The formula

$$\neg(p_1 \to \neg p_2) \to (p_3 \to (p_4 \to \neg p_5))$$

returns `false` for precisely one assignment to the propositions $p_1, \ldots, p_5$. Find this assignment using the laws of logical equivalence and *without* constructing a truth table for the formula.

Solution.

$$\begin{aligned}
&\neg(p_1 \to \neg p_2) \to (p_3 \to (p_4 \to \neg p_5)) \\
&\quad \equiv \neg(\neg p_1 \vee \neg p_2) \to (p_3 \to (\neg p_4 \vee \neg p_5)) && \text{implication law} \\
&\quad \equiv \neg(\neg p_1 \vee \neg p_2) \to (\neg p_3 \vee (\neg p_4 \vee \neg p_5)) && \text{implication law} \\
&\quad \equiv \neg\neg(\neg p_1 \vee \neg p_2) \vee (\neg p_3 \vee (\neg p_4 \vee \neg p_5)) && \text{implication law} \\
&\quad \equiv (\neg p_1 \vee \neg p_2) \vee (\neg p_3 \vee (\neg p_4 \vee \neg p_5)) && \text{double negation law} \\
&\quad \equiv \neg p_1 \vee \neg p_2 \vee \neg p_3 \vee \neg p_4 \vee \neg p_5
\end{aligned}$$

where the last step from the associative law for $\vee$. Considering this formula we have that the one assignment that returns `false` is when all the propositions $p_1, \ldots, p_5$ are `true`, i.e. all the formulae $\neg p_1, \ldots, \neg p_5$ are `false`.

Prove that $(p \wedge \neg q) \to q$ and $(p \wedge \neg q) \to \neg p$ are equivalent using laws of logical equivalence.

Solution.

$$
\begin{aligned}
(p \wedge \neg q) \to q &\equiv \neg(p \wedge \neg q) \vee q && \text{implication law} \\
&\equiv (\neg p \vee \neg\neg q) \vee q && \text{de Morgan law} \\
&\equiv (\neg p \vee q) \vee q && \text{double negation law} \\
&\equiv \neg p \vee (q \vee q) && \text{associative law} \\
&\equiv \neg p \vee q && \text{idempotency law} \\
&\equiv (\neg p \vee \neg p) \vee q && \text{idempotency law} \\
&\equiv \neg p \vee (\neg p \vee q) && \text{associative law} \\
&\equiv (\neg p \vee q) \vee \neg p && \text{commutative law} \\
&\equiv (\neg p \vee \neg\neg q) \vee \neg p && \text{double negation law} \\
&\equiv \neg(p \wedge \neg q) \vee \neg p && \text{de Morgan law} \\
&\equiv (p \wedge \neg q) \to \neg p && \text{implication law}
\end{aligned}
$$

Using the laws of logical equivalence show that the formula $(q \wedge (p \rightarrow \neg q)) \rightarrow \neg p$ is a tautology.

Solution. Below are two solutions to this question (others are possible).

$$
\begin{aligned}
(q \wedge (p \rightarrow \neg q)) \rightarrow \neg p &\equiv (q \wedge (\neg p \vee \neg q)) \rightarrow \neg p && \text{implication law} \\
&\equiv ((q \wedge \neg p) \vee (q \wedge \neg q)) \rightarrow \neg p && \text{distributive law} \\
&\equiv ((q \wedge \neg p) \vee \texttt{false}) \rightarrow \neg p && \text{contradiction law} \\
&\equiv (q \wedge \neg p) \rightarrow \neg p && \text{identity law} \\
&\equiv \neg(q \wedge \neg p) \vee \neg p && \text{implication law} \\
&\equiv (\neg q \vee p) \vee \neg p && \text{de Morgan law} \\
&\equiv \neg q \vee (p \vee \neg p) && \text{communtative law} \\
&\equiv \neg q \vee \texttt{true} && \text{tautology law} \\
&\equiv \texttt{true} && \text{domination law}
\end{aligned}
$$

$$
\begin{aligned}
(q \wedge (p \rightarrow \neg q)) \rightarrow \neg p &\equiv \neg(q \wedge (p \rightarrow \neg q)) \vee \neg p && \text{implication law} \\
&\equiv (\neg q \vee \neg(p \rightarrow \neg q)) \vee \neg p && \text{de Morgan law} \\
&\equiv (\neg q \vee \neg p) \vee \neg(p \rightarrow \neg q) && \text{associative law} \\
&\equiv (\neg p \vee \neg q) \vee \neg(p \rightarrow \neg q) && \text{commutative law} \\
&\equiv (p \rightarrow \neg q) \vee \neg(p \rightarrow \neg q) && \text{implication law} \\
&\equiv \texttt{true} && \text{tautology law}
\end{aligned}
$$

Suppose we have the following predicates:

- $P(x)$: x is prime
- $E(x)$: x is even
- $G(x, y)$: $x>y$
- $Eq(x, y)$: $x = y$
- $S(x, y, z) : x + y = z$

Express the following formula in good English (do not use variables and avoid the use of "there exists" and "for all").

- $\forall x \in \mathbb{Z}^+ . \exists x \in \mathbb{Z}^+ . G(y, x)$

  There is an integer greater than all integers.

- $\forall x \in \mathbb{Z}^+ . \exists y \in \mathbb{Z}^+ . \exists z \in \mathbb{Z}^+ . ((E(x) \wedge G(x, 3)) \rightarrow (P(y) \wedge P(z) \wedge S(y, z, x))$

  Any even positive integer greater than 3 can be expressed as the sum of two primes (this is the Goldbach conjecture).

- $\forall x \in \mathbb{Z}^+ . \forall y \in \mathbb{Z}^+ . ((\neg G(x, y) \wedge \neg G(y, x)) \rightarrow Eq(x, y))$

  Given any two positive integers, if neither is greater than the other then they are equal.

Suppose we have the following predicates:

- $P(x)$: x is prime
- $E(x)$: x is even
- $G(x, y)$: $x>y$
- $S(x, y, z) : x + y = z$
- $T(x, y, z)$: $x{\cdot}y = z$

Express the following English statements in logical formulae using the predicates given above over the domain of discourse $\mathbb{Z}^+$.

- Given any two positive integers there exists another positive integer which equals their product.

  $\forall x \in \mathbb{Z}^+ .\, \forall y \in \mathbb{Z}^+ .\, \exists z \in \mathbb{Z}^+ .\, T(x, y, z)$

- There is no largest prime.

  Two possible solutions are: $\forall x \in \mathbb{Z}^+ .\, \exists y \in \mathbb{Z}^+ .\, G(y, x)$ and $\neg\exists x \in \mathbb{Z}^+ .\, \forall y \in \mathbb{Z}^+ .\, (\neg Eq(x, y) \rightarrow G(x, y))$, other solutions are possible.

- If a positive integer is not prime, it is composite.

  One possible solution is:

  $$\forall x \in \mathbb{Z}^+ .\, (\neg P(x) \rightarrow \exists y \in \mathbb{Z}^+ .\, \exists z \in \mathbb{Z}^+ .\, (G(y, 1) \wedge G(z, 1) \wedge T(y, z, x))$$

  other solutions are possible.

