University of Glasgow

EXAMINATION FOR THE DEGREES OF
M.A. AND B.Sc.

Mathematics 2E - Introduction to Real Analysis

An electronic calculator may be used provided that it does not have a facility for either textual storage or display, or for graphical display.

Candidates must attempt all questions.

Question 1 and question 2 are multiple choice questions. Use the response form "2E Degree Exam Multiple Choice Section" to record your answers.

1. (i) Which of the following statements is the negative of the statement

$$\forall y \in \mathbb{R}, \forall x \in \mathbb{R}, \quad (x^2 > 2 \text{ and } y > x) \implies y^2 > 2\,. \quad ?$$

(A) $\exists y \in \mathbb{R}, \exists x \in \mathbb{R}, \quad (x^2 > 2 \text{ and } y > x) \implies y^2 > 2.$
(B) $\forall y \in \mathbb{R}, \forall x \in \mathbb{R}, \quad (x^2 \leqslant 2 \text{ or } y \leqslant x) \implies y^2 > 2.$
(C) $\exists y \in \mathbb{R}, \exists x \in \mathbb{R}, \quad (x^2 > 2 \text{ and } y > x) \text{ and } y^2 \leqslant 2.$
(D) $\forall y \in \mathbb{R}, \forall x \in \mathbb{R}, \quad y^2 \leqslant 2 \implies (x^2 \leqslant 2 \text{ and } y \leqslant x).$
(E) None of these statements.
C

(ii) Let m be a lower bound for the set $A \subset \mathbb{R}$. Which of the following is equivalent to the statement that m is the greatest lower bound for A?

(A) $\exists \epsilon > 0, \exists a \in A, a < m + \epsilon.$
(B) $\forall \epsilon > 0, \exists a \in A, a > m - \epsilon.$
(C) $\forall \epsilon > 0, \forall a \in A, a > m - \epsilon.$
(D) $\forall \epsilon > 0, \exists a \in A, a < m + \epsilon.$
(E) None of these statements.
D

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(iii) Let $(x_n)_{n=1}^{\infty}$ be a real sequence and consider the following statement

$$\forall \epsilon > 0,\ \forall n_0 \in \mathbb{N},\ \exists n \in \mathbb{N} \text{ such that } (n \geqslant n_0 \text{ and } |x_n - L| < \epsilon).$$

Which of the following choices of sequence $(x_n)_{n=1}^{\infty}$ demonstrates that this statement does not imply that the sequence $(x_n)_{n=1}^{\infty}$ converges to L?

(A) $x_n = L, \forall n \in \mathbb{N}$.
(B) $\forall n \in \mathbb{N}, x_{2n+1} = L$ and $x_{2n} = L + 1$.
(C) $x_n = L + \frac{1}{n}, \forall n \in \mathbb{N}$.
(D) $x_n = L + (-1)^n$.
(E) None of these statements.
B

(iv) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Which of the following statements is equivalent to the statement that f is not bounded above?

(A) $\exists K \in \mathbb{R}, \forall x \in \mathbb{R}, f(x) < K$.
(B) $\forall K \in \mathbb{R}, \exists x \in \mathbb{R}, f(x) > K$.
(C) $\exists x \in \mathbb{R}, \forall K \in \mathbb{R}, f(x) < K$.
(D) $\forall x \in \mathbb{R}, \exists K \in \mathbb{R}, f(x) < K$.
(E) None of these statements.
B

(v) Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n > 0, \forall n \in \mathbb{N}$. Which one of the following statements is equivalent to the statement that $\sum_{n=1}^{\infty} a_n$ converges.

(A) $a_n \to 0, n \to \infty$.
(B) $\{a_n | n \in \mathbb{N}\}$ is a bounded set.
(C) $\frac{a_{n+1}}{a_n} < 1, \forall n \in \mathbb{N}$.
(D) $\exists K \in \mathbb{R}, \forall n \in \mathbb{N}, \sum_{r=1}^{n} a_r \leqslant K$.
(E) None of these statements.
D

2. (i) Let $f : \mathbb{R} \to \mathbb{R}$ be a real function and $c \in \mathbb{R}$. Which of the following statements is equivalent to the statement that f is not continuous at c.

