2C Week 7 2020/21

Handout

Series

In this chapter we will study *series* of real numbers. This is the mathematical framework in which we can make precise the idea of *infinite sums*, like

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots .$$

Let's start with the following definition.

Definition 4.1. Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. The (infinite) series generated by $(a_n)_{n=1}^{\infty}$ is the sequence $(s_n)_{n=1}^{\infty}$ where

$$s_n = \sum_{j=1}^{n} a_j = a_1 + a_2 + \cdots + a_n.$$

The number s_n is called the n-th partial sum of the series. We also use the notation

$$\sum_{j=1}^{\infty} a_j$$

when referring to the sequence $(s_n)_{n=1}^{\infty}$ of partial sums.

In other words, series are a certain kind of sequences, obtained by summing the terms of a sequence. In particular, series are related to sequences in two ways. On the one hand, to define the series $\sum_{n=1}^{\infty} a_n$, we need the sequence $(a_n)_{n=1}^{\infty}$ as input. On the other hand, the series $\sum_{n=1}^{\infty} a_n$ itself is nothing but a sequence in disguise. By definition, it is the sequence $(s_n)_{n=1}^{\infty}$ of partial sums s_n.

In order to distinguish the sequences $(a_n)_{n=1}^{\infty}$ and $(s_n)_{n=1}^{\infty}$, the notation $\sum_{j=1}^{\infty} a_j$ introduced above as an abbreviation for $(s_n)_{n=1}^{\infty}$ is useful[1]. At this point, we should not interpret the symbol $\sum_{j=1}^{\infty} a_j$ as a real number, given by an actual (infinite) sum. This will only be done once we have discussed convergence of sequences below[2].

[1] Sometimes we use the shorthand notation $\sum a_j$ instead of $\sum_{j=1}^{\infty} a_j$.

[2] Unfortunately, the standard mathematical notation is ambiguous in this regard. Try not to get confused by this!

Let us examine in more detail the relationship between the sequences $(a_n)_{n=1}^{\infty}$ and $(s_n)_{n=1}^{\infty}$, that is, between a sequence $(a_n)_{n=1}^{\infty}$ and its associated series. By construction, the sequence $(s_n)_{n=1}^{\infty}$ is obtained from $(a_n)_{n=1}^{\infty}$ by summation. Conversely, one can entirely recover $(a_n)_{n=1}^{\infty}$ from $(s_n)_{n=1}^{\infty}$ since

$$\begin{aligned} a_1 &= s_1 \\ a_2 &= (a_1 + a_2) - a_1 = s_2 - s_1 \\ a_3 &= (a_1 + a_2 + a_3) - (a_1 + a_2) = s_3 - s_2, \end{aligned}$$

and more generally,

$$a_n = s_n - s_{n-1}$$

for any $n > 1$. In other words, the information contained in the sequences $(a_n)_{n=1}^{\infty}$ and $(s_n)_{n=1}^{\infty}$ is equivalent.

So why do we bother about series at all, and not just continue working with sequences? The main reason is that many familiar functions occur naturally in the form of series, and are best understood from this point of view[3]. Still, one should keep in mind that no information is lost by switching back and forth between series and sequences. When proving results about series, we will often use this to reduce matters to known results about sequences from chapter 3.

[3] For instance, to rigorously define and study the exponential function exp and trigonometric functions like sin and cos, one needs *power series*. This will be explained in the 3H course Analysis of Differentiation and Integration.

We will be flexible regarding the starting term of our series. For instance, it is sometimes useful to consider series of the form $\sum_{n=0}^{\infty} a_n$, starting at $n = 0$ instead of $n = 1$, or at other values.

Convergence of series

Since series are nothing but sequences in disguise, we define the notion of convergence for series as follows.

