Algorithmic Foundations 2 - Tutorial Sheet 8
Probability (and more Counting)

1. In roulette there is a wheel with 38 numbers of these 18 are red and 18 are black. The other two numbers are 0 and 00 which are neither red nor black. The probability that when the wheel is spun it lands an a particular number is 1/38.
   (a) What is the probability the wheel lands on a red number?
   (b) What is the probability the wheel lands on a black number twice in a row?
   (c) What is the probability the wheel lands on 0 or 00?
   (d) What is the probability in five spins the wheel neither lands on 0 nor 00?
   (e) What is the probability the wheel lands on one of the first six integers on one spin, but does not land on any of them on the next spin?

2. For each of the following pairs of events determine there probabilities and if they are independent or not when a coin is tossed three times.
   (a) E_1: the first coin comes up `tails`.
      E_2: the second coin comes up `heads`.
   (b) E_3: the first coin comes up `tails`.
      E_4: precisely two `heads` in a row.
   (c) E_5: the second coin comes up `tails`.
      E_6: precisely two `heads` in a row.

3. What probabilities should be assigned to the outcomes of a biased coin if the probability of heads equals four times the probability of tails.

4. What is the conditional probability that a randomly generated bit string of length four contain at least two consecutive 0's, given that the first bit is a `1`?

5. A *Bernoulli trial* is an experiment which can have only two possible outcomes (denoted *success* and *failure*).

   Find each of the probabilities when n independent Bernoulli trials are carried out, each with a probability of success equal to p.
   (a) The probability of no successes.
   (b) The probability of at least one success.
   (c) The probability of at most one success.
   (d) The probability of precisely two successes.
   (e) The probability of at least two successes.

6. Suppose there are two boxes of balls, the first box contains two white balls and three blue balls, while the second contains four white and one blue ball.

   Suppose you choose a box at random and then select a ball from that box at random, what is the probability that a ball from the first box was chosen, given you selected a blue ball.

7. Given three cards where:
   - the first is red on each side;
   - the second is green on each side;
   - and the third is red on one side and green on the other.

If we draw one card at random and look at one side only, what is the probability that given the side we are looking at is green that the other side is also green?

8. Suppose a test for opioids use has a 2% false positive rate and a 5% false negative rate. (More precisely, 2% of people who have not taken opioids test positive and 5% of people who have taken opioids test negative.) Assume that 1% of people have taken opioids.

   (a) Find the probability that someone who tests negative for opioids has not taken opioids.

   (b) Find the probability that someone who tests positive for opioids has actually taken opioids.

9. A box contains 3 yellow balls and 5 red balls. A ball is chosen at random from the box, then replaced in the box along with two other balls of the same colour.

   (a) If a second ball is now chosen at random from the box, what is the probability that it will be red?

   (b) Given that the second ball is red, what is the probability that the first ball was yellow?

10. An *octahedral die* has eight faces that are number 1 to 8. What are the expected value and the variance when a fair octahedral die is rolled.

Difficult/challenging questions.

11. Both undergraduates and postgraduates can use the university cafeteria. Each diner can choose between buying a meal or bringing a packed lunch. (Everyone has exactly one meal each, no more and no less). The cafeteria offers a daily choice between a hot meal or a cold meal. A survey of undergraduate diners finds that 40% of them bring their own food. Overall, only 25% of the diners bring their own food. Postgraduates make up one fifth of the diners in the cafeteria.

    (a) What is the probability that a diner is an undergraduate and buys a meal?

    (b) What is the probability that someone that buys a meal is a postgraduate?

12. The *Birthday Problem* asks what is the minimum number of people who need to be in a room so the probability at least two people have the same birthday is greater than $1/2$.

    Find this number under the assumption that the birthdays of the people in the room are independent, each birthday is equally likely and the number of days in a year is 366.

13. A space probe near Neptune communicates with Earth using bit strings. Suppose that in its transmissions it sends `1`'s one-third of the time and `0`'s the remaining two thirds. When a `0` is sent, the probability that a `0` is received is 0.9 (and a `1` is received with probability 0.1). On the other hand, when a `1` is sent, the probability that a `1` is received is 0.8 (and a `0` is received with probability 0.2).

    (a) What is the probability that a `0` is received.

    (b) What is the probability that a `0` was transmitted given that a `0` was received.