Algorithmic Foundations 2 - Tutorial Sheet 2

Predicate Logic and Sets

Predicates and Quantifiers

1. Suppose $P(x, y)$ is the statement $x + 2{\cdot}y = x{\cdot}y$, where the universe of discourse for both x and y is the set of integers $\mathbb{Z}$. What are the truth values of
   (a) $P(1, -1)$

   Solution: `true` - since $1 + 2{\cdot}(-1) = -1 = 1{\cdot}(-1)$.

   (b) $P(0, 0)$

   Solution: `true` - since $0 + 2{\cdot}0 = 0 = 0{\cdot}0$.

   (c) $P(2, 1)$

   Solution: `false` - since $2 + 2{\cdot}(1) = 4 \neq 2 = 1{\cdot}2$.

2. Suppose that $Q(x)$ is the statement $x{+}1 = 2{\cdot}x$. What are the truth values of
   (a) $Q(2)$

   Solution: `false` - since $2{+}1 = 3 \neq 4 = 2{\cdot}2$.

   (b) $\forall x \in \mathbb{R}.\, Q(x)$

   Solution: `false` - since, for example if $x{=}2$, then $2{+}1 = 3 \neq 4 = 2{\cdot}2$.

   (c) $\exists x \in \mathbb{R}.\, Q(x)$

   Solution: `true` - since, for example taking $x{=}1$ we have $1{+}1 = 2 = 2{\cdot}1$.

3. Let $P(m, n)$ be the statement $n \geq m$. What is the truth value of
   (a) $\forall n \in \mathbb{N}.\, P(0, n)$

   Solution: `true` - all natural numbers are greater than or equal to 0.

   (b) $\exists n \in \mathbb{N}.\, \forall m \in \mathbb{N}.\, P(m, n)$

   Solution: `false` - there is no largest natural number. For example, for any natural number n, letting $m = n{+}1$ we have $m \in \mathbb{N}$ and $P(m, n)$ does not hold.

   (c) $\forall m \in \mathbb{N}.\, \exists n \in \mathbb{N}.\, P(m, n)$

   Solution: `true` - for any $m \in \mathbb{N}$ letting $n = m{+}1$ we have $n \in \mathbb{N}$ and $P(m, n)$ holds.

4. Suppose $\mathcal{S}$ is the set of all students, $\mathcal{C}$ is the set of all courses, and we are given the following list of predicates:
   - $H(y)$: x is an honours course;
   - $C(x)$: x is a CS course;
   - $S(x)$: x is a second-year;
   - $P(x)$: x is a part-time student;
   - $F(x)$: x is a full-time student;
   - $T(x, y)$: x is taking course y.

5. Write each of the following statements using these predicates and quantifiers where necessary.
   (a) "Sarah is taking AF2"

   Solution: $T(\mathit{Sarah}, \mathit{AF2})$

   (b) "all students are second-years"

   Solution: $\forall x \in \mathcal{S}.\, S(x)$

   (c) "every second-year is a full-time student"

   Solution: $\forall x \in \mathcal{S}.\, (S(x) \to F(x))$

   (d) "no CS course is an honours course"

   Solution: $\forall y \in \mathcal{C}.\, (C(y) \to \neg H(y))$.

   Alternative (and equivalent) solutions include $\forall y \in \mathcal{C}.\, \neg(C(y) \wedge H(y))$ and $\neg\exists y \in \mathcal{C}.\, (C(y) \wedge H(y)))$.

