The formula

$$\neg(p_1 \to \neg p_2) \to (p_3 \to (p_4 \to \neg p_5))$$

returns `false` for precisely one assignment to the propositions $p_1, \ldots, p_5$. Find this assignment using the laws of logical equivalence and *without* constructing a truth table for the formula.

Solution.

$$\begin{aligned}
&\neg(p_1 \to \neg p_2) \to (p_3 \to (p_4 \to \neg p_5)) \\
&\qquad \equiv \neg(\neg p_1 \vee \neg p_2) \to (p_3 \to (\neg p_4 \vee \neg p_5)) && \text{implication law} \\
&\qquad \equiv \neg(\neg p_1 \vee \neg p_2) \to (\neg p_3 \vee (\neg p_4 \vee \neg p_5)) && \text{implication law} \\
&\qquad \equiv \neg\neg(\neg p_1 \vee \neg p_2) \vee (\neg p_3 \vee (\neg p_4 \vee \neg p_5)) && \text{implication law} \\
&\qquad \equiv (\neg p_1 \vee \neg p_2) \vee (\neg p_3 \vee (\neg p_4 \vee \neg p_5)) && \text{double negation law} \\
&\qquad \equiv \neg p_1 \vee \neg p_2 \vee \neg p_3 \vee \neg p_4 \vee \neg p_5
\end{aligned}$$

where the last step from the associative law for $\vee$. Considering this formula we have that the one assignment that returns `false` is when all the propositions $p_1, \ldots, p_5$ are `true`, i.e. all the formulae $\neg p_1, \ldots, \neg p_5$ are `false`.

Prove that $(p \wedge \neg q) \rightarrow q$ and $(p \wedge \neg q) \rightarrow \neg p$ are equivalent using laws of logical equivalence.

Solution.

$$
\begin{aligned}
(p \wedge \neg q) \rightarrow q &\equiv \neg(p \wedge \neg q) \vee q && \text{implication law} \\
&\equiv (\neg p \vee \neg\neg q) \vee q && \text{de Morgan law} \\
&\equiv (\neg p \vee q) \vee q && \text{double negation law} \\
&\equiv \neg p \vee (q \vee q) && \text{associative law} \\
&\equiv \neg p \vee q && \text{idempotency law} \\
&\equiv (\neg p \vee \neg p) \vee q && \text{idempotency law} \\
&\equiv \neg p \vee (\neg p \vee q) && \text{associative law} \\
&\equiv (\neg p \vee q) \vee \neg p && \text{commutative law} \\
&\equiv (\neg p \vee \neg\neg q) \vee \neg p && \text{double negation law} \\
&\equiv \neg(p \wedge \neg q) \vee \neg p && \text{de Morgan law} \\
&\equiv (p \wedge \neg q) \rightarrow \neg p && \text{implication law}
\end{aligned}
$$

Using the laws of logical equivalence show that the formula $(q \wedge (p \to \neg q)) \to \neg p$ is a tautology.

Solution. Below are two solutions to this question (others are possible).

$$
\begin{aligned}
(q \wedge (p \to \neg q)) \to \neg p &\equiv (q \wedge (\neg p \vee \neg q)) \to \neg p && \text{implication law} \\
&\equiv ((q \wedge \neg p) \vee (q \wedge \neg q)) \to \neg p && \text{distributive law} \\
&\equiv ((q \wedge \neg p) \vee \texttt{false}) \to \neg p && \text{contradiction law} \\
&\equiv (q \wedge \neg p) \to \neg p && \text{identity law} \\
&\equiv \neg(q \wedge \neg p) \vee \neg p && \text{implication law} \\
&\equiv (\neg q \vee p) \vee \neg p && \text{de Morgan law} \\
&\equiv \neg q \vee (p \vee \neg p) && \text{communtative law} \\
&\equiv \neg q \vee \texttt{true} && \text{tautology law} \\
&\equiv \texttt{true} && \text{domination law}
\end{aligned}
$$

$$
\begin{aligned}
(q \wedge (p \to \neg q)) \to \neg p &\equiv \neg(q \wedge (p \to \neg q)) \vee \neg p && \text{implication law} \\
&\equiv (\neg q \vee \neg(p \to \neg q)) \vee \neg p && \text{de Morgan law} \\
&\equiv (\neg q \vee \neg p) \vee \neg(p \to \neg q) && \text{associative law} \\
&\equiv (\neg p \vee \neg q) \vee \neg(p \to \neg q) && \text{commutative law} \\
&\equiv (p \to \neg q) \vee \neg(p \to \neg q) && \text{implication law} \\
&\equiv \texttt{true} && \text{tautology law}
\end{aligned}
$$