  Note. A positive number is composite if it equals the product of two positive integers both greater than 1.

Prove that $A \cap (B \cup A) = A$ using a containment proof. Explain your steps.

First we show $A \cap (B \cup A) \subseteq A$, therefore consider any $x \in A \cap (A \cup B)$, by definition of intersection we have:

$$\begin{aligned} x \in A \cap (A \cup B) &\Rightarrow\ x \in A \text{ and } x \in A \cup B \\ &\Rightarrow\ x \in A \end{aligned}$$

and hence, since $x \in A \cap (A \cup B)$ was arbitrary, we have $A \cap (B \cup A) \subseteq A$.

For the reverse direction, i.e. showing $A \subseteq A \cap (B \cup A)$, consider any $x \in A$, we have:

$$\begin{aligned} x \in A &\Rightarrow\ x \in A \text{ and } x \in A && \\ &\Rightarrow\ x \in A \text{ and } x \in A \cup B && \text{by definition of union} \\ &\Rightarrow\ x \in A \cap (A \cup B) && \text{by definition of intersection} \end{aligned}$$

since $x \in A$ was arbitrary, we have $A \subseteq A \cap (B \cup A)$ completing the proof.

Prove $(A \backslash C) \cup (B \backslash C) = (A \cup B) \backslash C$ using set builder notation and logical equivalences. Explain your steps.

$$
\begin{array}{ll}
(A{-}C) \cup (B{-}C) \;=\; \{x \mid x \in (A{-}C) \cup (B{-}C)\} & \\
=\; \{x \mid (x \in A{-}C) \vee (x \in B{-}C)\} & \text{definition of union} \\
=\; \{x \mid ((x \in A) \wedge (x \notin C)) \vee ((x \in B) \wedge (x \notin C))\} & \text{defn. of set difference} \\
=\; \{x \mid ((x \notin C) \wedge (x \in A)) \vee ((x \notin C) \wedge (x \in B))\} & \text{commutative law} \\
=\; \{x \mid (x \notin C) \wedge ((x \in A)) \vee (x \in B))\} & \text{distributive law} \\
=\; \{x \mid ((x \in A)) \vee (x \in B)) \wedge (x \notin C)\} & \text{commutative law} \\
=\; \{x \mid ((x \in A \cup B)) \wedge (x \notin C)\} & \text{defn. of union} \\
=\; \{x \mid x \in (A \cup B){-}C\} & \text{defn. of set difference} \\
= (A \cup B){-}C &
\end{array}
$$

Let a and b be integers and let m be a positive integer. Show that $a \equiv b \pmod{m}$ if and only if $a \pmod{m} = b \pmod{m}$. Explain your steps.

"If direction": if $a \pmod{m} = b \pmod{m}$, then $a = k{\cdot}m + r$ and $b = l{\cdot}m + r$ for some integers k, l, r such that $0 {\leq} r {<} m$. Hence

$$\begin{aligned} a - b = & \ (k{\cdot}m + r) - (l{\cdot}m + r) & \\ = & \ k{\cdot}m + r - l{\cdot}m - r & \text{rearranging} \\ = & \ m{\cdot}(k - l) & \text{rearranging} \end{aligned}$$

and therefore, since k and l are integers, we have $m|(a - b)$, and hence by definition $a \equiv b \pmod{m}$ as required.

"Only if direction": if $a \equiv b \pmod{m}$, then by definition m divides $a{-}b$, and hence $a{-}b = k{\cdot}m$ for some integer k which rearranging yields $a = b + k{\cdot}m$. Now by the division algorithm, we have $b = q{\cdot}m + r$ for some integers q and r, where $0{\leq}r{<}m$ and by definition $b \pmod{m} = r$. Combining these equations for a and b we have:

$$a = b + k{\cdot}m = q{\cdot}m + r + k{\cdot}m = (q + k){\cdot}m + r$$

Now since $0{\leq}r{<}m$, by definition $a \pmod{m} = r$, and since $b \pmod{m} = r$, we have $a \pmod{m} = b \pmod{m} = r$ as required.

For each of these arguments, explain which rules of inference are used for each step.

If I do the tutorial questions and attend the labs, then I can understand the material. If I understand the material, I will be an excellent computer scientist. Therefore, if I do the tutorial questions and attend the labs, then I will be an excellent computer scientist.

We need the following propositions:

- p – I do the tutorial exercises;
- q – I attend the labs;
- r – I understand the material;
- s – I will be an excellent computer scientist.

Using these we have the following argument.

1. $(p \wedge q) \to r$	premise
2. $r \to s$	premise
3. $(p \wedge q) \to s$	hypothetical syllogism using 1 and 2

Ella, a student in the AF2 class, knows how to prove an algorithm is correct. Every student who knows how to prove an algorithm is correct can get a job with Google. Therefore, someone in the AF2 class can get a job with Google.

We need the following:

- *Ella* - the student Ella;

- S_{AF2} - the set of students in the AF2 class;
- S - the set of students;
- $PA(x)$ - knows how to prove an algorithm is correct;
- $Google(x)$ - can get a job with Google.

Using these we have the following argument.

1. $PA(Ella)$	premise
2. $\forall x \in S.\,(PA(x) \to Google(x))$	premise
3. $PA(Ella) \to Google(Ella))$	universal instantiation of 2
4. $Google(Ella)$	modus ponens of 1 and 3
5. $\exists x \in S_{AF2}.\, Google(x)$	existential generalisation of 4

Using the universe of all students, write out the following argument using quantifiers, connectives, and symbols to stand for propositions as necessary, explaining which rules of inference are used for each step.

> *All AF2 students are second years. There exists an AF2 student from Glasgow. Therefore, there is a second year student from Glasgow.*

We need the following:

- S - the set of students;
- $AF2(x)$ - an AF2 student;
- $Sec(x)$ - a second year student;
- $Gla(x)$ - from Glasgow;

Using these we have the following argument.

1. $\forall x \in S.\,(AF2(x) \to Sec(x))$	premise
2. $\exists x \in S.\,(AF2(x) \wedge Gla(x))$	premise
3. $AF2(s) \wedge Gla(s)$	for some $s \in S$ existential instantiation of 2.
4. $AF2(s)$	simplification of 3
5. $AF2(s) \to Sec(s)$	universal instantiation of 1
6. $Sec(s)$	modus ponens of 4 and 5
7. $Gla(s)$	simplification of 3
8. $Sec(s) \wedge Gla(s)$	conjunction of 6 and 7
9. $\exists x \in S.\,(Sec(x) \wedge Gla(x))$	existential generalisation of 8

Use a direct proof to show that the sum of two even integers is even.

Proof. If a and b are even numbers, then $a = 2{\cdot}k$ and $b = 2{\cdot}l$ for some integers k and l. Therefore we have:

$$a + b = 2{\cdot}k + b = 2{\cdot}l = 2{\cdot}(k + 1)$$

and hence $a + b$ is even as requried.

Use an indirect proof to show that if $x + y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$.

