2B Linear Algebra

Ex Sheet 4

1 True/False

a) Let A be an $m \times n$ matrix over $\mathbb{R}$. Then dim(row(A))=dim(col(A^T)).

b) Let A be an $m \times n$ matrix over $\mathbb{R}$. Then the nonzero row vectors of A form a basis for its row space.

c) For any $m \times n$ matrix over $\mathbb{R}$, the row and column spaces have identical dimension.

d) The sum of the rank and the nullity of a given matrix equals the number of rows of the matrix.

e) The column space of the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ is $\operatorname{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.

f) Let $\{v_1, \ldots, v_n\}$ be a basis for $\mathbb{R}^n$. Then any vector in $\mathbb{R}^n$ can be expressed as a linear combination of the vectors $v_1, \ldots, v_n$.

g) Each linearly independent set of vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$.

h) If $\operatorname{Span}(S) = \mathbb{R}^n$ then S is a basis for $\mathbb{R}^n$.

i) The vector space consisting of just the zero vector has dimension 1.

j) If S and S' are bases for $\mathbb{R}^n$ then both S and S' contain n elements.

k) Any two vectors in $\mathbb{R}^2$ which are not scalar multiples of each other form a basis for $\mathbb{R}^2$.

l) Any two vectors in $\mathbb{R}^3$ which are not scalar multiples of each other form a basis for a 2-dimensional subspace of $\mathbb{R}^3$.

m) The rank of a matrix is the dimension of its column space.

n) For any matrix $A \in M_{3\times 2}(\mathbb{R})$, the following identity holds

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = 3.$$

o) If $\mathcal{B}$ is a basis for $\mathbb{R}^2$ then the matrix whose columns are the vectors in $\mathcal{B}$ must have determinant 0.

p) Let A be an $n \times n$ matrix. If $\operatorname{nullity}(A) = 0$ then the columns of A are linearly dependent.

q) Let A be an $n \times n$ matrix. Then $\operatorname{rank}(A) = n$ if and only if the rows of A are a basis for $\mathbb{R}^n$.

r) Let A be an $n \times n$ matrix. If the columns of A span $\mathbb{R}^n$ then A is invertible.

[1] **True/False Questions**
Every Exercise Sheet will have a section containing true/false questions. They are designed to test your understanding from lectures. You should look at your lecture notes and/or the textbook to help you answer these questions, but you should not need to write anything to work out the solution.

s) Let P be a plane through the origin in $\mathbb{R}^3$ and let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two non-parallel vectors in P. Suppose $\boldsymbol{w} = c\boldsymbol{u} + d\boldsymbol{v}$ and $\boldsymbol{w}' = c'\boldsymbol{u} + d'\boldsymbol{v}$, where $c, d, c', d' \in \mathbb{R}$. If $\boldsymbol{w} = \boldsymbol{w}'$ then $c = c'$ and $d = d'$.

t) If $\boldsymbol{v} = (v_1, v_2) \in \mathbb{R}^2$, and $\mathcal{E}$ is the standard basis for $\mathbb{R}^2$, then $[\boldsymbol{v}]_{\mathcal{E}} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

u) Let $\mathcal{B} : \boldsymbol{v}_1, \boldsymbol{v}_2$ be an ordered basis for $\mathbb{R}^2$. If $\boldsymbol{v} = 2\boldsymbol{v}_1 - 3\boldsymbol{v}_2$ then $[\boldsymbol{v}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

Solutions to True/False

(a) T (b) F (c) T (d) F (e) T (f) T (g) F (h) F (i) F (j) T (k) T (l) T (m) T (n) F o) F p) F q) T r) T s) T t) T u) F

Tutorial Exercises

T1 Find the dimension of the subspace in $\mathbb{R}^4$ consisting of all solutions of the homogeneous system of linear equations

$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & + & 2x_3 & + & x_4 & = & 0 \\ 2x_1 & - & x_2 & & & + & 3x_4 & = & 0 \\ x_1 & - & 2x_2 & - & 2x_3 & + & 2x_4 & = & 0 \end{array}$$

by first finding the general solution and then writing down a spanning set for the space of solutions that is also linearly independent.

