Algorithmic Foundations 2 - Tutorial Sheet 6

Induction and Recursive Definitions

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1. Use the principle of mathematical induction to show $\sum_{i=1}^{n} i{\cdot}(i!) = (n{+}1)! - 1$ for all $n \in \mathbb{N}$.

> **Solution:** Let $P(n)$ be the proposition $\sum_{i=1}^{n} i{\cdot}(i!) = (n{+}1)! - 1$.
>
> *Base case:* $P(1)$ holds since $1{\cdot}(1!) = 1 = 2{-}1 = (1{+}1)! - 1$.
>
> *Inductive step:* We now assume $P(n)$ is true for some $n \in \mathbb{N}$. Considering $n{+}1$ we have:
>
> $$\begin{aligned} \textstyle\sum_{i=1}^{n+1} i{\cdot}(i!) &= \textstyle\sum_{i=1}^{n} i{\cdot}(i!) + (n{+}1){\cdot}(n{+}1)! \\ &= \Big((n+1)! - 1\Big) + (n+1){\cdot}(n+1)! && \text{by the inductive hypothesis} \\ &= \Big(1 + (n{+}1)\Big){\cdot}(n+1)! - 1 && \text{rearranging} \\ &= (n+2){\cdot}(n+1)! - 1 && \text{simplifying} \\ &= (n+2)! - 1 && \text{by definition of factorial} \end{aligned}$$
>
> and hence $P(n{+}1)$ holds.
>
> Therefore by the principle of induction we have proved that $P(n)$ holds for all $n \in \mathbb{N}$.

2. Use the principle of mathematical induction to show $3^n < n!$ for all $n{>}6$.

> **Solution:** Let $P(n)$ be the proposition $3^n < n!$.
>
> *Base case:* $P(7)$ is true, since $37 = 2187 < 5040 = 7!$
>
> *Inductive step:* Assume that $P(n)$ is true for some $n{>}6$. Now considering $n{+}1$ we have:
>
> $$\begin{aligned} 3^{n+1} &= 3{\cdot}3^n && \text{rearranging} \\ &< 3{\cdot}n! && \text{by the inductive hypothesis} \\ &< (n+1){\cdot}n! && \text{since } n{>}6 \\ &= (n+1)! && \text{by definition of factorial} \end{aligned}$$
>
> and hence $P(n{+}1)$ holds.
>
> Therefore by the principle of induction we have proved that $P(n)$ holds for all $n{>}6$.

3. Use the principle of mathematical induction to show $n^3 > n^2 + 3$ for all $n \geq 2$.

> **Solution:** Let $P(n)$ be the proposition $n^3 > n^2 + 3$.
>
> *Base case:* $P(2)$ is true, since $2^3 = 8 > 7 = 2^2 + 3$.

Inductive step: Assume that $P(n)$ is true for some $n \geq 2$ and consider $n{+}1$. Now, expanding we have:

$$\begin{aligned}
(n+1)^3 &= n^3 + 3{\cdot}n^2 + 3{\cdot}n + 1 \\
&> (n^2+3) + 3n^2 + 3n + 1 && \text{by the inductive hypothesis} \\
&= 4n^2 + 3n + 1 + 3 && \text{rearranging} \\
&\geq n^2 + 2n + 1 + 3 && \text{since } n{\geq}0 \\
&= (n{+}1)^2 + 3 && \text{since } (n{+}1)^2 = n^2 + 2{\cdot}n + 1
\end{aligned}$$

and hence $P(n{+}1)$ holds.

Therefore by the principle of induction we have proved that $P(n)$ holds for all $n \geq 2$.

4. Suppose that

- $a_1 = 2$;
- $a_2 = 9$;
- $a_n = 2{\cdot}a_{n-1} + 3{\cdot}a_{n-2}$ for $n \geq 3$.

Use (the second principle of) mathematical induction to show $a_n \leq 3^n$ for all $n \in \mathbb{Z}^+$.

Solution: Let $P(n)$ be the proposition that $a_n \leq 3^n$.

