2A Multivariable Calculus 2020

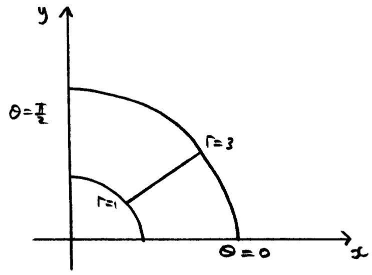
Ex Sheet **5**

T1 Evaluate

$$\int\int xy^2 \, dx \, dy$$

over the region in the first quadrant that lies outside the circle $x^2 + y^2 = 1$ but inside the circle $x^2 + y^2 = 9$.

Solution



In polar coordinates, the integral is

$$\int_0^{\pi/2} d\theta \int_1^3 r^4 \cos\theta \sin^2\theta \, dr = \int_0^{\pi/2} \cos\theta \sin^2\theta d\theta \int_1^3 r^4 \, dr$$
$$= \int_0^1 u^2 \, du \left[\frac{r^5}{5}\right]_1^3 = \frac{242}{15}.$$

T2 Evaluate

$$\int\int_R y(x^2 + y^2) \, dx \, dy$$

where R is

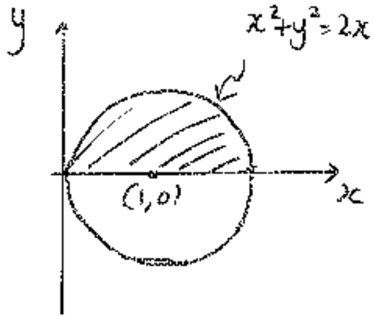
a) the part of the interior of the circle $x^2 + y^2 = 2x$ that lies in the first quadrant,

b) the part of the interior of the circle $x^2 + y^2 = 2x$ that lies above the line $y = x$.

c) the region in the first quadrant inside $x^2 + y^2 = 4ax$ but outside $x^2 + y^2 = 2ax$, where $a > O$.

Solution

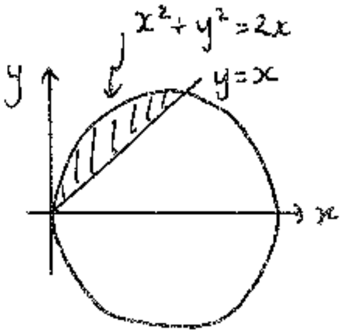
(a)



Since, $x^2 + y^2 = 2x$, in polar coordinates this is $r^2 = 2r\cos\theta$, i.e. $r = 2\cos\theta$.

$$\begin{aligned}\int_0^{\pi/2} d\theta \int_0^{2\cos\theta} r^4 \sin\theta\, dr &= \int_0^{\pi/2} \sin\theta\, d\theta \int_0^{2\cos\theta} r^4\, dr \\ &= \int_0^{\pi/2} \sin\theta\, [\frac{r^5}{5}]_0^{2\cos\theta}\, d\theta \\ &= \int_0^{\pi/2} \frac{32\sin\theta\cos^5\theta}{5}\, d\theta = \frac{32}{5}\int_1^0 -u^5\, du = \frac{16}{15}.\end{aligned}$$

(b)



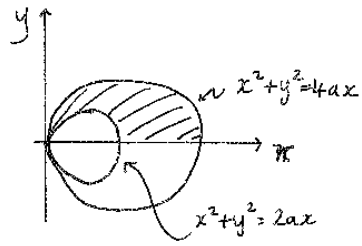
Since, $x^2 + y^2 = 2x$, in polar coordinates this is $r^2 = 2r\cos\theta$, i.e. $r = 2\cos\theta$. Also the line $y = x$ makes the angle $\pi/4$ with the y-axis, so the theta varies between $\pi/2$ and $\pi/4$.

