2A: Tutorial 4

School of Mathematics and Statistics

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Semester 1 2020–21

Instructions

Access your tutorial lecture

TU11,TU12,TU13,TU14,TU15: October 19th, 09:00-10:00

TU16,TU17,TU18,TU19,TU20: October 19th, 15:00-16:00

using the Zoom plug-in in the Moodle page of the course.

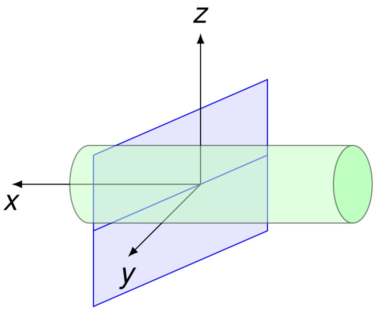
Please mute yourself when you are in the main room of the meeting.

Be ready to work in groups. Having a method of writing on screen, as mouse, tablet or similar is useful.

Ex Sheet 4, T11(a)

Find the volume of the solid bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y$, $x = 0$ and $z = 0$ in the first octant.

A sketch:

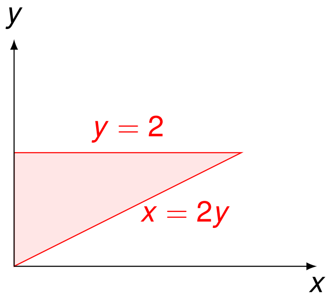


Need to cut the cylinder also by the planes $x = 0$ and $z = 0$ and consider $z \geq 0$ (since in the first octant we have that $x, y, z \geq 0$).

The solid consists of all the points lying under the surface

$$\begin{cases} z \geq 0 \\ y^2 + z^2 = 4 \end{cases} \Rightarrow z = \sqrt{4 - y^2}$$

and above the domain D (in red below) of the xy-plane.



Then, the volume is given by

$$\iint_D \sqrt{4 - y^2}\, dxdy\,.$$

D is both of type I and II, so you can compute this number with the formulas from Thursday's lecture.

From the Moodle forum: a question for you now

Find the formula representing the previous double integral as a type I iterated integral (it is obtained by writing D as a type I domain).

(A) $\displaystyle\int_0^2 dy \int_0^{2y} \sqrt{4-y^2}\,dx$

(B) $\displaystyle\int_0^4 dx \int_x^2 \sqrt{4-y^2}\,dy$

(C) $\displaystyle\int_0^4 dx \int_{x/2}^2 \sqrt{4-y^2}\,dy$

(D) $\displaystyle\int_0^4 dx \int_0^{x/2} \sqrt{4-y^2}\,dy$

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(A) $\int_0^2 dy \int_0^{2y} \sqrt{4-y^2}dx$ **(B)** $\int_0^4 dx \int_x^2 \sqrt{4-y^2}dy$

(C) $\int_0^4 dx \int_{x/2}^2 \sqrt{4-y^2}dy$ **(D)** $\int_0^4 dx \int_0^{x/2} \sqrt{4-y^2}dy$

ANSWER: **(C)** As a type I region D is described as the set of points $(x, y) \in \mathbb{R}^2$ such that

$$0 \le x \le 4\,, \quad x/2 \le y \le 2\,.$$

Hence, using the type I formulation,

$$\iint_D \sqrt{4-y^2}\,dxdy = \int_0^4 dx \int_{x/2}^2 \sqrt{4-y^2}dy\,.$$

Another question for you

Do you know how to compute

$$\int \sqrt{4 - y^2} dy \ \ ?$$

Yes **No**

If not, then try to change the order of integration and see if the computation becomes easier.

From the Moodle forum: finally the volume

Compute the previous integral by reversing the order of integration (namely write D as a type II domain and use the type II formula).

(A) 2020 **(B)** e

(C) 8 **(D)** $\frac{16}{3}$

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(A) 2020

(B) e

(C) 8

(D) $\dfrac{16}{3}$

ANSWER: **(D)** As a type II region D is described as the set of points $(x, y) \in \mathbb{R}^2$ such that

$$0 \le x \le 2y\,, \quad 0 \le y \le 2\,.$$

Hence, using the type I formulation,

$$\iint_D \sqrt{4 - y^2}\, dxdy = \int_0^2 dy \int_0^{2y} \sqrt{4 - y^2} dx\,.$$

From the Moodle forum: finally the volume

Compute the previous integral by reversing the order of integration (namely write D as a type II domain and use the type II formula).

