2C Intro to real analysis 2020/21

Ex Sheet 3

Solutions and Comments [1] [2]

Q1 *Show that the function $f : \mathbb{N} \to \mathbb{R}$ given by*

$$f(n) = \frac{3n^2 - 5n + 2}{2n^2 + 6n + 1}$$

is bounded above.

Remember first what we're trying to do here. By definition, a function $f : \mathbb{N} \to \mathbb{R}$ is bounded above if there exists $M \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, we have $f(n) \leq M$. Thus we need to find such an M. Since the domain of f is the natural numbers, it suffices to find an upper bound for $f(n)$ for large n, that is, we need to find some constant $M_0 \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \implies f(n) \leq M_0.$$

The point is that there are only finitely many values of $n \in \mathbb{N}$ with $n < n_0$, namely $n = 1, 2, \ldots, n_0 - 1$, so we can take the largest value of $f(1), \ldots, f(n_0 - 1)$ and M_0 for our upper bound M.

One possible approach is the systematic method via Lemma 1.9. More precisely, we can argue as follows.

By Lemma 1.9, there exists $n_1, n_2 \in \mathbb{N}$ such that

$$\begin{aligned} n \geq n_1 &\implies \frac{1}{2}3n^2 \leq 3n^2 - 5n + 2 \leq \frac{3}{2}3n^2, \\ n \geq n_2 &\implies \frac{1}{2}2n^2 \leq 2n^2 + 6n + 1 \leq \frac{3}{2}2n^2. \end{aligned}$$

Take $n_0 = \max(n_1, n_2)$. Then for $n \geq n_0$, we have

$$f(n) = \frac{3n^2 - 5n + 2}{2n^2 + 6n + 1} \leq \frac{\frac{3}{2}3n^2}{\frac{1}{2}2n^2} = \frac{9}{2}.$$

Take $M = \max(f(1), f(2), \ldots, f(n_0 - 1), \frac{9}{2})$. Then for all $n \in \mathbb{N}$, we have $f(n) \leq M$, so f is bounded above by M.

Alternatively, we can find an upper bound by a direct argument.

For $n \in \mathbb{N}$, we have

$$\frac{3n^2 - 5n + 2}{2n^2 + 6n + 1} \leq \frac{3n^2 - 5 \cdot 0 + 2n^2}{2n^2 + 0 + 0} = \frac{5}{2},$$

so $5/2$ is an upper bound for f.

[1] If you've not seriously tried the exercises, please don't look at these solutions and comments, until you have. You'll get the most benefit from reading these comments, when you've first thought hard about them yourself, even if you get really stuck — don't just try for a few minutes and then look at the solutions to work out how to proceed, you don't learn anywhere near as much that way.

[2] Note that I deliberately do not include formal answers for all questions.

Both methods are equally valid, and there are of course infinitely many other valid upper bounds[3]. We could also have noted that $-5n+2 \leq 0$ for all $n \in \mathbb{N}$ so that

$$\frac{3n^2-5n+2}{2n^2+6n+1} \leq \frac{3n^2+0}{2n^2+0+0} = \frac{3}{2},$$

for all n, and hence $\frac{3}{2}$ is also an upper bound for f.

Once we've learnt about convergent sequences, we could proceed in still another way. More precisely, using properties of limits we could argue that the sequence $(f(n))_{n=1}^{\infty}$ is convergent, and hence bounded.

Q2 *State the least upper bound and greatest lower bounds of the following sets, where they exist. You do not need to justify your answer.*

a) $A = (-5,2) \cup [3,7]$

b) $B = \{x^2+3 \mid x \in \mathbb{R}\}$;

c) $C = \{x \in \mathbb{R} \mid x^3 < 5\}$;

d) $D = \{m + 1/n \mid m, n \in \mathbb{N}\}$.

a) $\sup(A) = 7$ and $\inf(A) = -5$.

b) Note that B is not bounded above, so B does not have a least upper bound. We have $\inf(B) = 3$.

c) $\sup(C) = 5^{1/3}$, and C is not bounded below so $\inf(C)$ does not exist.

d) D is not bounded above, so does not have a least upper bound. We have $\inf(D) = 1$.