Proof. The proof is of the form $\forall x \in \mathbb{R}. \forall y \in \mathbb{R}. (P(x, y) \rightarrow Q(x, y))$ where $P(x, y) = (x + y \geq 2)$ and $Q(x, y) = (x \geq 1) \vee (y \geq 1)$. Since we are using an indirect proof we will consider arbitrary $x, y \in \mathbb{R}$ and show if $\neg Q(x, y)$ holds, then $\neg P(x, y)$ holds. Now

$$\begin{aligned} \neg Q(x, y) = \; & \neg((x \geq 1) \vee (y \geq 1)) & \\ \equiv \; & \neg(x \geq 1) \wedge \neg(y \geq 1) & \text{using de Morgan Law} \\ \equiv \; & (x < 1) \wedge (y < 1) & \text{rearranging} \end{aligned}$$

Therefore since $\neg Q(x, y)$ holds it follows that:

$$x + y < 1 + 1 = 2$$

and hence $\neg P(x, y)$ holds as required.

Show that if n is an integer and $n^3 + 5$ is odd, then n is even using an indirect proof.

Proof. This is an indirect proof, there we assume n is odd and try and show $n^3 + 5$ is even. Since n is odd we have $n = 2{\cdot}k + 1$ for some integer k and therefore:

$$\begin{aligned} n^3 + 5 = \; & (2{\cdot}k + 1)^3 + 5 & \\ = \; & (2{\cdot}k + 1)^2(2{\cdot}k + 1) + 5 & \text{rearranging} \\ = \; & (4{\cdot}k^2 + 4{\cdot}k + 1)(2{\cdot}k + 1) + 5 & \text{rearranging} \\ = \; & 8{\cdot}k^3 + 8{\cdot}k^2 + 2{\cdot}k + 4{\cdot}k^2 + 4{\cdot}4 + 1 + 5 & \text{rearranging} \\ = \; & 8{\cdot}k^3 + 12{\cdot}k^2 + 6{\cdot}4 + 6 & \text{rearranging} \\ = \; & 2{\cdot}(4{\cdot}k^3 + 6{\cdot}k^2 + 3{\cdot}4 + 3) & \text{rearranging} \end{aligned}$$

Therefore $n^3 + 5$ is odd as required.

Prove that if n is an integer, these four statements are equivalent:

(a) n is even;

(b) $n + 1$ is odd;

(c) $3n + 1$ is odd;

(d) $3n$ is even.

Proof. Here we show (a) $\rightarrow$ (b), (b) $\rightarrow$ (c), (c) $\rightarrow$ (d) and (d) $\rightarrow$ (a).

(a) $\rightarrow$ **(b)** If n is even, then $n = 2{\cdot}k$ for some integer k and hence $n + 1 = 2{\cdot}k + 1$ and is odd as required.

(b) $\rightarrow$ **(c)** If $n+1$ is odd, then $n+1 = 2{\cdot}k+1$ for some integer k. Therefore it follows $n = 2{\cdot}k$ by subtracting one 1 from both sides. Using this fact we have $3{\cdot}n+1 = 3{\cdot}(2{\cdot}k)+1$ which rearranging equals $2{\cdot}(3{\cdot}k)+1$, and hence $3{\cdot}n + 1$ is even as required.

(c) $\rightarrow$ (d) If $3{\cdot}n + 1$ is odd, then $n3{\cdot} + 1 = 2{\cdot}k + 1$ for some integer k. Therefore it follows $3{\cdot}n = 2{\cdot}k$ by subtracting one 1 from both sides as required.

(d) $\rightarrow$ (a) Here we will use an indirect proof, so we assume n is odd, and hence $n = 2{\cdot}k+1$ for some integer k. It follows that:

$$\begin{aligned} 3{\cdot}n &= 3{\cdot}(2{\cdot}k + 1) & \\ &= 6{\cdot}k + 3 & \text{rearranging} \\ &= 6{\cdot}k + 2 + 1 & \text{rearranging} \\ &= 2{\cdot}(3{\cdot}k + 1) + 1 & \text{rearranging} \end{aligned}$$

Therefore $3{\cdot}n$ is odd as required.

This completes the proof.

Show that $\sqrt{n}$ is irrational if n is a positive integer that is not a perfect square (an integer n is a perfect square if $n = k^2$ for some integer k).

Proof. Suppose for a contradiction that $\sqrt{n}$ is rational. Then $\sqrt{n} = a/b$ for some positive integers a and b, so that $a = b\sqrt{n}$, which implies that $a^2 = n{\cdot}b^2$.

Now by the Fundamental Theorem of Arithmetic any number can be expressed as the product of prime factors. It therefore follows that:

- when expressing the square of any number as the product of primes each power is even and, in particular when expressing a^2 and b^2 as the product of primes each power is even;
- since n is not a perfect square, expressing n as a product of powers of primes at least one of these prime factors must be raised to an odd power.

Thus when expressing $n{\cdot}b^2$ as the product of primes at least one prime is raised to an odd power, which contradicts the fact that $a^2 = n{\cdot}b^2$. Hence n cannot be rational and must be irrational.

Use a proof by cases to show that $\min(a, \min(b, c)) = \min(\min(a, b), c)$ whenever a, b, and c are real numbers.

Proof. There are three cases, depending on which of the three numbers is least.

- If $a \leq b, c$, then clearly $a \leq \min(b, c)$, and hence the left-hand side equals a. On the other hand, for the right-hand side we have $\min(a, b) = \min(a, c) = a$, and therefore the right hand side also equals a.
- if $b \leq a, c$, then similar reasoning shows us that both sides equal b.
- if $c \leq b, c$, then again similar reasoning shows us that both sides equal c.

The sum of the first n odd integers equals n^2.

Theorem. $\forall n \in \mathbb{Z}^+.\, P(n)$ where $P(n) : \sum\limits_{i=1}^{n}(2{\cdot}i - 1) = n^2$.

Base case: If $n = 1$, then $\sum_{i=1}^{1}(2{\cdot}i{-}1) = 1 = 1^2$ as required.

Inductive step: Suppose that $P(n)$ holds for some $n \geq 1$. Considering $n{+}1$ we have:

$$\begin{aligned} \sum_{i=1}^{n+1}(2{\cdot}i{-}1) &= \left(\sum_{i=1}^{n}(2{\cdot}i{-}1)\right) + 2{\cdot}(n+1){-}1 & \\ &= n^2 + 2{\cdot}(n+1){-}1 & \text{by induction} \\ &= n^2 + 2{\cdot}n + 1 & \text{rearranging} \\ &= (n+1)^2 & \text{rearranging} \end{aligned}$$

and hence $P(n{+}1)$ holds.

Therefore by the principle of induction we have proved that $P(n)$ holds for all $n \geq 1$.

Theorem. $\forall n \in \mathbb{Z}^+ .\, P(n)$ where $P(n) : \sum_{i=1}^{n} i = \frac{n\cdot(n+1)}{2}$

Base case: If $n = 1$, then $\sum_{i=1}^{1} i = 1 = (1)(2)/2$ as required.

