2A: Tutorial 7

School of Mathematics and Statistics

Dr. Ana Lecuona and Dr. Daniele Valeri

Semester 1 2020–21

INSTRUCTIONS

Access your tutorial lecture

TU01,TU02,TU03,TU04,TU05: November 9th, 09:00-10:00

TU06,TU07,TU08,TU09,TU10: November 9th, 15:00-16:00

using the Zoom plug-in in the Moodle page of the course.

Please mute yourself when you are in the main room of the meeting.

Be ready to work in groups. Having a method of writing on screen, as mouse, tablet or similar is useful. Screen sharing is allowed!

Multiple Choice Question 1

Ex Sheet 6, T4(a) (related)

Describe the region that lies in the first octant and is enclosed by the surface $x^2 + y^2 + z^2 = a^2$ in spherical polar coordinates.

(A) $0 \le r \le a$, $0 \le \theta \le \pi$, $0 \le \phi \le \pi$

(B) $0 \le r \le a$, $0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le \frac{\pi}{4}$

(C) $0 \le r \le a$, $0 \le \theta \le \frac{\pi}{4}$, $0 \le \phi \le \frac{\pi}{2}$

(D) $0 \le r \le a$, $0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le \frac{\pi}{2}$

Multiple Choice Question 1

Ex Sheet 6, T4(a) (related)

Describe the region that lies in the first octant and is enclosed by the surface $x^2 + y^2 + z^2 = a^2$ in spherical polar coordinates.

(A) $0 \leq r \leq a$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq \pi$

(B) $0 \leq r \leq a$, $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq \phi \leq \frac{\pi}{4}$

(C) $0 \leq r \leq a$, $0 \leq \theta \leq \frac{\pi}{4}$, $0 \leq \phi \leq \frac{\pi}{2}$

(D) $0 \leq r \leq a$, $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq \phi \leq \frac{\pi}{2}$

ANSWER: **(D)** The first octant is that part for which $x \geq 0$, $y \geq 0$ and $z \geq 0$. Using spherical polar coordinates we see that $z \geq 0$ means $r\cos\phi \geq 0$ and as $r \geq 0$ we must have $\cos\phi \geq 0$ too, this is satisfied for $0 \leq \phi \leq \frac{\pi}{2}$ (ϕ must lie in $[0, \pi]$). Now $\sin\phi \geq 0$ and $r \geq 0$ so with $x = r\sin\phi\cos\theta \geq 0$ and $y = r\sin\phi\sin\theta \geq 0$ we must have $\cos\theta \geq 0$ and $\sin\theta \geq 0$ and this is satisfied in the range $0 \leq \theta \leq \pi/2$. Enclosed by the sphere gives $0 \leq r \leq a$.

Multiple Choice Question 2

Unseen question

Let $\mathbf{r} = (x, y, z)$ and $\mathbf{c}$ be a constant vector. Let ϕ be a differentiable function of one variable, calculate

$$\nabla\phi\,(\mathbf{c}\cdot\mathbf{r})$$

(A) $\phi'\,(\mathbf{c}\cdot\mathbf{r})\,\mathbf{c}$

(B) $\phi'\,(\mathbf{c}\cdot\mathbf{r})\,\mathbf{r}$

(C) $(\mathbf{c}\cdot\mathbf{r})\,\phi'\,(\mathbf{c}\cdot\mathbf{r})$

(D) $\phi'\,(\mathbf{c}\cdot\mathbf{r})$

Multiple Choice Question 2

Unseen question

Let $\mathbf{r} = (x, y, z)$ and $\mathbf{c}$ be a constant vector. Let ϕ be a differentiable function of one variable, calculate

$$\nabla\phi\left(\mathbf{c}\cdot\mathbf{r}\right)$$

(A) $\phi'\left(\mathbf{c}\cdot\mathbf{r}\right)\mathbf{c}$ **(B)** $\phi'\left(\mathbf{c}\cdot\mathbf{r}\right)\mathbf{r}$

(C) $\left(\mathbf{c}\cdot\mathbf{r}\right)\phi'\left(\mathbf{c}\cdot\mathbf{r}\right)$ **(D)** $\phi'\left(\mathbf{c}\cdot\mathbf{r}\right)$

ANSWER: **(A)** We apply the chain rule to get

$$\frac{\partial}{\partial x}\left(\phi\left(\mathbf{c}\cdot\mathbf{r}\right)\right) = \frac{\partial}{\partial x}\left(\mathbf{c}\cdot\mathbf{r}\right)\phi'\left(\mathbf{c}\cdot\mathbf{r}\right) .$$