In (d), remember that in this course, the smallest natural number is 1; this is why we get $\inf(D) = 1$.

Q3 *Prove carefully that the least upper bound of*

$$\left\{ \frac{n-1}{n+1} \mid n \in \mathbb{N} \right\}$$

is 1.

We have two things to do. First we must show that 1 is an upper bound for the given set. This is straightforward, as $n-1 \leq n+1$ for all n: all we need to do is write something which makes it clear that we have checked this[4].

[3] Remember in particular that if M is an upper bound then any M' with $M' \geq M$ is an upper bound as well.

[4] Note that I start my answer by introducing some notation for the set in the question. This is always a good idea.

Let $A = \left\{ \frac{n-1}{n+1} \mid n \in \mathbb{N} \right\}$. For $n \in \mathbb{N}$, we have

$$\frac{n-1}{n+1} \leq 1,$$

so 1 is an upper bound for A.

For the second part of our answer we must show that 1 is the *least* upper bound of A. We do this by proving the quantified statement

$$\forall \varepsilon > 0, \exists a \in A \text{ s.t. } a > 1 - \varepsilon,$$

which was shown to be equivalent to 1 being the least upper bound of A in lemma 2.7. To remember this, think about what we need to do. To show that 1 is the least upper bound for A, we need to show that no number less than 1 can be an upper bound for A. This is precisely what the quantified statement does: it considers a general number less than 1, written in the form $1 - \varepsilon$ for $\varepsilon > 0$, and shows that $1 - \varepsilon$ is not an upper bound for A.

We start our answer by fixing an arbitrary value of $\varepsilon > 0$, for which we must find $a \in A$ with $a > 1 - \varepsilon$. It is vital to introduce our ε before starting to work with it.

Let $\varepsilon > 0$ be arbitrary.

We now need to find $a \in A$ with $a > 1 - \varepsilon$. The general form of an element of A is $\frac{n-1}{n+1}$ for $n \in \mathbb{N}$, so we need to find $n \in \mathbb{N}$ with $\frac{n-1}{n+1} > 1 - \varepsilon$. Thus we rearrange this inequality and demonstrate that there is a natural number satisfying it.

For $n \in \mathbb{N}$,

$$\frac{n-1}{n+1} > 1 - \varepsilon \Leftrightarrow n - 1 > n + 1 - \varepsilon(n+1) \Leftrightarrow \frac{2}{\varepsilon} < n + 1 \Leftrightarrow \frac{2}{\varepsilon} - 1 < n.$$

Thus we take $n \in \mathbb{N}$ with $n > \frac{2}{\varepsilon} - 1$, and then $\frac{n-1}{n+1} \in A$ and $\frac{n-1}{n+1} > 1 - \varepsilon$. Hence 1 is the least upper bound of A.

The point here is that we do know that there is some $n \in \mathbb{N}$ with $n > \frac{2}{\varepsilon} - 1$ (as the natural numbers are not bounded above), so it follows that the original inequality $\frac{n-1}{n+1} < 1 - \varepsilon$ has a solution. If we'd just said "take $n \in \mathbb{N}$ with $\frac{n-1}{n+1} < 1 - \varepsilon$" we wouldn't have addressed the question of why there was such an n.

Q4 *What is the greatest lower bound of*

$$\left\{ \frac{4n+3}{n-1} \mid n \in \mathbb{N}, n \geq 2 \right\}?$$

Carefully justify your answer.

Here we need to work out what the greatest lower bound is. Write $B = \left\{ \frac{4n+3}{n-1} \mid n \in \mathbb{N} \right\}$. Firstly, what happens for large n? We can see that as n gets large the fraction $\frac{4n+3}{n-1}$ is approximately 4, so we will be able to find elements in B arbitrary close[5] to 4. On the other hand, by inspection every element of B is at least 4, so 4 is a lower bound for B. Therefore 4 should be the greatest lower bound.

[5] In the next chapter we will define what it means to say that $\lim_{n\to\infty} \frac{4n+3}{n-1} = 4$.

Having found the right value, we now write up a formal answer in a similar fashion to the previous question.

