2A degree exam 2017–18, solutions

1. With $z(x,y) = \ln(x^2+y)$ we consider $Z(s,t) = z(x(s,t), y(s,t))$ with $x(s,t) = se^t$ and $y = te^{-s}$. Then the chain rule gives

$$\frac{\partial Z}{\partial s} = \frac{\partial x}{\partial s}\frac{\partial z}{\partial x} + \frac{\partial y}{\partial s}\frac{\partial z}{\partial y}$$
$$\frac{\partial Z}{\partial t} = \frac{\partial x}{\partial t}\frac{\partial z}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial z}{\partial y}.$$

The relevant partial derivatives are

$$\frac{\partial x}{\partial s} = e^t, \quad \frac{\partial x}{\partial t} = se^t, \quad \frac{\partial y}{\partial s} = -te^{-s}, \quad \frac{\partial y}{\partial t} = e^{-s}$$

and

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2+y}, \quad \frac{\partial z}{\partial y} = \frac{1}{x^2+y}.$$

Putting these together

$$\frac{\partial Z}{\partial s} = e^t \cdot \frac{2se^t}{(se^t)^2+te^{-s}} - te^{-s} \cdot \frac{1}{(se^t)^2+te^{-s}} = \frac{2se^{2t}-te^{-s}}{s^2e^{2t}+te^{-s}}$$
$$\frac{\partial Z}{\partial t} = e^t \cdot \frac{2se^t}{(se^t)^2+te^{-s}} + e^{-s} \cdot \frac{1}{(se^t)^2+te^{-s}} = \frac{2se^{2t}+e^{-s}}{s^2e^{2t}+te^{-s}}.$$

2. We seek to simplify the PDE by writing $f(x,y) = F(u(x,y), v(x,y))$. Then with $u = xy^4$ and $v = xy$ we have, using the chain rule,

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial F}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial F}{\partial v} = y^4\frac{\partial F}{\partial u} + y\frac{\partial F}{\partial v}$$
$$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial F}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial F}{\partial v} = 4xy^3\frac{\partial F}{\partial u} + x\frac{\partial F}{\partial v}.$$

Substitute into the PDE $xf_x - yf_y = 9xy$ to give

$$x\left(y^4F_u + \cancel{yF_v}\right) - y\left(4xy^3F_u + \cancel{xF_v}\right) = 9xy$$

which simplifies to $-3xy^4F_u = 9xy$ and using the change of variable to eliminate x and y we obtain the simplified PDE

$$\frac{\partial F}{\partial u} = -\frac{3v}{u}$$

a partial integration gives the general solution as

$$F(u,v) = -3v\log u + A(v)$$

where A is an arbitrary function of a single variable. The solution to the original PDE is therefore

$$f(x,y) = -3xy\log\left(xy^4\right) + A(xy).$$

3. (i) The Laplacian of f is

$$\nabla^2 f = \operatorname{div}(\operatorname{grad} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

(ii) The gradient of $1/r$ (using the chain rule and implicit differentiation to give $r_x = x/r$ etc.) is

$$\nabla\left(\frac{1}{r}\right) = -\left(\frac{\partial r}{\partial x}\frac{1}{r^2}, \frac{\partial r}{\partial y}\frac{1}{r^2}, \frac{\partial r}{\partial z}\frac{1}{r^2}\right) = -\frac{1}{r^2}\left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right) = -\frac{1}{r^2}\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = -\frac{\mathbf{x}}{r^3}$$

where $\mathbf{x} = (x, y, z)$. Now we calculate the divergence, using the nabla identity