Definition 4.2. Let $\sum_{n=1}^{\infty} x_n$ be a series of real numbers and let $L \in \mathbb{R}$. We say that the series $\sum_{n=1}^{\infty} x_n$ converges towards $L \in \mathbb{R}$ if the sequence $(s_n)_{n=1}^{\infty}$ of partial sums $s_n = \sum_{j=1}^{n} x_n$ converges towards L as n tends to infinity. In this case we also write $L = \sum_{n=1}^{\infty} x_n$ for the corresponding limit[4], and call it the *sum of the series*.
A series is called *divergent* if it does not converge to any limit.

[4] Here the symbol $\sum_{n=1}^{\infty} x_n$ stands for a real number, in contrast to the usage of this symbol as an abbreviation for the sequence of partial sums.

This definition encapsulates the intuitive idea that the limit L of the sequence of partial sums $s_n = x_1 + \cdots + x_n$ of a series is the "sum to infinity" of the terms x_n.

Let us point out again that we use the symbol $\sum_{n=1}^{\infty} x_n$ ambiguously to mean both the series, which always exists, and its sum, which may not exist.

In a similar fashion as we discussed convergence of sequences, Definition 4.2 can be written concisely as a quantified statement: the series $\sum_{n=1}^{\infty} x_n$ converges to $L \in \mathbb{R}$ if and only if

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, \left(n \geq n_0 \implies \left| \sum_{j=1}^{n} x_j - L \right| < \varepsilon \right).$$

Equivalently, we may write this condition as

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall (n \in \mathbb{N} \text{ with } n \geq n_0), \left| \sum_{j=1}^{n} x_j - L \right| < \varepsilon.$$

Notice that leaving out the first few terms in a series does not affect convergence, but it will usually affect the sum of a series[5].

[5] This should be compared with the situation for sequences: leaving out the first few terms in a convergent sequence $(x_n)_{n=1}^{\infty}$ neither affects convergence, nor the value of the limit $\lim_{n\to\infty} x_n$ of the sequence.

Properties of series

In this section we examine basic properties of series, and how convergent series interact with the algebraic operations on the real numbers. These rules provide methods for establishing convergence or divergence, and also tools for calculating sums of convergent series.

Theorem 4.3. *If a series $\sum_{n=1}^{\infty} x_n$ converges, then the sequence $(x_n)_{n=1}^{\infty}$ converges to 0, that is, $x_n \to 0$ as $n \to \infty$.*

Proof. Let $(s_n)_{n=1}^{\infty}$ the sequence of partial sums of the series $\sum_{n=1}^{\infty} x_n$. By assumption this sequence converges to S for some $S \in \mathbb{R}$. Then also the sequence[6] $(t_n)_{n=2}^{\infty}$ given by $t_n = s_{n-1}$ converges[7] to S. Using algebraic properties of limits, we deduce

$$x_n = s_n - s_{n-1} \to S - S = 0$$

as n tends to infinity. □

[6] Recall that we are flexible regarding the starting point of sequences. Here we start at $n = 2$ because $t_1 = s_0$ is not defined.

[7] For $\varepsilon > 0$ let $n_0 \in \mathbb{N}$ be chosen such that $|s_n - S| < \varepsilon$ for all $n \geq n_0$. Then we have $|t_n - S| = |s_{n-1} - S| < \varepsilon$ for all $n \geq n_0 + 1$. This means $t_n \to S$ as $n \to \infty$.

The contrapositive of the statement in Theorem 4.3 can be formulated as follows: if $a_n \not\to 0$ then $\sum_{n=1}^{\infty} a_n$ diverges. That is, the previous theorem provides us with a useful criterion to show that series do *not* converge: if the coefficient sequence fails to converge to 0, the series diverges.

Example 4.4. Show that the series

$$\sum_{n=1}^{\infty} \frac{3n+5}{7n+2}$$

diverges.

Solution. According to Theorem 4.3 it suffices to show that the sequence $(a_n)_{n=1}^{\infty}$ with $a_n = \frac{3n+5}{7n+2}$ does not converge to zero. By algebraic properties of sequence limits [8], we have

$$\frac{3n+5}{7n+2} \to \frac{3}{7} \neq 0.$$

Therefore[9], the series $\sum_{n=1}^{\infty} \frac{3n+5}{7n+2}$ diverges. □

[8] see Theorem 3.10 in Chapter 3.