   (e) "every student is taking at least one course"

   Solution: $\forall x \in \mathcal{S}.\, \exists y \in \mathcal{C}.\, T(x, y)$

   (f) "there is a part-time student who is not taking any CS course"

   Solution: $\exists x \in \mathcal{S}.\, \forall y \in \mathcal{C}.\, (P(x) \wedge (C(y) \to \neg T(x, y)))$ or alternatively
   $\exists x \in \mathcal{S}.\, (P(x) \wedge \forall y \in \mathcal{C}.\, (C(y) \to \neg T(x, y)))$

   (g) "every part-time second-year is taking some honours course"

   Solution: $\forall x \in \mathcal{S}.\, \exists y \in \mathcal{C}.\, ((P(x) \wedge S(x)) \to (H(y) \wedge T(x, y)))$ or alternatively
   $\forall x \in \mathcal{S}.\, ((P(x) \wedge S(x)) \to \exists y \in \mathcal{C}.\, (H(y) \wedge T(x, y)))$

6. Using the predicates from the previous question, write each of the following in good English without using variables in your answers.
   (a) $S(\mathit{Helen})$

> **Solution:** "Helen is a second-year student"

(b) $\neg\exists y \in \mathcal{C}.\, T(Joe, y)$

> **Solution:** "Joe is not taking any course"

(c) $\exists x \in \mathcal{S}.\, (P(x) \land \neg S(x))$

> **Solution:** "some part-time students are not second-years"

(d) $\exists x \in \mathcal{S}.\, \forall y \in \mathcal{C}.\, T(x, y)$

> **Solution:** "some student is taking every course"

(e) $\forall x \in \mathcal{S}.\, \exists y \in \mathcal{C}.\, ((F(x) \land S(x)) \to (C(y) \land T(x, y)))$

> **Solution:** "every full-time second year is taking a CS course"

7. Explain why the negation of "Some students in my class use e-mail" is not "Some students in my class do not use e-mail".

> **Solution:** Short answer: both statements can be true at the same time. Longer answer: the negation is "all students in my class do not use e-mail" which is not the same as saying "some students in my class do not use e-mail".

8. Let $\mathcal{S}$ be the set of all sets and consider the following predicates:

   - $F(x)$: x is a finite set;
   - $I(x)$: x is an infinite set;
   - $S(x, y)$: x is contained in y;
   - $E(x)$: x is the emptyset.

   Translate the following into logical expressions:

   (a) "not all sets are finite"

   > **Solution:** $\exists x \in \mathcal{S}.\, \neg F(x)$ or $\exists x \in \mathcal{S}.\, I(x)$

   (b) "every subset of a finite set is finite"

   > **Solution:** $\forall x \in \mathcal{S}.\, \forall y \in \mathcal{S}.\, ((F(y) \land S(x, y)) \to F(x))$

   (c) "no infinite set can be contained in a finite set"

   > **Solution:** $\neg\exists x \in \mathcal{S}.\, \exists y \in \mathcal{S}.\, (I(x) \land F(y) \land S(x, y))$
   > An alternatively would be $\forall x \in \mathcal{S}.\, (I(x) \to \neg(\exists y \in \mathcal{S}.\, (F(y) \land S(x, y))))$

Below is a proof showing these two formulae are logically equivalent:

$$
\begin{array}{lr}
\forall x \in \mathcal{S}.\,(I(x) \rightarrow \neg(\exists y \in \mathcal{S}.\,(F(y) \wedge S(x,y)))) & \\
\equiv\ \forall x \in \mathcal{S}.\,(\neg I(x) \vee \neg(\exists y \in \mathcal{S}.\,(F(y) \wedge S(x,y)))) & \text{implication law} \\
\equiv\ \forall x \in \mathcal{S}.\,\neg(I(x) \wedge (\exists y \in \mathcal{S}.\,(F(y) \wedge S(x,y)))) & \text{De Morgan law} \\
\equiv\ \neg\exists x \in \mathcal{S}.\,\neg\neg(I(x) \wedge (\exists y \in \mathcal{S}.\,(F(y) \wedge S(x,y)))) & \text{negation law} \\
\equiv\ \neg\exists x \in \mathcal{S}.\,(I(x) \wedge (\exists y \in \mathcal{S}.\,(F(y) \wedge S(x,y)))) & \text{double negation law} \\
\equiv\ \neg\exists x \in \mathcal{S}.\,\exists y \in \mathcal{S}.\,(I(x) \wedge F(y) \wedge S(x,y)) & \text{since } y \text{ does not appear in } I(x)
\end{array}
$$

(d) "the empty set is a subset of every finite set"

Solution: $\forall x \in \mathcal{S}.\,\forall y \in \mathcal{S}.\,((E(x) \wedge F(y)) \rightarrow S(x,y))$

Difficult/challenging questions (Predicate Logic).

