Algorithmic Foundations 2 - Tutorial Sheet 1

Propositional Logic and Logical Equivalences

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1. Which of the following are propositions and what are the truth values of those that are propositions?

   (a) Do not pass go

   > **Solution:** Not a proposition - this is a command

   (b) What time is it?

   > **Solution:** Not a proposition - this is a question

   (c) Glasgow is the largest city in Scotland

   > **Solution:** A proposition which is `true`

   (d) $4+x = 5$

   > **Solution:** Not a proposition - the truth value depends on the value of x (for example, if $x = 1$, then the statement is `true` and is `false` for any other value of x)

   (e) $4+1 = 6$

   > **Solution:** A proposition which is `false`

2. Let p, q and r be the propositions:

   - p: you have the flu;
   - q: you miss the final exam;
   - r: you pass the course.

3. Express each of the following propositions as an English sentence

   (a) $p \rightarrow q$

   > **Solution:** If you have the flu, then you miss the final exam.

   (b) $\neg q \leftrightarrow r$

   > **Solution:** You do not miss the final exam if and only if you pass the course.

   (c) $q \rightarrow \neg r$

   > **Solution:** If you miss the final exam, then you do not pass the course.

   (d) $p \vee q \vee r$

Solution: Either you have the flu, or you miss the final exam, or you pass the course.

(e) $(p \to \neg r) \vee (q \to \neg r)$

Solution: Either it holds that if you have the flu, then you do not pass the course, or it holds that if you miss the final exam, then you do not pass the course (or both).

(f) $(p \wedge q) \vee (\neg q \wedge r)$

Solution: Either you have the flu and miss the final exam, or you do not miss the final exam and pass the course.

4. Let p, q and r be the propositions:

- p: you get an A in the final exam;
- q: you do every tutorial exercise;
- r: you get an A for this module.

Write the following propositions using p, q, r and logical connectives.

(a) You get an A for this module, but you do not do every tutorial exercise.

Solution: $r \wedge \neg q$

(b) You get an A in the final exam, you do every tutorial exercise and you get an A for this module.

Solution: $p \wedge q \wedge r$

(c) To get an A for this module, it is necessary for you to get an A in the final exam.

Solution: $r \to p$

(d) You get an A in the final exam, but you do not do every tutorial exercise; nevertheless, you get an A for this module.

Solution: $p \wedge \neg q \wedge r$

(e) Getting an A in the final exam and doing every tutorial exercise is sufficient for getting an A for this module.

Solution: $(p \wedge q) \to r$

(f) You get an A for this module if and only if you either do every tutorial exercise or you get an A in the final exam.

Solution: $r \leftrightarrow (q \vee p)$

5. Construct a truth table for each of the following compound propositions.

(a) $\neg p \oplus \neg q$

Solution:

p	q	$\neg p$	$\neg q$	$\neg p \oplus \neg q$
0	0	1	1	0
0	1	1	0	1
1	0	0	1	1
1	1	0	0	0

(b) $(p \vee q) \wedge \neg r$

Solution:

p	q	r	$p \vee q$	$\neg r$	$(p \vee q) \wedge \neg r$
0	0	0	0	1	0
0	0	1	0	0	0
0	1	0	1	1	1
0	1	1	1	0	0
1	0	0	1	1	1
1	0	1	1	0	0
1	1	0	1	1	1
1	1	1	1	0	0

(c) $(p \to q) \to r$

Solution:

p	q	$p \to q$	r	$(p \to q) \to \neg r$
0	0	1	0	0
0	0	1	1	1
0	1	1	0	0
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	1	1	0	0
1	1	1	1	1

(d) $(p \wedge \neg q) \leftrightarrow (p \wedge r)$

Solution:

p	q	$\neg q$	$p \wedge \neg q$	r	$p \wedge r$	$(p \wedge \neg q) \leftrightarrow (p \wedge r)$
0	0	1	0	0	0	1
0	0	1	0	1	0	1
0	1	0	0	0	0	1
0	1	0	0	1	0	1
1	0	1	1	0	0	0
1	0	1	1	1	1	1
1	1	0	0	0	0	1
1	1	0	0	1	1	0

6. Show that each of the following implications is a tautology by using truth tables.

   (a) $(p \wedge q) \to q$

Solution:

p	q	$p \wedge q$	$(p \wedge q) \rightarrow q$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	1

(b) $(\neg p \wedge (p \vee q)) \rightarrow q$

Solution:

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$(\neg p \wedge (p \vee q)) \rightarrow q$
0	0	1	0	0	1
0	1	1	1	1	1
1	0	0	1	0	1
1	1	0	1	0	1

(c) $(p \wedge (p \rightarrow q)) \rightarrow q$

Solution:

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$(p \wedge (p \rightarrow q)) \rightarrow q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

7. Show that each implication in Exercise 6 is a tautology without using truth tables.

(a) $(p \wedge q) \rightarrow q$

Solution:

$$\begin{aligned}
(p \wedge q) \rightarrow q &\equiv \neg(p \wedge q) \vee q && \text{implication law} \\
&\equiv (\neg p \vee \neg q) \vee q && \text{de Morgan law} \\
&\equiv \neg p \vee (\neg q \vee q) && \text{associative law} \\
&\equiv \neg p \vee (q \vee \neg q) && \text{commutative law} \\
&\equiv \neg p \vee \texttt{true} && \text{tautology law} \\
&\equiv \texttt{true} && \text{domination law}
\end{aligned}$$

Alternative solution: If the hypothesis $p \wedge q$ is `false`, then the overall proposition is `true`. Therefore it remains to consider the case when the hypothesis is `true`. In this case, both p and q must be `true`, and hence the conclusion of the proposition is also `true`, again making the overall proposition `true`.

(b) $(\neg p \wedge (p \vee q)) \rightarrow q$

Solution:

$$
\begin{aligned}
(\neg p \wedge (p \vee q)) \to q &\equiv \neg(\neg p \wedge (p \vee q)) \vee q && \text{implication law}\\
&\equiv (\neg\neg p \vee \neg(p \vee q)) \vee q && \text{de Morgan law}\\
&\equiv (p \vee \neg(p \vee q)) \vee q && \text{double negation law}\\
&\equiv (p \vee (\neg p \wedge \neg q)) \vee q && \text{de Morgan law}\\
&\equiv ((p \vee \neg p) \wedge (p \vee \neg q)) \vee q && \text{distributive law}\\
&\equiv (\texttt{true} \wedge (p \vee \neg q)) \vee q && \text{tautology law}\\
&\equiv (p \vee \neg q) \vee q && \text{identity law}\\
&\equiv p \vee (\neg q \vee q) && \text{associative law}\\
&\equiv p \vee (q \vee \neg q) && \text{commutative law}\\
&\equiv p \vee \texttt{true} && \text{tautology law}\\
&\equiv \texttt{true} && \text{domination law}
\end{aligned}
$$

Alternative solution: If the hypothesis $\neg p \wedge (p \vee q)$ is `false`, then the overall proposition is `true`. Therefore, it remains to consider the case when the hypothesis is `true`. In this case, $\neg p$ must be `true`, so p is `false`. Since p is `false` and $(p \vee q)$ must be `true` for the hypothesis to hold, we have that q must be `true`. Hence the conclusion of the proposition is `true`, making the overall proposition `true`.

(c) $(p \wedge (p \to q)) \to q$

Solution:

$$
\begin{aligned}
(p \wedge (p \to q)) \to q &\equiv (p \wedge (\neg p \vee q)) \to q && \text{implication law}\\
&\equiv ((p \wedge \neg p) \vee (p \wedge q)) \to q && \text{distributive law}\\
&\equiv (\texttt{false} \vee (p \wedge q)) \to q && \text{contradiction law}\\
&\equiv (p \wedge q) \to q && \text{identity law}\\
&\equiv \neg(p \wedge q) \vee q && \text{implication law}\\
&\equiv (\neg p \vee \neg q) \vee q && \text{de Morgan law}\\
&\equiv \neg p \vee (\neg q \vee q) && \text{associative law}\\
&\equiv \neg p \vee (q \vee \neg q) && \text{commutative law}\\
&\equiv \neg p \vee \texttt{true} && \text{tautology law}\\
&\equiv \texttt{true} && \text{domination law}
\end{aligned}
$$

Alternative solution: If the hypothesis $p \wedge (p \to q)$ is `false`, then the overall proposition is `true`. Therefore it remains to consider the case when the hypothesis is `true`. In this case, we have both p and $p \to q$ are `true` which forces q to also be `true`. Hence the conclusion of the proposition is `true`, making the overall proposition `true`.

