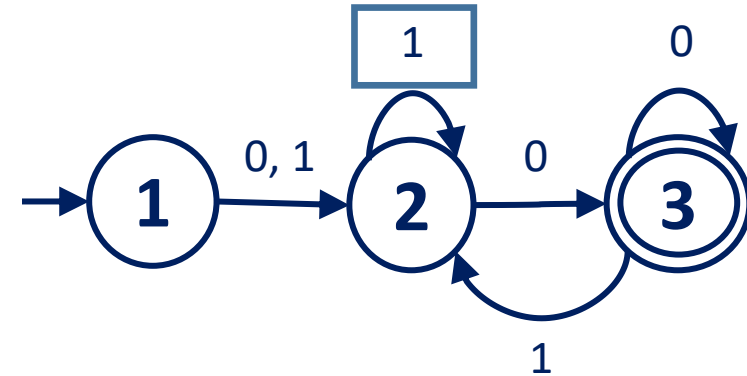
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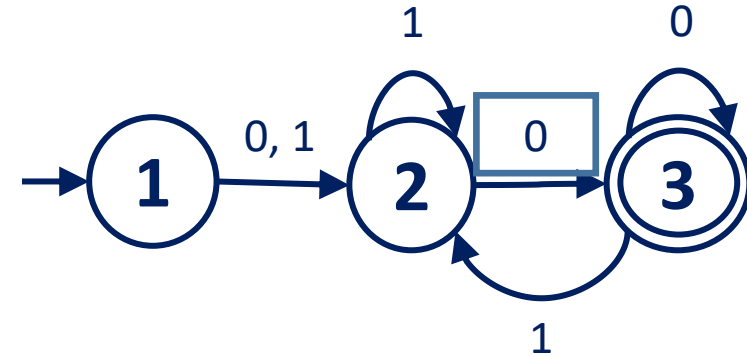
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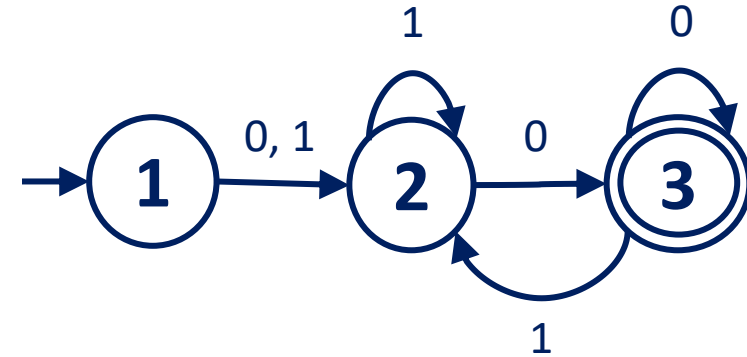
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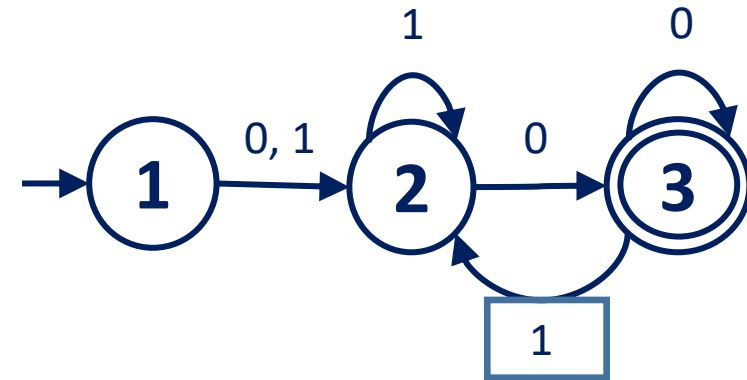
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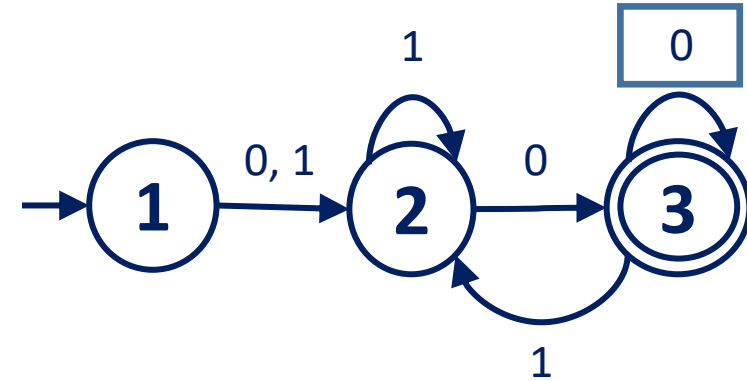
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$R_{13}^{(2)}$	$\emptyset + (0 + 1) \cdot (\varepsilon + 1)^* \cdot 0$	$(0 + 1)1^*0$
$R_{21}^{(2)}$	$\emptyset + (\varepsilon + 1) \cdot (\varepsilon + 1)^* \cdot \emptyset$	\emptyset
$R_{22}^{(2)}$	$(\varepsilon + 1) + (\varepsilon + 1) \cdot (\varepsilon + 1)^* \cdot (\varepsilon + 1)$	1^*
$R_{23}^{(2)}$		
$R_{31}^{(2)}$		
$R_{32}^{(2)}$		
$R_{33}^{(2)}$		

