2A: Tutorial 6

School of Mathematics and Statistics

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Semester 1 2019–20

Instructions

Access your tutorial lecture

TU11,TU12,TU13,TU14,TU15: November 2nd, 09:00-10:00

TU16,TU17,TU18,TU19,TU20: November 2nd, 15:00-16:00

using the Zoom plug-in in the Moodle page of the course.

Please mute yourself when you are in the main room of the meeting.

Be ready to work in groups. Having a method of writing on screen, as mouse, tablet or similar is useful. Screen sharing is allowed!

Multiple Choice Question 1

Ex Sheet 6, T1 (related)

The region W is bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x = 1$ and $y + z = 1$. Which one of the following integrals represents the volume of W?

(A) $\int_0^{1-y} \left(\int_0^1 \left(\int_0^1 dz \right) dy \right) dx$ **(B)** $\int_0^1 \left(\int_0^1 \left(\int_0^1 y\, dz \right) dx \right) dy$

(C) $\int_0^1 \left(\int_0^{1-y} \left(\int_0^1 dx \right) dz \right) dy$ **(D)** $\int_0^1 \left(\int_0^{1-z} \left(\int_0^{1-y} dx \right) dy \right) dz$

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(C) $\int_0^1\left(\int_0^{1-y}\left(\int_0^1 dx\right)dz\right)dy$ **(D)** $\int_0^1\left(\int_0^{1-z}\left(\int_0^{1-y} dx\right)dy\right)dz$

ANSWER: **(C)** The volume of W is $\iiint_W 1\,dxdydz$. The region W can be described as points (x, y, z) with

$$(x, y) \in D_1, \quad 0 \leq z \leq 1 - y\,,$$

where D_1 is the square region $[0, 1] \times [0, 1]$; alternatively by

$$(x, z) \in D_3, \quad 0 \leq y \leq 1 - z\,,$$

where D_3 is the square region $[0, 1] \times [0, 1]$;

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(C) $\int_0^1\left(\int_0^{1-y}\left(\int_0^1 dx\right)dz\right)dy$ **(D)** $\int_0^1\left(\int_0^{1-z}\left(\int_0^{1-y} dx\right)dy\right)dz$

or by

$$(y, z) \in D_2, \quad 0 \leq x \leq 1\,,$$

where D_2 is the triangular region $0 \leq y \leq 1$ and $0 \leq z \leq 1 - y$.

The order of integration is different with each description and choice **(C)** matches the last description above.

Multiple Choice Question 2

Ex Sheet 5, T18

For the region in the first quadrant enclosed by the parabolas $y^2 = x$, $y = x^2$ and $y^2 = 2x$, $y = 4x^2$; identify an appropriate change of variable for performing double integration.

(A) $u = xy$, $v = x/y$

(B) $u = y^2x$, $v = x^2y$

(C) $u = y^2/x$, $v = x^2/y$

(D) $u = y^2 - x$, $v = x^2 - y$

Multiple Choice Question 2

Ex Sheet 5, T18

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(A)	$u = xy$, $v = x/y$	**(B)**	$u = y^2x$, $v = x^2y$
(C)	$u = y^2/x$, $v = x^2/y$	**(D)**	$u = y^2 - x$, $v = x^2 - y$

ANSWER: **(C)** We see that the four boundaries of the region can be written as

$$\frac{y^2}{x} = 1, \quad \frac{y^2}{x} = 2, \qquad \frac{x^2}{y} = 1, \quad \frac{x^2}{y} = \frac{1}{4}$$

so that the change of variable $u = y^2/x$ and $v = x^2/y$ would transform the region of integration into a rectangle in the uv plane.

Tutorial questions

Ex Sheet 5, T6

Evaluate

$$\iint_D \sqrt{x^2+y^2}\, dA$$

where D is the disk with centre $(0,1)$ and radius 1.