Solution

Let W denote the subspace consisting of all solutions to the system of homogenous equations. The augmented matrix for this system is

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 \\ 2 & -1 & 0 & 3 & 0 \\ 1 & -2 & -2 & 2 & 0 \end{bmatrix}$$

which has reduced row echelon form

$$\begin{bmatrix} 1 & 0 & \frac{2}{3} & \frac{4}{3} & 0 \\ 0 & 1 & \frac{4}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

from which it follows that the general solution is

$$\begin{aligned} x_1 &= -\frac{2}{3}s - \frac{1}{3}t \\ x_2 &= -\frac{4}{3}s + \frac{1}{3}t \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

where $s, t \in \mathbb{R}$. In other words every vector $(x_1, x_2, x_3, x_4) \in W$ can be written in the form

$$(x_1, x_2, x_3, x_4) = (-\frac{2}{3}s - \frac{4}{3}t, -\frac{4}{3}s + \frac{1}{3}t, s, t) = s(-\frac{2}{3}, -\frac{4}{3}, 1, 0) + t(-\frac{4}{3}, \frac{1}{3}, 0, 1),$$

where $s, t \in \mathbb{R}$, i.e.

$$W = \operatorname{span}((-\frac{2}{3}, -\frac{4}{3}, 1, 0), (-\frac{4}{3}, \frac{1}{3}, 0, 1)).$$

This set is also linearly independent (look at the last two entries in each vector), and so

$$S = \{(-\frac{2}{3}, -\frac{4}{3}, 1, 0), (-\frac{4}{3}, \frac{1}{3}, 0, 1)\}$$

is a basis for W.

T2 Let

$$A = \begin{bmatrix} 1 & 5 & 0 & 8 & 2 \\ 2 & 0 & 3 & 2 & 1 \\ 0 & 0 & 4 & 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 2 & 1 & 4 \\ 1 & 2 & 5 & 2 \\ 3 & 1 & 2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 0 & 2 & 1 \\ 3 & 0 & 4 & 4 \\ 2 & 3 & 3 & 6 \\ 3 & 3 & 1 & 5 \end{bmatrix}.$$

For each of these matrices:

a) Find a basis for the row space.

b) Find a basis for the column space.

c) Find the rank, and then use the Rank Theorem to find the nullity.

Solution

a) The reduced row echelon form of A is

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -1/2 & -1/4 \\ 0 & 1 & 0 & 17/10 & 9/20 \\ 0 & 0 & 1 & 1 & 1/2 \end{bmatrix}.$$

Since these three rows are linearly independent (compare the first 3 entries in each row), a basis for $\operatorname{row}(A)$ is $\{[1, 0, 0, -1/2, -1/4], [0, 1, 0, 17/10, 9/20], [0, 0, 1, 1, 1/2]\}$. Alternatively, it would have been enough to just look at the row echelon form of A, since its rows will also be linearly independent.

The reduced row echelon form of

$$B = \begin{bmatrix} 2 & 2 & 1 & 4 \\ 1 & 2 & 5 & 2 \\ 3 & 1 & 2 & 2 \end{bmatrix}$$

is

$$\operatorname{rref}(B) = \begin{bmatrix} 1 & 0 & 0 & 6/19 \\ 0 & 1 & 0 & 36/19 \\ 0 & 0 & 1 & -8/19 \end{bmatrix}.$$

Since these three rows are linearly independent (compare the first 3 entries in each row), a basis for $\operatorname{row}(B)$ is $\{[1, 0, 0, 6/19], [0, 1, 0, 36/19], [0, 0, 1, -8/19]\}$. Alternatively, it would have been enough to just look at the row echelon form of B, since its rows will also be linearly independent.

The reduced row echelon form of

$$C = \begin{bmatrix} 3 & 0 & 2 & 1 \\ 3 & 0 & 4 & 4 \\ 2 & 3 & 3 & 6 \\ 3 & 3 & 1 & 5 \end{bmatrix}$$

is

$$\text{rref}(C) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since these four rows are linearly independent (they are in fact the standard basis for $\mathbb{R}^4$), a basis for $\text{row}(C)$ is $\{[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]\}$. Alternatively, it would have been enough to just look at the row echelon form of C, since its rows will also be linearly independent.

b) To find a basis for the column space, we find a basis for the row space of the transpose.

The reduced row echelon form of A^T is

$$\text{rref}(A^T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and so a basis for $\text{row}(A^T) = \text{col}(A)$ is $\{[1,0,0],[0,1,0],[0,0,1]\}$.

The reduced row echelon form of B^T is

$$\text{rref}(B^T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and so a basis for $\text{row}(B^T) = \text{col}(B)$ is $\{[1,0,0],[0,1,0],[0,0,1]\}$.

The reduced row echelon form of C^T is

$$\text{rref}(C^T) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and so a basis for $\text{row}(C^T) = \text{col}(C)$ is $\{[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]\}$.

c) We have $\text{rank}(A) = 3$ so since A has 5 columns, we have $\text{rank}(A) + \text{nullity}(A) = 5$ and so $\text{nullity}(A) = 2$.

We have $\text{rank}(B) = 3$ so since B has 4 columns, we have $\text{rank}(B) + \text{nullity}(B) = 4$ and so $\text{nullity}(B) = 1$.

We have $\text{rank}(C) = 4$ so since C has 4 columns, we have $\text{rank}(C) + \text{nullity}(C) = 4$ and so $\text{nullity}(C) = 0$.

