2B Linear Algebra

Ex Sheet 2

True/False

We will start the tutorial by going over these true false questions. Please make sure you've thought about them in advance of the tutorial and so are ready to answer.[1]

a) For a matrix A the (i, j)-entry of A in the entry in column i and row j.

b) Consider the matrix

$$C = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 7 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

Then C is diagonal with diagonal entries 2, 7 and 3.

c) Consider the matrix

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

Then D is diagonal with diagonal entries 4, 7 and 9.

d) For all matrices A and B of the same size, $A + B = B + A$.

e) For all square matrices A and B of the same size, $AB = BA$.

f) The transpose of an $m \times n$ matrix is an $n \times m$ matrix.

g) The matrices C and D above are both symmetric.

h) If the matrix A is symmetric then the matrix $-A$ is symmetric.

i) The matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is a linear combination of the matrices $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

j) The matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ span $M_{2\times 2}(\mathbb{R})$.

k) The matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are linearly independent.

[1] **True/False Questions**

Every Exercise Sheet will have a section containing true/false questions, with solutions at the end of the sheet. They are designed to test your understanding from lectures. The degree exam will have true/false questions selected from those on Exercise Sheets. You should look at your lecture notes and/or the textbook to help you answer these questions, but you should not need to write anything to work out the solution.

Solutions to True/False

(a) F (b) F (c) T (d) T (e) F (f) T (g) F (h) T (i) F (j) T (k) T

Tutorial Exercises

Before attempting these questions you should make sure you can do the questions on Matrices from Exercise Sheet 0.

T1 Give an example of a nonzero 2×2 matrix A so that A^2 is the zero matrix.

Solution

An example is $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

T2 Let $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Prove that

$$A^2 = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}.$$

Solution

We have

$$\begin{aligned} A^2 &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta - \sin^2\theta & -2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \end{aligned}$$

where we are applying double angle formulas to obtain the last equality.

T3 Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Prove by induction that for every positive integer k,

$$A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}.$$

Solution

When $k = 1$ we have $A^1 = A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, so the base case holds. Assume that for n a positive integer we have $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$. Then

$$A^{n+1} = A^n A = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n+1 \\ 0 & 1 \end{bmatrix}.$$

Therefore by induction the result holds for all positive integers k.

T4 Show that if A and B are 3×3 matrices,[2]

$$\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B).$$

[2] For a square matrix A, the *trace* of A, denoted by $\mathrm{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A (i.e., the diagonal from top left to bottom right).

Solution

The trace of A is

$$a_{11} + a_{22} + a_{33}$$

and the trace of B is

$$b_{11} + b_{22} + b_{33}.$$

The diagonal entries of $C = A + B$ are respectively

$$c_{11} = a_{11} + b_{11},$$

$$c_{22} = a_{22} + b_{22}$$

and

$$c_{33} = a_{33} + b_{33}.$$

The stated result is now immediate.

T5 Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ and let λ be a scalar.

Prove that:

a) $A + B = B + A$.

b) $\lambda(A + B) = \lambda A + \lambda B$.

c) $(A + B)^T = A^T + B^T$.

Solution

a) We have

$$\begin{aligned}
A+B &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} \\
&= \begin{bmatrix} b_{11}+a_{11} & b_{12}+a_{12} \\ b_{21}+a_{21} & b_{22}+a_{22} \end{bmatrix} \quad \text{by commutativity of scalar addition} \\
&= \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\
&= B+A
\end{aligned}$$

as required.

b) We have

$$\begin{aligned}
\lambda(A+B) &= \lambda \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} \\
&= \begin{bmatrix} \lambda(a_{11}+b_{11}) & \lambda(a_{12}+b_{12}) \\ \lambda(a_{21}+b_{21}) & \lambda(a_{22}+b_{22}) \end{bmatrix} \\
&= \begin{bmatrix} \lambda a_{11}+\lambda b_{11} & \lambda a_{12}+\lambda b_{12} \\ \lambda a_{21}+\lambda b_{21} & \lambda a_{22}+\lambda b_{22} \end{bmatrix} \quad \text{by the distributive law for scalars} \\
&= \begin{bmatrix} \lambda a_{11} & \lambda a_{12} \\ \lambda a_{21} & \lambda a_{22} \end{bmatrix} + \begin{bmatrix} \lambda b_{11} & \lambda b_{12} \\ \lambda b_{21} & \lambda b_{22} \end{bmatrix} \\
&= \lambda \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \lambda \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\
&= \lambda A + \lambda B
\end{aligned}$$

as required.

c) We have

$$\begin{aligned}
(A+B)^T &= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}^T \\
&= \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} \\ a_{12}+b_{12} & a_{22}+b_{22} \end{bmatrix}
\end{aligned}$$

while

$$\begin{aligned}
A^T+B^T &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^T + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}^T \\
&= \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} \\ a_{12}+b_{12} & a_{22}+b_{22} \end{bmatrix}.
\end{aligned}$$

Since $(A+B)^T$ and A^T+B^T are both equal to the same matrix, we have that $(A+B)^T = A^T + B^T$ as required.

T6 Write B as a linear combination of the other matrices, if possible:

a) $B = \begin{bmatrix} 2 & 5 \\ 0 & 3 \end{bmatrix}$, $A_1 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$.

b) $B = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $A_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Solution

a) We want to know if there are scalars c_1 and c_2 so that $c_1A_1 + c_2A_2 = B$. Now

$$c_1 \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 0 & 3 \end{bmatrix}$$

if and only if

$$\begin{bmatrix} c_1 & 2c_1 + c_2 \\ -c_1 + 2c_2 & c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 0 & 3 \end{bmatrix}$$

By comparing the $(1,1)$ entries on both sides we must have $c_1 = 2$. Then from the $(1,2)$ entries we obtain $c_2 = 1$. This is consistent with the remaining matrix entries, so the equation $c_1A_1 + c_2A_2 = B$ has (unique) solution $2A_1 + A_2 = B$.

