2A degree exam 2015–16, solutions

1. (i) With $f = e^x \sin 2y$, to calculate the second order derivatives, first calculate the first order derivatives

$$\frac{\partial f}{\partial x} = e^x \sin 2y, \qquad \frac{\partial f}{\partial y} = 2e^x \cos 2y.$$

Differentiate the first derivatives to find the second order derivatives (remember the mixed derivatives)

$$\frac{\partial^2 f}{\partial x^2} = e^x \sin 2y, \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 2e^x \cos 2y, \qquad \frac{\partial^2 f}{\partial y^2} = -4e^x \sin 2y$$

Check that f satisfies the Helmholtz equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + 3f = e^x \sin 2y - 4e^x \sin 2y + 3e^x \sin 2y = 0.$$

(ii) The chain rule for this composition gives

$$\frac{\partial F}{\partial x} = \frac{\partial r}{\partial x} g'(r), \qquad \frac{\partial F}{\partial y} = \frac{\partial r}{\partial y} g'(r)$$

If $r = \sqrt{x^2 + y^2}$ then

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \qquad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

and if $g(u) = \log u$ then $g'(u) = u^{-1}$. Using the chain rule as stated above we find

$$\frac{\partial F}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} \frac{1}{\sqrt{x^2 + y^2}} = \frac{x}{x^2 + y^2}$$

and

$$\frac{\partial F}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} \frac{1}{\sqrt{x^2 + y^2}} = \frac{y}{x^2 + y^2}$$

(iii) The chain rule for this *change of variable* is

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial F}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial F}{\partial v}$$
$$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial F}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial F}{\partial v}$$

For the change of variable given we have

$$\frac{\partial u}{\partial x} = y, \qquad \frac{\partial v}{\partial x} = -2x$$
$$\frac{\partial u}{\partial y} = x, \qquad \frac{\partial v}{\partial y} = 2y$$

Substitute into the PDE

$$y\left(y\frac{\partial F}{\partial u} - \cancel{2x\frac{\partial F}{\partial v}}\right) + x\left(x\frac{\partial F}{\partial u} + \cancel{2y\frac{\partial F}{\partial v}}\right) = \left(x^2 + y^2\right) F$$

so

$$\frac{\partial F}{\partial u} = F$$

this is a first order, *separable* PDE (see 1S for the theory of separable first order ODEs, the difference here is one of partial integration), we see the PDE can be written

$$\frac{\partial}{\partial u}(\log F) = 1$$

which has general solution

$$\log F = u + A(v)$$

where $A(v)$ is some arbitrary function of v, so

$$F = B(v)\exp u$$

where $B(v)$ is some arbitrary function of v. Then the solution to the original PDE is

$$f(x,y) = B\left(y^2 - x^2\right)\exp(xy).$$

2. (i) Let the components of $\mathbf{f}$ be $\mathbf{f} = (f_x, f_y, f_z)$ then (using product and chain rules)

$$\begin{aligned}
\operatorname{div}(\phi\mathbf{f}) &= \frac{\partial}{\partial x}(\phi f_x) + \frac{\partial}{\partial y}(\phi f_y) + \frac{\partial}{\partial z}(\phi f_z) \\
&= \phi\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}\right) + \phi'(r)\left(\frac{\partial r}{\partial x}f_x + \frac{\partial r}{\partial y}f_y + \frac{\partial r}{\partial z}f_z\right).
\end{aligned}$$

Now we have, $\partial r/\partial x = x/r$ and other partial derivatives have the same structure so the divergence becomes

$$\begin{aligned}
\operatorname{div}(\phi\mathbf{f}) &= \phi\operatorname{div}\mathbf{f} + \phi'(r)\left(\frac{x}{r}f_x + \frac{y}{r}f_y + \frac{z}{r}f_z\right) \\
&= \phi\operatorname{div}\mathbf{f} + \phi'(r)\hat{\mathbf{x}}\cdot\mathbf{f}
\end{aligned}$$

(ii) We apply the result from (i) with $\phi = r^2$, $\mathbf{f} = \boldsymbol{\omega}\times\mathbf{r}$. Now $\phi'(r) = 2r$ and

$$\boldsymbol{\omega}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = (\omega_2 z - \omega_3 y, \omega_3 x - \omega_1 z, \omega_1 y - \omega_2 x)$$

so

$$\begin{aligned}
\operatorname{div}(\boldsymbol{\omega}\times\mathbf{r}) &= \frac{\partial}{\partial x}(\omega_2 z - \omega_3 y) + \frac{\partial}{\partial y}(\omega_3 x - \omega_1 z) + \frac{\partial}{\partial z}(\omega_1 y - \omega_2 x) \\
&= 0 + 0 + 0 \\
&= 0
\end{aligned}$$

This means (using the result from (i))

$$\operatorname{div}\left(r^2\boldsymbol{\omega}\times\mathbf{r}\right) = 0 + 2r\hat{\mathbf{r}}\cdot(\boldsymbol{\omega}\times\mathbf{r}) = 0$$

using the fact that when two vectors are parallel in the scalar triple product, then this product is zero.