$R_{11}^{(1)}$	ε
$R_{12}^{(1)}$	$0 + 1$
$R_{13}^{(1)}$	\emptyset
$R_{21}^{(1)}$	\emptyset
$R_{22}^{(1)}$	$\varepsilon + 1$
$R_{23}^{(1)}$	0
$R_{31}^{(1)}$	\emptyset
$R_{32}^{(1)}$	1
$R_{33}^{(1)}$	$\varepsilon + 0$

Algorithm based on Path Construction

► $R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)} \cdot (R_{kk}^{(k-1)})^* \cdot R_{kj}^{(k-1)}$

$R_{11}^{(2)}$	$\varepsilon + (0 + 1) \cdot (\varepsilon + 1)^* \cdot \emptyset$	ε
$R_{12}^{(2)}$	$(0 + 1) + (0 + 1) \cdot (\varepsilon + 1)^* \cdot (\varepsilon + 1)$	$(0 + 1)1^*$
$R_{13}^{(2)}$	$\emptyset + (0 + 1) \cdot (\varepsilon + 1)^* \cdot 0$	$(0 + 1)1^*0$
$R_{21}^{(2)}$	$\emptyset + (\varepsilon + 1) \cdot (\varepsilon + 1)^* \cdot \emptyset$	\emptyset
$R_{22}^{(2)}$	$(\varepsilon + 1) + (\varepsilon + 1) \cdot (\varepsilon + 1)^* \cdot (\varepsilon + 1)$	1^*
$R_{23}^{(2)}$	$0 + (\varepsilon + 1) \cdot (\varepsilon + 1)^* \cdot 0$	1^*0
$R_{31}^{(2)}$		
$R_{32}^{(2)}$		
$R_{33}^{(2)}$		

$R_{11}^{(1)}$	ε
$R_{12}^{(1)}$	$0 + 1$
$R_{13}^{(1)}$	\emptyset
$R_{21}^{(1)}$	\emptyset
$R_{22}^{(1)}$	$\varepsilon + 1$
$R_{23}^{(1)}$	0
$R_{31}^{(1)}$	\emptyset
$R_{32}^{(1)}$	1
$R_{33}^{(1)}$	$\varepsilon + 0$

Algorithm based on Path Construction

► $R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)} \cdot (R_{kk}^{(k-1)})^* \cdot R_{kj}^{(k-1)}$

$R_{11}^{(2)}$	$\varepsilon + (0 + 1) \cdot (\varepsilon + 1)^* \cdot \emptyset$	ε
$R_{12}^{(2)}$	$(0 + 1) + (0 + 1) \cdot (\varepsilon + 1)^* \cdot (\varepsilon + 1)$	$(0 + 1)1^*$
$R_{13}^{(2)}$	$\emptyset + (0 + 1) \cdot (\varepsilon + 1)^* \cdot 0$	$(0 + 1)1^*0$
$R_{21}^{(2)}$	$\emptyset + (\varepsilon + 1) \cdot (\varepsilon + 1)^* \cdot \emptyset$	\emptyset
$R_{22}^{(2)}$	$(\varepsilon + 1) + (\varepsilon + 1) \cdot (\varepsilon + 1)^* \cdot (\varepsilon + 1)$	1^*
$R_{23}^{(2)}$	$0 + (\varepsilon + 1) \cdot (\varepsilon + 1)^* \cdot 0$	1^*0
$R_{31}^{(2)}$	$\emptyset + 1 \cdot (\varepsilon + 1)^* \cdot \emptyset$	\emptyset
$R_{32}^{(2)}$		
$R_{33}^{(2)}$		