> Write $B = \left\{ \frac{4n+3}{n-1} \mid n \in \mathbb{N}, n \geq 2 \right\}$. For $n \in \mathbb{N}$ with $n \geq 2$, we have
> $$\frac{4n+3}{n-1} \geq \frac{4n-4}{n-1} = 4,$$
> so 4 is a lower bound for B. Now let $\varepsilon > 0$ be arbitrary. For $n \in \mathbb{N}$ with $n \geq 2$, we have
> $$\frac{4n+3}{n-1} < 4 + \varepsilon \Leftrightarrow 4n+3 < 4n-4+\varepsilon(n-1) \Leftrightarrow 1 + \frac{7}{\varepsilon} < n,$$
> so take $n \in \mathbb{N}$ with $n > 1 + \frac{7}{\varepsilon}$ (this means in particular that $n \geq 2$), then $\frac{4n+3}{n-1} \in B$ satisfies $\frac{4n+3}{n-1} < 4 + \varepsilon$. Therefore 4 is the greatest lower bound for B.

Q5 *Give an example of a pair of non-empty subsets A and B of $\mathbb{R}$ with all of the following properties:*

a) $A \cap B = \emptyset$;

b) both A and B are bounded above;

c) $\sup(A) = \sup(B)$;

d) $\sup(A) \in A$ *and* $\sup(B) \notin B$.

The point to remember here is that the supremum of a set may or may not be in the set. We need to produce disjoint non-empty sets with the same suprema, so we can adopt a strategy of choosing any bounded set B which does not contain its suprema and then defining $A = \{\sup(B)\}$.

> Take $A = \{1\}$ and $B = (0, 1)$. Then $\sup(A) = \sup(B) = 1$, and the sets satisfy the required conditions.

Suppose the question was changed to ask for a pair of non-empty disjoint subsets A and B of $\mathbb{R}$ which are bounded above and satisfy

a) $\sup(A) = \sup(B)$ and

b) $\sup(A) \notin A$ and $\sup(B) \notin B$.

How would you proceed now?

Q6 *Let A be a non-empty subset of $\mathbb{R}$ and let m be a lower bound for A. Prove that m is the greatest lower bound*[6] *if and only if for all $\varepsilon > 0$, there exists $a \in A$ with $a < m + \varepsilon$.*

[6] that is, every lower bound m' for A has $m' \leq m$.

This asks you to adjust lemma 2.7 from the lectures. An answer is below.

> Let m be a lower bound for A. Suppose m is the greatest lower bound and let $\varepsilon > 0$ be arbitrary. Then $m + \varepsilon > m$, so $m + \varepsilon$ is not a lower bound for A. Therefore there exists $a \in A$ with $a < m + \varepsilon$.
>
> Conversely, suppose that m satisfies
>
> $$\forall \varepsilon > 0, \exists a \in A \text{ s.t. } a < m + \varepsilon.$$
>
> Given $m' > m$, define $\varepsilon = m' - m > 0$. Then the hypotheses ensure that there exists $a \in A$ with $a < m + \varepsilon = m'$, so m' is not a lower bound for A. Therefore m is the greatest lower bound for A.

Q7 *Let A be a non-empty subset of $\mathbb{R}$ which is bounded above. Prove that*

$$\sup\{5a + 3 \mid a \in A\} = 5 \sup(A) + 3.$$

It will certainly help to have some notation, so we start by giving the set in the question a name.

> Let $B = \{5a + 3 \mid a \in A\}$.

Here we need to prove that the least upper bound of B is $5 \sup(A) + 3$, so we show that this is an upper bound, and then that it satisfies the condition

$$\forall \varepsilon > 0, \exists b \in B \text{ s.t. } b > 5 \sup(A) + 3 - \varepsilon.$$

This is just the same as question 3, the only difference is that we will need to use the fact that $\sup(A)$ is the least upper bound of A to do these steps. Firstly, we do the easy bit by showing that $5 \sup(A) + 3$ is an upper bound for B. This uses the fact that $\sup(A)$ is an upper bound for A.

> For $a \in A$, we have $a \leq \sup(A)$, as $\sup(A)$ is an upper bound for A. Then $5a + 3 \leq 5 \sup(A) + 3$ so $5 \sup(A) + 3$ is an upper bound for B.

Now we come to second part. We need to prove a statement about all $\varepsilon > 0$, so we start in a way which should be familiar now.