[9] Alternatively, we could argue that

$$\frac{3n+5}{7n+2} \geq \frac{3n}{7n+2n} = \frac{3}{9}$$

for all $n \in \mathbb{N}$, which is enough to show that $(a_n)_{n=1}^{\infty}$ does not converge to zero.

We now come to an important example of a series.

Proposition 4.5. *The* geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

converges if and only if $|x| < 1$. For $|x| < 1$, the sum of the series is $\frac{1}{1-x}$.

Proof. Firstly, the geometric series $\sum_{n=0}^{\infty} x^n$ for $x = 1$ and $x = -1$ clearly diverges[10]. Now for any $x \in \mathbb{R}$ with $x \neq 1$ we have

$$\begin{aligned}(1-x)\sum_{j=0}^{n} x^j &= (1-x)(1 + x + x^2 + \cdots + x^n) \\ &= (1 + x + x^2 + \cdots + x^n) - (x + x^2 \cdots - x^{n+1}) \\ &= 1 - x^{n+1},\end{aligned}$$

or equivalently,

$$\sum_{j=0}^{n} x^n = \frac{1 - x^{n+1}}{1 - x}.$$

Moreover $x^n \to 0$ as $n \to \infty$ if and only if $|x| < 1$ by one of our standard limits, see Theorem 3.18. This has two consequences: firstly, for $|x| < 1$ the geometric series is convergent with sum $\sum_{j=0}^{n} x^n = 1/(1-x)$. Secondly, for $|x| > 1$ the series $\sum_{j=0}^{n} x^n$ diverges according to Theorem 4.3 since the sequence $a_n = x^n$ is unbounded[11]. □

[10] For $x = 1$, the geometric series has partial sums $s_n = n$, so is unbounded and therefore divergent. For $x = -1$, we have

$$s_n = \frac{1}{2}(1 - (-1)^{n+1}),$$

which is a divergent sequence since the subsequences $s_{2n} = 1$ and $s_{2n+1} = 0$ converge to different limits.

[11] and therefore in particular not converging to zero.

Proposition 4.5 shows in particular that, say, the series $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is convergent. This may come as a surprise at first sight: essentially, we are adding *infinitely many strictly positive terms*, and nonetheless get a *finite* total sum[12].

[12] This fact puzzled already famous ancient Greek philosophers. Indeed, the series $\sum_{n=0}^{\infty} \frac{1}{2^n}$ plays a central role in one version of *Zeno's paradox*. If you're curious to learn more about this, try to search on the internet!

Let us next discuss general properties of convergent series, and in particular how sums of series interact with the algebraic operations of addition, multiplication, and the order structure of the real numbers.

Theorem 4.6 (Properties of convergent series)**.** *Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be series with sums S and T respectively.*

a) For $\lambda \in \mathbb{R}$, we have $\sum_{n=1}^{\infty}(\lambda x_n) = \lambda S$;

b) The series $\sum_{n=1}^{\infty}(x_n + y_n)$ converges with sum $S + T$, that is,

$$\sum_{n=1}^{\infty}(x_n + y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n.$$

c) If $x_n \leq y_n$ for all $n \in \mathbb{N}$, then $S \leq T$, that is, $\sum_{n=1}^{\infty} x_n \leq \sum_{n=1}^{\infty} y_n$.

d) If $\sum_{n=1}^{\infty} |x_n|$ also converges with sum U, then $|S| \leq U$, that is,

$$|\sum_{n=1}^{\infty} x_n| \leq \sum_{n=1}^{\infty} |x_n|.$$

I'm going to prove only the first part, and encourage you to prove the remaining three statements. The strategy is similar to the ideas used to establish algebraic properties of convergent sequences.

