2C Week 6 2020/21

HANDOUT

Monotonic sequences

We now turn to some important consequences of completeness. Our first objective is the monotone convergence theorem. We start by introducing the definitions of monotonic sequences: those that are either increasing or decreasing.

Definition 3.19. Let $(x_n)_{n=1}^{\infty}$ be a sequence. We say that $(x_n)_{n=1}^{\infty}$ is

- *increasing* if and only if, for all $n \in \mathbb{N}, x_n \leq x_{n+1}$;
- *strictly increasing* if and only if, for all $n \in \mathbb{N}, x_n < x_{n+1}$;
- *eventually increasing* if and only if, there exists $N \in \mathbb{N}$, such that for all $n \geq N, x_n \leq x_{n+1}$;
- *eventually strictly increasing* if and only if, there exists $N \in \mathbb{N}$, such that for all $n \geq N, x_n < x_{n+1}$.

In a similar fashion, say that $(x_n)_{n=1}^{\infty}$ is

- *decreasing* if and only if, for all $n \in \mathbb{N}, x_n \geq x_{n+1}$;
- *strictly decreasing* if and only if, for all $n \in \mathbb{N}, x_n > x_{n+1}$;
- *eventually decreasing* if and only if, there exists $N \in \mathbb{N}$, such that for all $n \geq N, x_n \geq x_{n+1}$;
- *eventually strictly decreasing* if and only if, there exists $N \in \mathbb{N}$, such that for all $n \geq N, x_n > x_{n+1}$;

Note that by this definition a constant sequence is both increasing and decreasing. In fact, a sequence is constant if and only if it is both increasing and decreasing.

Definition 3.20. Let $(x_n)_{n=1}^{\infty}$ be a sequence. We say that $(x_n)_{n=1}^{\infty}$ is *monotonic* if and only if it is either increasing or decreasing. We say that $(x_n)_{n=1}^{\infty}$ is *eventually monotonic* if it is eventually increasing or eventually decreasing.

There are two standard strategies to show that a sequence $(x_n)_{n=1}^{\infty}$ is (eventually) monotonic. Firstly, to decide whether $(x_n)_{n=1}^{\infty}$ is eventually increasing or eventually decreasing you can look at the difference $x_{n+1} - x_n$, simplify this, and see if you can show that $x_{n+1} - x_n$ is eventually positive (when $(x_n)_{n=1}^{\infty}$ will be increasing) or eventually negative (when $(x_n)_{n=1}^{\infty}$ will be decreasing). Let's see an example in action.

Example 3.21. Let $x_n = \frac{n^2-3n}{n^2-5}$ and $y_n = \frac{n!}{n^n}$. Show that $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are eventually monotonic.

Solution. We have[1]

$$x_{n+1} - x_n = \frac{3n^2 - 7n + 10}{(n^2 - 5)(n^2 + 2n - 4)} > 0,$$

for $n \geq 3$. Therefore $(x_n)_{n=1}^{\infty}$ is eventually increasing.

A more systematic approach to this example would be to use Lemma 1.9. If you end up with an expression of the form

$$x_{n+1} - x_n = \frac{p(n)}{q(n)}$$

for polynomials $p(n)$ and $q(n)$ whose leading terms have positive coefficients, then applying Lemma 1.9 suitably will show you[2] that $x_{n+1} - x_n > 0$ for sufficiently large n.

Let us now turn to the sequence $(y_n)_{n=1}^{\infty}$. We have $y_n = \frac{n!}{n^n} > 0$ for all n. Moreover

$$\frac{y_{n+1}}{y_n} = \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n < 1,$$

so the sequence $(y_n)_{n=1}^{\infty}$ is strictly decreasing.

We see here another key method for deciding whether a sequence is monotonic, which applies when all the terms are positive[3]: when $y_n > 0$ for all n, then we can consider the ratio $\frac{y_{n+1}}{y_n}$ as $y_{n+1} > y_n$ if and only if $\frac{y_{n+1}}{y_n} > 1$.