Base cases: $P(1)$ and $P(2)$ are true, since $a_1 = 2 \leq 3 = 3^1$ and $a_2 = 9 = 3^2$.

Inductive step: Let $n{\geq}2$ and assume that $P(k)$ is true for all $1{\leq}k{\leq}n$. Now by definition we have

$$\begin{aligned}
a_{n+1} &= 2{\cdot}a_n + 3{\cdot}a_{n-1} \\
&\leq 2{\cdot}3^n + 3{\cdot}3^{n-1} && \text{by the inductive hypothesis (using both } P(n) \text{ and } P(n{-}1)) \\
&= 2{\cdot}3^n + 3^n && \text{rearranging} \\
&= 3{\cdot}3^n && \text{rearranging} \\
&= 3^{n+1} && \text{and hence } P(n{+}1) \text{ holds.}
\end{aligned}$$

Therefore by the principle of induction we have proved that $P(n)$ holds for all $n \in \mathbb{Z}^+$.

5. Use the principle of mathematical induction to show a function f defined by specifying $f(0)$ and a rule for obtaining $f(n{+}1)$ from $f(n)$ (for each $n \geq 0$) is well-defined.

Solution: Let $P(n)$ be the proposition that $f(n)$ is well-defined.

Base case: $P(0)$ is true, since $f(0)$ is well-defined.

Inductive step: Assume that $P(n)$ is true for some $n \in \mathbb{Z}^+$. Now $f(n{+}1)$ is defined in terms of $f(n)$ and by the inductive hypothesis, $f(n)$ is well-defined. Therefore $f(n{+}1)$ is well-defined and $P(n{+}1)$ holds.

Therefore by the principle of induction we have proved that $P(n)$ holds for all $n \in \mathbb{Z}^+$.

6. Find $f(i)$ for $i = 1, 2, 3, 4$ given $f(n)$ is defined recursively by $f(0) = 3$ and for each $n \geq 0$:

   (a) $f(n{+}1) = -2{\cdot}f(n)$;

   Solution: -6, 12, -24, 48

   (b) $f(n{+}1) = 3{\cdot}f(n) + 7$;

   Solution: 16, 55, 172, 523

   (c) $f(n{+}1) = f(n)^2 - 2{\cdot}f(n) - 2$;

   Solution: 1, -3, 13, 141

   (d) $f(n{+}1) = 3{\cdot}f(n)/3$.

   Solution: 3, 3, 3, 3

7. Give a recursive definition for each of the following non-recursive definitions:

   (a) $g_1(n) = 4{\cdot}7^n$ for all $n \geq 0$;

   Solution: $g_1(0) = 4$ and $g_1(n{+}1) = 7{\cdot}g_2(n)$ for $n \geq 0$

   This can derived as follows: by definition we have $g_1(0) = 4{\cdot}7^0 = 4{\cdot}1 = 4$, while expanding $g_1(n{+}1)$ yields:

$$
\begin{aligned}
g_1(n{+}1) &= 4{\cdot}7^{n+1} && \text{by definition} \\
&= 7{\cdot}(4{\cdot}7^n) && \text{rearranging} \\
&= 7{\cdot}g_1(n) && \text{by definition of } g_1
\end{aligned}
$$

   (b) $g_2(n) = 3{\cdot}n + 5$ for all $n \geq 0$;

   Solution: $g_2(0) = 5$ and $g_2(n{+}1) = g_2(n) + 3$ for $n \geq 0$ This can derived as follows: by definition we have $g_2(0) = 3{\cdot}0 + 5 = 0 + 5 = 5$, while expanding $g_2(n{+}1)$ yields:

$$
\begin{aligned}
g_2(n{+}1) &= 3{\cdot}(n{+}1) + 5 && \text{by definition} \\
&= 3{\cdot}n + 3 + 5 && \text{rearranging} \\
&= (3{\cdot}n + 5) + 3 && \text{rearranging} \\
&= g_2(n) + 3 && \text{by definition of } g_2
\end{aligned}
$$

   (c) $g_3(n) = n!$ for all $n \geq 1$;

   Solution: $g_3(1) = 1$ and $g_3(n{+}1) = (n+1){\cdot}g_3(n)$ for $n \geq 1$ This can derived as follows: by definition we have $g_3(1) = 1! = 1$, while expanding $g_2(n{+}1)$ yields:

$$
\begin{aligned}
g_2(n{+}1) &= (n{+}1)! && \text{by definition} \\
&= (n{+}1){\cdot}n! && \text{rearranging since } n \geq 1 \\
&= (n{+}1){\cdot}g_3(n) && \text{by definition of } g_3
\end{aligned}
$$

(d) $g_4(n) = n^2$ for all $n \geq 0$.