T3 Find a basis for the null space of the matrices

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 2 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence find nullity(A) and nullity(B), and then use the Rank Theorem to find rank(A) and rank(B).

Solution

The reduced row echelon form of $(A|\mathbf{0})$ is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The general solution is $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that

$$x_3 = 0; \quad x_1 + x_2 = 0$$

hence

$$x_1 = -x_2; \quad x_3 = 0.$$

Thus

$$\operatorname{null}(A) = \{(-x_2, x_2, 0) : x_2 \in \mathbb{R}\} = \{x_2(-1, 1, 0) : x_2 \in \mathbb{R}\} = \operatorname{Span}((-1, 1, 0)).$$

Therefore a basis for the null space of A is $S = \{(-1, 1, 0)\}$.

The reduced row echelon form of $(B|\mathbf{0})$ is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The general solution is $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that

$$x_3 = 0; \quad x_1 + x_2 = 0$$

hence

$$x_1 = -x_2; \quad x_3 = 0.$$

Thus

$$\operatorname{null}(B) = \{(-x_2, x_2, 0) : x_2 \in \mathbb{R}\} = \{x_2(-1, 1, 0) : x_2 \in \mathbb{R}\} = \operatorname{Span}((-1, 1, 0)).$$

Therefore a basis for the null space of B is $S = \{(-1, 1, 0)\}$.

Note that even though A and B are different sized matrices, they have the same null space, which is a subspace of $\mathbb{R}^3$.

Since the null space for both A and B has basis consisting of a single vector, we have nullity$(A) =$ nullity$(B) = 1$. Both A and B have 3 columns so by the Rank Theorem we also have rank$(A) =$ rank$(B) = 3 - 1 = 2$.

T4 In this question we work in $\mathbb{R}^4$.

a) Find a basis for each of the following subspaces:

i) $W_1 = \{(w,x,y,z) : x - 2y + 3z = 0\}$,

ii) $W_2 = \{(w,x,y,z) : w - x + y + z = 0\}$,

iii) $W_3 = \{(w,x,y,z) : w - y + 4z = 0\}$,

iv) $W_4 = \{(w,x,y,z) : w - 2x + 5z = 0\}$.

b) Find a basis for the subspaces $W_1 \cap W_2$, $W_1 \cap W_2 \cap W_3$ and $W_1 \cap W_2 \cap W_3 \cap W_4$ in $\mathbb{R}^4$. (See T7(a) below for the definition of $\cap$ why these intersections are subspaces.)

Solution

a) We begin with

i) $W_1 = \{(w,x,y,z) : x - 2y + 3z = 0\}$. Here we have

$$\begin{aligned} W_1 &= \{(w,x,y,z) \mid x = 2y - 3z\} \\ &= \{(w, 2y - 3z, y, z) \mid w,y,z \in \mathbb{R}\} \\ &= \{w(1,0,0,0) + y(0,2,1,0) + z(0,-3,0,1) \mid w,y,z \in \mathbb{R}\} \\ &= \text{Span}\left((1,0,0,0),(0,2,1,0),(0,-3,0,1)\right). \end{aligned}$$

So we have found vectors that span W_1. To check that they are linearly independent, consider

$$\alpha(1,0,0,0) + \beta(0,2,1,0) + \gamma(0,-3,0,1) = (0,0,0,0)$$

leading to the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

which some simple computations with EROs shows is row equivalent to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence the only solutions is $\alpha = 0, \beta = 0, \gamma = 0$ showing that the vectors are linearly independent. So we have a set of vectors that spans W_1 and is linearly independent. Thus the set $S = \{(1,0,0,0),(0,2,1,0),(0,-3,0,1)\}$ is a basis for W_1.

ii) $W_2 = \{(w,x,y,z) : w - x + y + z = 0\}$. Here we have

$$\begin{aligned} W_2 &= \{(w,x,y,z) \mid w = x - y - z\} \\ &= \{(x - y - z, x, y, z) \mid x,y,z \in \mathbb{R}\} \\ &= \{x(1,1,0,0) + y(-1,0,1,0) + z(-1,0,0,1) \mid x,y,z \in \mathbb{R}\} \\ &= \text{Span}\left((1,1,0,0),(-1,0,1,0),(-1,0,0,1)\right). \end{aligned}$$

By the method used for W_1 we find that these are linearly independent.

iii) $W_3 = \{(w,x,y,z) : w - y + 4z = 0\}$. Here we have

$$\begin{aligned} W_3 &= \{(w,x,y,z) \mid w = y - 4z\} \\ &= \{(y-4z,x,y,z) \mid x,y,z \in \mathbb{R}\} \\ &= \{x(0,1,0,0) + y(1,0,1,0) + z(-4,0,0,1) \mid x,y,z \in \mathbb{R}\} \\ &= \operatorname{Span}\big((0,1,0,0),(1,0,1,0),(-4,0,0,1)\big). \end{aligned}$$