So which method should you use? I'd suggest looking at the ratio $\frac{y_{n+1}}{y_n}$ in those cases where the form of y_n is such that there will be lots of cancellation in the expression $\frac{y_{n+1}}{y_n}$. For example, this happens when y_n is defined using exponential functions of n, like $y_n = 7^n$, where we have $\frac{y_{n+1}}{y_n} = 7$. Similarly things like factorials[4] and binomial coefficients (which can be expressed using factorials) naturally bring ratios to mind. □

The key reason for focusing on monotonic sequences is the monotone convergence theorem, which in turn is a consequence of the completeness axiom[5]. This theorem says that **bounded monotonic sequences converge** — a slogan you should know, and which should come to mind whenever you see a monotonic sequence.

Theorem 3.22 (Monotone convergence theorem)**.** *Let $(x_n)_{n=1}^{\infty}$ be an eventually increasing sequence which is bounded above, or an eventually decreasing sequence which is bounded below. Then $(x_n)_{n=1}^{\infty}$ converges.*

Proof. I will prove the theorem in the case that $(x_n)_{n=1}^{\infty}$ is eventually increasing and bounded above. The other case is similar[6].

Suppose then that $N \in \mathbb{N}$ has the property that for $n \geq N$ we have $x_{n+1} \geq x_n$. Define $S = \{x_n \mid n \geq N\}$, and note that S is clearly non-empty. Moreover S is bounded above since the sequence $(x_n)_{n=1}^{\infty}$ is bounded above by assumption[7]. Therefore, by the completeness axiom[8], $\sup(S)$ exists. Write $L = \sup(S)$.

We claim that $x_n \to L$ as $n \to \infty$, so let $\varepsilon > 0$ be arbitrary. By the defining property of the supremum, there exists $n_0 \geq N$ such that $x_{n_0} > L - \varepsilon$. Therefore, for $n \geq n_0$, we have

$$L - \varepsilon < x_{n_0} \leq x_n \leq L < L + \varepsilon \implies |x_n - L| < \varepsilon,$$

[1] How did I arrive at the last inequality? Firstly, for $n \geq 3$, both $n^2 - 5 \geq 0$ and $n^2 + 2n - 4 \geq 0$, so the denominator is positive for $n \geq 3$. For the numerator, we have $3n^2 - 7n \geq n(3n - 7) \geq 0$ for $n \geq 3$, so the numerator is also positive when $n \geq 3$.

[2] I leave the details to you. Similarly, you can check that if $x_{n+1} - x_n = -\frac{p(n)}{q(n)}$ for polynomials $p(n)$ and $q(n)$ whose leading terms have positive coefficients, Lemma 1.9 will show that $(x_n)_{n=1}^{\infty}$ is eventually decreasing.

[3] or at least when they all have the same sign.

[4] Note that $\frac{(n+1)!}{n!} = n + 1$, but be careful: $\frac{(2(n+1))!}{(2n)!} = (2n+2)(2n+1)$; think about where you are putting brackets when you're using factorials.

[5] It is in fact equivalent to the completeness axiom: an ordered field satisfying the axioms (a)-(l) listed in Axioms 2.1 and 2.2 which satisfies the monotone convergence theorem has the property that suprema exist for every non-empty subset which is bounded above. For more on this theme see Körner's excellent book "A companion to analysis: a first second course and second first course in analysis".

[6] You can also note that if $(x_n)_{n=1}^{\infty}$ is eventually decreasing and bounded below, then defining $y_n = -x_n$, we obtain an eventually increasing sequence $(y_n)_{n=1}^{\infty}$ which is bounded above. Convergence of $(y_n)_{n=1}^{\infty}$ implies convergence of $(x_n)_{n=1}^{\infty}$ by algebraic properties of limits.

[7] To prove that $(x_n)_{n=1}^{\infty}$ converges we will have to identify the proposed limit, and the natural candidate is the supremum of the set S. Draw a picture to persuade yourself that x_n should converge to the supremum of S.

[8] From a writing style view point, note how I check both conditions in the supremum axiom explicitly before using it.

as the sequence $(x_m)_{m=1}^\infty$ is increasing for $m \geq N$. Hence $x_n \to L$ as $n \to \infty$. $\square$

Let's turn to some applications of the monotone convergence theorem. First note that it gives us a method for showing that sequences converge without having to find the proposed limit first: simply check that the sequence is eventually monotonic by, say, using the methods in Example 3.21) and bounded, then it will converge. After you know the sequence converges, you may be able to find the limit directly using other means, such as properties of limits. We will see an example of this below.

