University of Glasgow

Wednesday, 13 December 2017
1.00 pm – 2.30 pm
(1 hour 30 minutes)

DEGREES OF MSci, MEng, BEng, BSc, MA and MA (Social Sciences)

ALGORITHMIC FOUNDATIONS 2: COMPSCI2003

Answer all questions

This examination paper is worth a total of 60 marks.

The use of calculators is not permitted in this examination.

INSTRUCTIONS TO INVIGILATORS: Please collect all exam question papers and exam answer scripts and retain for school to collect. Candidates must not remove exam question papers.

1. **(a)** The formula

$$\neg(p_1 \rightarrow \neg p_2) \rightarrow (p_3 \rightarrow (p_4 \rightarrow \neg p_5))$$

returns `false` for precisely one assignment to the propositions $p_1, \ldots, p_5$. Find this assignment using the laws of logical equivalence and *without* constructing a truth table for the formula. (Justify your answer.) [5]

(b) Define what it means for a function to be injective, surjective and bijective. [3]

(c) Is the function $f : \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = x + 2$ injective and/or surjective? (Justify your answer.) [2]

2. Let $P(x, y)$ be the predicate *variable x has value y* (i.e. $x = y$), where the universe of discourse is the set of program variables V and set of integers $\mathbb{Z}$. Write each of the following statements using only the above predicate, quantifiers and logical operators:

(a) if program variable x has value 3, then program variable y has value 4; [2]

(b) every variable has a value between 4 and 6. [2]

Let $Q(x, y)$ be the predicate $x^2 = y$. Express in English, and determine the truth value of, each of the following propositions:

(c) $Q(4, 2)$ [2]

(d) $\exists x \in \mathbb{Z}^+ . \forall y \in \mathbb{Z}^+ . Q(x, y)$ [2]

(e) $\exists x \in \mathbb{Z}^+ . \forall y \in \mathbb{Z}^+ . Q(y, x)$ [2]

3. **(a)** Use mathematical induction to show that the following holds for all $n \in \mathbb{Z}^+$ (positive integers):

$$\sum_{k=1}^{n} k^2 = \frac{n \cdot (n+1) \cdot (2 \cdot n+1)}{6}$$

Justify each step. [5]

(b) Prove the triangular inequality, which states that if x and y are real numbers then $|x + y| \leq |x| + |y|$ (where $|z|$ is the absolute value of z). State what method of proof you used. [5]

4. **(a)** A car number plate has two forms:

(i) three digits followed by three letters;

(ii) four letters followed by two digits.

How many car number plates are there (you can leave your answer in powers of 10 and 26). Explain your answer. [2]

(b) How may ways are there to choose 4 pieces of fruit from a bowl that contains 6 bananas, 5 apples, 7 pears, and 4 oranges? Explain your answer. [3]

(c) How many sequences of 8 bits are there, that start with a 0 or end with 111 (you can leave you answer in powers of 2)? Explain your answer. [3]

(d) How many students must be in a class to guarantee that at least 5 were born on the same day of the week? Explain your answer. [2]

5. Suppose you have one deck of playing cards consisting of 8 playing cards with 4 aces, 2 kings and 2 queens and a second deck of 8 playing cards which has 1 ace, 4 kings and 3 queens.

(a) What is the probability you randomly select an ace from the first deck. [1]

(b) Suppose you randomly select a card from both decks, what is the probability you select two queens. [2]

(c) Suppose you random select two cards from the first deck without replacement, what is the probability the second card you select is a king. [3]

(d) Suppose you randomly choose a card from the first deck, add it to the second deck and then random choose a card from the second deck. If the card drawn from the second deck was an ace, what is the probability that the card selected from the first deck was also an ace? [4]

6. Define what it means for a graph $G = (V,E)$ to be:

(a) connected; [2]

(b) a clique (complete). [2]

For each relation below, determine if the relation is reflexive, symmetric and/or antisymmetric, explain your answer.

(c) $R_1 = \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid x+y = 0\}$ [3]

(d) $R_2 = \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid x=1\}$ [3]