Mathematics 2A - Lecture 1

• Functions of several variables

<u>Recall</u>: a function $f : D \longrightarrow R$ is a process
domain   codomain
which assigns to each element of D a unique element of R.
We have also:

- $Im(f) = \{ f(d) \mid d \in D \} \subseteq R$ is the image of f.
- $graph(f) = \{ (d, f(d)) \mid d \in D \} \subseteq D \times R$ is the graph of f.

To define a function we need:
- domain/codomain
- some expression for f

<u>Example</u>: $f(x) = \sqrt{1 - x^2}$

in order for $f(x)$ to make sense, $1 - x^2 \geqslant 0 \implies x \in [-1, 1]$

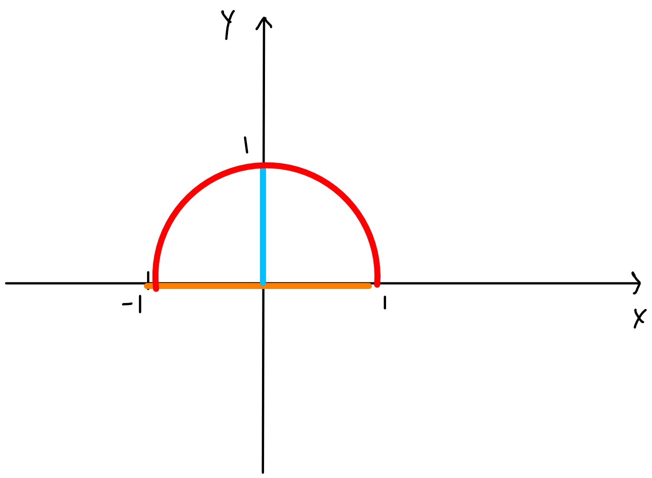
Hence our $D \subseteq [-1, 1]$

maximal domain for f: largest domain for which the expression for f makes sense

so we have a function
$$\begin{aligned} f : D &\longrightarrow R = \mathbb{R} \\ x &\longrightarrow f(x) = \sqrt{1 - x^2} \end{aligned}$$

Convention: if D is not specified we assume it is the max domain, so in this case $D = [-1, 1]$

In a picture:



$D = [-1,1]$ domain

$R = \mathbb{R}$ the y-axis is the codomain

$\operatorname{Im} f = [0,1]$ is the image of f

graph(f) is part of a curve

In this course:

$D \subseteq \mathbb{R}^n = \{ (x_1, \dots, x_n) \mid x_i \in \mathbb{R} \}$

$R \subseteq \mathbb{R}^k$ (n and k may be different)

To start we consider $n = 2,\ k = 1$:

$$f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto f(x,y)$$

Example: volume of a cylinder (a tube) of radius x and height y: $f(x,y) = \pi x^2 y$

So we have $f : (0,+\infty) \times (0,+\infty) \longrightarrow \mathbb{R}$

$$(x, y) \longmapsto f(x,y) = \pi x^2 y$$

Note that $f(x,y)$ makes sense also for x, y negative.

According to our convention, if we wrote

$$f : D \longrightarrow \mathbb{R}, \quad f(x,y) = \pi x^2 y$$

then $D = \mathbb{R}^2$ is the maximal domain.

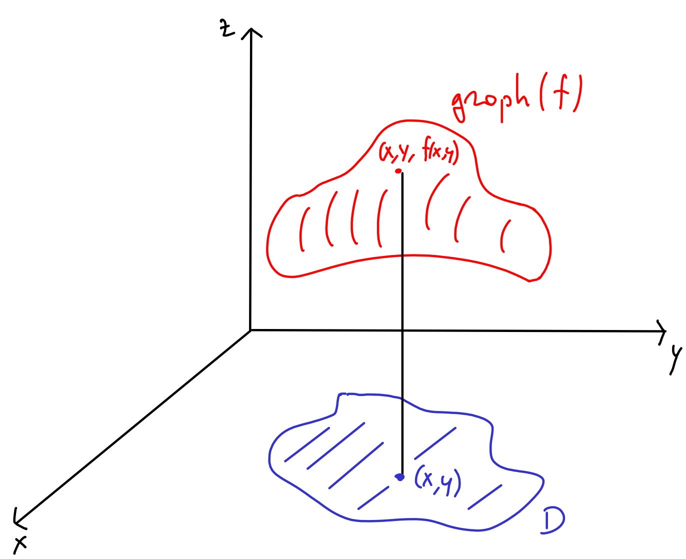
- Graphs of functions of two variables and surfaces

Let $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$. Then the graph of f is
$\quad (x,y) \longrightarrow f(x,y)$

$$\text{graph}(f) = \{ (x,y,f(x,y)) \mid (x,y) \in D \} \subseteq \mathbb{R}^3$$ and it is

now part of a surface!