Example 3.23. Let $x \in \mathbb{R}$ and define $a_n = \frac{x^n}{n!}$.

a) For $x > 0$, show that $(a_n)_{n=1}^\infty$ is eventually decreasing.

b) For all $x \in \mathbb{R}$, show $a_n \to 0$.

Solution. *a*) When $x > 0$, we have $a_n > 0$ for all n. Then

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n} = \frac{x}{n+1}.$$

Thus for $n > x - 1$, we have $a_{n+1} < a_n$. Hence $(a_n)_{n=1}^\infty$ is eventually decreasing.

When $x > 0$, we certainly have $a_n > 0$ for all n, so that $(a_n)_{n=1}^\infty$ is bounded below, and hence there exists $L \in \mathbb{R}$ with $a_n \to L$ as $n \to \infty$ by the monotone convergence theorem. Now note that $a_{n+1} = \frac{x}{n+1}a_n$. Since $a_{n+1} \to L$ as $n \to \infty$ and $\frac{x}{n+1} \to 0$ as $n \to \infty$, uniqueness of limits gives

$$L = 0 \times L = 0,$$

as required[9].

b) Note that if $x = 0$, then $a_n = 0$ for all n, when the result is immediate, while if $x < 0$, then $|a_n| = \frac{|x|^n}{n!} \to 0$, by part *a*), and hence $a_n \to 0$. $\square$

[9] Note that $a_{n+1} = \frac{x}{n+1}a_n$ is a *recursion relation* defining a_{n+1} in terms of n, a_n, and the fixed constant x.

Note the technique of *taking limits in the recursion relation* $a_{n+1} = \frac{x}{n+1}a_n$ above. We see this technique again in the next example.

Example 3.24. Let $\alpha \in (2, 3)$ and define a sequence $(x_n)_{n=1}^\infty$ recursively by

$$x_1 = \alpha, \quad x_{n+1} = x_n^2 - 4x_n + 6, \quad n \in \mathbb{N}.$$

Show

a) $x_n \in (2, 3)$ for all n;

b) $(x_n)_{n=1}^\infty$ is decreasing;

c) $x_n \to 2$ as $n \to \infty$.

Solution. *a*) Since the sequence is defined recursively, it makes sense to prove *a*) by induction[10].

To carry out the proof by induction[11], we start by observing that

[10] We are told that $x_1 = \alpha \in (2, 3)$, and have a formula for x_{n+1} in terms of x_n.

[11] Note that I'll not set my induction proof out by introducing the statement $P(n)$ to be $x_n \in (2, 3)$, and then using a dummy variable k. Setting out inductions using $P(n)$ and a dummy variable k is perfectly valid, though rather cumbersome — it's essentially designed to prevent people writing "let $n = n + 1$" which can't be true (but is a very popular thing to write in proofs by induction).

$x_1 \in (2,3)$ according to the assumptions. Assuming inductively that $x_n \in (2,3)$ for some $n \in \mathbb{N}$, we have

$$x_{n+1} - 2 = x_n^2 - 4x_n + 4 = (x_n - 2)^2 > 0$$

and

$$x_{n+1} - 3 = x_n^2 - 4x_n + 3 = (x_n - 3)(x_n - 1) < 0,$$

as $x_n > 1$ and $x_n < 3$. Therefore, by induction, $x_n \in (2,3)$ for all $n \in \mathbb{N}$.

b) We note that the recursion formula won't give us that much useful information about the ratio $\frac{x_{n+1}}{x_n}$, so we look at $x_{n+1} - x_n$ to see the sequence is decreasing. For $n \in \mathbb{N}$, we have

$$x_{n+1} - x_n = x_n^2 - 5x_n + 6 = (x_n - 2)(x_n - 3) < 0,$$

as $2 < x_n < 3$. Therefore $(x_n)_{n=1}^{\infty}$ is decreasing.

