2B Linear Algebra

Ex Sheet 1

True/False

We will start the tutorial by going over these true false questions. Please make sure you've thought about them in advance of the tutorial and so are ready to answer.[1]

a) The vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ can be written as a linear combination of the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$.

b) A system of linear equations with augmented matrix $[A|\boldsymbol{b}]$ has a unique solution if and only if $\boldsymbol{b}$ is a linear combination of the columns of A.

c) The vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in Span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \end{bmatrix} \right\}$.

d) If S is any non-empty set of vectors then $\mathbf{0}$ is in Span(S).

e) If a vector $\boldsymbol{v}$ is in the span of some set of vectors S, then so is its additive inverse $-\boldsymbol{v}$.

f) The set of vectors $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \end{bmatrix} \right\}$ is a spanning set for $\mathbb{R}^2$.

g) For any two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^3$, Span$(\boldsymbol{u}, \boldsymbol{v})$ is a plane through the origin.

h) In $\mathbb{R}^n$, the vectors $(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ are linearly independent.

i) A set of vectors in linearly dependent if at least one vector in the set can be expressed as a linear combination of the others.

j) The zero vector is contained in any linearly independent set of vectors.

k) A set of vectors $S = \{\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}\}$ is linearly independent if and only if there are no solutions to the equation $\lambda_1 \boldsymbol{v_1} + \cdots + \lambda_n \boldsymbol{v_n} = \mathbf{0}$ for scalars $\lambda_1, \ldots, \lambda_n$.

l) A set of vectors $S = \{\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}\}$ is linearly independent if $\boldsymbol{v}_1$ is in Span$\{\boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\}$.

m) A homogeneous system of linear equations with augmented matrix $[A|\mathbf{0}]$ has a nontrivial solution if and only if the columns of A are linearly dependent.

n) Any set of 2 vectors in $\mathbb{R}^3$ is linearly dependent.

[1] **True/False Questions**

Every Exercise Sheet will have a section containing true/false questions, with solutions at the end of the sheet. They are designed to test your understanding from lectures. You should look at your lecture notes and/or the textbook to help you answer these questions, but you should not need to write anything to work out the solution.

o) Any set of 4 vectors in $\mathbb{R}^3$ is linearly dependent.

Solutions to True/False

(a) F (b) F (c) F (d) T (e) T (f) F (g) F (h) T (i) T (j) F (k) F (l) F (m) T (n) F (o) T

Tutorial Exercises

Before attempting these questions you should make sure you do all of the questions on Vectors and Systems of Linear Equations from Exercise Sheet 0.

T1 Which of the vectors $(1,2)$ and $(0,0)$ can be written as a linear combination of $\boldsymbol{u} = (1,-1)$ and $\boldsymbol{v} = (2,-1)$?

Solution

Both vectors can be written as a linear combination of $\boldsymbol{u} = (1,-1)$ and $\boldsymbol{v} = (2,-1)$ since

$$(1,2) = -5(1,-1) + 3(2,-1) = -5\boldsymbol{u} + 3\boldsymbol{v}.$$

and

$$(0,0) = 0(1,-1) + 0(2,-1) = 0\boldsymbol{u} + 0\boldsymbol{v}.$$

T2 Which of the vectors $(0,-3,6)$, $(3,-9,-2)$, $(0,0,0)$ and $(7,8,9)$ can be written as a linear combination of $\boldsymbol{u} = (2,1,4)$, $\boldsymbol{v} = (1,-1,3)$ and $\boldsymbol{w} = (3,2,5)$?

Solution

Linear combinations of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are all vectors of the form

$$x(2,1,4) + y(1,-1,3) + z(3,2,5)$$

for some $x, y, z \in \mathbb{R}$, that is, any vector of the form

$$(2x + y + 3z, x - y + 2z, 4x + 3y + 5z).$$

The zero vector $(0,0,0)$ can be written in this form by taking $x = y = z = 0$. As for the rest, rather than do three separate calculations, we just do one: the vector (b_1, b_2, b_3) is of the above form if and only if the following system of equations has a solution:

$$\begin{aligned} 2x + y + 3z &= b_1 \\ x - y + 2z &= b_2 \\ 4x + 3y + 5z &= b_3. \end{aligned}$$

Perform EROs on the augmented matrix

$$\begin{pmatrix} 2 & 1 & 3 & b_1 \\ 1 & -1 & 2 & b_2 \\ 4 & 3 & 5 & b_3 \end{pmatrix}$$

to obtain the augmented matrix in echelon form (this result isn't unique, yours may differ)

$$\begin{pmatrix} 1 & -1 & 2 & b_1 \\ 0 & 1 & -1 & b_3 - 2b_1 \\ 0 & 0 & 1 & \frac{7}{2}b_1 - b_2 - \frac{3}{2}b_3 \end{pmatrix}$$

This system is consistent, that is, we can solve the system by back substitution for any b_1, b_2, b_3. This means that the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ span the whole of $\mathbb{R}^3$. In particular, each of the vectors $(0, -3, 6)$, $(3, -9, -2)$, $(0, 0, 0)$ and $(7, 8, 9)$ can be written as a linear combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

T3 Give a geometric description of the span of the following pairs of vectors in $\mathbb{R}^3$. Is the span a plane or a line?

a) $\boldsymbol{v}_1 = [3, -4, 5]$, $\boldsymbol{v}_2 = [4, 2, 7]$

b) $\boldsymbol{v}_1 = [3, 5, 2]$, $\boldsymbol{v}_2 = [6, 10, 4]$

Solution

a) We have

$$\mathrm{Span}(\boldsymbol{v}_1, \boldsymbol{v}_2) = \{\lambda_1[3, -4, 5] + \lambda_2[4, 2, 7] : \lambda_1, \lambda_2 \in \mathbb{R}\}.$$

As $\boldsymbol{v}_1 = [3, -4, 5]$ is not a scalar multiple of $\boldsymbol{v}_2 = [4, 2, 7]$, the span is the plane through the origin spanned by $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

b) We have

$$\mathrm{Span}(\boldsymbol{v}_1, \boldsymbol{v}_2) = \{\lambda_1[3, 5, 2] + \lambda_2[6, 10, 4] : \lambda_1, \lambda_2 \in \mathbb{R}\}.$$

Since $[6, 10, 4] = 2[3, 5, 2]$, the span is the line through the origin in the direction of $[3, 5, 2]$.

T4 Which of the following sets are spanning sets for $\mathbb{R}^3$?

a) $S_1 = \{(2, 1, 0), (1, 1, -1)\}$;

b) $S_2 = \{(1, 0, 1), (2, 1, 0), (1, 1, -1)\}$;

c) $S_3 = \{(1, 0, 1), (2, 1, 0), (1, 1, -1), (3, -1, 3)\}$;

d) $S_4 = \{(2, 1, 0), (1, 1, -1), (3, -1, 3)\}$.

Solution

We examine which of these sets span $\mathbb{R}^3$.

a) $S_1 = \{(2, 1, 0), (1, 1, -1)\}$.

If this set spans $\mathbb{R}^3$ then any $(x,y,z) \in \mathbb{R}^3$ van be written as

$$(x,y,z) = \lambda(2,1,0) + \mu(1,1,-1)$$

for some $\lambda, \mu \in \mathbb{R}$. This vector equation is equivalent to a system of three equations that we can represent in the augmented matrix

$$\begin{pmatrix} 2 & 1 & x \\ 1 & 1 & y \\ 0 & -1 & z \end{pmatrix}.$$

Simple EROs show that the *reduced* row echelon form of this matrix (remember that the reduced row echelon form of a matrix is unique — you should get this answer!) is

$$\begin{pmatrix} 1 & 0 & x-y \\ 0 & 1 & 2y-x \\ 0 & 0 & z+2y-x \end{pmatrix}.$$

The final line shows that the system is consistent only when the equation $z + 2y - x = 0$ holds. In other words, we cannot find $\lambda, \mu \in \mathbb{R}$ for every $(x,y,z) \in \mathbb{R}^3$. For example, if $(x,y,z) = (1,1,1)$ then the equation becomes $2 = 0$ which is not true. Therefore $(1,1,1)$ does not lie in the span of the set $S_1 = \{(2,1,0),(1,1,-1)\}$, so S_1 does not span $\mathbb{R}^3$.

b) $S_2 = \{(1,0,1),(2,1,0),(1,1,-1)\}$.
If this set spans $\mathbb{R}^3$ then any $(x,y,z) \in \mathbb{R}^3$ van be written as

$$(x,y,z) = \lambda(1,0,1) + \mu(2,1,0) + \nu(1,1,-1)$$

for some $\lambda, \mu, \nu \in \mathbb{R}$. As in (i) we consider the augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 & x \\ 0 & 1 & 1 & y \\ 1 & 0 & -1 & z \end{bmatrix}$$

which has reduced row echelon form

$$\begin{bmatrix} 1 & 0 & -1 & x-2y \\ 0 & 1 & 1 & y \\ 0 & 0 & 0 & z+2y-x \end{bmatrix}.$$

As in the previous example we require $z + 2y - x = 0$ which does not hold for all $(x,y,z) \in \mathbb{R}^3$. Hence S_2 does not span $\mathbb{R}^3$.

c) $S_3 = \{(1,0,1),(2,1,0),(1,1,-1),(3,-1,3)\}$.
If this set spans $\mathbb{R}^3$ then any $(x,y,z) \in \mathbb{R}^3$ van be written as

$$(x,y,z) = \lambda(1,0,1) + \mu(2,1,0) + \nu(1,1,-1) + \delta(3,-1,3)$$

for some $\lambda, \mu, \nu, \delta \in \mathbb{R}$. As above we consider the augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 & 3 & x \\ 0 & 1 & 1 & -1 & y \\ 1 & 0 & -1 & 3 & z \end{bmatrix}$$

and find that it has reduced row echelon form

$$\begin{bmatrix} 1 & 0 & -1 & 0 & -\frac{3}{2}x + 3y + \frac{5}{2}z \\ 0 & 1 & 1 & 0 & \frac{1}{2}x - \frac{1}{2}z \\ 0 & 0 & 0 & 1 & \frac{1}{2}x - y - \frac{1}{2}z \end{bmatrix}.$$

In this case the equations always have a real solution in $\lambda, \mu, \nu, \delta$ given by

$$\begin{aligned} \lambda &= -\frac{3}{2}x + 3y + \frac{5}{2}z + \nu \\ \mu &= \frac{1}{2}x - \frac{1}{2}z + \nu \\ \delta &= \frac{1}{2}x - y - \frac{1}{2}z. \end{aligned}$$

So given any $(x, y, z) \in \mathbb{R}$ we can let ν to be any real number and calculate λ, μ, δ from the equations. So in fact, there are infinitely many solutions to the original equations. So S_3 does span $\mathbb{R}^3$.

d) $S_4 = \{(2, 1, 0), (1, 1, -1), (3, -1, 3)\}$.
If this set spans $\mathbb{R}^3$ then any $(x, y, z) \in \mathbb{R}^3$ van be written as

$$(x, y, z) = \lambda(2, 1, 0) + \mu(1, 1, -1) + \nu(3, -1, 3)$$

for some $\lambda, \mu, \nu, \in \mathbb{R}$. We consider the augmented matrix

$$\begin{bmatrix} 2 & 1 & 3 & x \\ 1 & 1 & -1 & y \\ 0 & -1 & 3 & z \end{bmatrix}$$

which has reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & -x + 3y + 2z \\ 0 & 1 & 0 & \frac{3}{2}x - 3y - \frac{5}{2}z \\ 0 & 0 & 1 & \frac{x}{2} - y - \frac{z}{2} \end{bmatrix}.$$

In this case the equations always have a real solution in $\lambda, \mu, \nu, \delta$ given by

$$\begin{aligned} \lambda &= -x + 3y + 2z \\ \mu &= \frac{3}{2}x - 3y - \frac{5}{2}z \\ \nu &= \frac{x}{2} - y - \frac{z}{2}. \end{aligned}$$

So given any $(x, y, z) \in \mathbb{R}$ we calculate λ, μ, δ from the equations above. This time the values will be unique and there is exactly one solution to the original equations. So, S_4 does span $\mathbb{R}^3$.

T5 Prove that $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are all in $\mathrm{Span}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$.

Solution

The key point is that the coefficient 0 is allowed in the definition of span. Since

$$\boldsymbol{u} = 1\boldsymbol{u} + 0\boldsymbol{v} + 0\boldsymbol{w}, \quad \boldsymbol{v} = 0\boldsymbol{u} + 1\boldsymbol{v} + 0\boldsymbol{w}, \quad \boldsymbol{w} = 0\boldsymbol{u} + 0\boldsymbol{v} + 1\boldsymbol{w}$$

the result follows.

T6 Determine whether the following sets of vectors are linearly independent, giving reasons for your answer. For any sets that are linearly dependent, find a nontrivial linear combination of the vectors in that set which equals $\mathbf{0}$. (Such a linear combination is called a dependence relationship.)

a) In $\mathbb{R}^2$:
$$\boldsymbol{u} = [1, 1], \quad \boldsymbol{v} = [-2, -2]$$

b) In $\mathbb{R}^2$:
$$\boldsymbol{u} = [1, 1], \quad \boldsymbol{v} = [-2, 2]$$

c) In $\mathbb{R}^2$:
$$\boldsymbol{u} = [1, 1], \quad \boldsymbol{v} = [-2, 1], \quad \boldsymbol{w} = [4, 1]$$

d) In $\mathbb{R}^3$:
$$\boldsymbol{u} = [1, 0, 1], \quad \boldsymbol{v} = [-2, -2, 1], \quad \boldsymbol{w} = [1, 2, -3]$$

e) In $\mathbb{R}^3$:
$$\boldsymbol{u} = [1, 0, 1], \quad \boldsymbol{v} = [-2, -2, 1], \quad \boldsymbol{w} = [-1, -2, 2]$$

f) In $\mathbb{R}^7$:
$$\boldsymbol{u} = [1, 2, 3, 4, 5, 6, 7], \boldsymbol{v} = [8, 6, 7, 5, 3, 0, 9], \boldsymbol{w} = [0, 0, 0, 0, 0, 0, 0]$$

Solution

a) These vectors are not linearly independent since they are scalar multiples of each other. Since $\boldsymbol{v} = -2\boldsymbol{u}$ a dependence relationship is $2\boldsymbol{u} + \boldsymbol{v} = \mathbf{0}$.

b) These vectors are linearly independent since they are not scalar multiples of each other.

c) These vectors are not linearly independent since one of them can be written as a linear combination of the others, for instance
$$\boldsymbol{w} = [4, 1] = 2[1, 1] - [-2, 1] = 2\boldsymbol{u} - \boldsymbol{v}.$$
Alternatively, no set of 3 vectors in $\mathbb{R}^2$ is linearly independent. Using the previous equation, a dependence relationship is $2\boldsymbol{u} - \boldsymbol{v} - \boldsymbol{w} = \mathbf{0}$.

d) This set of vectors is linearly independent. We want to solve the equation $a\boldsymbol{u} + b\boldsymbol{v} + c\boldsymbol{w} = \mathbf{0}$. The corresponding system of linear equations has augmented matrix
$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 1 & 1 & -3 & 0 \end{pmatrix}.$$

After applying EROs we obtain a row echelon form

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix}.$$

This system has a unique solution $a = b = c = 0$, hence the vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly independent.

e) These vectors are not linearly independent since one of them can be written as a linear combination of the others, for instance

$$\boldsymbol{w} = [-1, -2, 2] = [1, 0, 1] + [-2, -2, 1] = \boldsymbol{u} + \boldsymbol{v}.$$

Using this equation, a dependence relationship is $\boldsymbol{u} + \boldsymbol{v} - \boldsymbol{w} = \boldsymbol{0}$.

f) These vectors are not linearly independent since this set contains the zero vector. A dependence relationship is

$$0\boldsymbol{u} + 0\boldsymbol{v} + 1\boldsymbol{w} = \boldsymbol{0}.$$

T7

a) Draw diagrams to illustrate properties (a), (d) and (e) of Theorem 1.1.

b) Give an algebraic proof of properties (d) and (e) of Theorem 1.1.

Solution

a) Ask your tutor or lecturer.

b) For (d), let $\boldsymbol{u} \in \mathbb{R}^n$. We want to show that $\boldsymbol{u} + (-\boldsymbol{u}) = \boldsymbol{0}$. For this, let $\boldsymbol{u} = [u_1, u_2, \ldots, u_n]$. Then $-\boldsymbol{u} = [-u_1, -u_2, \ldots, -u_n]$, so

$$\begin{aligned} \boldsymbol{u} + (-\boldsymbol{u}) &= [u_1, u_2, \ldots, u_n] + [-u_1, -u_2, \ldots, -u_n] \\ &= [u_1 + (-u_1), u_2 + (-u_2), \ldots, u_n + (-u_n)] \\ &= [u_1 - u_1, u_2 - u_2, \ldots, u_n - u_n] \\ &= [0, 0, \ldots, 0] \\ &= \boldsymbol{0} \end{aligned}$$

as required.

For (e), let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ and let $c \in \mathbb{R}$. We want to show that $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$. For this, let $\boldsymbol{u} = [u_1, u_2, \ldots, u_n]$ and $\boldsymbol{v} = [v_1, v_2, \ldots, v_n]$. Then

$$\begin{aligned} c(\boldsymbol{u} + \boldsymbol{v}) &= c([u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n]) \\ &= c[u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n] \\ &= [c(u_1 + v_1), c(u_2 + v_2), \ldots, c(u_n + v_n)]. \end{aligned}$$

On the other hand

$$\begin{aligned} c\boldsymbol{u} + c\boldsymbol{v} &= c[u_1, u_2, \ldots, u_n] + c[v_1, v_2, \ldots, v_n] \\ &= [cu_1, cu_2, \ldots, cu_n] + [cv_1, cv_2, \ldots, cv_n] \\ &= [cu_1 + cv_1, cu_2 + cv_2, \ldots, cu_n + cv_n] \\ &= [c(u_1 + v_1), c(u_2 + v_2), \ldots, c(u_n + v_n)]. \end{aligned}$$

Since both $c(\boldsymbol{u} + \boldsymbol{v})$ and $c\boldsymbol{u} + c\boldsymbol{v}$ are equal to $[c(u_1 + v_1), c(u_2 + v_2), \ldots, c(u_n + v_n)]$, we have that $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$ as required.

General comment: sometimes it is easier to get two expressions to equal the same thing by manipulating both of them, rather than trying to work on just one of them to obtain the other.

T8 Suppose $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ are vectors in $\mathbb{R}^n$, and let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ and $T = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$. Prove that $\mathrm{Span}(S) \subseteq \mathrm{Span}(T)$. That is, prove that $\mathrm{Span}(S)$ is a subset of $\mathrm{Span}(T)$. [Hint: What you need to show is that every vector in $\mathrm{Span}(S)$ is also contained in $\mathrm{Span}(T)$. So your proof should probably start with: "Let $\boldsymbol{v}$ be a vector in $\mathrm{Span}(S)$."]

Solution

Let $\boldsymbol{v}$ be a vector in $\mathrm{Span}(S)$. Then by definition of span, there are scalars λ_1, λ_2 so that $\boldsymbol{v} = \lambda_1 \boldsymbol{u}_1 + \lambda_2 \boldsymbol{u}_2$. Hence

$$\boldsymbol{v} = \lambda_1 \boldsymbol{u}_1 + \lambda_2 \boldsymbol{u}_2 + 0\boldsymbol{u}_3$$

and so $\boldsymbol{v}$ is in $\mathrm{Span}(T)$ as well. Therefore $\mathrm{Span}(S) \subseteq \mathrm{Span}(T)$.

T9 Suppose that vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly independent. Are the vectors $\boldsymbol{u} + \boldsymbol{v}$, $\boldsymbol{v} + \boldsymbol{w}$ and $\boldsymbol{u} + \boldsymbol{w}$ linearly independent? What about the vectors $\boldsymbol{u} - \boldsymbol{v}$, $\boldsymbol{v} - \boldsymbol{w}$ and $\boldsymbol{u} - \boldsymbol{w}$? Justify your answers.

Solution

Suppose that

$$a(\boldsymbol{u} + \boldsymbol{v}) + b(\boldsymbol{v} + \boldsymbol{w}) + c(\boldsymbol{u} + \boldsymbol{w}) = \boldsymbol{0}$$

where a, b, c are scalars. This rearranges to give

$$(a + c)\boldsymbol{u} + (a + b)\boldsymbol{v} + (b + c)\boldsymbol{w} = \boldsymbol{0}.$$

Since $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly independent, the only solution to this equation is $a + c = 0$, $a + b = 0$ and $b + c = 0$. Check that the unique solution to this system is $a = 0$, $b = 0$ and $c = 0$. Thus the vectors $\boldsymbol{u} + \boldsymbol{v}$, $\boldsymbol{v} + \boldsymbol{w}$ and $\boldsymbol{u} + \boldsymbol{w}$ are linearly independent.

Now suppose that

$$a(\boldsymbol{u} - \boldsymbol{v}) + b(\boldsymbol{v} - \boldsymbol{w}) + c(\boldsymbol{u} - \boldsymbol{w}) = \boldsymbol{0}$$

where a, b, c are scalars. This rearranges to give

$$(a + c)\boldsymbol{u} + (-a + b)\boldsymbol{v} + (-b - c)\boldsymbol{w} = \boldsymbol{0}.$$

Since $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly independent, the only solution to this equation is $a + c = 0$, $-a + b = 0$ and $-b - c = 0$. However this system has infinitely many solutions and in particular has nonzero solutions, for example $a = 1$, $b = 1$ and $c = -1$. Therefore the vectors $\boldsymbol{u} - \boldsymbol{v}$, $\boldsymbol{v} - \boldsymbol{w}$ and $\boldsymbol{u} - \boldsymbol{w}$ are not linearly independent.

2B Linear Algebra

Ex Sheet **2**

True/False

We will start the tutorial by going over these true false questions. Please make sure you've thought about them in advance of the tutorial and so are ready to answer.[1]

a) For a matrix A the (i, j)-entry of A in the entry in column i and row j.

b) Consider the matrix

$$C = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 7 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

Then C is diagonal with diagonal entries 2, 7 and 3.

c) Consider the matrix

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

Then D is diagonal with diagonal entries 4, 7 and 9.

d) For all matrices A and B of the same size, $A + B = B + A$.

e) For all square matrices A and B of the same size, $AB = BA$.

f) The transpose of an $m \times n$ matrix is an $n \times m$ matrix.

g) The matrices C and D above are both symmetric.

h) If the matrix A is symmetric then the matrix $-A$ is symmetric.

i) The matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is a linear combination of the matrices $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

j) The matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ span $M_{2\times 2}(\mathbb{R})$.

k) The matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are linearly independent.

[1] **True/False Questions**

Every Exercise Sheet will have a section containing true/false questions, with solutions at the end of the sheet. They are designed to test your understanding from lectures. The degree exam will have true/false questions selected from those on Exercise Sheets. You should look at your lecture notes and/or the textbook to help you answer these questions, but you should not need to write anything to work out the solution.

Solutions to True/False

(a) F (b) F (c) T (d) T (e) F (f) T (g) F (h) T (i) F (j) T (k) T

Tutorial Exercises

Before attempting these questions you should make sure you can do the questions on Matrices from Exercise Sheet 0.

T1 Give an example of a nonzero 2×2 matrix A so that A^2 is the zero matrix.

Solution

An example is $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

T2 Let $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Prove that

$$A^2 = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}.$$

Solution

We have

$$\begin{aligned} A^2 &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta - \sin^2\theta & -2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \end{aligned}$$

where we are applying double angle formulas to obtain the last equality.

T3 Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Prove by induction that for every positive integer k,

$$A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}.$$

Solution

When $k = 1$ we have $A^1 = A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, so the base case holds. Assume that for n a positive integer we have $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$. Then

$$A^{n+1} = A^n A = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n+1 \\ 0 & 1 \end{bmatrix}.$$

Therefore by induction the result holds for all positive integers k.

T4 Show that if A and B are 3×3 matrices,[2]

$$\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B).$$

[2] For a square matrix A, the *trace* of A, denoted by $\mathrm{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A (i.e., the diagonal from top left to bottom right).

Solution

The trace of A is

$$a_{11} + a_{22} + a_{33}$$

and the trace of B is

$$b_{11} + b_{22} + b_{33}.$$

The diagonal entries of $C = A + B$ are respectively

$$c_{11} = a_{11} + b_{11},$$

$$c_{22} = a_{22} + b_{22}$$

and

$$c_{33} = a_{33} + b_{33}.$$

The stated result is now immediate.

T5 Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ and let λ be a scalar.

Prove that:

a) $A + B = B + A$.

b) $\lambda(A + B) = \lambda A + \lambda B$.

c) $(A + B)^T = A^T + B^T$.

Solution

a) We have

$$\begin{aligned} A+B &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} \\ &= \begin{bmatrix} b_{11}+a_{11} & b_{12}+a_{12} \\ b_{21}+a_{21} & b_{22}+a_{22} \end{bmatrix} \quad \text{by commutativity of scalar addition} \\ &= \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= B+A \end{aligned}$$

as required.

b) We have

$$\begin{aligned} \lambda(A+B) &= \lambda \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} \\ &= \begin{bmatrix} \lambda(a_{11}+b_{11}) & \lambda(a_{12}+b_{12}) \\ \lambda(a_{21}+b_{21}) & \lambda(a_{22}+b_{22}) \end{bmatrix} \\ &= \begin{bmatrix} \lambda a_{11}+\lambda b_{11} & \lambda a_{12}+\lambda b_{12} \\ \lambda a_{21}+\lambda b_{21} & \lambda a_{22}+\lambda b_{22} \end{bmatrix} \quad \text{by the distributive law for scalars} \\ &= \begin{bmatrix} \lambda a_{11} & \lambda a_{12} \\ \lambda a_{21} & \lambda a_{22} \end{bmatrix} + \begin{bmatrix} \lambda b_{11} & \lambda b_{12} \\ \lambda b_{21} & \lambda b_{22} \end{bmatrix} \\ &= \lambda \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \lambda \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ &= \lambda A + \lambda B \end{aligned}$$

as required.

c) We have

$$\begin{aligned} (A+B)^T &= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}^T \\ &= \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} \\ a_{12}+b_{12} & a_{22}+b_{22} \end{bmatrix} \end{aligned}$$

while

$$\begin{aligned} A^T+B^T &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^T + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}^T \\ &= \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} \\ a_{12}+b_{12} & a_{22}+b_{22} \end{bmatrix}. \end{aligned}$$

Since $(A+B)^T$ and A^T+B^T are both equal to the same matrix, we have that $(A+B)^T = A^T+B^T$ as required.