By the method used for W_1 we find that $S = \{(0,1,0,0),(1,0,1,0),(-4,0,0,1)\}$ is a basis.

iv) $W_4 = \{(w,x,y,z) : w - 2x + 5z = 0\}$. Here we have

$$\begin{aligned} W_4 &= \{(w,x,y,z) \mid w = 2x - 5z\} \\ &= \{(2x-5z,x,y,z) \mid x,y,z \in \mathbb{R}\} \\ &= \{x(2,1,0,0) + y(0,0,1,0) + z(-5,0,0,1) \mid x,y,z \in \mathbb{R}\} \\ &= \operatorname{Span}\big((2,1,0,0),(0,0,1,0),(-5,0,0,1)\big). \end{aligned}$$

By the method used for W_1 we find that $S = \{(2,1,0,0),(0,0,1,0),(-5,0,0,1)\}$ is a basis.

b) Next we consider some of the intersections of these subspaces.

i) We consider $W_1 \cap W_2 = \{(w,x,y,z) \mid x - 2y + 3z = 0 \text{ and } w - x + y + z = 0\}$. We need to find the solutions to these equations and so consider the matrix

$$\begin{bmatrix} 0 & 1 & -2 & 3 & 0 \\ 1 & -1 & 1 & 1 & 0 \end{bmatrix}.$$

It is easy to see that this has reduced row echelon form

$$\begin{bmatrix} 1 & 0 & -1 & 4 & 0 \\ 0 & 1 & -2 & 3 & 0 \end{bmatrix}.$$

Hence we need $w - y + 4z = 0$ and $x - 2y + 3z = 0$. So

$$\begin{aligned} W_1 \cap W_2 &= \{(w,x,y,z) \mid w = y - 4z, x = 2y - 3z\} \\ &= \{(y-4z, 2y-3z, y, z) \mid y,z \in \mathbb{R}\} \\ &= \{y(1,2,1,0) + z(-4,-3,0,1) \mid y,z \in \mathbb{R}\} \\ &= \operatorname{Span}\big((1,2,1,0),(-4,-3,0,1)\big). \end{aligned}$$

The two vectors are linearly independent since they are not multiples of each other. So $S = \{(1,2,1,0),(-4,-3,0,1)\}$ is a basis for $W_1 \cap W_2$.

ii) We consider

$$W_1 \cap W_2 \cap W_3 = \{(w,x,y,z) \mid x - 2y + 3z = 0,\ w - x + y + z = 0,\ w - y + 4z = 0\}.$$

We need to find the solutions to these equations and so consider the matrix

$$\begin{bmatrix} 0 & 1 & -2 & 3 & 0 \\ 1 & -1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 4 & 0 \end{bmatrix}.$$

This has reduced row echelon form

$$\begin{bmatrix} 1 & 0 & -1 & 4 & 0 \\ 0 & 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So the solution here is identical to the one found above when considering $W_1 \cap W_2$. Hence

$$\begin{aligned} W_1 \cap W_2 \cap W_3 &= W_1 \cap W_2 \\ &= \mathrm{Span}\big((1,2,1,0),(-4,-3,0,1)\big). \end{aligned}$$

and thus $S = \{(1,2,1,0),(-4,-3,0,1)\}$ is a basis for $W_1 \cap W_2 \cap W_3$.

iii) We consider

$$W_1 \cap W_2 \cap W_3 \cap W_4 = \left\{ (w,x,y,z) \;\middle|\; \begin{array}{rcl} x - 2y + 3z & = & 0, \\ w - x + y + z & = & 0, \\ w - y + 4z & = & 0, \\ w - 2x + 5z & = & 0 \end{array} \right\}.$$

We need to find the solutions to these equations and so consider the matrix

$$\begin{bmatrix} 0 & 1 & -2 & 3 & 0 \\ 1 & -1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 4 & 0 \\ 1 & -2 & 0 & 5 & 0 \end{bmatrix}.$$

This has reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & \frac{5}{3} & 0 \\ 0 & 1 & 0 & -\frac{5}{3} & 0 \\ 0 & 0 & 1 & -\frac{7}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So we need $w + \frac{5}{3}z = 0$, $x - \frac{5}{3}z = 0$ and $y - \frac{7}{3}z = 0$. Hence

$$\begin{aligned} W_1 \cap W_2 \cap W_3 \cap W_4 &= \left\{ \left(-\frac{5}{3}z, \frac{5}{3}z, \frac{7}{3}z, z \right) \mid z \in \mathbb{R} \right\} \\ &= \{ \frac{1}{3}z(-5,5,7,3) \mid z \in \mathbb{R} \} \\ &= \mathrm{Span}\big((-5,5,7,3)\big). \end{aligned}$$

Since a single non-zero vector is necessarily linearly independent, this single vector suffices. That is, a basis for $W_1 \cap W_2 \cap W_3 \cap W_4$ is $S = \{(-5,5,7,1)\}$.