c) Finally, we use the monotone convergence theorem to learn that $(x_n)_{n=1}^{\infty}$ converges, then take limits in the recursion formula $x_{n+1} = x_n^2 - 4x_n + 6$ to find this limit. Since $(x_n)_{n=1}^{\infty}$ is decreasing and bounded below, it converges by the monotone convergence theorem to some number $L \in \mathbb{R}$. As $2 < x_n \leq x_1 < 3$ for all n, we have $2 \leq L \leq x_1 < 3$ by the order properties of limits. On the other hand, as $x_{n+1} \to L$ as $n \to \infty$, taking limits in the recursion formula gives $L = L^2 - 4L + 6$, so $L^2 - 5L + 6 = 0$, and hence $L = 2$ or $L = 3$. Since we have already noted that $L < 3$, we must have $L = 2$. □

Subsequences

Recall that a sequence is an infinite list of numbers, for example

$$1, 3, 5, 2, 7, 9, 15, 16, 18, 25, \ldots.$$

Subsequences of this sequence are obtained by deleting some (possibly infinitely many) of the numbers in the list, to obtain a new infinite list of numbers[12]. For example,

$$1, \cancel{3}, 5, 2, \cancel{7}, \cancel{9}, 15, 16, 18, \cancel{25}, \ldots$$

gives a subsequence of the sequence above.

[12] So, deleting *all* the numbers is not allowed, and neither is deleting all the numbers after some fixed point, for instance.

Of course, we should give a precise definition of what a subsequence is, and we shall prepare for this as follows. Each term in the subsequence corresponds to a term in the original sequence. In the example above, 1 is the first term in the subsequence and the first term of the sequence; 5 is the second term of the subsequence and the third term of the sequence; 2 the third term of the subsequence and the fourth term of the sequence, and so forth.

Let us now write our original sequence more formally as $(x_n)_{n=1}^{\infty}$, and our subsequence as $(y_n)_{n=1}^{\infty}$. For each $n \in \mathbb{N}$, the n-th term y_n of the subsequence must be some term in the original sequence, say the k_n-th term in the original sequence. So we should have $y_n = x_{k_n}$ for some $k_n \in \mathbb{N}$. What properties must the indices k_n have? For each

n, the $n+1$-st term in the subsequence $x_{k_{n+1}}$ must be located further along the original sequence than the n-th term x_{k_n}. That is, we must have $k_{n+1} > k_n$ for all $n \in \mathbb{N}$. We use this to make the following precise definition.

Definition 3.25. A *subsequence* of a sequence $(x_n)_{n=1}^{\infty}$ is a sequence of the form $(x_{k_n})_{n=1}^{\infty}$ for some strictly increasing natural numbers $k_1 < k_2 < k_3 < \dots$.

Note that for strictly increasing natural numbers $k_1 < k_2 < k_3 < \dots$ we have[13] $k_n \geq n$ for each $n \in \mathbb{N}$. In the example above, we have $k_1 = 1, k_2 = 3, k_3 = 4$, and so on.

[13] A formal proof of this statement can be given by induction.

There are a number of standard subsequences. Letting $(x_n)_{n=1}^{\infty}$ be a sequence, we have the subsequence $(x_{2n})_{n=1}^{\infty}$ consisting of the even terms of $(x_n)_{n=1}^{\infty}$; similarly, $(x_{2n-1})_{n=1}^{\infty}$ is the subsequence consisting of the odd terms of $(x_n)_{n=1}^{\infty}$; we also have a shifted sequence $(x_{n+1})_{n=1}^{\infty}$ which is obtained by removing the first term. You should be able to work out descriptions for other subsequences: how would you write down the subsequence which starts with the second term of $(x_n)_{n=1}^{\infty}$ and takes every fifth subsequence term? Finally, note that by taking $k_n = n$ for all $n \in \mathbb{N}$, we see that $(x_n)_{n=1}^{\infty}$ is a subsequence of itself.

Thinking about what convergence means, it should be clear that we expect that if $(x_n)_{n=1}^{\infty}$ converges to L, then every subsequence of $(x_n)_{n=1}^{\infty}$ converges too[14].

[14] The converse is also true, and actually devoid of content: every sequence is a subsequence of itself.