Proof of a). Let $\sum_{n=1}^{\infty} x_n$ be a convergent series with sum S. That is, the sequences of partial sums $(s_n)_{n=1}^{\infty}$, given by $s_n = x_1 + \cdots + x_n$, converges. The n-th partial sum for the series $\sum_{n=1}^{\infty} \lambda x_n$ is

$$\sum_{j=1}^{n} \lambda x_j = \lambda \sum_{j=1}^{n} x_j = \lambda s_n,$$

so by algebraic properties of limits[13], it is a convergent sequence. By definition, this means that the series $\sum_{n=1}^{\infty}(\lambda x_n)$ converges, with sum λS. □

[13] for convergent sequences, compare Theorem 3.10 in Chapter 3.

Note that the key idea in this proof was to reduce the claim to a familiar statement about convergent sequences.

Let us now have a look at further examples of series. A prominent example of a divergent series is the *harmonic series*.

Proposition 4.7. *The harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent.

Proof. Let us write

$$s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

for the n-th partial sum of the harmonic series. By way of contradiction, we shall assume that $\sum_{n=1}^{\infty} \frac{1}{n}$ is convergent, and write $S = \lim_{n\to\infty} s_n$ for its sum.

Consider in addition the series $\sum_{n=1}^{\infty} y_n$ where

$$y_n = \begin{cases} 1 & n = 1 \\ 1/2 & n = 2 \\ 1/(n+1) & n \geq 3 \text{ and odd} \\ 1/n & n \geq 4 \text{ and even} \end{cases}$$

We claim that our assumption implies that $\sum_{n=1}^{\infty} y_n$ converges as well. Firstly, if t_n denotes the n-th partial sum of the series $\sum_{n=1}^{\infty} y_n$, then we have

$$\begin{aligned} t_{2n} &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} + \cdots + \frac{1}{2n} + \frac{1}{2n} \\ &= \frac{1}{2} + 1 + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \cdots + \left(\frac{1}{2n} + \frac{1}{2n}\right) \\ &= \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \\ &= \frac{1}{2} + s_n \end{aligned}$$

for all $n \in \mathbb{N}$. Since the strictly increasing sequence $(s_n)_{n=1}^{\infty}$ is convergent by assumption, it is bounded above by its limit S. Hence the sequence $(t_{2n})_{n=1}^{\infty}$ is bounded above by $\frac{1}{2} + S$. Moreover, the sequence $(t_n)_{n=1}^{\infty}$ is clearly strictly increasing as well, which means that $\frac{1}{2} + S$ is in fact an upper bound[14] for all terms t_n.

According to the monotone convergence theorem, the sequence $(t_n)_{n=1}^{\infty}$ is therefore convergent. Moreover, its limit T must satisfy $T \leq \frac{1}{2} + S$. Using the formula $t_{2n} = \frac{1}{2} + s_n$ and properties of limits, we obtain in fact[15] $T = \frac{1}{2} + S$.

However, by the very construction of y_n we have $y_n \leq 1/n$ for all n, which implies $T \leq S$ according to Theorem 4.6 c). Hence $\frac{1}{2} + S \leq S$, which is a contradiction. $\square$

[14] For any $n \in \mathbb{N}$, we have $t_n < t_{2n} \leq \frac{1}{2} + S$.

[15] Recall that if $(x_n)_{n=1}^{\infty}$ is a convergent sequence, then also every subsequence of $(x_n)_{n=1}^{\infty}$ is convergent with the same limit. In particular, the subsequence $(t_{2n})_{n=1}^{\infty}$ of even terms of the sequence $(t_n)_{n=1}^{\infty}$ converges to T.

How can we decide whether an infinite series converges or not? In the following paragraphs we discuss several methods to answer this question.

Tests for convergence I: the comparison test

Let us start with the comparison test.

Theorem 4.8 (Comparison Test). *Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be sequences, and assume that there exists an $N \in \mathbb{N}$ such that $0 \leq x_n \leq y_n$ for all $n \geq N$. Then*

$$\sum_{n=1}^{\infty} y_n \text{ converges} \implies \sum_{n=1}^{\infty} x_n \text{ converges},$$

$$\sum_{n=1}^{\infty} x_n \text{ diverges} \implies \sum_{n=1}^{\infty} y_n \text{ diverges}.$$

Proof. The second statement is the contrapositive of the first, so it's enough to prove the first statement.

