Mathematics 2A, brief summary. Please note, **not** a complete description of the course contents.

Chapter 1

The graph of a function of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is a subset of $\mathbb{R}^3$ with graph$(f) = \{(x, y, z)\,;\,(x, y) \in D,\ z = f(x, y)\}$, this set is a surface. A *cross-section* of a surface is the intersection of the surface with a plane. Level-curves are obtained by particular cross-sections $z = c$.

Partial derivatives of $f : \mathbb{R}^n \to \mathbb{R}$,

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, x_2, \ldots, x_i + h, \ldots, x_n) - f(x_1, x_2, \ldots, x_i, \ldots, x_n)}{h}$$

this is "differentiation with respect to one argument, treating the others as fixed". For higher order derivatives we have Clairaut's theorem regarding the symmetry of mixed partial derivatives. The chain rule takes specific forms for certain compositions, for example with $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ we have the composition $(f \circ g) : \mathbb{R}^n \to \mathbb{R}$ with $(f \circ g)(\mathbf{x}) = f(g(\mathbf{x}))$ and

$$\frac{\partial}{\partial x_i}(f \circ g) = \frac{\partial g}{\partial x_i} f'(g(\mathbf{x})).$$

For compositions of $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{g} : \mathbb{R} \to \mathbb{R}^n$ we have $(f \circ \mathbf{g}) : \mathbb{R} \to \mathbb{R}$ with $(f \circ \mathbf{g})(t) = f(\mathbf{g}(t))$ and

$$\frac{d}{dt}(f \circ \mathbf{g}) = \mathbf{g}'(t) \cdot \nabla f(\mathbf{g}(t)) = g_1'(t)\frac{\partial f}{\partial x_1} + \ldots + g_n'(t)\frac{\partial f}{\partial x_n}.$$

For compositions of $F(u, v)$ and $u(x, y)$ and $v(x, y)$ defined by $f(x, y) = F(u(x, y), v(x, y))$ we have

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial F}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial F}{\partial v}, \qquad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial F}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial F}{\partial v}$$

this is called a change of variable and can be used to solve PDEs.

Chapter 2

For a region $V \subset \mathbb{R}^3$ for which $0 \le z \le f(x, y)$ and $(x, y) \in D \subset \mathbb{R}^2$, the volume of V is

$$\iint_D f(x, y)\, dxdy.$$

If the region D is *rectangular*, say $D = [a, b] \times [c, d]$, then

$$\iint_D f(x, y)\, dxdy = \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx.$$

The region D is *type I* if it can be expressed as $f(x) \le y \le g(x)$ and $x \in [a, b]$ and it is *type II* if it can be expressed as $F(y) \le x \le G(y)$ and $y \in [c, d]$, in these cases

$$\iint_D I(x, y)\, dxdy = \int_a^b \left(\int_{f(x)}^{g(x)} I(x, y)\, dy \right) dx, \qquad \iint_D I(x, y)\, dxdy = \int_c^d \left(\int_{F(y)}^{G(y)} I(x, y)\, dx \right) dy.$$

A *regular* region is the union of finitely many type I or type II regions. For a region that is both type I and type II "changing the order of integration" means converting between types I and II.

For integrals in polar coordinates

$$\iint_D f(x, y)\, dxdy = \iint_{D'} f(r\cos\theta, r\sin\theta) r\, drd\theta$$

where D' is the region D expressed in polar coordinates (in the r-θ plane). Note the extra factor of r is the *Jacobian* of the transformation. For a change of variable $x(u, v)$ and $y(u, v)$ we have the general result

$$\iint_D f(x, y)\, dxdy = \iint_{D'} f(x(u, v), y(u, v)) |J|\, dudv$$

where D' is the region D expressed in the u-v plane and J is the *Jacobian* of the transformation,

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}, \qquad J = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^{-1}.$$

The latter expression is for when the transformation is more naturally expressed as $u(x,y)$ and $v(x,y)$. For triple integrals, if the region V can be expressed as $f(x,y) \le z \le g(x,y)$ and $(x,y) \in D$ then

$$\iiint_V F(x,y,z)\,dxdydz = \iint_D \left(\int_{f(x,y)}^{g(x,y)} F(x,y,z)\,dz\right) dxdy$$

where D is the projection of the region V onto the x-y plane. Standard results for double integrals can then be used after the first integration. In spherical polars we have

$$\iiint_V f(x,y,z)\,dxdydz = \iiint_{V'} f\left(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi\right) r^2 \sin\phi\,drd\phi d\theta$$

where V' is the region V expressed in spherical polars.

Chapter 3

A scalar field $f : \mathbb{R}^n \to \mathbb{R}$ and a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$. We define the gradient of f and the divergence of $\mathbf{F}$ to be

$$\mathrm{grad}\,(f) = \nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right), \qquad \mathrm{div}\,(\mathbf{F}) = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \ldots + \frac{\partial F_n}{\partial x_n}.$$

In the case where $n = 3$ we define the curl of the vector field

$$\mathrm{curl}\,(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

The directional derivative is $\partial f/\partial \mathbf{u} = \mathbf{u} \cdot \nabla f$ where $\mathbf{u}$ is a unit vector. The *Laplacian* of a scalar field is $\nabla \cdot (\nabla f) = \nabla^2 f = \partial^2 f/\partial x_1^2 + \ldots + \partial^2 f/\partial x_n^2$. There are a number of *nabla identities*, e.g. $\nabla \cdot (f\mathbf{F}) = \nabla f \cdot \mathbf{F} + f \nabla \cdot \mathbf{F}$.

Chapter 4

A parametric curve is a function $\mathbf{r} : [a,b] \to \mathbb{R}^n$ and the image of this function is called the trace. To perform line integrals along curves we use a parameterisation

$$\int_C f(\mathbf{x})\; ds = \int_a^b f(\mathbf{r}(t)) \left|\frac{d\mathbf{r}}{dt}\right| dt, \qquad \int_C \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\,dt.$$

The latter result (for vector fields) is called the work done by $\mathbf{F}$ moving along C. If the work done is independent of the path taken then the vector field is *conservative*. A test for conservative vector fields is, in $\mathbb{R}^3$, to check $\nabla \times \mathbf{F} = \mathbf{0}$. If a vector field is conservative then there exists a potential such that $\mathbf{F} = \nabla\phi$ (note some people, especially in physics use $\mathbf{F} = -\nabla\phi$).

A parametric surface is $\mathbf{r} : D \to \mathbb{R}^3$ and the image is a surface S. To peform surface integrals we use the parameterisation and we have

$$\iint_S f(\mathbf{x})\; dS = \iint_D f(\mathbf{r}(u,v)) \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| dudv, \qquad \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) dudv.$$

We have a number of integral theorems. Green's theorem for a simple, closed and postively oriented plane curve C that encloses a region A,

$$\int_C P\,dx + Q\,dy = \iint_A \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dxdy.$$

The divergence theorem for a bounded volume V with closed surface S,

$$\iint_S \mathbf{F} \cdot \mathbf{n}\,dS = \iiint_V \nabla \cdot \mathbf{F}\,dV,$$

where $\mathbf{n}$ is the unit outward-pointing normal.