Inductive step: Suppose that $P(n)$ holds for some $n \geq 1$. Considering $n{+}1$ we have:

$$\begin{aligned}
\sum_{i=1}^{n+1} i &= \left(\sum_{i=1}^{n} i\right) + (n+1) \\
&= \frac{n\cdot(n+1)}{2} + (n+1) && \text{by induction} \\
&= \frac{(n+1)}{2}(n+2) && \text{rearranging} \\
&= \frac{(n+1)((n+1)+1)}{2} && \text{rearranging}
\end{aligned}$$

and hence $P(n{+}1)$ holds.

Therefore by the principle of induction we have proved that $P(n)$ holds for all $n \geq 1$.

Theorem. For any $a, r \in \mathbb{Z}$ such that $r \neq 1$ and $n \in \mathbb{N}$: $\sum_{i=0}^{n} a\cdot r^i = \frac{a\cdot(r^{n+1}-1)}{(r-1)}$.

Let $P(n)$ be the proposition $\sum_{i=0}^{n} a\cdot r^i = \frac{a\cdot(r^{n+1}-1)}{(r-1)}$ and consider any

Base case: If $n = 0$, then for $a, r \in \mathbb{Z}$ such that $r \neq 1$

$$\sum_{i=1}^{1} a\cdot r^i = a\cdot r^0 = a\cdot 1 = \frac{a\cdot(r^1-1)}{(r-1)}$$

as required.

Inductive step: Suppose that $P(n)$ holds for some $n \geq 1$. Considering $n{+}1$ we have:

$$\begin{aligned}
\sum_{i=1}^{n+1} a\cdot r^i &= \left(\sum_{i=1}^{n} a\cdot r^i\right) + a\cdot r^{n+1} \\
&= \frac{a\cdot(r^{n+1}-1)}{(r-1)} + a\cdot r^{n+1} && \text{by induction} \\
&= \frac{a}{(r-1)}(r^{n+1} - 1 + (r-1)\cdot r^{n+1}) && \text{rearranging} \\
&= \frac{a}{(r-1)}(r^{n+1} - 1 + r^{n+1} - r^{n+1}) && \text{rearranging} \\
&= \frac{a}{(r-1)}(r^{n+2} - 1) && \text{rearranging} \\
&= \frac{a}{(r-1)}(r^{(n+1)+1} - 1) && \text{rearranging}
\end{aligned}$$

and hence $P(n{+}1)$ holds.

Therefore by the principle of induction we have proved that $P(n)$ holds for all $n \geq 1$.

The Fibonacci numbers $f_0, f_1, f_2, \ldots$ and Lucas numbers $l_0, l_1, l_2, \ldots$ are defined by the equations:

- $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1}+f_{n-2}$ for all $n \geq 2$;
- $l_0 = 2$, $l_1 = 1$ and $l_n = l_{n-1}+l_{n-2}$ for all $n \geq 2$

respectively. Prove that $f_n+f_{n+2} = l_{n+1}$ for all $n \geq 1$.

Let $P(n)$ be the proposition $f_n + f_{n+2} = l_{n+1}$ for all $n \geq 1$. We use the second principle of mathematical induction.

Base cases: $P(1)$ and $P(2)$ hold since:

$$\begin{aligned} f_1 + f_3 = 1 + 2 &= l_1 + l_0 = l_2 \\ f_2 + f_4 = 1 + 3 = 3 + 1 &= l_2 + l_1 = l_3 \end{aligned}$$

Inductive step: Suppose $n \geq 2$ and $P(k)$ is true for all $1 \leq k \leq n$, Now, by definition of the Fibonacci numbers we have:

$$\begin{aligned} & f_{n+1} + f_{n+3} \;=\; (f_{n-1} + f_n) + (f_{n+1} + f_{n+2}) && \\ & = (f_{n-1} + f_{n+1}) + (f_n + f_{n+2}) && \text{rearranging} \\ & = l_n + l_{n+1} && \text{by induction (using } P(n-1) \text{ and } P(n)) \\ & = l_{n+2} && \text{by definition of the Lucas numbers} \end{aligned}$$

and hence $P(n+1)$ holds.

Therefore by the principle of induction we have proved that $P(n)$ holds for all $n \geq 1$.

The set of bit strings $\mathbb{B}^*$ are be defined recursively by:

- $\varepsilon \in \mathbb{B}^*$ (where ε is the empty string);
- if $w \in \mathbb{B}^*$ and $x \in \{0, 1\}$, then $wx \in \mathbb{B}^*$.

We can define concatenation of two bit strings denoted ++, recursively as follows:

- if $w \in \mathbb{B}^*$, then $w{+}{+}\varepsilon = w$;
- if $w, v \in \mathbb{B}^*$ and $x \in \{0, 1\}$, then $w{+}{+}(vx) = (w{+}{+}v)x$.

Give a recursive definition of the function $\texttt{ones} : \mathbb{B}^* \to \mathbb{N}$ which counts the number of ones in a bit string. The function $\texttt{ones} : \mathbb{B}^* \to \mathbb{N}$ is defined as follows. For any $v \in \mathbb{B}^*$:

$$\texttt{ones}(v) = \begin{cases} 0 & \text{if } v = \varepsilon \\ \texttt{ones}(w) & \text{if } v = wx \text{ and } x{=}0 \\ 1 + \texttt{ones}(w) & \text{if } v = wx \text{ and } x{=}1 \end{cases}$$

The use structural induction to prove that $\texttt{ones}(w{+}{+}v) = \texttt{ones}(w) + \texttt{ones}(v)$ for all $w, v \in \mathbb{B}^*$.

We will prove $\texttt{ones}(w{+}{+}v) = \texttt{ones}(w) + \texttt{ones}(v)$ for all $w, v \in \mathbb{B}^*$ by induction on the structure of v.

Base cases: in this case we have $v = \varepsilon$, and hence by definition of concatenation:

$$\begin{aligned} \texttt{ones}(w{+}{+}v) &= \texttt{ones}(w) && \\ &= \texttt{ones}(w) + 0 && \text{rearranging} \\ &= \texttt{ones}(w) + \texttt{ones}(v) && \text{by definition of } \texttt{ones} \text{ and since } v{=}\varepsilon. \end{aligned}$$

Induction step: in this case we have $v = v'x$ for some $v' \in \mathbb{B}^*$ and $x \in \{0, 1\}$. We have two cases to consider.

