FB1: Suppose that $f: A \to B$ is a surjective function. Define the following relation on *A*:

$$a_1 \sim a_2 \text{ if and only if } f(a_1) = f(a_2).$$

Show that this is an equivalence relation. Denote by $A/\sim$ the set of equivalence classes of $\sim$. Prove that

$$|A/\sim| = |B|.$$

The relation on *A* is **reflexive** because $f(a_1) = f(a_1)$; thus,

$$a_1 \sim a_1.$$

Furthermore, the relation is **symmetric** because $f(a_1) = f(a_2)$ implies that $f(a_2) = f(a_1)$; thus,

$$a_2 \sim a_1.$$

Finally, the relation is **transitive** because, if for an element $a_3 \in A$ there is a relation $a_2 \sim a_3$, then the condition $f(a_2) = f(a_3)$ applies, which combined with $f(a_1) = f(a_2)$ implies that $f(a_1) = f(a_3)$; thus,

$$a_1 \sim a_3.$$

By surjectivity, each element in B is a distinct value $f(a)$, where $a \in A$. Since the relation $\sim$ partitions the set *A* into distinct subsets (each containing a distinct collection of $f(a)$), each equivalence class corresponds to exactly one element in B. Thus, $|A/\sim| = |B|$.

FB2: Suppose that G is a group with identity element e. Let $\alpha, \beta, \gamma \in G$ be arbitrary. Prove the following statements.

(i) $\alpha\beta\gamma = e$ implies $\beta\gamma\alpha = e$.
(ii) $\beta\alpha\gamma = \alpha^{-1}$ implies $\gamma\alpha\beta = \alpha^{-1}$.

(i)

$\alpha\beta\gamma = e$ implies that either $\beta\gamma = \alpha^{-1}$ or $\alpha\beta = \gamma^{-1}$. If $\beta\gamma = \alpha^{-1}$, then

$$\begin{aligned} &\alpha^{-1}\alpha = e \\ &\Leftrightarrow \beta\gamma\alpha = e. \end{aligned}$$

Because of associativity, $(\alpha\beta)\gamma = \alpha(\beta\gamma) = e$; thus, $\alpha\beta = \gamma^{-1}$ also implies $\beta\gamma\alpha = e$. Therefore, $\alpha\beta\gamma = e$ implies $\beta\gamma\alpha = e$.

(ii)

Using the law of inverses and 'multiplying' on the left both sides of the equation by α,

$$\begin{aligned} &\beta\alpha\gamma = \alpha^{-1} \\ &\Leftrightarrow \alpha\beta\alpha\gamma = \alpha\alpha^{-1} \\ &\Leftrightarrow (\alpha\beta)(\alpha\gamma) = e, \end{aligned}$$

where $\alpha\beta = (\alpha\gamma)^{-1}$ or $\alpha\gamma = (\alpha\beta)^{-1}$. By associativity, any other case is equivalent. In either case, they can swapped to find

$$\begin{aligned} &(\alpha\gamma)(\alpha\beta) = e \\ &\Leftrightarrow \alpha\gamma\alpha\beta = \alpha\alpha^{-1} \\ &\Leftrightarrow \gamma\alpha\beta = \alpha^{-1}. \end{aligned}$$

Therefore, $\beta\alpha\gamma = \alpha^{-1}$ implies $\gamma\alpha\beta = \alpha^{-1}$.

FB3: A parametric curve is described by the following equations

$$\frac{dx}{dt} = x, y = \cos t, z = \sin t,$$

and passes through $\langle 1,1,0 \rangle$ when $t = 0$. By solving the ODE for $x(t)$, or otherwise, find an expression for x in terms of t and use this to write the space curve as a vector function. Hence, find the unit tangent to the curve $\mathbf{T}(t)$ at the point $\langle 1, 1, 0 \rangle$.

To solve the separable differential equation, the variables need to be separated into

$$\frac{1}{x} dx = dt.$$

Then by integrating both sides one gets

$$\ln|x| = t + C.$$

Since the curve passes through $\langle 1,1,0 \rangle$ when $t = 0$, the constant C is found by

$$\ln(1) = 0 + C$$
$$\Rightarrow C = 0.$$

Thus,

$$ln(x) = t$$
$$\Rightarrow x = e^t$$

for all $x > 0$.

The parametric curve is written as a vector function $\boldsymbol{r}(t)$ by

$$\mathbf{r}(t) = \langle e^t, \cos(t), \sin(t) \rangle.$$

The tangent to the curve at t is

$$\mathbf{T}(t) = \frac{r'(t)}{|r'(t)|}$$

$$= \frac{\langle e^t, -\sin(t), \cos(t) \rangle}{\sqrt{e^{2t} + \sin^2(t) + \cos^2(t)}}$$

$$= \frac{\langle e^t, -\sin(t), \cos(t) \rangle}{\sqrt{e^{2t} + 1}}.$$

Hence, when $t = 0$,

$$\mathbf{T}(0) = \frac{\langle 1,0,1 \rangle}{\sqrt{2}}.$$