FB1: Find the remainder *r* (between 0 and 8) that we get when we divide 7^{96} by 9. Be sure to show your work.

From the problem above, a congruence equation can be written down as

$$x \equiv 7^{96} \ (mod\ 9).$$

The value *x* can then be found by starting with

$$7 \equiv 7 \ (mod\ 9)$$

and squaring both sides multiple times modulo 9 like this:

$$7^2 \equiv 49 \ (mod\ 9) = 4 \ (mod\ 9)$$

$$\Rightarrow 7^4 \equiv 16 \ (\mathrm{mod}\ 9) = 7 \ (\mathrm{mod}\ 9)$$

$$\Rightarrow 7^8 \equiv 4 \ (mod\ 9)$$

$$\Rightarrow 7^{16} \equiv 7 \ (mod\ 9)$$

$$\Rightarrow 7^{32} \equiv 4 \ (mod\ 9)$$

$$\Rightarrow 7^{64} \equiv 7 \ (mod\ 9).$$

Additionally, the power of 7 in the question is 96, which means that *x* can be found by multiplying already found values of powers of 7 like this:

$$7^{96} \equiv 7^{64} \cdot 7^{32} \equiv 7 \cdot 4 \equiv 28 \equiv 1 \ (mod\ 9).$$

This shows that the remainder is 1. It can also be explicitly stated as

$$9k = 7^{96} - 1$$

for some integer *k*. Then 7^{96} can be expressed as

$$7^{96} = 9k + 1,$$

the term $9k$ being a multiple of 9, showing that the right-hand side has a remainder of 1 when divided by 9 and that 7^{96} also has a remainder of 1 when divided by 9.

FB2: Does the congruence equation $6x \ \equiv 7$ (mod 25) have a solution for *x*? If it does, find the solution. If it does not, prove that it does not.

The congruence equation does have a solution because $\mathrm{hcf}(6, 25) = 1$, which divides 7. This is proven by Euclid's algorithm:

$$25 = 4 \cdot 6 + 1$$

$$6 = 6 \cdot 1 + 0.$$

To find *x*, the highest common factor must be written out as

$$1 = 25t + 6s$$

for some integers *s* and *t*. Rearranging it, one gets

$$6s = 1 - 25t,$$

for which the congruence equation in modulo 25 is

$$6s \equiv 1\ (mod\ 25)$$

because of 25*t* being a multiple of 25. When multiplying the congruence by 7, it is

$$6 \cdot 7s \equiv 7\ (mod\ 25).$$

By comparing this to the given congruence equation, it can be seen that

$$x = 7s.$$

To get *s*, the first equation in the Euclidean algorithm can be rearranged to make

$$1 = 25 - 4 \cdot 6,$$

making $t = 1$ and $s = -4$.

Therefore,

$$x = 7 \cdot (-4)\ (mod\ 25) = -28\ (mod\ 25) = 22\ (mod\ 25).$$

This can also be written as

$$x = 25n\ + 22$$

for some integer *n*.

FB3: Find the acute angle between the lines $3x + y = 5$ and $x - 2y = 4$.

By rewriting the first equation as

$$y = 5 - 3x$$

and the second equation as

$$y = \frac{1}{2}x - 2$$

and looking at the gradients, two arbitrary vectors can be made:

$$\boldsymbol{a} = \langle 1, -3 \rangle$$

$$\boldsymbol{b} = \left\langle 1, \frac{1}{2} \right\rangle,$$

each of which belongs to line 1 and line 2 respectively. Their magnitude and positive/negative direction do not matter because the angle is not dependent on either.

Let θ be the angle between the vectors ***a*** and ***b***. Then the angle is

$$\theta = arccos\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

$$= arccos\frac{1 - \frac{3}{2}}{\sqrt{10 \cdot \frac{5}{4}}}$$

$$\approx 98°.$$

Then the acute angle between vectors ***a*** and ***b*** is

$$180° - \theta = 180° - 98 = 82°.$$