- If x=0, then

$$\begin{aligned}
\texttt{ones}(w{+}{+}v) &= \texttt{ones}(w{+}{+}(v'x)) \\
&= \texttt{ones}((w{+}{+}v')x) && \text{by definition of concatenation} \\
&= \texttt{ones}(w{+}{+}v') && \text{by definition of } \texttt{ones} \text{ \& since } x{=}0 \\
&= \texttt{ones}(w) + \texttt{ones}(v') && \text{by the induction hypothesis} \\
&= \texttt{ones}(w) + \texttt{ones}(v'x) && \text{by definition of } \texttt{ones} \text{ \& since } x{=}0 \\
&= \texttt{ones}(w) + \texttt{ones}(v) && \text{by construction.}
\end{aligned}$$

- If x=1, then

$$\begin{aligned}
\texttt{ones}(w{+}{+}v) &= \texttt{ones}(w{+}{+}(v'x)) \\
&= \texttt{ones}((w{+}{+}v')x) && \text{by definition of concatenation} \\
&= 1 + \texttt{ones}(w{+}{+}v') && \text{by definition of } \texttt{ones} \text{ \& since } x{=}0 \\
&= 1 + \texttt{ones}(w) + \texttt{ones}(v') && \text{by the induction hypothesis} \\
&= \texttt{ones}(w) + (1 + \texttt{ones}(v')) && \text{rearranging} \\
&= \texttt{ones}(w) + \texttt{ones}(v'x) && \text{by definition of } \texttt{ones} \text{ \& since } x{=}1 \\
&= \texttt{ones}(w) + \texttt{ones}(v) && \text{by construction.}
\end{aligned}$$

Since these are the other cases to consider the inductive step holds.

Therefore by the principle of structural induction we have proved that

$$\texttt{ones}(w{+}{+}v) = \texttt{ones}(w) + \texttt{ones}(v)$$

for all $w, v \in \mathbb{B}^*$.

Non-empty proper binary trees over X (where X is a data set):

- base case: if $x \in X$, then $\texttt{node}(\texttt{nil}, \texttt{nil}, x)$ is a tree over X;
- inductive step: if t_1 and t_2 are non-empty proper binary trees over X and $x \in X$, then $\texttt{node}(t_1, t_2, x)$ is a tree over X.

Define recursive functions for the number of nodes $\texttt{n}(t)$ and height $\texttt{h}(t)$ of a complete non-empty binary tree.

The height of a true is the length of the longest path from the root and the length of a path is the number of edges in the path.

The functions $\texttt{n}$ and $\texttt{h}$ are defined as follows. For any tree t:

$$\texttt{n}(t) = \begin{cases} 1 & \text{if } t = \texttt{node}(\texttt{nil}, \texttt{nil}, x) \\ 1 + \texttt{n}(t_1) + \texttt{n}(t_2) & \text{if } t = \texttt{node}(t_1, t_2, x) \end{cases}$$

$$\texttt{h}(t) = \begin{cases} 0 & \text{if } t = \texttt{node}(\texttt{nil}, \texttt{nil}, x) \\ 1 + \max\{\texttt{n}(t_1), \texttt{n}(t_2)\} & \text{if } t = \texttt{node}(t_1, t_2, x) \end{cases}$$

Use structural induction to show $\texttt{n}(t) \geq 2{\cdot}\texttt{h}(t){+}1$ for any complete non-empty binary tree t.

Base case: In the base case we have that $t = \varepsilon$ and by definition of $\texttt{n}$:

$$\begin{aligned}
\texttt{n}(\varepsilon) &= 1 \\
&= 2{\cdot}0 + 1 && \text{rearranging} \\
&= 2{\cdot}\texttt{h}(\varepsilon) + 1 && \text{by definition of } \texttt{h}
\end{aligned}$$

as required.

Inductive step: Noe assume $\mathtt{n}(t_i) \geq 2{\cdot}\mathtt{h}(t_i){+}1$ for $i = 1, 2$ and consider an arbitrary $x \in X$. By definition of $\mathtt{n}$:

$$\begin{aligned}
\mathtt{n}(\mathtt{node}(t_1, t_2, x)) &= 1 + \mathtt{n}(t_1) + \mathtt{n}(t_2) \\
&\geq 1 + 2{\cdot}\mathtt{h}(t_1){+}1 + 2{\cdot}\mathtt{h}(t_2){+}1 && \text{by the inductive hypothesis} \\
&= 1 + 2{\cdot}(1 + \mathtt{h}(t_1) + 2{\cdot}\mathtt{h}(t_2)) && \text{rearranging} \\
&\geq 1 + 2{\cdot}(1 + \max\{\mathtt{h}(t_1) + 2{\cdot}\mathtt{h}(t_2)\}) && \text{since } l + m \geq \max\{l, m\} \text{ for any } l, m \in \mathbb{N} \\
&\geq 1 + 2{\cdot}\mathtt{h}(\mathtt{node}(t_1, t_2, x)) && \text{by definition of } \mathtt{h}
\end{aligned}$$

and therefore the the inductive step holds.

Therefore by the principle of structural induction we have proved that $\mathtt{n}(t) \geq 2{\cdot}\mathtt{h}(t){+}1$ for any complete non-empty binary tree t.

How many ways are there to choose 6 items from 10 distinct items when ...

(a) ... the items in the choices are ordered and repetition is not allowed?

There are the possible permutations of size r=6 from a set of size n=10. Therefore there are:

$$P(10,6) = \frac{10!}{(10-6)!} = \frac{10!}{4!} = 151,200$$

ways.

(b) ... the items in the choices are ordered and repetition is allowed?

Here using the product rule there are 10 choices for each position and therefore 10^6 ways.

(c) ... the items in the choices are unordered and repetition is not allowed?

Here we are choosing r=6 items out of a set of size n=10. Therefore there are:

$$C(10,6) = \frac{10!}{6!{\cdot}(10-6)!} = \frac{10!}{6!{\cdot}4!} = 210$$

ways.

(d) ... the items in the choices are unordered and repetition is allowed?

Here we can use the stars and bars approach as repetition is allowed. There are 6 stars and 9 bars (since 10 distinct items) and therefore there are: $15!/(6!{\cdot}9!)$ ways.

How many strings of length 10 over the alphabet $\{a,b,c\}$ have either exactly three a's or exactly four b's?

First we consider the number of strings of length 10 over the alphabet $\{a,b,c\}$ have exactly three a's. First if we consider the options for the a's we have three a's that can be placed in 10 different positions so we have $C(10,3)$ options. For the remaining 7 positions we can either choose a b or a c (there are precisely 3 a's that we have already specified), so 2 options for 7 positions, and hence 2^7 options. Since we need to do both there the product rule there are $C(10,3){\cdot}2^7$ ways.

Next we consider the number of strings of length 10 over the alphabet $\{a,b,c\}$ have exactly four b's. First if we consider the options for the b's we have four b's that can be placed in 10 different positions so we have $C(10,4)$ options. For the remaining 6 positions we can either choose an a or a c (there are precisely 4 b's that we have already specified), so 2 options for 6 positions, and hence 2^6 options. Since we need to do both there the product rule there are $C(10,4){\cdot}2^6$ ways.

To prevent over counting we also need to calculate the number of strings of length 10 over the alphabet $\{a,b,c\}$ have exactly three a's and four b's. First if we consider the options for the a's we have three a's that can be placed in 10 different positions so we have $C(10,3)$ options. For the b's we have four b's that be in any of the remaining 7 positions, so we have $C(7,4)$. For the remaining 6 positions we only choose a c (the a's and b's have already specified), so 1 option for 3 positions, and hence 1 options. Since we need to do both there the product rule there are $C(10,3){\cdot}C(7,4){\cdot}1$ ways.