8. State the converse and contrapositive of each of the following implications:

(a) If it snows tonight, then I will stay at home.

Solution: *Converse* $(q \to p)$: If I stay at home, then it will snow tonight.
Contrapositive $(\neg q \to \neg p)$: If I do not stay at home, then it will not snow tonight.

(b) I go to the beach whenever it is a sunny summer day.

> **Solution:** *Converse* $(q \to p)$: If I go to the beach, then it is a sunny summer day.
> *Contrapositive* $(\neg q \to \neg p)$: If I do not go to the beach, then it is not a sunny summer day.

(c) When I stay up late, it is necessary that I sleep until noon.

> **Solution:** *Converse* $(q \to p)$: If I sleep until noon, then I stay up late.
> *Contrapositive* $(\neg q \to \neg p)$: If I do not sleep until noon, then I do not stay up late.

Difficult/challenging questions.

9. Suppose that a truth table in n Boolean variables is specified. Show that a compound proposition with this truth table can be formed by taking the disjunction of conjunctions of the variables or their negations. (Hint: there should be one conjunction included for each combination of values for which the proposition is true.) Note that the resulting compound proposition is said to be in disjunctive normal form.

> **Solution:** Each line of the truth table corresponds to exactly one combination of truth values for the n Boolean variables involved. We can write down a conjunction that is `true` precisely in this case, namely the conjunction of all the Boolean variables that are `true` together with the negation of all the Boolean variables that are `false`. If we do this for each line of the truth table for which the value of the compound proposition is `true` and take the disjunction of the resulting propositions, then we have the desired proposition in its disjunctive normal form. For example, consider the following truth table involving three Boolean variables, p, q and r:

p	q	r	truth value	corresponding conjunction
0	0	0	0	-
0	0	1	1	$\neg p \land \neg q \land r$
0	1	0	0	-
0	1	1	0	-
1	0	0	1	$p \land \neg q \land \neg r$
1	0	1	0	-
1	1	0	0	-
1	1	1	1	$p \land q \land r$

The desired proposition in *Disjunctive Normal Form* is therefore:

$$(\neg p \land \neg q \land r) \lor (p \land \neg q \land \neg r) \lor (p \land q \land r).$$

Recall from the lecture that, due to the associativity laws, there is no ambiguity in writing a sequence of conjunctions (or disjunctions) as $p_1 \land p_2 \land \cdots \land p_n$ without parentheses.

10. The proposition $p\,NOR\,q$ is true when both p and q are false and is denoted by the formula $p \downarrow q$. Write the truth table for the logical operator $\downarrow$ and then find a logical proposition equivalent to $p \to q$ using only the operator $\downarrow$.

If you want a further exercise express all logical connectives using just $\downarrow$ without using the logical equivalences you have been given.

Hint. First consider constructing the formula $\neg(p \to q)$, second consider how to define negation using only $\downarrow$ and finally combine these results.

Solution:

p	q	$p \downarrow q$
0	0	1
0	1	0
1	0	0
1	1	0

and a formula equivalent to $p \to q$ is $((p \downarrow p) \downarrow q) \downarrow ((p \downarrow p) \downarrow q)$.

The remaining operators can be expressed as follows:

$$\begin{array}{rcl}
\neg p & \equiv & p \downarrow p \\
p \wedge q & \equiv & (p \downarrow p) \downarrow (q \downarrow q) \\
p \vee q & \equiv & (p \downarrow q) \downarrow (p \downarrow q) \\
p \oplus q & \equiv & (p \downarrow q) \downarrow ((p \downarrow p) \downarrow (q \downarrow q)) \\
p \leftrightarrow q & \equiv & (p \downarrow p) \downarrow q) \downarrow ((q \downarrow q) \downarrow p)
\end{array}$$