T6 Write B as a linear combination of the other matrices, if possible:

a) $B = \begin{bmatrix} 2 & 5 \\ 0 & 3 \end{bmatrix}$, $A_1 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$.

b) $B = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $A_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Solution

a) We want to know if there are scalars c_1 and c_2 so that $c_1A_1 + c_2A_2 = B$. Now

$$c_1 \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 0 & 3 \end{bmatrix}$$

if and only if

$$\begin{bmatrix} c_1 & 2c_1 + c_2 \\ -c_1 + 2c_2 & c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 0 & 3 \end{bmatrix}$$

By comparing the $(1,1)$ entries on both sides we must have $c_1 = 2$. Then from the $(1,2)$ entries we obtain $c_2 = 1$. This is consistent with the remaining matrix entries, so the equation $c_1A_1 + c_2A_2 = B$ has (unique) solution $2A_1 + A_2 = B$.

Alternatively, an augmented matrix corresponding to the system of equations obtained by comparing the matrix entries is

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 2 & 1 & 5 \\ -1 & 2 & 0 \\ 1 & 1 & 3 \end{array}\right]$$

and by carrying out EROs then solving by back-substitution this sytem has unique solution $c_1 = 2$, $c_2 = 1$, so $2A_1 + A_2 = B$.

b) We want to know if there are scalars c_1, c_2 and c_3 so that $c_1A_1 + c_2A_2 + c_3A_3 = B$. Now

$$c_1 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

if and only if

$$\begin{bmatrix} c_1 - c_2 + c_3 & 2c_2 + c_3 & -c_1 + c_3 \\ 0 & c_1 + c_2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

A corresponding system of equations is:

$$\begin{array}{rcrcrcl} c_1 & - & c_2 & + & c_3 & = & 3 \\ & & 2c_2 & + & c_3 & = & 1 \\ -c_1 & & & + & c_3 & = & 1 \\ c_1 & + & c_2 & & & = & 1 \end{array}.$$

(We don't need to write down equations for the entries which are 0 on both sides.) The augmented matrix for this system is:

$$\begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 2 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

and after applying EROs we get an echelon form

$$\begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

This system does not have a solution, so it is not possible to write B as a linear combination of the other matrices.

T7 Determine whether each of the sets of matrices in question T6 is linearly independent.

Solution

a) The matrices B, A_1 and A_2 are not linearly independent since B can be expressed as a linear combination of A_1 and A_2.

b) Suppose there are scalars c_1, c_2, c_3 and c_4 so that $c_1A_1 + c_2A_2 + c_3A_3 + c_4B = 0$. Then

$$\begin{bmatrix} c_1 - c_2 + c_3 + 3c_4 & 2c_2 + c_3 + c_4 & -c_1 + c_3 + c_4 \\ 0 & c_1 + c_2 + c_4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A corresponding homogeneous system of equations is:

$$\begin{array}{rcrcrcrl} c_1 & - & c_2 & + & c_3 & + & 3c_4 = & 0 \\ & & 2c_2 & + & c_3 & + & c_4 = & 0 \\ -c_1 & & & + & c_3 & + & c_4 = & 0 \\ c_1 & + & c_2 & & & + & c_4 = & 0 \end{array}.$$

The augmented matrix for this system is:

$$\begin{bmatrix} 1 & -1 & 1 & 3 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

and after applying EROs we get an echelon form

$$\begin{bmatrix} 1 & -1 & 1 & 3 & 0 \\ 0 & 1 & -2 & 4 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

This system has unique solution $c_1 = c_2 = c_3 = c_4 = 0$. Therefore the matrices A_1, A_2, A_3, B are linearly independent.

T8

a) Prove that if A and B are symmetric matrices, then $A + B$ is symmetric.

b) Prove that if A is a symmetric matrix and c is a scalar, then cA is symmetric.

Solution

a) Suppose A and B are symmetric. By Theorem 3.4(b) we have $(A + B)^T = A^T + B^T$. Since A and B are symmetric, $A^T = A$ and $B^T = B$. Thus $(A + B)^T = A + B$, hence $A + B$ is symmetric.

b) Suppose A is symmetric and c is a scalar. By Theorem 3.4(c), $(cA)^T = cA^T$. Since A is symmetric, $A^T = A$. So $(cA)^T = cA$, hence cA is symmetric.

T9 Use Theorem 3.4 to prove that for any $m \times n$ matrix A, the matrices AA^T and $A^T A$ are symmetric. [Hint: take the transpose.]

Solution

By Theorem 3.4 part (d) and then part (a)

$$(AA^T)^T = (A^T)^T A^T = AA^T$$

so AA^T is symmetric. The proof is similar for $A^T A$.

T10 A square matrix A is defined to be *skew-symmetric* if $A^T = -A$.

a) Prove that if A and B are skew-symmetric, then $A + B$ is skew-symmetric.

b) Prove that if A is a skew-symmetric matrix and c is a scalar, then cA is skew-symmetric.

c) Prove that for all square matrices A, the matrix $A - A^T$ is skew-symmetric.

Solution

a) Suppose A and B are skew-symmetric. By Theorem 3.4(b) we have $(A + B)^T = A^T + B^T$. Since A and B are skew-symmetric, $A^T = -A$ and $B^T = -B$. Thus $(A + B)^T = -A - B = -(A + B)$, hence $A + B$ is skew-symmetric.

b) Suppose A is skew-symmetric and c is a scalar. By Theorem 3.4(c), $(cA)^T = cA^T$. Since A is skew-symmetric, $A^T = -A$. So $(cA)^T = c(-A) = -cA$, hence cA is skew-symmetric.

T11 Using Theorem 3.5(a) and several of the previous exercises in this section, prove that any square matrix A can be written as the sum of a symmetric matrix and a skew-symmetric matrix.

Solution

Let A be a square matrix. By Theorem 3.5(a) the matrix $A + A^T$ is symmetric, and by F3(c) the matrix $A - A^T$ is skew-symmetric. Notice that

$$(A + A^T) + (A - A^T) = 2A.$$

So if we divide through this equation by 2 we can make A the subject:

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

By F1(b), the matrix $\frac{1}{2}(A + A^T)$ is symmetric, and by F3(b) the matrix $\frac{1}{2}(A - A^T)$ is skew-symmetric. Therefore we have written A as a sum of a symmetric matrix and a skew-symmetric matrix.

2B Linear Algebra

Ex Sheet 3

True/False

[1]

a) If A, B and C are $n \times n$ matrices and $AB = AC$ then $B = C$.

b) If A and B are $n \times n$ matrices and $AB = I$ then $B = A^{-1}$.

c) If A is an invertible matrix then the inverse of A can be written as $\frac{1}{A}$.

d) If A is invertible then the system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution.

e) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then A is invertible if and only if $ad - bc > 0$.

f) If A and B are invertible then AB is invertible and $(AB)^{-1} = A^{-1}B^{-1}$.

g) If A is invertible then A^T is invertible, and the inverse of A^T is the transpose of A^{-1}.

h) Performing an elementary row operation is equivalent to left multiplication by an elementary matrix.

i) Every invertible matrix A can be expressed uniquely as a product of elementary matrices.

j) Let A be a non-invertible $n \times n$ matrix. Then the matrix obtained by swapping rows one and two of A is also non-invertible.

k) Let A be an invertible matrix. Then the matrix obtained by adding twice the first row to the second may not be invertible.

l) Let A be a non-invertible $n \times n$ matrix. Then the linear system $A\boldsymbol{x} = \boldsymbol{0}$ has infinitely many solutions.

m) For matrices A, B and C, if A is row equivalent to B and B is row equivalent to C then A is row equivalent to C.

n) Every elementary matrix is invertible.

o) The product of two $n \times n$ elementary matrices must also be an elementary matrix.

p) If E_1 and E_2 are both $n \times n$ elementary matrices then $E_1E_2 = E_2E_1$.

q) A square matrix A is invertible if and only if its reduced row echelon form is the identity.

r) Any line in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.

[1] **True/False Questions**

Every Exercise Sheet will have a section containing true/false questions. They are designed to test your understanding from lectures. You should look at your lecture notes and/or the textbook to help you answer these questions, but you should not need to write anything to work out the solution.

s) The set of solutions to the equation $3v - 10w + 3x + 2y - 5z = 0$ is a subspace of $\mathbb{R}^5$.

t) The set $\{(a, b, c, 0, 0) : a, b, c \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^5$.

u) Let A be an $m \times n$ matrix over $\mathbb{R}$. Then the row space of A is the subspace of $\mathbb{R}^n$ given by the solutions to $A\mathbf{v} = \mathbf{0}$.

v) Let A be an $m \times n$ matrix over $\mathbb{R}$. Then the null space of A is a subspace of $\mathbb{R}^n$.

Solutions to True/False

(a) F (b) T (c) F (d) T (e) F (f) F (g) T (h) T (i) F (j) T (k) F (l) T (m) T (n) T (o) F (p) F (q) T (r) F (s) T (t) T (u) F (v) T

Tutorial Exercises

T1 Let

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \ B = \begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix}.$$

Work out $A^{-1}B$.

Solution

First we calculate the inverse of A to be

$$\frac{1}{-2}\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$$

Then the required product is

$$A^{-1}B = \begin{bmatrix} -\frac{9}{2} & 5 \\ \frac{7}{2} & -2 \end{bmatrix}.$$

T2 Use Theorem 3.8 to determine whether the matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is invertible, and if it is invertible finds its inverse.

Solution

The matrix has determinant $\cos^2\theta + \sin^2\theta = 1 \neq 0$ so it is invertible (no matter what the value of θ). Its inverse is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

T3 Let A and P be $n \times n$ matrices; P being invertible. Simplify each of the following expressions:

a) $(P^{-1}AP)^2$, b) $(P^{-1}AP)^3$.

Solution

a) $(P^{-1}AP)^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A(PP^{-1})AP = P^{-1}A(I)AP = P^{-1}A^2P$.

b) $(P^{-1}AP)^3 = (P^{-1}AP)^2(P^{-1}AP) = P^{-1}A^2(PP^{-1})AP = P^{-1}A^2(I)AP = P^{-1}A^3P$.

T4 Find the inverse of the given matrix if it exists[2], or justify why it does not exist. For 2×2 matrices you can use either Theorem 3.8 or the Gauss–Jordan method.

[2] and check your answer!

$$A = \begin{pmatrix} -3 & 6 \\ 4 & 5 \end{pmatrix}; \quad B = \begin{pmatrix} 6 & -4 \\ -3 & 2 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 4 & 4 \\ 1 & 2 & 4 \\ 1 & 3 & 2 \end{pmatrix}.$$

Solution

The Gauss–Jordan method is written out for the 2×2 matrices here.

Let's first consider A: the Gauss–Jordan method tells us that we need to find the reduced row echelon form of the matrix

$$(A \mid \mathbb{I}_2) = \begin{pmatrix} -3 & 6 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{pmatrix}$$

Some calculation shows that the reduced row echelon form is

$$\begin{pmatrix} 1 & 0 & -\frac{5}{39} & \frac{2}{13} \\ 0 & 1 & \frac{4}{39} & \frac{1}{13} \end{pmatrix}$$

This is of the form $(\mathbb{I}_2 \mid X)$ and so

$$X = \begin{pmatrix} -\frac{5}{39} & \frac{2}{13} \\ \frac{4}{39} & \frac{1}{13} \end{pmatrix}$$

is the inverse of A.

Now let's consider B: we need to find the reduced row echelon form of the matrix

$$(B \mid \mathbb{I}_2) = \begin{pmatrix} 6 & -4 & 1 & 0 \\ -3 & 2 & 0 & 1 \end{pmatrix}$$

Again some calculation shows that the reduced row echelon form is

$$\begin{pmatrix} 1 & -\frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

This is not of the form $(\mathbb{I}_2 \mid Y)$ and so B is not invertible.

Finally we consider C: we need to find the reduced row echelon form of the matrix

$$(C \mid \mathbb{I}_3) = \begin{pmatrix} 1 & 4 & 4 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{pmatrix}$$

Again some calculation shows that this is the matrix

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 2 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \end{array}\right)$$

This is of the form $(\mathbb{I}_3 \mid Z)$ and so

$$Z = \begin{pmatrix} -2 & 1 & 2 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \end{pmatrix}$$

is the inverse of C.

T5 For which real values of a and b is the matrix $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ invertible? For these values, find A^{-1}.

Solution

We have $\det(A) = a^2 + b^2$. Since a and b are real numbers, $a^2 + b^2 \geq 0$ with $a^2 + b^2 = 0$ if and only if $a = b = 0$. So A is invertible unless a and b both equal 0. For all other real values of a and b, $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \frac{1}{a^2+b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

T6 Let A be as in T4 and let $\boldsymbol{b} = \begin{pmatrix} 39 \\ 13 \end{pmatrix}$. Use A^{-1} to solve the system $A\boldsymbol{x} = \boldsymbol{b}$.

Solution

The unique solution to this system is

$$A^{-1}\boldsymbol{b} = \begin{pmatrix} -\frac{5}{39} & \frac{2}{13} \\ \frac{4}{39} & \frac{1}{13} \end{pmatrix} \begin{pmatrix} 39 \\ 13 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}.$$

T7 In the context of 3×3 matrices, write down the elementary matrix E corresponding to the ERO

$$R_2 \to R_2 + 5R_3.$$

Verify that pre-multiplication by E has the same effect on a 3×3 matrix as the ERO

$$R_2 \to R_2 + 5R_3.$$

Solution

The elementary matrix corresponding to the ERO $R_2 \to R_2 + 5R_3$ is

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$

For $a, b, c, d, e, f, g, h, i \in \mathbb{R}$, let F be the matrix

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Then

$$EF = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d+5g & e+5h & f+5i \\ g & h & i \end{bmatrix},$$

which is the same as performing the ERO $R_2 \to R_2 + 5R_3$ on F.

T8 Consider the matrices

$$A = \begin{pmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 4 & 7 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 0 & -5 & 25 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 \\ -5 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}.$$

a) Find elementary matrices E_1 and E_2 such that $E_1 A = B$ and $E_2 B = A$.

b) Write the matrix C as a product of elementary matrices.

c) Is it possible to repeat part (b) for the matrix D? Justify your answer.

Solution

a) We first find an elementary row operation (ERO) which takes the matrix A to the matrix B:

$$\begin{pmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 4 & 7 & 9 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} \begin{pmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 0 & -5 & 25 \end{pmatrix}$$

Clearly to get back to the matrix A from the matrix B we need to reverse this ERO:

$$\begin{pmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 0 & -5 & 25 \end{pmatrix} \xrightarrow{R_3 \to R_3 + 2R_1} \begin{pmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 4 & 7 & 9 \end{pmatrix}$$

Now recall that doing an ERO to a matrix is the same thing as left multiplying by the *elementary matrix* corresponding to the ERO. What is the elementary matrix E_1 corresponding to the ERO

$R_3 \to R_3 - 2R_1$? This is just the matrix we get by applying this ERO to the 3×3 identity matrix $\mathbb{I}_3$, in other words

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

Therefore $E_1 A = B$ (since B is obtained from A by doing the ERO corresponding to E_1). Likewise the elementary matrix E_2 corresponding to the ERO $R_3 \to R_3 + 2R_1$ is

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

Therefore $E_2 B = A$ (since A is obtained from B by doing the ERO corresponding to E_2). Since the two EROs reverse the effect of each other (i.e. doing one and then the other one gets you back to where you started) the matrices E_1 and E_2 are inverse to one another, i.e. $E_1 E_2 = \mathbb{I}_3 = E_2 E_1$.

b) We can write the matrix C as a product of elementary matrices if and only if we can find a sequence of EROs which make C row equivalent to $\mathbb{I}_2$, i.e. if and only if the reduced row echelon form of C is $\mathbb{I}_2$. We have

$$\begin{pmatrix} 1 & 0 \\ -5 & 2 \end{pmatrix} \xrightarrow{R_2 \to R_2 + 5R_1} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The EROs $R_2 \to R_2 + 5R_1$ and $R_2 \to \frac{1}{2}R_2$ correspond to elementary matrices E_1 and E_2 given by

$$E_1 = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

respectively. Therefore $E_2 E_1 C = \mathbb{I}_2$ (note the order), and so

$$C = E_1^{-1} E_2^{-1} \mathbb{I}_2 = E_1^{-1} E_2^{-1}.$$

By Theorem 3.11 the matrices E_1^{-1} and E_2^{-1} are elementary, and it is easy to see that

$$E_1^{-1} = \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix} \quad \text{and} \quad E_2^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

c) By Theorem 3.12, D can be written as a product of elementary matrices if and only if D is invertible. But D is not invertible, since $\det(D) = 0$. So we cannot repeat part (b) for D.

T9 Suppose A and B are $n \times n$ matrices such that $(AB)^2 = I$. Prove that A is invertible and find A^{-1} in terms of A and B. Show also that $(BA)^2 = I$.

Solution

In this and subsequent questions we have to know what ‘invertible’ means.

Advice: *realising that definitions are central to understanding mathematics is often the key to doing well in mathematics at Level 2 and beyond. It’s impossible to get started on many questions without knowing the definitions, but on the plus side, it can become easy to solve such questions when you get the hang of it. This* *word of advice could be the difference between a summer holiday and a resit, so please take note!*

The equation $(AB)^2 = I$ can be expanded as $ABAB = I$, hence $A(BAB) = I$. Therefore by Theorem 3.13, A is invertible and $A^{-1} = BAB$. Now $(BA)^2 = BABA = (BAB)A = A^{-1}A = I$ as required.

T10 Let A and B be $n \times n$ matrices. Prove that if A and B are invertible then AB is invertible.

Solution

By definition, since A and B are invertible there exist matrices M and N such that

$$AM = \mathbb{I}_n = MA, \tag{1}$$
$$BN = \mathbb{I}_n = NB. \tag{2}$$

To show that the product AB is invertible, we must find a matrix P such that

$$(AB)P = \mathbb{I}_n = P(AB).$$

[Aside: How do we choose P? The *only* information we've got is from the equations (1) and (2) above, and here you just have to guess and check. You might guess that P equals M, or N, or MN or NM or,.... so just try to compute $(AB)P$ for each of these guesses, and you'll see that only one of them is the identity matrix.] Define $P = NM$, that is, $P = B^{-1}A^{-1}$. Then

$$\begin{aligned} (AB)P &= A(BN)M && \text{since matrix multiplication is associative} \\ &= A\mathbb{I}_n M && \text{by (2)} \\ &= AM \\ &= \mathbb{I}_n && \text{by (1).} \end{aligned}$$

You can now check similarly that $P(AB) = \mathbb{I}_n$, so P is indeed the inverse of AB, that is,

$$(AB)^{-1} = B^{-1}A^{-1}.$$

T11 Prove by induction on $k \geq 2$ that the product of k invertible matrices, all of size $n \times n$, is an invertible matrix.

Solution

We've just proved the result for two matrices, so suppose by induction that the analogous formula holds for a product of $k - 1$ matrices:

$$(A_1 A_2 \cdots A_{k-1})^{-1} = A_{k-1}^{-1} \cdots A_2^{-1} A_1^{-1}. \tag{3}$$

If A_k is also invertible, then A_k^{-1} exists and we can define

$$P := A_k^{-1} A_{k-1}^{-1} \cdots A_2^{-1} A_1^{-1}. \tag{4}$$

Now, just as for two matrices, we compute that

$$\begin{aligned}(A_1A_2\cdots A_k)P &= (A_1A_2\cdots A_k)(A_k^{-1}A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}) \\ &= A_1A_2\cdots A_{k-1}(A_k\cdot A_k^{-1})A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1} && \text{as matrix mult is associative} \\ &= A_1A_2\cdots A_{k-1}\mathbb{I}_n A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1} && \text{as } A_k^{-1} \text{ is inverse of } A_k \\ &= (A_1A_2\cdots A_{k-1})(A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}) \\ &= \mathbb{I}_n && \text{by (3).}\end{aligned}$$

Again, one shows similarly that $PA_1A_2\cdots A_k = \mathbb{I}_n$, so the matrix P defined in (4) is the inverse of the matrix $A_1A_2\cdots A_k$. This completes the proof.

T12 Let A and B be $n\times n$ matrices. Prove that if AB is invertible then both A and B are invertible.

Solution

Since AB is invertible there is an $n\times n$ matrix C so that $(AB)C = I = C(AB)$. Therefore $A(BC) = I$ and $(CA)B = I$. Hence by Theorem 3.13, both A and B are invertible (with $A^{-1} = BC$ and $B^{-1} = CA$).

T13 For which (if any) elementary matrices E is it true that $E^{-1} = E$? Justify your answer.

Solution

If E corresponds to the row operation of multiplying row i by λ then E^{-1} corresponds to the row operation of multiplying row i by $\frac{1}{\lambda}$. So E is equal to E^{-1} if and only if $\lambda = \frac{1}{\lambda}$, equivalently $\lambda^2 = 1$, equivalently $\lambda = \pm 1$. Note that if $\lambda = 1$ then the row operation does nothing, that is, $E = I$, but the equation $E = E^{-1}$ still holds since $I = I^{-1}$. The other case is when $\lambda = -1$, and so there is a nontrivial case in which $E = E^{-1}$ for this type of row operation.

If E is the elementary matrix corresponding to swapping row i and row j, then $E^2 = I$ since doing this swap twice results in the identity matrix. Hence $E^{-1} = E$ for all such row operations.

If E is the elementary matrix corresponding to the row operation $R_i \to R_i + \lambda R_j$ then E^{-1} is the elementary matrix corresponding to the row operation $R_i \to R_i - \lambda R_j$. Thus $E = E^{-1}$ if and only if $\lambda = 0$. Note that if $\lambda = 0$ then the row operation does nothing, that is, $E = I$, but the equation $E = E^{-1}$ still holds since $I = I^{-1}$.

T14 For each of the following subsets of $\mathbb{R}^2$, sketch the region of the plane described by the set U, then determine whether the set U contains $\mathbf{0}$, whether U is closed under vector addition and whether U is closed under scalar multiplication, giving reasons for your answers (either proofs or a counterexample). Hence determine whether U is a subspace of $\mathbb{R}^2$.

a) $U = \{(x,y)\in\mathbb{R}^2 : y = 0\}$

b) $U = \{(x,y)\in\mathbb{R}^2 : y = x\}$

c) $U = \{(x,y) \in \mathbb{R}^2 : x = y+1\}$

d) $U = \{(x,y) \in \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0\}$

e) $U = \{(x,y) \in \mathbb{R}^2 : xy \geq 0\}$

Solution

For the sketches, ask your tutor or lecturer.

a) The set U contains $\mathbf{0} = (0,0)$ since it contains all points $(x,0)$ with $x \in \mathbb{R}$. It is closed under vector addition since if $(x,0), (x',0) \in U$ then $(x,0) + (x',0) = (x+x', 0+0) = (x+x',0)$ is also in U. It is closed under scalar multiplication since if $(x,0) \in U$ and $c \in \mathbb{R}$ then $c(x,0) = (cx, c0) = (cx,0)$ is in U. Therefore U is a subspace of $\mathbb{R}^2$. (Geometrically, it is the x-axis.)

b) The set U contains $\mathbf{0} = (0,0)$ since it contains all points (x,x) with $x \in \mathbb{R}$. It is closed under vector addition since if $(x,x), (x',x') \in U$ then $(x,x) + (x',x') = (x+x', x+x')$ is also in U. It is closed under scalar multiplication since if $(x,x) \in U$ and $c \in \mathbb{R}$ then $c(x,x) = (cx,cx)$ is in U. Therefore U is a subspace of $\mathbb{R}^2$. (Geometrically, it is the line through the origin $y = x$.)

c) The set U does not contain $\mathbf{0}$ since $0 \neq 0+1$. It is not closed under vector addition since the points $(2,1)$ and $(3,2)$ both lie in U, but $(2,1) + (3,2) = (5,3)$ is not in U because $5 \neq 3+1$. It is not closed under scalar multiplication because $(2,1) \in U$ but $2(2,1) = (4,2)$ is not in U since $4 \neq 2+1$. Thus U is not a subspace; in fact it does not satisfy any of the conditions required for a subspace. (Geometrically, U is the line $y = x-1$, which does not pass through the origin.)

d) The set U contains $\mathbf{0} = (0,0)$ since it contains all points (x,y) with $x \geq 0$ and $y \geq 0$. It is closed under vector addition since if $(x,y), (x',y') \in U$ then $x, y, x', y' \geq 0$ so $x + y \geq 0$ and $x' + y' \geq 0$. Thus $(x,y) + (x',y') = (x+x', y+y')$ is also in U. It is not closed under scalar multiplication since $(1,1)$ is in U but $-1(1,1) = (-1,-1)$ is not in U. Therefore U is not a subspace of $\mathbb{R}^2$. (Geometrically, it is the first quadrant.)

e) The set U contains $\mathbf{0} = (0,0)$ since it contains all points (x,y) with $xy \geq 0$, and $0 \cdot 0 = 0$. It is not closed under vector addition since $(2,2), (-1,-3) \in U$ but $(2,2) + (-1,-3) = (2-1, 2-3) = (1,-1)$ is not in U. It is closed under scalar multiplication since if $(x,y) \in U$ and $c \in \mathbb{R}$ then as $xy \geq 0$ and $c^2 \geq 0$ we have $(cx)(cy) = c^2xy \geq 0$, hence $c(x,y) = (cx,cy) \in U$. Therefore U is not a subspace. (Geometrically, it is the first and third quadrants.)

