2B Linear Algebra

Ex Sheet 10

1 True/False

a) The polynomial $q(x) = 0$ is a quadratic form.

b) The polynomial $q(x_1, x_2) = x_1 - x_2$ is a quadratic form.

c) Any quadratic form can be represented by an upper triangular matrix.

d) The determinant of the matrix representing a quadratic form in n variables is $a_{11}a_{22}\ldots a_{nn}$.

e) Two non-zero quadratic forms in one variable are equivalent if and only if one is a scalar multiple of the other.

f) Two non-zero quadratic forms in one variable are equivalent if and only if one is a positive scalar multiple of the other.

g) If A is an invertible matrix in $M_{n\times n}(\mathbb{C})$, then the conjugate transpose A^* is invertible.

h) A symmetric matrix in $M_{n\times n}(\mathbb{C})$ is automatically Hermitian.

i) A symmetric matrix in $M_{n\times n}(\mathbb{R})$ is automatically Hermitian.

j) The diagonal entries of any Hermitian matrix in $M_{n\times n}(\mathbb{C})$ must be real- valued.

k) For any $A \in M_{n\times n}(\mathbb{C})$, $B = A + A^*$ is Hermitian.

l) The sum of two unitary matrices in $M_{n\times n}(\mathbb{C})$ is again a unitary matrix.

m) If $A \in M_{n\times n}(\mathbb{C})$ is both Hermitain and unitary, then $A^2 = \mathbb{I}$.

n) In a unitary matrix, the sum of the entries of the first row must add up to 1.

o) In a unitary matrix, the sum of the squares of the entries of the first row must add up to 1.

p) A diagonal matrix that is unitary must have all of its entries be either 1 or -1.

q) If $A \in M_{n\times n}(\mathbb{C})$ is Hermitian, then A^2 is positive-definite.

r) If $U \in M_{n\times n}(\mathbb{C})$ is unitary, then $U\mathbf{v} \cdot U\mathbf{w} = \mathbf{v} \cdot \mathbf{w}$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$.

[1] **True/False Questions**

Every Exercise Sheet will have a section containing true/false questions. They are designed to test your understanding from lectures. You should look at your lecture notes and/or the textbook to help you answer these questions, but you should not need to write anything to work out the solution.

Solutions to True/False

a) T b) F c) T d) T e) F f) T g) T h) F i) T j) T k)T l) F m) T n)F o)T p)T q) F r) T

Tutorial Exercises

T1 (i) Find the rank and signature of each of the following quadratic forms:

(a) $q(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 5x_3^2 - 2x_1x_2$,

(b) $q(x_1, x_2) = 5x_1^2 + 5x_2^2 - 4x_1x_2$,

(c) $q(x_1, x_2) = -(x_1 - x_2)^2$,

(d) $q(x_1, x_2, x_3) = x_1x_2$.

(ii) State whether each of the forms is positive-definite, positive-semidefinite, negative-definite, negative-semidefinite or indefinite.

Solution

a) The quadratic form can be written as $\mathbf{x}^t A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^3$ and

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

The eigenvalues are of A are $\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = 5$, as these are all positive the quadratic form has (i) rank 3, signature 3, (ii) is positive definite .

b) The quadratic form can be written as $\mathbf{x}^t A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^2$ and

$$A = \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}.$$

The eigenvalues are of A are $\lambda_1 = 3, \lambda_2 = 7$, as these are both positive the quadratic form has (i) rank 2, signature 2, (ii) is positive definite.

c) The quadratic form can be written as $\mathbf{x}^t A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^2$ and

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The eigenvalues are of A are $\lambda_1 = 0, \lambda_2 = -2$, hence the quadratic form has (i) rank 1, signature -1 (ii) is negative semi-definite.

d) The quadratic form can be written as $\mathbf{x}^t A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^2$ and

$$A = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}.$$

The eigenvalues are of A are $\lambda_1 = 1/2$, $\lambda_2 = -1/2$, hence the quadratic form has (i) rank 2, signature 0, (ii) is indefinite.