Finally using the inclusion exclusion principle the number strings of length 10 over the alphabet $\{a,b,c\}$ have either exactly three a's or exactly four b's equals"

$$C(10,3){\cdot}2^7 + C(10,4){\cdot}2^6 - C(10,3){\cdot}C(7,4){\cdot}1$$

A drawer contains a dozen black, a dozen red socks and a dozen white socks, all unmatched. If you take socks out at random in the dark ...

(a) ... how many socks must you take out to be sure that you have at least two socks of the same color?

This can be solved using the pigeon hole principle. There are three containers representing the colours black, red and white. We want to calculate the fewest number of objects (socks) needed to ensure that at least one of the containers contains 2 objects. By the Generalised Pigeonhole Principle, we need the smallest n such that $\texttt{ceil}(n/3) = 2$, and hence 4 socks are required.

(b) ... how many socks must you take out to be sure that you have at least two black socks?

Here you can pick all the red and white socks before getting a pair of black socks, and therefore $12 + 12 + 2 = 26$ socks are required

Show that there are at least seven people in Glasgow (population: 1.7 million) with the same two initials who were born on the same day of the year (but not necessarily in the same year). Assume that everyone has two initials.

This can be solved using the pigeon hole principle. The contains are for people with the same two initials who were born on the same day of the year. There are 26^2 different initials using the product rule and 366 days (remember the extra day in a leap year). By the Generalised Pigeonhole Principle we have that one container has at least $\texttt{ceil}(1,700,000/(26^2{\cdot}366)) = 7$ objects are required.

How many different strings can be made from the letters in *MISSISSIPPI*, using all the letters?

There are 11 letters, but the 4 I's are indistinguishable, as are the 4 S's and 2 P's. Therefore there are $11!/(5!{\cdot}5!{\cdot}2!)$ different strings.

How many ways are there to choose a dozen donuts from 20 varieties...

(a) ... if there are no two donuts of the same variety?

These is a permutation of size r=12 from 20 objects, and hence $P(20, 12) = 20!/(20 - 12)! = 20!/8!$ ways.

(b) ... if all donuts are of the same variety?

Here there are 20 ways, as there are 20 varieties and all must be of the same variety.

(c) ... if there are no restrictions?

This is a combination with repetition and therefore we can use the stars and bars approach. There are 12 stars and 19 bars (since 20 distinct varieties) and therefore there are: $31!/(12!{\cdot}19!)$ ways.

(d) ... if there are at least two varieties among the dozen donuts chosen?

Here we can just take the answer for (c) and subtract those that do not meet the requirement, i.e. when all the donuts are of the same variety which is the answer to part (a), i.e. there are $1!/(12!{\cdot}19!) - 20$ ways.

(e) ... if there must be at least six blueberry-filled donuts?

Here we have fixed 6 donuts, and therefore there are $12 - 6 = 6$ left to choose and these can be of any variety. This is therefore a combination with repetition, using the stars and bars approach, there are 6 stars and 19 bars (since 20 distinct varieties) and therefore there are: $25!/(6!{\cdot}19!)$ ways.

What is the probability of these events when we randomly select a permutation of $\{\mathtt{1}, \mathtt{2}, \mathtt{3}, \mathtt{4}\}$?
Hint: Use symmetry.

(a) `1` comes before `4` in the permutation.

Here `1` coming before `4` and `4` comes before `1` are equally likely and one must occur, and therefore the probability is $\frac{1}{2}$.

(b) `4` comes before `1` in the permutation.

For the same reason as for (a) the probability is $\frac{1}{2}$.

(c) `4` comes before `1` and `4` comes before `2` in the permutation.

Here there are three possible equally likely outcomes either `1`, `2` or `4` comes first and hence the probability is $\frac{1}{3}$.

(d) `4` comes before `1`, `4` comes before `2`, and `4` comes before `3` in the permutation.

Here there are four possible equally likely outcomes either `1`, `2`, `3` or `4` comes first and hence the probability is $\frac{1}{4}$.

(e) `4` comes before `3` and `2` comes before `1` in the permutation.

The probabilities `4` comes before `3` and `2` comes before `1` are each $\frac{1}{2}$ as for (a) and (b) and sence the events are indepedent the probability of both occuring is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

What is the probability of these events when we randomly select a permutation of the 26 lowercase letters of the English alphabet?

First let us think about all the possible permutations, here we r-permutations from a set of size n where $n = 26$ and $r = 26$ and therefore the total number is $26!$.

- The permutation consists of the letters in reverse alphabetic order.

  Only one permutation meets this requirement, and hence the probability is $\frac{1}{26!}$.

- `z` is the first letter of the permutation.

  If we fix `z` as the first element then we have a permutation of size 25 with 25 characters so get the probability $\frac{25!}{26!}$.

- `z` comes before `a` in the permutation.

  Here we can just use the fact `z` coming before `a` is as likely as `a` comes before `z` and one of these must occur. Therefore the probability is $\frac{1}{2}$.

- `a` immediately comes before `z` in the permutation.

  If we think of `az` as a single character then we have a permutation of size 25 with 25 characters so get the probability $\frac{25!}{26!}$.

- `a` immediately comes before `m`, which immediately comes before `z` in the permutation.

  If we think of `amz` as a single character then we have a permutation of size 25 with 25 characters so get the probability $\frac{24!}{26!}$.

- `m`, `n`, and `o` are in their original places in the permutation.

  This means we have 23 spaces to fill using 23 chararacters, and hence the probability is (23!)/(26!).

Monty Hall puzzle. A prize behind one of the three doors (d_1, d_2 and d_3) each with probability $\frac{1}{3}$.

1. You select a door.
2. Monty Hall opens one of the two doors you did not select that he knows is a door without the prize behind, selecting at random if neither has a prize behind.
3. Monty then asks you whether you would like to switch doors.

Suppose that:

- W is the random variable whose value is the winning door;
- M denote the random variable corresponding to the door that Monty opens
- you choose door d_i.

(a) What is the probability that you will win the prize if you never switch doors?

This has probability $\frac{1}{3}$ as the prize behind one of the three doors each with probability $\frac{1}{3}$.

(b) What is the probability that you will win the prize if you always switch doors?

This has probability 2/3 as if you choose the right door initially after swapping you will have chosen the wrong door, while if you selected one of the doors without the prize, Monty will then open the other door without the prize and by switching you will be switching to the door with the prize.

(c) Find $\mathbf{P}[M = d_k \mid W = d_j]$ for $j = 1, 2, 3$ and $k = 1, 2, 3$ when $d_i \neq d_j$ i.e. when you did not initially choose the winning door.

Here since you have chosen one door and the prize is behind one other, there is only one door Monty can choose i.e. we have:

$$\mathbf{P}[M = d_k \mid W = d_j] = \begin{cases} 1 & \text{if } k \neq i \text{ and } k \neq j \\ 0 & \text{otherwise} \end{cases}$$

(d) Find $\mathbf{P}[M = d_k \mid W = d_j]$ for $j = 1, 2, 3$ and $k = 1, 2, 3$ when $d_i = d_j$ i.e. when you did choose the winning door.