T15 Determine which of the following subsets are subspaces of $\mathbb{R}^4$.

$$\begin{aligned}
W_1 &= \{(w,x,y,z) \in \mathbb{R}^4 : x = y,\ w = 2z\},\\
W_2 &= \{(w,x,y,z) \in \mathbb{R}^4 : w + x - y = 0\},\\
W_3 &= \{(w,x,y,z) \in \mathbb{R}^4 : y = 1\},\\
W_4 &= \{(w,x,y,z) \in \mathbb{R}^4 : y = x^2\},\\
W_5 &= \{(w,x,y,z) \in \mathbb{R}^4 : wy = xz\}.
\end{aligned}$$

In each case, justify fully your conclusion.

Solution

- There are two approaches. One is to notice that $W_1 = \{(w, x, y, z) \in \mathbb{R}^4 : x = y,\ w = 2z\}$ can be written as the set of solutions to a homogeneous system of linear equations (which ones?) and hence by Theorem 3.21 is a subspace.

  Alternatively, we can check the definition of a subspace. First, the zero vector of $\mathbb{R}^4$ is $(0, 0, 0, 0)$ and $(0, 0, 0, 0) \in W_1$ since $0 = 0$ and $0 = 2 \cdot 0$.

  Now suppose that $(w_1, x_1, y_1, z_1), (w_2, x_2, y_2, z_2) \in W_1$ and $\lambda \in \mathbb{R}$ is arbitrary. Then

$$\begin{aligned} x_1 &= y_1 \quad \text{and} \quad w_1 = 2z_1 \quad \text{since} \quad (w_1, x_1, y_1, z_1) \in W_1 \quad \text{and} \\ x_2 &= y_2 \quad \text{and} \quad w_2 = 2z_2 \quad \text{since} \quad (w_2, x_2, y_2, z_2) \in W_1. \end{aligned}$$

  We first consider the sum

$$(w_1, x_1, y_1, z_1) + (w_2, x_2, y_2, z_2) = (w_1 + w_2, x_1 + x_2, y_1 + y_2, z_1 + z_2).$$

  We need to see whether this is also an element of W_1. Since $x_1 = y_1$ and $x_2 = y_2$ we have

$$x_1 + x_2 = y_1 + y_2$$

  and since $w_1 = 2z_1$ and $w_2 = 2z_2$ we have

$$w_1 + w_2 = 2z_1 + 2z_2 = 2(z_1 + z_2).$$

  Therefore the vector $(w_1 + w_2, x_1 + x_2, y_1 + y_2, z_1 + z_2)$ is in W_1.

  We now consider the scalar multiple

$$\lambda(w_1, x_1, y_1, z_1) = (\lambda w_1, \lambda x_1, \lambda y_1, \lambda z_1).$$

  We need to see whether this is also an element of W_1. Since $x_1 = y_1$ we have

$$\lambda x_1 = \lambda y_1$$

  and since $w_1 = 2z_1$ we have

$$\lambda w_1 = \lambda(2z_1) = 2(\lambda z_1).$$

  Therefore the vector $(\lambda w_1, \lambda x_1, \lambda y_1, \lambda z_1)$ is in W_1. We conclude that W_1 is a subspace of $\mathbb{R}^4$.

- Again, one approach is to notice that $W_2 = \{(w, x, y, z) \in \mathbb{R}^4 : w + x - y = 0\}$ can be written as the set of solutions to a homogeneous system of linear equations (which ones?) and hence by Theorem 3.21 is a subspace.

  Alternatively, one checks the definition of subspace. First, the zero vector $(0, 0, 0, 0)$ is in W_2 since $0 + 0 - 0 = 0$.

  Now suppose that $(w_1, x_1, y_1, z_1), (w_2, x_2, y_2, z_2) \in W_2$ and $\lambda \in \mathbb{R}$ is arbitrary. Then

$$w_1 + x_1 - y_1 = 0 \quad \text{and} \quad w_2 + x_2 - y_2 = 0.$$

  We consider first the sum

$$(w_1, x_1, y_1, z_1) + (w_2, x_2, y_2, z_2) = (w_1 + w_2, x_1 + x_2, y_1 + y_2, z_1 + z_2).$$

We need to see if this is also an element of W_2. We have

$$(w_1 + w_2) + (x_1 + x_2) - (y_1 + y_2) = \qquad = (w_1 + x_1 - y_1) + (w_2 + x_2 - y_2)$$
$$= 0 + 0 = 0.$$

So we have that $(w_1 + w_2, x_1 + x_2, y_1 + y_2, z_1 + z_2)$ is in W_2.
We now consider the scalar multiple

$$\lambda(w_1, x_1, y_1, z_1) = (\lambda w_1, \lambda x_1, \lambda y_1, \lambda z_1).$$

We need to see whether this is also an element of W_2. We have

$$\lambda w_1 + \lambda x_1 - \lambda y_1 = \lambda(w_1 + x_1 - y_1) = \lambda \cdot 0 = 0.$$

Therefore the vector $(\lambda w_1, \lambda x_1, \lambda y_1, \lambda z_1)$ is in W_2. We conclude that W_2 is a subspace of $\mathbb{R}^4$.

- $W_3 = \{(w, x, y, z) \in \mathbb{R}^4 : y = 1\}$.
  This time the zero vector $(0, 0, 0, 0)$ is not in W_3, so W_3 is not a subspace.
- $W_4 = \{(w, x, y, z) \in \mathbb{R}^4 : y = x^2\}$.
  This is not a subspace of $\mathbb{R}_4$. Consider $(0, 2, 4, 0) \in W_4$ and $2 \in \mathbb{R}$. Then

$$2(0, 2, 4, 0) = (0, 4, 8, 0) \notin W_4 \qquad (\text{since } 8 \neq 4^2).$$

  Hence W_4 is not closed under scalar multiplication and so is not a subspace.
- $W_5 = \{(w, x, y, z) \in \mathbb{R}^4 : wy = xz\}$.
  This is not a subspace of $\mathbb{R}^4$. Consider

$$\begin{aligned} (1, 1, 1, 1) &\in W_5 && (\text{since } 1 \times 1 = 1 \times 1) \\ (1, -1, 1, -1) &\in W_5 && (\text{since } 1 \times 1 = (-1) \times (-1)). \end{aligned}$$

  Then

$$(1, 1, 1, 1) + (1, -1, 1, -1) = (2, 0, 2, 0) \notin W_5 \qquad (\text{since } 2 \times 2 = 0 \times 0).$$

  Hence W_5 is not closed under vector addition and so is not a subspace.

T16 In $\mathbb{R}^6$, let

$$\boldsymbol{u} = (3, 200, 456, -188, 87, 900000000) \quad \text{and} \quad \boldsymbol{v} = (1, 2, 4, 8, 16, 32)$$

Use a theorem from lectures to explain why $\mathrm{Span}(\boldsymbol{u}, \boldsymbol{v})$ is a subspace of $\mathbb{R}^6$.

Solution

By Theorem 3.19, the span of any collection of vectors in $\mathbb{R}^6$ is a subspace of $\mathbb{R}^6$. (So the particular values of $\boldsymbol{u}$ and $\boldsymbol{v}$ don't matter.)

T17 Show that the vector $\boldsymbol{v} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ is in the null space of

$$A = \begin{pmatrix} 1 & 267584 & 3 \\ 4 & 3765417 & 12 \end{pmatrix}$$

by carrying out a matrix multiplication.

Solution

Check that $A\boldsymbol{v} = \mathbf{0}$. Therefore $\boldsymbol{v} \in \text{null}(A)$.

2B Linear Algebra

Ex Sheet 4

1 True/False

[1] **True/False Questions**
Every Exercise Sheet will have a section containing true/false questions. They are designed to test your understanding from lectures. You should look at your lecture notes and/or the textbook to help you answer these questions, but you should not need to write anything to work out the solution.

a) Let A be an $m \times n$ matrix over $\mathbb{R}$. Then dim(row(A))=dim(col(A^T)).

b) Let A be an $m \times n$ matrix over $\mathbb{R}$. Then the nonzero row vectors of A form a basis for its row space.

c) For any $m \times n$ matrix over $\mathbb{R}$, the row and column spaces have identical dimension.

d) The sum of the rank and the nullity of a given matrix equals the number of rows of the matrix.

e) The column space of the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ is $\operatorname{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.

f) Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be a basis for $\mathbb{R}^n$. Then any vector in $\mathbb{R}^n$ can be expressed as a linear combination of the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$.

g) Each linearly independent set of vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$.

h) If $\operatorname{Span}(S) = \mathbb{R}^n$ then S is a basis for $\mathbb{R}^n$.

i) The vector space consisting of just the zero vector has dimension 1.

j) If S and S' are bases for $\mathbb{R}^n$ then both S and S' contain n elements.

k) Any two vectors in $\mathbb{R}^2$ which are not scalar multiples of each other form a basis for $\mathbb{R}^2$.

l) Any two vectors in $\mathbb{R}^3$ which are not scalar multiples of each other form a basis for a 2-dimensional subspace of $\mathbb{R}^3$.

m) The rank of a matrix is the dimension of its column space.

n) For any matrix $A \in M_{3\times 2}(\mathbb{R})$, the following identity holds

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = 3.$$

o) If $\mathcal{B}$ is a basis for $\mathbb{R}^2$ then the matrix whose columns are the vectors in $\mathcal{B}$ must have determinant 0.

p) Let A be an $n \times n$ matrix. If $\operatorname{nullity}(A) = 0$ then the columns of A are linearly dependent.

q) Let A be an $n \times n$ matrix. Then $\operatorname{rank}(A) = n$ if and only if the rows of A are a basis for $\mathbb{R}^n$.

r) Let A be an $n \times n$ matrix. If the columns of A span $\mathbb{R}^n$ then A is invertible.

s) Let P be a plane through the origin in $\mathbb{R}^3$ and let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two non-parallel vectors in P. Suppose $\boldsymbol{w} = c\boldsymbol{u} + d\boldsymbol{v}$ and $\boldsymbol{w}' = c'\boldsymbol{u} + d'\boldsymbol{v}$, where $c, d, c', d' \in \mathbb{R}$. If $\boldsymbol{w} = \boldsymbol{w'}$ then $c = c'$ and $d = d'$.

t) If $\boldsymbol{v} = (v_1, v_2) \in \mathbb{R}^2$, and $\mathcal{E}$ is the standard basis for $\mathbb{R}^2$, then $[\boldsymbol{v}]_{\mathcal{E}} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

u) Let $\mathcal{B} : \boldsymbol{v}_1, \boldsymbol{v}_2$ be an ordered basis for $\mathbb{R}^2$. If $\boldsymbol{v} = 2\boldsymbol{v}_1 - 3\boldsymbol{v}_2$ then $[\boldsymbol{v}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

Solutions to True/False

(a) T (b) F (c) T (d) F (e) T (f) T (g) F (h) F (i) F (j) T (k) T (l) T (m) T (n) F o) F p) F q) T r) T s) T t) T u) F

Tutorial Exercises

T1 Find the dimension of the subspace in $\mathbb{R}^4$ consisting of all solutions of the homogeneous system of linear equations

$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & + & 2x_3 & + & x_4 & = & 0 \\ 2x_1 & - & x_2 & & & + & 3x_4 & = & 0 \\ x_1 & - & 2x_2 & - & 2x_3 & + & 2x_4 & = & 0 \end{array}$$

by first finding the general solution and then writing down a spanning set for the space of solutions that is also linearly independent.

Solution

Let W denote the subspace consisting of all solutions to the system of homogenous equations. The augmented matrix for this system is

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 \\ 2 & -1 & 0 & 3 & 0 \\ 1 & -2 & -2 & 2 & 0 \end{bmatrix}$$

which has reduced row echelon form

$$\begin{bmatrix} 1 & 0 & \frac{2}{3} & \frac{4}{3} & 0 \\ 0 & 1 & \frac{4}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

from which it follows that the general solution is

$$\begin{aligned} x_1 &= -\frac{2}{3}s - \frac{1}{3}t \\ x_2 &= -\frac{4}{3}s + \frac{1}{3}t \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

where $s, t \in \mathbb{R}$. In other words every vector $(x_1, x_2, x_3, x_4) \in W$ can be written in the form

$$(x_1, x_2, x_3, x_4) = (-\frac{2}{3}s - \frac{4}{3}t, -\frac{4}{3}s + \frac{1}{3}t, s, t) = s(-\frac{2}{3}, -\frac{4}{3}, 1, 0) + t(-\frac{4}{3}, \frac{1}{3}, 0, 1),$$

where $s, t \in \mathbb{R}$, i.e.

$$W = \operatorname{span}((-\frac{2}{3}, -\frac{4}{3}, 1, 0), (-\frac{4}{3}, \frac{1}{3}, 0, 1)).$$

This set is also linearly independent (look at the last two entries in each vector), and so

$$S = \{(-\frac{2}{3}, -\frac{4}{3}, 1, 0), (-\frac{4}{3}, \frac{1}{3}, 0, 1)\}$$

is a basis for W.

T2 Let

$$A = \begin{bmatrix} 1 & 5 & 0 & 8 & 2 \\ 2 & 0 & 3 & 2 & 1 \\ 0 & 0 & 4 & 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 2 & 1 & 4 \\ 1 & 2 & 5 & 2 \\ 3 & 1 & 2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 0 & 2 & 1 \\ 3 & 0 & 4 & 4 \\ 2 & 3 & 3 & 6 \\ 3 & 3 & 1 & 5 \end{bmatrix}.$$

For each of these matrices:

a) Find a basis for the row space.

b) Find a basis for the column space.

c) Find the rank, and then use the Rank Theorem to find the nullity.

Solution

a) The reduced row echelon form of A is

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -1/2 & -1/4 \\ 0 & 1 & 0 & 17/10 & 9/20 \\ 0 & 0 & 1 & 1 & 1/2 \end{bmatrix}.$$

Since these three rows are linearly independent (compare the first 3 entries in each row), a basis for $\operatorname{row}(A)$ is $\{[1, 0, 0, -1/2, -1/4], [0, 1, 0, 17/10, 9/20], [0, 0, 1, 1, 1/2]\}$. Alternatively, it would have been enough to just look at the row echelon form of A, since its rows will also be linearly independent.

The reduced row echelon form of

$$B = \begin{bmatrix} 2 & 2 & 1 & 4 \\ 1 & 2 & 5 & 2 \\ 3 & 1 & 2 & 2 \end{bmatrix}$$

is

$$\operatorname{rref}(B) = \begin{bmatrix} 1 & 0 & 0 & 6/19 \\ 0 & 1 & 0 & 36/19 \\ 0 & 0 & 1 & -8/19 \end{bmatrix}.$$

Since these three rows are linearly independent (compare the first 3 entries in each row), a basis for $\operatorname{row}(B)$ is $\{[1, 0, 0, 6/19], [0, 1, 0, 36/19], [0, 0, 1, -8/19]\}$. Alternatively, it would have been enough to just look at the row echelon form of B, since its rows will also be linearly independent.

The reduced row echelon form of

$$C = \begin{bmatrix} 3 & 0 & 2 & 1 \\ 3 & 0 & 4 & 4 \\ 2 & 3 & 3 & 6 \\ 3 & 3 & 1 & 5 \end{bmatrix}$$

is

$$\text{rref}(C) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since these four rows are linearly independent (they are in fact the standard basis for $\mathbb{R}^4$), a basis for $\text{row}(C)$ is $\{[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]\}$. Alternatively, it would have been enough to just look at the row echelon form of C, since its rows will also be linearly independent.

b) To find a basis for the column space, we find a basis for the row space of the transpose.

The reduced row echelon form of A^T is

$$\text{rref}(A^T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and so a basis for $\text{row}(A^T) = \text{col}(A)$ is $\{[1,0,0],[0,1,0],[0,0,1]\}$.

The reduced row echelon form of B^T is

$$\text{rref}(B^T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and so a basis for $\text{row}(B^T) = \text{col}(B)$ is $\{[1,0,0],[0,1,0],[0,0,1]\}$.

The reduced row echelon form of C^T is

$$\text{rref}(C^T) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and so a basis for $\text{row}(C^T) = \text{col}(C)$ is $\{[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]\}$.

c) We have $\text{rank}(A) = 3$ so since A has 5 columns, we have $\text{rank}(A) + \text{nullity}(A) = 5$ and so $\text{nullity}(A) = 2$.

We have $\text{rank}(B) = 3$ so since B has 4 columns, we have $\text{rank}(B) + \text{nullity}(B) = 4$ and so $\text{nullity}(B) = 1$.

We have $\text{rank}(C) = 4$ so since C has 4 columns, we have $\text{rank}(C) + \text{nullity}(C) = 4$ and so $\text{nullity}(C) = 0$.

T3 Find a basis for the null space of the matrices

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 2 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence find nullity(A) and nullity(B), and then use the Rank Theorem to find rank(A) and rank(B).

Solution

The reduced row echelon form of $(A|\mathbf{0})$ is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The general solution is $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that

$$x_3 = 0; \quad x_1 + x_2 = 0$$

hence

$$x_1 = -x_2; \quad x_3 = 0.$$

Thus

$$\text{null}(A) = \{(-x_2, x_2, 0) : x_2 \in \mathbb{R}\} = \{x_2(-1, 1, 0) : x_2 \in \mathbb{R}\} = \text{Span}((-1, 1, 0)).$$

Therefore a basis for the null space of A is $S = \{(-1, 1, 0)\}$.

The reduced row echelon form of $(B|\mathbf{0})$ is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The general solution is $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that

$$x_3 = 0; \quad x_1 + x_2 = 0$$

hence

$$x_1 = -x_2; \quad x_3 = 0.$$

Thus

$$\text{null}(B) = \{(-x_2, x_2, 0) : x_2 \in \mathbb{R}\} = \{x_2(-1, 1, 0) : x_2 \in \mathbb{R}\} = \text{Span}((-1, 1, 0)).$$

Therefore a basis for the null space of B is $S = \{(-1, 1, 0)\}$.

Note that even though A and B are different sized matrices, they have the same null space, which is a subspace of $\mathbb{R}^3$.

Since the null space for both A and B has basis consisting of a single vector, we have nullity$(A) =$ nullity$(B) = 1$. Both A and B have 3 columns so by the Rank Theorem we also have rank$(A) =$ rank$(B) = 3 - 1 = 2$.

T4 In this question we work in $\mathbb{R}^4$.

a) Find a basis for each of the following subspaces:

i) $W_1 = \{(w,x,y,z) : x - 2y + 3z = 0\}$,

ii) $W_2 = \{(w,x,y,z) : w - x + y + z = 0\}$,

iii) $W_3 = \{(w,x,y,z) : w - y + 4z = 0\}$,

iv) $W_4 = \{(w,x,y,z) : w - 2x + 5z = 0\}$.

b) Find a basis for the subspaces $W_1 \cap W_2$, $W_1 \cap W_2 \cap W_3$ and $W_1 \cap W_2 \cap W_3 \cap W_4$ in $\mathbb{R}^4$. (See T7(a) below for the definition of $\cap$ why these intersections are subspaces.)

Solution

a) We begin with

i) $W_1 = \{(w,x,y,z) : x - 2y + 3z = 0\}$. Here we have

$$\begin{aligned} W_1 &= \{(w,x,y,z) \mid x = 2y - 3z\} \\ &= \{(w, 2y-3z, y, z) \mid w,y,z \in \mathbb{R}\} \\ &= \{w(1,0,0,0) + y(0,2,1,0) + z(0,-3,0,1) \mid w,y,z \in \mathbb{R}\} \\ &= \text{Span}\left((1,0,0,0),(0,2,1,0),(0,-3,0,1)\right). \end{aligned}$$

So we have found vectors that span W_1. To check that they are linearly independent, consider

$$\alpha(1,0,0,0) + \beta(0,2,1,0) + \gamma(0,-3,0,1) = (0,0,0,0)$$

leading to the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

which some simple computations with EROs shows is row equivalent to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence the only solutions is $\alpha = 0, \beta = 0, \gamma = 0$ showing that the vectors are linearly independent. So we have a set of vectors that spans W_1 and is linearly independent. Thus the set $S = \{(1,0,0,0),(0,2,1,0),(0,-3,0,1)\}$ is a basis for W_1.

ii) $W_2 = \{(w,x,y,z) : w - x + y + z = 0\}$. Here we have

$$\begin{aligned} W_2 &= \{(w,x,y,z) \mid w = x - y - z\} \\ &= \{(x-y-z, x, y, z) \mid x,y,z \in \mathbb{R}\} \\ &= \{x(1,1,0,0) + y(-1,0,1,0) + z(-1,0,0,1) \mid x,y,z \in \mathbb{R}\} \\ &= \text{Span}\left((1,1,0,0),(-1,0,1,0),(-1,0,0,1)\right). \end{aligned}$$

By the method used for W_1 we find that these are linearly independent.

iii) $W_3 = \{(w,x,y,z) : w - y + 4z = 0\}$. Here we have

$$\begin{aligned} W_3 &= \{(w,x,y,z) \mid w = y - 4z\} \\ &= \{(y-4z,x,y,z) \mid x,y,z \in \mathbb{R}\} \\ &= \{x(0,1,0,0) + y(1,0,1,0) + z(-4,0,0,1) \mid x,y,z \in \mathbb{R}\} \\ &= \operatorname{Span}\left((0,1,0,0),(1,0,1,0),(-4,0,0,1)\right). \end{aligned}$$

By the method used for W_1 we find that $S = \{(0,1,0,0),(1,0,1,0),(-4,0,0,1)\}$ is a basis.

iv) $W_4 = \{(w,x,y,z) : w - 2x + 5z = 0\}$. Here we have

$$\begin{aligned} W_4 &= \{(w,x,y,z) \mid w = 2x - 5z\} \\ &= \{(2x-5z,x,y,z) \mid x,y,z \in \mathbb{R}\} \\ &= \{x(2,1,0,0) + y(0,0,1,0) + z(-5,0,0,1) \mid x,y,z \in \mathbb{R}\} \\ &= \operatorname{Span}\left((2,1,0,0),(0,0,1,0),(-5,0,0,1)\right). \end{aligned}$$

By the method used for W_1 we find that $S = \{(2,1,0,0),(0,0,1,0),(-5,0,0,1)\}$ is a basis.

b) Next we consider some of the intersections of these subspaces.

i) We consider $W_1 \cap W_2 = \{(w,x,y,z) \mid x - 2y + 3z = 0 \text{ and } w - x + y + z = 0\}$. We need to find the solutions to these equations and so consider the matrix

$$\begin{bmatrix} 0 & 1 & -2 & 3 & 0 \\ 1 & -1 & 1 & 1 & 0 \end{bmatrix}.$$

It is easy to see that this has reduced row echelon form

$$\begin{bmatrix} 1 & 0 & -1 & 4 & 0 \\ 0 & 1 & -2 & 3 & 0 \end{bmatrix}.$$

Hence we need $w - y + 4z = 0$ and $x - 2y + 3z = 0$. So

$$\begin{aligned} W_1 \cap W_2 &= \{(w,x,y,z) \mid w = y - 4z, x = 2y - 3z\} \\ &= \{(y-4z, 2y-3z, y, z) \mid y,z \in \mathbb{R}\} \\ &= \{y(1,2,1,0) + z(-4,-3,0,1) \mid y,z \in \mathbb{R}\} \\ &= \operatorname{Span}\left((1,2,1,0),(-4,-3,0,1)\right). \end{aligned}$$

The two vectors are linearly independent since they are not multiples of each other. So $S = \{(1,2,1,0),(-4,-3,0,1)\}$ is a basis for $W_1 \cap W_2$.

ii) We consider

$$W_1 \cap W_2 \cap W_3 = \{(w,x,y,z) \mid x - 2y + 3z = 0,\ w - x + y + z = 0,\ w - y + 4z = 0\}.$$

We need to find the solutions to these equations and so consider the matrix

$$\begin{bmatrix} 0 & 1 & -2 & 3 & 0 \\ 1 & -1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 4 & 0 \end{bmatrix}.$$

This has reduced row echelon form

$$\begin{bmatrix} 1 & 0 & -1 & 4 & 0 \\ 0 & 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So the solution here is identical to the one found above when considering $W_1 \cap W_2$. Hence

$$\begin{aligned} W_1 \cap W_2 \cap W_3 &= W_1 \cap W_2 \\ &= \operatorname{Span}\big((1,2,1,0),(-4,-3,0,1)\big). \end{aligned}$$

and thus $S = \{(1,2,1,0),(-4,-3,0,1)\}$ is a basis for $W_1 \cap W_2 \cap W_3$.

iii) We consider

$$W_1 \cap W_2 \cap W_3 \cap W_4 = \left\{ (w,x,y,z) \;\middle|\; \begin{array}{rcl} x-2y+3z &=& 0, \\ w-x+y+z &=& 0, \\ w-y+4z &=& 0, \\ w-2x+5z &=& 0 \end{array} \right\}.$$

We need to find the solutions to these equations and so consider the matrix

$$\begin{bmatrix} 0 & 1 & -2 & 3 & 0 \\ 1 & -1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 4 & 0 \\ 1 & -2 & 0 & 5 & 0 \end{bmatrix}.$$

This has reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & \frac{5}{3} & 0 \\ 0 & 1 & 0 & -\frac{5}{3} & 0 \\ 0 & 0 & 1 & -\frac{7}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So we need $w + \frac{5}{3}z = 0$, $x - \frac{5}{3}z = 0$ and $y - \frac{7}{3}z = 0$. Hence

$$\begin{aligned} W_1 \cap W_2 \cap W_3 \cap W_4 &= \left\{ \left(-\frac{5}{3}z, \frac{5}{3}z, \frac{7}{3}z, z \right) \mid z \in \mathbb{R} \right\} \\ &= \{ \frac{1}{3}z(-5,5,7,3) \mid z \in \mathbb{R} \} \\ &= \operatorname{Span}\big((-5,5,7,3)\big). \end{aligned}$$

Since a single non-zero vector is necessarily linearly independent, this single vector suffices. That is, a basis for $W_1 \cap W_2 \cap W_3 \cap W_4$ is $S = \{(-5,5,7,1)\}$.

