Chapter 1: Logic and Inequalities

This chapter is a brief background on two key topics which will appear throughout the course: mathematical logic and inequalities.[1]

Logic and Writing

Our aim is to understand the structure of written mathematics, so that you can write your own mathematical arguments in a logically correct fashion. I recommend reading [H] Chapters 6-13 alongside this section. Some material can also be found in Sections 0.1 and 0.2 of [ERA].

Statements

The building blocks of written mathematics are statements, by which we mean the following.

Definition 1.1. *A* statement *is a sentence which is either true or false, but not both. Statements may contain* free variables, *for example x, where the truth of the statement depends on the value of the variable.*

Examples

- "$2+7=9$" is a statement, which is true.
- "Dogs can fly" is a statement, this one happens to be false.[2]
- "Turn on the lights" is not a statement from our point of view (it is a request or command).
- "There are infinitely many prime numbers" is a true statement.
- "There are infinitely many pairs of prime numbers with difference 2" is also a statement — it is either true or false, though at the time of writing we do not know which.[3]
- "$x^2+2x+1=0$" is a statement, with *free variable* x; the truth of the statement depends on the value of the variable (it's true if and only if $x=-1$).
- "There is some $x \in \mathbb{R}$ with $x^2+2x+1=0$" is also a statement. This time there are no free variables, and the statement is true (take $x=-1$). This last example of constructing a new statement by *quantifying* over a free variable of another statement is an important concept.

We can combine and negate statements using the *logical connectives* "and", "or" and "not". In what follows we often use symbols like P and Q to denote abstract statements, and $P(x)$ to denote an abstract statement depending on the free variable x.

[1] Both of these topics are major reasons why some students find mathematical analysis difficult — and regularly the source of lost marks in the exam. To succeed in this course, it's important to be able to confidently manipulate inequalities, and logical expressions, and to be able to write clearly in a logical fashion.

[2] Context is always important. In one sense I am no more able to fly than a dog but you *know what I mean* when I say that I "fly down to Stansted". Wittgenstein says: "A language is a way of life."

[3] This is the twin prime conjecture, which having been an open question for centuries has recently seen much progress. In 2013 it was proved that there exists a constant $N \in \mathbb{N}$ such that there are infinitely many pairs of primes whose difference is at most N. The first bound on this constant was $N \approx 70$ million, which subsequently could be reduced significantly.

Definition 1.2. *Let P and Q be statements.*

- *The negation of P is the statement that is true when P is false and false when P is true. We write* not(P) *for the negation of P.*
- *The statements $(P$ and $Q)$ and $(P$ or $Q)$*[4] *are defined by the truth tables*

P	Q	$(P$ and $Q)$	$(P$ or $Q)$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

Note how to negate joined statements:

- "not(P or Q)" is equivalent to "not(P) and not(Q)"[5].
- "not(P and Q)" is equivalent to "not(P) or not (Q)".

[4] In mathematics we *always* use the inclusive or: so that "P or Q" is defined to be true when both P and Q are true (as well as when P is true and Q is false, and when Q is true and P is false). This isn't always the case in every day English — if you're asked do you want a baked potato or chips with your steak, it's understood that both is not an allowed option. Be careful when trying to check logic with examples from every day speech.

[5] that is, these two statements have the same truth table (which you should check). We'll discuss equivalence of statements again later.

Implications

Implications are a key logical connective. Again, we combine two statements to produce a new statement.

Definition 1.3. *Let P and Q be statements. The statement $P \implies Q$*[6] *has truth table*

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

The statement $P \implies Q$ can be written in English in a number of ways: "P implies Q", "If P is true, then Q is true", "if P, then Q"[7], "P only if Q" (i.e. "P is true only if Q is true.")[8], "Q whenever P" (i.e. "Q is true whenever P is true"), amongst others.

[6] Note that the symbol $\implies$ is reserved for this meaning. It should only ever be used to connect two mathematical statements, the first of which implies the second. Any other use of $\implies$ is at best a serious mathematical grammar error. In particular, I've noted in homeworks and exams the symbols $=, \to, \implies$ being used interchangeably: each of these has it's own meaning, and needs to be used correctly.

[7] This is essentially shorthand for "If P is true, then Q is true"

[8] It is harder to convince yourself that this is another logically equivalent way of stating the implication. See the discussion on page 67 of [H].

> Note that "$P \implies Q$" is true whenever P is false.

For example "If pigs can fly, then there is a man on the moon" is a true statement as is "If pigs can fly, then there is not a man on the moon".

The crucial point about the truth table for $\implies$ is simply that it disallows us to deduce a falsehood from a truth.

