2C Intro to real analysis 2020/21

Ex Sheet 3

Feedback and solutions

Q1 *Define a sequence* $(x_n)_{n=1}^{\infty}$ *by* $x_1 = 2$ *and* $x_{n+1} = \frac{1}{3}x_n^2 - \frac{1}{3}x_n + 1$ *for* $n \in \mathbb{N}$.

a) *Show that* $1 < x_n < 3$ *for all* $n \in \mathbb{N}$.

b) *Show that* $(x_n)_{n=1}^{\infty}$ *is decreasing.*

c) *Prove that* $(x_n)_{n=1}^{\infty}$ *converges and find its limit.*

This question is pretty similar to an example from lectures involving recursive sequences. The first two parts of the question show that the sequence is decreasing and bounded below, then the monotone convergence theorem will show that the sequence converges. Finally, we take limits in the recursion relation to obtain the value of the limit. With that, I'll now go straight to the solution.

(a) We have $1 < x_1 < 3$. Suppose inductively that $1 < x_n < 3$ for some $n \in \mathbb{N}$. Then

$$x_{n+1} - 1 = \frac{1}{3}x_n^2 - \frac{1}{3}x_n = \frac{1}{3}x_n(x_n - 1) > 0,$$

so $x_{n+1} > 1$. Also

$$x_{n+1} - 3 = \frac{1}{3}x_n^2 - \frac{1}{3}x_n - 2 = \frac{1}{3}(x_n + 2)(x_n - 3) < 0,$$

as $x_n > 0$ and $x_n < 3$, so $x_{n+1} < 3$. By induction, we have $1 < x_n < 3$ for all $n \in \mathbb{N}$.

(b) For $n \in \mathbb{N}$, we have

$$x_{n+1} - x_n = \frac{1}{3}x_n^2 - \frac{4}{3}x_n + 1 = \frac{1}{3}(x_n - 3)(x_n - 1) < 0,$$

as $x_n - 1 > 0$ and $x_n - 3 < 0$. Therefore $x_n > x_{n+1}$ for all n, so the sequence $(x_n)_{n=1}^{\infty}$ is (strictly) decreasing.

(c) The sequence $(x_n)_{n=1}^{\infty}$ is decreasing by (b) and bounded below by (a), so converges by the monotone convergence theorem. Write $L = \lim_{n\to\infty} x_n$. Since $x_{n+1} \to L$, we have $L = \frac{1}{3}L^2 - \frac{1}{3}L + 1$, i.e. $L^2 - 4L + 3 = 0$. Therefore $L = 3$ or $L = 1$. Since $x_n < x_1 = 2$ for all n, we cannot have $L = 3$, so $L = 1$.

Q2 *Determine whether the series*

$$\sum_{n=1}^{\infty} \frac{n!}{n^n + 3^n}$$

converges or diverges. Carefully justify your answer making sure to indicate which results of the course you use[1].

[1] You may want to do this in two steps, using both the comparison test and ratio test. What is the dominating term in the denominator?

If you try to use the ratio test, things will get messy. For example, let $a_n = \frac{n!}{n^n+3^n}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!(n^n+3^n)}{((n+1)^{n+1}+3^{n+1})n^n},$$

and there are no very obvious cancellations.

So we need to simplify matters using some comparison series. Note that

$$\frac{n!}{n^n+3^n} < \frac{n!}{n^n} = \frac{n}{n}\frac{n-1}{n}\frac{n-2}{n}\cdots\frac{3}{n}\frac{2}{n}\frac{1}{n}.$$

For any n this is a product of terms less than 1. But in particular

$$\frac{n}{n}\frac{n-1}{n}\frac{n-2}{n}\cdots\frac{3}{n}\frac{2}{n}\frac{1}{n} \leq \frac{2}{n^2}$$

for $n \geq 3$.

By comparison with $\sum_1^\infty \frac{2}{n^2}$ we have convergence.

Hence:

For $n \in \mathbb{N}$, we have

$$0 \leq \frac{n!}{n^n+3^n} \leq \frac{n!}{n^n} \leq \frac{2}{n^2}$$

for $n \geq 3$.

Since $\sum \frac{1}{n^2}$ converges we have convergence of the original series by the comparison test.