T5 Answer the following questions by considering the matrix with the given vectors as its columns.

a) Do the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ form a basis for $\mathbb{R}^3$?

b) Do the vectors $\begin{bmatrix}1\\1\\1\\0\end{bmatrix}$, $\begin{bmatrix}1\\1\\0\\1\end{bmatrix}$, $\begin{bmatrix}1\\0\\1\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\\1\\1\end{bmatrix}$ form a basis for $\mathbb{R}^4$?

Solution

a) Let

$$A = \begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\\0 & 1 & 1\end{bmatrix}$$

This has row echelon form

$$\begin{bmatrix}1 & 1 & 0\\0 & 1 & -1\\0 & 0 & 1\end{bmatrix}$$

Since the rows of this row echelon form are linearly independent, $\text{rank}(A) = 3$. Therefore the column space of A has dimension 3, and so the columns of A span $\mathbb{R}^3$. To show that they are a basis, we need to establish linear independence. For this we can observe that the homogeneous system corresponding to the augmented matrix $(A|\mathbf{0})$ has a unique solution, since A has row echelon form as above, thus the columns of A are linearly independent.

b) Let

$$A = \begin{bmatrix}1 & 1 & 1 & 0\\1 & 1 & 0 & 1\\1 & 0 & 1 & 1\\0 & 1 & 1 & 1\end{bmatrix}$$

This has row echelon form

$$\begin{bmatrix}1 & 1 & 1 & 0\\0 & 1 & 0 & -1\\0 & 0 & 1 & -1\\0 & 0 & 0 & 1\end{bmatrix}$$

Since the rows of this row echelon form are linearly independent, $\text{rank}(A) = 4$. Therefore the column space of A has dimension 4, and so the columns of A span $\mathbb{R}^4$. To show that they are a basis, we need to establish linear independence. For this we can observe that the homogeneous system corresponding to the augmented matrix $(A|\mathbf{0})$ has a unique solution, since A has row echelon form as above, thus the columns of A are linearly independent.

T6 Without doing any unnecessary calculations, determine whether S is a basis for $\text{Span}(S)$:

a) $S = \{(1,0,1,1),(2,0,0,3)\}$

b) $S = \{(1,0,1,1),(2,0,0,3),(4,0,0,6)\}$

c) $S = \{(1,0,1,1),(2,0,0,3),(2,0,2,2)\}$

d) $S = \{(1,0,1,1),(2,0,0,3),(2,0,2,2),(-1,9,8,0),(3,0,2,1)\}$

In all cases, find a basis for $\text{Span}(S)$. Are any of these sets S a basis for $\mathbb{R}^4$? Justify your answer.

Solution

We consider whether the following sets are bases for the space they span and for the real vector space $\mathbb{R}^4$.

a) $S = \{(1,0,1,1),(2,0,0,3)\}$

Both vectors have 0 as their second entry. This means that any vector $(w,x,y,z) \in \mathbb{R}^4$ with $x \neq 0$ cannot possibly be written as

$$(w,x,y,z) = a(1,0,1,1) + b(2,0,0,3),$$

so S does not span $\mathbb{R}^4$. However, S is a spanning set for $\mathrm{Span}(S)$ (by definition!), so they'll give a basis for $\mathrm{Span}(S)$ if they are linearly independent. Since there are only two vectors, we can check linear independence by just making sure they are not scalar multiples of each other. This is the case, so they are a basis for $\mathrm{Span}(S)$.

b) $S = \{(1,0,1,1),(2,0,0,3),(4,0,0,6)\}$.

Since the second and third vectors are scalar multiples of each other we know that this set is not linearly independent. Indeed, the equation

$$\lambda_1(1,0,1,1) + \lambda_2(2,0,0,3) + \lambda_3(4,0,0,6) = 0$$

has solution $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = -1$. Hence it cannot be a basis for either $\mathrm{Span}(S)$ or for $\mathbb{R}^4$. If we were to discard the third vector then the two remaining ones are linearly independent by the answer to (a) above. Hence they are a basis for $\mathrm{Span}(S)$.

c) $S = \{(1,0,1,1),(2,0,0,3),(2,0,2,2)\}$. Since $(2,0,2,2) = 2(1,0,1,1)$ we know that the vectors are not linearly independent because the equation

$$\lambda_1(1,0,1,1) + \lambda_2(2,0,0,3) + \lambda_3(2,0,2,2) = 0$$

has solution $\lambda_1 = 2$, $\lambda_2 = 0$, $\lambda_3 = -1$. Hence it cannot be a basis for either $\mathrm{Span}(S)$ or for $\mathbb{R}^4$. If we were to discard the third vector then the two remaining ones are linearly independent by the answer to (a) above, so they are a basis for $\mathrm{Span}(S_3)$.

d) $S = \{(1,0,1,1),(2,0,0,3),(2,0,2,2),(-1,9,8,0),(3,0,2,1)\}$.

As above, we have

$$(2,0,2,2) = 2(1,0,1,1).$$

so the vectors are not linearly independent. We discard $(2,0,2,2)$ to leave

$$S' = \{(1,0,1,1),(2,0,0,3),(-1,9,8,0),(3,0,2,1)\}.$$

We need to check whether this set is linearly independent. We consider

$$\alpha(1,0,1,1) + \beta(2,0,0,3) + \gamma(-1,9,8,0) + \delta(3,0,2,1) = (0,0,0,0).$$

leading to the matrix

$$\begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 0 & 0 & 9 & 0 & 0 \\ 1 & 0 & 8 & 2 & 0 \\ 1 & 3 & 0 & 1 & 0 \end{bmatrix}$$

which has reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Hence the solution is $\alpha = 0, \beta = 0, \gamma = 0, \delta = 0$, showing that S' is linearly independent. To see that it is a basis for $\mathbb{R}^4$, we need to check that for every $(w, x, y, z) \in \mathbb{R}^4$, there exist constants $\alpha, \beta, \gamma, \delta$ such that

$$\alpha(1,0,1,1) + \beta(2,0,0,3) + \gamma(-1,9,8,0) + \delta(3,0,2,1) = (w, x, y, z).$$

leading to the matrix

$$\begin{bmatrix} 1 & 2 & -1 & 3 & w \\ 0 & 0 & 9 & 0 & x \\ 1 & 0 & 8 & 2 & y \\ 1 & 3 & 0 & 1 & z \end{bmatrix}.$$

The left hand 4×4 matrix here is the same as that in the check for linear independence above, so the same EROs give the same left hand 4×4 matrix in the reduced row echelon matrix as above, namely, the identity. This means we can solve for each of $\alpha, \beta, \gamma, \delta$ without imposing constraints on w, x, y, z, so S' spans $\mathbb{R}^4$.

The standard basis $\{e_1, e_2, e_3, e_4\}$ for $\mathbb{R}^4$ has four elements, so $dim(\mathbb{R}^4) = 4$ and hence every basis of $\mathbb{R}^4$ has four elements. None of the sets S has four elements, so none of them give a basis for $\mathbb{R}^4$.

T7 Let U and V be subspaces of $\mathbb{R}^n$.

a) Prove that the intersection $U \cap V = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \in U \text{ and } \boldsymbol{x} \in V\}$ is also a subspace of $\mathbb{R}^n$.

b) Is the union $U \cup V = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \in U \text{ or } \boldsymbol{x} \in V\}$ always a subspace? Justify your answer.

Solution

a) Since U and V are subspaces, we have $\mathbf{0} \in U$ and $\mathbf{0} \in V$, hence $\mathbf{0} \in U \cap V$.

Let $\boldsymbol{x}, \boldsymbol{y}$ be in $U \cap V$. Then since U and V are subspaces, we have $\boldsymbol{x} + \boldsymbol{y} \in U$ and $\boldsymbol{x} + \boldsymbol{y} \in V$, hence $\boldsymbol{x} + \boldsymbol{y} \in U \cap V$.

Let $\boldsymbol{x}$ be in $U \cap V$ and let c be a scalar. Then since U and V are subspaces, we have $c\boldsymbol{x} \in U$ and $c\boldsymbol{x} \in V$, hence $c\boldsymbol{x} \in U \cap V$.

Therefore $U \cap V$ is a subspace of $\mathbb{R}^n$.

b) No. For example let $U = \text{Span}([1,0])$ and $V = \text{Span}([0,1])$ in $\mathbb{R}^2$ (so U is the x-axis and V is the y-axis). Then $[1,0] \in U$ and $[0,1] \in V$ but $[1,0] + [0,1] = [1,1]$ is not in U or V. So $U \cup V$ is not closed under vector addition, hence $U \cup V$ is not a subspace. (Note that $U \cup V$ does contain $\mathbf{0}$, and is closed under scalar multiplication.)

T8 Let $a \in \mathbb{R}$ and consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4a \end{pmatrix}.$$

Find all values of a such that:

a) $\operatorname{nullity}(A) = 0$;

b) $\operatorname{nullity}(A) = 1$;

c) $\operatorname{nullity}(A) = 2$.

Solution

After applying $R_2 \to R_2 - 2R_1$ to the augmented matrix $(A|\mathbf{0})$ we obtain

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 4a-4 & 0 \end{pmatrix}. \qquad (\star)$$

Suppose first that $4a - 4 = 0$, that is, that $a = 1$. Then $(\star)$ equals

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and so is already the reduced row echelon form of $(A|\mathbf{0})$. The general solution is $(x_1, x_2) \in \mathbb{R}^2$ such that

$$x_1 + 2x_2 = 0 \implies x_1 = -2x_2$$

and so in this case

$$\operatorname{null}(A) = \{(-2x_2, x_2) : x_2 \in \mathbb{R}\} = \{x_2(-2, 1) : x_2 \in \mathbb{R}\} = \operatorname{Span}((-2, 1)).$$

Thus if $a = 1$ we have $\operatorname{nullity}(A) = 1$.

Now suppose that $4a - 4 \neq 0$, that is, that $a \neq 1$. Then applying $R_2 \to \frac{1}{4a-4}R_2$ and then $R_1 \to R_1 - 2R_2$ to $(\star)$ above we obtain that the reduced row echelon form of $(A|\mathbf{0})$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The unique solution is $x_1 = x_2 = 0$ and so in this case, $\operatorname{null}(A) = \{\mathbf{0}\}$ and hence $\operatorname{nullity}(A) = 0$.

Therefore $\operatorname{nullity}(A) = 0$ if $a \neq 1$, $\operatorname{nullity}(A) = 1$ if $a = 1$, and there are no values of a for which $\operatorname{nullity}(A) = 2$.

T9 Let $b \in \mathbb{R}$ and consider the matrix

$$B = \begin{pmatrix} 1 & b \\ 3b & 3 \end{pmatrix}.$$

Find all values of b such that:

a) $\operatorname{rank}(B) = 0$;

b) $\operatorname{rank}(B) = 1$;

c) $\text{rank}(B) = 2$.

Solution

A row echelon form of $(B|\mathbf{0})$ is

$$\begin{pmatrix} 1 & b & 0 \\ 0 & 3-3b^2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & b & 0 \\ 0 & 3(1-b^2) & 0 \end{pmatrix}. \qquad (\star\star)$$

Suppose first that $3(1-b^2) = 0$, that is, that $b = \pm 1$. Then $(\star\star)$ equals

$$\begin{pmatrix} 1 & \pm 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and so is already the reduced row echelon form of $(B|\mathbf{0})$. Since this matrix contains one non-zero row, in this case we have $\text{rank}(B) = 1$.

Now suppose that $3(1-b^2) \neq 0$, that is, that $b \neq \pm 1$. Then applying $R_2 \to \frac{1}{3(b^2-1)} R_2$ and then $R_1 \to R_1 - bR_2$ to $(\star\star)$ above we obtain that the reduced row echelon form of $(B|\mathbf{0})$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since this matrix has two non-zero rows, in this case we have $\text{rank}(B) = 2$.

Therefore $\text{rank}(B)$ never equals 0, $\text{rank}(B) = 1$ if $b = \pm 1$, and $\text{rank}(B) = 2$ if $b \neq \pm 1$.

T10

a) Let $A \in M_{3\times 7}(\mathbb{R})$. Can the columns of A be linearly independent? Explain your answer.

b) Let $A \in M_{15\times 14}(\mathbb{R})$. Can the rows of A be linearly independent? Explain your answer.

c) Let $A \in M_{3\times 7}(\mathbb{R})$. Can the row space of A be equal to $\mathbb{R}^7$? Explain your answer.

d) Let $A \in M_{2014\times 2014}(\mathbb{R})$. If the dimension of the row space of A is 2014, what is the null space of A? Explain your answer.

Solution

a) No, the columns of A cannot be linearly independent. The greatest possible value of $\text{rank}(A)$ is 3, the number of rows of A, since $\text{rank}(A)$ is equal to the dimension of the row space of A. Thus the dimension of the column space of A is at most 3. By Theorem 2.8, any set of more than 3 vectors in $\mathbb{R}^3$ must be linearly dependent. Since A has 7 columns, this means the columns of A are linearly dependent.

b) No, the rows of A cannot be linearly independent. The greatest possible value of $\text{rank}(A)$ is 14, the number of columns of A, since $\text{rank}(A)$ is equal to the dimension of the column space of A. Thus the dimension of the row space of A is at most 14. By Theorem 2.8, any set in $\mathbb{R}^{14}$ having more than 14 vectors is linearly dependent. Since A has 15 rows, this means the rows of A are linearly dependent.

c) No. The rank of A is at most 3 since this is the number of rows of A.

d) In this case $\mathrm{null}(A) = \{\mathbf{0}\}$, where $\mathbf{0}$ denotes the zero vector in $\mathbb{R}^{2014}$. The dimension of the row space is 2014 so we have $\mathrm{rank}(A) = 2014$. Since A has 2014 columns, the Rank Theorem then implies that $\mathrm{nullity}(A) = 2014 - 2014 = 0$. Thus $\mathrm{null}(A)$ is the trivial subspace $\{\mathbf{0}\}$.

T11 Show that the columns of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ are a basis for $\mathbb{R}^2$. Then find the coordinate vector of $v = \begin{pmatrix} -2 \\ -8 \end{pmatrix}$ with respect to the ordered basis $\mathcal{B}$ given by the columns of A.

Solution

There are many ways to show that the columns of A form a basis for $\mathbb{R}^2$. One method is to observe that a row echelon form of A is

$$R = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Since both rows of R are non-zero, we have $\mathrm{rank}(R) = \mathrm{rank}(A) = 2$. Therefore the columns of A are a basis for $\mathbb{R}^2$.

To find $[v]_{\mathcal{B}}$, we need to find the scalars c_1, c_2 such that

$$c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -8 \end{pmatrix}.$$

The augmented matrix for this system is $(A|v)$. This augmented matrix has reduced row echelon form

$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & \frac{1}{2} \end{pmatrix}.$$

Thus the unique solution to the system is $c_1 = -3$, $c_2 = \frac{1}{2}$, and the coordinate vector of v with respect to the basis $\mathcal{B}$ is

$$[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -3 \\ \frac{1}{2} \end{bmatrix}.$$

T12 Show that the columns of the matrix $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 3 & 2 & 0 & -2 \\ 0 & 1 & -1 & 1 \\ 5 & 3 & 0 & -1 \end{pmatrix}$ are a basis for $\mathbb{R}^4$. Then find the coordinate vector of $v = \begin{pmatrix} -2 \\ -6 \\ -4 \\ -2 \end{pmatrix}$ with respect to the ordered basis $\mathcal{B}$ given by the columns of A.

Solution

Again, there are many correct answers to the first part. For instance, a row echelon form of A is

$$R = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since R has no all-zero rows, there will be a unique solution to $A\boldsymbol{x} = \mathbf{0}$, so the columns of A are a basis for $\mathbb{R}^4$.

To find $[\boldsymbol{v}]_{\mathcal{B}}$, we need to find the scalars c_1, c_2, c_3, c_4 such that

$$c_1 \begin{pmatrix} 1 \\ 3 \\ 0 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 2 \\ 1 \\ 3 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ -2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \\ -4 \\ -2 \end{pmatrix}.$$

The augmented matrix for this system is $(A|\boldsymbol{v})$. This augmented matrix has reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 12 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}.$$

Thus the unique solution to the system is $c_1 = -2$, $c_2 = 4$, $c_3 = 12$, $c_4 = 4$ and the coordinate vector of $\boldsymbol{v}$ with respect to the basis $\mathcal{B}$ is

$$[\boldsymbol{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 12 \\ 4 \end{bmatrix}.$$

T13 Find the coordinates of the vector $\boldsymbol{v}$ with respect to the given ordered basis $\mathcal{B}$:

a) $\boldsymbol{v} = (3,4,3)$ and $\mathcal{B} : (3,2,1), (2,-1,0), (5,0,0)$ on $\mathbb{R}^3$.

b) $\boldsymbol{v} = (-1,0,5)$ and $\mathcal{B} : (1,2,3), (1,1,-1)$ on $\operatorname{Span}((1,2,3),(1,1,-1))$.

Solution

a) We solve for c_1, c_2, c_3 in

$$c_1(3,2,1) + c_2(2,-1,0) + c_3(5,0,0) = (3,4,3).$$

The system of equations is

$$\begin{aligned} 3c_1 + 2c_2 + 5c_3 &= 3 \\ 2c_1 - c_2 &= 4 \\ c_1 &= 3 \end{aligned}$$

Back substitution shows that $c_1 = 3$, that $c_2 = 2$ and that $c_3 = -2$, so the coordinate vector of $\boldsymbol{v}$ in the given basis is

$$[\boldsymbol{v}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}.$$

b) We solve for c_1, c_2 in

$$c_1(1,2,3) + c_2(1,1,-1) = (-1,0,5).$$

The system of equations is

$$\begin{aligned} c_1 + c_2 &= -1 \\ 2c_1 + c_2 &= 0 \\ 3c_1 - c_2 &= 5. \end{aligned}$$

This system has solution $c_1 = 1$, $c_2 = -2$, so the coordinate vector of $\boldsymbol{v}$ in the given basis is

$$[\boldsymbol{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Note that even though $\boldsymbol{v}$ is in $\mathbb{R}^3$, its coordinate vector has only 2 entries. This is because $\operatorname{Span}((1,2,3),(1,1,-1))$ is a 2-dimensional subspace of $\mathbb{R}^3$.

T14 You are given that

$$\mathcal{B}: \begin{bmatrix} 0 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 1 \\ 4 \end{bmatrix}$$

is a basis for $\mathbb{R}^4$. Find the vector $\boldsymbol{v} \in \mathbb{R}^4$ such that

$$[\boldsymbol{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -3 \\ 4 \\ 2 \end{bmatrix}.$$

Solution

We have

$$\boldsymbol{v} = 1\begin{bmatrix} 0 \\ 1 \\ 3 \\ -1 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 5 \\ -3 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 4 \\ -1 \\ 0 \\ 2 \end{bmatrix} + 2\begin{bmatrix} -6 \\ 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ -14 \\ 14 \\ 12 \end{bmatrix}.$$

2B Linear Algebra

Ex Sheet **5**

True/False [1]

a) Every $n \times n$ matrix A is the change-of-basis matrix for some change of basis for $\mathbb{R}^n$.

b) If the change-of-basis matrix from a basis $\mathcal{B}$ to another basis $\mathcal{B}'$ is diagonal, then the coordinate vector of each vector with respect to $\mathcal{B}'$ is a scalar multiple of its coordinate vector with respect to $\mathcal{B}$.

c) If $\mathcal{B}$ is an ordered basis for a vector space V, then the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{B}$ is the identity.

d) Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then T_A is the function that assigns to each vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ the vector $\mathbf{y} = (x_1 + x_2, x_2) \in \mathbb{R}^2$.

e) Let $A \in M_{2\times 3}(\mathbb{R})$. Then T_A is a function from $\mathbb{R}^2$ to $\mathbb{R}^3$.

f) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then $\text{range}(T) = \mathbb{R}^m$.

g) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear then $T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) - T(\boldsymbol{v})$.

h) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear then $T(2\boldsymbol{u}) = T(\boldsymbol{u}) + T(\boldsymbol{u})$.

i) For all linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^n$, we have $S \circ T = T \circ S$.

j) Let A and B be two $n \times n$ matrices over $\mathbb{R}$. If $AB = BA$ then $T_A \circ T_B = T_B \circ T_A$.

k) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Then $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is linear.

l) A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if it is a matrix transformation.

[1] **True/False Questions**

Every Exercise Sheet will have a section containing true/false questions. They are designed to test your understanding from lectures. You should look at your lecture notes and/or the textbook to help you answer these questions, but you should not need to write anything to work out the solution.

Solutions to True/False

a) F b) F c) T d) T e) F f) F g) F h) T i) F j) T k) T l) T

Tutorial Exercises

T1

a) You are given that $\mathcal{B} : \boldsymbol{e}_1, \boldsymbol{e}_1 + \boldsymbol{e}_2, \boldsymbol{e}_1 + \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_4$ is an ordered basis for the vector space $V = \mathbb{R}^4$. Find the vector $\boldsymbol{v} \in \mathbb{R}^4$ so that the coordinate vector of $\boldsymbol{v}$ with respect to the basis $\mathcal{B}$ is

$$[\boldsymbol{v}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 4 \end{bmatrix}$$

b) Find the coordinate vector for the vector $\boldsymbol{w} = (-2, 3, -5, 1)$ with respect to the ordered basis $\mathcal{B}$ for $\mathbb{R}^4$ given in (a).

c) Find the change-of-basis matrix $\mathcal{P}_{\mathcal{E}\leftarrow\mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{E}$ where $\mathcal{E}$ is the standard basis for V. Use the matrix $\mathcal{P}_{\mathcal{E}\leftarrow\mathcal{B}}$ to check your answer to (a). Now find the change-of-basis matrix from $\mathcal{E}$ to $\mathcal{B}$, and use this matrix to check your answer to (b).

Solution

a) This coordinate vector means that

$$\begin{aligned} \boldsymbol{v} &= 2\boldsymbol{e}_1 + 0(\boldsymbol{e}_1 + \boldsymbol{e}_2) + 1(\boldsymbol{e}_1 + \boldsymbol{e}_3) + 4(\boldsymbol{e}_1 + \boldsymbol{e}_4) \\ &= 2(1,0,0,0) + 0(1,1,0,0) + 1(1,0,1,0) + 4(1,0,0,1) \\ &= (2,0,0,0) + (0,0,0,0) + (1,0,1,0) + (4,0,0,4) \\ &= (7,0,1,4). \end{aligned}$$

b) We need to find the scalars c_1, c_2, c_3, c_4 so that

$$\boldsymbol{w} = c_1\boldsymbol{e}_1 + c_2(\boldsymbol{e}_1 + \boldsymbol{e}_2) + c_3(\boldsymbol{e}_1 + \boldsymbol{e}_3) + c_4(\boldsymbol{e}_1 + \boldsymbol{e}_4)$$

This equation is

$$(-2, 3, -5, 1) = c_1(1,0,0,0) + c_2(1,1,0,0) + c_3(1,0,1,0) + c_4(1,0,0,1)$$

which holds if and only if

$$(-2, 3, -5, 1) = (c_1 + c_2 + c_3 + c_4, c_2, c_3, c_4).$$

By comparing components, we get immediately that $c_2 = 3$, $c_3 = -5$ and $c_4 = 1$. Substituting these values into the equation $c_1 + c_2 + c_3 + c_4 = -2$ from the first component we get $c_1 = -1$. Thus the coordinate vector of $\boldsymbol{w}$ with respect to the basis $\mathcal{B}$ is

$$[\boldsymbol{w}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 1 \end{bmatrix}.$$

c) We want to write the vectors in $\mathcal{B}$ in terms of the standard basis $\mathcal{E}$, and the change of basis matrix from $\mathcal{B}$ to $\mathcal{E}$ that we obtain is

$$\mathcal{P}_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The calculations on which this answer is based is:

We have

$$\begin{aligned} e_1 &= 1e_1 + 0e_2 + 0e_3 + 0e_4 \\ e_1 + e_2 &= 1e_1 + 1e_2 + 0e_3 + 0e_4 \\ e_1 + e_3 &= 1e_1 + 0e_2 + 1e_3 + 0e_4 \\ e_1 + e_4 &= 1e_1 + 0e_2 + 0e_3 + 1e_4 \end{aligned}$$

so the columns of $\mathcal{P}_{\mathcal{E}\leftarrow\mathcal{B}}$ are given by

$$[e_1]_\mathcal{E} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad [e_1+e_2]_\mathcal{E} = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \quad [e_1+e_3]_\mathcal{E} = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \quad [e_1+e_4]_\mathcal{E} = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}$$

To use $\mathcal{P}_{\mathcal{E}\leftarrow\mathcal{B}}$ to check (a), the key calculation is that

$$\begin{pmatrix}1&1&1&1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{bmatrix}2\\0\\1\\4\end{bmatrix} = \begin{bmatrix}7\\0\\1\\4\end{bmatrix}. \tag{1}$$

Use the fact that $\mathcal{P}_{\mathcal{E}\leftarrow\mathcal{B}}[v]_\mathcal{B} = [v]_\mathcal{E}$. By equation (**??**), we get $[v]_\mathcal{E} = \begin{bmatrix}7\\0\\1\\4\end{bmatrix}$. Since $\mathcal{E}$ is the standard basis for $\mathbb{R}^4$ this means $v = (7, 0, 1, 4)$.

The change-of-basis matrix from $\mathcal{E}$ to $\mathcal{B}$ is the inverse of $\mathcal{P}_{\mathcal{E}\leftarrow\mathcal{B}}$, so

$$\mathcal{P}_{\mathcal{B}\leftarrow\mathcal{E}} = \mathcal{P}_{\mathcal{E}\leftarrow\mathcal{B}}^{-1} = \begin{pmatrix}1&1&1&1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}^{-1} = \begin{pmatrix}1&-1&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

To use $\mathcal{P}_{\mathcal{B}\leftarrow\mathcal{E}}$ to check your answer to (b), the key calculation is that

$$\begin{pmatrix}1&-1&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{bmatrix}-2\\3\\-5\\1\end{bmatrix} = \begin{bmatrix}-1\\3\\-5\\1\end{bmatrix}. \tag{2}$$

Use the fact that $\mathcal{P}_{\mathcal{B}\leftarrow\mathcal{E}}[w]_\mathcal{E} = [w]_\mathcal{B}$. We have $[w]_\mathcal{E} = \begin{bmatrix}-2\\3\\-5\\1\end{bmatrix}$ so by equation (**??**), $[w]_\mathcal{B} = \begin{bmatrix}-1\\3\\-5\\1\end{bmatrix}$.

