EXAMINATION FOR THE DEGREES OF
M.A. AND B.Sc.

Mathematics 2B - Linear Algebra

An electronic calculator may be used provided that it does not have a facility for either textual storage or display, or for graphical display.

Candidates must attempt all questions.

Specimen written section of the exam paper - 30 marks

1. We consider a subset of $\mathbb{R}^3$ in the form

$$\mathcal{C} = \{[1,1,0],[0,1,1],[0,0,1]\}$$

(i) Show that $\mathcal{C}$ forms a basis for $\mathbb{R}^3$.

(ii) Construct the change-of-basis matrix $P_{\mathcal{C}\leftarrow\mathcal{E}}$, where $\mathcal{E}$ is the standard basis for $\mathbb{R}^3$

(iii) Consider the vector $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \in \mathbb{R}^3$. Using your solution to part(ii), or otherwise, compute the coordinates of $\mathbf{w}$ with respect to the basis $\mathcal{C}$.

2. Consider two $n \times n$ matrices C and D, where D is invertible. Prove that the matrix C is non-invertible if and only if the matrix DC is non-invertible.

3. Consider an $n \times n$ matrix A. Suppose that A has an eigenvalue λ. Show that the eigenspace E_λ is a subspace of $\mathbb{R}^n$.

4. Let U be the subspace of $\mathbb{R}^4$ spanned by

$$(1,1,0,0), \quad (2,0,1,0), \quad (1,0,2,1)$$

Use the Gram-schmidt Process to find an orthogonal basis for U.

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5. Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$(x_1, x_2) \to (x_1 + 2x_2, -x_1).$$

Show that T is a linear transformation.

6. Let q be the quadratic form defined by

$$q(x_1, x_2) = -3x_1^2 + 5x_2^2 - 6x_1x_2.$$

(i) Find a non-singular change of variables $\mathbf{x} = Q\mathbf{y}$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

such that $q(y_1, y_2) = \lambda_1 y_1^2 + \lambda_2 y_2^2$, for some $\lambda_1, \lambda_2 \in \mathbb{R}$.

(ii) Find a diagonal matrix D such $Q^T AQ = D$, where A is the matrix associated to q and determine the rank and the signature of q.

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