FB1:

Using the definition of the derivative find the derivative of the function

$$\boldsymbol{f(x) = x^{5/2}}.$$

State the domain of the function and the domain of its derivative.

The definition of the derivative of a function is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

When applied to the given function, its derivative can be expressed as

$$\lim_{h\to 0} \frac{\sqrt{(x+h)^5} - \sqrt{x^5}}{h},$$

and it can be simplified by multiplying both the numerator and the denominator with the numerator's expression with an inverted sign, which is the same as multiplying by 1:

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{(x+h)^5} - \sqrt{x^5}}{h} \cdot \frac{\sqrt{(x+h)^5} + \sqrt{x^5}}{\sqrt{(x+h)^5} + \sqrt{x^5}} =$$
$$= \lim_{h\to 0} \frac{(x+h)^5 - x^5}{h\left(\sqrt{(x+h)^5} + \sqrt{x^5}\right)} =$$
$$= \lim_{h\to 0} \frac{x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5 - x^5}{h\left(\sqrt{(x+h)^5} + \sqrt{x^5}\right)}$$

Because both the numerator and the denominator are multiples of *h*, they can be removed from the expression by dividing both by *h*.

$$\lim_{h\to 0} \frac{5x^4 + 10x^3h + 10x^2h^2 + 5xh^3 + h^4}{\left(\sqrt{(x+h)^5} + \sqrt{x^5}\right)} =$$
$$= \frac{5x^4}{2\sqrt{x^5}} =$$
$$= \frac{5}{2}x^{\frac{3}{2}}$$

The domains for both the function *f(x)* and its derivative *f'(x)* are $x \in [0, +\infty)$ because the square root is defined only for non-negative numbers in the real set.

FB2: For $\boldsymbol{z} \in$ C, find all solutions to the equality $\boldsymbol{z^6 = 1 + \sqrt{3}i}$

To find the solutions, it would be easier to express the right-hand side in polar form. This can be done by finding the modulus and the argument of the complex number. The modulus *r* is

$$r = \sqrt{1^2 + \sqrt{3}^2} = \sqrt{4} = 2.$$

The argument θ can be found as the angle the complex number makes in an Argand diagram:

$$\theta = arctan\frac{\sqrt{3}}{1} = \frac{\pi}{3}$$

Therefore, the equality can be restated as

$$z^6 = 2e^{\frac{\pi}{3}i}.$$

By taking the sixth root of both sides, the result is

$$z = 2^{\frac{1}{6}}e^{\frac{\pi}{18}i}.$$

All of the other roots of the equality can be found by multiplying the obtained root by sixth roots of unity, which themselves can be found by finding the sixth roots of 1. The first sixth root of unity *w* is therefore

$$w = 1^{\frac{1}{6}} = e^{\frac{2\pi}{6}i} = e^{\frac{\pi}{3}i}.$$

Additionally, all of the roots of the given equality can be expressed as

$$zw^0, zw, zw^2, zw^3, zw^4, zw^5,$$

which evaluates to

$$2^{\frac{1}{6}}e^{\frac{\pi}{18}i}, 2^{\frac{1}{6}}e^{\frac{7\pi}{18}i}, 2^{\frac{1}{6}}e^{\frac{13\pi}{18}i}, 2^{\frac{1}{6}}e^{\frac{19\pi}{18}i}, 2^{\frac{1}{6}}e^{\frac{25\pi}{18}i}, 2^{\frac{1}{6}}e^{\frac{31\pi}{18}i}.$$

FB3: For *x*, *y* ∈ $\mathbf{R}$, prove that $\big||x| - |y|\big| \leq |x - y|$, (Hint: consider $x = y + (x - y)$).

The inequality can be proven most easily by squaring both sides, which can be done because both sides of the inequality are positive; therefore, no solutions may be lost because the square roots of the squares would be the same as the given expressions.

$$\big||x| - |y|\big|^2 \leq |x - y|^2$$

$$\Rightarrow (|x| - |y|)^2 \leq (x - y)^2$$
$$\Rightarrow |x|^2 - 2|x||y| + |y|^2 \leq x^2 - 2xy + y^2$$

Furthermore, $|x|^2$ is the same as x^2 because $(-x)^2 = x^2$, so the inequality can be expressed as

$$x^2 - 2|x||y| + y^2 \leq x^2 - 2xy + y^2.$$

It can be simplified further by subtracting $(x^2 + y^2)$ from both sides and then dividing by (-2), which reverses the inequality because (-2) is a negative multiplier.

$$|x||y| \geq xy$$

Additionally, the product of two moduli is itself a modulus, which changes the expression to

$$|xy| \geq xy.$$

The resulting inequality is always true for any product *xy* because there are two different cases: 1) *xy* is non-negative and its modulus is the same value, or 2) *xy* is negative and the modulus (a positive value) is greater than *xy*. Hence, the given inequality $\big||x| - |y|\big| \leq |x - y|$ is proven true.