EXAMINATION FOR THE DEGREES OF
M.A. AND B.Sc.

Mathematics 2C - Introduction to Real Analysis

This exam has two parts, consisting of a moodle quiz and the questions below. Candidates must attempt all questions.

1. (i) Show that the set

$$A = \{3x - 2y + \frac{1}{2z} \mid x, y, z \in (2,4)\}$$

is bounded. **3**

(ii) Show that

$$\inf \left\{ \frac{2n+5}{7n+7} \mid n \in \mathbb{N} \right\} = \frac{2}{7}.$$ **3**

2. Prove the following statements *directly from the definition.*

(i) Every convergent real sequence is bounded above. **4**

(ii) The function $f : (0, \infty) \to \mathbb{R}$ given by

$$f(x) = \frac{x^2 - 1}{x + 2}$$

is continuous at $x = 1$. **4**

3. (i) Prove *directly from the definition of convergence* that a sequence $(x_n)_{n=1}^{\infty}$ with $|x_n - x_{n+1}| > 1$ for all $n \in \mathbb{N}$ does not converge to any limit. **3**

(ii) Calculate

$$\lim_{n\to\infty} \frac{4n^3 + 2n^2}{3n^4 + (-1)^n},$$

stating clearly all properties of limits used. **4**

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4. For each of the series below, determine whether they converge or diverge. Justify your answers, clearly referring to any results or tests you use from the course. Answers without a justification will receive zero marks.

   (i)
   $$\sum_{n=1}^{\infty}(-1)^n \frac{2n^2+1}{n^2}$$ **3**

   (ii)
   $$\sum_{n=1}^{\infty}\frac{\cos(n)}{n^2}$$ **3**

   (iii)
   $$\sum_{n=1}^{\infty}\frac{5^n}{3^n+3n!}$$ **3**

5. (i) Give an example of a continuous function $f:(-1,1)\to\mathbb{R}$ that is both unbounded above and unbounded below. Justify your answer. **2**

   (ii) Suppose that $f:[0,1]\to[0,1]$ is a continuous function. Show that there exists $c\in[0,1]$ such that $f(c)=1-c^2$. **3**

6. (i) Let $(x_n)_{n=1}^{\infty}$ be a real sequence, let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function and assume $\lim_{n\to\infty}x_n=L$. Show that
   $$\forall\varepsilon>0\,\exists n_0\in\mathbb{N}\text{ such that }(n\geqslant n_0\Rightarrow|f(x_n)-f(L)|<\varepsilon).$$ **3**

   (ii) Give an example of a function $f:\mathbb{R}\to\mathbb{R}$ and a convergent real sequence $(x_n)_{n=1}^{\infty}$ such that $(f(x_n))_{n=1}^{\infty}$ does not converge. **2**

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