3. (i) The region $\mathcal{D}$ is type I (it is *not* type II). In order to be regular a region must be the union of finitely many type I or type II regions. As the region is simply a type I region then it is regular. In polar coordinates we have the region described by

$$\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}, \qquad 1 \leq r \leq 3.$$

The integral is then

$$\begin{aligned}
\iint_{\mathcal{D}} xy\, dxdy &= \int_{\pi/4}^{3\pi/4} \int_{1}^{3} (r\cos\theta)(r\sin\theta)\, r\, drd\theta \\
&= \left(\int_1^3 r^3\, dr\right)\left(\int_{\pi/4}^{3\pi/4} \cos\theta\sin\theta\, d\theta\right) \\
&= 10 \int_{\pi/4}^{3\pi/4} \sin 2\theta\, d\theta \\
&= -5\left[\cos 2\theta\right]_{\pi/4}^{3\pi/4} \\
&= 0
\end{aligned}$$

(ii) The notation

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

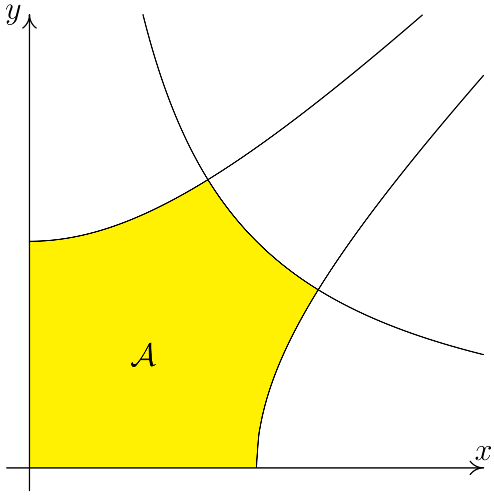
and is related to the Jacobian of the change of variables. In the case

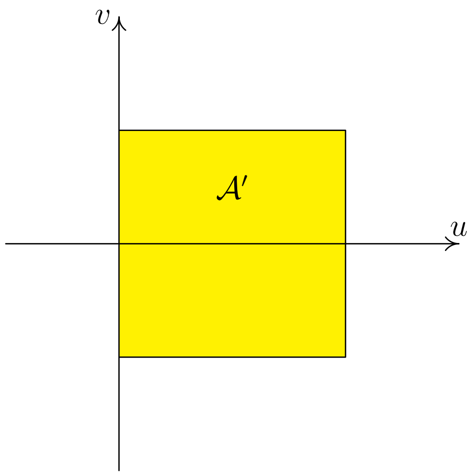
$$u = xy, \qquad v = \frac{1}{2}\left(y^2 - x^2\right)$$

the Jacobian is

$$J^{-1} = \frac{\partial(u,v)}{\partial(x,y)} = y(y) - x(-x) = x^2 + y^2$$

the different curves that bound the region of integration are: the axes ($u = 0$), the curve $y = 1/x$ ($u = 1$), the curve $y = \sqrt{x^2 - 1}$ ($v = -1/2$) and the curve $y = \sqrt{x^2 + 1}$ ($v = 1/2$). The regions in the xy and uv planes are shown below.





The change of variables gives (where $\mathcal{A}'$ is the region of integration in the uv-plane and note that $J \geq 0$ in our region of integration so $|J| = J$.)

$$\begin{aligned}
\iint_{\mathcal{A}} yx^3 + xy^3 \, dxdy &= \iint_{\mathcal{A}'} \left(yx^3 + xy^3\right) |J| \, dudv \\
&= \iint_{\mathcal{A}'} \frac{yx^3 + xy^3}{x^2 + y^2} \, dudv \\
&= \iint_{\mathcal{A}'} xy \, dudv = \int_0^1 \int_{-1/2}^{1/2} u \, dvdu \\
&= \int_0^1 u \, [v]_{-1/2}^{1/2} \, du = \frac{1}{2} \left[u^2\right]_0^1 = \frac{1}{2}
\end{aligned}$$