Proposition 3.26. *Let $(x_n)_{n=1}^{\infty}$ be a sequence and $L \in \mathbb{R}$. Then $x_n \to L$ if and only if every subsequence of $(x_n)_{n=1}^{\infty}$ converges to L.*

Proof. As every sequence is a subsequence of itself the implication from right to left is immediate. For the implication from left to right, let $(x_n)_{n=1}^{\infty}$ be a sequence with $x_n \to L$ as $n \to \infty$. Let $k_1 < k_2 < \dots$ be an arbitrary strictly increasing sequence of natural numbers defining a subsequence $(x_{k_n})_{n=1}^{\infty}$. Let $\varepsilon > 0$ be arbitrary, so by definition of convergence, there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0 \implies |x_n - L| < \varepsilon$. For $n \geq n_0$, we have $k_n \geq n \geq n_0$, and hence $|x_{k_n} - L| < \varepsilon$. That is, we have $n \geq n_0 \implies |x_{k_n} - L| < \varepsilon$. Hence we conclude that $(x_{k_n})_{n=1}^{\infty}$ converges to L as $n \to \infty$. □

The previous proposition provides a tool for quickly proving that various sequences do *not* converge: find either a non-convergent subsequence or two subsequences converging to different limits.

Example 3.27. Show that $(x_n)_{n=1}^{\infty}$ does not converge when $x_n = \frac{n-1}{n}(-1)^n$.

Solution. We consider the subsequences $(x_{2n})_{n=1}^{\infty}$ and $(x_{2n-1})_{n=1}^{\infty}$ of even and odd terms of $(x_n)_{n=1}^{\infty}$, respectively. Then we have

$$x_{2n} = \frac{2n-1}{2n} \to 1, \text{ and } x_{2n-1} = -\frac{2n-2}{2n-1} \to -1$$

as $n \to \infty$. As these two subsequences converge to different values, the sequence $(x_n)_{n=1}^{\infty}$ does not converge. □

The Bolzano–Weierstrass Theorem

I now want to turn to another consequence of the completeness axiom: the Bolzano–Weierstrass Theorem[15]. The key idea of the proof that we will study below is found in the following combinatorial lemma.

[15] Which, like the monotone convergence theorem is in fact equivalent to completeness.

Lemma 3.28 (A combinatorial lemma)**.** *Every real sequence has a monotone subsequence.*

Proof. Let $(x_n)_{n=1}^{\infty}$ be a real sequence. Define[16] a natural number n to be *farseeing*[17] if for all $m > n$, we have $x_m < x_n$.

[16] Definitions in proofs are normally made just for the purpose of the proof, and shouldn't be considered a general mathematical concept. That certainly applies here.

[17] The picture I imagine here is a series of towers, one at each natural number n, of height x_n. Then the condition that n is farseeing means that I can stand on the n-th tower, and when look to the right, my view is not obstructed by any subsequent towers.

Suppose that there are infinitely many farseeing natural numbers, and write $k_1 < k_2 < k_3 < \dots$ for these farseeing numbers. Then the subsequence $(x_{k_n})_{n=1}^{\infty}$ is (strictly) decreasing, as for each n we have $k_{n+1} > k_n$, so $x_{k_{n+1}} < x_{k_n}$ as k_n is farseeing.

Otherwise there are only finitely many farseeing natural numbers, so we can fix N such that for all $n \geq N$, the natural number n is not farseeing. We will inductively construct an increasing subsequence of $(x_n)_{n=1}^{\infty}$, say $(x_{k_n})_{n=1}^{\infty}$ by specifying a suitable strictly increasing sequence of naturals $k_1 < k_2 < k_3 < \dots$. Set $k_1 = N$. Suppose inductively that for some $n \in \mathbb{N}$, we have defined natural numbers $N = k_1 < k_2 < k_3 < \cdots < k_n$ such that $x_{k_1} \leq x_{k_2} \leq \cdots \leq x_{k_n}$. Then, as $k_n \geq N$, the number k_n is not farseeing. By definition, this means that there exists a natural number $k_{n+1} > k_n$ such that $x_{k_{n+1}} \geq x_{k_n}$, and so we obtain the required increasing subsequence of $(x_n)_{n=1}^{\infty}$ by induction. □

Theorem 3.29 (Bolzano–Weierstrass)**.** *Every bounded real sequence has a convergent subsequence.*

Proof. Let $(x_n)_{n=1}^{\infty}$ be a bounded real sequence. By the combinatorial lemma above, there exists a monotonic subsequence $(x_{k_n})_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$. Since this subsequence is bounded it converges by the monotone convergence theorem. □

We will see one quick application of Bolzano–Weierstrass in the following section, and then again right at the end of the course.

