# 2C Week 11 2020/21

Handout

## Uniform continuity

In this subsection we discuss *uniform continuity*. Uniform continuity is a stronger property than continuity, its definition is obtained by slightly reordering the ingredients in the definition of continuity.

**Definition 5.24.** A real function $f : \mathrm{dom}(f) \to \mathbb{R}$ is called uniformly continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $c, x \in \mathrm{dom}(f)$ we have that $|c - x| < \delta$ implies $|f(c) - f(x)| < \epsilon$.

We can rewrite this using quantifiers: the function $f$ is uniformly continuous if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x, c \in \mathrm{dom}(f), |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon.$$

As before, the value of $\delta$ will usually depend on $\varepsilon$. However, $\delta$ needs to be independent of $c \in \mathrm{dom}(f)$. In contrast, in the definition of continuity the number $\delta$ may depend not only on $\varepsilon$, but also on the point $c$.

From this discussion we obtain the following theorem[1].

[1] I'll leave it to you to write down a detailed proof: this is a good excercise in working with quantifiers!

**Theorem 5.25.** *Any uniformly continuous function is continuous.*

Let us have a look at some examples.

**Example.** *Check* directly from the definition *that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 7$ is uniformly continuous.*

*Solution.* In order to check the definition of uniform continuity let $\varepsilon > 0$. Then for $c, x \in \mathbb{R}$ we have

$$|f(x) - f(c)| = |3x + 7 - (3c + 7)| = |3x - 3c| = 3|x - c|.$$

Therefore, $|f(x) - f(c)| < \varepsilon$ is equivalent to $3|x - c| < \varepsilon$, or $|x - c| < \varepsilon/3$. If we chose $\delta = \varepsilon/3$ we conclude that $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$. Thus $f$ is uniformly continuous. □

We already know that all polynomial functions are continuous. However, such functions need not be uniformly continuous, as the following example illustrates.

**Example.** *Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not uniformly continuous.*

*Solution.* Let $\varepsilon = 1$. We claim that there is no $\delta > 0$ such that $|x - c| < \delta$ for $x, c \in \mathbb{R}$ implies $|f(x) - f(c)| < 1$. Indeed, notice that

$$|f(x) - f(c)| = |x^2 - c^2| = |x - c||x + c|.$$

So if we set $x = \frac{1}{\delta}, c = \frac{1}{\delta} + \delta/2$ we obtain

$$|f(x) - f(c)| = \frac{\delta}{2}\left(\frac{2}{\delta} + \frac{\delta}{2}\right) = 1 + \frac{\delta^2}{4} > 1,$$

and this shows that $f$ is not uniformly continuous. $\square$

If we restrict attention to functions defined on *bounded intervals* the situation changes significantly: continuous functions on bounded intervals are automatically uniformly continuous.

**Theorem 5.26.** *Any continuous function $f : [a,b] \to \mathbb{R}$ is uniformly continuous.*

*Proof.* Suppose that $f$ is not uniformly continuous. Then there exists an $\varepsilon > 0$ such that for each $\delta > 0$ we find $x, c \in [a,b]$ such that $|c - x| < \delta$ and $|f(c) - f(x)| \geq \varepsilon$. In particular, applying this to $\delta = 1/n$ we find $x_n, c_n \in [a,b]$ such that $|c_n - x_n| < 1/n$ and $|f(c) - f(x)| \geq \varepsilon$. Since the sequence $(x_n)_{n=1}^{\infty}$ is bounded, the Bolzano-Weierstrass theorem shows that there exists a subsequence $(x_{k_n})_{n=1}^{\infty}$ such that $x_{k_n} \to c$ as $n \to \infty$ for some $c \in [a,b]$.

We claim that $(c_{k_n})_{n=1}^{\infty}$ also converges to $c$. To see this, consider

$$0 \leq |c_{k_n} - c| \leq |c_{k_n} - x_{k_n}| + |x_{k_n} - c| < \frac{1}{k_n} + |x_{k_n} - c| \to 0$$

as $n \to \infty$, using the triangle inequality in the first step. By the sandwich principle, the sequence $|c_{k_n} - c|$ converges to zero, which means precisely $c_{k_n} \to c$ as $n \to \infty$.

Since $f$ is continuous at $c$, we have $f(x_{k_n}) \to f(c)$ and $f(c_{k_n}) \to f(c)$ as $n \to \infty$. This contradicts the fact that $|f(x_{k_n}) - f(c_{k_n})| \geq \varepsilon$ for all $n$ according to our initial construction. Hence our assumption that $f$ is not uniformly continuous was wrong. $\square$