T2 Identify the conic section represented by the equation:

a) $2x^2 - 2y^2 = 20$

b) $5x^2 + 3y^2 - 15 = 0$

c) $x^2 - 3 = -y^2$

d) $4y^2 - x^2 = 20$

Solution

(a) Hyperbola, (b) Ellipse (c) Circle (d) Hyperbola

T3 Identify the conic section represented by the equation by rotating axes to place the conic in standard position. Find an equation of the conic in the rotated coordinates, and find the angle of rotation.

a) $x^2 - 4xy - 2y^2 + 8 = 0$

b) $5x^2 + 4xy + 5y^2 = 9$

c) $2x^2 - 12xy - 3y^2 - 7 = 0$

Solution

a) Rewriting the quadratic as $-x^2 + 4xy + 2y^2 = 8$, the left hand side can be written as $\mathbf{x}^T A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^2$ and

$$A = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = 3$, $\lambda_2 = -2$. Applying orthogonal diagonalisation, the equation of the conic in the rotated coordinates is an hyperbola opening left and right

$$3(x')^2 - 2(y')^2 = 8,$$

where the rotated coordinates are $\mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$ and satisfy $Q\mathbf{x}' = \mathbf{x}$ where Q is the orthogonal matrix

$$Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Hence, the change of coordinates corresponds to a rotation (counterclockwise) by the angle θ where $\theta = \tan^{-1} 1 = \pi/4$ or 45^o. The hyperbola intersects the x'-axis at $\pm\sqrt{8/3}$.

b) The left hand side of the quadratic can be written as $\mathbf{x}^T A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^2$ and

$$A = \begin{bmatrix} 2 & -6 \\ -6 & -3 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = -7$, $\lambda_2 = 3$. Applying orthogonal diagonalisation, the equation of the conic in the rotated coordinates is the ellipse

$$7(x')^2 + 3(y')^2 = 8,$$

where the rotated coordinates are $\mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$ and satisfy $Q\mathbf{x}' = \mathbf{x}$ where Q is the orthogonal matrix

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Hence, the change of coordinates corresponds to a rotation (counterclockwise) by the angle θ where $\theta = \tan^{-1} 1 = \pi/4$ or 45^o. The ellipse intersects the x'-axis at $\pm\sqrt{8/7}$ and the y'-axis at $\pm\sqrt{8/3}$.

c) Rewriting the quadratic as $2x^2 - 12xy - 3y^2 = 7$, the left hand side can be written as $\mathbf{x}^T A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^2$ and

$$A = \begin{bmatrix} 2 & -6 \\ -6 & -3 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = -7$, $\lambda_2 = 6$. Applying orthogonal diagonalisation, the equation of the conic in the rotated coordinates the hyperbola opening up and down

$$-7(x')^2 + 6(y')^2 = 7,$$

where the rotated coordinates are $\mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$ and satisfy $Q\mathbf{x}' = \mathbf{x}$ where Q is the orthogonal matrix

$$Q = \frac{1}{\sqrt{13}} \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Hence, the change of coordinates corresponds to a rotation (counterclockwise) by the angle θ where $\theta = \tan^{-1}(3/2)$, or 56.3^o. The hyperbola intersects the y'-axis at $\pm\sqrt{7/6}$.

T4 Let a, b, c be real numbers. (i) Prove that the quadratic form

$$q(x, y) = ax^2 + by^2 + 2cxy$$

is positive-definite if and only if $a > 0$ and $ab > c^2$. Find similar necessary and sufficient conditions for $q(x, y)$ to be (ii) negative-definite, (iii) indefinite. Try to find necessary and sufficient conditions for $q(x, y)$ to be (iv) positive-semidefinite, (v) negative-semidefinite.

Solution

(i) First suppose that $a \neq 0$. Then

$$\begin{aligned} q(x, y) &= ax^2 + by^2 + 2cxy \\ &= a\left[x^2 + 2\frac{c}{a}xy\right] + by^2 \end{aligned}$$

$$= a\left[\left(x + \frac{c}{a}y\right)^2 - \frac{c^2}{a^2}y^2\right] + by^2$$
$$= a\left(x + \frac{c}{a}y\right)^2 + \frac{ab - c^2}{a}y^2.$$

Therefore, provided $a \neq 0$,

$$\begin{aligned} q \text{ is positive-definite} &\iff a > 0 \text{ and } \frac{ab - c^2}{a} > 0 \\ &\iff a > 0 \text{ and } ab - c^2 > 0 \\ &\iff a > 0 \text{ and } ab > c^2. \end{aligned}$$

Now suppose that $a = 0$ and $b \neq 0$. Then

$$q(x, y) = by^2 + 2cxy = b\left(y + \frac{c}{b}x\right)^2 - \frac{c^2}{b}x^2.$$

In this case, q cannot be positive-definite because if $b > 0$ then $-\dfrac{c^2}{b} \leqslant 0$.

Finally, suppose that $a = b = 0$. Then

$$q(x, y) = 2cxy = \frac{c}{2}(x + y)^2 - \frac{c}{2}(x - y)^2.$$

In this case, q cannot be positive-definite because if $\dfrac{c}{2} > 0$ then $-\dfrac{c}{2} < 0$.

Hence, from the three cases,

$$q \text{ is positive-definite} \iff a > 0 \text{ and } ab > c^2.$$

(Note that, from $ab > c^2$, it follows that $ab > 0$ and so a, b have the same sign. This means that

$$\begin{aligned} q \text{ is positive-definite} &\iff b > 0 \text{ and } ab > c^2 \\ &\iff a + b > 0 \text{ and } ab > c^2. \end{aligned}$$

The symmetry of the latter condition is attractive. But we have focused on the first of our conditions because of its relevance for Chapter 5.)

(ii) Similarly, provided $a \neq 0$,

$$\begin{aligned} q \text{ is negative-definite} &\iff a < 0 \text{ and } \frac{ab - c^2}{a} < 0 \\ &\iff a < 0 \text{ and } ab - c^2 > 0 \\ &\iff a < 0 \text{ and } ab > c^2. \end{aligned}$$

Furthermore, q cannot be negative-definite in the second or third cases because if $b < 0$ then $-\dfrac{c^2}{b} \geqslant 0$, and if $\dfrac{c}{2} < 0$ then $-\dfrac{c}{2} > 0$.

Hence, from the three cases,

$$q \text{ is negative-definite} \iff a < 0 \text{ and } ab > c^2.$$

(Note that, from $ab > c^2$, it follows that $ab > 0$ and so a, b have the same sign. This means that

$$\begin{aligned} q \text{ is negative-definite} &\iff b < 0 \text{ and } ab > c^2 \\ &\iff a + b < 0 \text{ and } ab > c^2. \end{aligned}$$

Once again, we have focused on the first of our conditions because of its relevance for Chapter 5.)

(iii) Provided $a \neq 0$,

$$\begin{aligned} q \text{ is indefinite} &\iff a \text{ and } \frac{ab - c^2}{a} \text{ have opposite signs} \\ &\iff ab - c^2 < 0 \\ &\iff ab < c^2. \end{aligned}$$

In the case $a = 0$ and $b \neq 0$,

$$\begin{aligned} q \text{ is indefinite} &\iff b \text{ and } -\frac{c^2}{b} \text{ have opposite signs} \\ &\iff -c^2 < 0 \\ &\iff 0 < c^2 \\ &\iff ab < c^2. \end{aligned}$$

In the case $a = 0$ and $b = 0$,

$$\begin{aligned} q \text{ is indefinite} &\iff \frac{c}{2} \text{ and } -\frac{c}{2} \text{ have opposite signs} \\ &\iff c \neq 0 \\ &\iff 0 < c^2 \\ &\iff ab < c^2. \end{aligned}$$

Hence, from the three cases,

$$q \text{ is indefinite} \iff ab < c^2.$$

It is already clear that if $ab = c^2$ then q must be positive-semidefinite or negative-semidefinite.

