7c Diagonalisation and similarity

$$P^{-1}AP = D.$$

Proof 4.23

Suppose A is $n\times n$ diagonisable and so exists ~~that~~ an invertible matrix P such that $P^{-1}AP = D$, equivalently $AP = PD$.

Let $P = \left[\, p_1 \mid p_2 \mid \cdots \mid p_n \,\right]$, where p_i is the ith column of P, and $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$.

where $\lambda_i \in \mathbb{R}$,

Then $AP = PD \Rightarrow A\left[\, p_1 \mid p_2 \cdots \mid p_n \,\right] = \left[\, p_1 \mid \cdots \mid p_n \,\right] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$\Rightarrow \left[\, Ap_1 \mid Ap_2 \mid \cdots \mid Ap_n \,\right] = \left[\, \lambda_1 p_1 \mid \cdots \mid \lambda_n p_n \,\right].$

Equating columns gives:

$$Ap_1 = \lambda_1 p_1, \ldots, Ap_n = \lambda_n p_n$$

The columns of P are eigenvectors of A with corresponding eigenvalues being the entries in D, written in the same order.

The columns of P are linearly independent since P is invertible. So A has n linearly independent eigenvectors.

Suppose A has n linearly independent eigenvectors, $p_1, \dots, p_n \in \mathbb{R}^n$, with corresponding eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

Then $Ap_1 = \lambda_1 p_1, \dots, Ap_n = \lambda_n p_n$.

Let P be the matrix with columns $p_1, \dots, p_n$.

Then we reverse the steps from the first part of the proof, to get $AP = PD$, where D is the diagonal matrix with entries $\lambda_1, \dots, \lambda_n$.

Since columns of P are linearly independent so P is invertible. So $P^{-1}AP = D$.

So A is diagonalisable as it is similar to a diagonal matrix. $\square$.

Solution 1

Recall from Ex 1. of lecture 6d.

A has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 0$.

and eigenspaces $E_1 = \text{span}\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$, $E_0 = \text{span}\left(\begin{bmatrix}2\\-1\\1\end{bmatrix}\right)$

Since there are only 2 linearly independent eigenvectors A is <u>not</u> diagonalisable.

Solution 2

We must find all λ such that $\det(A - \lambda I) = 0$.

$$\det(A-\lambda I) = \det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}$$

$$= \lambda^2 - 4\lambda + 3$$

$$= (\lambda-3)(\lambda-1).$$

Setting $\det(A-\lambda I)=0$ and solving for λ gives the eigenvalues of A as $\lambda = 3$, $\lambda = 1$.

<u>Eigenspace for $\lambda = 1$</u>

Consider $(A - 1 \cdot I \mid 0) = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The general solution for vector $\underline{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} \in \text{null}(A-I)$ is $x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$.

Hence $E_1 = \left\{ x_2 \begin{bmatrix}-1\\1\end{bmatrix} : x_2 \in \mathbb{R} \right\} = \text{span}\left(\begin{bmatrix}-1\\1\end{bmatrix}\right)$.

Show $E_3 = \text{span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$. Exercise.

Since A has 2 linearly independent eigenvectors A is diagonalisable,
where $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

and $P^{-1}AP = D$. Check : $\underline{\underline{AP = PD}}$.

ASIDE

$\mathcal{B} = \{ \underline{e}_1, \dots, \underline{e}_n \}$ standard basis for $\mathbb{R}^n$

$\mathcal{C} = \{ \underline{v}_1, \dots, \underline{v}_n \}$ eigenvector of an $n \times n$ matrix A.
$\underline{v}_n \in \mathbb{R}^n$.

- P^{-1} is the change of basis matrix $\mathcal{C} \to \mathcal{B}$. ie. $P^{-1}_{\mathcal{B} \leftarrow \mathcal{C}}$
- P is the change of basis matrix. $\mathcal{B} \to \mathcal{C}$

$$\mathcal{C} \xrightarrow{P^{-1}} \mathcal{B} \xrightarrow{A} \mathcal{B} \xrightarrow{P} \mathcal{C} \qquad \mathcal{C} \xrightarrow{D} \mathcal{C}$$

Solution 3.

From example 2 we showed $P^{-1}AP = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

where $P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

Let $D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$, then for any $k \in \mathbb{N}$ we have

$$A^k = (PDP^{-1})^k = P D^k P^{-1}$$
$$= P \begin{pmatrix} 3^k & 0 \\ 0 & 1^k \end{pmatrix} P^{-1}$$

Computing $P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ we conclude

$$A^k = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3^k & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 3^k & -1 \\ 3^k & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3^k+1 & 3^k-1 \\ 3^k-1 & 3^k+1 \end{pmatrix}$$