$R_{11}^{(1)}$	ε
$R_{12}^{(1)}$	$0 + 1$
$R_{13}^{(1)}$	\emptyset
$R_{21}^{(1)}$	\emptyset
$R_{22}^{(1)}$	$\varepsilon + 1$
$R_{23}^{(1)}$	0
$R_{31}^{(1)}$	\emptyset
$R_{32}^{(1)}$	1
$R_{33}^{(1)}$	$\varepsilon + 0$

Algorithm based on Path Construction

► $R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)} \cdot (R_{kk}^{(k-1)})^* \cdot R_{kj}^{(k-1)}$

$R_{11}^{(2)}$	$\varepsilon + (0 + 1) \cdot (\varepsilon + 1)^* \cdot \emptyset$	ε
$R_{12}^{(2)}$	$(0 + 1) + (0 + 1) \cdot (\varepsilon + 1)^* \cdot (\varepsilon + 1)$	$(0 + 1)1^*$
$R_{13}^{(2)}$	$\emptyset + (0 + 1) \cdot (\varepsilon + 1)^* \cdot 0$	$(0 + 1)1^*0$
$R_{21}^{(2)}$	$\emptyset + (\varepsilon + 1) \cdot (\varepsilon + 1)^* \cdot \emptyset$	\emptyset
$R_{22}^{(2)}$	$(\varepsilon + 1) + (\varepsilon + 1) \cdot (\varepsilon + 1)^* \cdot (\varepsilon + 1)$	1^*
$R_{23}^{(2)}$	$0 + (\varepsilon + 1) \cdot (\varepsilon + 1)^* \cdot 0$	1^*0
$R_{31}^{(2)}$	$\emptyset + 1 \cdot (\varepsilon + 1)^* \cdot \emptyset$	\emptyset
$R_{32}^{(2)}$	$1 + 1 \cdot (\varepsilon + 1)^* \cdot (\varepsilon + 1)$	$1+$
$R_{33}^{(2)}$		

$R_{11}^{(1)}$	ε
$R_{12}^{(1)}$	$0 + 1$
$R_{13}^{(1)}$	\emptyset
$R_{21}^{(1)}$	\emptyset
$R_{22}^{(1)}$	$\varepsilon + 1$
$R_{23}^{(1)}$	0
$R_{31}^{(1)}$	\emptyset
$R_{32}^{(1)}$	1
$R_{33}^{(1)}$	$\varepsilon + 0$

Algorithm based on Path Construction

► $R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)} \cdot (R_{kk}^{(k-1)})^* \cdot R_{kj}^{(k-1)}$

$R_{11}^{(2)}$	$\varepsilon + (0 + 1) \cdot (\varepsilon + 1)^* \cdot \emptyset$	ε
$R_{12}^{(2)}$	$(0 + 1) + (0 + 1) \cdot (\varepsilon + 1)^* \cdot (\varepsilon + 1)$	$(0 + 1)1^*$
$R_{13}^{(2)}$	$\emptyset + (0 + 1) \cdot (\varepsilon + 1)^* \cdot 0$	$(0 + 1)1^*0$
$R_{21}^{(2)}$	$\emptyset + (\varepsilon + 1) \cdot (\varepsilon + 1)^* \cdot \emptyset$	\emptyset
$R_{22}^{(2)}$	$(\varepsilon + 1) + (\varepsilon + 1) \cdot (\varepsilon + 1)^* \cdot (\varepsilon + 1)$	1^*
$R_{23}^{(2)}$	$0 + (\varepsilon + 1) \cdot (\varepsilon + 1)^* \cdot 0$	1^*0
$R_{31}^{(2)}$	$\emptyset + 1 \cdot (\varepsilon + 1)^* \cdot \emptyset$	\emptyset
$R_{32}^{(2)}$	$1 + 1 \cdot (\varepsilon + 1)^* \cdot (\varepsilon + 1)$	$1+$
$R_{33}^{(2)}$	$(\varepsilon + 0) + 1 \cdot (\varepsilon + 1)^* \cdot 0$	$\varepsilon + 0 + 11^*0$