(A) $\forall \epsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R}, |x - c| < \delta \implies |f(x) - f(c)| < \epsilon$.
(B) $\exists \epsilon > 0, \forall \delta > 0, \forall x \in \mathbb{R}, |x - c| \geqslant \delta$ and $|f(x) - f(c)| < \epsilon$.
(C) $\exists \epsilon > 0, \forall \delta > 0, \forall x \in \mathbb{R}, |x - c| < \delta \implies |f(x) - f(c)| < \epsilon$.
(D) $\exists \epsilon > 0, \forall \delta > 0, \exists x \in \mathbb{R}, |x - c| < \delta$ and $|f(x) - f(c)| \geqslant \epsilon$.
(E) None of these statements.
D

(ii) Let $f, g : \mathbb{R} \to \mathbb{R}$ be real functions. Which of the following statements is correct?

(A) If $f \circ g$ and f are continuous then g is continuous.
(B) If $f \circ g$ and g are continuous then f is continuous.
(C) There are functions $f, g : \mathbb{R} \to \mathbb{R}$ such that f and $g \circ f$ are not continuous.
(D) If f and g are not continuous then $f \circ g$ s not continuous.
(E) None of these statements.
C

(iii) Let $a < b$ and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Which of the following statements is the conclusion of the extremal value theorem?

(A) $\exists u, v \in [a, b]$ such that $\forall x \in [a, b]$, $f(u) \leqslant f(x) \leqslant f(v)$.
(B) $\exists u, v \in (a, b)$ such that $\forall x \in [a, b]$, $f(u) \leqslant f(x) \leqslant f(v)$.
(C) $\exists u, v \in [a, b]$ such that $\forall x \in [a, b]$, $f(u) < f(x) < f(v)$.
(D) $\exists u, v \in [a, b]$ such that $\forall x \in (a, b)$, $f(u) < f(x) < f(v)$.
(E) None of these statements.
A

(iv) Let $\epsilon > 0$ and $n \in \mathbb{N}$. Which of the following conditions on $n_0 \in \mathbb{N}$ ensure that the implication

$$(n \in \mathbb{N} \text{ and } n \geqslant n_0) \implies \left| \frac{n^2}{2n^2 + 2n + 1} - \frac{1}{2} \right| < \epsilon$$

holds?

(A) $n_0 > \frac{1}{2\epsilon}$.
(B) $n_0 > \frac{1}{2}$.
(C) $n_0 > \frac{1}{12}\epsilon$.
(D) $n_0 < \frac{1}{2\epsilon}$.
(E) None of these statements.
A

(v) Let $(x_n)_{n=1}^{\infty}$ be a sequence such that $x_n > 0, \forall n \in \mathbb{N}$. Which of the following statements is a consequence of the statement that $\sum_{n=1}^{\infty} x_n$ is convergent?

(A) $\exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant n_0, x_{n+1} < x_n$.
(B) $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 0$.
(C) $\forall \epsilon > 0, \exists N \in \mathbb{N}, x_N < \epsilon$.
(D) $\sum_{n=1}^{\infty} n x_n$ is convergent.
(E) None of these statements.
C

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3. (i) Show that the set

$$A = \{x^2 - y^2 | x \in \mathbb{R}, y \in [-1, 1]\}$$

is bounded below and unbounded above.

