FB1: Find y'' using implicit differentiation of

$$\boldsymbol{x^2 - y^2 = 5}.$$

The second derivative y'' can be found by implicit differentiation, i.e., differentiating both sides of the equation twice. The first derivative is found to be

$$2x - 2yy' = 0,$$

where both chain rule and difference rule is used on the left-hand side, while the right-hand side is found to be 0 because the derivative of a constant is always 0. Furthermore, the second derivative can be found using the product rule on the $-2yy'$ term and simple differentiation on $2x$ like this:

$$2 - 2(y'^2 + yy'') = 0.$$

Finally, rearranging for *y''* and dividing both sides by 2 gives

$$y'' = \frac{1 - y'^2}{y}.$$

FB2: Prove by induction that $\boldsymbol{n! \leq n^n}$ for all $\boldsymbol{n \in \mathbb{N}}$.

Let the statement P(n) be that $n! \leq n^n$ for all $n \in \mathbb{N}$. To prove a statement by induction, two steps have to be taken. Firstly, the base case has to be proven, which in this case is most easily done with $n = 1$ because 1 is the smallest natural number. With the base case, the statement becomes

$$1! \leq 1^1$$
$$\Rightarrow 1 \leq 1,$$

which results in a true statement. Secondly, the statement P(n) is assumed to be true. It can be restated as

$$n! \leq n^n.$$

The last step is to check whether P(n+1) can be proven by P(n). To do that one needs to keep in mind what the right-hand side for P(n+1) is supposed to be, which is $(n+1)^{n+1}$. Therefore, the right-hand side of P(n) can be compared with a simplified version of this, which can be gained by comparing *n* and (*n*+1), and then taking them both to the *n*th power (which can be done without falling into fallacies because both sides are positive (*n* being a natural number)) like this:

$$n \leq n + 1$$
$$\Rightarrow n^n \leq (n+1)^n.$$

This can then be combined with P(n) to make

$$n! \leq n^n \leq (n+1)^n,$$

from which it can be concluded that

$$n! \leq (n+1)^n.$$

When multiplying both sides by $(n + 1)$, which does not affect the inequality because it is a positive number, the statement becomes

$$(n+1)n! \leq (n+1)(n+1)^n$$
$$\Rightarrow (n+1)! \leq (n+1)^{n+1}.$$

This is the same as statement P(n+1), acquired from P(n).

Therefore, since both steps of induction (the base case for $n = 1$ and the induction step from P(n) to P(n+1)) hold true, the statement P(n), or $n! \leq n^n$, is proven true by the mathematical principle of induction.

FB3: Suppose the roots of the equation $2x^2 - 5x - 6 = 0$ are α and β. Find the quadratic equation with roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

To find the sum and product of α and β, Vieta's formulas can be used, which state that

$$\alpha + \beta = \frac{5}{2}$$

and

$$\alpha\beta = -\frac{6}{2} = -3.$$

Furthermore, Vieta's formulas can also be used to find the coefficients of the new quadratic equation. The *x* term will be the negative value of the sum, which can be expressed as

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}.$$

And the constant term will be the product, which can be expressed as

$$\frac{1}{\alpha} \cdot \frac{1}{\beta} = \frac{1}{\alpha\beta}.$$

When the sum and product of α and β are inserted into these equations, the sum evaluates to

$$\frac{\frac{5}{2}}{-3} = -\frac{5}{6},$$

and the product becomes

$$\frac{1}{-3} = -\frac{1}{3}.$$

With these values, the new quadratic equation can be found, and it is

$$x^2 + \frac{5}{6}x - \frac{1}{3} = 0,$$

which can be simplified a bit by multiplying both sides by 6 and getting

$$6x^2 + 5x - 2 = 0.$$