$$\begin{aligned}I = \int_{\pi/4}^{\pi/2} d\theta \int_0^{2\cos\theta} r^4 \sin\theta\, dr &= \int_{\pi/4}^{\pi/2} \sin\theta\, [\frac{r^5}{5}]_0^{2\cos\theta}\, d\theta \\ &= \int_{\pi/4}^{\pi/2} \frac{32}{5} \sin\theta\cos^5\theta\, d\theta\end{aligned}$$

Making a change of variables, $u = \cos\theta$, so $du = -\sin\theta\, d\theta$.

$$I = \frac{32}{5}[\frac{-\cos^6\theta}{6}]_{\pi/4}^{\pi/2} = \frac{32}{30}\cdot\frac{1}{8} = \frac{2}{15}.$$

(c)



Since, $x^2+y^2=4ax$ on the outer circle, in polar coordinates this is $r=4a\cos\theta$. The inner circle gives $x^2+y^2=2ax$, in polar coordinates $r=2a\cos\theta$.

$$\begin{aligned}
I=\int_0^{\pi/2}d\theta\int_{2a\cos\theta}^{4a\cos\theta}r^4\sin\theta\,dr &= \int_0^{\pi/2}\sin\theta\left[\frac{r^5}{5}\right]_{2a\cos\theta}^{4a\cos\theta}d\theta\\
&=\frac{1}{5}\int_0^{\pi/2}(1024a^5\cos^5\theta-32a^5\cos^5\theta)\sin\theta\,d\theta\\
&=\frac{a^5}{5}(1024-32)\int_0^{\pi/2}\sin\theta\cos^5\theta\,d\theta\\
&=\frac{a^5 992}{5}\int_1^0 -u^5\,du=\frac{496a^5}{15}.
\end{aligned}$$

The last line was found by using the change of variables $u=\cos\theta$.

T3 Evaluate the following integrals by converting to polar coordinates

$$\text{(a)}\ \int_0^1 dy\int_y^{\sqrt{2-y^2}}3(x+y)\,dx,\quad \text{(b)}\ \int_0^2 dx\int_0^{\sqrt{2x-x^2}}\sqrt{x^2+y^2}\,dy.$$

Solution

(a)

$$\begin{aligned}
\int_0^1 dy\int_y^{\sqrt{2-y^2}}3(x+y)\,dx &= \int_0^{\pi/4}d\theta\int_0^{\sqrt{2}}3r^2(\cos\theta+\sin\theta)\,dr\\
&=\int_0^{\pi/4}2\sqrt{2}(\cos\theta+\sin\theta)\,d\theta=2\sqrt{2}.
\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^2 dx\int_0^{\sqrt{2x-x^2}}\sqrt{x^2+y^2}\,dy &= \int_0^{\pi/2}d\theta\int_0^{2\cos\theta}r^2\,dr\\
&=\frac{8}{3}\int_0^{\pi/2}\cos^3\theta\,d\theta=16/9
\end{aligned}$$

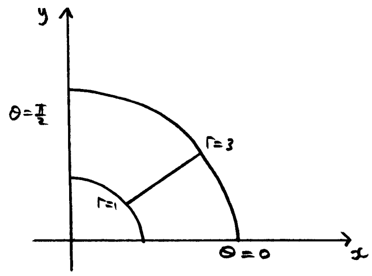
T4 Find the volume of the section of the cylinder $x^2+y^2=1$, between the planes $z=0$ and $x+y+z=2$.

Solution

The required volume is
$$V = \iint_D 2 - x - y \, dxdy,$$
where D is the circle $x^2 + y^2 = 1$. In polar coordinates, this is
$$V = \int_0^{2\pi} \left(\int_0^1 (2 - r\cos\theta - r\sin\theta) r \, dr \right) d\theta = \int_0^{2\pi} \left[r^2 - \tfrac{1}{3} r^3 \cos\theta - \tfrac{1}{3} r^3 \sin\theta \right]_0^1 d\theta$$
$$= \int_0^{2\pi} 1 - \tfrac{1}{3}\cos\theta - \tfrac{1}{3}\sin\theta \, d\theta = 2\pi.$$

T5 Use polar coordinates to evaluate
$$\iint_D \cos\left(x^2 + y^2\right) \, dA$$
where D is the region in the first quadrant between the circles with centre $(0,0)$ and radii 1 and 3 respectively.