9. A statement is in *prenex normal form* when it is of the form:

$$\nabla_1 x_1.\,\nabla_2 x_2 \ldots \nabla_n x_n.\,P(x_1, x_2, \ldots, x_n)$$

where $\nabla_i \in \{\forall, \exists\}$ for $1 \leq i \leq n$ and $P(x_1, x_2, \ldots, x_n)$ is a predicate involving no quantifiers. For example we have that $\exists x.\,\forall y.\,(P(x,y) \vee Q(y))$ is in prenex normal form, while $\forall x.\,P(x) \wedge \exists y.\,Q(y)$ is not. Using the rules for logical equivalence write the following formulae in prenex normal form.

(a) $\exists x.\,P(x) \vee \exists x.\,Q(x) \vee R$ where R is a propositional formula, containing no variables or quantifiers;

Solution: Changing the variable x to y in the subforumula $\exists x.\,Q(x)$ we have:

$$
\begin{aligned}
\exists x.\,P(x) \vee \exists x.\,Q(x) \vee R &\equiv\ \exists x.\,P(x) \vee \exists y.\,Q(y) \vee R \\
&\equiv \exists x.\,\exists y.\,(P(x) \vee Q(y) \vee R)
\end{aligned}
$$

since y does not appear free in $\exists x.\,P(x)$, x does not appear free in $\exists y.\,Q(y)$ and neither x nor y appear free in R.

(b) $\neg(\forall x.\,P(x) \vee \forall x.\,Q(x))$

Solution: Changing the variable x to y in the subforumula $\forall x.\,Q(x)$ we have:

$$
\begin{array}{lr}
\neg(\forall x.\,P(x) \vee \forall x.\,Q(x)) \equiv\ \neg(\forall x.\,P(x) \vee \forall y.\,Q(y)) & \\
\equiv\ \neg\forall x.\,P(x) \wedge \neg\forall y.\,Q(y) & \text{De Morgan law} \\
\equiv\ \neg\forall x.\,\neg\neg P(x) \wedge \neg\forall y.\,\neg\neg Q(y) & \text{double negation law (twice)} \\
\equiv\ \exists x.\,\neg P(x) \wedge \exists y.\,\neg Q(y) & \text{quantifier law} \\
\equiv\ \exists x.\,\exists y.\,(\neg P(x) \wedge \neg Q(y)) &
\end{array}
$$

since y does not appear free in $\forall x.\,\neg P(x)$ and x does not appear free in $\exists y.\,\neg Q(y)$.

(c) $\exists x.\, P(x) \to \exists x.\, Q(x)$

Solution: Changing the variable x to y in the second sub-forumula we have:

$$\begin{aligned} \exists x.\, P(x) \to \exists x.\, Q(x) \;&\equiv\; \exists x.\, P(x) \to \exists y.\, Q(y) \\ \equiv\; \neg\exists x.\, P(x) \vee \exists y.\, Q(y) & \qquad \text{implication law} \\ \equiv\; \neg\exists x.\, \neg\neg P(x) \vee \exists y.\, Q(y) & \qquad \text{double negation law} \\ \equiv\; \forall x.\, \neg P(x) \vee \exists y.\, Q(y) & \qquad \text{quantifier law} \\ \equiv\; \forall x.\, \exists y.\, (\neg P(x) \vee Q(y)) & \end{aligned}$$

since y does not appear free in $\forall x.\, \neg P(x)$ and x does not appear free in $\exists y.\, Q(y)$.