In the implication $P \implies Q$ we call P the *hypothesis* of the implication and Q the *conclusion*. Note that pretty much every mathematical theorem can be stated in the form of an implication, though sometimes the hypotheses are not explicitly stated. For example "There are no real solutions to $x^2 + 2x + 1 = -1$" can be written as "$x \in \mathbb{R} \implies x^2 + 2x + 1 \neq -1$".

Definition 1.4. *Let P and Q be statements and consider the implication $P \implies Q$.*

- *The* converse *of $P \implies Q$ is the implication $Q \implies P$.*
- *The* contrapositive *of $P \implies Q$ is the implication $(\text{not } Q) \implies (\text{not } P)$.*

The implication $P \implies Q$ is equivalent to its contrapositive $(\text{not } Q) \implies (\text{not } P)$[9]. If we want to prove these equivalent statements, we can prove which ever is easiest. It's critical not to get an implication $P \implies Q$ confused with the converse $Q \implies P$. These two statements are not in general equivalent[10]. This is another point where precise mathematical argument often differs from every day speech. Consider the implication "If you don't tidy your room, then you can't go out." From our precise mathematical view point, this statement says nothing about what happens if you do tidy your room, whereas the unspoken assumption is that you can go out if you do tidy your room. In mathematics we must write down what we mean, not leave it to be inferred: the correct statement here is "you can go out if and only if you tidy your room". Always check to make you use an implication in the direction you state it!

[9] To justify this claim, work out the truth table for the contrapositive.

[10] Compare the two truth tables

Negating implications

Looking at the truth table

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

we can see that the implication $P \implies Q$ is false in just the second row of the table, that is when P is true and Q is false. Therefore

"not($P \implies Q$)" is equivalent to "P and not(Q)".

This is another source of errors. In more involved questions I often see the negation of "$P \implies Q$" written as "not $(P) \implies Q$" or "$P \implies$ not (Q). Be very careful when negating an implication to make sure what you've written is genuinely the negation. In particular, if you've ended up with another implication then you've probably made a logical mistake — remember there is only one way $P \implies Q$ can be false: when P is true **and** Q is not.

Equivalent statements

Definition 1.5. *Let P and Q be statements (with the same free variables). Then P and Q are* equivalent *precisely when they have the same truth values. We write $P \iff Q$ in this case.*

Note that $P \iff Q$ is equivalent to $((P \implies Q)$ and $(Q \implies P))$. This can be seen using the truth table, which shows that they have the same truth values.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q)$ and $(Q \Rightarrow P)$	$P \Leftrightarrow Q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

When you want to prove an equivalence $P \iff Q$, you can either provide separate arguments for $P \implies Q$ and $Q \implies P$ or look for an argument proving the equivalence $P \iff Q$ directly.

We have already seen a number of examples of equivalences:

- The implication $P \implies Q$ is equivalent to its contrapositive "not $(Q) \implies$ not (P)".
- The implication $P \implies Q$ is equivalent to the statement "((P and Q) or not(P))"[11].

[11] Check this claim by writing down a truth table.

How many logical connectives on two statements are there? Since we can choose the value of the connective in only two ways but there are four possible arguments, there must be $2^4 = 16$ such functions. We can write out all the possibilities and assign them to the known connectives. Thus (using a bar to denote negation)

P	Q	T	or	$\Longleftarrow$	$\Longrightarrow$	$\overline{and}$	P	Q	$\Longleftrightarrow$	$\overline{\Longleftrightarrow}$	$\bar{Q}$	$\bar{P}$	$\overline{or}$	$\overline{\Longleftarrow}$	$\overline{\Longrightarrow}$	and	F
T	T	T	T	T	T	F	T	T	T	F	F	F	F	F	F	T	F
T	F	T	T	T	F	T	T	F	F	T	T	F	F	F	T	F	F
F	T	T	T	F	T	T	F	T	F	T	F	T	F	T	F	F	F
F	F	T	F	T	T	T	F	F	T	F	T	T	T	F	F	F	F

Note that the list includes a "terminal" function which always returns the value T (and the negation of this function) and projections P and Q onto the first and second arguments (and their negations).

Quantification

We often consider statements containing free variable(s), such as "$x^2 + 2x + 1 = 0$" whose truth depends on the value of the variable x. There are two fundamental ways of *quantifying* the free variable x so as to obtain a statement with no free variables: we can ask for $P(x)$ to be true for every possible value of x, or we can ask for $P(x)$ to be true for at least one value of x. For example $x^2 + 2x + 1 = 0$ is not true for every real value of x, but there is a solution to the equation $x^2 + 2x + 1 = 0$, that is, there exists at least one value of x for which this statement is true.