Graphically:



More generally, a surface $S = \{ (x,y,z) \mid F(x,y,z) = 0 \}$ for some function $F(x,y,z)$.

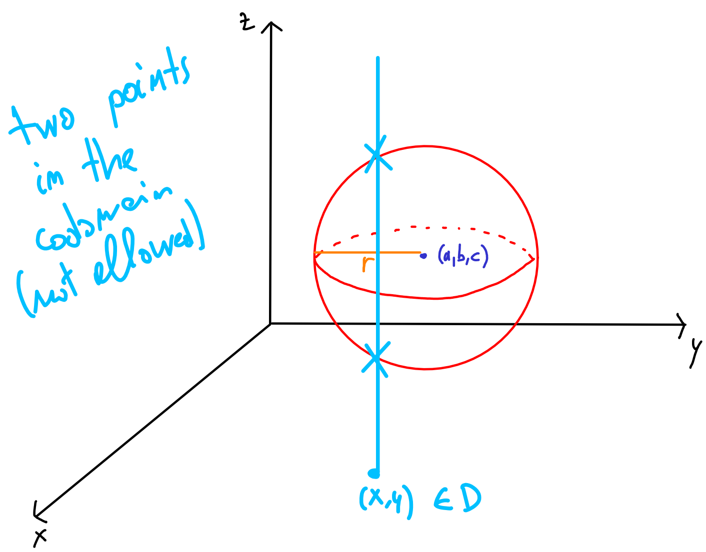
Example: the equation defining a sphere of radius r and centre (a,b,c) is:

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

The sphere is a surface: indeed let

$$F(x,y,z) = (x-a)^2 + (y-b)^2 + (z-c)^2 - r^2,$$

then the sphere can be defined as $S = \{ (x,y,z) \in \mathbb{R}^3 \mid F(x,y,z) = 0 \}$



the sphere is not the graph of a function!

We can rearrange the equation of the sphere as follows:

$$(z-c)^2 = r^2 - (x-a)^2 - (y-b)^2$$

$$\Rightarrow \underset{f(x,y)}{\underset{\parallel}{z}} = c \pm \sqrt{r^2 - (x-a)^2 - (y-b)^2}$$

upper hemishpere

lower hemisphere

the maximal domain for this function is

$$D = \{ (x,y) \mid r^2 - (x-a)^2 - (y-b)^2 \geq 0 \}$$

$$= \{ (x,y) \in \mathbb{R}^2 \mid (x-a)^2 + (y-b)^2 \leq r^2 \}$$

(disc centered in (a,b) of radius r)

Summarising

- the graph of a function f is a surface : just consider

$$F(x,y,z) = z - f(x,y)$$

- not all surfaces are graphs of some functions : as the spheres for example.

Example 1.1 from lecture notes :

Sketch the graph of $f(x,y) = -\sqrt{1-2x-x^2-y^2}$:

from the previous formula $f(x,y) = c - \sqrt{r^2-(x-a)^2-(y-b)^2}$ describes a lower hemisphere whose centre is (a,b,c) and radius r.

We immediately get $c = 0$ and $r^2-(x-a)^2-(y-b)^2 = 1-2x-\cancel{x^2}-\cancel{y^2}$

$$= r^2 - \cancel{x^2} + 2ax - a^2 - \cancel{y^2} + 2by - b^2$$

$$\Rightarrow \begin{cases} 2a = -2 \\ 2b = 0 \\ r^2 - a^2 - b^2 = 1 \end{cases} \Rightarrow \begin{cases} a = -1 \\ b = 0 \\ r^2 = 2 \end{cases} \Rightarrow \begin{cases} a = -1 \\ b = 0 \\ r = \sqrt{2} \end{cases}$$

(r being a radius should be positive)

The graph of $f(x,y) = -\sqrt{1-2x-x^2-y^2}$ is a lower hemisphere with centre $(-1,0,0)$ and radius $\sqrt{2}$

Graphically: