NABLA IDENTITIES

$$\left.\begin{matrix}\text{grad}\\ \text{div}\\ \text{curl}\end{matrix}\right\}(a+b) = \begin{matrix}\text{grad}(a)+\text{grad}(b)\\ \text{div}(a)+\text{div}(b)\\ \text{curl}(a)+\text{curl}(b)\end{matrix}$$

<u>likewise</u> $\text{grad}(cf) = c\cdot\text{grad}(f)$

$\text{div}(cF) = c\cdot\text{div}(F)$

<u>Product rules:</u> $\nabla(f\cdot g) = \nabla f\cdot g + f\cdot\nabla g$ (gradient)

$$\text{div}(fF) = \nabla\cdot(\underset{\text{scalar field}}{f}\ \underset{\text{vect. field}}{F}) = \nabla f\cdot F + f\,\nabla\cdot F = \text{grad}\, f\cdot F + f\cdot\text{div}\, F \quad (*)$$

$$\text{curl}(f\cdot F) = \nabla\times(fF) = \nabla f\times F + f(\nabla\times F) =$$

<u>Proof of (*)</u> : $\text{div}(fF) = \nabla\cdot(fF) = \frac{\partial}{\partial x_1} fF_1 + \frac{\partial}{\partial x_2} fF_2 + \dots + \frac{\partial}{\partial x_n} fF_n =$

$$= \left(f\cdot\frac{\partial F_1}{\partial x_1} + F_1\cdot\frac{\partial f}{\partial x_1}\right) + \cdots + \left(f\cdot\frac{\partial F_n}{\partial x_n} + F_n\cdot\frac{\partial f}{\partial x_n}\right)$$

$$= f\cdot\left(\frac{\partial F_1}{\partial x_1} + \dots + \frac{\partial F_n}{\partial x_n}\right) + \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)\cdot(F_1, \dots, F_n)$$

$$= f\cdot\nabla\cdot F + \nabla f\cdot F$$

EXAMPLE 3.10

$f: \mathbb{R}^3 \to \mathbb{R}^3$ with continuous 2nd partial derivatives:

① $\operatorname{curl} \operatorname{grad} f = 0$; $\operatorname{curl}(fF) = f \cdot \operatorname{curl} F + \operatorname{grad} f \times F$; $\operatorname{div}(fF) = f \operatorname{div} F + (\operatorname{grad} f) \cdot F$

$\nabla \times (fF) = f \cdot (\nabla \times F) + \nabla f \times F$ $\qquad$ $\nabla \cdot (fF) = f \cdot (\nabla \cdot F) + \nabla f \cdot F$

Sol. ①

Clairaut's thm

$$\operatorname{curl} \operatorname{grad} f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = (f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy}) =$$

vector product between ∇ & $\downarrow$.

$$= (0,0,0) = \underline{0}.$$

EXAMPLE 3.11

$\underline{r} = (x,y,z) \in \mathbb{R}^3$

$r = |\underline{r}| = \sqrt{x^2+y^2+z^2}$

$\underline{c} \in \mathbb{R}^3$ vector

(i) $\operatorname{div}(r^n \cdot (\underline{c} \times \underline{r}))$

(ii) $\operatorname{curl}(r^n (\underline{c} \times \underline{r}))$

(i) $$\operatorname{div}(r^n(\underline{c}\times\underline{r})) = \underbrace{\nabla r^n}_{(a)} \cdot \underbrace{(\underline{c}\times\underline{r})}_{(b)} + r^n \cdot \underbrace{(\nabla\cdot(\underline{c}\times\underline{r}))}_{(c)} = n \cdot r^{n-2} \cdot \underline{r} \cdot \overbrace{(\underline{c}\times\underline{r})}^{\text{vector } \perp \text{ to } \underline{c} \,\&\, \underline{r}} + 0 = 0 + 0 = 0$$

$$\nabla \cdot (\overbrace{f}\ \overbrace{F})$$

$$= 0$$

(a) ∇r^n was computed on Ex 3.2: $\underbrace{f(\underline{r}) = f(x,y,z) = \phi(r)}_{\text{setting Ex. 3.2.}}$ $\rightsquigarrow$ $\nabla\phi(r) = \dfrac{\phi'(r)}{r}(x,y,z)$.

In our setting $\phi(r) = r^n \rightsquigarrow \nabla r^n = \dfrac{n \cdot r^{n-1}}{r} \cdot \underline{r} = n \cdot r^{n-2}\,\underline{r}$

(b) $$\underline{c}\times\underline{r} = \begin{vmatrix} i & j & k \\ c_1 & c_2 & c_3 \\ x & y & z \end{vmatrix} = (c_2 z - y c_3,\ x c_3 - c_1 z,\ c_1 y - x c_2)$$

(c) $$\nabla \cdot (\underline{c}\times\underline{r}) = \frac{\partial}{\partial x}(c_2 z - y c_3) + \frac{\partial}{\partial y}(x c_3 - c_1 z) + \frac{\partial}{\partial z}(c_1 y - x c_2) = 0.$$

(ii) $\operatorname{curl}(r^n(\underline{c}\times\underline{r})) = \underbrace{\nabla r^n}_{n\cdot r^{n-2}\cdot\underline{r}} \times (\underline{c}\times\underline{r}) + r^n\,\nabla\times(\underline{c}\times\underline{r})$ $\circled{=}$

$\nabla \times (\underbrace{\;}_{\;}\;\; \underbrace{F}_{\;})$

vector triple product

$a\times(b\times c) = (\underline{a}\cdot\underline{c})\cdot\underline{b} - (\underline{a}\cdot\underline{b})\cdot\underline{c}$

$* \; \nabla r^n \times (\underline{c}\times\underline{r}) = n\cdot r^{n-2}\cdot\underline{r}\times(\underline{c}\times\underline{r}) = n\cdot r^{n-2}\,((\underbrace{\underline{r}\cdot\underline{r}}_{r^2})\underline{c} - (\underline{r}\cdot\underline{c})\underline{r})$

$* \; \operatorname{curl}(\underline{c}\times\underline{r}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ c_2 z - y c_3 & x c_3 - c_1 z & c_1 y - x c_2 \end{vmatrix} = (c_1 + c_1, c_2 + c_2, c_3 + c_3) = 2\underline{c}$

$= n\cdot r^{n-2}\cdot(r^2\underline{c} - (\underline{r}\cdot\underline{c})\cdot\underline{r}) + r^n 2\underline{c}$

$= n\,r^n\cdot\underline{c} - n\cdot r^{n-2}\cdot(\underline{r}\cdot\underline{c})\underline{r} + r^n 2\underline{c} = (n+2)r^n\cdot\underline{c} - n\cdot r^{n-2}\cdot(\underline{r}\cdot\underline{c})\cdot\underline{r}$