Solution

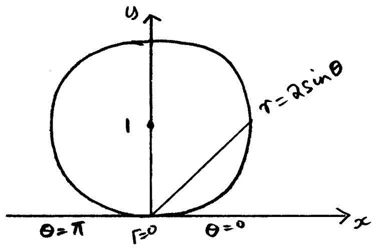


In polar coordinates, the integral is
$$\int_0^{\pi/2} \left(\int_1^3 r\cos(r^2) \, dr \right) d\theta = \int_0^{\pi/2} d\theta \int_1^9 \frac{1}{2} \cos u \, du$$
$$= \frac{\pi}{4} (\sin 9 - \sin 1).$$

T6 Evaluate
$$\iint_D \sqrt{x^2 + y^2} \, dA$$
where D is the disk with centre $(0,1)$ and radius 1.

Solution

The circle is $x^2 + (y-1)^2 = 1$ i.e. $x^2 + y^2 = 2y$, which is $r = 2\sin\theta$ in polar coordinates.



Hence the integral is

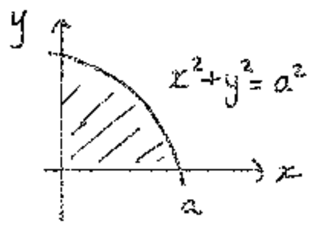
$$\int_0^\pi \left(\int_0^{2\sin\theta} r^2\,dr\right) d\theta = \frac{8}{3}\int_0^\pi \sin^3\theta\,d\theta = \frac{8}{3}\int_0^\pi (1-\cos^2\theta)\sin\theta\,d\theta$$
$$= -\frac{8}{3}\int_1^{-1}(1-u^2)\,du \text{ (where } u=\cos\theta)$$
$$= \frac{32}{9}.$$

T7 Evaluate

$$\int\int \frac{y^2}{x^2+y^2}\,dx\,dy$$

over the region in the first quadrant that lies inside the circle $x^2+y^2 = a^2$, where $a > 0$. What is the value of the same integral over the entire disc enclosed by this circle?

Solution



$$\int_0^{\pi/2} d\theta \int_0^a \frac{r^2\sin^2\theta}{r}\,dr = \int_0^{\pi/2}\sin^2\theta\,d\theta \int_0^a r\,dr$$
$$= \frac{1}{2}\cdot\frac{\pi}{2}\left[\frac{r^2}{2}\right]_0^a = \frac{\pi a^2}{8}.$$

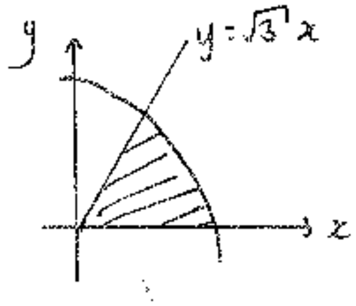
The integral over the entire disc is 4 times $\frac{\pi a^2}{8}$ by the symmetry of the sine function. The function $sin^2\theta$ is positive on the interval $[0, 2\pi]$ and the area under the curve between 0 and 2π is 4 times the area between 0 and $\pi/2$.

T8 Evaluate

$$\int\int x\sqrt{x^2+y^2}\,dx\,dy$$

over the finite region in the first quadrant enclosed by the x-axis, the line $y = \sqrt{3}x$ and the circle $x^2 + y^2 = a^2$, where $a > O$.

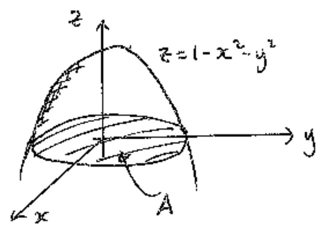
Solution



$$\int_0^{\pi/3} d\theta \int_0^a r^3 \cos\theta \, dr = \int_0^{\pi/3} \cos\theta \, d\theta \int_0^a r^3 \, dr$$
$$= [\sin\theta]_0^{\pi/3} [\frac{r^4}{4}]_0^a = (\frac{\sqrt{3}}{2} - 0)\frac{a^4}{4} = \frac{\sqrt{3}a^4}{8}.$$

T9 An inflatable rubber tent takes the form of the paraboloid $z = 1 - x^2 - y^2$ for $z \geq 0$. Find the volume of air which it encloses.

Solution

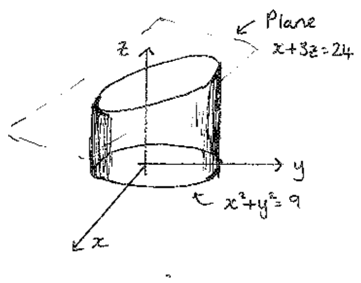


The paraboloid cuts the xy-plane where $z = 0$, i.e. when $x^2 + y^2 = 1$. Hence,

$$\text{Volume of the tent} = \int\int_A z \, dxdy \quad (\text{where } A \text{ is the interior of the circle } x^2 + y^2 = 1)$$
$$= \int_0^{2\pi} d\theta \int_0^1 (1 - r^2) r \, dr = 2\pi \int_0^1 r - r^3 \, dr$$
$$= 2\pi [\frac{r^2}{2} - \frac{r^4}{4}]_0^1 = 2\pi(1/2 - 1/4) = \pi/2.$$

T10 A dummy funnel on a passenger steamer is to be used as a water tank. The tank is to have vertical sides, a horizontal base and slanting plane top. Find the volume of the tank if the base is the plane $z = 0$, the top is the plane $x + 3z = 24$ and the sides are determined by the circular cylinder $x^2 + y^2 = 9$.

Solution



$$\begin{aligned}
\text{Volume } &= \int\int_A z\,dxdy \quad \text{(where } A \text{ is the interior of the circle } x^2+y^2=9)\\
&= \int\int_A \frac{1}{3}(24-x)\,dxdy = \int\int_A 8\,dxdy - \frac{1}{3}\int\int_A x\,dxdy\\
&= 8\int\int_A 1\,dxdy \quad \text{(the last integral is zero due to symmetry of cosine)}\\
&= 8(\text{Area of the disc } x^2+y^2 \le 9) = 8\pi 3^2 = 72\pi.
\end{aligned}$$

T11 (a) A cylindrical drill with radius r_1 is used to bore a hole through the center of a sphere of radius r_2. Find the volume of the ring shaped solid that remains.
(b) Express the volume in part (a) in terms of the height h of the ring. Notice that the volume depends only on h not on r_1 or r_2.

Solution

(a) The volume of sphere is $(4/3)\pi r_2^3$. We use the symmetry of the problem and calculate the volume of the cylinder is which is removed from the top hemisphere. This volume is bounded by the surface the sphere, namely by $z = \sqrt{r_2^2 - x^2 - y^2}$ and the domain of integration is a circle of radius r_1. Thus the following integral gives the portion of the top hemisphere which removed,

$$\int_0^{2\pi}\int_0^{r_1}\sqrt{(r_2^2 - r^2)}\,r\,dr\,d\theta = \frac{2\pi}{3}\left((r_2^3 - (r_2^2 - r_1^2)^{3/2}\right).$$

By symmetry the same amount is removed from the lower hemisphere. Thus the volume remaining after removing the cylindrical region is $\frac{4\pi}{3}\left(r_2^2 - r_1^2\right)^{3/2}$.
(b) The height of the ring h, is found by calculating when the cylinder ($x^2 + y^2 = r_1^2$) and sphere ($z^2 = r_2^2 - x^2 - y^2$) intersect. We find this be substituting the equation of the cylinder into the equation for the sphere, giving $z^2 = (r_2^2 - r_1^2) = h^2$. Substituting this definition for h into our solution for part (a) gives the volume is $\frac{4\pi}{3}h^3$, which is independent of the radii as required.

T12 Given the change of variables

$$u = \frac{1}{3}(x+y) \qquad v = \frac{1}{3}(x-2y)$$

express x and y in terms of u and v.

Solution

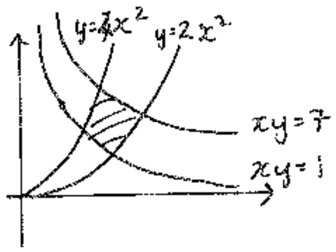
$x = 2u + v$, $y = u - v$.

T13 By making a suitable change of variables, evaluate

$$\iint xy \, dx \, dy$$

over the region enclosed by the two hyperbolas $xy = 1$ and $xy = 7$ and the two parabolas $y = 2x^2$ and $y = 4x^2$.

Solution



Let $u = xy$ and $v = y/x^2$, so $1 \le u \le 7$ and $2 \le v \le 4$. The Jacobian is

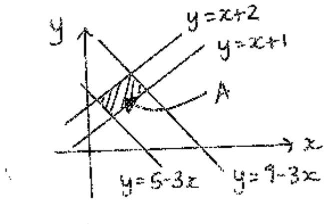
$$\frac{\partial(u,v)}{\partial(x,y)} = (y).(1/x^2) - x.(-2y/x^3) = 3y/x^2,$$

and so $\dfrac{\partial(x,y)}{\partial(u,v)} = x^2/3y$. Hence the integral is

$$\begin{aligned} I &= \int_1^7 du \int_2^4 xy.\left|\frac{x^2}{3y}\right| dv = \frac{1}{3}\int_1^7 du \int_2^4 \frac{u}{v}\, dv \\ &= \frac{1}{3}\int_1^7 u\, du \int_2^4 \frac{1}{v}\, dv = \frac{1}{3}\left[\frac{u^2}{2}\right]_1^7 [\log v]_2^4 \\ &= \frac{1}{6}.48.(\log 4 - \log 2) = 8\log 2. \end{aligned}$$

T14 Use double integration and an appropriate change of variables to find the area of the parallelogram enclosed by the lines $y = x + 1$, $y = x + 2$, $y = 5 - 3x$, $y = 9 - 3x$

Solution



Let $u = y - x$ and $v = y + 3x$, so $1 \le u \le 2$ and $5 \le v \le 9$. The Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1).(1) - 1.(3) = -4,$$

and so $\dfrac{\partial(x,y)}{\partial(u,v)} = -1/4$. Hence this area, A, is

$$\iint_A dA = \int_1^2 du \int_5^9 1. \left|\frac{-1}{4}\right| dv = \frac{1}{4}[u]_1^2 [v]_5^9 = 1.$$

T15 Evaluate the integral

$$\int_0^3 dx \int_{x/4}^{x/4+2} \left(\frac{x+y}{4}\right) dy,$$

using the change of variables $u = \frac{x}{4}, \quad v = \frac{x+y}{2}$.

Solution

Upon rearranging we have $x = 4u$ and $y = 2v - 4u$. Since $0 \le x \le 3$, we then have $0 \le u \le 3/4$. Similarly, given $x/4 \le y \le x/4 + 2$ we obtain $\frac{5}{2}u \le v \le \frac{5}{2}u + 1$. The Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = (\frac{1}{4})(\frac{1}{2}) - \frac{1}{2}(0) = \frac{1}{8},$$

