Wednesday, 9th December, 2020

EXAMINATION FOR THE DEGREES OF
M.A. AND B.Sc.

Mathematics 2C - Introduction to Real Analysis

This exam has two parts, consisting of a moodle quiz and the questions below. Candidates must attempt all questions.

1. (i) For each of the following sets, state whether the infimum exists and whether the supremum exists, and give their values when they do. You do not need to justify your answers.

   (a) $X = [-2, 4) \cup (-3, 2) \cup [3, 7]$.

   (b) $Y = \left\{ \frac{1}{x+4} + 2 \mid x < -4 \right\}$.

   (c) $Z = \{x \in \mathbb{R} \mid (x+1)^2 < 2\}$. **3**

   (ii) Let A be a set which has an infimum and supremum, with $\inf(A) > 0$. Define the set B by:

$$B = \left\{ a^2 + 2a + \frac{5}{a} \mid a \in A \right\}.$$

   Prove that $\sup(B)$ exists. **4**

2. (i) Prove *directly from the definition* that the following sequence converges:

$$x_n = \frac{2}{n} + 3.$$

2

   (ii) Calculate the limit

$$\lim_{n\to\infty} \left(\frac{n}{1+\frac{1}{n}} - n \right),$$

   stating clearly all properties of limits used. **2**

   (iii) Prove *directly from the definition* that the following sequence does not converge to any limit:

$$y_n = \begin{cases} 0 & \text{if } n \text{ is divisible by 3;} \\ 1/2 & \text{otherwise.} \end{cases}$$

3

3. (i) Prove that the sequence $x_n = (2^n + 3^n)^{1/n}$ is convergent. **2**

   (ii) Suppose $(x_n)_{n=1}^{\infty}$ is a sequence such that $\frac{x_n - 2}{x_n + 1} \to 0$ as $n \to \infty$. Show that $x_n \to 2$ as $n \to \infty$. **2**

4. For each of the series below, determine whether they converge or diverge. Justify your answers clearly, referring to any results or tests you use from the course. Any answer with no justification will receive no marks.

   (i)
$$\sum_{n=1}^{\infty} \frac{7^n}{n!}$$
3

   (ii)
$$\sum_{n=1}^{\infty} \frac{2}{3^n - \cos(2n)}$$
3

   (iii)
$$\sum_{n=1}^{\infty} \frac{n^3 - n}{3n^4 + 3n^2 + 2}$$
3

5. (i) Give an example of a conditionally convergent series with sum zero. Prove that your series is convergent. **3**

   (ii) Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series. Must $\sum_{n=1}^{\infty} a_n^2$ also converge? Either give a proof or give a counterexample. **2**

6. (i) Prove, directly from the definition of continuity, that the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = 4x - 7$ is continuous. **3**

   (ii) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function
$$f(x) = \begin{cases} x\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0. \end{cases}$$
   Show that f is continuous at $0 \in \mathbb{R}$. **2**

   (iii) Let $f\colon [-1, 1] \to [0, 2]$ be a continuous function. Show that there exists a point $x \in [-1, 1]$ for which $f(x) = 1 - x$. **3**

END]