Q1 Let A, B be subsets of R such that sup(A) and inf(B) exist and sup(A) < 0. Define

$$C = \{\frac{1}{a} + b | a \in A, b \in B\}$$

Explain why this makes sense (i.e. why $a \neq 0$ for every $a \in A$), and

prove that inf(C) exists.

By definition of supremum,

$$\forall a \in A, a \leq sup(A) < 0.$$

Thus, $a \neq 0$.

Let $m = \frac{1}{sup(A)} + inf(B)$. For $a \in A$, we have $a \leq sup(A)$; thus, $\frac{1}{a} \geq \frac{1}{sup(A)}$. For $b \in B$, we have $b \geq inf(B)$. Therefore,

$$\frac{1}{a} + b \geq \frac{1}{sup(A)} + inf(B) = m.$$

Thus, m is a lower bound for C, proving that C is bounded below. By the theorem derived from the completeness axiom, $inf(C)$ exists, as required.

Q2 Show directly from the definition that

$$\lim_{n\to\infty} \frac{4n^4 + 5n^3 + 1}{2n^4 - n^2 + 3} = 2.$$

By the definition of convergence,

$$\forall \varepsilon > 0 \, \exists n_0 \in \mathbb{N} \, s.t. \forall (n \in \mathbb{N} \text{ with } n \geq n_0), \left| \frac{4\mathrm{n}^4 + 5\mathrm{n}^3 + 1}{2\mathrm{n}^4 - \mathrm{n}^2 + 3} - 2 \right| < \varepsilon$$

Let $\varepsilon > 0$ be arbitrary. For $n \in \mathbb{N}$, we have

$$|\frac{4n^4 + 5n^3 + 1}{2n^4 - n^2 + 3} - 2| = |\frac{5n^3 + 2n^2 - 5}{2n^4 - n^2 + 3}|.$$

By the polynomial estimation lemma, there exist $n_1, n_2 \in \mathbb{N}$ such that

$$n \geq n_1 \Rightarrow \frac{1}{2} 5n^3 \leq 5n^3 + 2n^2 - 5 \leq \frac{3}{2} 5n^3$$
$$n \geq n_2 \Rightarrow \frac{1}{2} 2n^4 \leq 2n^4 - n^2 + 3 \leq \frac{3}{2} 2n^4.$$

Particularly, when $n \geq max(n_1, n_2)$, we have

$$|\frac{5n^3+2n^2-5}{2n^4-n^2+3}| = \frac{5n^3+2n^2-5}{2n^4-n^2+3} \le \frac{\frac{15n^3}{2}}{n^4} = \frac{15}{2n} < \varepsilon,$$

provided n also satisfies $n > \frac{15}{2\varepsilon}$. Therefore, take $n_0 \in \mathbb{N}$ with $n_0 > max\left(n_1, n_2, \frac{15}{2\varepsilon}\right)$. For $n \in \mathbb{N}$ with $n \ge n_0$, we have $\left|\frac{4\text{n}^4+5\text{n}^3+1}{2\text{n}^4-\text{n}^2+3} - 2\right| < \varepsilon$, so $\lim_{n\to\infty} \frac{4n^4+5n^3+1}{2n^4-n^2+3} = 2$, as required.

Q3: For $x > 0$ and $n \in \mathbb{N}$, the quantity $x^{\frac{1}{n}}$ is defined to be the unique positive real numbers which has $\left(x^{\frac{1}{n}}\right)^n = x$.

a) For $n \in \mathbb{N}$, use the binomial expansion for $\left(1+\frac{2}{n}\right)^n$ to show that $\left(1+\frac{2}{n}\right)^n \ge 3$, and deduce that $1 \le 3^{\frac{1}{n}} \le 1 + \frac{2}{n}$.

By the binomial theorem and for $r \in (\mathbb{Z}^+ + \{0\})$ such that $r \le n$, we have

$$\left(1+\frac{2}{n}\right)^n = \sum_{r=0}^{n} \frac{n!}{r!\,(n-r)!}\left(\frac{2}{n}\right)^r.$$

When $n = 1$, $\left(1+\frac{2}{n}\right)^n = 3$. For any $n > 1$, $\sum_{r=0}^{n} \frac{n!}{r!(n-r)!}\left(\frac{2}{n}\right)^r > 3$ because all subsequent terms in the sum are positive for naturals r and n. Thus,

$$\left(1+\frac{2}{n}\right)^n \ge 3.$$

Since both sides of the inequality are positive,

$$\left(\left(1+\frac{2}{n}\right)^n\right)^{\frac{1}{n}} \ge 3^{\frac{1}{n}}$$
$$\Rightarrow 1+\frac{2}{n} \ge 3^{\frac{1}{n}}.$$

Furthermore,

$$1 \le 3.$$

Again, they can be exponentiated to make

$$1 \le 3^{\frac{1}{n}}.$$

Combining both inequalities gives us

$$1 \le 3^{\frac{1}{n}} \le 1 + \frac{2}{n},$$

as required.

b) Use the previous part to show, directly from the definition, that $\lim_{n\to\infty} 3^{1/n} = 1$.

Let $\varepsilon > 0$ be arbitrary. For $n \in \mathbb{N}$, we have

$$|3^{1/n} - 1| < \varepsilon$$
$$\Rightarrow 3^{1/n} - 1 < \varepsilon$$

(the absolute value does not change anything since the LHS is non-negative).

Since $3^{\frac{1}{n}} \leq 1 + \frac{2}{n}$ (from Q3 part a),

$$3^{1/n} - 1 \leq 1 + \frac{2}{n} - 1 = \frac{2}{n}.$$

Furthermore,

$$\frac{2}{n} < \varepsilon,$$

provided n satisfies $n > \frac{2}{\varepsilon}$. From combining the two inequalities, we get

$$3^{1/n} - 1 \leq \frac{2}{n} < \varepsilon.$$
$$\Rightarrow 3^{1/n} - 1 < \varepsilon.$$

Take $n_0 \in \mathbb{N}$ such that $n_0 > \frac{2}{\varepsilon}$. Then $|3^{1/n} - 1| < \varepsilon$ for all $n \geq n_0$. That is, $3^{1/n} \to 1$ as $n \to \infty$, as required.