Let $\mathbf{c} = (c_1, c_2, c_3)$. Then

$$\frac{\partial}{\partial x}\left(\mathbf{c}\cdot\mathbf{r}\right) = \frac{\partial}{\partial x}\left(c_1 x + c_2 y + c_3 z\right) = c_1 .$$

Multiple Choice Question 2

Unseen question

Let $\mathbf{r} = (x, y, z)$ and $\mathbf{c}$ be a constant vector. Let ϕ be a differentiable function of one variable, calculate

$$\nabla\phi\left(\mathbf{c}\cdot\mathbf{r}\right)$$

(A) $\phi'\left(\mathbf{c}\cdot\mathbf{r}\right)\mathbf{c}$

(B) $\phi'\left(\mathbf{c}\cdot\mathbf{r}\right)\mathbf{r}$

(C) $\left(\mathbf{c}\cdot\mathbf{r}\right)\phi'\left(\mathbf{c}\cdot\mathbf{r}\right)$

(D) $\phi'\left(\mathbf{c}\cdot\mathbf{r}\right)$

Similarly for the other partial derivatives.

Hence, we see that

$$\operatorname{grad}\,\phi\left(\mathbf{c}\cdot\mathbf{r}\right) = \left(c_1, c_2, c_3\right)\phi'\left(\mathbf{c}\cdot\mathbf{r}\right) = \phi'\left(\mathbf{c}\cdot\mathbf{r}\right)\mathbf{c}.$$

Tutorial questions

Ex Sheet 7, T8(a)

Find the directional derivative of $f(x, y, z) = \exp(2x - y + z)$ at $(x, y, z) = (1, 1, -1)$ in the direction $\mathbf{d} = (-1, -3, -5)$.

Unseen question

Let $\mathbf{r} = (x, y, z)$ and $r^2 = x^2 + y^2 + z^2$. Let f be a differentiable scalar function of one variable. Compute

$$\operatorname{div} f(r)\mathbf{r}\,.$$

Ex Sheet 7, T8(a)

Find the directional derivative of $f(x, y, z) = \exp(2x - y + z)$ at $(x, y, z) = (1, 1, -1)$ in the direction $\mathbf{d} = (-1, -3, -5)$.

The unit vector in the direction of $\mathbf{d}$ is $\hat{\mathbf{d}} = (-1, -3, -5)/\sqrt{35}$ and $\nabla f(x, y, z) = e^{2x-y+z}(2, -1, 1)$. Hence, $\nabla f(1, 1, -1) = (2, -1, 1)$. Therefore the directional derivative is

$$\nabla f \cdot \hat{\mathbf{d}} = \frac{1}{\sqrt{35}}(2, -1, 1) \cdot (-1, -3, -5) = \frac{-2 + 3 - 5}{\sqrt{35}} = -\frac{4}{\sqrt{35}}.$$

Tutorial questions

Unseen question

Let $\mathbf{r} = (x, y, z)$ and $r^2 = x^2 + y^2 + z^2$. Let f be a differentiable scalar function of one variable. Compute

$$\operatorname{div} f(r)\mathbf{r}\,.$$

The divergence is

$$\begin{aligned}
\operatorname{div} f(r)\mathbf{r} &= \frac{\partial}{\partial x}(xf(r)) + \frac{\partial}{\partial y}(yf(r)) + \frac{\partial}{\partial z}(zf(r)) \\
&= 3f(r) + \left(x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right) f'(r) \\
&= 3f(r) + \left(x\frac{x}{r} + y\frac{y}{r} + z\frac{z}{r}\right) f'(r) \\
&= 3f(r) + rf'(r)\,.
\end{aligned}$$

Tutorial question

Bonus question

This is Pac-Man, a very famous influencer from the 80's.

Its head is the sphere $x^2 + y^2 + z^2 = a^2$ and the angle formed by its mouth is $2\pi b$. What is the volume of Pac-Man?

Tutorial question

Let V_a be the interior of the sphere of radius a. If you do not remember the formula for its volume you may use a triple integral and spherical coordinates to get

$$
\begin{aligned}
Vol(V_a) &= \iiint_{V_a} dxdydz = \left(\int_0^a r^2 dr\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\phi d\phi\right) \\
&= \left[\frac{r^3}{3}\right]_0^a [\theta]_0^{2\pi} [-\cos\phi]_0^{\pi} = \frac{4\pi}{3} a^3 .
\end{aligned}
$$

The mouth of Pac-Man takes away $\frac{2\pi b}{2\pi} Vol(V_a) = bVol(V_a)$ from the volume of the sphere. The remaining part is the volume of Pac-Man. Hence,

$$Vol(\text{Pac-Man}) = \frac{4\pi}{3} a^3 (1-b) .$$

For example, when eating a ghost, Pac-Man's mouth opens by an angle of $\frac{\pi}{2}$. Then its volume is πa^3.