> Let $\varepsilon > 0$ be arbitrary.

Now we need to find $a \in A$ with $5a + 3 > 5\sup(A) + 3 - \varepsilon$. Rearranging this, you can see that we need a to satisfy $a > \sup(A) - \varepsilon/5$. We can do this, as $\sup(A)$ is the least upper bound for A, so satisfies

$$\forall \eta > 0, \exists a \in A \text{ s.t. } a > \sup(A) - \eta.$$

Note that I don't use ε here — ε is fixed, so it makes no sense for it to be a dummy variable in a quantified statement. Moreover, we want to use a certain value of η depending on ε, namely $\eta = \varepsilon/5$, in this statement.

> As $\sup(A)$ is the least upper bound for A and $\varepsilon/5 > 0$, there exists $a \in A$ with $a > \sup(A) - \varepsilon/5$. Then $5a + 3 \in B$ and has $5a + 3 > 5(\sup(A) - \varepsilon/5) + 3 = 5\sup(A) + 3 - \varepsilon$. Therefore $5\sup(A) + 3$ is the least upper bound for B, as required.

Can you see why it wouldn't work if we started by taking $a \in A$ with $a > \sup(A) - \varepsilon$?

Alternatively, we could also verify directly the condition that if N is any upper bound for B then $5\sup(A) + 3 \leq N$. Recall that, by definition, this means that $5\sup(A) + 3$ is the least upper bound for B.

More precisely, assume that N is any upper bound for B. Then we have $5a + 3 \leq N$ for all $a \in A$, which implies $a \leq \frac{1}{5}(N - 3)$ for all $a \in A$. That is, $\frac{1}{5}(N - 3)$ is an upper bound for A. Since $\sup(A)$ is the least upper bound for A we obtain $\sup(A) \leq \frac{1}{5}(N - 3)$. Now we can rewrite this condition again and obtain $5\sup(A) + 3 \leq N$ as desired.

Notice that the second proof is shorter and simpler — it is always good to keep in mind that there are usually several ways of proving statements, and sometimes finding a quicker argument can save you work.

Q8 *Let A and B be non-empty bounded subsets of $\mathbb{R}$. Let*

$$C = \{3a - 2b \mid a \in A, b \in B\}.$$

Find a formula for $\inf(C)$ in terms of $\sup(A), \sup(B), \inf(A)$ and $\inf(B)$ (not all these expressions need appear in your formula) and carefully prove that your formula holds.

The first thing we need to do here is identify $\inf(C)$. If you struggle with this, why not proceed in stages? What is the greatest lower bound of $\{3a \mid a \in A\}$ going to be? Can you see why this should be $3\inf(A)$? Now what is the greatest lower bound of $\{-2b \mid b \in B\}$? Note that for $-2b$ to be small we need to take b as large as possible; this leads us to see that the greatest lower bound of $\{-2b \mid b \in B\}$ is $-2\sup(B)$. Combining these, we guess that $\inf(C) = 3\inf(A) - 2\sup(B)$. Now we prove this formally[7].

[7] Can you see why I use $\varepsilon/5$ in the following answer?

For $a \in A$ and $b \in B$ we have $a \geq \inf(A)$ and $b \leq \sup(B)$, so $3a - 2b \geq 3\inf(A) - 2\sup(B)$. Thus $3\inf(A) - 2\sup(B)$ is a lower bound for C.

Now, let $\varepsilon > 0$ be arbitrary. As $\varepsilon/5 > 0$, there exists $a \in A$ and $b \in B$ with $a < \inf(A) + \varepsilon/5$ and $b > \sup(B) - \varepsilon/5$ by the defining properties of $\inf(A)$ and $\sup(B)$, respectively. Then $3a - 2b \in C$ and $3a - 2b < 3(\inf(A) + \varepsilon/5) - 2(\sup(B) - \varepsilon/5) = 3\inf(A) - 2\sup(B) + \varepsilon$. Thus $3\inf(A) - 2\sup(B)$ is the greatest lower bound of C.

Alternatively, in the same way as in the previous exercise you can try to verify directly the condition for a greatest lower bound for $3\inf(A) - 2\sup(B)$. I'll leave it to you to work out the details.