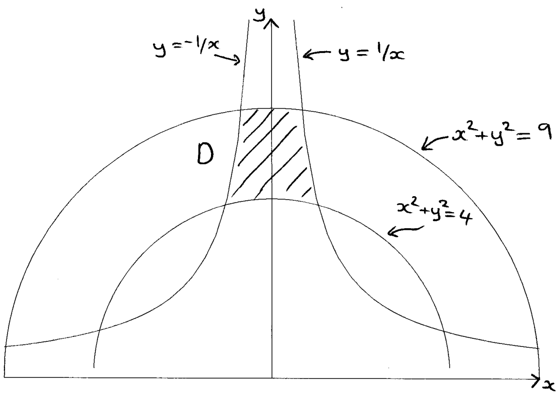
and so $\dfrac{\partial(x,y)}{\partial(u,v)} = 8$. Hence the integral is

$$I = \int_0^{\frac{3}{4}} du \int_{\frac{5}{2}u}^{\frac{5}{2}u+1} \frac{v}{2}. |8|\ dv == 2\int_0^{\frac{3}{4}} 1 + 5u\ du = \frac{69}{16}$$

T16 Evaluate

$$\iint_D x^4 - y^4\, dxdy$$

where D is the region illustrated below.



Solution

Let $u = x^2 + y^2$ and $v = xy$. The Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = (2x).x - (2y).y = 2(x^2 - y^2),$$

and so $\dfrac{\partial(x,y)}{\partial(u,v)} = 1/(2(x^2 - y^2))$. Now, since in D $y > |x|$ and hence $x^2 - y^2 < 0$,

$$(x^4 - y^4)\left|\frac{1}{2(x^2 - y^2)}\right| = -\frac{x^2 + y^2}{2} = -\frac{u}{2},$$

hence the integral is

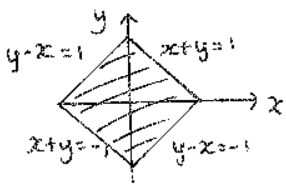
$$-\iint_D \frac{1}{2}u\, dudv = -\int_4^9 \left(\int_{-1}^{1} \frac{1}{2}u\, dv\right) du = -\int_4^9 u\, du = -\frac{65}{2}.$$

T17 By making a suitable change of variables, evaluate

$$\iint x^2\, dx\, dy$$

over the square enclosed by the lines $x + y = -1$, $x + y = 1$, $y - x = -1$, $y - x = 1$.

Solution



Let $u = x + y$ and $v = y - x$, so $-1 \le u \le 1$ and $-1 \le v \le 1$. Solving for x and y in terms of u and v we get $x = (u - v)/2$. The Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = (1).(1) - 1.(-1) = 2,$$

and so $\dfrac{\partial(x,y)}{\partial(u,v)} = 1/2$. Hence the integral is

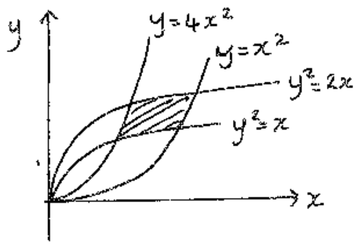
$$I = \int_{-1}^{1} du \int_{-1}^{1} x^2 . \left|\frac{1}{2}\right| dv = \frac{1}{2}\int_{-1}^{1} du \int_{-1}^{1} (\frac{1}{2}(u-v))^2 \, dv$$
$$= \frac{1}{8}\int_{-1}^{1} du \int_{-1}^{1} u^2 - 2uv + v^2 \, dv = \frac{1}{8}\int_{-1}^{1} [u^2 v - uv^2 + \frac{v^3}{3}]_{-1}^{1} du$$
$$= \frac{1}{8}\int_{-1}^{1} 2u^2 - \frac{2}{3}\, du = \frac{1}{8}\left[2\frac{u^3}{3} + \frac{2}{3}u\right]_{-1}^{1} = \frac{1}{3}.$$

T18 By making a suitable change of variables, evaluate

$$\iint \frac{y^2}{x}\, dxdy$$

over the region in the first quadrant enclosed by the four parabolas
$y^2 = x$, $y^2 = 2x$, $y = x^2$, $y = 4x^2$