Sets and Set Operations

10. List the members of the following sets (recall that $\mathbb{Z}$ is the set of integers and $\mathbb{N}$ is the set of natural numbers).

(a) $\{x \mid x \in \mathbb{Z} \wedge x^2{=}5\}$

Solution: $\varnothing$

(b) $\{5{\cdot}x \mid x \in \mathbb{Z} \wedge (-2{\leq}x{\leq}2)\}$

Solution: $\{-10, -5, 0, 5, 10\}$

(c) $\{x \mid x \in \mathbb{N} \wedge x^2 \in \{1, 4, 9\}\}$

Solution: $\{1, 2, 3\}$

(d) $\{x \mid x \in \mathbb{Z} \wedge x^2 \in \{1, 4, 9\}\}$

Solution: $\{-3, -2, -1, 1, 2, 3\}$

11. Use set builder notation to give a description of each of the following sets.

(a) $\{0, 3, 6, 9, 12\}$

Solution: $\{3{\cdot}x \mid x \in \mathbb{N} \wedge 0{\leq}x{\leq}4\}$

(b) $\{-3, -2, -1, 0, 1, 2, 3\}$

Solution: $\{x \mid x \in \mathbb{Z} \wedge -3{\leq}x{\leq}3\}$

(c) $\{1, 4, 9, 16, 25, 36, 49\}$

Solution: $\{x^2 \mid x \in \mathbb{N} \wedge 1 \leq x \leq 7\}$

12. Suppose $A = \{a, b, c\}$ and $B = \{b, \{c\}\}$. Mark each of the following `true` or `false`.

(a) $\{a, c\} \in A$

Solution: `false` ($\{a, c\}$ is actually a strict subset of A, i.e. $\{a, b\} \subset A$)

(b) $\{c\} \subseteq B$

Solution: `false` (actually we have $\{c\} \in B$)

(c) $B \subseteq A$

Solution: `false` (for example, $\{c\} \in B$ and $\{c\} \notin A$)

(d) $\{b, c\} \in \mathcal{P}(A)$

Solution: `true` (b and c are elements of A, and hence $\{b, c\}$ is a subset of A)

(e) $\{\{a\}\} \subseteq \mathcal{P}(A)$

Solution: `true` ($\{a\}$ is an element of $\mathcal{P}(A)$ so the set containing $\{a\}$ is a subset of $\mathcal{P}(A)$)

(f) $\{b, \{c\}\} \in \mathcal{P}(B)$

Solution: `true` (since a set is element of its powerset)

(g) $\{\{\{c\}\}\} \subseteq \mathcal{P}(B)$

Solution: `true` ($\{c\} \in B$ implies $\{\{c\}\} \in \mathcal{P}(B)$ which implies $\{\{\{c\}\}\} \subseteq \mathcal{P}(B)$)

(h) $|\mathcal{P}(A \times B)| = 32$

Solution: `false` ($|A \times B| = 3 \cdot 2 = 6$ so the power set is of size $2^6 = 64$)

(i) $\{a, b\} \in A \times A$

Solution: `false` (the set $A \times A$ contains ordered pair, but $\{a, b\}$ is the set containing the elements a and b)

(j) $\varnothing \subseteq A \times A$

Solution: `true` (the emptyset is a subset of any set - to prove a set A is a subset of B we need to show any element of A is in B, when A is the empty set this holds vacuously as there are no elements in A)

(k) $(c,c) \in A \times A$

> **Solution:** `true` - since $c \in A$

13. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ by giving

(a) a containment proof;

