*Tuesday*, 15*th December*, 2015 4.30 pm *to* 6.00 pm

EXAMINATION FOR THE DEGREES OF
M.A. AND B.Sc.

# Mathematics 2A - Multivariable Calculus

*An electronic calculator may be used provided that it does not have a facility for either textual storage or display, or for graphical display.*

*Candidates must attempt all questions.*

1. (i) The function $f$ is defined by $f(x, y) = e^x \sin 2y$. Compute all second order partial derivatives of $f$ and verify that $f$ solves the Helmholtz equation **3**

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + 3f = 0.$$

(ii) Let $F(x, y) = g\big(r(x, y)\big)$. Using the chain rule, express the partial derivatives of $F$ in terms of the derivatives of $g$ and $r$. Use your result to calculate the derivatives $\dfrac{\partial F}{\partial x}$ and $\dfrac{\partial F}{\partial y}$ in the case $r(x, y) = \sqrt{x^2 + y^2}$ and $g(u) = \log u$. **3**

(iii) Consider the following partial differential equation

$$y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = (x^2 + y^2)f,$$

for $f : \mathbb{R}^2 \to \mathbb{R}$. By writing $f(x, y) = F\big(u(x, y), v(x, y)\big)$ with $u(x, y) = xy$ and $v(x, y) = y^2 - x^2$, construct the general solution to the PDE. **6**

2. (i) Let $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ and $r : \mathbb{R}^3 \to \mathbb{R}$ be a vector and scalar field respectively, with $r(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a differentiable function of one variable. Establish the result

$$\nabla \cdot (\phi(r)\mathbf{f}) = \phi(r)\nabla \cdot \mathbf{f} + \phi'(r)\hat{\mathbf{r}} \cdot \mathbf{f}$$

where $\hat{\mathbf{r}} = (x, y, z)/r$. **5**

(ii) Use your result from part (i) to evaluate to a real number the expression

$$\nabla \cdot \big(r^2\,(\boldsymbol{\omega} \times \mathbf{r})\big)$$

where $\boldsymbol{\omega}$ is a constant vector. **5**

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3. (i) Let $\mathcal{D}$ be the region of the annulus between circles centred at the origin with radii 1 and 3, lying between the lines $y = x$ and $y = -x$ with $y \geqslant 0$. Decide whether $\mathcal{D}$ is regular or not and give reasons for your answer. By making a change to polar coordinates, evaluate the integral **6**

$$\iint_{\mathcal{D}} xy\,dxdy.$$

(ii) Define what is meant by the notation, $\dfrac{\partial(u,v)}{\partial(x,y)}$. Let $\mathcal{A}$ be the region between the $y$-axis, the $x$-axis, the curve $y = 1/x$ and the curves $y = \sqrt{x^2-1}$ and $y = \sqrt{x^2+1}$. Sketch $\mathcal{A}$ in the $x$-$y$ plane. Change variables to $u = xy$ and $v = (y^2 - x^2)/2$ and sketch $\mathcal{A}$ in the $u$-$v$ plane. Hence evaluate the integral **8**

$$\iint_{\mathcal{A}} \left(x^3y + y^3x\right) dxdy.$$

(iii) Evaluate the volume integral

$$\iiint_{\mathcal{V}} z\,dxdydz,$$

where $\mathcal{V}$ is the region between the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$. **4**

4. (i) State Green's Theorem. Use it to evaluate the integral

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r},$$

where the vector field $\mathbf{F}(x,y) = (e^{-x} + y^2, e^{-y} + x^2))$ and the curve $\mathcal{C}$ consists of the arc of $y = \sin x$ from $(0,0)$ to $(\pi,0)$ and the line segment from $(\pi,0)$ to $(0,0)$. **8**

(ii) Evaluate the surface integral

$$\iint_{\mathcal{S}} \sqrt{x^2+y^2}\,dS,$$

where $\mathcal{S}$ is the cone-shaped surface given by the equations $z = 4 - 2\sqrt{x^2+y^2}$, $0 \leqslant z \leqslant 4$. **5**

(iii) State Gauss' Divergence Theorem. Use it to evaluate the integral

$$\iint_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,dS,$$

where the vector field $\mathbf{F}(x,y,z) = (xz^2, x^2y - z^3, y^2z + z^2/2)$ and $\mathbf{n}$ is the outward pointing unit normal to the surface $\mathcal{S}$ of the region bounded by the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ with radius $a$ and the planes $x = 0$, $y = 0$ and $z = 0$. **7**

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