(iv) Provided $a \neq 0$,

$$\begin{aligned} q \text{ is positive-semidefinite} &\iff a \geqslant 0 \text{ and } \frac{ab - c^2}{a} \geqslant 0 \\ &\iff a \geqslant 0 \text{ and } ab - c^2 \geqslant 0 \\ &\iff a \geqslant 0 \text{ and } ab \geqslant c^2 \\ &\iff a + b \geqslant 0 \text{ and } ab \geqslant c^2. \end{aligned}$$

(Here, since $a \neq 0$,

$$a \geqslant 0 \iff a > 0.$$

Also, from $ab \geqslant c^2$, it follows that $ab \geqslant 0$, i.e. a, b cannot have opposite signs.)

In the case $a = 0$ and $b \neq 0$,

$$\begin{aligned} q \text{ is positive-semidefinite} &\iff b \geqslant 0 \text{ and } -\frac{c^2}{b} \geqslant 0 \\ &\iff b \geqslant 0 \text{ and } -c^2 \geqslant 0 \\ &\iff a + b \geqslant 0 \text{ and } ab \geqslant c^2. \end{aligned}$$

(Here, since $a = 0$ and $b \neq 0$, we have

$$b \geqslant 0 \iff b > 0,$$

$a + b = b$ and $ab = 0$.)

In the case $a = 0$ and $b = 0$,

$$\begin{aligned} q \text{ is positive-semidefinite} &\iff \frac{c}{2} \geqslant 0 \text{ and } -\frac{c}{2} \geqslant 0 \\ &\iff c \geqslant 0 \text{ and } c \leqslant 0 \\ &\iff c = 0 \\ &\iff a + b \geqslant 0 \text{ and } ab \geqslant c^2. \end{aligned}$$

Hence, from the three cases,

$$q \text{ is positive-semidefinite} \iff a + b \geqslant 0 \text{ and } ab \geqslant c^2.$$

(v) Provided $a \neq 0$,

$$\begin{aligned} q \text{ is negative-semidefinite} &\iff a \leqslant 0 \text{ and } \frac{ab - c^2}{a} \leqslant 0 \\ &\iff a \leqslant 0 \text{ and } ab - c^2 \geqslant 0 \\ &\iff a \leqslant 0 \text{ and } ab \geqslant c^2 \\ &\iff a + b \leqslant 0 \text{ and } ab \geqslant c^2. \end{aligned}$$

(Here, since $a \neq 0$,

$$a \leqslant 0 \iff a < 0.$$

Also, from $ab \geqslant c^2$, it follows that $ab \geqslant 0$, i.e. a, b cannot have opposite signs.)

In the case $a = 0$ and $b \neq 0$,

$$\begin{aligned} q \text{ is negative-semidefinite} &\iff b \leqslant 0 \text{ and } -\frac{c^2}{b} \leqslant 0 \\ &\iff b \leqslant 0 \text{ and } -c^2 \geqslant 0 \\ &\iff a + b \leqslant 0 \text{ and } ab \geqslant c^2. \end{aligned}$$

(Here, since $a = 0$ and $b \neq 0$, we have

$$b \geqslant 0 \iff b > 0,$$

$a + b = b$ and $ab = 0$.)

In the case $a = 0$ and $b = 0$,

$$\begin{aligned} q \text{ is negative-semidefinite} &\iff \frac{c}{2} \geqslant 0 \text{ and } -\frac{c}{2} \geqslant 0 \\ &\iff c \geqslant 0 \text{ and } c \leqslant 0 \\ &\iff c = 0 \\ &\iff a + b \leqslant 0 \text{ and } ab \geqslant c^2. \end{aligned}$$

Hence, from the three cases,

$$q \text{ is negative-semidefinite} \iff a + b \leqslant 0 \text{ and } ab \geqslant c^2.$$

T5 For each of the following matrices A, find A^*:

a)

$$A = \begin{bmatrix} 2+i & 1 \\ 2 & -3i \\ 0 & 1+5i \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 2i & 1-i & -1+i \\ 4 & 5-7i & -i \end{bmatrix}.$$