T5 Answer the following questions by considering the matrix with the given vectors as its columns.

a) Do the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ form a basis for $\mathbb{R}^3$?

b) Do the vectors $\begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\\1\\1\end{bmatrix}$ form a basis for $\mathbb{R}^4$?

Solution

a) Let

$$A = \begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\\0 & 1 & 1\end{bmatrix}$$

This has row echelon form

$$\begin{bmatrix}1 & 1 & 0\\0 & 1 & -1\\0 & 0 & 1\end{bmatrix}$$

Since the rows of this row echelon form are linearly independent, $\operatorname{rank}(A) = 3$. Therefore the column space of A has dimension 3, and so the columns of A span $\mathbb{R}^3$. To show that they are a basis, we need to establish linear independence. For this we can observe that the homogeneous system corresponding to the augmented matrix $(A|\mathbf{0})$ has a unique solution, since A has row echelon form as above, thus the columns of A are linearly independent.

b) Let

$$A = \begin{bmatrix}1 & 1 & 1 & 0\\1 & 1 & 0 & 1\\1 & 0 & 1 & 1\\0 & 1 & 1 & 1\end{bmatrix}$$

This has row echelon form

$$\begin{bmatrix}1 & 1 & 1 & 0\\0 & 1 & 0 & -1\\0 & 0 & 1 & -1\\0 & 0 & 0 & 1\end{bmatrix}$$

Since the rows of this row echelon form are linearly independent, $\operatorname{rank}(A) = 4$. Therefore the column space of A has dimension 4, and so the columns of A span $\mathbb{R}^4$. To show that they are a basis, we need to establish linear independence. For this we can observe that the homogeneous system corresponding to the augmented matrix $(A|\mathbf{0})$ has a unique solution, since A has row echelon form as above, thus the columns of A are linearly independent.

T6 Without doing any unnecessary calculations, determine whether S is a basis for $\operatorname{Span}(S)$:

a) $S = \{(1,0,1,1),(2,0,0,3)\}$

b) $S = \{(1,0,1,1),(2,0,0,3),(4,0,0,6)\}$

c) $S = \{(1,0,1,1),(2,0,0,3),(2,0,2,2)\}$

d) $S = \{(1,0,1,1),(2,0,0,3),(2,0,2,2),(-1,9,8,0),(3,0,2,1)\}$

In all cases, find a basis for $\operatorname{Span}(S)$. Are any of these sets S a basis for $\mathbb{R}^4$? Justify your answer.

Solution

We consider whether the following sets are bases for the space they span and for the real vector space $\mathbb{R}^4$.

a) $S = \{(1,0,1,1),(2,0,0,3)\}$

Both vectors have 0 as their second entry. This means that any vector $(w,x,y,z) \in \mathbb{R}^4$ with $x \neq 0$ cannot possibly be written as

$$(w,x,y,z) = a(1,0,1,1) + b(2,0,0,3),$$

so S does not span $\mathbb{R}^4$. However, S is a spanning set for $\mathrm{Span}(S)$ (by definition!), so they'll give a basis for $\mathrm{Span}(S)$ if they are linearly independent. Since there are only two vectors, we can check linear independence by just making sure they are not scalar multiples of each other. This is the case, so they are a basis for $\mathrm{Span}(S)$.

b) $S = \{(1,0,1,1),(2,0,0,3),(4,0,0,6)\}$.

Since the second and third vectors are scalar multiples of each other we know that this set is not linearly independent. Indeed, the equation

$$\lambda_1(1,0,1,1) + \lambda_2(2,0,0,3) + \lambda_3(4,0,0,6) = 0$$

has solution $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = -1$. Hence it cannot be a basis for either $\mathrm{Span}(S)$ or for $\mathbb{R}^4$. If we were to discard the third vector then the two remaining ones are linearly independent by the answer to (a) above. Hence they are a basis for $\mathrm{Span}(S)$.

c) $S = \{(1,0,1,1),(2,0,0,3),(2,0,2,2)\}$. Since $(2,0,2,2) = 2(1,0,1,1)$ we know that the vectors are not linearly independent because the equation

$$\lambda_1(1,0,1,1) + \lambda_2(2,0,0,3) + \lambda_3(2,0,2,2) = 0$$

has solution $\lambda_1 = 2$, $\lambda_2 = 0$, $\lambda_3 = -1$. Hence it cannot be a basis for either $\mathrm{Span}(S)$ or for $\mathbb{R}^4$. If we were to discard the third vector then the two remaining ones are linearly independent by the answer to (a) above, so they are a basis for $\mathrm{Span}(S_3)$.

d) $S = \{(1,0,1,1),(2,0,0,3),(2,0,2,2),(-1,9,8,0),(3,0,2,1)\}$.