Cauchy Sequences

The last topic in this section is of Cauchy sequences, and this is really laying the scene for the 3H course Metric spaces and Basic Topology next year. In that course you will learn about an abstract setting for the study of convergence, just as the earlier courses this year developed the abstract notion of a vector space as a general setting for the study of vectors and linear maps. In order to define the notion of completeness in this setting one has to use properties which can be defined only using the distance function[18] of a metric space[19]. Cauchy sequences are the tool that will be used to do this.

Let us start with the formal definition.

[18] In the case of the real numbers the distance function is $d(x, y) = |x - y|$.

[19] The completeness axiom we use for the real numbers is defined in terms of the order structure, and this will not be an ingredient in the theory of metric spaces: indeed, there is no total ordering on $\mathbb{C}$ which is compatible with the algebraic axioms — still the complex numbers are a perfectly nice example of a metric space.

Definition 3.30. Let $(x_n)_{n=1}^{\infty}$ be a sequence. We say that $(x_n)_{n=1}^{\infty}$ is *Cauchy* if

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall (n, m \in \mathbb{N} \text{ with } n, m \geq n_0), |x_n - x_m| < \varepsilon.$$

You should think of the condition of being Cauchy as meaning that as n gets large, all the terms of the sequence get arbitrarily close together[20]. I'd encourage you to spend time unpacking this definition in just the same way I did with the definition of convergence earlier in the section to try to see why I think of Cauchy sequences in this way.

[20] It's important to note that this is not the same as saying that eventually x_n gets arbitrarily close to x_{n+1}. The statement

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t.}$$
$$\forall (n \in \mathbb{N} \text{ with } n \geq n_0), |x_n - x_{n+1}| < \varepsilon$$

is *not* equivalent to $(x_n)_{n=1}^{\infty}$ being Cauchy: a counterexample is $x_n = \sum_{r=1}^{n} \frac{1}{r}$, as we'll see in the next chapter on series.

The process of verifying that a certain sequence $(x_n)_{n=1}^{\infty}$ is Cauchy directly from the definition is very much like verifying that a sequence converges to L directly from the definition. First, introduce the symbol ε with "Let $\varepsilon > 0$ be arbitrary". You will then need to manipulate the inequality $|x_n - x_m| < \varepsilon$, and see if you can find a suitably large n_0 which will guarantee that $|x_n - x_m| < \varepsilon$ for $n, m \geq n_0$. When you do this, you can always assume say that $m \geq n$ as the other case is handled by symmetry. More precisely, the statement

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall (n, m \in \mathbb{N} \text{ with } m \geq n \geq n_0), |x_n - x_m| < \varepsilon.$$

is equivalent to $(x_n)_{n=1}^{\infty}$ is Cauchy.

Example. *Show directly from the definition that the sequence $(x_n)_{n=1}^{\infty}$ given by $x_n = \frac{n+1}{n+2}$ is Cauchy.*

Solution. Let $\varepsilon > 0$ be arbitrary. For natural numbers $m \geq n$, we have

$$\begin{aligned} |x_m - x_n| &= \left| \frac{m+1}{m+2} - \frac{n+1}{n+2} \right| \\ &= \left| \frac{(m+1)(n+2) - (n+1)(m+2)}{(m+2)(n+2)} \right| = \left| \frac{m-n}{(m+2)(n+2)} \right| \end{aligned}$$

Now, we estimate[21]

$$\left| \frac{m-n}{(m+2)(n+2)} \right| \leq \frac{m}{mn} = \frac{1}{n},$$

so if we take $n_0 \in \mathbb{N}$ such that $n_0 > \frac{1}{\varepsilon}$, it follows that if $m \geq n \geq n_0$, then $|x_m - x_n| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, it follows that $(x_n)_{n=1}^{\infty}$ is Cauchy. □

[21] I'm no longer trying to keep track of $|x_m - x_n|$ exactly, but am happy to make this quantity larger, and simpler to work with, provided I can bound the simpler quantity by ε. Thus I use the estimates $|m - n| = m - n \leq m$ as $m \geq n$ and $(m+2)(n+2) \geq mn$. Other estimates would work here.