Definition 1.6. *Let $P(x)$ be a statement conditional on the variable x. We form two quantified statements:*

- *"For all x, $P(x)$ is true", written symbolically "$\forall x, P(x)$". We can express this in a number of ways, for example "For every $x, P(x)$ (is true)", "For each $x, P(x)$".*
- *"There exists x such that $P(x)$ is true", written symbolically "$\exists x$ s.t. $P(x)$".*

$\forall$ is a called *universal quantifier* and $\exists$ is called *existential quantifier*.

When working with universal or existential quantifiers, we should be clear what the range of possible x we are considering is. For example "for all x, $x^2 \neq -1$" is true when we consider the variable x to be ranging over the real numbers and false if we consider x to be ranging over the complex numbers. We make this more precise, by writing quantified statements like "$\forall x \in X, P(x)$" for the statement "for all $x \in X, P(x)$", and similarly for existential quantifiers[12]. Here X is the set of possible values of the variable x. The statement "$\exists x \in \mathbb{Q}$ s.t. $x^2 = 2$" is unambiguous (and false), whereas in $\exists x$ s.t. $x^2 = 2$ it is not clear from the context what are the allowed values of x. We should always specify the range of a quantification unless it is completely clear from the context.

[12] Recall the membership symbol $\in$. Make sure you only use it as allowed in that course; $a \in A$ means that a is a member of the set A.

> In this course, we shall take the convention that $x > 0$ means that x is a real number and $x > 0$, and write quantified statements like "$\forall x > 0, P(x)$" to mean "For all strictly positive real numbers $x, P(x)$ is true".

You're free to relabel quantified variables in statements: "$\forall x, P(x)$" is equivalent to "$\forall y, P(y)$" as both claim that the statement $P(\cdot)$ is true no matter what the value of the variable is.

This is often particularly important when working with multiple statements. Suppose we know that $\exists N \in \mathbb{N}$ s.t. $P(N)$ and $\exists N \in \mathbb{N}$ s.t. $Q(N)$ are both true. This does not mean that there is an individual $N \in \mathbb{N}$ such that $P(N)$ and $Q(N)$ are both true[13]. If we want to use our hypotheses we must introduce two different symbols, and take $N_1 \in \mathbb{N}$ such that $P(N_1)$ is true and $N_2 \in \mathbb{N}$ such that $Q(N_2)$ is true, we can't just use N for both[14].

[13] For example, let $P(n)$ be "n is even" and $Q(n)$ be "n is odd."

[14] I'll give concrete examples of errors caused by this later.

We can combine quantifiers to form more complicated statements. For example if $S \subseteq \mathbb{R}$, the statement that S is bounded above[15] is

$$\exists M \in \mathbb{R} \text{ s.t. } \forall x \in S, x \leq M. \tag{1}$$

[15] We'll make this the definition that S is bounded above in the next chapter.

The order of the quantifiers matters: the statement above has been built up as follows:

- starting with the statement "$x \leq M$" in two free (real) variables x and M;
- we quantify over $x \in S$ to form "for all $x \in S$, $x \leq M$" a statement in one free variable M;
- finally we quantify over $M \in \mathbb{R}$ to obtain $\exists M \in \mathbb{R}$ s.t. $\forall x \in S, x \leq M$.

Decomposing statements in this fashion can help in understanding what is going on. For instance, "for all $x \in S, x \leq M$" is true when M is greater than or equal to every element in S. Thus the full statement "$\exists M \in \mathbb{R}$ s.t. $\forall x \in S, x \leq M$" is true when there is a real number M which is greater than or equal to every element in S. When $S = [1, 3]$

the statement is true: we can take $M = 3$, or $M = \pi$, $M = 4, \ldots$, whereas if $S = \mathbb{N}$ the statement is false — there is no real number greater than or equal to every natural number[16]. If you find yourself confused by a long quantified statement, try to understand it piece by piece, *starting at the right*.

Note that the order we perform quantification matters[17]. Continuing with the previous example, the statement

$$\forall x \in S, \exists M \in \mathbb{R}, x \leq M \tag{2}$$

is not always equivalent to "$\exists M \in \mathbb{R}$ s.t. $\forall x \in S, x \leq M$." To see this think about what the statement (2) means. For any real value x, the statement "$\exists M \in \mathbb{R}, x \leq M$" is true — the point is that M is allowed to depend on x, so we can take $M = x + 1$. Thus we have proved that "$\forall x \in \mathbb{R}, \exists M \in \mathbb{R}$ s.t. $x \leq M$". In particular (2) will be true no matter what the subset $S \subseteq \mathbb{R}$ is. The key difference between (1) and (2) is that in (2) the value of M is allowed to depend on x as M is specified after x in the statement; whereas in (1) the value of M must be specified first, and for the statement to be true, this value of M must be greater than or equal to every element of S. You may find it helpful to record the possible dependence of M on x in (2) explicitly and write this statement as

$$\forall x \in S, \exists M_x \in \mathbb{R}, x \leq M_x.$$

[16] This rather obvious looking statement is actually not as obvious as it looks. It is known as Archimedes axiom, and we'll come back to this in the next chapter.