$$\nabla\cdot(f\mathbf{F}) = \mathbf{F}\cdot\nabla f + f\nabla\cdot\mathbf{F}$$

and that $\nabla\cdot\mathbf{x} = 3$,

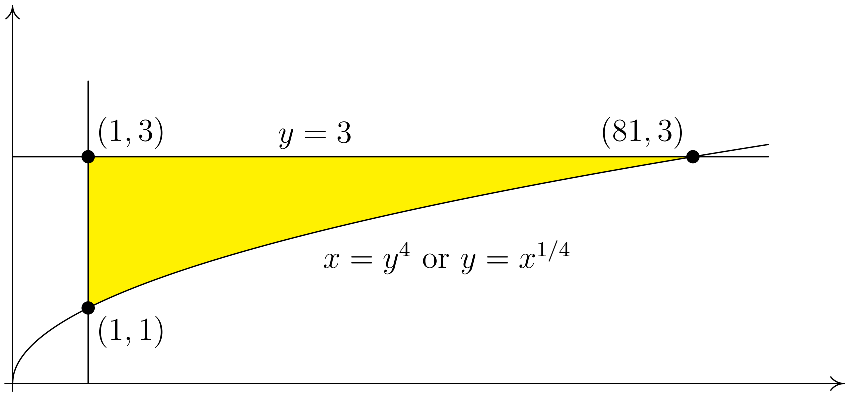
$$\nabla^2\left(\frac{1}{r}\right) = -\nabla\cdot\left(\frac{\mathbf{x}}{r^3}\right) = -\frac{1}{r^3}\nabla\cdot\mathbf{x} - \mathbf{x}\cdot\nabla\left(\frac{1}{r^3}\right) = -\frac{3}{r^3} - \mathbf{x}\cdot\left(-3\frac{\mathbf{x}}{r^5}\right) = -\frac{3}{r^3} + \frac{3}{r^3} = 0$$

(iii) The curl of the gradient is

$$\nabla\times\nabla\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ \phi_x & \phi_y & \phi_z \end{vmatrix} = (\phi_{zy} - \phi_{yz}, \phi_{xz} - \phi_{zx}, \phi_{yx} - \phi_{xy})$$

If ϕ has continuous second derivatives then the order of differentiation doesn't matter and so by Clairaut's theorem $\nabla\times\nabla\phi = \mathbf{0}$.

4. The sketch is shown below.



The integral is written as a type-II integral (x integral first) and so we convert to type-I. The type-I description of the region is $x^{1/4} \le y \le 3$ and $1 \le x \le 81$, therefore the integral can be written

$$I = \int_1^3 \int_1^{y^4} \frac{y^2}{x}\,dx\,dy = \int_1^{81}\int_{x^{1/4}}^3 \frac{y^2}{x}\,dy\,dx.$$

Performing the iterated integral gives

$$\int_1^{81}\left[\frac{y^3}{3x}\right]_{x^{1/4}}^3 dx = \int_1^{81} \frac{9}{x} - \frac{1}{3x^{1/4}}\,dx = \left[9\log x - \frac{4x^{3/4}}{9}\right]_1^{81} = 9\log 81 - \frac{104}{9}$$

5. In polar coordinates, in the r-θ plane the region is rectangular with $(r, \theta) \in [0, 2] \times [\pi/4, \pi/2]$: the y-axis is the line $\theta = \pi/2$ and the line $y = x$, in the first quadrant, is $\theta = \pi/4$); the circle $x^2 + y^2 = 4$ corresponds to $r = 2$. Therefore, substitution of $x = r\cos\theta$ and $y = r\sin\theta$ in the integrand and use of the Jacobian $J = r$ (scale factor of areas) gives

$$\iint_D xy\,dxdy = \int_{\pi/4}^{\pi/2}\left(\int_0^2 (r\cos\theta)\,(r\sin\theta)\,r\,dr\right)\,d\theta$$

note that the integrand is separable and the region is rectangular so the integral can be written as the product of two one-dimensional integrals

$$\left[\int_{\pi/4}^{\pi/2} \sin\theta\cos\theta\, d\theta\right]\left[\int_0^2 r^3\, dr\right] = \left[-\frac{1}{4}\cos 2\theta\right]_{\pi/4}^{\pi/2}\left[\frac{1}{4}r^4\right]_0^2 = 1$$