Solution: First we show $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Considering any $x \in A \cap (B \cup C)$, by definition of intersection we have:

$$\begin{array}{ll} x \in A \cap (B \cup C) \;\Rightarrow\; x \in A \text{ and } x \in B \cup C & \\ \Rightarrow\; x \in A \text{ and either } x \in B \text{ or } x \in C & \text{by definition of union} \\ \Rightarrow\; \text{either } x \in A \text{ and } x \in B, \text{ or } x \in A \text{ and } x \in C & \text{rearranging} \\ \Rightarrow\; \text{either } x \in A \cap B \text{ or } x \in A \cap C & \text{by definition of intersection} \\ \Rightarrow\; x \in (A \cap B) \cup (A \cap C) & \text{by definition of union} \end{array}$$

and hence, since $x \in A \cap (B \cup C)$ was arbitrary, we have $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ as required.

To complete the proof we show $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Considering any $x \in (A \cap B) \cup (A \cap C)$, by definition of union we have:

$$\begin{array}{ll} x \in (A \cap B) \cup (A \cap C) \;\Rightarrow\; \text{either } x \in A \cap B \text{ or } x \in A \cap C & \\ \Rightarrow\; \text{either } x \in A \text{ and } x \in B, \text{ or } x \in A \text{ and } x \in C & \text{by definition of intersection} \\ \Rightarrow\; x \in A \text{ and either } x \in B \text{ or } x \in C & \text{rearranging} \\ \Rightarrow\; x \in A \text{ and } x \in B \cup C & \text{by definition of union} \\ \Rightarrow\; x \in A \cap (B \cup C) & \text{by definition of intersection} \end{array}$$

and hence, since $x \in (A \cap B) \cup (A \cap C)$ was arbitrary, we have $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ completing the proof.

(b) an element table proof;

Solution:

A	B	C	$A \cap B$	$A \cap C$	$B \cup C$	$A \cap (B \cup C)$	$(A \cap B) \cup (A \cap C)$
0	0	0	0	0	0	**0**	**0**
0	0	1	0	0	1	**0**	**0**
0	1	0	0	0	1	**0**	**0**
0	1	1	0	0	1	**0**	**0**
1	0	0	0	0	0	**0**	**0**
1	0	1	0	1	1	**1**	**1**
1	1	0	1	0	1	**1**	**1**
1	1	1	1	1	1	**1**	**1**

Each set has the same values in the element table: the value is 1 if and only if A has the value 1 and either B or C has the value 1.

(c) a proof using logical equivalence.

Solution:

$$\begin{aligned}
A \cap (B \cup C) &= \{x \mid x \in A \cap (B \cup C)\} && \text{by definition} \\
&= \{x \mid (x \in A) \wedge (x \in (B \cup C))\} && \text{by definition of } \cap \\
&= \{x \mid (x \in A) \wedge ((x \in B) \vee (x \in C))\} && \text{by definition of } \cup \\
&= \{x \mid ((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C))\} && \text{distributive law} \\
&= \{x \mid (x \in A \cap B) \vee (x \in A \cap C)\} && \text{by definition of } \cap \\
&= \{x \mid x \in (A \cap B) \cup (A \cap C)\} && \text{by definition of } \cup \\
&= (A \cap B) \cup (A \cap C).
\end{aligned}$$

14. Prove or disprove: $A - (B \cap C) = (A - B) \cup (A - C)$.

Solution: Proof. By definition of set difference:

$$\begin{aligned}
A - (B \cap C) &= A \cap (\overline{B \cap C}) \\
&= A \cap (\overline{B} \cup \overline{C}) && \text{de Morgan} \\
&= (A \cap \overline{B}) \cup (A \cap \overline{C}) && \text{distributivity} \\
&= (A - B) \cup (A - C) && \text{definition of set difference}
\end{aligned}$$

15. Prove or disprove: $A - (B \cap C) = (A - B) \cap (A - C)$.

Solution: `false` - for example, if $A = \{1, 2\}$, $B = \{1\}$, $C = \{2\}$, then $A - (B \cap C) = A$ while $(A - B) \cap (A - C) = \varnothing$

16. Prove or disprove: $A \oplus (B \oplus C) = (A \oplus B) \oplus C$.

Solution: `true` - this is easiest with a membership table showing each set has the same values: the value is 1 if and only if exactly one of A, B and C has the value 1, or all three have value 1.