Alternatively, an augmented matrix corresponding to the system of equations obtained by comparing the matrix entries is

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 5 \\ -1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$

and by carrying out EROs then solving by back-substitution this sytem has unique solution $c_1 = 2$, $c_2 = 1$, so $2A_1 + A_2 = B$.

b) We want to know if there are scalars c_1, c_2 and c_3 so that $c_1A_1 + c_2A_2 + c_3A_3 = B$. Now

$$c_1 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

if and only if

$$\begin{bmatrix} c_1 - c_2 + c_3 & 2c_2 + c_3 & -c_1 + c_3 \\ 0 & c_1 + c_2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

A corresponding system of equations is:

$$\begin{array}{rcrcrcl} c_1 & - & c_2 & + & c_3 & = & 3 \\ & & 2c_2 & + & c_3 & = & 1 \\ -\ c_1 & & & + & c_3 & = & 1 \\ c_1 & + & c_2 & & & = & 1 \end{array}.$$

(We don't need to write down equations for the entries which are 0 on both sides.) The augmented matrix for this system is:

$$\begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 2 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

and after applying EROs we get an echelon form

$$\begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

This system does not have a solution, so it is not possible to write B as a linear combination of the other matrices.

T7 Determine whether each of the sets of matrices in question T6 is linearly independent.

Solution

a) The matrices B, A_1 and A_2 are not linearly independent since B can be expressed as a linear combination of A_1 and A_2.

b) Suppose there are scalars c_1, c_2, c_3 and c_4 so that $c_1A_1 + c_2A_2 + c_3A_3 + c_4B = 0$. Then

$$\begin{bmatrix} c_1 - c_2 + c_3 + 3c_4 & 2c_2 + c_3 + c_4 & -c_1 + c_3 + c_4 \\ 0 & c_1 + c_2 + c_4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A corresponding homogeneous system of equations is:

$$\begin{array}{rcrcrcrl} c_1 & - & c_2 & + & c_3 & + & 3c_4 = & 0 \\ & & 2c_2 & + & c_3 & + & c_4 = & 0 \\ -\ c_1 & & & + & c_3 & + & c_4 = & 0 \\ c_1 & + & c_2 & & & + & c_4 = & 0 \end{array}.$$

The augmented matrix for this system is:

$$\begin{bmatrix} 1 & -1 & 1 & 3 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

and after applying EROs we get an echelon form

$$\begin{bmatrix} 1 & -1 & 1 & 3 & 0 \\ 0 & 1 & -2 & 4 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

This system has unique solution $c_1 = c_2 = c_3 = c_4 = 0$. Therefore the matrices A_1, A_2, A_3, B are linearly independent.

T8

a) Prove that if A and B are symmetric matrices, then $A + B$ is symmetric.

b) Prove that if A is a symmetric matrix and c is a scalar, then cA is symmetric.

Solution

a) Suppose A and B are symmetric. By Theorem 3.4(b) we have $(A+B)^T = A^T + B^T$. Since A and B are symmetric, $A^T = A$ and $B^T = B$. Thus $(A+B)^T = A + B$, hence $A + B$ is symmetric.

b) Suppose A is symmetric and c is a scalar. By Theorem 3.4(c), $(cA)^T = cA^T$. Since A is symmetric, $A^T = A$. So $(cA)^T = cA$, hence cA is symmetric.

T9 Use Theorem 3.4 to prove that for any $m \times n$ matrix A, the matrices AA^T and A^TA are symmetric. [Hint: take the transpose.]

Solution

By Theorem 3.4 part (d) and then part (a)

$$(AA^T)^T = (A^T)^T A^T = AA^T$$

so AA^T is symmetric. The proof is similar for A^TA.

T10 A square matrix A is defined to be *skew-symmetric* if $A^T = -A$.

a) Prove that if A and B are skew-symmetric, then $A + B$ is skew-symmetric.

b) Prove that if A is a skew-symmetric matrix and c is a scalar, then cA is skew-symmetric.

c) Prove that for all square matrices A, the matrix $A - A^T$ is skew-symmetric.

Solution

a) Suppose A and B are skew-symmetric. By Theorem 3.4(b) we have $(A+B)^T = A^T + B^T$. Since A and B are skew-symmetric, $A^T = -A$ and $B^T = -B$. Thus $(A+B)^T = -A - B = -(A+B)$, hence $A + B$ is skew-symmetric.

b) Suppose A is skew-symmetric and c is a scalar. By Theorem 3.4(c), $(cA)^T = cA^T$. Since A is skew-symmetric, $A^T = -A$. So $(cA)^T = c(-A) = -cA$, hence cA is skew-symmetric.

T11 Using Theorem 3.5(a) and several of the previous exercises in this section, prove that any square matrix A can be written as the sum of a symmetric matrix and a skew-symmetric matrix.

Solution

Let A be a square matrix. By Theorem 3.5(a) the matrix $A + A^T$ is symmetric, and by F3(c) the matrix $A - A^T$ is skew-symmetric. Notice that

$$(A + A^T) + (A - A^T) = 2A.$$

So if we divide through this equation by 2 we can make A the subject:

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

By F1(b), the matrix $\frac{1}{2}(A + A^T)$ is symmetric, and by F3(b) the matrix $\frac{1}{2}(A - A^T)$ is skew-symmetric. Therefore we have written A as a sum of a symmetric matrix and a skew-symmetric matrix.