Solution

a)

$$A = \begin{bmatrix} 2-i & 2 & 0 \\ 1 & 3i & 1-5i \end{bmatrix}$$

b)

$$A = \begin{bmatrix} -2i & 4 \\ 1+i & 5+7i \\ -1-i & i \end{bmatrix}.$$

T6 For each of the following matrices A, substitute numbers for the ?s so that A is Hermitian:

a)

$$A = \begin{bmatrix} 3 & ? & ? \\ 3+2i & -2 & ? \\ 7 & 1-5i & 6 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 2 & 0 & 3+5i \\ ? & -4 & -i \\ ? & ? & 6 \end{bmatrix}.$$

Solution

a)

$$A = \begin{bmatrix} 3 & 3-2i & 7 \\ 3+2i & -2 & 1+5i \\ 7 & 1-5i & 6 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 2 & 0 & 3+5i \\ 0 & -4 & -i \\ 3-5i & i & 6 \end{bmatrix}.$$

T7 For each of the following matrices A, show that A is not Hermitian for any choice of the ?s:

a)

$$A = \begin{bmatrix} 2 & ? & 3-7i \\ ? & 1+i & 2i \\ 3+7i & ? & 0 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 2 & 4+i & ? \\ -4+i & -1 & ? \\ ? & ? & ? \end{bmatrix}$$

c)

$$A = \begin{bmatrix} 1 & 1+i & ? \\ 1+i & 7 & ? \\ 6-2i & ? & 0 \end{bmatrix}$$

d)

$$A = \begin{bmatrix} 1 & ? & 3+5i \\ ? & 3 & 1-i \\ 3-5i & ? & 2+i \end{bmatrix}.$$

Solution

a) Non-real entry on the diagonal.

b) $(1,2)$- and $(2,1)$-entries are not conjugate.

c) $(1,2)$- and $(2,1)$-entries are not conjugate.

d) Non-real entry on the diagonal.

T8 For each of the following matrices A, show that A is unitary and find A^{-1}:

a)

$$A = \begin{bmatrix} \frac{4}{5} & \frac{3i}{5} \\ -\frac{3i}{5} & -\frac{4}{5} \end{bmatrix}$$

b)

$$A = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

Solution

In each case, it is enough to show that $A^*A = I$, and then A is unitary with $A^{-1} = A^*$.

a)

$$A^{-1} = \begin{bmatrix} \frac{4}{5} & \frac{3i}{5} \\ -\frac{3i}{5} & -\frac{4}{5} \end{bmatrix} = A.$$

b)

$$A^{-1} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \\ -\frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

T9 For each of the following Hermitian matrices A, find a unitary matrix P that diagonalises A, and determine $P^{-1}AP$:

a)

$$A = \begin{bmatrix} 9 & 12i \\ -12i & 16 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 0 & 3+i \\ 3-i & -3 \end{bmatrix}$$

c)

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & -1+i \\ 0 & -1-i & 0 \end{bmatrix}$$

d)

$$A = \begin{bmatrix} 2 & \frac{1}{\sqrt{2}}i & -\frac{1}{\sqrt{2}}i \\ -\frac{1}{\sqrt{2}}i & 2 & 0 \\ \frac{1}{\sqrt{2}}i & 0 & 2 \end{bmatrix}.$$

Solution

a) $P = \begin{bmatrix} \frac{3i}{5} & -\frac{4i}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$ gives $P^{-1}AP = \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix}$.

b) $P = \frac{1}{\sqrt{14}} \begin{bmatrix} 2 & 3+i \\ -3+i & 2 \end{bmatrix}$ gives $P^{-1}AP = \begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}$.

c) $P = \frac{1}{\sqrt{6}} \begin{bmatrix} 0 & 0 & 1 \\ 2 & -1+i & 0 \\ 1+i & 2 & 0 \end{bmatrix}$ gives $P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

d) $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{i}{2} & \frac{1}{\sqrt{2}} & -\frac{i}{2} \\ -\frac{i}{2} & \frac{1}{\sqrt{2}} & \frac{i}{2} \end{bmatrix}$ gives $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

T10 Let A be any $n \times n$ matrix with complex entries, and define the matrices B and C by

$$B = \frac{1}{2}(A + A^*), \qquad C = \frac{1}{2i}(A - A^*).$$

a) Show that B and C are Hermitian.

b) Show that $A = B + iC$ and $A^* = B - iC$.

c) What condition must B and C satisfy for A to be normal?