Proving with the other methods is possible, but is more involved.

17. Let $A_i = \{1, 2, \ldots, i\}$ for $i \in \mathbb{Z}^+$, find $\cup_{i=1}^n A_i$ and $\cap_{i=1}^n A_i$ for $n \in \mathbb{Z}^+$.

Solution: We have $\cup_{i=1}^n A_i = A_n$ and $\cap_{i=1}^n A_i = \{1\}$

18. Mark each of the following `true` or `false`:

(a) $A - (B - C) = (A - B) - C$

Solution: `false` - for example, consider $A = B = \{a, b\}$ and $C = \{b\}$, then $A - (B - C) = \{a, b\} - \{a\} = \{b\}$ while $(A - B) - C = \varnothing - \{b\} = \varnothing$

(b) $(A-C)-(B-C) = A-B$

Solution: `false` - for example, take $A = \{a\}$, $B = \{b\}$ and $C = \{a, b\}$, then $(A-C)-(B-C) = \varnothing - \varnothing = \varnothing$ while $A-B = \{a\}$

(c) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Solution: `true` - below is a containment proof:
First we show $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. Therefore, considering any $x \in A \cup (B \cap C)$, by definition of union:

$$
\begin{aligned}
& x \in A \cup (B \cap C) \;\Rightarrow\; x \in A \text{ or } x \in B \cap C \\
& \Rightarrow\; x \in A \text{ or both } x \in B \text{ and } x \in C && \text{by definition of intersection} \\
& \Rightarrow\; x \in A \text{ or } x \in B, \text{ and } x \in A \text{ or } x \in C && \text{rearranging} \\
& \Rightarrow\; x \in A \cup B, \text{ and } x \in A \cup C && \text{by definition of union} \\
& \Rightarrow\; x \in (A \cup B) \cap (A \cup C) && \text{by definition of intersection}
\end{aligned}
$$

and, since $x \in A \cup (B \cap C)$ was arbitrary, we have $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ are required.
To complete the proof we show $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. Considering any $x \in (A \cup B) \cap (A \cup C)$, definition of intersection we have:

$$
\begin{aligned}
& x \in (A \cup B) \cap (A \cup C) \;\Rightarrow\; x \in A \cup B, \text{ and } x \in A \cup C \\
& \Rightarrow\; x \in A \text{ or } x \in B, \text{ and } x \in A \text{ or } x \in C && \text{by definition of union} \\
& \Rightarrow\; x \in A \text{ or both } x \in B \text{ and } x \in C && \text{rearranging} \\
& \Rightarrow\; x \in A \text{ or } x \in B \cap C && \text{by definition of intersection} \\
& \Rightarrow\; x \in A \cup (B \cap C) && \text{by definition of union.}
\end{aligned}
$$

Hence, since $x \in (A \cup B) \cap (A \cup C)$ was arbitrary, we have $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ completing the proof.

(d) $A \cap (B \cup C) = (A \cup B) \cap (A \cup C)$

Solution: `false` - for example, if $A = \{a\}$ and $B = C = \{b\}$, then $A \cap (B \cup C) = \{a\} \cap \{b\} = \varnothing$ while $(A \cup B) \cap (A \cup C) = \{a, b\} \cap \{a, b\} = \{a, b\}$

(e) If $A \cup C = B \cup C$, then $A = B$

Solution: `false` - for example, consider $A = \{a\}$, $B = \{b\}$ and $C = \{a, b\}$

(f) If $A \cap C = B \cap C$, then $A = B$

Solution: `false` - for example, consider $A = \{a, c\}$, $B = \{b, c\}$ and $C = \{c\}$

(g) If $A \cap B = A \cup B$, then $A = B$

Solution: `true`. Below we give a containment proof showing $A = B$ using the hypothesis $A \cap B = A \cup B$. First we show $A \subseteq B$, by definition of union we have:

$$\begin{aligned} x \in A \Rightarrow\ & x \in A \cup B \\ \Rightarrow\ & x \in A \cap B && \text{by the hypothesis} \\ \Rightarrow\ & x \in B && \text{by the definition of intersection} \end{aligned}$$

and hence $A \subseteq B$.