Here since you have chosen the same dorr the prize is behind, there are two doors Monty can choose and he chooses them at random when there is choice we have:

$$\mathbf{P}[M = d_k \mid W = d_j] = \begin{cases} \frac{1}{2} & \text{if } k \neq i = j \\ 0 & \text{otherwise} \end{cases}$$

(e) Use Bayes' theorem to find $\mathbf{P}[W = d_j \mid M = d_k]$ where j and k are distinct and $d_i \neq d_j$ i.e. when you did not choose the winning door. **Note.** Monty will never choose the winning door to reveal so the probability j and k are equal is aways 0.

First we choose specific values for i, j and k. Since in this casde i, j and k are distinct let us choose without loss of generality $i = 1$, $j = 2$ and $k = 3$. Now, using Bayes' rule we have:

$$\mathbf{P}[W = d_2 \mid M = d_3] = \frac{\mathbf{P}[M = d_3 \mid W = d_2] \cdot \mathbf{P}[W = d_2]}{\mathbf{P}[M = d_3]}$$

Now we know $\mathbf{P}[M = d_3 \mid W = d_2] = 1$ from part (c) and $\mathbf{P}[W = d_2] = \frac{1}{3}$. It remains to compute $\mathbf{P}[M = d_3]$ which we can do using Partition theorem (the law of total probability) considering the events of the prize being behind the three different doors (the prize must be behind one of the doors and the prize cannot be behind two doors). Therefore we have $\mathbf{P}[M = d_3]$ equals

$$\begin{aligned}&\mathbf{P}[M = d_3|W = d_1]\cdot\mathbf{P}[W = d_1] + \mathbf{P}[M = d_3|W = d_2]\cdot\mathbf{P}[W = d_2] + \mathbf{P}[M = d_3|W = d_3]\cdot\mathbf{P}[W = d_3]\\ =\ & \frac{1}{2}\cdot\frac{1}{3} + 1\cdot\frac{1}{3}0\cdot\frac{1}{3} = \frac{1}{2}\end{aligned}$$

The fact $\mathbf{P}[M = d_3|W = d_1] = \frac{1}{2}$ follows from (d), $\mathbf{P}[M = d_3|W = d_2] = \frac{1}{2}$ follows from (c), $\mathbf{P}[M = d_3|W = d_3] = 0$ as Monty never chooses the door that the prize is behind. The facts that $\mathbf{P}[W = d_j] = \frac{1}{3}$ for $j = 1, 2, 3$ follow from the fact the prize is placed behind the door at random.

We therefore have that:

$$\mathbf{P}[W = d_2 \mid M = d_3] = \frac{\mathbf{P}[M = d_3 \mid W = d_2]\cdot\mathbf{P}[W = d_2]}{\mathbf{P}[M = d_3]} = \frac{1\cdot\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}.$$

(f) Use Bayes' theorem to find $\mathbf{P}[W = d_j \mid M = d_k]$ where j and k are distinct and $d_i = W$ i.e. when you did choose the winning door.

First we choose specific values for i, j and k. Since in this casde $i = j$ and k is distinct let us choose without loss of generality $i = j = 1$ and $k = 2$. Now, using Bayes' rule we have:

$$\mathbf{P}[W = d_1 \mid M = d_2] = \frac{\mathbf{P}[M = d_2 \mid W = d_1]\cdot\mathbf{P}[W = d_1]}{\mathbf{P}[M = d_2]}$$

Now we know $\mathbf{P}[M = d_2 \mid W = d_1] = \frac{1}{2}$ from part (d) and $\mathbf{P}[W = d_1] = \frac{1}{3}$. It remains to compute $\mathbf{P}[M = d_2]$ which we can do using Partition theorem (the law of total probability) considering the events of the prize being behind the three different doors (the prize must be behind one of the doors and the prize cannot be behind two doors). Therefore we have that $\mathbf{P}[M = d_2]$ equals

$$\begin{aligned}&\mathbf{P}[M = d_2|W = d_1]\cdot\mathbf{P}[W = d_1] + \mathbf{P}[M = d_2|W = d_2]\cdot\mathbf{P}[W = d_2] + \mathbf{P}[M = d_2|W = d_3]\cdot\mathbf{P}[W = d_3]\\ =\ & \frac{1}{2}\cdot\frac{1}{3} + 0\cdot\frac{1}{3}1\cdot\frac{1}{3} = \frac{1}{2}\end{aligned}$$

The fact $\mathbf{P}[M = d_2|W = d_1] = \frac{1}{2}$ follows from (d), $\mathbf{P}[M = d_2|W = d_2] = 0$ as Monty never chooses the door that the prize is behind and $\mathbf{P}[M = d_3|W = d_2] = 1$ follows from (c). The facts that $\mathbf{P}[W = d_j] = \frac{1}{3}$ for $j = 1, 2, 3$ follow from the fact the prize is placed behind the door at random.

We therefore have that:

$$\mathbf{P}[W = d_1 \mid M = d_2] = \frac{\mathbf{P}[M = d_3 \mid W = d_2]\cdot\mathbf{P}[W = d_2]}{\mathbf{P}[M = d_3]} = \frac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}.$$

Suppose that we roll a fair die until a 6 comes up.

(a) What is the probability that we roll the die n times?

Here we need to first roll $n-1$ without getting a 6 and then roll a siz so the probability is $(\frac{5}{6})^{n-1}\cdot\frac{1}{6}$.

(b) What is the probability the game ends?

If $\mathbf{P}[end]$ is the probability of ending we have:

$$\mathbf{P}[end] = \frac{1}{6} + \frac{5}{6}\cdot\mathbf{P}[end]$$

since with probability $\frac{1}{6}$ we finish after one throw or we continue. Solving this equation we get $\mathbf{P}[end] = 1$.

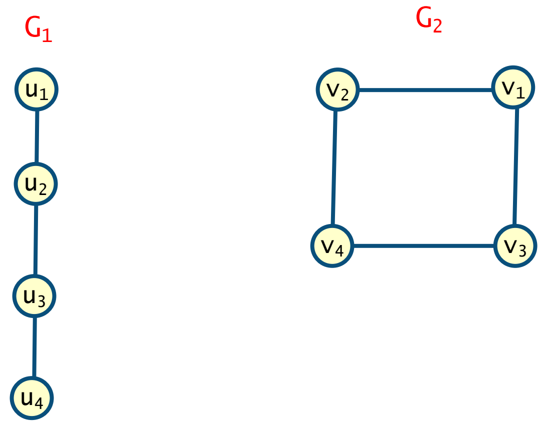
(c) What is the expected number of times we roll the die?

If $\mathbf{E}[end]$ is the expected number of rolls we have:

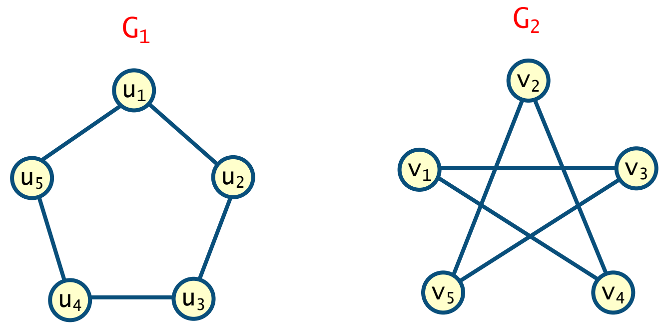
$$\mathbf{E}[end] = 1 + \frac{5}{6} \cdot \mathbf{E}[end]$$

since with probability $\frac{5}{6}$ we continue and roll again. Solving this equation we get $\mathbf{E}[end] = 6$.