T2 Consider ordered bases $\mathcal{B} : (1, 2), (3, -1)$ and $\mathcal{C} : (2, -2), (4, 3)$ for $\mathbb{R}^2$.

a) Find the change of basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{C}$.

b) Find the coordinate vector of $(5, -1)$ with respect to the old basis

$\mathcal{B}$.

c) Find the coordinate vector of $(5,-1)$ with respect to the new basis $\mathcal{C}$, and verify that your answer can be obtained by multiplying together your answers to (a) and (b).

Solution

a) Write the old basis vectors in terms of the new to produce the columns of the change-of-basis matrix. That is, we solve the equations

$$\begin{aligned}(1,2) &= a_1(2,-2)+a_2(4,3)\\ (3,-1) &= b_1(2,-2)+b_2(4,3).\end{aligned}$$

Each equation gives a system of two equations in two unknowns which we can solve to give

$$[(1,2)]_{\mathcal{C}} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{14} \\ \frac{3}{7} \end{bmatrix} \quad \text{and} \quad [(3,-1)]_{\mathcal{C}} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \frac{13}{14} \\ \frac{2}{7} \end{bmatrix}, \quad \text{hence} \quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} -\frac{5}{14} & \frac{13}{14} \\ \frac{3}{7} & \frac{2}{7} \end{pmatrix}$$

b) Here we solve the system of two equations in two unknowns coming from the equation

$$(5,-1) = c_1(1,2)+c_2(3,-1).$$

You can show that

$$[(5,-1)]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} \\ \frac{11}{7} \end{bmatrix}.$$

c) Here we solve the system of two equations in two unknowns coming from the equation

$$(5,-1) = c_1'(2,-2)+c_2'(4,3).$$

You can show that

$$[(5,-1)]_{\mathcal{C}} = \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} \frac{19}{14} \\ \frac{4}{7} \end{bmatrix}.$$

It remains to notice that

$$[(5,-1)]_{\mathcal{C}} = \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} \frac{19}{14} \\ \frac{4}{7} \end{bmatrix} = \begin{pmatrix} -\frac{5}{14} & \frac{13}{14} \\ \frac{3}{7} & \frac{2}{7} \end{pmatrix} \begin{bmatrix} \frac{2}{7} \\ \frac{11}{7} \end{bmatrix} = P_{\mathcal{C}\leftarrow\mathcal{B}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P_{\mathcal{C}\leftarrow\mathcal{B}}[(5,-1)]_{\mathcal{B}}$$

as required.

T3 Suppose that $\mathcal{B}$ and $\mathcal{C}$ are ordered bases for a 3-dimensional vector space V and that $[\boldsymbol{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. If the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

find $[\boldsymbol{v}]_{\mathcal{C}}$.

Solution

We just need to compute that

$$[v]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[v]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ 9 \end{bmatrix}.$$

T4 Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

a) Prove that $T(\mathbf{0}) = \mathbf{0}$.

b) Prove that for all $\boldsymbol{v}, \boldsymbol{v}' \in \mathbb{R}^n$, $T(\boldsymbol{v} - \boldsymbol{v}') = T(\boldsymbol{v}) - T(\boldsymbol{v}')$.

c) Prove by induction on k that for all $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ and all $c_1, \ldots, c_k \in \mathbb{R}$,

$$T(c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_k) = c_1T(\boldsymbol{v}_1) + \cdots + c_kT(\boldsymbol{v}_k).$$

Solution

a) Since T is linear, $T(\lambda\mathbf{0}) = \lambda T(\mathbf{0})$ for all $\lambda \in \mathbb{R}$. But $\lambda\mathbf{0} = \mathbf{0}$ for any $\lambda \in \mathbb{R}$, and $0\boldsymbol{w} = \mathbf{0}$ for any $\boldsymbol{w} \in \mathbb{R}^m$. Thus putting $\lambda = 0$ we obtain

$$T(\mathbf{0}) = T(0\mathbf{0}) = 0T(\mathbf{0}) = \mathbf{0}.$$

Alternatively,

$$T(\mathbf{0}) = T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0}).$$

Now subtract $T(\mathbf{0})$ from both sides to obtain $\mathbf{0} = T(\mathbf{0})$.

b) The key is to note that $\boldsymbol{v} - \boldsymbol{v}' = \boldsymbol{v} + (-1)\boldsymbol{v}'$. Then as T is linear we have

$$T(\boldsymbol{v} - \boldsymbol{v}') = T(\boldsymbol{v} + (-1)\boldsymbol{v}') = T(\boldsymbol{v}) + T((-1)\boldsymbol{v}') = T(\boldsymbol{v}) + (-1)T(\boldsymbol{v}') = T(\boldsymbol{v}) - T(\boldsymbol{v}').$$

c) When $k = 1$ the statement is that $T(c_1\boldsymbol{v}_1) = c_1T(\boldsymbol{v}_1)$, which holds since T is linear. Assume the statement is true for $k \geq 1$. Then, for any scalars $c_1, \ldots, c_{k+1}$ and vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1}$ we have

$$\begin{aligned} T(c_1\boldsymbol{v}_1 + \cdots + c_{k+1}\boldsymbol{v}_{k+1}) &= T((c_1\boldsymbol{v}_1 + \cdots c_k\boldsymbol{v}_k) + c_{k+1}\boldsymbol{v}_{k+1}) \\ &= T(c_1\boldsymbol{v}_1 + \cdots c_k\boldsymbol{v}_k) + T(c_{k+1}\boldsymbol{v}_{k+1}) \end{aligned}$$

since T is linear and so $T(\boldsymbol{v} + \boldsymbol{v}') = T(\boldsymbol{v}) + T(\boldsymbol{v}')$ for any vectors $\boldsymbol{v}$ and $\boldsymbol{v}'$, in particular for $\boldsymbol{v} = c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k$ and $\boldsymbol{v}' = c_{k+1}\boldsymbol{v}_{k+1}$. The inductive hypothesis implies that

$$T(c_1\boldsymbol{v}_1 + \cdots c_k\boldsymbol{v}_k) = c_1T(\boldsymbol{v}_1) + \cdots + c_kT(\boldsymbol{v}_k)$$

and the linearity of T implies that

$$T(c_{k+1}\boldsymbol{v}_{k+1}) = c_{k+1}T(\boldsymbol{v}_{k+1}).$$

Therefore

$$T(c_1v_1 + \cdots + c_{k+1}v_{k+1}) = c_1T(v_1) + \cdots + c_kT(v_k) + c_{k+1}T(v_{k+1})$$

as required.

T5 For each of the the following functions, determine whether it is a linear transformation. If it is a linear transformation you should prove this, and if it is not a linear transformation you should give a counterexample.

a) $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = ax$, where $a \in \mathbb{R}$.

b) $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = ax + b$, where $a, b \in \mathbb{R}$ and $b \neq 0$.

c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (|x|, |y|)$.

d) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x, y, z) = (y, z, x)$.

e) $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by $T(w, x, y, z) = (3w, 2x, y)$.

f) $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by $T(x, y, z) = (z^2, x + y)$.

g) $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by $T(x, y) = (y - 1, x + 2y, 2x + y)$.

h) $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ given by $T(x, y) = (7x, x - y, 2y, 2x - 5y)$.

Solution

a) Let $x, y \in \mathbb{R}$. Then $T(x + y) = a(x + y) = ax + ay = T(x) + T(y)$. Now let $x \in \mathbb{R}$ and let c be a scalar. Then $T(cx) = a(cx) = c(ax) = cT(x)$. Therefore T is linear.

b) We have $T(1) = a + b$ and $T(2) = 2a + b$. But $2T(1) = 2(a + b) = 2a + 2b \neq 2a + b = T(2)$ since $b \neq 0$. Thus T is not linear. Alternatively, use T6(a): since $T(0) = b \neq 0$, the function T is not linear.

c) We have $T(1, 1) = (1, 1)$ and $T(-1, -1) = (1, 1)$, so $T(1, 1) + T(-1, -1) = (2, 2)$. But

$$T((1, 1) + (-1, -1)) = T(0, 0) = (0, 0) \neq (2, 2) = T(1, 1) + T(-1, -1).$$

So T is not linear.

d) Let $(x, y, z), (x', y', z') \in \mathbb{R}^3$. Then

$$\begin{aligned} T((x, y, z) + (x', y', z')) &= T(x + x', y + y', z + z') \\ &= (y + y', z + z', x + x') \\ &= (y, z, x) + (y', z', x') \\ &= T(x, y, z) + (x', y', z'). \end{aligned}$$

Now let $(x, y, z) \in \mathbb{R}^3$ and let $c \in \mathbb{R}$. Then

$$T(c(x, y, z)) = T(cx, cy, cz) = (cy, cz, cx) = c(y, z, x) = cT(x, y, z).$$

Therefore T is a linear transformation.

e) Let $(w,x,y,z),(w',x',y',z') \in \mathbb{R}^4$. Then

$$\begin{aligned}
T((w,x,y,z)+(w',x',y',z')) &= T(w+w',x+x',y+y',z+z') \\
&= (3(w+w'),2(x+x'),y+y') \\
&= (3w+3w',2x+2x',y+y') \\
&= (3w,2x,y)+(3w',2x',y') \\
&= T(w,x,y,z)+T(w',x',y',z').
\end{aligned}$$

Now let $(w,x,y,z) \in \mathbb{R}^4$ and let $c \in \mathbb{R}$. Then

$$T(c(w,x,y,z)) = T(cw,cx,cy,cz) = (3(cw),2(cx),cy) = (c(3w),c(2x),cy) = c(3w,2x,y) = cT(w,x,y,z).$$

Therefore T is a linear transformation.

f) We have

$$T\big(2(0,\,0,\,1)\big) = T(0,\,0,\,2) = (4,0),$$

however

$$2T(0,\,0,\,1) = 2(1,\,0) = (2,\,0).$$

So T is not a linear mapping since $T\big(2(0,\,0,\,1)\big) \neq 2T(0,\,0,\,1)$.

g) Note that $T(0,\,0) = (-1,\,0,\,0) \neq (0,0,0)$, so T is not a linear map by T6(a).

h) For $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ we have

$$\begin{aligned}
T\big(\lambda(x_1,\,y_1)+(x_2,\,y_2)\big) &= T(\lambda x_1+x_2,\,\lambda y_1+y_2) \\
&= \big(7(\lambda x_1+x_2),\,(\lambda x_1+x_2)-(\lambda y_1+y_2),\,2(\lambda y_1+y_2),\,2(\lambda x_1+x_2)-5(\lambda y_1+y_2)\big) \\
&= (7\lambda x_1+7x_2,\,\lambda x_1+x_2-\lambda y_1-y_2,\,2\lambda y_1+2y_2,\,2\lambda x_1+2x_2-5\lambda y_1-5y_2) \\
&= (7\lambda x_1,\,\lambda x_1-\lambda y_1,\,2\lambda y_1,\,2\lambda x_1-5\lambda y_1)+(7x_2,\,x_2-y_2,\,2y_2,\,2x_2-5y_2) \\
&= \lambda(7x_1,\,x_1-y_1,\,2y_1,\,2x_1-5y_1)+(7x_2,\,x_2-y_2,\,2y_2,\,2x_2-5y_2) \\
&= \lambda T(x_1,\,y_1)+T(x_2,\,y_2).
\end{aligned}$$

This is enough to show that T is a linear map, because special cases include the two defining properties of a linear map, namely

$$\begin{aligned}
T\big((x_1,y_1)+(x_2,y_2)\big) &= T(x_1,y_1)+T(x_2,y_2) \\
T\big(\lambda(x_1,y_1)\big) &= \lambda T(x_1,y_1).
\end{aligned}$$

T6 Find the standard matrix $[T]$ for each function T in T5 which is a linear transformation.

Solution

The linear maps are from parts a), d), e) and h).

For part a), we have $T:\mathbb{R}\to\mathbb{R}$ so $[T]$ will be the 1×1 matrix $[a]$.

For part d), we have $T(e_1) = T(1,0,0) = (0,0,1)$, $T(e_2) = T(0,1,0) = (1,0,0)$ and $T(e_3) =$

$T(0,0,1) = (0,1,0)$. The standard matrix for T is the matrix $[T]$ with ith column given by $T(\boldsymbol{e}_i)$:

$$[T] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

For part e), we have $T(\boldsymbol{e}_1) = T(1,0,0,0) = (3,0,0)$, $T(\boldsymbol{e}_2) = T(0,1,0,0) = (0,2,0)$, $T(\boldsymbol{e}_3) = T(0,0,1,0) = (0,0,1)$ and $T(\boldsymbol{e}_4) = T(0,0,0,1) = (0,0,0)$. The standard matrix for T is the matrix $[T]$ with ith column given by $T(\boldsymbol{e}_i)$:

$$[T] = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

For part h), we have $T(\boldsymbol{e}_1) = T(1,0) = (7,1,0,2)$ and $T(\boldsymbol{e}_2) = T(0,1) = (0,-1,2,-5)$. The standard matrix for T is the matrix $[T]$ with ith column given by $T(\boldsymbol{e}_i)$:

$$[T] = \begin{bmatrix} 7 & 0 \\ 1 & -1 \\ 0 & 2 \\ 2 & -5 \end{bmatrix}.$$

T7 Let

$$A = \begin{pmatrix} 4 & 3 \\ 2 & -1 \\ 0 & 9 \end{pmatrix}$$

and let T_A be the corresponding matrix transformation.

a) Determine the m and n so that $T_A : \mathbb{R}^n \to \mathbb{R}^m$.

b) For $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, find a formula for $T_A(\boldsymbol{x}) \in \mathbb{R}^m$.

Now repeat this question for the following matrices:

$$B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & -5 & 8 \\ 1 & -2 & 17 & 6 \\ 8 & 2 & 3 & 4 \end{pmatrix}.$$

Solution

For the matrix A:

a) $m = 3$ and $n = 2$.

b) For $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$ we have

$$T_A(\boldsymbol{x}) = A\boldsymbol{x} = \begin{pmatrix} 4 & 3 \\ 2 & -1 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (4x_1 + 3x_2, 2x_1 - x_2, 9x_2) \in \mathbb{R}^3.$$

For the matrix B:

a) $m = 2$ and $n = 2$.

b) For $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$ we have

$$T_B(\boldsymbol{x}) = B\boldsymbol{x} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (2x_1 - x_2, -x_1 + 2x_2) \in \mathbb{R}^2.$$

For the matrix C:

a) $m = 3$ and $n = 4$.

b) For $\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ we have

$$T_C(\boldsymbol{x}) = C\boldsymbol{x} = \begin{pmatrix} 1 & 2 & -5 & 8 \\ 1 & -2 & 17 & 6 \\ 8 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$= (x_1 + 2x_2 - 5x_3 + 8x_4, x_1 - 2x_2 + 17x_3 + 6x_4, 8x_1 + 2x_2 + 3x_3 + 4x_4) \in \mathbb{R}^3.$$

T8 Let A and B be as in T7.

a) Find the matrix AB.

b) Determine the k and l so that $T_{AB} : \mathbb{R}^l \to \mathbb{R}^k$. For $\boldsymbol{x} = (x_1, \ldots, x_l) \in \mathbb{R}^l$, find a formula for $T_{AB}(\boldsymbol{x}) \in \mathbb{R}^k$.

c) Determine the p and q so that $T_A \circ T_B : \mathbb{R}^q \to \mathbb{R}^p$. For $\boldsymbol{x} = (x_1, \ldots, x_q) \in \mathbb{R}^q$, find a formula for $(T_A \circ T_B)(\boldsymbol{x}) \in \mathbb{R}^p$ using the formulas for T_A and T_B in exercise T8. Is your final answer the same as part b)?

Solution

a)

$$AB = \begin{pmatrix} 4 & 3 \\ 2 & -1 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 5 & -4 \\ -9 & 18 \end{pmatrix}.$$

b) $k = 3$ and $l = 2$, and for $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$ we have

$$T_{AB}(\boldsymbol{x}) = (AB)\boldsymbol{x} = \begin{pmatrix} 5 & 2 \\ 5 & -4 \\ -9 & 18 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (5x_1 + 2x_2, 5x_1 - 4x_2, -9x_1 + 18x_2) \in \mathbb{R}^3.$$

c) $p = 3$ and $q = 2$, and for $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$ we have

$$\begin{aligned} (T_A \circ T_B)(\boldsymbol{x}) &= T_A(T_B(\boldsymbol{x})) \\ &= T_A(2x_1 - x_2, -x_1 + 2x_2) \\ &= (4(2x_1 - x_2) + 3(-x_1 + 2x_2), 2(2x_1 - x_2) - (-x_1 + 2x_2), 9(-x_1 + 2x_2)) \\ &= (5x_1 + 2x_2, 5x_1 - 4x_2, -9x_1 + 18x_2) \in \mathbb{R}^3. \end{aligned}$$

Yes, this final answer is the same as in part b).

T9 Let B be as in T12.

a) Find the matrix B^{-1} and hence find a formula for $T_{B^{-1}}(\boldsymbol{x}) \in \mathbb{R}^2$, where $\boldsymbol{x} \in \mathbb{R}^2$.

b) Use the formulas for T_B and $T_{B^{-1}}$ to show that $(T_B \circ T_{B^{-1}})(\boldsymbol{x}) = \boldsymbol{x}$ and $(T_{B^{-1}} \circ T_B)(\boldsymbol{x}) = \boldsymbol{x}$, for all $\boldsymbol{x} \in \mathbb{R}^2$. (This shows that T_B is invertible with inverse $(T_B)^{-1} = T_{B^{-1}}$.)

Solution

a) We have

$$B^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

and so for $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$

$$T_{B^{-1}}(\boldsymbol{x}) = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(\frac{2}{3}x_1 + \frac{1}{3}x_2, \frac{1}{3}x_1 + \frac{2}{3}x_2 \right) \in \mathbb{R}^2.$$

b) For the first composition we have

$$\begin{aligned}
(T_B \circ T_{B^{-1}})(\boldsymbol{x}) &= T_B(T_{B^{-1}}(\boldsymbol{x})) \\
&= T_B\left(\frac{2}{3}x_1 + \frac{1}{3}x_2, \frac{1}{3}x_1 + \frac{2}{3}x_2 \right) \\
&= \left(2\left(\frac{2}{3}x_1 + \frac{1}{3}x_2 \right) - \left(\frac{1}{3}x_1 + \frac{2}{3}x_2 \right), - \left(\frac{2}{3}x_1 + \frac{1}{3}x_2 \right) + 2\left(\frac{1}{3}x_1 + \frac{2}{3}x_2 \right) \right) \\
&= (x_1, x_2) \\
&= \boldsymbol{x}.
\end{aligned}$$

For the second composition we have

$$\begin{aligned}
(T_{B^{-1}} \circ T_B)(\boldsymbol{x}) &= T_{B^{-1}}(T_B(\boldsymbol{x})) \\
&= T_{B^{-1}}(2x_1 - x_2, -x_1 + 2x_2) \\
&= \left(\frac{2}{3}(2x_1 - x_2) + \frac{1}{3}(-x_1 + 2x_2), \frac{1}{3}(2x_1 - x_2) + \frac{2}{3}(-x_1 + 2x_2) \right) \\
&= (x_1, x_2) \\
&= \boldsymbol{x}.
\end{aligned}$$

T10 Consider the real vector space $\mathbb{R}^3$ and the ordered basis

$$\mathcal{B}\colon (1, -1, 1), (1, 1, 0), (2, 1, 0).$$

Find a formula for the coordinates of a vector $\boldsymbol{x} = (x, y, z)$ with respect to $\mathcal{B}$.

Solution

The coordinate vector of $\boldsymbol{x}$ with respect to the basis $\mathcal{B}$ is

$$[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

where the coordinates λ_i are the unique scalars which satisfy

$$(x, y, z) = \lambda_1(1, -1, 1) + \lambda_2(1, 1, 0) + \lambda_3(2, 1, 0),$$

in other words the λ_i are the solutions of the system

$$\begin{aligned} \lambda_1 + \lambda_2 + 2\lambda_3 &= x \\ -\lambda_1 + \lambda_2 \quad + \lambda_3 &= y \\ \lambda_1 \qquad\qquad &= z. \end{aligned}$$

Elementary row operations show that

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & x \\ -1 & 1 & 1 & y \\ 1 & 0 & 0 & z \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 0 & 0 & z \\ 0 & 1 & 0 & -x + 2y + 3z \\ 0 & 0 & 1 & x - y - 2z \end{array}\right],$$

so the coordinate vector that we're looking for is

$$[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} z \\ -x + 2y + 3z \\ x - y - 2z \end{bmatrix}.$$

T11 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix $A = [T]$. Prove that $\text{range}(T) = \text{col}(A)$.

Solution

We first show that $\text{range}(T) \subseteq \text{col}(A)$. For this, let $\boldsymbol{w} \in \mathbb{R}^m$ be in $\text{range}(T)$. Then by definition of the range, $\boldsymbol{w} = T(\boldsymbol{v})$ for some $\boldsymbol{v} \in \mathbb{R}^n$. By definition of A, we have $T(\boldsymbol{v}) = A\boldsymbol{v}$ and so $A\boldsymbol{v} = \boldsymbol{w}$. Let $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$. Then $\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + \cdots + v_n\boldsymbol{e}_n$. So

$$\boldsymbol{w} = A\boldsymbol{v} = A(v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + \cdots + v_n\boldsymbol{e}_n) = v_1 A\boldsymbol{e}_1 + v_2 A\boldsymbol{e}_2 + \cdots + v_n A\boldsymbol{e}_n.$$

Now $A\boldsymbol{e}_i$ is the ith column of the matrix A, so we have expressed $\boldsymbol{w}$ as a linear combination of the columns of A. Therefore $\boldsymbol{w}$ is in $\text{col}(A)$ as required.

We now show that $\text{col}(A) \subseteq \text{range}(T)$. For this, let $\boldsymbol{w} \in \mathbb{R}^m$ be in $\text{col}(A)$ and let the columns of A be $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$. Then by definition of the column space, there are scalars $c_1, c_2, \ldots, c_n$ so that

$$\boldsymbol{w} = c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2 + \cdots + c_n\boldsymbol{a}_n.$$

Now the ith column of A is $A\boldsymbol{e}_i$, hence we have $\boldsymbol{a}_i = A\boldsymbol{e}_i = T(\boldsymbol{e}_i)$. Thus

$$\boldsymbol{w} = c_1 T(\boldsymbol{e}_1) + c_2 T(\boldsymbol{e}_2) + \cdots + c_n T(\boldsymbol{e}_n).$$

As T is a linear map, the right-hand side is equal to $T(c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + \cdots + c_n\boldsymbol{e}_n)$. Let $\boldsymbol{v} = c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + \cdots + c_n\boldsymbol{e}_n$, then we have $\boldsymbol{w} = T(\boldsymbol{v})$. Thus $\boldsymbol{w}$ is in range(T) as required.

We conclude that range$(T) = $ col(A).

T12 Let T be a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. Prove that T is linear if and only if for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ and all scalars $\lambda \in \mathbb{R}$,

$$T(\lambda\boldsymbol{u} + \boldsymbol{v}) = \lambda T(\boldsymbol{u}) + T(\boldsymbol{v}).$$

Solution

Assume that T is linear. Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then by definition of linearity,

$$T(\lambda\boldsymbol{u} + \boldsymbol{v}) = T(\lambda\boldsymbol{u}) + T(\boldsymbol{v}) = \lambda T(\boldsymbol{u}) + T(\boldsymbol{v}).$$

Now assume that for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ and all scalars $\lambda \in \mathbb{R}$,

$$T(\lambda\boldsymbol{u} + \boldsymbol{v}) = \lambda T(\boldsymbol{u}) + T(\boldsymbol{v}).$$

Then in the special case that $\lambda = 1$, we have

$$T(\boldsymbol{u} + \boldsymbol{v}) = T(1\boldsymbol{u} + \boldsymbol{v}) = 1T(\boldsymbol{u}) + T(\boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v}).$$

Thus for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, we have $T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$. Now in the special case that $\boldsymbol{v} = \boldsymbol{0}$, we have

$$T(\lambda\boldsymbol{u}) = T(\lambda\boldsymbol{u} + \boldsymbol{0}) = \lambda T(\boldsymbol{u}) + T(\boldsymbol{0}).$$

We would like to deduce that $T(\lambda\boldsymbol{u}) = \lambda T(\boldsymbol{u})$ since $T(\boldsymbol{0}) = \boldsymbol{0}$, but we cannot use T6(a) since we have not yet proved that T is linear. However observe that in the special case $\lambda = 1$ and $\boldsymbol{u} = \boldsymbol{v} = \boldsymbol{0}$, we get

$$T(\boldsymbol{0}) = T(1\boldsymbol{0} + \boldsymbol{0}) = 1T(\boldsymbol{0}) + T(\boldsymbol{0}) = 2T(\boldsymbol{0}).$$

Subtract $T(\boldsymbol{0})$ from both sides of this to get $T(\boldsymbol{0}) = \boldsymbol{0}$ as desired. Therefore for all $\boldsymbol{u} \in \mathbb{R}^n$ and all scalars $\lambda \in \mathbb{R}$, we have $T(\lambda\boldsymbol{u}) = \lambda T(\boldsymbol{u})$. We conclude that T is linear.

T13 Answer the following questions using the criterion for linearity in T12, rather than any results about matrix transformations.

a) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^p$ be linear transformations. Prove that $S \circ T$ is linear.

b) Let $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. Define a map $S + T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ by

$$(S + T)(\boldsymbol{x}) = S(\boldsymbol{x}) + T(\boldsymbol{x}).$$

Prove that $S + T$ is linear.

c) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $c \in \mathbb{R}$ be a scalar. Define a map cT from $\mathbb{R}^n$ to $\mathbb{R}^m$ by

$$(cT)(\boldsymbol{x}) = c(T(\boldsymbol{x})).$$

Prove that cT is linear.