Since convergence of a series does not depend on the first N terms, we may assume $N = 1$. Let's write s_n and t_n for the n-th partial sums of $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$, respectively. Then our assumption gives $0 \leq s_n \leq t_n$ for all $n \in \mathbb{N}$. Since (t_n) converges it is bounded above by M. Therefore $(s_n)_{n=1}^{\infty}$ is bounded as well. Moreover, we have

$$s_{n+1} = s_n + x_{n+1} \geq s_n,$$

so that $(s_n)_{n=1}^{\infty}$ is increasing. By the monotone convergence theorem, the sequence $(s_n)_{n=1}^{\infty}$ converges. That is, $\sum_{n=1}^{\infty} x_n$ is convergent. □

As a consequence of Theorem 4.8 we obtain the following variant of the comparison test.

Corollary 4.9 (Limit version of comparison test)**.** *Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be sequences of eventually positive*[16] *terms, and suppose that $x_n/y_n \to L$ as $n \to \infty$ for some $L \in (0, \infty)$. Then*

$$\sum_{n=1}^{\infty} x_n \text{ converges} \iff \sum_{n=1}^{\infty} y_n \text{ converges.}$$

[16] A sequence $(a_n)_{n=1}^{\infty}$ is called eventually positive if there exists $n_0 \in \mathbb{N}$ such that $a_n > 0$ for all $n \geq n_0$.

Proof. Since convergence of a series is not affected by the first few terms, we may assume that $x_n, y_n > 0$ for all n. By assumption, the sequence $\frac{x_n}{y_n}$ converges and hence is bounded above, say by M. Then we have $0 \leq \frac{x_n}{y_n} \leq M$, or equivalently $0 \leq x_n \leq My_n$ for all $n \in \mathbb{N}$. Thus convergence of $\sum_{n=1}^{\infty} y_n$ implies convergence of $\sum_{n=1}^{\infty} x_n$ by the comparison test.

Conversely, by algebraic properties of sequence limits we have $\frac{y_n}{x_n} \to L^{-1}$. In particular, the sequence $\frac{y_n}{x_n}$ is bounded above as well. In the same way as above, this shows that convergence of $\sum_{n=1}^{\infty} x_n$ implies convergence of $\sum_{n=1}^{\infty} y_n$. □

Let's turn to some applications of the comparison test. Note that it gives us a method for showing that a series converges without having to find the proposed limit first. After having established that the series converges you may be able to find the limit by other means, such as properties of limits.

Proposition 4.10. *Let $p \in \mathbb{N}$. Then*[17] *the series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$.

[17] This result holds in fact for any real number $p > 1$. Since we have not discussed how to rigorously define exponentials n^p for $p \in \mathbb{R}$ we restrict ourselves to the case $p \in \mathbb{N}$.

Proof. The case $p = 1$ corresponds to the harmonic series, which we already know to be divergent by Proposition 4.7.

Let us next consider $p = 2$ and define $x_n = \frac{1}{n(n+1)}$. Since

$$\begin{aligned} x_k = \frac{1}{k(k+1)} &= \frac{k+1}{k(k+1)} - \frac{k}{k(k+1)} \\ &= \frac{1}{k} - \frac{1}{k+1} \end{aligned}$$

for all $k \in \mathbb{N}$ we see that

$$x_1 + x_2 + \cdots + x_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= 1 - \frac{1}{n+1}.$$

It follows that $\sum_{n=1}^{\infty} x_n$ is convergent with limit

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

Since $\frac{1}{(n+1)^2} < \frac{1}{n(n+1)}$ for all $n \in \mathbb{N}$, the comparison test shows that the series $\sum_{n=1} \frac{1}{(n+1)^2}$ is convergent. We conclude that

$$\sum_{n=1} \frac{1}{n^2} = 1 + \sum_{n=1} \frac{1}{(n+1)^2}$$

is convergent as well[18].