$R_{11}^{(1)}$	ε
$R_{12}^{(1)}$	$0 + 1$
$R_{13}^{(1)}$	\emptyset
$R_{21}^{(1)}$	\emptyset
$R_{22}^{(1)}$	$\varepsilon + 1$
$R_{23}^{(1)}$	0
$R_{31}^{(1)}$	\emptyset
$R_{32}^{(1)}$	1
$R_{33}^{(1)}$	$\varepsilon + 0$

Algorithm based on Path Construction

► $R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)} \cdot (R_{kk}^{(k-1)})^* \cdot R_{kj}^{(k-1)}$

$R_{11}^{(3)}$	$\varepsilon + ((0 + 1)1^*0) \cdot (\varepsilon + 0 + 11^*0)^* \cdot \emptyset$	ε
$R_{12}^{(3)}$		
$R_{13}^{(3)}$		
$R_{21}^{(3)}$		
$R_{22}^{(3)}$		
$R_{23}^{(3)}$		
$R_{31}^{(3)}$		
$R_{32}^{(3)}$		
$R_{33}^{(3)}$		

$R_{11}^{(2)}$	ε
$R_{12}^{(2)}$	$(0 + 1)1^*$
$R_{13}^{(2)}$	$(0 + 1)1^*0$
$R_{21}^{(2)}$	\emptyset
$R_{22}^{(2)}$	1^*
$R_{23}^{(2)}$	1^*0
$R_{31}^{(2)}$	\emptyset
$R_{32}^{(2)}$	$1+$
$R_{33}^{(2)}$	$\varepsilon + 0 + 11^*0$

Algorithm based on Path Construction

► $R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)} \cdot (R_{kk}^{(k-1)})^* \cdot R_{kj}^{(k-1)}$

$R_{11}^{(3)}$	$\varepsilon + ((0 + 1)1^*0) \cdot (\varepsilon + 0 + 11^*0)^* \cdot \emptyset$	ε
$R_{12}^{(3)}$	$(0 + 1)1^* + ((0 + 1)1^*0) \cdot (\varepsilon + 0 + 11^*0)^* \cdot 1+$	
$R_{13}^{(3)}$		
$R_{21}^{(3)}$		
$R_{22}^{(3)}$		
$R_{23}^{(3)}$		
$R_{31}^{(3)}$		
$R_{32}^{(3)}$		
$R_{33}^{(3)}$		

$R_{11}^{(2)}$	ε
$R_{12}^{(2)}$	$(0 + 1)1^*$
$R_{13}^{(2)}$	$(0 + 1)1^*0$
$R_{21}^{(2)}$	\emptyset
$R_{22}^{(2)}$	1^*
$R_{23}^{(2)}$	1^*0
$R_{31}^{(2)}$	\emptyset
$R_{32}^{(2)}$	$1+$
$R_{33}^{(2)}$	$\varepsilon + 0 + 11^*0$

Algorithm based on Path Construction

$$\blacktriangleright R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)} \cdot (R_{kk}^{(k-1)})^* \cdot R_{kj}^{(k-1)}$$

$R_{11}^{(3)}$	$\varepsilon + ((0 + 1)1^*0) \cdot (\varepsilon + 0 + 11^*0)^* \cdot \emptyset$	ε
$R_{12}^{(3)}$	$(0 + 1)1^* + ((0 + 1)1^*0) \cdot (\varepsilon + 0 + 11^*0)^* \cdot 1 +$	
$R_{13}^{(3)}$	$(0 + 1)1^*0 + ((0 + 1)1^*0) \cdot (\varepsilon + 0 + 11^*0)^* \cdot (\varepsilon + 0 + 11^*0)$	

$R_{s,f1}^{(n)} + \dots + R_{s,fk}^{(n)}$ is the regular expression of the DFA

RE: $R_{13}^{(3)} = (0 + 1)1^*0 + ((0 + 1)1^*0) \cdot (\varepsilon + 0 + 11^*0)^* \cdot (\varepsilon + 0 + 11^*0)$

Simplified: $(0+1)1^*0(0+11^*0)^*$

$R_{11}^{(2)}$	ε
$R_{12}^{(2)}$	$(0 + 1)1^*$
$R_{13}^{(2)}$	$(0 + 1)1^*0$
$R_{21}^{(2)}$	\emptyset
$R_{22}^{(2)}$	1^*
$R_{23}^{(2)}$	1^*0
$R_{31}^{(2)}$	\emptyset
$R_{32}^{(2)}$	$1 +$
$R_{33}^{(2)}$	$\varepsilon + 0 + 11^*0$