Solution

a) Let $\boldsymbol{u}, \boldsymbol{u}' \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then since both T and S are linear

$$(S \circ T)(\lambda \boldsymbol{u} + \boldsymbol{u}') = S(T(\lambda \boldsymbol{u} + \boldsymbol{u}')) = S(\lambda T(\boldsymbol{u}) + T(\boldsymbol{u}')) = \lambda S(T(\boldsymbol{u})) + S(T(\boldsymbol{u}')) = \lambda((S \circ T)(\boldsymbol{u})) + (S \circ T)(\boldsymbol{u}').$$

Hence $S \circ T$ is linear.

b) Let $\boldsymbol{u}, \boldsymbol{u}' \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then since both T and S are linear

$$\begin{aligned}
(S+T)(\lambda \boldsymbol{u} + \boldsymbol{u}') &= S(\lambda \boldsymbol{u} + \boldsymbol{u}') + T(\lambda \boldsymbol{u} + \boldsymbol{u}') \\
&= \lambda S(\boldsymbol{u}) + S(\boldsymbol{u}') + \lambda T(\boldsymbol{u}) + T(\boldsymbol{u}') \\
&= \lambda(S(\boldsymbol{u}) + T(\boldsymbol{u})) + (S(\boldsymbol{u}') + T(\boldsymbol{u}')) \\
&= \lambda((S+T)(\boldsymbol{u})) + (S+T)(\boldsymbol{u}').
\end{aligned}$$

Hence $S + T$ is linear.

c) Let $\boldsymbol{u}, \boldsymbol{u}' \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then since T is linear

$$\begin{aligned}
(cT)(\lambda \boldsymbol{u} + \boldsymbol{u}') &= c(T(\lambda \boldsymbol{u} + \boldsymbol{u}')) \\
&= c(\lambda T(\boldsymbol{u}) + T(\boldsymbol{u}')) \\
&= c(\lambda T(\boldsymbol{u})) + cT(\boldsymbol{u}') \\
&= \lambda(cT(\boldsymbol{u})) + cT(\boldsymbol{u}') \\
&= \lambda((cT)(\boldsymbol{u})) + (cT)(\boldsymbol{u}').
\end{aligned}$$

Hence cT is linear.

2B Linear Algebra

Ex Sheet 6

1 True/False

a) The determinant of $\begin{pmatrix} 1 & 0 & 0 \\ -67 & 2 & 0 \\ -10 & 10 & 1 \end{pmatrix}$ is 2.

b) The determinant of an upper triangular matrix is equal to the product of the entries on the main diagonal.

c) The determinant of an invertible matrix is sometimes zero.

d) For any square matrix A, $\det(A) = \det(A^T)$.

e) Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $B = \begin{pmatrix} 2a_{21} & 2a_{22} & 2a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ two 3×3 matrices. Then $\det(B) = -2\det(A)$.

f) Let E be the elementary matrix corresponding to an ERO that swaps two rows. Then $\det(E) = 1$.

g) Let E be the elementary matrix corresponding to an ERO that multiplies a row by 1000. Then $\det(E) = 1000$.

h) Let A and B be $n \times n$ matrices. Then $\det(AB) = \det(B)\det(A)$.

i) Any non-zero vector $\mathbf{v} \in \mathbb{R}^n$ is an eigenvector of the identity I.

j) If $\mathbf{v}$ is an eigenvector of $A \in M_{n\times n}(\mathbb{R})$ corresponding to λ, then for $\lambda \neq 0$ it follows that $\lambda\mathbf{v}$ is an eigenvector of A corresponding to λ.

k) If $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors of $A \in M_{n\times n}(\mathbb{R})$ corresponding to λ (such that $\mathbf{v} \neq -\mathbf{w}$), then $\mathbf{v} + \mathbf{w}$ is an eigenvector of A corresponding to λ.

l) The characteristic polynomial of the $n \times n$ identity matrix I is λ^n.

m) The characteristic polynomial of $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 3 & 3 \end{pmatrix}$ is $(1-\lambda)(2-\lambda)(3-\lambda)$.

n) If $\det(A + 3I) = 0$ then 3 is an eigenvalue of A.

o) If λ is an eigenvalue of a matrix A then $A\boldsymbol{x} + \lambda\boldsymbol{x} = \mathbf{0}$ for some $\boldsymbol{x} \neq \mathbf{0}$.

p) Suppose a matrix A has eigenvalue 1. Then the 1-eigenspace of A consists of all vectors $\boldsymbol{x}$ so that $A\boldsymbol{x} = \boldsymbol{x}$.

[1] **True/False Questions**

Every Exercise Sheet will have a section containing true/false questions. They are designed to test your understanding from lectures. You should look at your lecture notes and/or the textbook to help you answer these questions, but you should not need to write anything to work out the solution.

q) An eigenvalue can equal 0 but an eigenvector can never equal $\mathbf{0}$.

r) The numbers 0 and 2 are eigenvalues of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

s) The vector $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$.

t) The vector $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ corresponding to the eigenvalue $\lambda = 1$.

u) The vector $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ corresponding to the eigenvalue $\lambda = 2$.

Solutions to True/False

a) T b) T c) F d) T e) T f) F g) T h) T i) T j) T k) T l) F m) T n) F o) F (p) T (q) T (r) T (s) F (t) F (u) T

Tutorial Exercises

T1 Find the determinants of the matrices

$$A = \begin{bmatrix} 3 & 4 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 5 & -10 \\ 2 & 3 & 1 \end{bmatrix}$$

Solution

Using the formula for a 2×2 determinant we have

$$\det A = 3 \times 1 - 4 \times (-1) = 7.$$

Using the definition of determinant of an $n \times n$ matrix, (expanding along the top row), gives

$$\begin{aligned} \det B &= 1 \times \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - 4 \times \begin{vmatrix} 0 & 3 \\ -1 & 2 \end{vmatrix} + 6 \times \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} \\ &= 1(1) - 4(3) + 6(2) \\ &= 1. \end{aligned}$$

Similarly, expanding along the top row of C gives

$$\begin{aligned} \det C &= 1 \times \begin{vmatrix} 5 & -10 \\ 3 & 1 \end{vmatrix} - 2 \times \begin{vmatrix} 5 & -10 \\ 2 & 1 \end{vmatrix} + 3 \times \begin{vmatrix} 5 & 5 \\ 2 & 3 \end{vmatrix} \\ &= 1(35) - 2(25) + 3(5) \\ &= 0. \end{aligned}$$

T2 Consider the following matrices:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a+1 & 1 \\ 1 & 1 & a^2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & 2 \\ 2 & 2 & a \end{bmatrix}.$$

In each case:

a) calculate its determinant; and

b) find the values of a for which it is not invertible.

Solution

For (a) we expand along the top row

$$\begin{aligned} \begin{vmatrix} 1 & 1 & 1 \\ 1 & a+1 & 1 \\ 1 & 1 & a^2 \end{vmatrix} &= 1 \times \begin{bmatrix} a+1 & 1 \\ 1 & a^2 \end{bmatrix} - 1 \times \begin{bmatrix} 1 & 1 \\ 1 & a^2 \end{bmatrix} + 1 \times \begin{bmatrix} 1 & a+1 \\ 1 & 1 \end{bmatrix} \\ &= (a^3 + a^2 - 1) - (a^2 - 1) + (1 - a - 1) \\ &= a^3 - a \\ &= a(a^2 - 1) \\ &= a(a+1)(a-1). \end{aligned}$$

So the matrix is not invertible when $a = -1, 0, +1$.

For (b), expanding along the top row

$$\begin{aligned} \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 2 \\ 2 & 2 & a \end{vmatrix}. &= 1 \times \begin{bmatrix} a & 2 \\ 2 & a \end{bmatrix} - 1 \times \begin{bmatrix} 1 & 2 \\ 2 & a \end{bmatrix} + 1 \times \begin{bmatrix} 1 & a \\ 2 & 2 \end{bmatrix} \\ &= (a^2 - 4) - (a - 4) + (2 - 2a) \\ &= a^2 - 3a + 2 \\ &= (a-1)(a-2). \end{aligned}$$

So the matrix is not invertible when $a = 1$ or 2.

T3 Find the determinants of the matrices B, C, D, E and F, given that the matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

satisfies $\det(A) = 2$.

$$B = \begin{pmatrix} 3a & 3b & 3c \\ d & e & f \\ g & h & i \end{pmatrix}, \quad C = \begin{pmatrix} a+d+g & b+e+h & c+f+i \\ d & e & f \\ g & h & i \end{pmatrix},$$

$$D = \begin{pmatrix} a & b & c \\ g & h & i \\ d & e & f \end{pmatrix}, \quad E = \begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix},$$

$$F = \begin{pmatrix} 2a & 2b & 2c \\ -d & -e & -f \\ g+4a & h+4b & i+4c \end{pmatrix}.$$

Solution

- $\det(B) = 3\det(A) = 6$ since multiplying a row by $\lambda = 3$ multiplies the determinant by $\lambda = 3$.
- $\det(C) = \det(A) = 2$ since adding multiples of one row to another does not change the determinant.
- $\det(D) = -\det(A) = -2$ since swapping two rows multiplies the determinant by -1.
- $\det(E) = -\det(D) = -(-2) = 2$ since E is obtained from D by swapping two rows, and swapping two rows multiplies the determinant by -1. Alternatively, E is obtained by A by carrying out two row-swaps, and so $\det(E) = -(-\det(A)) = 2$.
- $\det(F) = 2(-1)\det(A) = -4$ since we are multiplying row 1 by 2 and row 2 by -1, hence we must multiply the determinant by $2(-1)$. Adding 4 times row 1 to row 4 does not change the determinant.

T4 Solve each of the following equations. Here, $|A|$ means $\det(A)$.

(a) $\begin{vmatrix} x+5 & 2 & -1 \\ 3 & x & x+2 \\ 24 & 8 & -3 \end{vmatrix} = 0,$ (b) $\begin{vmatrix} 1 & 3 & -2 \\ 3 & x+5 & -4 \\ 0 & 4 & x+6 \end{vmatrix} = 0.$

Solution

We solve these equations by expanding the determinants and solving the polynomial equations in x. For (a), expanding along the top row

$$\begin{aligned} \begin{vmatrix} x+5 & 2 & -1 \\ 3 & x & x+2 \\ 24 & 8 & -3 \end{vmatrix} &= (x+5)\begin{vmatrix} x & x+2 \\ 8 & -3 \end{vmatrix} - 2\begin{vmatrix} 3 & x+2 \\ 24 & -3 \end{vmatrix} - 1\begin{vmatrix} 3 & x \\ 24 & 8 \end{vmatrix} \\ &= (x+5)(-3x-8x-16) - 2(-9-24x-48) - (24-24x) \\ &= (x+5)(-11x-16) + 72x + 90 \\ &= -11x^2 + x + 10. \end{aligned}$$

We use the quadratic formula to solve the equation $-11x^2 + x + 10 = 0$, giving

$$x = \frac{-1 \pm \sqrt{1+440}}{(-22)}$$

Since $\sqrt{441} = 21$ we have

$$x = \frac{20}{-22} = -\frac{10}{11}, \quad \text{or} \quad x = \frac{-22}{-22} = 1.$$

For (b) we illustrate a different approach using the elementary column operations $C_2 \longrightarrow C_2 - 3C_1$

and $C_3 \longrightarrow C_3 + 2C_1$ to introduce zeros along the top row.

$$\begin{vmatrix} 1 & 0 & 0 \\ 3 & x-4 & 2 \\ 0 & 4 & x+6 \end{vmatrix} = 0.$$

Then expanding the determinant along the top row we have

$$\begin{aligned} 1 \times \left[(x-4)(x+6) - 8 \right] &= 0 \\ x^2 + 2x - 32 &= 0. \end{aligned}$$

Hence

$$x = \frac{-2 \pm \sqrt{4+128}}{2} = -1 \pm \sqrt{33}.$$

T5 Let $A \in M_{n\times n}(\mathbb{R})$.

a) Suppose $A^2 = I$. Find all possible values of $\det(A)$. Must A be invertible?

b) Suppose $A^2 = A$. Find all possible values of $\det(A)$. Must A be invertible?

c) Suppose $AA^T = I$. Find all possible values of $\det(A)$. Must A be invertible?

d) Suppose $A^k = O$ for some positive integer k, where O is the zero matrix. Find all possible values of $\det(A)$. Can A be invertible?

Solution

a) Since $A^2 = I$, we have $\det(A^2) = \det(I)$ and so $\det(A)\det(A) = 1$. Put $d = \det(A)$ then $d^2 = 1$. Thus the possible values of $\det(A)$ are 1 and -1. Since both these values are non-zero, A must be invertible.

b) Since $A^2 = A$, we have $\det(A^2) = \det(A)$ and so $\det(A)\det(A) = \det(A)$. Put $d = \det(A)$ then $d^2 = d$ and so $d(d-1) = 0$, thus $d = 0$ or $d = 1$. Thus the possible values of $\det(A)$ are 0 and 1. Since $\det(A)$ could be zero, the matrix A does not have to be invertible.

c) Since $AA^T = I$, we have $\det(AA^T) = \det(I)$ and so $\det(A)\det(A^T) = \det(A)\det(A) = 1$. Put $d = \det(A)$ then $d^2 = 1$ and so $d = \pm 1$. Thus the possible values of $\det(A)$ are -1 and 1. Since $\det(A)$ cannot be zero, the matrix A must be invertible.

d) Since $A^k = O$, we have $\det(A^k) = \det(O)$ and so $\det(A)^k = 0$. Thus $\det(A) = 0$. The matrix A is never invertible.

T6 Show that the vector $v = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}$ is an eigenvector of the matrix

$$A = \begin{pmatrix} 2 & -2 & 2 & 2 \\ -2 & 2 & 2 & 2 \\ 2 & 2 & 2 & -2 \\ 2 & 2 & -2 & 2 \end{pmatrix}$$

and find the corresponding eigenvalue.[2]

[2] Hint: compute Av.

Solution

We have

$$Av = \begin{pmatrix} 2 & -2 & 2 & 2 \\ -2 & 2 & 2 & 2 \\ 2 & 2 & 2 & -2 \\ 2 & 2 & -2 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \\ 12 \\ 12 \end{pmatrix} = 4 \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix} = 4v.$$

Thus v is an eigenvector of A with corresponding eigenvalue 4.

T7 Find the eigenvalues over $\mathbb{R}$ of each of the following matrices, and give bases for each of the corresponding eigenspaces.

(a) $A = \begin{pmatrix} 1 & 3 \\ 0 & -4 \end{pmatrix}$, (b) $B = \begin{pmatrix} 1 & -9 \\ 1 & -5 \end{pmatrix}$, (c) $C = \begin{pmatrix} 2 & 1 \\ -6 & -3 \end{pmatrix}$

Solution

a) We have

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 3 \\ 0 & -4-\lambda \end{pmatrix} = (1-\lambda)(-4-\lambda)$$

so the eigenvalues of A are $\lambda = 1, -4$.

Consider $\lambda = 1$: We need to find the null space of the matrix $A - 1I = A + I$. The augmented matrix $(A - 1I|\mathbf{0})$ is

$$\begin{pmatrix} 0 & 3 & 0 \\ 0 & -5 & 0 \end{pmatrix}.$$

We perform EROs on the augmented matrix to get it into reduced row echelon form

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the general solution to $(A - 1I)\boldsymbol{x} = \mathbf{0}$ is $x_2 = 0$, $x_1 = t$ with $t \in \mathbb{R}$. So

$$\text{null}(A - 1I) = \left\{ \begin{pmatrix} t \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

Hence the 1-eigenspace of A is

$$E_1 = \text{Span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$$

and so E_1 has basis $\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$.

Consider $\lambda = -4$: We need to find the null space of the matrix $A-(-4)I = A+4I$. The augmented matrix $(A-(-4)I|\mathbf{0})$ is

$$\left(\begin{array}{cc|c}5 & 3 & 0\\0 & 0 & 0\end{array}\right).$$

We perform EROs on the augmented matrix to get it into reduced row echelon form

$$\left(\begin{array}{cc|c}1 & \frac{3}{5} & 0\\0 & 0 & 0\end{array}\right).$$

Thus the general solution to $(A-(-4)I)\boldsymbol{x} = \mathbf{0}$ is $x_2 = t$, $x_1 = -\frac{3}{5}t$ with $t \in \mathbb{R}$. So

$$\text{null}(A-(-4)I) = \left\{\begin{pmatrix}-\frac{3}{5}t\\t\end{pmatrix} : t \in \mathbb{R}\right\} = \text{Span}\left\{\begin{pmatrix}-\frac{3}{5}\\1\end{pmatrix}\right\}.$$

Hence the (-4)-eigenspace of A is

$$E_{-4} = \text{Span}\left\{\begin{pmatrix}-\frac{3}{5}\\1\end{pmatrix}\right\}$$

and so E_{-4} has basis $\left\{\begin{pmatrix}-\frac{3}{5}\\1\end{pmatrix}\right\}$.

b) We have

$$\begin{aligned}\det(B-\lambda I) &= \det\begin{pmatrix}1-\lambda & -9\\1 & -5-\lambda\end{pmatrix}\\&= (1-\lambda)(-5-\lambda)+9\\&= \lambda^2+4\lambda-5+9\\&= \lambda^2+4\lambda+4\\&= (\lambda+2)^2\end{aligned}$$

so B has only one eigenvalue $\lambda = -2$ (repeated twice).

Consider $t = -2$: We need to find the null space of the matrix $B-(-2)I = B+2I$. The augmented matrix $(B-(-2)I|\mathbf{0})$ is

$$\left(\begin{array}{cc|c}3 & -9 & 0\\1 & -3 & 0\end{array}\right).$$

We perform EROs on the augmented matrix to get it into reduced row echelon form

$$\left(\begin{array}{cc|c}1 & -3 & 0\\0 & 0 & 0\end{array}\right).$$

Thus the general solution to $(B-(-2)I)\boldsymbol{x}=\mathbf{0}$ is $x_2=t$, $x_1=3t$ with $t\in\mathbb{R}$. So

$$\operatorname{null}(B-(-2)I)=\left\{\begin{pmatrix}3t\\t\end{pmatrix}:t\in\mathbb{R}\right\}=\operatorname{Span}\left\{\begin{pmatrix}3\\1\end{pmatrix}\right\}.$$

Hence the (-2)-eigenspace of B is

$$E_{-2}=\operatorname{Span}\left\{\begin{pmatrix}3\\1\end{pmatrix}\right\}$$

and so E_{-2} has basis $\left\{\begin{pmatrix}3\\1\end{pmatrix}\right\}$.

c) We have

$$\begin{aligned}\det(C-\lambda I)&=\det\begin{pmatrix}2-\lambda & 1\\-6 & -3-\lambda\end{pmatrix}\\&=(2-\lambda)(-3-\lambda)+6\\&=\lambda^2+\lambda\\&=\lambda(\lambda+1)\end{aligned}$$

so the eigenvalues of C are $\lambda=0,-1$.

Consider $\lambda=0$: We need to find the null space of the matrix $C-0I=C$. The augmented matrix $(C-0I|\mathbf{0})$ is

$$\left(\begin{array}{cc|c}2 & 1 & 0\\-6 & -3 & 0\end{array}\right).$$

We perform EROs on the augmented matrix to get it into reduced row echelon form

$$\left(\begin{array}{cc|c}1 & \frac{1}{2} & 0\\0 & 0 & 0\end{array}\right).$$

Thus the general solution to $(C-0I)\boldsymbol{x}=\mathbf{0}$ is $x_2=t$, $x_1=-\frac{1}{2}t$ with $t\in\mathbb{R}$. So

$$\operatorname{null}(C-0I)=\left\{\begin{pmatrix}-\frac{1}{2}t\\t\end{pmatrix}:t\in\mathbb{R}\right\}=\operatorname{Span}\left\{\begin{pmatrix}-\frac{1}{2}\\1\end{pmatrix}\right\}.$$

Hence the 0-eigenspace of C is

$$E_0=\operatorname{Span}\left\{\begin{pmatrix}-\frac{1}{2}\\1\end{pmatrix}\right\}$$

and so E_0 has basis $\left\{\begin{pmatrix}-\frac{1}{2}\\1\end{pmatrix}\right\}$.

Consider $\lambda=-1$: We need to find the null space of the matrix $C-(-1)I=C+I$. The augmented matrix $(C-(-1)I|\mathbf{0})$ is

$$\left(\begin{array}{cc|c}3 & 1 & 0\\-6 & -2 & 0\end{array}\right).$$

We perform EROs on the augmented matrix to get it into reduced row echelon form

$$\begin{pmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the general solution to $(C-(-1)I)\boldsymbol{x} = \mathbf{0}$ is $x_2 = t$, $x_1 = -\frac{1}{3}t$ with $t \in \mathbb{R}$. So

$$\text{null}(C-(-1)I) = \left\{ \begin{pmatrix} -\frac{1}{3}t \\ t \end{pmatrix} : t \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} \right\}.$$

Hence the (-1)-eigenspace of C is

$$E_{-1} = \text{Span}\left\{ \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} \right\}$$

and so E_{-1} has basis $\left\{ \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} \right\}$.

T8 Find the eigenvalues over $\mathbb{C}$ of the following matrix A, and give bases for each of the corresponding eigenspaces.

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Solution

We have

$$\det(A-\lambda I) = \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1$$

so the eigenvalues of A over $\mathbb{C}$ are $\lambda = i, -i$. (Note that a matrix with all entries real may have complex eigenvalues.)

Consider $\lambda = i$: We need to find the null space of the matrix $A - iI$. The augmented matrix $(A-iI|\mathbf{0})$ is

$$\begin{pmatrix} -i & -1 & 0 \\ 1 & -i & 0 \end{pmatrix}.$$

We perform EROs on the augmented matrix to get it into reduced row echelon form, as follows:

$$\begin{pmatrix} -i & -1 & 0 \\ 1 & -i & 0 \end{pmatrix} \xrightarrow{R_1 \to iR_1} \begin{pmatrix} 1 & -i & 0 \\ 1 & -i & 0 \end{pmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus the general solution to $(A-iI)\boldsymbol{x} = \mathbf{0}$ is $x_2 = t$, $x_1 = it$ with $t \in \mathbb{C}$ (the scalars are now $\mathbb{C}$). So

$$\text{null}(A-iI) = \left\{ \begin{pmatrix} it \\ t \end{pmatrix} : t \in \mathbb{C} \right\} = \text{Span}\left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} \right\}.$$

Hence the i-eigenspace of A is

$$E_i = \text{Span}\left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} \right\}$$

and so E_i has basis $\left\{\begin{pmatrix} i \\ 1 \end{pmatrix}\right\}$.

Consider $\lambda = -i$: We need to find the null space of the matrix $A - (-i)I = A + iI$. The augmented matrix $(A - (-i)I|\mathbf{0})$ is

$$\left(\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right).$$

We perform EROs on the augmented matrix to get it into reduced row echelon form, as follows:

$$\left(\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right) \xrightarrow{R_1 \to (-i)R_1} \left(\begin{array}{cc|c} 1 & i & 0 \\ 1 & i & 0 \end{array}\right) \xrightarrow{R_2 \to R_2 - R_1} \left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Thus the general solution to $(A - (-i)I)\boldsymbol{x} = \mathbf{0}$ is $x_2 = t$, $x_1 = -it$ with $t \in \mathbb{C}$ (the scalars are now $\mathbb{C}$). So

$$\text{null}(A - (-i)I) = \left\{\begin{pmatrix} -it \\ t \end{pmatrix} : t \in \mathbb{C}\right\} = \text{Span}\left\{\begin{pmatrix} -i \\ 1 \end{pmatrix}\right\}.$$

Hence the $(-i)$-eigenspace of A is

$$E_{-i} = \text{Span}\left\{\begin{pmatrix} -i \\ 1 \end{pmatrix}\right\}$$

and so E_{-i} has basis $\left\{\begin{pmatrix} -i \\ 1 \end{pmatrix}\right\}$.

T9 Suppose $A \in M_{7\times 7}(\mathbb{R})$ has characteristic polynomial

$$(1-\lambda)^2(3+\lambda)(17+\lambda)(9-\lambda)^3$$

Write down the eigenvalues of A.

Solution

The eigenvalues of A are the roots of the characteristic polynomial, that is, $\lambda = 1, -3, -17, 9$.

T10 Find the characteristic polynomial and the eigenvalues of each of the following matrices, then find a basis for each of the corresponding eigenspaces.

(a) $A = \begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix}$, (b) $B = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 2 \\ -2 & -2 & 0 \end{pmatrix}$, (c) $C = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

Solution

a) The characteristic polynomial is

$$\begin{aligned}\det(A-\lambda I) &= \det\begin{pmatrix}-1-\lambda & 2 & 2\\ 2 & 2-\lambda & 2\\ -3 & -6 & -6-\lambda\end{pmatrix}\\ &= (-1-\lambda)[(2-\lambda)(-6-\lambda)+12]-2[2(-6-\lambda)+6]+2[-12+3(2-\lambda)]\\ &= -\lambda^3-5\lambda^2-6\lambda\\ &= -\lambda(\lambda^2+5\lambda+6)\\ &= -\lambda(\lambda+2)(\lambda+3),\end{aligned}$$

so the eigenvalues are $\lambda = 0, -2, -3$.

Consider $t = 0$: We need to find the null space of the matrix $A - 0I = A$. The augmented matrix $(A - 0I|\mathbf{0})$ is just the matrix $(A|\mathbf{0})$:

$$\begin{pmatrix}-1 & 2 & 2 & 0\\ 2 & 2 & 2 & 0\\ -3 & -6 & -6 & 0\end{pmatrix}.$$

We perform EROs on this augmented matrix to get it into reduced row echelon form

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Thus the general solution to $(A-0I)\boldsymbol{x} = \mathbf{0}$ is $x_1 = 0$, $x_2 = -t$, $x_3 = t$ with $t \in \mathbb{R}$. So

$$\operatorname{null}(A-0I) = \left\{\begin{pmatrix}0\\ -t\\ t\end{pmatrix} : t \in \mathbb{R}\right\} = \operatorname{Span}\left\{\begin{pmatrix}0\\ -1\\ 1\end{pmatrix}\right\}.$$

Hence the 0-eigenspace of A is

$$E_0 = \operatorname{Span}\left\{\begin{pmatrix}0\\ -1\\ 1\end{pmatrix}\right\}$$

and so E_0 has basis $\left\{\begin{pmatrix}0\\ -1\\ 1\end{pmatrix}\right\}$.

