FB 1: Using the Euclidean algorithm, find hcf(86, 100), and use this to find integers *s, t* such that

hcf(86, 100) = 86s + 100t**.**

The Euclidean algorithm uses quotients and remainders of two integers recursively to find their greatest common factor. When dividing 100 by 86, the expression is

$$100 = 1 \cdot 86 + 14.$$

Then the previous divisor becomes the dividend, and the remainder becomes the divisor, to make

$$86 = 6 \cdot 14 + 2.$$

By doing the same step of the algorithm again, the new expression is

$$14 = 7 \cdot 2 + 0,$$

where the remainder is 0. Since 2 divides 14, 2 also divides 86, and it also divides 100, making it the highest common factor of numbers 100 and 86. To achieve the second part of the task, the second equation can be used to express 2 like this:

$$2 = 86 - 6 \cdot 14.$$

Furthermore, 14 can be expressed from the first equation like this:

$$2 = 86 - 6 \cdot (100 - 86)$$
$$\Leftrightarrow 2 = 7 \cdot 86 - 6 \cdot 100,$$

making $s = 7$ and $t = -6$.

FB 2: Which positive integers have exactly three positive divisors?

All positive integers have at least two positive divisors: the number 1 and the number itself. Furthermore, the only positive integers to have exactly two divisors are prime numbers because they are by definition not divisible by anything else.

Let *n* be a positive integer. Then *n* with only two divisors can be expressed as

$$n = p,$$

where *p* is a prime. If there is another prime *q* added as a factor, i.e.,

$$n = pq,$$

the positive integer *n* has four divisors:

$$1, p, q, pq.$$

Any non-prime has at least four divisors: 1, itself, and its two factors (if they are different numbers). Hence, multiplying the first definition of *n* by the same prime *p* takes it to the power of two, making

$$n = p^2,$$

for which there are three divisors:

$$1, p, p^2.$$

Therefore, the positive integers which have exactly three positive divisors can be expressed as

$$n = p^2,$$

where *p* is a prime number.

FB 3: Let $f(x) = 2 - |4x - 2|$. Show that there is no value of *c* such that $f(3) - f(0) = f'(c)(3 - 0)$. Why does this not contradict the Mean Value Theorem?

The left-hand side can be easily computed by putting in the values 3 and 0 into the function, so it becomes

$$f(3) - f(0) = 2 - |4 \cdot 3 - 2| - (2 - |4 \cdot 0 - 2|) = -10 + 2 = -8.$$

Furthermore, since the function includes a modulus expression, it can be represented as a piecewise function

$$f(x) = \begin{cases} 4x \text{ if } x < \dfrac{1}{2} \\ 4 - 4x \text{ if } x \geq \dfrac{1}{2} \end{cases}.$$

When looking at the first case $x < \frac{1}{2}$, the derivative of the function is

$$f'(x) = 4,$$

a constant number. Since it is a constant number, the derivative is independent of *x*; therefore, any value of *c* will give the same value, making the right-hand side of the equation provided

$$3f'(c) = 3 \cdot 4 = 12,$$

which does not equal -8. Furthermore, the derivative of the function in the case $x > \frac{1}{2}$ is

$$f'(x) = -4,$$

again a constant number, making the right-hand side equal to

$$3f'(c) = 3 \cdot (-4) = -12,$$

again not equal to -8. Since both cases fail to make the equation true and *f(x)* is not differentiable at $x = \frac{1}{2}$ (proven later), there is no value of *c* such that $f(3) - f(0) = f'(c)(3 - 0)$.

This equation looks like the definition of the Mean Value Theorem because it could be rearranged to make

$$f'(c) = \frac{f(3) - f(0)}{3 - 0}$$

for the function *f(x)* in the interval (0,3). The second condition of the Mean Value Theorem states that *f(x)* needs to be differentiable in the open interval (a,b), in this case (0,3). Looking at *f(x)* at point $\left(\frac{1}{2}, 2\right)$, the differential limit from the positive side is

$$\lim_{h\to 0^+} \frac{f\left(\frac{1}{2}+h\right)-f\left(\frac{1}{2}\right)}{h} = \lim_{h\to 0^+} \frac{-4h}{h} = -4,$$

while the limit from the negative side is

$$\lim_{h\to 0^-} \frac{f\left(\frac{1}{2}+h\right)-f\left(\frac{1}{2}\right)}{h} = \lim_{h\to 0^-} \frac{4h}{h} = 4.$$

Since both limits are not equal, *f(x)* is not differentiable at $x = \frac{1}{2}$, a value in the interval (0,3); therefore, the Mean Value Theorem does not apply to *f(x)*.