Q1: Find a value of $K > 0$ such that the implication

$$|x+4| < 2 \Rightarrow |\frac{x^2+4}{x-1}+4| \leq K|x+4|$$

is true. Make sure you justify your choice of *K*, with a proof of the implication for your value of *K*.

By simplifying the expression on the RHS, we get

$$\frac{x^2+4}{x-1}+4 = \frac{x^2+4+4x-4}{x-1} = \frac{x(x+4)}{x-1}.$$

From this we can see that

$$K = \frac{|x|}{|x-1|}.$$

To get the required parts of K, we need to analyse the LHS:

$$\begin{aligned}
&|x+4| < 2 \Leftrightarrow -2 < x+4 < 2\\
&\Leftrightarrow -6 < x < -2\\
&\Leftrightarrow -7 < x-1 < -3\\
&\Leftrightarrow -\frac{1}{3} < \frac{1}{x-1} < -\frac{1}{7}.
\end{aligned}$$

By converting those parts to absolute values, we get

$$2 < |x| < 6,$$

$$\frac{1}{7} < \frac{1}{|x-1|} < \frac{1}{3}.$$

Once multiplied, they are

$$\frac{2}{7} < \frac{|x|}{|x-1|} < 2.$$

As the inequality is a strict one, we need to choose

$$K > 2$$

in order to satisfy it. By taking $K = 3$ as the nearest positive integer, the implication is satisfied, as required.

Q2: Let $x, y \in \mathbb{R}$. Use the triangle inequality to prove the reverse triangle inequality

$$\big||x|-|y|\big| \leq |x-y|.$$

The term $|x|$ can be expressed as

$$|x| = |x - y + y|.$$

According to the triangle inequality, dividing the RHS into two terms, $(x - y)$ and y, we get

$$\begin{aligned}
&|x - y + y| \leq |x - y| + |y| \\
&\Leftrightarrow |x| \leq |x - y| + |y| \\
&\Leftrightarrow |x| - |y| \leq |x - y|,
\end{aligned}$$

which proves one case. By expressing the term |y| as

$$|y| = |y - x + x|,$$

the triangle equality can be applied to the RHS to get

$$\begin{aligned}
&|y - x + x| \leq |y - x| + |x| \\
&\Leftrightarrow |y| \leq |y - x| + |x| \\
&\Leftrightarrow |y| - |x| \leq |y - x|.
\end{aligned}$$

Because $|y - x| = |-(x - y)| = |x - y|$, the above inequality can also be expressed as

$$|x - y| \geq |y| - |x|,$$

thus proving the other case. Taking them together, we get

$$|x - y| \geq \big||x| - |y|\big|,$$

as required.

Q3: Show that the function $f: \mathbb{N} \to \mathbb{R}$ given by

$$f(n) = \frac{8n^2 - 7n + 1}{4n^2 + 8n + 3}$$

is bounded above.

By using the polynomial estimation lemma, there exist $N_1, N_2 \in \mathbb{N}$

$$n \geq N_1 \Rightarrow \frac{1}{2}8n^2 \leq 8n^2 - 7n + 1 \leq \frac{3}{2}8n^2;$$

$$n \geq N_2 \Rightarrow \frac{1}{2}4n^2 \leq 4n^2 + 8n + 3 \leq \frac{3}{2}4n^2.$$

Define $N = \max(N_1, N_2)$ so that for $n \geq N$, we have

$$f(n) \leq \frac{\frac{3}{2}8n^2}{\frac{1}{2}4n^2} = 6.$$

Since the domain of the function is $\mathbb{N}$, there are only finitely many natural numbers *n* with $n \leq N$.

Thus, we can define $M = \max(f(1), f(2), \dots, f(N-1), 6)$. Then, for any $n \in \mathbb{N}$, we have $f(n) \leq M$, so that f is bounded above, as required.