Consider $\lambda = -2$: This time we jump straight to the matrix $(A - \lambda I)$ with $\lambda = -2$ and the column of zeros.

$$\begin{pmatrix}1 & 2 & 2 & 0\\ 2 & 4 & 2 & 0\\ -3 & -6 & -4 & 0\end{pmatrix}$$

which has reduced row echelon form

$$\begin{pmatrix}1 & 2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Thus the general solution to $(A-(-2)I)\boldsymbol{x}=\mathbf{0}$ is $x_1=-2t$, $x_2=t$, $x_3=0$ with $t\in\mathbb{R}$. So

$$\operatorname{null}(A-(-2)I)=\left\{\begin{pmatrix}-2t\\t\\0\end{pmatrix}:t\in\mathbb{R}\right\}=\operatorname{Span}\left\{\begin{pmatrix}-2\\1\\0\end{pmatrix}\right\}.$$

Hence the (-2)-eigenspace of A is

$$E_{-2}=\operatorname{Span}\left\{\begin{pmatrix}-2\\1\\0\end{pmatrix}\right\}$$

and so E_{-2} has basis $\left\{\begin{pmatrix}-2\\1\\0\end{pmatrix}\right\}$.

Consider $\lambda=-3$**:** This time we jump straight to the matrix $(A-\lambda I)$ with $\lambda=-3$ and the column of zeros.

$$\begin{pmatrix}2&2&2&0\\2&5&2&0\\-3&-6&-3&0\end{pmatrix}$$

which has reduced row echelon form

$$\begin{pmatrix}1&0&1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}.$$

Thus the general solution to $(A-(-3)I)\boldsymbol{x}=\mathbf{0}$ is $x_1=-t$, $x_2=0$, $x_3=t$ with $t\in\mathbb{R}$. So

$$\operatorname{null}(A-(-3)I)=\left\{\begin{pmatrix}-t\\0\\t\end{pmatrix}:t\in\mathbb{R}\right\}=\operatorname{Span}\left\{\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}.$$

Hence the (-3)-eigenspace of A is

$$E_{-3}=\operatorname{Span}\left\{\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$$

and so E_{-3} has basis $\left\{\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$.

b) The characteristic polynomial is

$$\begin{aligned}\det(B-\lambda I)&=\det\begin{pmatrix}-\lambda&1&2\\-1&-\lambda&2\\-2&-2&-\lambda\end{pmatrix}\\&=-\lambda(\lambda^2+4)-1(\lambda+4)+2(2-2\lambda)\\&=-\lambda^3-4\lambda-\lambda-4+4-4\lambda\\&=-\lambda^3-9\lambda\\&=-\lambda(\lambda^2+9).\end{aligned}$$

The eigenvalues are the roots of the characteristic polynomial and so are $\lambda = 0, \pm 3i$.

Consider $t = 0$: We need to consider the matrix $(B - \lambda I)$ with $\lambda = 0$ and a column of zeros:

$$\begin{pmatrix} 0 & 1 & 2 & 0 \\ -1 & 0 & 2 & 0 \\ -2 & -2 & 0 & 0 \end{pmatrix}.$$

which has reduced row echelon matrix

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus the general solution to $(B - 0I)\boldsymbol{x} = \mathbf{0}$ is $x_1 = 2t$, $x_2 = -2t$, $x_3 = t$ with $t \in \mathbb{C}$. So

$$\text{null}(B - 0I) = \left\{ \begin{pmatrix} 2t \\ -2t \\ t \end{pmatrix} : t \in \mathbb{C} \right\} = \text{Span}\left\{ \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

Hence the 0-eigenspace of B is

$$E_0 = \text{Span}\left\{ \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}$$

and so E_0 has basis $\left\{ \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}$.

Consider $t = 3i$: We need to consider the matrix $(B - \lambda I)$ with $\lambda = 3i$ and a column of zeros:

$$\begin{pmatrix} -3i & 1 & 2 & 0 \\ -1 & -3i & 2 & 0 \\ -2 & -2 & -3i & 0 \end{pmatrix}.$$

We swap R_1 and R_2 to give

$$\begin{pmatrix} -1 & -3i & 2 & 0 \\ -3i & 1 & 2 & 0 \\ -2 & -2 & -3i & 0 \end{pmatrix}$$

and multiply R_1 by -1 to give

$$\begin{pmatrix} 1 & 3i & -2 & 0 \\ -3i & 1 & 2 & 0 \\ -2 & -2 & -3i & 0 \end{pmatrix}.$$

Then we use $R_2 \longrightarrow R_2 + 3iR_1$ and $R_3 \longrightarrow R_3 + 2R_1$ to give

$$\begin{pmatrix} 1 & 3i & -2 & 0 \\ 0 & -8 & 2 - 6i & 0 \\ 0 & -2 + 6i & -3i - 4 & 0 \end{pmatrix}.$$

Using $R_2 \longrightarrow -\frac{1}{8}R_2$ gives

$$\begin{pmatrix} 1 & 3i & -2 & 0 \\ 0 & 1 & \frac{3i-1}{4} & 0 \\ 0 & -2 + 6i & -3i - 4 & 0 \end{pmatrix}.$$

Using $R_1 \longrightarrow R_1 - 3iR_2$ and $R_3 \longrightarrow R_3 - (-2+6i)R_2$ we have that the reduced row echelon form is

$$\begin{pmatrix} 1 & 0 & \frac{1+3i}{4} & 0 \\ 0 & 1 & \frac{3i-1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus the general solution to $(B - 3iI)\mathbf{x} = \mathbf{0}$ is $x_1 = -\frac{1+3i}{4}t$, $x_2 = -\frac{3i-1}{4}t$, $x_3 = t$ with $t \in \mathbb{C}$. So

$$\mathrm{null}(B-3iI) = \left\{ \begin{pmatrix} -\frac{1+3i}{4}t \\ -\frac{3i-1}{4}t \\ t \end{pmatrix} : t \in \mathbb{C} \right\} = \mathrm{Span}\left\{ \begin{pmatrix} -\frac{1+3i}{4} \\ -\frac{3i-1}{4} \\ 1 \end{pmatrix} \right\}.$$

Hence the $3i$-eigenspace of B is

$$E_{3i} = \mathrm{Span}\left\{ \begin{pmatrix} -\frac{1+3i}{4} \\ -\frac{3i-1}{4} \\ 1 \end{pmatrix} \right\}$$

and so E_{3i} has basis $\left\{ \begin{pmatrix} -\frac{1+3i}{4} \\ -\frac{3i-1}{4} \\ 1 \end{pmatrix} \right\}$. Another correct answer, which avoids fractions, would be that a basis for E_{3i} is $\left\{ \begin{pmatrix} 1+3i \\ 3i-1 \\ -4 \end{pmatrix} \right\}$. (Multiply through by -4.)

Consider $\lambda = -3i$: We could proceed as in the $\lambda = +3i$ case. However, it is quicker to note that any eigenvector for this case will be a complex conjugate of an eigenvector for the one above. Indeed, from the working above we know that

$$\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 2 \\ -2 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1+3i \\ 3i-1 \\ -4 \end{pmatrix} = 3i \begin{pmatrix} 1+3i \\ 3i-1 \\ -4 \end{pmatrix}.$$

If we take the complex conjugate of this equation we have

$$\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 2 \\ -2 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1-3i \\ -3i-1 \\ -4 \end{pmatrix} = -3i \begin{pmatrix} 1-3i \\ -3i-1 \\ -4 \end{pmatrix}.$$

Thus a basis for the $(-3i)$-eigenspace is $\left\{ \begin{pmatrix} 1-3i \\ -3i-1 \\ -4 \end{pmatrix} \right\}$. (You should fill in the details.)

c) The characteristic polynomial is

$$\begin{aligned} \det(C - \lambda I) &= \det \begin{pmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{pmatrix} \\ &= (2-\lambda)[(2-\lambda)(2-\lambda) - 0] \\ &= (2-\lambda)^3. \end{aligned}$$

So there is only one eigenvalue, $\lambda = 2$, repeated three times.

For the eigenspace, we need to consider the matrix $(C - \lambda I)$ with $\lambda = 2$ and a column of zeros.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix is already in reduced row echelon form. Thus the general solution to $(C - 2I)\boldsymbol{x} = \mathbf{0}$ is $x_1 = t$, $x_2 = 0$, $x_3 = 0$ with $t \in \mathbb{R}$. So

$$\text{null}(C - 2I) = \left\{ \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Hence the 2-eigenspace of C is

$$E_2 = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

and so E_2 has basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

T11 Show that the eigenvalues of $A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ are a and d. Assuming that $a \neq d$, find a basis for the corresponding eigenspaces.[3]

[3] Hint: Consider the cases $c = 0$ and $c \neq 0$ separately.

Solution

We have

$$\det(A - \lambda I) = \det \begin{pmatrix} a - \lambda & 0 \\ c & d - \lambda \end{pmatrix} = (a - \lambda)(d - \lambda).$$

This has solutions $\lambda = a$ and $\lambda = d$, so the eigenvalues of A are a and d.

Suppose first that $c = 0$, so that $A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. Then

$$(A - aI|\mathbf{0}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & d - a & 0 \end{pmatrix}$$

which, since $d - a \neq 0$, has reduced row echelon form

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence the a-eigenspace has basis $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.

We also have

$$(A - dI|\mathbf{0}) = \begin{pmatrix} a - d & 0 & 0 \\ c & 0 & 0 \end{pmatrix}$$

which, since $a - d \neq 0$, has reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence the d-eigenspace has basis $\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. Note that this does not depend upon the value of c!

Now suppose that $c \neq 0$. Then

$$(A - aI|\mathbf{0}) = \begin{pmatrix} 0 & 0 & 0 \\ c & d-a & 0 \end{pmatrix}$$

which, since $c \neq 0$, has reduced row echelon form

$$\begin{pmatrix} 1 & \frac{d-a}{c} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence the a-eigenspace has basis $\left\{ \begin{pmatrix} -\frac{d-a}{c} \\ 1 \end{pmatrix} \right\}$.

By the same calculation as in the case $c = 0$, the d-eigenspace again has basis $\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

T12 Let A be an $n \times n$ matrix with real entries. Show that A is invertible if and only if 0 is *not* an eigenvalue of A.

Solution

a) We have $A\boldsymbol{v} = \mathbf{0} = 0\boldsymbol{v}$. Since $\boldsymbol{v}$ is non-zero, this implies that 0 is an eigenvalue of A.

b) Many solutions are possible. Here are some.

The matrix A is invertible if and only if the unique solution to $A\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$. But 0 is an eigenvalue of A if and only if there is a nontrivial vector $\boldsymbol{x}$ so that $A\boldsymbol{x} = 0\boldsymbol{x} = \mathbf{0}$. Therefore A is invertible if and only if 0 is not an eigenvalue of A.

Alternatively, suppose that A is invertible. Assume by way of contradiction that 0 is an eigenvalue of A, with corresponding eigenvector $\boldsymbol{v}$. Then

$$\begin{aligned} \boldsymbol{v} &= I\boldsymbol{v} && \text{since multiplying by } I \text{ changes nothing} \\ &= (A^{-1}A)\boldsymbol{v} && \text{since } A^{-1}A = I \\ &= A^{-1}(A\boldsymbol{v}) && \text{since matrix multiplication is associative} \\ &= A^{-1}\mathbf{0} && \text{since } A\boldsymbol{v} = 0\boldsymbol{v} = \mathbf{0} \\ &= \mathbf{0}. \end{aligned}$$

But since $\boldsymbol{v}$ is an eigenvector we have $\boldsymbol{v} \neq \mathbf{0}$, a contradiction. Therefore 0 is not an eigenvalue of A. A similar computation can be used to establish the converse.

T13 Let A be an $n \times n$ matrix with real entries, and let $\boldsymbol{v}$ be an eigenvector for A with corresponding eigenvalue λ.

a) Show that $\boldsymbol{v}$ is also an eigenvector for the matrix A^2 with corresponding eigenvalue λ^2.

b) Show tutorial that $\boldsymbol{v}$ is an eigenvector for A^3 with corresponding eigenvalue λ^3.

c) Generalise this!

Solution

a) By the definition of eigenvector and eigenvalue, $\boldsymbol{v} \neq \mathbf{0}$ and $A\boldsymbol{v} = \lambda\boldsymbol{v}$. Then we have

$$A^2\boldsymbol{v} = A(A\boldsymbol{v}) = A(\lambda\boldsymbol{v}) = \lambda A\boldsymbol{v} = \lambda(A\boldsymbol{v}) = \lambda(\lambda\boldsymbol{v}) = \lambda^2\boldsymbol{v}.$$

We know that $\boldsymbol{v} \neq \mathbf{0}$, so the equation $A^2\boldsymbol{v} = \lambda^2\boldsymbol{v}$ shows that $\boldsymbol{v}$ is an eigenvector of A^2 with corresponding eigenvalue λ^2.

b) Consider

$$\begin{aligned} A^3\boldsymbol{v} &= A(A^2\boldsymbol{v}) = A(\lambda^2\boldsymbol{v}) && \text{by part (i) above} \\ &= \lambda^2 A\boldsymbol{v} = \lambda^2(A\boldsymbol{v}) \\ &= \lambda^2(\lambda\boldsymbol{v}) \\ &= \lambda^3\boldsymbol{v}. \end{aligned}$$

It is still true that $\boldsymbol{v} \neq \mathbf{0}$, but now $A^3\boldsymbol{v} = \lambda^3\boldsymbol{v}$ shows that $\boldsymbol{v}$ is an eigenvector of A^3 with corresponding eigenvalue λ^3.

c) The generalisation is that *$\boldsymbol{v}$ is an eigenvector of the matrix A^n with corresponding eigenvalue λ^n* for all positive integers n. We will prove this by induction. Let $P(n)$ be the statement in italics above. We know that $P(1)$ is true as this was the starting point. (We also showed $P(2)$ and $P(3)$ are true above, although this isn't required for the induction.) Assume by induction that $P(k)$ is true for some $k \geq 1$. Then

$$\begin{aligned} A^{k+1}\boldsymbol{v} = A(A^k\boldsymbol{v}) &= A(\lambda^k\boldsymbol{v}) && \text{by the inductive hypothesis } P(k) \\ &= \lambda^k A\boldsymbol{v} \\ &= \lambda^k(\lambda\boldsymbol{v}) && \text{since } \lambda \text{ is an eigenvalue of } A \text{ with eigenvector } \boldsymbol{v} \\ &= \lambda^{k+1}\boldsymbol{v}. \end{aligned}$$

Since $\boldsymbol{v} \neq \mathbf{0}$ and $A^{k+1}\boldsymbol{v} = \lambda^{k+1}\boldsymbol{v}$, we deduce that $\boldsymbol{v}$ is an eigenvector of A^{k+1} corresponding to eigenvalue λ^{k+1}. Hence $P(k+1)$ is true. It follows by induction that $P(n)$ is true for all $n \geq 1$.

T14 Let A and B be $n \times n$ matrices. Prove the following statements using determinants.

a) AB is invertible if and only if both A and B are invertible.

b) If A is invertible then A^{-1} is invertible.

Solution

a) If A and B are invertible then $\det(A) \neq 0$ and $\det(B) \neq 0$. Thus $\det(A)\det(B) \neq 0$. But $\det(A)\det(B) = \det(AB)$, hence $\det(AB) \neq 0$ and so AB is invertible.

If AB is invertible then $\det(AB) \neq 0$. Now $\det(AB) = \det(A)\det(B)$, so neither $\det(A)$ nor $\det(B)$ can equal 0. Hence A and B are both invertible.

b) If A is invertible then $\det(A) \neq 0$, and $\det(A^{-1}) = \frac{1}{\det(A)}$. Since $\det(A) \neq 0$, we have $\frac{1}{\det(A)} \neq 0$, hence $\det(A^{-1}) \neq 0$, and so A^{-1} is invertible.

T15

a) Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$. Show that

$$\det(A - \lambda I) = \lambda^2 - (\operatorname{tr}(A))\lambda + \det A.$$

b) Consider an arbitrary 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

By expanding $\det(A - \lambda I)$ along the first row, verify that $\det(A - \lambda I)$ is a polynomial of degree 3 in λ in which the coefficient of λ^3 is -1, the coefficient of λ^2 is $\operatorname{tr} A$, and the constant term is $\det A$.

Solution

a) We have

$$\begin{aligned}
\det(A - \lambda I) &= \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \\
&= (a-\lambda)(d-\lambda) - bc \\
&= \lambda^2 - (a+d)\lambda + (ad - bc) \\
&= \lambda^2 - (\operatorname{tr}(A))\lambda + \det(A).
\end{aligned}$$

b) We have

$$\det(A - \lambda I) = \det \begin{pmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{pmatrix}.$$

Expanding along the top row gives

$$\begin{aligned}
\det(A - \lambda I) = (a_{11} - \lambda)&[(a_{22}-\lambda)(a_{33}-\lambda) - a_{23}a_{32}] \\
&- a_{12}[a_{21}(a_{33}-\lambda) - a_{23}a_{31}] + a_{13}[a_{21}a_{32} - a_{31}(a_{22}-\lambda)]
\end{aligned}$$

Expanding the brackets and collecting together the terms with the same powers of λ we have

$$\begin{aligned}\det(A-\lambda I) = -\lambda^3 &+ \lambda^2(a_{11}+a_{22}+a_{33}) \\ &- \lambda(a_{11}a_{22}+a_{11}a_{33}+a_{22}a_{33}-a_{13}a_{31}-a_{23}a_{32}-a_{12}a_{21}) \\ &+ (a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33})\end{aligned}$$

We now have a polynomial of degree 3 where the coefficient of λ^3 is -1. We can check from the matrix A that

$$\mathrm{tr}(A) = a_{11}+a_{22}+a_{33}$$

and so the coefficient of λ^2 is $\mathrm{tr}(A)$. Compute directly that the formula for the determinant is

$$\det(A) = a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}$$

and so the constant term is $\det(A)$ as required.

2B Linear Algebra

Ex Sheet 7

True/False [1]

a) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a diagonal matrix.

b) Any matrix $A \in M_{n\times n}(\mathbb{R})$ is similar to itself.

c) Similar matrices have the same eigenvalues.

d) A square matrix is diagonalisable if it is similar to a diagonal matrix.

e) For all diagonalisable matrices A, there is a unique diagonal matrix D and a unique invertible matrix P so that $P^{-1}AP = D$.

f) k eigenvectors corresponding to k distinct eigenvalues are linearly independent.

g) An $n \times n$ matrix with real entries is diagonalisable if and only if it has n distinct real eigenvalues.

h) $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ is diagonalisable.

i) The geometric multiplicity of an eigenvalue λ of the square matrix A is the number of vectors in the λ-eigenspace of A.

j) The sum of the algebraic multiplicities of the eigenvalues of an $n \times n$ real matrix is n.

[1] **True/False Questions** Every Exercise Sheet will have a section containing true/false questions. They are designed to test your understanding from lectures. You should look at your lecture notes and/or the textbook to help you answer these questions, but you should not need to write anything to work out the solution.

Solutions to True/False

a) T b) T c) T d) T e) F f) T g) F h) T i) F j) T

Tutorial Exercises

T1 Let $A, B \in M_{n\times n}$.

a) Show that if A and B are similar, then A is invertible if and only if B is invertible.

b) Prove that if A and B are similar and both invertible, then A^{-1} and B^{-1} are similar.

Solution

a) The matrix A is invertible if and only if $\det(A) \neq 0$ and the matrix B is invertible if and only if $\det(B) \neq 0$. Since $\det(A) = \det(B)$ the result follows.

b) We have $B^{-1} = (P^{-1}AP)^{-1} = P^{-1}A^{-1}(P^{-1})^{-1} = P^{-1}A^{-1}P$. Since P is invertible, this means that A^{-1} and B^{-1} are similar.

T2 Consider the matrices

$$A = \begin{bmatrix} 2 & 3 & 2 & 4 \\ -1 & 2 & 1 & 1 \\ 2 & 3 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

By considering determinants, show that A and B are *not* similar.

Solution

Suppose by way of contradiction that A and B are similar. Then $\det(A) = \det(B)$. Now we need only notice that $\det(A) = 0$ (expand along the bottom row), and $\det(B) = 4$ (it's upper triangular, so the determinant is the product of the entries on the main diagonal). These numbers are different, so A and B cannot be similar.

T3 For each of the matrices in T7 and T9 on Exercise Sheet 6:

- Determine whether the matrix is diagonalisable, and if it is diagonalisable, find a diagonal matrix D and an invertible matrix P such that $P^{-1}AP = D$ (replace A by B or C as appropriate).
- Find the algebraic and geometric multiplicities of each of the eigenvalues.

Solution

For T7 on Exercise Sheet 6:

a) Since A has two distinct eigenvalues, A is diagonalisable. Using the answers to T7(a), if

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix} \text{ and } P = \begin{pmatrix} 1 & -\frac{3}{5} \\ 0 & 1 \end{pmatrix}$$

then $P^{-1}BP = D$.

Each of the eigenvalues of A has algebraic multiplicity 1 and geometric multiplicity 1.

b) The only eigenvalue is $\lambda = -2$ and we have

$$E_{-2} = \text{Span}\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$$

which is one-dimensional. Thus $\mathbb{R}^2$ does not have a basis consisting of eigenvectors of B, hence B is not diagonalisable.

The eigenvalue $\lambda = 2$ has algebraic multiplicity 2 and geometric multiplicity 1.

c) Since C has two distinct eigenvalues, C is diagonalisable. Using the answers to T7(c), if

$$D = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } P = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{3} \\ 1 & 1 \end{pmatrix}$$

then $P^{-1}CP = D$.

Each of the eigenvalues of C has algebraic multiplicity 1 and geometric multiplicity 1.

For T9 on Exercise Sheet 6:

a) Since A has three distinct eigenvalues, A is diagonalisable. Using the answers to T9(a), if

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \text{ and } P = \begin{pmatrix} 0 & -2 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

then $P^{-1}AP = D$.

Each of the eigenvalues of A has algebraic multiplicity 1 and geometric multiplicity 1.

b) Since B has three distinct eigenvalues, B is diagonalisable. Using the answers to T9(b), if

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3i & 0 \\ 0 & 0 & -3i \end{pmatrix} \text{ and } P = \begin{pmatrix} 2 & 1+3i & 1-3i \\ -2 & 3i-1 & -3i-1 \\ 1 & -4 & -4 \end{pmatrix}$$

then $P^{-1}AP = D$.

Each of the eigenvalues of B has algebraic multiplicity 1 and geometric multiplicity 1.

c) The only eigenvalue is $\lambda = 2$ and we have

$$E_2 = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

which is one-dimensional. Thus $\mathbb{R}^3$ does not have a basis consisting of eigenvectors of C, hence C is not diagonalisable.

The eigenvalue $\lambda = 2$ has algebraic multiplicity 3 and geometric multiplicity 1.

T4 Let $A, B \in M_{3\times 3}(\mathbb{C})$ and suppose the eigenvalues of both A and B are $1, 2+i$ and 4.

a) Write down a diagonal matrix D to which both A and B are similar.

b) Hence prove that A is similar to B.

Solution

a) Since A and B have the same three distinct eigenvalues, they are both similar to the diagonal matrix

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2+i & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

b) Since A is similar to D, there is an invertible matrix $P \in M_{3\times 3}(\mathbb{C})$ such that $P^{-1}AP = D$. Since B is similar to D, there is an invertible matrix $Q \in M_{3\times 3}(\mathbb{C})$ such that $Q^{-1}BQ = D$. Now this latter equation means $B = QDQ^{-1}$, so we have

$$B = QDQ^{-1} = QP^{-1}APQ^{-1} = (PQ^{-1})^{-1}A(PQ^{-1}).$$

Therefore A and B are similar matrices.

T5 Construct the matrix A which has eigenvalues 0 and -1, with corresponding eigenspaces

$$E_0 = \mathrm{Span}\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right), \qquad E_{-1} = \mathrm{Span}\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right).$$

Solution

Since A has distinct eigenvalues, we know that it is similar to a diagonal matrix D. Hence, we have $A = PDP^{-1}$ where

$$D = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}.$$

We compute

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 \\ -6 & -3 \end{bmatrix}.$$

T6 Let $A \in M_{n\times n}(\mathbb{R})$ be invertible. Recall that the eigenvalues of an invertible matrix are non-zero.

a) Suppose that $\lambda \in \mathbb{R}$ is an eigenvalue of A. Prove that λ^{-1} is an eigenvalue of A^{-1}.

b) Show that if $D = (d_{ij}) \in M_{n\times n}(\mathbb{R})$ is diagonal, with all diagonal entries $d_{ii} \neq 0$, then D is invertible, with D^{-1} the diagonal matrix with diagonal entries d_{ii}^{-1}.

c) Prove that if A is diagonalisable, then A^{-1} is diagonalisable.