Let us now assume $p \geq 3$ is arbitrary. Since in this case $\frac{1}{n^p} \leq \frac{1}{n^2}$, the comparison test shows that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent as well. □

[18] If you're unsure why the last equality holds remind yourself that both sides of the equation are defined as limits of certain sequences of partial sums. Compare the partial sums on both sides and then take limits.

Here are some further applications of the comparison test.

Example 4.11. Discuss the convergence of

$$\sum_{n=1}^{\infty} \frac{n^2 - 5n + 1}{n^4 + n^2 + 1}.$$

Solution. For $n \geq 5$ we have

$$0 \leq \frac{n^2 - 5n + 1}{n^4 + n^2 + 1} \leq \frac{n^2 + 1}{n^4} \leq \frac{n^2 + n^2}{n^4} = \frac{2}{n^2}.$$

Using the comparison test and Proposition 4.10 we conclude that $\sum_{n=1}^{\infty} \frac{n^2-5n+1}{n^4+n^2+1}$ is convergent.

Alternatively, set $x_n = \frac{n^2-5n+1}{n^4+n^2+1}$ and $y_n = \frac{1}{n^2}$. Then the terms x_n and y_n are eventually positive, and we have

$$\frac{x_n}{y_n} = n^2 \cdot \frac{n^2 - 5n + 1}{n^4 + n^2 + 1} = \frac{n^4 - 5n^3 + n^2}{n^4 + n^2 + 1} \to 1$$

as $n \to \infty$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by Proposition 4.10, the limit version of the comparison test shows that $\sum_{n=1}^{\infty} \frac{n^2-5n+1}{n^4+n^2+1}$ is convergent. □

Tests for convergence II: the ratio test

A useful test for convergence is obtained by comparison with the geometric series. More precisely, we have the following result.

Theorem 4.12 (Ratio Test)**.** *Let $(x_n)_{n=1}^{\infty}$ be a sequence.*

a) Assume there exists $\lambda < 1$ and $n_0 \in \mathbb{N}$ such that $x_n > 0$ and $\frac{x_{n+1}}{x_n} \leq \lambda$ for all $n \geq n_0$. Then $\sum_{n=1}^{\infty} x_n$ converges.

b) *Assume there exists $n_0 \in \mathbb{N}$ such that $x_n > 0$ and $\frac{x_{n+1}}{x_n} \geq 1$ for all $n \geq n_0$. Then $\sum_{n=1}^{\infty} x_n$ diverges.*

Proof. *a*) Since convergence of a series is not affected by the first few terms, we may assume that $n_0 = 1$. Then for $n \in \mathbb{N}$ we have

$$x_{n+1} \leq \lambda x_n \leq \lambda^2 x_{n-1} \leq \cdots \leq \lambda^n x_1.$$

That is, we have $0 < x_n \leq x_1 \lambda^{n-1}$ for every $n \in \mathbb{N}$. Since the geometric series $\sum_{n=1}^{\infty} \lambda^{n-1}$ converges, the same holds for $\sum_{n=1}^{\infty} \lambda^{n-1} x_1$, and hence $\sum_{n=1}^{\infty} x_n$ converges by the comparison test.

b) Again we may assume $n_0 = 1$. If $\frac{x_{n+1}}{x_n} \geq 1$ for all n we have

$$x_{n+1} \geq x_n \geq \cdots \geq x_1 > 0$$

It follows that the sequence $(x_n)_{n=1}^{\infty}$ does not converge to zero. Therefore, $\sum_{n=1}^{\infty} x_n$ is divergent by Theorem 4.3. □

Note that in order to apply the ratio test for showing convergence it is *not* sufficient to have $\frac{x_{n+1}}{x_n} < 1$ for all $n \geq n_0$. Indeed, we have $\frac{x_{n+1}}{x_n} < 1$ for $x_n = 1/n$, but according to Proposition 4.7 the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges.

Example 4.13. Show that

$$\sum_{n=1}^{\infty} \frac{n}{3^n}$$

converges.

