# 2B 2020 - Homework 2 Solutions.

**(Q1)** We conclude that $V$ is a subspace of $\mathbb{R}^3$ and that $W$ is not a subspace of $\mathbb{R}^3$ and outline our reasoning as follows.

(i) We check the subspace definition for $V = \{(x, 2x, 3x) \mid x \in \mathbb{R}\}$.

(a) We have $0 \in \mathbb{R}$ and so $(0,0,0) = \underline{0} \in \mathbb{R}^3$.

(b) Consider the vectors $\underline{v}_1 = (x_1, 2x_1, 3x_1) \in V$, $\underline{v}_2 = (x_2, 2x_2, 3x_2) \in V$. We have

$$\underline{v}_1 + \underline{v}_2 = (x_1 + x_2,\ 2x_1 + 2x_2,\ 3x_1 + 3x_2)$$
$$= (x_1 + x_2,\ 2(x_1+x_2),\ 3(x_1+x_2)).$$

We see that $\underline{v}_1 + \underline{v}_2 \in V$. since $x_1 + x_2 \in \mathbb{R}$.

(c) Consider the vector $\underline{v} = (x, 2x, 3x) \in V$ and let $c \in \mathbb{R}$ be an arbitrary scalar. We have

$$c\underline{v} = c(x, 2x, 3x)$$
$$= (cx, 2cx, 3cx) = ((cx), 2(cx), 3(cx))$$

And so deduce that $c\underline{v} \in V$. since $cx \in \mathbb{R}$.

As $\underline{0} \in V$ and $V$ is closed under addition and scalar multiplication then we conclude that $V$ is a subspace of $\mathbb{R}^3$.

(ii) For $W = \{(x, x^2, x^3) \mid x \in \mathbb{R}\}$, we consider closure under addition. Consider $\underline{w}_1 = (x_1, x_1^2, x_1^3) \in W$ and $\underline{w}_2 = (x_2, x_2^2, x_2^3) \in W$,

$$\underline{w}_1 + \underline{w}_2 = (x_1, x_1^2, x_1^3) + (x_2, x_2^2, x_2^3)$$
$$= (x_1 + x_2,\ x_1^2 + x_2^2,\ x_1^3 + x_2^3) \notin W$$

We see $\underline{w}_1 \in W$, $\underline{w}_2 \in W$ but $\underline{w}_1 + \underline{w}_2 \notin W$ (as $x_1^2 + x_2^2 \neq (x_1+x_2)^2$ and $x_1^3 + x_2^3 \neq (x_1+x_2)^3$ in general) and so conclude that as $W$ is not closed under addition then it is not a subspace of $\mathbb{R}^3$

Assume, $a \neq 0$, if $a=0$ then interchange row 1 and 2 and proceed similarly. Note since $ad-bc \neq 0$, we cannot have both $a$ and $c = 0$.

Q2) WE PROCEED AS FOLLOWS:

$$(A|\mathbb{I}_2) = \left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right)$$

$$\underset{R_2 \to R_2 - \frac{c}{a}R_1}{\sim} \left(\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & d-\frac{cb}{a} & -\frac{c}{a} & 1 \end{array}\right)$$

$$\underset{R_1 \to \frac{1}{a}R_1}{\sim} \left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right)$$

$$\underset{R_2 \to \frac{a}{ad-bc}R_2}{\sim} \left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)$$

$$\underset{R_1 \to R_1 - \frac{b}{a}R_2}{\sim} \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{a}+\frac{bc}{a(ad-bc)} & -\frac{b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)$$

Now,

$$\left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{a}+\frac{bc}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right) = \left(\begin{array}{cc|cc} 1 & 0 & \frac{ad-bc+bc}{a(ad-bc)} & -\frac{b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)$$

So WE DEDUCE THAT THE INVERSE OF $A$ IS GIVEN BY

$$A^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

$$= \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Q3) We have

$$A = \begin{pmatrix} 1 & 2 & a \\ 1 & 4 & 1 \\ -3a & 4 & a+2 \end{pmatrix}$$

Performing row operations we have

$$\begin{pmatrix} 1 & 2 & a \\ 1 & 4 & 1 \\ -3a & 4 & a+2 \end{pmatrix} \underset{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 3aR_1}}{\sim} \begin{pmatrix} 1 & 2 & a \\ 0 & 2 & 1-a \\ 0 & 4+6a & 3a^2+a+2 \end{pmatrix}$$

$$\underset{R_2 \to \frac{1}{2}R_2}{\sim} \begin{pmatrix} 1 & 2 & a \\ 0 & 1 & \frac{1}{2}(1-a) \\ 0 & 4+6a & 3a^2+a+2 \end{pmatrix}$$

$$\underset{R_3 \to R_3 - (4+6a)R_2}{\sim} \begin{pmatrix} 1 & 2 & a \\ 0 & 1 & \frac{1}{2}(1-a) \\ 0 & 0 & 3a^2+a+2-(2+3a)(1-a) \end{pmatrix}.$$

We then have

$$\begin{pmatrix} 1 & 2 & a \\ 0 & 1 & \frac{1}{2}(1-a) \\ 0 & 0 & 3a^2+a+2-(2+3a)(1-a) \end{pmatrix} = \begin{pmatrix} 1 & 2 & a \\ 0 & 1 & \frac{1}{2}(1-a) \\ 0 & 0 & 6a^2 \end{pmatrix}.$$

If $a=0$ then $\text{rank}(A)=2$ and if $a \neq 0$ then $\text{rank}(A)=3$.

By the Rank-Nullity theorem we have

$$\text{Rank}(A) + \text{Nullity}(A) = 3$$

So it follows that when $a=0$, $\text{nullity}(A)=1$ and when $a \neq 0$ then $\text{Nullity}(A)=0$. The nullity is precisely the definition of the dimension of the null space.