Solution. $y \in [-1,1] \implies 0 \leqslant y^2 \leqslant 1 \implies -1 \leqslant -y^2 \leqslant 0$. *Also* $x^2 \geqslant 0$. *Hence* $x^2 - y^2 \geqslant -1$ *and* A *is bounded below. Let* $K > 0$. *Let* $y = 0$ *and* $x > \sqrt{K}$. *Then* $x^2 - y^2 > K$ *and hence* A *is unbounded above.*

(ii) Show that

$$\sup\{\frac{n^2 - 1}{n^2 + 1} | n \in \mathbb{N}\} = 1.$$

Solution. Firstly, $1 - \frac{n^2-1}{n^2+1} = \frac{2}{n^2+1} > 0, \forall n \in \mathbb{N}$. *Hence* 1 *is an upper bound. Let* $\epsilon > 0$. *Choose* $n_0 \in \mathbb{N}, n_0 > \sqrt{\frac{2}{\epsilon}}$. *then* $n \geqslant n_0 \implies n^2 \geqslant n_0^2 > \frac{2}{\epsilon} - 1 \implies \frac{n^2+1}{2} > \frac{1}{\epsilon} \implies \frac{n^2-1}{n^2+1} = 1 - \frac{2}{n^2+1} > 1 - \epsilon$. *Hence* 1 *is the least upper bound.*

4. Using the definition of convergence for sequences, prove that:

(i) The limit of a convergent sequence is unique.

Solution. Suppose not. Then the sequence $(x_n)_{n=1}^{\infty}$ *has distinct limits* L *and* M. *Define* $\epsilon = \frac{|L-M|}{2} > 0$. *By convergence there exist* $n_1, n_2 \in \mathbb{N}$ *such that for* $n \geqslant max\{n_1, n_2\}$, *both* $|x_n - L| < \epsilon$ *and* $|x_n - M| < \epsilon$. *Hence* $|L - M| = |L - x_n - (M - x_n)| \leqslant |x_x - L| + |x_n - M| < |L - M|$. *Contradiction. Hence* $L = M$.

(ii) If there exists a $\delta > 0$ such that $|x_{n+1} - x_n| > \delta$ for all $n \in \mathbb{N}$ then $(x_n)_{n=1}^{\infty}$ is divergent.

Solution. Suppose $x_n \to L, n \to \infty$. *Choose* $\epsilon = \frac{\delta}{2}$. *Then there exist* $n_0 \in \mathbb{N}$ *such that* $n \geqslant n_0 \implies |x_n - L| < \epsilon$. *So for* $n \geqslant n_0$, $|x_{n+1} - L - x_n + L| \leqslant |x_{n+1} - L| + |x_n - L| < \delta$. *This contradicts the hypothesis for* $n \geqslant n_0$. *Hence the sequence is divergent.*

5. (i) State the sandwich principle for sequences.

Solution. Let $(x_n)_{n=1}^{\infty}$ *and* $(z_n)_{n=1}^{\infty}$ *be real sequences converging to the same limit,* L. *If a third sequence* $(y_n)_{n=1}^{\infty}$ *satisfies, for* $n > N \in \mathbb{N}$, *the inequalities*

$$x_n \leqslant y_n \leqslant z_n$$

then $(y_n)_{n=1}^{\infty}$ *also converges to* L.

(ii) Prove that the sequence $y_n = (2^n + 3^n + 4^n)^{\frac{1}{n}}$ converges with limit equal to 4.

Solution. Let $x_n = 4$ *and* $z_n = 4 \times 3^{\frac{1}{n}}, \forall n \in \mathbb{N}$ *then*

$$x_n = (4^n)^{\frac{1}{n}} < (2^n + 3^n + 4^n)^{\frac{1}{n}} < (3 \times 4^n)^{\frac{1}{n}} = z_n.$$

But by the standard limit $a^{\frac{1}{n}} \to 1, m \to \infty$ *for* $a > 0$ *both* $(x_n)_{n=1}^{\infty}$ *and* $(z_n)_{n=1}^{\infty}$ *have limit* 4. *By the sandwich principle, therefore,* $(y_n)_{n=1}^{\infty}$ *has limit* 4.

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6. For each of the series below, determine whether they converge or diverge. Be careful to justify your conclusions by referring to clearly stated results and tests from the course. Answers with no such justification will receive no marks.