Solution. Writing $x_n = \frac{n}{3^n}$ we have

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)3^n}{3^{n+1}n} = \frac{n+1}{3n} \leq \frac{2}{3} < 1.$$

for all $n \in \mathbb{N}$. Hence $\sum_{n=1}^{\infty} \frac{n}{3^n}$ converges by the ratio test. □

Sometimes the following variant of the ratio test is useful.

Theorem 4.14 (Limit version of the Ratio Test)**.** *Let $(x_n)_{n=1}^{\infty}$ be a sequence. Assume that there exists $n_0 \in \mathbb{N}$ such that $x_n > 0$ for all $n \geq n_0$ and that $\frac{x_{n+1}}{x_n} \to L$ as $n \to \infty$ for some $L \in [0, \infty)$.*

a) *If $L < 1$ then $\sum_{n=1}^{\infty} x_n$ converges.*

b) *If $L > 1$ then $\sum_{n=1}^{\infty} x_n$ diverges.*

Proof. Again, we may assume $n_0 = 1$. Let $L \in [0, 1)$ and pick $\lambda \in (L, 1)$. Then $\varepsilon = \lambda - L > 0$, so by the definition of convergence, there exists $n_1 \in \mathbb{N}$ such that

$$n \geq n_1 \implies \left| \frac{x_{n+1}}{x_n} - L \right| < \varepsilon = \lambda - L.$$

In particular, we conclude $\frac{x_{n+1}}{x_n} - L < \lambda - L$, or equivalently $\frac{x_{n+1}}{x_n} < \lambda$ for $n \geq n_1$. Since $\lambda < 1$ we see that $\sum_{n=1}^{\infty} x_n$ converges according to the ratio test.

Now let $L \in (1, \infty)$. Then since $\epsilon = L - 1 > 0$, there exists $n_2 \in \mathbb{N}$ such that

$$n \geq n_2 \implies \left| \frac{x_{n+1}}{x_n} - L \right| < \varepsilon = L - 1.$$

This means $1 - L = -(L-1) < \frac{x_{n+1}}{x_n} - L$, or equivalently, $1 < \frac{x_{n+1}}{x_n}$ for $n \geq n_2$. It follows that $\sum_{n=1}^{\infty} x_n$ diverges according to the ratio test. $\square$

Example 4.15. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{2^n}{n\sqrt{n}}, \quad \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ for } x > 0, \quad \sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}}$$

are convergent or divergent.

Solution. Consider the first series and write $x_n = \frac{2^n}{n\sqrt{n}}$. Then for any $n \in \mathbb{N}$ we have $x_n > 0$, and

$$\frac{x_{n+1}}{x_n} = \frac{2^{n+1}}{(n+1)\sqrt{n+1}} \frac{n\sqrt{n}}{2^n} = 2\left(\frac{n}{n+1}\right)^{3/2} = 2\left(\frac{1}{1+\frac{1}{n}}\right)^{3/2} \to 2$$

as $n \to \infty$. We conclude that $\sum_{n=1}^{\infty} \frac{2^n}{n\sqrt{n}}$ diverges by the limit version of the ratio test.

For the second series write $y_n = \frac{x^n}{n!}$. Then we have

$$\frac{y_{n+1}}{y_n} = \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} = \frac{x}{n+1} = \frac{\frac{x}{n}}{1+\frac{1}{n}} \to 0$$

as $n \to \infty$. Hence $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges by the limit version of the ratio test.

Finally, for the third series we write $z_n = \frac{4^n}{\binom{2n}{n}}$. Then for all $n \in \mathbb{N}$,

$$\begin{aligned} \frac{z_{n+1}}{z_n} &= \frac{4^{n+1}}{\binom{2(n+1)}{(n+1)}} \frac{\binom{2n}{n}}{4^n} = \frac{4(n+1)^2}{(2n+2)(2n+1)} = \frac{(2n+2)^2}{(2n+2)(2n+1)} \\ &= \frac{2n+2}{2n+1} \geq 1. \end{aligned}$$

Hence $\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}}$ diverges by the ratio test. $\square$