Solution: $g_4(0) = 0$ and $g_4(n{+}1) = g_4(n) + 2{\cdot}n + 1$ for $n \geq 0$

This can derived as follows: by definition we have $g_4(0) = n^2 = 0$, while expanding $g_4(n{+}1)$ yields:

$$\begin{aligned} g_4(n{+}1) &= (n{+}1)^2 && \text{by definition} \\ &= n^2 + 2{\cdot}n + 1 && \text{rearranging} \\ &= g_4(n) + 2{\cdot}n + 1 && \text{by definition of } g_4 \end{aligned}$$

8. Give recursive definitions of the functions max and min, so that $\max(a_1, a_2, \ldots, a_n)$ and $\min(a_1, a_2, \ldots, a_n)$ are the maximum and minimum of the n real numbers $a_1, a_2, \ldots, a_n$ respectively.

Solution: The recursive definitions of the max and min functions are denoted here by $\max_r$ and $\min_r$ respectively.

$$\begin{aligned} \max_r(a_1) &= a_1 \\ \max_r(a_1, a_2, \ldots, a_n, a_{n+1}) &= \max(\max_r(a_1, a_2, \ldots, a_n), a_{n+1}) \\ \min_r(a_1) &= a_1 \\ \min_r(a_1, a_2, \ldots, a_n, a_{n+1}) &= \min(\min_r(a_1, a_2, \ldots, a_n), a_{n+1}) \end{aligned}$$

where

$$\max(x, y) = \begin{cases} y & \text{if } x \leq y \\ x & \text{if } x > y \end{cases} \quad \text{and} \quad \min(x, y) = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } x > y \end{cases}$$

9. Give a recursive definition of the following sets:

(a) the odd positive integers;

Solution: $1 \in S$ and if $x \in S$, then $x{+}2 \in S$

(b) the positive integer powers of 3;

Solution: $3 \in S$ and if $x \in S$, then $3{\cdot}x \in S$

(c) the polynomials with integer coefficients.

Solution: $q \in S$ for any $q \in \mathbb{Z}$ and if $p(x) \in S$, then $x{\cdot}p(x) + q \in S$ for any $q \in \mathbb{Z}$.

10. Give recursive definitions with initial condition(s) for each of the following sets:

(a) $\{0.1, 0.01, 0.001, \ldots\}$

Solution: $0.1 \in S$ and if $x \in S$, then $x/10 \in S$

(b) the set of positive integers congruent to 4 (mod 7)

Solution: $4 \in S$ and if $x \in S$, then $x{+}7 \in S$

(c) the set of integers not divisible by 3

Solution: $1 \in S$, $2 \in S$ and if $x \in S$, then $x{+}3 \in S$ and $x{-}3 \in S$

11. Assume that we have a list l, and are given the functions:

- `head`(l) which returns the first element of a non-empty list;
- `tail`(l) which returns the tail of a non-empty list;
- `isEmpty`(l) returns `true` if the list is empty and `false` otherwise.

For example if l equals $\langle 5,3,4,2,7,8,3,4\rangle$, then `head`$(l)$ would deliver 5, `tail`(l) would deliver $\langle 3,4,2,7,8,3,4\rangle$, and `isEmpty`$(l)$ would deliver `false`.

Using the above functions, in a pseudo code of your choice:

(a) write a recursive function `length`(l) that returns the length of the list l as an integer.

For example, `length`$(\langle 1,5,2,9,8,3,2\rangle)$ would return 7.