As above, we have

$$(2,0,2,2) = 2(1,0,1,1).$$

so the vectors are not linearly independent. We discard $(2,0,2,2)$ to leave

$$S' = \{(1,0,1,1),(2,0,0,3),(-1,9,8,0),(3,0,2,1)\}.$$

We need to check whether this set is linearly independent. We consider

$$\alpha(1,0,1,1) + \beta(2,0,0,3) + \gamma(-1,9,8,0) + \delta(3,0,2,1) = (0,0,0,0).$$

leading to the matrix

$$\begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 0 & 0 & 9 & 0 & 0 \\ 1 & 0 & 8 & 2 & 0 \\ 1 & 3 & 0 & 1 & 0 \end{bmatrix}$$

which has reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Hence the solution is $\alpha = 0, \beta = 0, \gamma = 0, \delta = 0$, showing that S' is linearly independent. To see that it is a basis for $\mathbb{R}^4$, we need to check that for every $(w, x, y, z) \in \mathbb{R}^4$, there exist constants $\alpha, \beta, \gamma, \delta$ such that

$$\alpha(1,0,1,1) + \beta(2,0,0,3) + \gamma(-1,9,8,0) + \delta(3,0,2,1) = (w,x,y,z).$$

leading to the matrix

$$\begin{bmatrix} 1 & 2 & -1 & 3 & w \\ 0 & 0 & 9 & 0 & x \\ 1 & 0 & 8 & 2 & y \\ 1 & 3 & 0 & 1 & z \end{bmatrix}.$$

The left hand 4×4 matrix here is the same as that in the check for linear independence above, so the same EROs give the same left hand 4×4 matrix in the reduced row echelon matrix as above, namely, the identity. This means we can solve for each of $\alpha, \beta, \gamma, \delta$ without imposing constraints on w, x, y, z, so S' spans $\mathbb{R}^4$.

The standard basis $\{e_1, e_2, e_3, e_4\}$ for $\mathbb{R}^4$ has four elements, so $dim(\mathbb{R}^4) = 4$ and hence every basis of $\mathbb{R}^4$ has four elements. None of the sets S has four elements, so none of them give a basis for $\mathbb{R}^4$.

T7 Let U and V be subspaces of $\mathbb{R}^n$.

a) Prove that the intersection $U \cap V = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \in U \text{ and } \boldsymbol{x} \in V\}$ is also a subspace of $\mathbb{R}^n$.

b) Is the union $U \cup V = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \in U \text{ or } \boldsymbol{x} \in V\}$ always a subspace? Justify your answer.

Solution

a) Since U and V are subspaces, we have $\mathbf{0} \in U$ and $\mathbf{0} \in V$, hence $\mathbf{0} \in U \cap V$.

Let $\boldsymbol{x}, \boldsymbol{y}$ be in $U \cap V$. Then since U and V are subspaces, we have $\boldsymbol{x} + \boldsymbol{y} \in U$ and $\boldsymbol{x} + \boldsymbol{y} \in V$, hence $\boldsymbol{x} + \boldsymbol{y} \in U \cap V$.

Let $\boldsymbol{x}$ be in $U \cap V$ and let c be a scalar. Then since U and V are subspaces, we have $c\boldsymbol{x} \in U$ and $c\boldsymbol{x} \in V$, hence $c\boldsymbol{x} \in U \cap V$.

Therefore $U \cap V$ is a subspace of $\mathbb{R}^n$.

b) No. For example let $U = \text{Span}([1,0])$ and $V = \text{Span}([0,1])$ in $\mathbb{R}^2$ (so U is the x-axis and V is the y-axis). Then $[1,0] \in U$ and $[0,1] \in V$ but $[1,0] + [0,1] = [1,1]$ is not in U or V. So $U \cup V$ is not closed under vector addition, hence $U \cup V$ is not a subspace. (Note that $U \cup V$ does contain $\mathbf{0}$, and is closed under scalar multiplication.)

T8 Let $a \in \mathbb{R}$ and consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4a \end{pmatrix}.$$

Find all values of a such that:

a) $\mathrm{nullity}(A) = 0$;

b) $\mathrm{nullity}(A) = 1$;

c) $\mathrm{nullity}(A) = 2$.

Solution

After applying $R_2 \to R_2 - 2R_1$ to the augmented matrix $(A|\mathbf{0})$ we obtain

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 4a-4 & 0 \end{pmatrix}. \qquad (\star)$$

Suppose first that $4a - 4 = 0$, that is, that $a = 1$. Then $(\star)$ equals

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and so is already the reduced row echelon form of $(A|\mathbf{0})$. The general solution is $(x_1, x_2) \in \mathbb{R}^2$ such that

$$x_1 + 2x_2 = 0 \implies x_1 = -2x_2$$

and so in this case

$$\mathrm{null}(A) = \{(-2x_2, x_2) : x_2 \in \mathbb{R}\} = \{x_2(-2, 1) : x_2 \in \mathbb{R}\} = \mathrm{Span}((-2, 1)).$$

Thus if $a = 1$ we have $\mathrm{nullity}(A) = 1$.

Now suppose that $4a - 4 \neq 0$, that is, that $a \neq 1$. Then applying $R_2 \to \frac{1}{4a-4} R_2$ and then $R_1 \to R_1 - 2R_2$ to $(\star)$ above we obtain that the reduced row echelon form of $(A|\mathbf{0})$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The unique solution is $x_1 = x_2 = 0$ and so in this case, $\mathrm{null}(A) = \{\mathbf{0}\}$ and hence $\mathrm{nullity}(A) = 0$.

Therefore $\mathrm{nullity}(A) = 0$ if $a \neq 1$, $\mathrm{nullity}(A) = 1$ if $a = 1$, and there are no values of a for which $\mathrm{nullity}(A) = 2$.

T9 Let $b \in \mathbb{R}$ and consider the matrix

$$B = \begin{pmatrix} 1 & b \\ 3b & 3 \end{pmatrix}.$$

Find all values of b such that:

a) $\mathrm{rank}(B) = 0$;

b) $\mathrm{rank}(B) = 1$;

c) $\operatorname{rank}(B) = 2$.

Solution

A row echelon form of $(B|\mathbf{0})$ is

$$\begin{pmatrix} 1 & b & 0 \\ 0 & 3-3b^2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & b & 0 \\ 0 & 3(1-b^2) & 0 \end{pmatrix}. \qquad (\star\star)$$

Suppose first that $3(1-b^2) = 0$, that is, that $b = \pm 1$. Then $(\star\star)$ equals

$$\begin{pmatrix} 1 & \pm 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and so is already the reduced row echelon form of $(B|\mathbf{0})$. Since this matrix contains one non-zero row, in this case we have $\operatorname{rank}(B) = 1$.

Now suppose that $3(1-b^2) \neq 0$, that is, that $b \neq \pm 1$. Then applying $R_2 \to \frac{1}{3(b^2-1)} R_2$ and then $R_1 \to R_1 - bR_2$ to $(\star\star)$ above we obtain that the reduced row echelon form of $(B|\mathbf{0})$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since this matrix has two non-zero rows, in this case we have $\operatorname{rank}(B) = 2$.

Therefore $\operatorname{rank}(B)$ never equals 0, $\operatorname{rank}(B) = 1$ if $b = \pm 1$, and $\operatorname{rank}(B) = 2$ if $b \neq \pm 1$.

T10

a) Let $A \in M_{3\times 7}(\mathbb{R})$. Can the columns of A be linearly independent? Explain your answer.

b) Let $A \in M_{15\times 14}(\mathbb{R})$. Can the rows of A be linearly independent? Explain your answer.

c) Let $A \in M_{3\times 7}(\mathbb{R})$. Can the row space of A be equal to $\mathbb{R}^7$? Explain your answer.

d) Let $A \in M_{2014\times 2014}(\mathbb{R})$. If the dimension of the row space of A is 2014, what is the null space of A? Explain your answer.

Solution

a) No, the columns of A cannot be linearly independent. The greatest possible value of $\operatorname{rank}(A)$ is 3, the number of rows of A, since $\operatorname{rank}(A)$ is equal to the dimension of the row space of A. Thus the dimension of the column space of A is at most 3. By Theorem 2.8, any set of more than 3 vectors in $\mathbb{R}^3$ must be linearly dependent. Since A has 7 columns, this means the columns of A are linearly dependent.

b) No, the rows of A cannot be linearly independent. The greatest possible value of $\operatorname{rank}(A)$ is 14, the number of columns of A, since $\operatorname{rank}(A)$ is equal to the dimension of the column space of A. Thus the dimension of the row space of A is at most 14. By Theorem 2.8, any set in $\mathbb{R}^{14}$ having more than 14 vectors is linearly dependent. Since A has 15 rows, this means the rows of A are linearly dependent.

c) No. The rank of A is at most 3 since this is the number of rows of A.

d) In this case $\text{null}(A) = \{\mathbf{0}\}$, where $\mathbf{0}$ denotes the zero vector in $\mathbb{R}^{2014}$. The dimension of the row space is 2014 so we have $\text{rank}(A) = 2014$. Since A has 2014 columns, the Rank Theorem then implies that $\text{nullity}(A) = 2014 - 2014 = 0$. Thus $\text{null}(A)$ is the trivial subspace $\{\mathbf{0}\}$.

T11 Show that the columns of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ are a basis for $\mathbb{R}^2$. Then find the coordinate vector of $v = \begin{pmatrix} -2 \\ -8 \end{pmatrix}$ with respect to the ordered basis $\mathcal{B}$ given by the columns of A.