[17] You can interchange the order of quantifiers of the same type: $\forall x, \forall y, P(x,y)$ and $\forall y, \forall x P(x,y)$ are equivalent; similarly $\exists x$ s.t. $\exists y$ s.t. $P(x,y)$ and $\exists y$ s.t. $\exists x$ s.t. $P(x,y)$ are equivalent. We write these statements as $\forall x, y, P(x,y)$ and $\exists x, y$ s.t. $P(x,y)$ respectively.

Negating Quantified Statements

The statement $\forall x \in X, P(x)$ is defined to be true precisely when $P(x)$ is true for every $x \in X$. Therefore the statement $\forall x \in X, P(x)$ is false precisely when there is at least one $x \in X$ such that $P(x)$ is false[18]. This gives

$$\boxed{\text{not}(\forall x \in X, P(x)) \text{ is equivalent to } \exists x \in X \text{ s.t. not}(P(x)).} \tag{3}$$

Similarly,

$$\boxed{\text{not}(\exists x \in X \text{ s.t. } P(x)) \text{ is equivalent to } \forall x \in X, \text{not}(P(x)).} \tag{4}$$

We negate nested quantified statements by applying these two procedures in turn.

Example To find the negation of (1), we proceed by negating the quantifiers in turn. The statement (1) is "$\exists M \in \mathbb{R}$ s.t. $\forall x \in S, x \leq M$" and so can be written "$\exists M \in \mathbb{R}$ s.t. $P(M)$", where $P(M)$ is the statement "$\forall x \in S, x \leq M$". The first equivalence below obtained by

[18] Note that we do not change the range of quantification. An error I've seen many times is to write "$\exists \varepsilon \leq 0$ s.t. not $P(\varepsilon)$" for the negation of "$\forall \varepsilon > 0, P(\varepsilon)$". The range of quantification of $\forall \varepsilon > 0, P(\varepsilon)$ is all strictly positive real numbers. This range does not change: if it is not true that for every strictly positive ε, $P(\varepsilon)$ is true, then there must be at least one strictly positive ε such that $P(\varepsilon)$ is not true.

applying (4) to "not($\exists M \in \mathbb{R}$ s.t. $P(M)$", and the second by applying (3) to find an expression for "not $P(M)$". We have[19]

$$\begin{aligned} & \text{not}(\exists M \in \mathbb{R} \text{ s.t. } \forall x \in S, x \leq M) \\ \Leftrightarrow & \forall M \in \mathbb{R}, \text{not}(\forall x \in S, x \leq M) \\ \Leftrightarrow & \forall M \in \mathbb{R},\ \exists x \in S \text{ s.t. not}(x \leq M) \\ \Leftrightarrow & \forall M \in \mathbb{R},\ \exists x \in S \text{ s.t. } x > M. \end{aligned}$$

[19] It is not necessary to include all these steps if you are asked to write down a negated form of a quantified statement. What matters is that you can do so accurately, as you'll often need to do this in order to work out what needs to be proved.

Take particular care when negating complicated quantified statements which involve implications.

Direct proofs of quantified statements

Suppose we are given a quantified statement

$$\forall a \in A, \exists b \in B \text{ s.t. } \forall c \in C, P(a,b,c). \tag{5}$$

What structure should a direct proof of this statement take[20]? The statement (5) makes a claim about all $a \in A$; we must prove "$\exists b \in B$ s.t. $\forall c \in C, P(a,b,c)$" for every value of a. We start our proof by fixing the symbol a, to indicate to the reader that it represents an arbitrary element of A (but won't change in the rest of the argument). I would open the formal proof with "Let $a \in A$ be arbitrary".

[20] We will see that the statement $\lim_{n\to\infty} x_n = L$ is exactly of this form, so it's a very relevant example for this course.

With our fixed value of a, we now have to prove "$\exists b \in B$ s.t. $\forall c \in C, P(a,b,c)$", so we should show the reader that there is some $b \in B$ which has the property "$\forall c \in C, P(a,b,c)$." Note that the b is allowed to depend on a, but the same b must work for all values of $c \in C$, i.e. we can think of b as a function of a. One way to do this, is to state a formula, or expression for b, in terms of a[21]. You'll probably need to find such a condition by means of a load of rough computations, which don't need to appear in your final proof; instead the next step of the formal proof, would be to take an arbitrary $c \in C$ and then carefully check that $P(a,b,c)$ is true. Thus the proof has the following structure.