To complete the proof we show $B \subseteq A$. Considering any $x \in B$, by definition of union we have:

$$\begin{aligned} x \in B \Rightarrow\ & x \in A \cup B \\ \Rightarrow\ & x \in A \cap B && \text{by the hypothesis} \\ \Rightarrow\ & x \in A && \text{by the definition of intersection} \end{aligned}$$

and hence $B \subseteq A$ completing the proof.

(h) If $A \oplus B = A$, then $A = B$

Solution: `false` - for example, if $A = \{a\}$ and $B = \varnothing$, then $A \oplus B = (A - B) \cup (B - A) = \{a\} \cup \varnothing = \{a\}$

(i) there is a set A such that $|P(A)| = 12$

Solution: `false` - from the lectures we have that the size of the power set equals 2^n where n is the size of the set

(j) $A \oplus A = A$

Solution: `false` - for example, if $A = \{a\}$, then $A \oplus A = (A - A) \cup (A - A) = \varnothing \cup \varnothing = \varnothing$.

We actually have that $A \oplus A = \varnothing$ for all sets A. Below is a proof of this fact using using set comprehension and logical equivalences.

$$\begin{aligned} A \oplus A =\ & \{x \mid x \in A \oplus A\} && \text{by definition} \\ =\ & \{x \mid x \in (A - A) \cup (A - A)\} && \text{by definition of symmetric difference} \\ =\ & \{x \mid (x \in A - A) \vee (x \in A - A)\} && \text{by definition of } \cup \\ =\ & \{x \mid x \in A - A\} && \text{idempotent law} \\ =\ & \{x \mid (x \in A) \wedge (x \notin A)\} && \text{by definition of set difference} \\ =\ & \{x \mid (x \in A) \wedge \neg(x \in A)\} && \text{by definition of negation} \\ =\ & \{x \mid \texttt{false}\} && \text{contradiction law} \\ =\ & \varnothing \end{aligned}$$

Difficult/challenging questions (Sets and Set Operations).

19. Prove or disprove: $A \oplus (B \oplus C) = (A \oplus B) \oplus C$.

> **Solution:** `true` - this is easiest with a membership table showing each set has the same values: the value is 1 if and only if exactly one of A, B and C has the value 1, or all three have value 1.
>
> Proving with the other methods is possible, but is more involved.

20. Suppose that A, B and C are sets such that $A \oplus C = B \oplus C$, does it follow that $A = B$.

> **Solution:** The answer is yes. We will prove using a containment proof.
>
> First we show $A \subseteq B$. Considering any $x \in A$ we split the proof into the following two cases.
>
> - If $x \in C$, then by definition of set difference $x \notin A \oplus C$, and hence since $A \oplus C = B \oplus C$ it follows that $x \notin B \oplus C$. Now since $x \in C$ and $x \notin B \oplus C$, by definition of set difference it must be the case that $x \in B$.
> - If $x \notin C$, then by definition of set difference $x \in A \oplus C$, and hence since $A \oplus C = B \oplus C$ it follows that $x \in B \oplus C$. Now since $x \notin C$ and $x \in B \oplus C$, by definition of set difference it must be the case that $x \in B$.
>
> Since these are all the cases to consider ot follows that $x \in B$ and $A \subseteq B$.
>
> The proof that $B \subseteq A$ follows similarly, and therefore we have that $A = B$.