(iii) The triple integral can be written as an iterated integral as

$$\begin{aligned}
\int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} z \, dz \right) dy \right) dx &= \int_0^1 \left(\int_0^{1-x} \left[\frac{1}{2} z^2 \right]_0^{1-x-y} dy \right) dx \\
&= \frac{1}{2} \int_0^1 \left(\int_0^{1-x} (1-x-y)^2 \, dy \right) dx \\
&= \frac{1}{6} \int_0^1 \left[-(1-x-y)^3 \right]_0^{1-x} dx \\
&= \frac{1}{6} \int_0^1 (1-x)^3 \, dx \\
&= \frac{1}{24} \left[-(1-x)^4 \right]_0^1 \\
&= \frac{1}{24}
\end{aligned}$$

4. (i) Green's theorem for a positively oriented simple closed curve C is

$$\int_C P(x,y) \, dx + Q(x,y) \, dy = \iint_A \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dxdy$$

where A is the region enclosed by C. To calculate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

recognise that $d\mathbf{r} = (dx, dy)$ and $\mathbf{F} = (P, Q)$. In the example given

$$P = e^{-x} + y^2, \qquad Q = e^{-y} + x^2, \qquad \frac{\partial P}{\partial y} = 2y, \qquad \frac{\partial Q}{\partial x} = 2x$$

so we calculate

$$-\iint_A 2x - 2y \, dxdy$$

where A is the region bounded by the x-axis and the curve $y = \sin x$ between 0 and π, and the minus sign outside the integral comes from the fact that the curve given is negatively oriented (clockwise). The region A is a type I and a type II region, we treat it as a type

I region for integration

$$\int_0^\pi \left(\int_0^{\sin x} 2y - 2x \, dy \right) dx = \int_0^\pi \left[y^2 - 2xy \right]_0^{\sin x} dx$$
$$= \int_0^\pi \sin^2 x - 2x \sin x \, dx$$
$$= \int_0^\pi \frac{1}{2}(1 - \cos 2x) - 2\cos x \, dx + [2x \cos x]_0^\pi$$
$$= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x - 2\sin x + 2x \cos x \right]_0^\pi$$
$$= \frac{1}{2}\pi - 2\pi = -\frac{3\pi}{2}$$

(where we have used integration by parts and a trigonometric identity).

(ii) Recall for a surface $\mathcal{S}$ that is the graph of a function $z(x, y)$ and whose projection onto the xy-plane is $\mathcal{D}$ we have

$$\iint_{\mathcal{S}} f(x, y, z) \, dS = \iint_{\mathcal{D}} f(x, y, z(x, y)) \sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2} \, dxdy$$

In this case

$$\frac{\partial z}{\partial x} = -\frac{2x}{x^2 + y^2}, \qquad \frac{\partial z}{\partial y} = -\frac{2y}{x^2 + y^2}$$

so

$$\sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2} = \sqrt{1 + \frac{4x^2}{x^2 + y^2} + \frac{4y^2}{x^2 + y^2}} = \sqrt{5}.$$

The projection of the surface is the disc centred at the origin radius 2, call this A, so the integral becomes

$$\iint_A \sqrt{x^2 + y^2}\sqrt{5} \, dxdy = \sqrt{5} \int_0^2 dr \int_0^{2\pi} r^2 \, d\theta$$
$$= 2\pi\sqrt{5} \left[\frac{1}{3}r^3 \right]_0^2$$
$$= \frac{16\pi\sqrt{5}}{3}$$

(iii) The divergence theorem for a vector field $\mathbf{F}$ and a volume V bounded by the closed orientable surface S with outward pointing unit normal $\mathbf{n}$ is

$$\iiint_V \nabla \cdot \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS.$$

For the given vector field

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}\left(xz^2 \right) + \frac{\partial}{\partial y}\left(x^2 y - z^3 \right) + \frac{\partial}{\partial z}\left(y^2 z + \frac{1}{2}z^2 \right)$$
$$= z^2 + x^2 + y^2 + z$$

then the surface intgeral (using the divergence theorem) becomes the integral over the sphere centred at the origin radius a in the first octant. Spherical symmetry suggest a change to spherical polar coordinates, for which $x^2 + y^2 + z^2 = \rho^2$ and $z = \rho\cos\phi$, using the limits $0 \le \rho \le a$ and $0 \le \theta \le \pi/2$ and $0 \le \phi \le \pi/2$, then

$$\begin{aligned}
\int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi \int_0^a \big(\rho^2 + \rho\cos\phi\big)\,\rho^2 \sin\phi\, d\rho &= \int^{\pi/2} d\theta \int_0^{\pi/2} \frac{a^5}{5}\sin\phi + \frac{a^4}{8}\sin 2\phi\, d\phi \\
&= \left[-\frac{\pi a^5}{10}\cos\phi - \frac{\pi a^4}{32}\cos 2\phi\right]_0^{\pi/2} \\
&= \pi a^4\left(\frac{a}{10} + \frac{1}{16}\right)
\end{aligned}$$