6. In spherical polar coordinates the hemisphere is described by $r \in [0, \sqrt{3}]$, $\theta \in [0, 2\pi]$ and $\phi \in [0, \pi/2]$. Using spherical polar coordinates we note that $x^2 + y^2 + z^2 = r^2$ and the Jacobian of the change of variables is $J = r^2 \sin\phi$, so the integral becomes

$$\iiint_{V'} \left(2 + 5r^2\right) r^2 \sin\phi\, dr d\theta d\phi$$

and since the integrand is separable and the region is cuboidal we can write the integral as the product of three one-dimensional integrals as follows

$$\left[\int_0^{\sqrt{3}} (2+5r^2)r^2\, dr\right]\left[\int_0^{\pi/2} \sin\phi\, d\phi\right]\left[\int_0^{2\pi} 1\, d\theta\right] = \left[\frac{2r^3}{3} + r^5\right]_0^{\sqrt{3}} \left[-\cos\phi\right]_0^{\pi/2} \left[\theta\right]_0^{2\pi} = 22\pi\sqrt{3}.$$

7. The projection of the surface onto the xy-plane is the quadrilateral given ($0 \le x \le 1$ and $0 \le y \le x+3$). The surface is the graph $z = 25 - 3x - 2y$, to convert the surface integral into a double integral we calculate

$$\sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + (-3)^2 + (-2)^2} = \sqrt{14}.$$

The quadrilateral is a type-I region so the surface integral becomes

$$\int_0^1 \left(\int_0^{x+3} y\sqrt{14}\, dy\right) dx = \int_0^1 \frac{1}{2}(x+3)^2\sqrt{14}\, dx = \left[\frac{1}{6}(x+3)^3\sqrt{14}\right]_0^1 = \frac{37\sqrt{14}}{6}.$$

8. A parametric description of the given curve is $\mathbf{r}(t) = (\cos t, \sin t)$ for $t \in [-\pi/2, \pi/2]$. We can check that all points on this curve lie on the circle $x^2 + y^2 = 1$, check that $|\mathbf{r}|^2 = 1$ (the calculation $\cos^2 t + \sin^2 t = 1$ confirms this). The start, when $t = -\pi/2$ gives the point $(0, -1)$ and the end, when $t = \pi/2$, gives the point $(0, 1)$. The orientation is anticlockwise (the positive direction), we can confirm this by calculating $\dot{\mathbf{r}}(-\pi/2) = (1, 0)$ (as $\dot{\mathbf{r}} = (-\sin t, \cos t)$). Now we have

$$\int_C \mathbf{F}\cdot d\mathbf{x} = \int_{-\pi/2}^{\pi/2} (-\sin t, \cos t)\cdot(-\sin t, \cos t)\ dt = \int_{-\pi/2}^{\pi/2} \sin^2 t + \cos^2 t\, dt = \pi.$$

9. The divergence says that, given $\mathbf{F}$, a vector field in $\mathbb{R}^3$ we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_V \nabla\cdot\mathbf{F}\, dV$$

where S is a closed orientable surface enclosing the region V, with outward pointing unit normal $\mathbf{n}$.

In our case $\nabla \cdot \mathbf{F} = -3x^2 - 2y + 6z$ and we can write the triple integral as the following iterated integral

$$\begin{aligned}
\int_0^1 \left(\int_{-1}^1 \left(\int_{-1}^2 -3x^2 - 2y + 6z \, dz \right) dy \right) dx &= \int_0^1 \left(\int_{-1}^1 \left[-3x^2 z - 2yz + 3z^2 \right]_{-1}^2 dy \right) dx \\
&= \int_0^1 \left(\int_{-1}^1 -9x^2 - 6y + 9 \, dy \right) dx \\
&= \int_0^1 \left[-9x^2 y - 3y^2 + 9y \right]_{-1}^1 dx \\
&= \int_0^1 -18x^2 + 18 \, dx \\
&= \left[-6x^3 + 18x \right]_0^1 = 12
\end{aligned}$$