Solution:

$$\texttt{length}(l) = \textbf{if } \texttt{isEmpty}(l) \textbf{ then } 0 \textbf{ else } 1 + \texttt{length}(\texttt{tail}(l))$$

(b) write a recursive function `sum`(l), that returns the summation of the elements in a list.

For example, `sum`$(\langle 1,5,2,3\rangle)$ returns $1+5+2+3 = 11$.

Solution:

$$\texttt{sum}(l) = \textbf{if } \texttt{isEmpty}(l) \textbf{ then } 0 \textbf{ else } \texttt{head}(l) + \texttt{sum}(\texttt{tail}(l))$$

(c) write a recursive function `present`(e, l), that delivers `true` if e appears in the list l and `false` otherwise.

For example, `present`$(6, \langle 1,5,2,3\rangle)$ returns `false` and `present`$(4, \langle 1,2,3,1,2,4,2\rangle)$ returns `true`.

Solution:

$$\begin{aligned}\texttt{present}(e,l) = \ & \textbf{if } \texttt{isEmpty}(l) \textbf{ then } \texttt{false} \textbf{ else} \\ & \qquad \mathit{Equals}(e, \texttt{head}(l)) \vee \texttt{present}(e, \texttt{tail}(l))\end{aligned}$$

where $\mathit{Equals}(x, y)$ is the predicate that returns `true` if and only if $x{=}y$.

(d) write a recursive function `remove`(e, l) that removes all occurrences of e from the list l.

For example, `remove`$(5, \langle 1,5,2,3,5\rangle)$ returns $\langle 1,2,3\rangle$.

Solution:

$$\begin{aligned}\texttt{remove}(e,l) = &\textbf{if } \texttt{isEmpty}(l) \textbf{ then } l\\ &\textbf{else if } \mathit{Equals}(e,\texttt{head}(l)) \textbf{ then } \texttt{remove}(e,\texttt{tail}(l))\\ &\textbf{else } \langle \texttt{head}(l), \texttt{remove}(e,\texttt{tail}(l))\rangle\end{aligned}$$

Difficult/challenging questions.

12. Show that the set S defined by:

    - $5 \in S$;
    - if $s \in S$ and $t \in S$, then $s + t \in S$

    is the set of positive integers divisible by 5.

Solution: Let T be the set of positive integers divisible by 5. In order to show that $S = T$, we prove that $S \subseteq T$ and $T \subseteq S$.

- In order to prove that $S \subseteq T$, we use the following method of mathematical induction over the recursively defined set S:

  Let $P(s)$ be the proposition that $s \in T$, for each $s \in S$. The proof by induction consists of establishing the following:

  Base case: $P(5)$ holds;
  Inductive step: if $P(s)$ and $P(t)$ hold for $s \in S$ and $t \in S$, then $P(s{+}t)$ holds.

  Notice that this is a different form of induction from the one we have used previously; however, in view of the recursive definition of S, establishing each of these steps corresponds exactly to showing that $S \subseteq T$.

  Clearly the base case holds, since $5 = 5{\cdot}1$. For the inductive step, assume that $P(s)$ is true and $P(t)$ is true, for some $s \in S$ and $t \in S$. Then each of s and t is divisible by 5, so that $s{+}t$ is divisible by 5, and hence $P(s{+}t)$ is true.

  Thus by unduction $P(s)$ holds for all $s \in T$, and hence $S \subseteq T$.

- In order to prove that $T \subseteq S$, we again use induction again, but this time over $\mathbb{N}$ rather than over the recursive set S. Let $Q(n)$ be the proposition that $5{\cdot}n \in S$, for each $n \in \mathbb{Z}^+$.

  Base case: $Q(1)$ is true since $5 \in S$.

  Inductive step: Assume that $Q(n)$ is true for some $n \in \mathbb{Z}^+$. Now combining the facts:

  - using the inductive hypothesis we have $5{\cdot}n \in S$;
  - using the initial conditions of S we have $5 \in S$.
  - $5{\cdot}(n{+}1) = 5{\cdot}n + 5$;

– by the definition of S, if $s, t \in S$, then $s + t \in S$;

we have $5{\cdot}n + 5 \in S$, and hence $Q(n{+}1)$ is true.