   (i) $\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$
   Solution. We use the comparison test. Note that all terms are positive and that

$$\frac{n+1}{n^2+1} > \frac{n}{n^2+1} > \frac{n}{2n^2} = \frac{1}{2n}.$$

   Since the terms of the series in question dominate those of the series $\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ *which is divergent, it must be divergent also.*

   (ii) $\sum_{n=1}^{\infty} \frac{n!^2}{(2n)!}$
   Solution. We use the ratio test. Consider

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!^2}{(2n+2)!}\frac{(2n)!}{n!^2} = \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{2}\frac{n+1}{2n+1} < \frac{1}{2}, \forall n \in \mathbb{N}.$$

   Since $\frac{1}{2} < 1$ *the ratio test determines that the series is convergent.*

   (iii) $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1}$
   Solution. If this is to converge it may do so by the Leibniz test. Let $a_n = \frac{n}{n^2+1} > 0$. *The terms alternate in sign,* $a_n < \frac{n}{n^2} = \frac{1}{n} \to 0$, *and*

$$a_{n+1} - a_n = \frac{(n+1)(n^2+1) - n(n^2+2n+2)}{(n^2+1)(n^2+2n+2)} = \frac{-n^2-n+1}{(n^2+1)(n^2+2n+2)} < 0,$$

   $\forall n \in \mathbb{N}$. *The three conditions for convergence in the Leibniz test are therefore satisfied.*

7. (i) State the intermediate value theorem.
   Let $f : [a,b] \to \mathbb{R}$ *be a continuous function and assume that* d *is a number such that* $f(a) < d < f(b)$ *or* $f(b) < d < f(a)$. *Then there exists a point* $c \in (a,b)$ *such that* $f(c) = d$.

   (ii) Suppose that $f : [-1,1] \to [-1,1]$ is a continuous function. Show there exists $c \in [-1,1]$ such that $f(c) = 1 - c - c^2$.
   Solution. Consider the function $g : [-1,1] \to \mathbb{R}$ *defined by* $g(x) = f(x) - 1 + x + x^2$. *This function is continuous. Also* $g(-1) = f(-1) - 1 \leqslant 0$ *and* $g(1) = f(1) + 1 \geqslant 0$ *since* $-1 \leqslant f(x) \leqslant 1$ *for* $x \in [-1,1]$. *The intermediate value theorem then implies that* $\exists c \in [-1,1], g(c) = 0$, *i.e.* $f(c) = 1 - c - c^2$.

8. (i) Let $(x_n)_{n=1}^{\infty}$ be a real sequence converging to the limit L and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that

$$\lim_{n\to\infty} f(x_n) = f(L).$$

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Solution. $\epsilon > 0$. *By continuity of* f *at* L, $\exists \delta > 0, \forall x \in \mathbb{R}, |x - L| < \delta \implies |f(x) - f(L)| < \epsilon$. *Now let* $(x_n)_{n+1}^{\infty}$ *be a sequence converging to* L. *Then* $\exists n_0 \in \mathbb{N}, n > n_0 \implies |x_n - L| < \delta$. *Therefore* $n > n_0 \implies |f(x_n) - f(L)| < \epsilon$.

(ii) Give an example of a continuous function $f : (-1, 1) \to \mathbb{R}$ that is both unbounded above and unbounded below. Justify your answer.

Solution. $f(x) = \frac{2x}{1-x^2}$. *Let* $K > 0$. *Consider* $x \in [\frac{1}{2}, 1)$. *Then* $x \geqslant \frac{1}{2}$ *and* $\frac{1}{1+x} > \frac{1}{2}$. *Therefore* $\frac{2x}{1-x^2} = \frac{2x}{(1+x)(1-x)} > \frac{1}{2}\frac{1}{1-x}$. *So for* $x > x_0 = \max\{\frac{1}{2}, 1 - \frac{1}{2K}\}$ *we have* $f(x) > K$. *A Similar argument for* $x \in (-1, -\frac{1}{2}]$ *will give* $f(x) < -K$ *for* $x < x_1 = \min\{-\frac{1}{2}, -1 + \frac{1}{2K}.\}$

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