2C Intro to real analysis 2020/21

Ex Sheet 1

Feedback and solutions

Q1 *Find a value of $K > 0$ such that the implication*

$$|x+4| < 2 \implies \left|\frac{x^2+4}{x-1} + 4\right| \leq K|x+4|$$

is true. Make sure you justify your choice of K, with a proof of the implication for your value of K.

This is similar to example 1.8 in the lecture notes[1]. I start by simplifying the expression $\left|\frac{x^2+4}{x-1} + 4\right|$.

[1] There the question was divided into two parts; here we need to work out what the first part of the question should be, and then do it.

For $x \in \mathbb{R}$ (with $x \neq 1$), we have

$$\begin{aligned}\left|\frac{x^2+4}{x-1} + 4\right| &= \left|\frac{x^2+4x}{x-1}\right| \\ &= \left|\frac{(x+4)x}{x-1}\right| = \frac{|x|}{|x-1|}|x+4|.\end{aligned}$$

So what we need to do is to find an upper bound for $\frac{|x|}{|x-1|}$ subject to the condition that $|x+4| < 2$. We do this in a similar fashion to the first part of example 1.8.

For $x \in \mathbb{R}$, we have

$$\begin{aligned}|x+4| < 2 &\implies -2 < x+4 < 2 \\ &\implies -6 < x < -2 \text{ and } -7 < x-1 < -3 \\ &\implies 2 < |x| < 6 \text{ and } 3 < |x-1| < 7 \\ &\implies \frac{2}{7} < \frac{|x|}{|x-1|} < \frac{6}{3} = 2\end{aligned}$$

One other difference between this example and example 1.8 in the notes, is that here the initial condition is $|x+2| < 1$, whereas the corresponding example involved a $\leq$ sign[2].

[2] It turns out that in this case this makes no difference, but I've found over the years that there's a considerable amount of confusion as to when it matters whether we use $<$ or $\leq$. The key point here is that if $a < b$, then it is also true that $a \leq b$, i.e. $a < b \implies a \leq b$.

Therefore

$$|x+4| < 2 \implies \left|\frac{x^2+4}{x-1} + 4\right| = \frac{|x|}{|x-1|}|x+4| \leq 2|x-3|.$$

and so we can take $K = 2$.

Of course any value of K with $K \geq 2$ is also an equally correct answer[3].

[3] Think about what would have happened had the question asked you to find a value of K such that for some $L \neq 4$

$$|x+4| \leq 2 \implies \left|\frac{x^2+4}{x-1} + L\right| < K|x+4|$$

is true. Would $K = 2$ still be a correct answer?

Q2 *Let $x, y \in \mathbb{R}$. Use the triangle inequality to prove the reverse triangle inequality* [4]

$$\big||x| - |y|\big| \leq |x - y|.$$

[4] Hint: You may wish to do this in two cases, by proving both $|x| - |y| \leq |x - y|$ and $|y| - |x| \leq |x - y|$. Writing $x = x - y + y$ might help.

In this question it is important to be clear when we use the triangle inequality, and to introduce all our symbols. Hopefully the hint led you to the right idea.

For $x, y \in \mathbb{R}$, we have

$$|x| = |(x - y) + y| \leq |x - y| + |y|,$$

by the triangle inequality. Therefore $|x| - |y| \leq |x - y|$. Similarly,

$$|y| = |(y - x) + x| \leq |y - x| + |x| = |x - y| + |x|,$$

by the triangle inequality. Therefore $|y| - |x| \leq |x - y|$. Since $\big||x| - |y|\big| = \max(|x| - |y|, |y| - |x|)$, we have

$$\big||x| - |y|\big| \leq |x - y|,$$

as required.

Q3 *Show that the function $f : \mathbb{N} \to \mathbb{R}$ given by*

$$f(n) = \frac{8n^2 - 7n + 1}{4n^2 + 8n + 3}$$

is bounded above.

Note that it is important to persuade the person reading your work that the answer is correct, so you should indicate any results you have used (you can assume that the reader has access to lecture notes and the ERA book). I'll first do the question using the polynomial estimation lemma. Note also that the denominator, $4n^2 + 8n + 3$, is nonvanishing on $\mathbb{N}$ so that f is indeed a function.

By Lemma 1.9 (and the note following the lemma), there exist $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned} n \geq N_1 &\implies \frac{1}{2}8n^2 \leq 8n^2 - 7n + 1 \leq \frac{3}{2}8n^2 \\ n \geq N_2 &\implies \frac{1}{2}4n^2 \leq 4n^2 + 8n + 3 \leq \frac{3}{2}4n^2. \end{aligned}$$

Define $N = \max(N_1, N_2)$, so that for $n \geq N$, we have

$$f(n) \leq \frac{\frac{3}{2}8n^2}{\frac{1}{2}4n^2} = 6.$$

Let $M = \max\{f(1), f(2), \ldots, f(N-1), 6\}$. Then M is an upper bound for f, so f is bounded above.

This isn't the only way to proceed; it is equally reasonable to find some estimates to show that f is bounded above directly. An alternative answer might be as follows.

For $n \in \mathbb{N}$, we have $1 \leq 7n$ so that $8n^2 - 7n + 1 \leq 8n^2$. Also, $4n^2 + 8n + 3 \geq 4n^2$ for all $n \in \mathbb{N}$. Therefore, for any $n \in \mathbb{N}$, we have

$$f(n) \leq \frac{8n^2}{4n^2} = 2,$$

and so f is bounded above by 2.