Therefore by mathematical induction $Q(n)$ holds for all $n \in \mathbb{Z}^+$. Now suppose $t \in T$, by definition $t = 5{\cdot}k$ for some positive integer k and since $Q(k)$ holds, it follows that $t \in S$, and hence $T \subseteq S$ completing the proof.

13. Prove that

$$\sum_{j=0}^{n} \left(-\frac{1}{2}\right)^j = \frac{2^{n+1} + (-1)^n}{3{\cdot}2^n}$$

for all $n \in \mathbb{N}$.

Solution: Let $P(n)$ be the proposition that $\sum_{j=0}^{n}(-\frac{1}{2})^j = \frac{2^{n+1}+(-1)^n}{3\cdot 2^n}$, for each $n \in \mathbb{N}$.

Base case: For $P(0)$ we have:

$$\sum_{j=0}^{0} \left(-\frac{1}{2}\right)^j = \left(-\frac{1}{2}\right)^0 = 1 = \frac{3}{3} = \frac{2+1}{3{\cdot}1} = \frac{2^{0+1} + (-1)^n}{3{\cdot}2^0}$$

Inductive step: Assume that $P(n)$ holds for some $n \in \mathbb{N}$. To prove that $P(n{+}1)$ holds we will split into two cases: when n is even and when n odd.

- If n is even, then considering $n{+}1$ we have that:

$$\begin{aligned}
\sum_{j=0}^{n+1} \left(-\frac{1}{2}\right)^j &= \sum_{j=0}^{n} \left(-\frac{1}{2}\right)^j + \left(-\frac{1}{2}\right)^{n+1} && \text{rearranging} \\
&= \frac{2^{n+1} + (-1)^n}{3{\cdot}2^n} + \left(-\frac{1}{2}\right)^{n+1} && \text{by induction} \\
&= \frac{2^{n+1} + 1}{3{\cdot}2^n} - \frac{1}{2^{n+1}} && \text{since } n \text{ is even (and } n{+}1 \text{ is odd)} \\
&= \frac{2^{n+2} + 2 - 3}{3{\cdot}2^{n+1}} && \text{rearranging} \\
&= \frac{2^{(n+1)+1} - 1}{3{\cdot}2^{n+1}} && \text{rearranging} \\
&= \frac{2^{(n+1)+1} + (-1)^{n+1}}{3{\cdot}2^{n+1}} && \text{since } n \text{ is even (and } n{+}1 \text{ is odd)}
\end{aligned}$$

and hence $P(n{+}1)$ holds in this case.

- If n is odd, then considering $n{+}1$ we have:

$$\begin{aligned}
\sum_{j=0}^{n+2}\left(-\frac{1}{2}\right)^j &= \sum_{j=0}^{n}\left(-\frac{1}{2}\right)^j + \left(-\frac{1}{2}\right)^{n+1} && \text{rearranging}\\
&= \frac{2^{n+1}+(-1)^n}{3{\cdot}2^n} + \left(-\frac{1}{2}\right)^{n+1} && \text{by induction}\\
&= \frac{2^{n+1}-1}{3{\cdot}2^n} + \frac{1}{2^{n+1}} && \text{since } n \text{ is odd (and } n{+}1 \text{ is even)}\\
&= \frac{2^{n+2}-2+3}{3{\cdot}2^{n+1}} && \text{rearranging}\\
&= \frac{2^{n+2}+1}{3{\cdot}2^{n+1}} && \text{rearranging}\\
&= \frac{2^{((n+1)+1)+1}+1}{3{\cdot}2^{n+1}} && \text{rearranging}\\
&= \frac{2^{((n+1)+1)+1}+(-1)^{n+2}}{3{\cdot}2^{n+1}} && \text{since } n \text{ is odd (and } n{+}1 \text{ is even)}
\end{aligned}$$

  and hence $P(n{+}1)$ holds in this case.

Since these are the only cases to consider we have proved $P(n{+}1)$ holds.

Therefore by the principle of induction we have proved that $P(n)$ holds for all $n \in \mathbb{N}$.