Solution

There are many ways to show that the columns of A form a basis for $\mathbb{R}^2$. One method is to observe that a row echelon form of A is

$$R = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Since both rows of R are non-zero, we have $\text{rank}(R) = \text{rank}(A) = 2$. Therefore the columns of A are a basis for $\mathbb{R}^2$.

To find $[v]_{\mathcal{B}}$, we need to find the scalars c_1, c_2 such that

$$c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -8 \end{pmatrix}.$$

The augmented matrix for this system is $(A|v)$. This augmented matrix has reduced row echelon form

$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & \frac{1}{2} \end{pmatrix}.$$

Thus the unique solution to the system is $c_1 = -3$, $c_2 = \frac{1}{2}$, and the coordinate vector of v with respect to the basis $\mathcal{B}$ is

$$[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -3 \\ \frac{1}{2} \end{bmatrix}.$$

T12 Show that the columns of the matrix $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 3 & 2 & 0 & -2 \\ 0 & 1 & -1 & 1 \\ 5 & 3 & 0 & -1 \end{pmatrix}$ are a basis for $\mathbb{R}^4$. Then find the coordinate vector of $v = \begin{pmatrix} -2 \\ -6 \\ -4 \\ -2 \end{pmatrix}$ with respect to the ordered basis $\mathcal{B}$ given by the columns of A.

Solution

Again, there are many correct answers to the first part. For instance, a row echelon form of A is

$$R = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since R has no all-zero rows, there will be a unique solution to $A\boldsymbol{x} = \mathbf{0}$, so the columns of A are a basis for $\mathbb{R}^4$.

To find $[\boldsymbol{v}]_{\mathcal{B}}$, we need to find the scalars c_1, c_2, c_3, c_4 such that

$$c_1 \begin{pmatrix} 1 \\ 3 \\ 0 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 2 \\ 1 \\ 3 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ -2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \\ -4 \\ -2 \end{pmatrix}.$$

The augmented matrix for this system is $(A|\boldsymbol{v})$. This augmented matrix has reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 12 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}.$$

Thus the unique solution to the system is $c_1 = -2$, $c_2 = 4$, $c_3 = 12$, $c_4 = 4$ and the coordinate vector of $\boldsymbol{v}$ with respect to the basis $\mathcal{B}$ is

$$[\boldsymbol{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 12 \\ 4 \end{bmatrix}.$$

T13 Find the coordinates of the vector $\boldsymbol{v}$ with respect to the given ordered basis $\mathcal{B}$:

a) $\boldsymbol{v} = (3,4,3)$ and $\mathcal{B} : (3,2,1), (2,-1,0), (5,0,0)$ on $\mathbb{R}^3$.

b) $\boldsymbol{v} = (-1,0,5)$ and $\mathcal{B} : (1,2,3), (1,1,-1)$ on $\mathrm{Span}((1,2,3),(1,1,-1))$.

Solution

a) We solve for c_1, c_2, c_3 in

$$c_1(3,2,1) + c_2(2,-1,0) + c_3(5,0,0) = (3,4,3).$$

The system of equations is

$$\begin{aligned} 3c_1 + 2c_2 + 5c_3 &= 3 \\ 2c_1 - c_2 &= 4 \\ c_1 &= 3 \end{aligned}$$

Back substitution shows that $c_1 = 3$, that $c_2 = 2$ and that $c_3 = -2$, so the coordinate vector of v in the given basis is

$$[v]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}.$$

b) We solve for c_1, c_2 in

$$c_1(1,2,3) + c_2(1,1,-1) = (-1,0,5).$$

The system of equations is

$$\begin{aligned} c_1 + c_2 &= -1 \\ 2c_1 + c_2 &= 0 \\ 3c_1 - c_2 &= 5. \end{aligned}$$

This system has solution $c_1 = 1$, $c_2 = -2$, so the coordinate vector of v in the given basis is

$$[v]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Note that even though v is in $\mathbb{R}^3$, its coordinate vector has only 2 entries. This is because $\text{Span}((1,2,3),(1,1,-1))$ is a 2-dimensional subspace of $\mathbb{R}^3$.

T14 You are given that

$$\mathcal{B}: \begin{bmatrix} 0 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 1 \\ 4 \end{bmatrix}$$

is a basis for $\mathbb{R}^4$. Find the vector $v \in \mathbb{R}^4$ such that

$$[v]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -3 \\ 4 \\ 2 \end{bmatrix}.$$

Solution

We have

$$v = 1\begin{bmatrix} 0 \\ 1 \\ 3 \\ -1 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 5 \\ -3 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 4 \\ -1 \\ 0 \\ 2 \end{bmatrix} + 2\begin{bmatrix} -6 \\ 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ -14 \\ 14 \\ 12 \end{bmatrix}.$$