(A) 2020 **(B)** e

(C) 8 **(D)** $\frac{16}{3}$

Keep computing to get

$$\int_0^2 dy \int_0^{2y} \sqrt{4-y^2}dx = \int_0^2 \left[x\sqrt{4-y^2}\right]_{x=0}^{x=2y} dy$$

$$= 2\int_0^2 y\sqrt{4-y^2}dy = -\frac{2}{3}\left[(4-y^2)^{3/2}\right]_0^2 = \frac{16}{3}\,.$$

Multiple Choice Question

Unseen question

Find the general solution of the PDE

$$\frac{\partial \phi}{\partial x} = \frac{x}{x^2 + y^2},$$

where $\phi = \phi(x, y)$.

(A) $\phi = \cot^{-1}(\frac{x}{y}) + c(y)$

(B) $\phi = \tan^{-1}(y) + c(x)$

(C) $\phi = \log(\sqrt{x^2 + y^2}) + c(y)$

(D) $\phi = \frac{1}{2}\log(x^2 + y^2) + c(x, y)$

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(C) $\phi = \log(\sqrt{x^2+y^2}) + c(y)$ **(D)** $\phi = \frac{1}{2}\log(x^2+y^2)+c(x, y)$

ANSWER: **(C)** Partial integration with respect to x gives

$$\phi = \frac{1}{2}\log(x^2+y^2) + c(y) = \log(\sqrt{x^2+y^2}) + c(y)\,.$$

(Partial integration introduces arbitrary functions of the remaining variables.)

Tutorial questions

Ex Sheet 3, T4(a)

Find the general solution of the following partial differential equation:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3y(y^2 - x^2) \qquad \text{(change to } u = x \text{ and } v = y/x\text{).}$$

Ex Sheet 4, T7

Evaluate $\iint_D xy\, dxdy$, where D is the bounded region lying between $y = x^2$ and $x = y^2$.

Tutorial questions

Ex Sheet 3, T4(a)

Find the general solution of the following partial differential equation:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3y(y^2 - x^2) \qquad \text{(change to } u = x \text{ and } v = y/x\text{).}$$

Write the function $f(x, y) = F(u(x, y), v(x, y))$. Then the chain rule gives

$$\frac{\partial f}{\partial x} = \frac{\partial F}{\partial u} - \frac{y^2}{x}\frac{\partial F}{\partial v}, \qquad \frac{\partial f}{\partial y} = \frac{1}{x}\frac{\partial F}{\partial v}.$$

Substitution into the PDE gives

$$x\left(\frac{\partial F}{\partial u} - \cancel{\frac{y}{x^2}\frac{\partial F}{\partial v}} +\right) + y\left(\cancel{\frac{1}{x}\frac{\partial F}{\partial v}}\right) = 3y(y^2 - x^2).$$

Now simplify and write y in terms of u and v to get

$$\frac{\partial F}{\partial u} = 3\frac{y}{x}(y^2 - x^2) = 3v((uv)^2 - u^2) = 3u^2v^3 - 3u^2v\,.$$

Partial integration with respect to u gives

$$F = u^3v^3 - u^3v + A(v)\,,$$

where A is an arbitrary function of one variable. Finally the general solution is

$$f(x,y) = F(u(x,y), v(x,y)) = y^3 - x^2y + A\left(\frac{y}{x}\right)\,.$$

Ex Sheet 4, T7

Evaluate $\iint_D xy\, dxdy$, where D is the bounded region lying between $y = x^2$ and $x = y^2$.

D is both of type I and II. As a type II region it is described as the set of points $(x, y) \in \mathbb{R}^2$ such that

$$y^2 \leq x \leq \sqrt{y}\,, \quad 0 \leq y \leq 1\,.$$

Hence, using the type II formulation,

$$\iint_D xy\, dxdy = \int_0^1 \left(\int_{y^2}^{\sqrt{y}} xy\, dx \right) dy = \frac{1}{2} \int_0^1 \left[x^2 y \right]_{x=y^2}^{x=\sqrt{y}} dy$$

$$= \frac{1}{2} \int_0^1 (y^2 - y^5)\, dy = \frac{1}{12}.$$