Q3 *In this question you may use any properties of the natural logarithm function* $\log(x)$ *and integration you know freely, but please do state the properties you use.*

a) *Explain why* $\frac{1}{n+1} \leq \int_n^{n+1} \frac{1}{x}dx$ *for each* $n \in \mathbb{N}$.

b) *Define a sequence* $(t_n)_{n=1}^\infty$ *by* $t_n = (\sum_{r=1}^n \frac{1}{r}) - \log(n)$. *Show that this sequence is decreasing and that* $0 \leq t_n \leq 1$ *for all* $n \in \mathbb{N}$.

c) *Why does* $\lim_{n\to\infty} t_n$ *exist?*[2]

Note that the hint makes it clear that we are using the natural log in this question. In essentially all pure mathematics courses[3] log is used for the natural log and not $\log_{10}$.

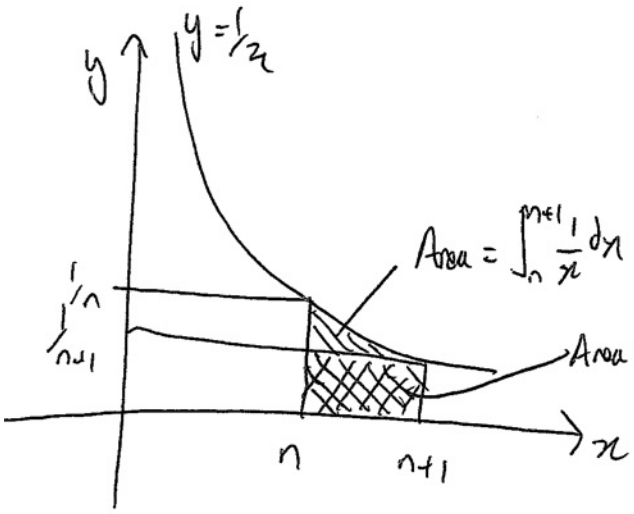
For part (a), it's useful to remember what the integral represents[4] rather than just integrate the function: the integral $\int_a^b f(x)dx$ represents the area under the curve $y = f(x)$ between $x = a$ and $x = b$.

(a) Recall that $\int_n^{n+1} \frac{1}{x}dx$ represents the area under the curve $y = \frac{1}{x}$ between $x = n$ and $x = n+1$, as shaded in the diagram below. The smaller cross shaded rectangle has area $\frac{1}{n+1}$ and hence $\frac{1}{n+1} \leq \int_n^{n+1} \frac{1}{x}dx$.

[2] The limit is called γ and is known as the Euler–Mascheroni constant or the Euler constant (not to be confused with e). It is not known whether γ is rational or irrational!

[3] The reason for this is there is essentially no good reason for using $\log_{10}$ now that we're in an era which doesn't need to use log tables and slide rules (if you've never heard of that, look it up on wikipedia) to multiply numbers, so we certainly don't want to reserve log for log base 10.

[4] Of course we have not covered integration from a precise view point yet, which is why the question allows you to use any facts about the integral you know.



For part (b), in order to try and work out what is going on I look at the first few terms. Note that $t_1 = 1$ and $t_2 = 1 + \frac{1}{2} - \log(2)$. Here $t_2 = t_1 + \frac{1}{2} - \log(2)$, and noting that $\log(2) = \int_1^2 \frac{1}{x} dx$, part (a) gives $\frac{1}{2} \leq \log(2)$, so $t_2 \leq t_1$. Carrying on, we get $t_3 = 1 + \frac{1}{2} + \frac{1}{3} - \log(3)$, and $t_2 - t_3 = \log(3) - \log(2) - \frac{1}{3} = \int_2^3 \frac{1}{x} dx - \frac{1}{3}$, hence $t_3 \leq t_2$. Now I can see why the series is going to be decreasing.

(b) For $n \in \mathbb{N}$, using the fact that $\int_n^{n+1} \frac{1}{x} dx = \log(n+1) - \log(n)$, we have

$$t_n - t_{n+1} = \log(n+1) - \log(n) - \frac{1}{n+1} = \int_n^{n+1} \frac{1}{x} dx - \frac{1}{n+1} \geq 0,$$

so that $(t_n)_{n=1}^{\infty}$ is decreasing. Since $t_1 = 1$, we have $t_n \leq 1$ for all $n \in \mathbb{N}$.

Note that

$$\log(n) = \int_1^n \frac{1}{x} dx = \sum_{r=1}^{n-1} \int_r^{r+1} \frac{1}{x} dx \leq \sum_{r=1}^{n-1} \frac{1}{r} \leq \sum_{r=1}^{n} \frac{1}{r},$$

where the first inequality comes from $\int_r^{r+1} \frac{1}{x} dx \leq \frac{1}{r}$, obtained in the same way as the estimate in (a). Therefore

$$t_n = \sum_{r=1}^{n} \frac{1}{r} - \log(n) \geq 0.$$

(c) The sequence $(t_n)_{n=1}^{\infty}$ converges by the monotone convergence theorem.