Solution

a) $B^* = \frac{1}{2}(A^* + A) = B$ and $C^* = \frac{-1}{2i}(A^* - A) = C$.

b) Substitute.

c) $AA^* = B^2 + C^2 + (CB - BC)i$ and $A^*A = B^2 + C^2 + (BC - CB)i$, so A is normal if and only if $CB - BC = BC - CB$, i.e. $BC = CB$, i.e. B and C commute.

T11 Let A be an $n \times n$ matrix with complex entries, and let $\mathbf{u}$ and $\mathbf{v}$ be column vectors in $\mathbb{C}^n$. Prove that

$$A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^*\mathbf{v} \qquad \text{and} \qquad \mathbf{u} \cdot A\mathbf{v} = A^*\mathbf{u} \cdot \mathbf{v}.$$

Solution

$A\mathbf{u} \cdot \mathbf{v} = (A\mathbf{u})^*\mathbf{v} = \mathbf{u}^*A^*\mathbf{v} = \mathbf{u} \cdot A^*\mathbf{v}$ and $\mathbf{u} \cdot A\mathbf{v} = \mathbf{u}^*A\mathbf{v} = (A^*\mathbf{u})^*\mathbf{v} = A^*\mathbf{u} \cdot \mathbf{v}$.

T12 Prove that the eigenvalues of a unitary matrix have modulus 1.

Solution

Let λ be an eigenvalue of a unitary matrix A. Choose an eigenvector $\mathbf{u}$.

First solution: putting $\mathbf{v} = A\mathbf{u}$ in the first equation of the last question gives $\|A\mathbf{u}\|^2 = \|\mathbf{u}\|^2$ (since $A^*A = I$). But $A\mathbf{u} = \lambda\mathbf{u}$, so $|\lambda|^2\|\mathbf{u}\|^2 = \|\mathbf{u}\|^2$. Since $\mathbf{u} \neq \mathbf{0}$, this implies that $|\lambda| = 1$.

Second solution: we have

$$(A\mathbf{u}) \cdot (A\mathbf{u}) = (\lambda\mathbf{u}) \cdot (\lambda\mathbf{u}) = \bar{\lambda}\lambda(\mathbf{u} \cdot \mathbf{u}) = |\lambda|^2\|\mathbf{u}\|^2,$$

but also

$$(A\mathbf{u}) \cdot (A\mathbf{u}) = (A\mathbf{u})^*A\mathbf{u} = \mathbf{u}^*A^*A\mathbf{u} = \mathbf{u}^*\mathbf{u} = \|\mathbf{u}\|^2,$$

so $|\lambda|^2\|\mathbf{u}\|^2 = \|\mathbf{u}\|^2$. Since $\mathbf{u} \neq \mathbf{0}$, this implies that $|\lambda| = 1$.

T13 Let $\mathbf{u}$ be a nonzero column vector in $\mathbb{C}^n$. Prove that the matrix $P = \mathbf{u}\mathbf{u}^*$ is Hermitian.

Solution

$P^* = (\mathbf{u}\mathbf{u}^*)^* = \mathbf{u}^{**}\mathbf{u}^* = \mathbf{u}\mathbf{u}^* = P$.

T14 Let A be an invertible matrix. Prove that A^* is invertible and that $(A^*)^{-1} = (A^{-1})^*$.

Solution

$(A^{-1})^*A^* = (AA^{-1})^* = I^* = I$ and $A^*(A^{-1})^* = (A^{-1}A)^* = I^* = I$, so by definition of inverse matrix, A^* is invertible with inverse $(A^{-1})^*$.