Ex Sheet 6, T6(b)

Use triple integration to express the volume of the region bounded by surfaces

$$y = x^2, \quad z = -y+4, \quad z = 0.$$

Ex Sheet 6, T3

A solid shell of variable density is in the form of a region between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=9$. The density at (x,y,z) is $\rho(x,y,z)=\sqrt{x^2+y^2+z^2}$. Calculate the mass of the shell.

EX SHEET 5, T6

Evaluate

$$\iint_D \sqrt{x^2 + y^2}\, dA$$

where D is the disk with centre $(0, 1)$ and radius 1.

The equation of the circle is $x^2 + (y-1)^2 = 1$, namely $x^2 + y^2 = 2y$. In polar coordinates it becomes $r = 2\sin\theta$. Since $r \geq 0$ this also forces $0 \leq \theta \leq \pi$ (or you can sketch a picture to see this directly).

Hence the integral is

$$\int_0^\pi \left(\int_0^{2\sin\theta} r^2\, dr \right) d\theta = \frac{8}{3}\int_0^\pi \sin^3\theta\, d\theta = \frac{8}{3}\int_0^\pi (1-\cos^2\theta)\sin\theta\, d\theta$$

$$= -\frac{8}{3}\int_1^{-1} (1-u^2)\, du = \frac{32}{9}. \quad \text{(We used the substitution } u = \cos\theta.\text{)}$$

Ex Sheet 6, T6(b)

Use triple integration to express the volume of the region bounded by surfaces

$$y = x^2, \quad z = -y + 4, \quad z = 0.$$

The region of integration can be described as $0 \leq z \leq 4 - y$ and $(x, y) \in D$ where D is the region that lies above the parabola $y = x^2$ and below $y = 4$ (this is where the plane $z = 4 - y$ intersects $z = 0$). The curves $y = 4$ and $y = x^2$ intersect at $x^2 = 4$ so $x = \pm 2$.

Therefore the triple integral that represents the volume is

$$\int_{-2}^{2} \left(\int_{x^2}^{4} \left(\int_{0}^{4-y} dz \right) dy \right) dx.$$

A patient computation gives

$$\int_{-2}^{2}\left(\int_{x^2}^{4}[z]_0^{4-y}\ dy\right)dx=\int_{-2}^{2}\left(\int_{x^2}^{4}4-y\,dy\right)dx$$

$$=\int_{-2}^{2}\left[-\frac{1}{2}(4-y)^2\right]_{x^2}^{4}dx=\int_{-2}^{2}\frac{1}{2}\left(4-x^2\right)^2\ dx$$

$$=\int_{0}^{2}\left(4-x^2\right)^2\ dx\quad\text{(even integrand)}$$

$$=\left[16x-\frac{8}{3}x^3+\frac{1}{5}x^5\right]_0^2=\frac{256}{15}\,.$$

Ex Sheet 6, T3

A solid shell of variable density is in the form of a region between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=9$. The density at (x,y,z) is $\rho(x,y,z)=\sqrt{x^2+y^2+z^2}$. Calculate the mass of the shell.

The triple integral representing the mass is

$$\iiint_V \sqrt{x^2+y^2+z^2}\,dxdydz\,.$$

In spherical polars the region V is described as $1\le r\le 3$, $0\le\phi\le\pi$ and $0\le\theta\le 2\pi$ (a cuboid in spherical polar coordinates). Hence,

$$\iiint_V \sqrt{x^2+y^2+z^2}\,dxdydz = \iiint_{\tilde{V}} r\,r^2\sin\phi\,drd\phi d\theta$$

and as the integrand is separable and the region $\tilde{V}$ is a cuboid we have that the mass is

$$\left(\int_1^3 r^3\,dr\right)\left(\int_0^\pi \sin\phi\,d\phi\right)\left(\int_0^{2\pi} 1\,d\theta\right) = \left[\frac{1}{4}r^4\right]_1^3 [-\cos\phi]_0^\pi\,[\theta]_0^{2\pi} = 80\pi\,.$$