[21] This isn't the only way to show things exist; there are some beautiful non-constructive existence proofs in mathematics which demonstrate that certain objects exist without exhibiting them. You'll see some of these in future courses, but in this course, most of the time we need to show some thing exists, we'll be able to find some inequality, and then take anything satisfying this inequality.

- "Let $a \in A$ be arbitrary,"
- "Define $b \in B$ to be (some expression in terms of a", or perhaps "Take $b \in B$ to satisfy (some condition in terms of a)" where you can see the condition has at least one solution $b \in B$. For example, if B is the set $\mathbb{N}$ and a is a real number, a typical condition might be $b > a^2$, which we know has a solution[22].
- "Let $c \in C$ be arbitrary."
- Proof that $P(a,b,c)$ is true.

[22] This looks very obvious; it will actually be a theorem, known as Archimedes axiom, in the next chapter, that it is always possible to find a natural number greater than a specified real number.

Let's see this in practice, with an example[23] we'll come back to in Chapter 2.

[23] albeit one with only two quantifiers.

Example 1.7. *Prove*

$$\forall \varepsilon > 0, \exists n \in \mathbb{N} \text{ s.t. } \frac{3n}{n+1} > 3 - \varepsilon.$$

We start our proof with "Let $\varepsilon > 0$ be arbitrary." Now we need to work out what value of $n \in \mathbb{N}$ we should take in terms of ε. Let's look at the condition that we need to satisfy and work out when it's true. We have

$$\frac{3n}{n+1} > 3 - \varepsilon \Leftrightarrow \varepsilon(n+1) > 3 \Leftrightarrow n > \frac{3}{\varepsilon} - 1.$$

Thus we should take any $n \in \mathbb{N}$ satisfying this last inequality. The point is that we know this last inequality has solutions in the natural numbers; this is not so obvious for the first form of the inequality.[24] I might write "Take $n \in \mathbb{N}$ with $n > 3/\varepsilon - 1$" as the next line of my proof, and then say "for such an n, we have $(n+1)\varepsilon > 3$ and hence $\frac{3n}{n+1} > 3 - \varepsilon$, as required" to complete my proof.

[24] If you end up with an inequality of the form "$n < K$" it could be that something has gone wrong. In particular, if K is negative, this will not in general have solutions. If something like this happens in one of your answers, look carefully to see if you have multiplied an inequality by a negative number without reversing the direction of the inequality.

Proof. Let $\varepsilon > 0$ be arbitrary. Take $n \in \mathbb{N}$ with $n > 3/\varepsilon - 1$. For such an n, we have $(n+1)\varepsilon > 3$ and hence $\frac{3n}{n+1} > 3 - \varepsilon$, as required. □

This isn't the only way such a proof can be written. You might think it is better to keep the indication of how the n was found, and write[25]:

[25] If you use the structure below, make sure you get the logic the 'right way round'. A standard error is to 'solve the inequality' and write

$$\begin{aligned} &\frac{3n}{n+1} > 3 - \varepsilon \\ \Longrightarrow &\varepsilon(n+1) > 3 \\ \Longrightarrow &n > \frac{3}{\varepsilon} - 1. \end{aligned}$$

and then say "So take $n > \frac{3}{\varepsilon} - 1$". Of course the implications above are correct; they're just not what is being used. What matters in this proof is that the implication $n > \frac{3}{\varepsilon} - 1 \Longrightarrow \frac{3n}{n+1} > 3 - \varepsilon$ is true — we can see that there is an n with $n > \frac{3}{\varepsilon} - 1$ and hence there is an n with $\frac{3n}{n+1} > 3 - \varepsilon$. If you are using the implication $n > \frac{3}{\varepsilon} - 1 \Longrightarrow \frac{3n}{n+1} > 3 - \varepsilon$ make sure you state this implication, not it's converse.

Proof. Let $\varepsilon > 0$ be arbitrary. For $n \in \mathbb{N}$, we have

$$\frac{3n}{n+1} > 3 - \varepsilon \Leftrightarrow \varepsilon(n+1) > 3 \Leftrightarrow n > \frac{3}{\varepsilon} - 1.$$

Therefore take $n \in \mathbb{N}$ with $n > \frac{3}{\varepsilon} - 1$, so that this n satisfies $\frac{3n}{n+1} > 3 - \varepsilon$. □