Solution



Let $u = y^2/x$ and $v = y/x^2$, so $1 \le u \le 2$ and $1 \le v \le 4$. The Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = (-y^2/x^2).(1/x^2) - (2y/x).(-2y/x^3) = 3y^2/x^4,$$

and so $\dfrac{\partial(x,y)}{\partial(u,v)} = x^4/(3y^2)$. Hence the integral is

$$I = \int_1^2 du \int_1^4 \frac{y^2}{x}.\left|\frac{x^4}{3y^2}\right| dv = \frac{1}{3}\int_1^2 du \int_1^4 x^3\, dv.$$

Now, $y = vx^2$, substituting this into the expression for u gives, $u = v^2x^4/x = v^2x^3$. Hence, $x^3 = u/v^2$. Thus,

$$I = \frac{1}{3}\int_1^2 du \int_1^4 \frac{u}{v^2}\, dv = \frac{1}{3}[\frac{u^2}{2}]_1^2[-\frac{1}{v}]_1^4 = \frac{3}{8}.$$

T19 Evaluate $\iint_R (x^2 + y^2) dA$, where R is the parallelogram with vertices $(0,0)$, $(2,0)$, $(3,1)$ and $(1,1)$.

Solution

The parallelogram is bounded by the lines $y = 0$, $y = 1$, $y = x$ and $y = x - 2$. Letting $u = x - y$ and $v = y$ the domain can be described by $0 \leq u \leq 2$ and $0 \leq v \leq 1$.

Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = (1).(1) - 0.(-1) = 1,$$

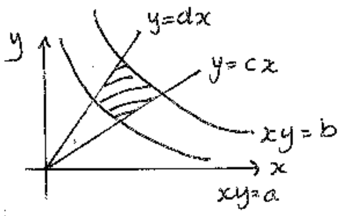
and so $\dfrac{\partial(x,y)}{\partial(u,v)} = 1$. Hence the integral is

$$\iint_R dA = \int_0^1 dv \int_0^2 (u+v)^2 + v^2 \, |1| \, du = \int_0^1 \left[\frac{1}{3}(u+v)^3 + v^2 u\right]_0^2 dv = 6.$$

T20 Show that the area of the region in the first quadrant enclosed by the two hyperbolas $xy = a$, $xy = b$ and the two lines $y = cx$, $y = dx$, where $b > a > 0$ and $d > c > 0$ is

$$\frac{1}{2}(b-a)\log\left(\frac{d}{c}\right).$$

Solution



Let $u = xy$ and $v = y/x$, so $a \leq u \leq b$ and $c \leq v \leq d$. The Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = (y).(1/x) - x.(-y/x^2) = 2y/x,$$

and so $\dfrac{\partial(x,y)}{\partial(u,v)} = x/(2y)$. Hence this area, A, is

$$\iint_A dA = \int_a^b du \int_c^d 1.\left|\frac{x}{2y}\right| dv = \frac{1}{2}[u]_a^b [\log v]_c^d = \frac{1}{2}(b-a)\log(d/c).$$

T21 Use change of variables to evaluate

$$\iint x^4 + y^4 \, dxdy$$

over the interior of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution

Let $x = au$ and $y = bv$, so the ellipse becomes $u^2 + v^2 \leq 1$ and so $0 \leq u \leq 1$ and $0 \leq v \leq 1$. The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = (a).(b) - 0.(0) = ab,$$

Hence the integral is

$$\begin{aligned}
I &= \int\int_{u^2+v^2 \leq 1} (a^4u^4 + b^4v^4)ab\,du dv \\
&= ab(a^4+b^4)\int_0^{2\pi} d\theta \int_0^1 r^5 \cos^4\theta\, dr, \quad \text{(By symmetry } \int\int_{\text{circle}} u^4\, du dv = \int\int_{\text{circle}} v^4\, du dv) \\
&= ab(a^4+b^4)\int_0^{2\pi} \cos^4\theta\, d\theta \left[\frac{r^6}{6}\right]_0^1 \\
&= ab(a^4+b^4)4\frac{3.1}{4.2}\frac{\pi}{2}\frac{1}{6} = \frac{\pi ab}{8}(a^4+b^4).
\end{aligned}$$