Now we turn to our main theoretical result, the general principle of convergence. As a preparation, we need two propositions which I'll leave as exercises. The first is a good exercise in making sure you understand the definitions of convergence and Cauchy and can manipulate these definitions[22]. For the second, look at the proof of Theorem **??** and make sure you get a bound which does not depend on n or m. We note that neither result uses the completeness axiom.

[22] As a hint, you'll probably want to consider $\frac{\varepsilon}{2}$ in the definition of convergence for the first one.

Proposition 3.31. *Every convergent sequence is Cauchy.*

Proposition 3.32. *Every Cauchy sequence is bounded.*

Now we reach the general principle of convergence, which says that convergent sequences and Cauchy sequences are actually the same thing. This will be used in later courses to exhibit elements of metric spaces with certain properties as limits of Cauchy sequences[23]. The key strategy in the hard direction of the proof is to use Bolzano–Weierstass to obtain a convergent subsequence of a Cauchy sequence. We then show that the entire sequence converges to the same limit as the convergent subsequence[24].

[23] For example, we can show that there exist solutions to various differential equations in this way.

[24] This should not come as a surprise: large terms in the subsequence get close to the limit, and large terms in the sequence get close together, suggesting that all large terms in the sequence should be close to the limit. You can see from that sentence that we're estimating the distance in two steps, so we use an "$\frac{\varepsilon}{2}$ argument" in our proof.

Theorem 3.33 (The general principle of convergence)**.** *Let $(x_n)_{n=1}^{\infty}$ be a real sequence. Then $(x_n)_{n=1}^{\infty}$ is Cauchy if and only if it converges.*

Proof. If $(x_n)_{n=1}^{\infty}$ converges, then it is Cauchy by Proposition . Conversely, let $(x_n)_{n=1}^{\infty}$ be Cauchy, so that $(x_n)_{n=1}^{\infty}$ is bounded by Proposition . By the Bolzano–Weierstrass Theorem, $(x_n)_{n=1}^{\infty}$ has a subsequence, say $(x_{k_n})_{n=1}^{\infty}$ converging to $L \in \mathbb{R}$, say (where $k_1 < k_2 < \dots$ is a strictly increasing sequence of natural numbers, and so satisfy $n \leq k_n$ for all n).

We claim that $x_n \to L$ as $n \to \infty$, so let $\varepsilon > 0$ be arbitrary. As $(x_n)_{n=1}^{\infty}$ is Cauchy, there exists $n_0 \in \mathbb{N}$ such that $m, n \geq n_0 \implies |x_n - x_m| < \frac{\varepsilon}{2}$. Since $(x_{k_n})_{n=1}^{\infty}$ converges to L, there exists $n_1 \in \mathbb{N}$ such that $n \geq n_1 \implies |x_{k_n} - L| < \frac{\varepsilon}{2}$. Now take $n \geq n_0$, and let $m = \max(n_0, n_1)$, so that $m \geq n_1$. Then $|x_{k_m} - L| < \frac{\varepsilon}{2}$, and $k_m \geq m \geq n_0$, hence $|x_{k_m} - x_n| < \frac{\varepsilon}{2}$. Therefore

$$|x_n - L| < |x_n - x_{k_m}| + |x_{k_m} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we conclude $x_n \to L$, as claimed. □

As a final note, let us remark that the general principle of convergence is not quite equivalent to the completeness axiom[25]. The completeness axiom is equivalent to the statement that both Archimedes Axiom[26] and the general principle of convergence holds.

[25] unlike the monotone convergence theorem, and the Bolzano–Weierstrass Theorem which are.

[26] which says that the natural numbers are not bounded above.