Are the following two graphs are isomorphic.

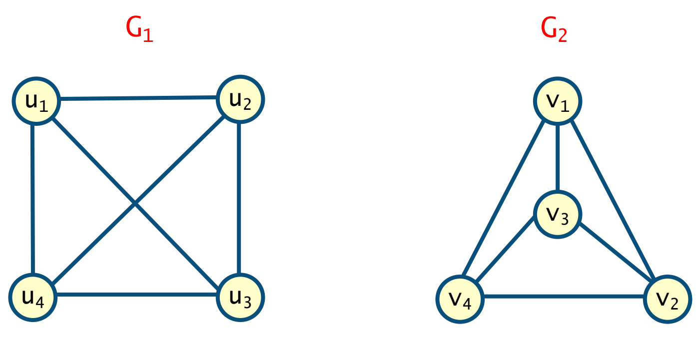


These graphs are not isomorphic since G_1 has 3 edges while G_2 has 4 edges (also the first has two vertices with degree 1 and all the vertices in G_2 have degree 2.



These graphs are isomorphic, first the have the same number of vertices and edges and the degrees of the vertices are all equal. One such isomorphism is given by the bijection:

$$u_1 \mapsto v_1,\ u_2 \mapsto v_3,\ u_3 \mapsto v_5,\ u_4 \mapsto v_2,\ u_5 \mapsto v_1,$$



These graphs are isomorphic, first the have the same number of vertices and edges and the degrees of the vertices are all equal. In this case since in both graphs all vertices are adjacent to all others any bijection between the vertices will work. One such isomorphism is given by the bijection:

$$u_1 \mapsto v_1,\ u_2 \mapsto v_2,\ u_3 \mapsto v_3,\ u_4 \mapsto v_4,$$

Which of these relations on the set of all functions from $\mathbb{Z}$ to $\mathbb{Z}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

(a) $\{(f, g) \mid f(1) = g(1)\}$

- **Reflexive.** It is reflexive since $f(1) = f(1)$ for any function f.
- **Symmetric.** It is symmetric since if $f(1) = g(1)$, then clearly $g(1) = f(1)$ for any functions f, g.
- **Transitive.** It is transitive since if $f(1) = g(1)$ and $g(1) = h(1)$, then $f(1) = h(1)$ for any functions f, g, h.

(b) $\{(f, g) \mid f(0) = g(0) \vee f(1) = g(1)\}$

- **Reflexive.** It is reflexive since $f(0) = f(0)$ and $f(1) = f(1)$ for any function f.
- **Symmetric.** It is symmetric since if $f(0) = g(0)$ or $f(1) = g(1)$, then clearly $g(0) = f(0)$ or $g(1) = f(1)$ for any functions f, g.
- **Transitive.** It is not transitive for example if we consider any functions f, gv and h where $f(0) = g(0) = 3$ and $h(0) = 4$ and $f(1) = 4$ and $g(1) = h(1) = 3$, (f, g) and (g, h) are in the relation but (f, h) is not in the relation.

(c) $\{(f, g) \mid f(x) - g(x) = 1 \text{ for all } x \in \mathbb{Z}\}$

- **Reflexive.** It is not reflexive since $f(x) - f(x) = 0 \neq 1$ for any function f.
- **Symmetric.** It is not symmetric since if $f(x) - g(x) = 1$, then $g(1) - f(1) = -(f(x) - g(x)) = -1 \neq 1$ for any functions f, g.
- **Transitive.** It is not transitive since if $f(x) - g(x) = 1$ and $g(x) = h(x) = 1$, then:

$$\begin{aligned}
f(x) - h(x) &= f(x) + (g(x) - g(x)) - h(x) && \text{since } g(x) - g(x) = 0 \\
&= (f(x) - g(x)) + (g(x) - h(x)) && \text{rearranging} \\
&= 1 + 1 && \text{since } f(x) - g(x) = 1 \text{ and } g(x) = h(x) = 1 \\
&= 2 \neq 1
\end{aligned}$$

(d) $\{(f, g) \mid \exists C \in \mathbb{Z}. \forall x \in \mathbb{Z}. (f(x) - g(x) = C)\}$

- **Reflexive.** It is reflexive since $f(x) - f(x) = 0$ for any function f.
- **Symmetric.** It is symmetric since if $f(x) - g(x) = C$, then clearly $g(x) - f(x) = -(f(x) - g(x)) = -C$ for any functions f, g.
- **Transitive.** It is transitive since if $f(x) - g(x) = C_1$ and $g(x) - h(x) = C_2$, then:

$$\begin{aligned}
f(x) - h(x) &= f(x) + (g(x) - g(x)) - h(x) && \text{since } g(x) - g(x) = 0 \\
&= (f(x) - g(x)) + (g(x) - h(x)) && \text{rearranging} \\
&= C_1 + C_2 && \text{since } f(x) - g(x) = C_1 \text{ and } g(x) = h(x) = C_2
\end{aligned}$$

Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $a + d = b + c$. Show that R is an equivalence relation.

- **Reflexive.** It is reflexive since $a + b = b + a$ for any (a, b).
- **Symmetric.** It is symmetric since if $a + d = b + c$, then clearly $c + b = d + a$ for any functions (a, b) and (b, a).
- **Transitive.** It is transitive since if $a + d = b + c$, then clearly $c + e = d + f$, then adding both sides of these equations we preserve equality and therefore:

$$(a + d) + (c + e) = (b + c) + (d + f)$$

Regarraning we have

$$(a + e) + (c + d) = (b + f) + (c + d)$$

which simplifying yields $(a + e) = (b + f)$ as required.

Find the lexicographic ordering of the bit strings `0`, `01`, `11`, `001`, `010`, `011`, `0001`, and `0101` based on the ordering $\texttt{0} \sqsubseteq \texttt{1}$.

First `11` is the only element with a `1` in the first position, and therefore this is the "largest element". From the remaining the elements:

$$\texttt{01},\ \texttt{010},\ \texttt{011},\ \texttt{0101}$$

have a `1` in the second position. From these only `011` has a `1` in the third so this comes next. The remaining are prefixes of each other so we have $\texttt{01} sqsubset \texttt{010} \sqsubset \texttt{0101}$. What is left are `0`, `001` and `0001`. The string `001` comes next as it has a `1` in the third position and finally $\texttt{0} \sqsubset \texttt{0001}$ since the first is a prefix of the second. This yields the final ordering:

$$\texttt{0} \sqsubset \texttt{0001} \sqsubset \texttt{001} \sqsubset \texttt{01} \sqsubset \texttt{010} \sqsubset \texttt{0101} \sqsubset \texttt{011} \sqsubset \texttt{11}$$