Solution

a) Since λ is an eigenvalue of A, there is a non-zero vector $\boldsymbol{v}$ so that $A\boldsymbol{v} = \lambda\boldsymbol{v}$. Now multiply both sides of this equation by A^{-1} to get

$$A^{-1}A\boldsymbol{v} = A^{-1}(\lambda\boldsymbol{v}) \implies \mathbb{I}_n\boldsymbol{v} = \lambda(A^{-1}\boldsymbol{v}) \implies \boldsymbol{v} = \lambda(A^{-1}\boldsymbol{v}).$$

Since $\lambda \neq 0$, we can then multiply both sides of this last equation by λ^{-1} to get

$$\lambda^{-1}\boldsymbol{v} = 1(A^{-1}\boldsymbol{v}) \implies \lambda^{-1}\boldsymbol{v} = A^{-1}\boldsymbol{v}.$$

Since the vector $\boldsymbol{v}$ is non-zero, this means that λ^{-1} is an eigenvalue of A^{-1}.

b) The determinant of D is the product of its diagonal entries. Since each $d_{ii} \neq 0$, we have that

$\det(D) \neq 0$, so D is invertible. If E is the diagonal matrix with $e_{ii} = d_{ii}^{-1}$ then a direct computation shows that $DE = ED = \mathbb{I}_n$. Hence $D^{-1} = E$ as required.

c) If A is diagonalisable there exists an invertible matrix P and a diagonal matrix D so that $P^{-1}AP = D$. The diagonal entries of D are the eigenvalues of A, so the diagonal entries of D are non-zero as A is invertible. Hence D is invertible and D^{-1} is diagonal, by (ii). So we may take the inverse of both sides of the equation $P^{-1}AP = D$ to get

$$(P^{-1}AP)^{-1} = D^{-1} \implies P^{-1}A^{-1}(P^{-1})^{-1} = D^{-1}.$$

Put $Q = P^{-1}$ then we have $QA^{-1}Q^{-1} = D^{-1}$, with D^{-1} diagonal, hence A^{-1} is diagonalisable.

T7 Let A, B and C be $n \times n$ matrices and suppose that A is similar to B, with $P^{-1}AP = B$, and B is similar to C, with $Q^{-1}BQ = C$, where P and Q are $n \times n$ invertible matrices. Prove that A is similar to C.

Solution

Since P and Q are invertible, PQ is invertible. Now

$$(PQ)^{-1}A(PQ) = Q^{-1}P^{-1}APQ = Q^{-1}BQ = C$$

and so A and C are similar.

T8 Let A and B be $n \times n$ matrices and suppose that A is similar to B, with $AP = PB$ for an $n \times n$ invertible matrix P.

a) Recall that row-equivalent matrices have the same row space. Use this to show that $\operatorname{rank}(B) = \operatorname{rank}(PB)$ and that $\operatorname{rank}((AP)^T) = \operatorname{rank}(A^T)$.

b) Deduce that $\operatorname{rank}(A) = \operatorname{rank}(B)$.

Solution

a) Since P is invertible, P is a product of elementary matrices. So PB is row-equivalent to B, hence $\operatorname{row}(PB) = \operatorname{row}(B)$ and thus $\operatorname{rank}(PB) = \operatorname{rank}(B)$.
Now $(AP)^T = P^T A^T$ and P^T is invertible since P is invertible. Thus by the same argument $\operatorname{rank}((AP)^T) = \operatorname{rank}(A^T)$.

b) Using a) and the equation $AP = PB$, as well as results about rank, we have

$$\operatorname{rank}(A) = \operatorname{rank}(A^T) = \operatorname{rank}((AP)^T) = \operatorname{rank}(AP) = \operatorname{rank}(PB) = \operatorname{rank}(B)$$

and so $\operatorname{rank}(A) = \operatorname{rank}(B)$ as required.

In the remaining exercises you will prove the Diagonalisation Theorem, Theorem 4.27. The notation is as follows. Suppose $A = (a_{ij}) \in M_{n\times n}(\mathbb{R})$ has eigenvalues $\lambda_1, \ldots, \lambda_k$. For $1 \leq i \leq k$ let

$$m_i = \text{the algebraic multiplicity of the eigenvalue } \lambda_i$$

and

d_i = the geometric multiplicity of the eigenvalue λ_i.

The aim is to show that A is diagonalisable if and only, for each $1 \le i \le k$, we have $d_i = m_i$.

T9

a) Prove by induction on $n \ge 2$ if $B = (b_{ij}) \in M_{n\times n}(\mathbb{R})$ then $\det(B)$ is a polynomial in the entries of B of degree at most n. This means that for each monomial term of $\det(B)$, the sum of the powers of the b_{ij} appearing in that monomial is at most n.

b) Using (a), prove by induction on $n \ge 2$ that

$$\det(A - tI) = (a_{11} - t)(a_{22} - t)\cdots(a_{nn} - t) + g(t)$$

where $g(t)$ is a polynomial in the variable t with coefficients in $\mathbb{R}$ and degree strictly less than n.

c) Conclude that $\det(A - tI)$ has degree equal to n, hence

$$m_1 + m_2 + \cdots + m_k = n.$$

Solution

a) In the case $n = 2$ we have

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

so $\det(B) = b_{11}b_{22} - b_{12}b_{21}$. Thus $\det(B)$ is a polynomial in the entries of B of degree at most 2. Now assume that for any $k \times k$ real matrix B, $\det(B)$ is a polynomial in the entries of B of degree at most k.

Let $B = (b_{ij})$ be a $(k+1) \times (k+1)$ real matrix. Then

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1,k+1} \\ b_{21} & b_{22} & \cdots & b_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k+1,1} & b_{k+1,2} & \cdots & b_{k+1,k+1} \end{pmatrix}.$$

We calculate $\det(B)$ by expanding along the first row. We have

$$\det(B) = \sum_{j=1}^{k+1} (-1)^{j+1} b_{1j} \det(B_{1j}).$$

Now each cofactor B_{1j} is a $k \times k$ matrix, so by the inductive assumption, $\det(B_{1j})$ is a polynomial in the entries of B of degree at most k. Hence each summand $(-1)^{j+1} b_{1j} \det(B_{1j})$ has degree at most $k+1$, and thus $\det(B)$ has degree at most $k+1$ as required.

b) In the case $n = 2$ we have

$$\det(A - tI) = \begin{vmatrix} a_{11} - t & a_{12} \\ a_{21} & a_{22} - t \end{vmatrix} = (a_{11} - t)(a_{22} - t) - a_{12}a_{21}$$

Let $g(t) = -a_{12}a_{21}$. That is, $g(t)$ is the constant polynomial $-a_{12}a_{21} \in \mathbb{R}$. Then we have $\det(A - tI) = (a_{11} - t)(a_{22} - t) + g(t)$ with $g(t)$ a real polynomial of degree 0. Since $0 < 2$, this proves the statement in the case $n = 2$.

Now assume that if A is $k \times k$ then

$$\det(A - tI) = (a_{11} - t)(a_{22} - t) \cdots (a_{kk} - t) + g(t)$$

where $g(t)$ is a polynomial in the variable t with coefficients in $\mathbb{R}$ and degree less than k.

Let A be $(k+1) \times (k+1)$. Then

$$\det(A - tI) = \det \begin{pmatrix} a_{11} - t & a_{12} & \cdots & a_{1,k+1} \\ a_{21} & a_{22} - t & \cdots & a_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+1,1} & a_{k+1,2} & \cdots & a_{k+1,k+1} - t \end{pmatrix}.$$

We compute this determinant by expanding along the top row. This gives

$$\det(A - tI) = (a_{11} - t) \det((A - tI)_{11}) + \sum_{j=2}^{k+1} (-1)^{j+1} a_{1j} \det((A - tI)_{1j}).$$

Consider the term $(a_{11} - t) \det((A - tI)_{11})$. The cofactor $(A - tI)_{11} = (A - tI_{k+1})_{11}$ is equal to $(A_{11} - tI_k)$, and so

$$\det((A - tI_{k+1})_{11}) = \det(A_{11} - tI_k).$$

This is the characteristic polynomial of A_{11}, so by inductive assumption, since A_{11} is $k \times k$ we have

$$\det(A - tI_k) = (a_{22} - t) \cdots (a_{k+1,k+1} - t) + g(t)$$

where $g(t)$ is a polynomial in the variable t with coefficients in $\mathbb{R}$ and degree less than k. Thus

$$(a_{11} - t) \det((A - tI_{k+1})_{11}) = (a_{11} - t)(a_{22} - t) \cdots (a_{k+1,k+1} - t) + (a_{11} - t)g(t).$$

Since $g(t)$ has degree less than k, the term $(a_{11} - t)g(t)$ has degree less than $(k+1)$.

It now suffices to show that

$$\sum_{j=2}^{k+1} (-1)^{j+1} a_{1j} \det((A - tI)_{1j})$$

has degree less than $(k+1)$. For this, it suffices to show that for each $2 \leq j \leq n$, the polynomial in t given by $\det((A - tI)_{1j})$ has degree at most k. By part (a), since $(A - tI)_{1j}$ is a $k \times k$ matrix, $\det(A - tI)_{1j})$ is a polynomial in the entries of $(A - tI)_{1j}$ of degree at most k. This completes the proof.

c) By part (b), we have

$$\det(A - tI) = (a_{11} - t)(a_{22} - t) \cdots (a_{nn} - t) + g(t)$$

where $g(t)$ has degree less than n. Since the expression $(a_{11} - t)(a_{22} - t) \cdots (a_{nn} - t)$ is a polynomial in t of degree equal to n (the coefficient of t^n is $(-1)^n \neq 0$), it follows that $\det(A - tI)$ has degree n.

Since the eigenvalues of A are the roots of $\det(A - tI)$, the polynomial $\det(A - tI)$ factors as

$$\det(A - tI) = (-1)^n (t - \lambda_1)^{m_1} (t - \lambda_2)^{m_2} \cdots (t - \lambda_k)^{m_k}.$$

As $\det(A - tI)$ has degree n, the sum of the m_i must equal n, that is,

$$m_1 + m_2 + \cdots + m_k = n.$$

T10 Let $\lambda = \lambda_i$ be an eigenvalue of A, let $m = m_i$ be the algebraic multiplicity of λ and let $d = d_i$ be the geometric multiplicity of λ. Let $S : \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_d$ be an ordered basis for the λ-eigenspace E_λ.

a) Let U be the $n \times d$ matrix which has $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_d$ as its columns. Explain why $AU = \lambda U$.

b) Let Q be any invertible matrix in $M_{n\times n}(\mathbb{R})$ which has $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_d$ as its first d columns. Then we can write Q as a "partitioned matrix" $Q = (U \mid V)$, where V is $n \times (n - d)$. By considering the product $Q^{-1}Q$, prove that if Q^{-1} is the partitioned matrix

$$Q^{-1} = \begin{pmatrix} C \\ D \end{pmatrix}$$

where C is $d \times n$ and D is $(n - d) \times n$, then the following equations hold:

$$CU = \mathbb{I}_d \quad CV = \mathbb{O}_{d,n-d} \quad DU = \mathbb{O}_{n-d,d} \quad CU = \mathbb{I}_{n-d}.$$

Here, $\mathbb{O}_{k,l}$ is the $k \times l$ matrix with all entries 0.

c) Hence prove that

$$\det(Q^{-1}AQ - tI) = (\lambda - t)^d \det(DAV - tI).$$

d) Conclude that $d \leq m$. That is, for each eigenvalue of A, the geometric multiplicity is less than or equal to the algebraic multiplicity.

Solution

a) Since $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_d$ are all eigenvectors of A with corresponding eigenvector λ, we have $A\boldsymbol{v}_i = \lambda \boldsymbol{v}_i$ for each $1 \leq i \leq d$. Thus AU is the $n \times d$ matrix with i^{th} column $A\boldsymbol{v}_i$, and so $AU = \lambda U$.

b) We have

$$Q^{-1}Q = \begin{pmatrix} C \\ D \end{pmatrix} (U \mid V) = \begin{pmatrix} CU & CV \\ DU & DV \end{pmatrix}$$

But also $Q^{-1}Q = \mathbb{I}_n$ which we partition as

$$Q^{-1}Q = \mathbb{I}_n = \begin{pmatrix} \mathbb{I}_d & \mathbb{O}_{d,n-d} \\ \mathbb{O}_{n-d,d} & \mathbb{I}_{n-d} \end{pmatrix}.$$

By considering the sizes of the products CU, CV, DU and DV we obtain the required equations

$$CU = \mathbb{I}_d \quad CV = \mathbb{O}_{d,n-d} \quad DU = \mathbb{O}_{n-d,d} \quad DV = \mathbb{I}_{n-d}.$$

c) We first compute $Q^{-1}AQ$:

$$Q^{-1}AQ = \begin{pmatrix} C \\ D \end{pmatrix} A(U \mid V) = \begin{pmatrix} CAU & CAV \\ DAU & DAV \end{pmatrix}$$

Now $AU = \lambda U$ by part (a), and $CU = \mathbb{I}_d$ and $DU = \mathbb{O}_{n-d,d}$ by part (b). So

$$Q^{-1}AQ = \begin{pmatrix} C\lambda U & CAV \\ D\lambda U & DAV \end{pmatrix} = \begin{pmatrix} \lambda CU & CAV \\ \lambda DU & DAV \end{pmatrix} = \begin{pmatrix} \lambda \mathbb{I}_d & CAV \\ \lambda \mathbb{O}_{n-d,d} & DAV \end{pmatrix} = \begin{pmatrix} \lambda \mathbb{I}_d & CAV \\ \mathbb{O}_{n-d,d} & DAV \end{pmatrix}.$$

Hence

$$\det(Q^{-1}AQ - tI) = \det \begin{pmatrix} (\lambda - t)\mathbb{I}_d & CAV \\ \mathbb{O}_{n-d,d} & DAV - t\mathbb{I}_{n-d} \end{pmatrix} = (\lambda - t)^d \det(DAV - t\mathbb{I}_{n-d})$$

as required.

d) Since A and $Q^{-1}AQ$ are similar matrices, they have the same characteristic polynomial. Therefore by part (c),

$$\det(A - tI) = \det(Q^{-1}AQ - tI) = (\lambda - t)^d \det(DAV - tI).$$

Thus the algebraic multiplicity of λ is at least d, and so $d \leq m$ as required.

T11 For $1 \leq i \leq k$, let

$$S_i : \boldsymbol{v}_{i1}, \boldsymbol{v}_{i2}, \ldots, \boldsymbol{v}_{id_i}$$

be an ordered basis for the λ_i-eigenspace $\mathrm{Eig}_{\lambda_i}(A)$.

a) Prove that

$$S : \boldsymbol{v}_{11}, \boldsymbol{v}_{12}, \ldots, \boldsymbol{v}_{1d_1}, \boldsymbol{v}_{21}, \boldsymbol{v}_{22}, \ldots, \boldsymbol{v}_{2d_2}, \ldots, \boldsymbol{v}_{k1}, \boldsymbol{v}_{k2}, \ldots, \boldsymbol{v}_{kd_k}$$

obtained by taking the union of the S_i is linearly independent. [Hint: remember that eigenspaces are subspaces, and use Theorem 4.20.]

b) Hence prove that $\mathbb{R}^n$ has a basis consisting of eigenvectors of A if and only if $d_1 + d_2 + \cdots + d_k = n$.

Solution

a) Suppose that

$$\lambda_{11}\boldsymbol{v}_{11} + \lambda_{12}\boldsymbol{v}_{12} + \cdots + \lambda_{1d_1}\boldsymbol{v}_{1d_1} + \lambda_{21}\boldsymbol{v}_{21} + \lambda_{22}\boldsymbol{v}_{22} + \cdots + \lambda_{2d_2}\boldsymbol{v}_{2d_2} + \cdots + \lambda_{k1}\boldsymbol{v}_{k1} + \lambda_{k2}\boldsymbol{v}_{k2} + \cdots + \lambda_{kd_k}\boldsymbol{v}_{kd_k} = \boldsymbol{0}$$

where each $\lambda_{ij} \in \mathbb{R}$. Let

$$\begin{aligned} \boldsymbol{v}_1 &= \lambda_{11}\boldsymbol{v}_{11} + \lambda_{12}\boldsymbol{v}_{12} + \cdots + \lambda_{1d_1}\boldsymbol{v}_{1d_1} \\ \boldsymbol{v}_2 &= \lambda_{21}\boldsymbol{v}_{21} + \lambda_{22}\boldsymbol{v}_{22} + \cdots + \lambda_{2d_2}\boldsymbol{v}_{2d_2} \\ \vdots \;&\;\vdots\;\; \vdots \\ \boldsymbol{v}_k &= \lambda_{k1}\boldsymbol{v}_{k1} + \lambda_{k2}\boldsymbol{v}_{k2} + \cdots + \lambda_{kd_k}\boldsymbol{v}_{kd_k}. \end{aligned}$$

Then

$$v_1 + v_2 + \cdots + v_k = \mathbf{0}. \tag{1}$$

Now for each $1 \leq i \leq k$, we have that $v_i \in \mathrm{Eig}_{\lambda_i}(A)$, since eigenspaces are subspaces. So either $v_i = \mathbf{0}$ or v_i is an eigenvector of A. If there is some $v_i \neq \mathbf{0}$ then the collection $\{v_i \mid v_i \neq \mathbf{0}\}$ is a collection of eigenvectors of A corresponding to distinct eigenvalues. By Theorem 4.20, this collection is linearly independent. However Equation (1) above gives a linear dependence between these eigenvectors, a contradiction. Therefore each $v_i = \mathbf{0}$. Now as each S_i is a basis, and thus linearly independent, the k equations above defining the v_i mean that $\lambda_{ij} = 0$ for all i, j. Hence S is linearly independent.

b) Suppose $d_1 + d_2 + \cdots + d_k = n$. Then the set S is a linearly independent set in $\mathbb{R}^n$ containing n vectors, hence S is a basis. So $\mathbb{R}^n$ has a basis consisting of eigenvectors of A.

Suppose $\mathbb{R}^n$ has a basis consisting of eigenvectors of A. Now by F5 we have $d_i \leq m_i$ for each i, and by T10 we have $m_1 + m_2 + \cdots + m_k = n$. So $d_1 + d_2 + \cdots + d_k \leq n$. If $d_1 + d_2 + \cdots + d_k < n$ then the set S does not span $\mathbb{R}^n$, since it contains fewer than n vectors. However every eigenvector in the basis of eigenvectors belongs to $\mathrm{Eig}_{\lambda_i}(A)$ for some i, and so must be in the span of S_i. This is a contradiction. Hence $d_1 + d_2 + \cdots + d_k = n$.

T12 It follows from Theorem 4.23 that A is diagonalisable if and only if $\mathbb{R}^n$ has a basis consisting of eigenvectors of A. Use this together with results above to prove that A is diagonalisable if and only if $d_i = m_i$ for each $1 \leq i \leq k$.

Solution

Suppose A is diagonalisable. Then it follows from Theorem 4.23 that $\mathbb{R}^n$ has a basis consisting of eigenvectors of A. By T11, this means $d_1 + d_2 + \cdots + d_k = n$. Now by T11, we have $m_1 + m_2 + \cdots + m_k = n$. Thus

$$0 = (m_1 + m_2 + \cdots + m_k) - (d_1 + d_2 + \cdots + d_k) = (m_1 - d_1) + (m_2 - d_2) + \cdots (m_k - d_k).$$

By F5, $d_i \leq m_i$ for each i, hence $m_i - d_i \geq 0$ for each i. Thus we must have $m_i - d_i = 0$ for all i. That is, the geometric and algebraic multiplicities of each eigenvalue are equal.

Suppose that $d_i = m_i$ for each i. Then using F4 it follows that

$$m_1 + m_2 + \cdots + m_k = n = d_1 + d_2 + \cdots + d_k.$$

Hence by F6, $\mathbb{R}^n$ has a basis consisting of eigenvectors, and so by Theorem 4.23, A is diagonalisable.

T13 Let

$$A = \begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix}.$$

Find an expression for A^n, where n is a positive integer, in a form that displays the entries of the matrix explicitly.

Solution

In **T9** exercise sheet 6, we found that

$$P^{-1}AP = D,$$

where

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad D = \operatorname{diag}(4, 3).$$

Hence

$$A = PDP^{-1}.$$

So, for any positive integer n,

$$\begin{aligned} A^n &= PD^nP^{-1} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4^n & 0 \\ 0 & 3^n \end{bmatrix} \frac{1}{1} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2(4^n) & 3^n \\ 4^n & 3^n \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2(4^n) - 3^n & -2(4^n) + 2(3^n) \\ 4^n - 3^n & -4^n + 2(3^n) \end{bmatrix}. \end{aligned}$$

T14 Consider the matrix

$$A(x) = \begin{bmatrix} (x-2) & 2 \\ -1 & (x+1) \end{bmatrix},$$

where $x \in \mathbb{R}$. Find an invertible matrix P and a diagonal matrix $D(x)$ (which depends on x) such that $A(x) = P^{-1}D(x)P$. Calculate $A(0)^8 + A(1)^9$.

Solution

The eigenvalues λ satisfy the quadratic equation $\lambda^2 + (1-2x)\lambda + x^2 - x = 0$. This has solutions $\lambda_1 = x$ and $\lambda_2 = x - 1$.

Solving $A(x)\mathbf{y} = x\mathbf{y}$ yields the eigenvector $\mathbf{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Solving $A(x)\mathbf{y} = (x-1)\mathbf{y}$ yields the eigenvector $\mathbf{y} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Hence $D(x) = \begin{pmatrix} x & 0 \\ 0 & x-1 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$.

Now

$$\begin{aligned}
A(0)^8 + A(1)^9 &= P^{-1}D(0)^8P + P^{-1}D(1)^9P \\
&= P^{-1}(D(0)^8 + D(1)^9)P \\
&= P^{-1}\left(\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}^8 + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^9\right)P \\
&= P^{-1}IP \\
&= I.
\end{aligned}$$

2B Linear Algebra

Ex Sheet 8

1 True/False

a) For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with $\mathbf{x} \neq 0$, we have that $\frac{|\mathbf{x}\cdot\mathbf{y}|}{\|\mathbf{x}\|} \leq \|\mathbf{y}\|$.

b) For any vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and any scalar d, we have $\mathbf{x} \cdot (d\mathbf{y}) = (d\mathbf{x}) \cdot \mathbf{y}$.

c) If θ is the angle between vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, and $\theta > \frac{\pi}{2}$, then $\mathbf{x} \cdot \mathbf{y} > 0$.

d) There exists three non-zero mutually orthogonal vectors in $\mathbb{R}^2$.

e) The standard unit vectors in $\mathbb{R}^n$ are mutually orthogonal.

f) If $\mathbf{v}$ is orthogonal to the vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ in $\mathbb{R}^m$, then $\mathbf{v}$ is orthogonal to $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

g) Starting from any five given vectors in $\mathbb{R}^5$, the Gram-Schmidt process can produce five orthonormal vectors.

h) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a set of non-zero mutually orthogonal vectors in $\mathbb{R}^m$, the Gram-Schmidt process would just produce the same set of vectors.

i) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a set of orthonormal vectors in $\mathbb{R}^m$, the Gram-Schmidt process would just produce the same set of vectors.

j) Let Q be an orthogonal $n \times n$ matrix, and $\mathbf{x}$ and $\mathbf{y}$ vectors in $\mathbb{R}^n$. Then $Q\mathbf{x} \cdot Q\mathbf{y} \leq \|\mathbf{x}\|\|\mathbf{y}\|$.

k) Let Q_1, Q_2, Q_3, Q_4 be orthogonal $n \times n$ matrices, all with the same determinant. Then the determinant of $Q_1Q_2Q_3Q_4$ is 1.

l) The sum of two orthogonal $n \times n$ matrices is an orthogonal $n \times n$ matrix.

m) The subset of all orthogonal matrices is a subspace of the space of all matrices.

[1] **True/False Questions**
Every Exercise Sheet will have a section containing true/false questions. They are designed to test your understanding from lectures. You should look at your lecture notes and/or the textbook to help you answer these questions, but you should not need to write anything to work out the solution.

Solutions to True/False

a) T b) T c) F d) F e) T f) T g) F h) F i) T j) T k) T l) F m)F

Tutorial Exercises

T1 Suppose that $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors such that

$$\mathbf{u}\cdot\mathbf{v} = 2,\ \mathbf{v}\cdot\mathbf{w} = -3,\ \mathbf{u}\cdot\mathbf{w} = 5,$$

$$\|\mathbf{u}\| = 1,\ \|\mathbf{v}\| = 2,\ \|\mathbf{w}\| = 7.$$

Evaluate

a) $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{v}+\mathbf{w})$

b) $(2\mathbf{v}-\mathbf{w})\cdot(3\mathbf{u}+2\mathbf{w})$

c) $(\mathbf{u}-\mathbf{v}-2\mathbf{w})\cdot(4\mathbf{u}+\mathbf{v})$

d) $\|\mathbf{u}+\mathbf{v}\|$

e) $\|2\mathbf{w}-\mathbf{v}\|$

f) $\|\mathbf{u}-2\mathbf{v}+4\mathbf{w}\|$

Solution

a) Expanding, we get $\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}+\|\mathbf{v}\|^2+\mathbf{v}\cdot\mathbf{w}=2+5+4-3=8$.

b) Expanding, we get $6\mathbf{v}\cdot\mathbf{u}+4\mathbf{v}\cdot\mathbf{w}-3\mathbf{w}\cdot\mathbf{u}-2\|\mathbf{w}\|^2=12-12-15-98=-113$.

c) Expanding, and gathering like terms, we get $4\|\mathbf{u}\|^2-3\mathbf{u}\cdot\mathbf{v}-\|\mathbf{v}\|^2-8\mathbf{w}\cdot\mathbf{u}-2\mathbf{w}\cdot\mathbf{v}=4-6-4-40+6=-40$.

d) $\|\mathbf{u}+\mathbf{v}\|^2=(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})=\|\mathbf{u}\|^2+2\mathbf{u}\cdot\mathbf{v}+\|v\|^2=1+4+4=9$. Hence $\|\mathbf{u}+\mathbf{v}\|=3$.

e) $\|2\mathbf{w}-\mathbf{v}\|^2=(2\mathbf{w}-\mathbf{v})\cdot(2\mathbf{w}-\mathbf{v})=4\|\mathbf{w}\|^2-4\mathbf{w}\cdot\mathbf{v}+\|\mathbf{v}\|^2=196+12+4=212$. Hence $\|2\mathbf{w}-\mathbf{v}\|=\sqrt{212}$.

f) $\|\mathbf{u}-2\mathbf{v}+4\mathbf{w}\|^2=\|\mathbf{u}\|^2-4\mathbf{u}\cdot\mathbf{v}+8\mathbf{u}\cdot\mathbf{w}+4\|\mathbf{v}\|^2-16\mathbf{v}\cdot\mathbf{w}+16\|\mathbf{w}\|^2=1-8+40+16+48+784=881$. Hence $\|u-2v+4w\|=\sqrt{881}$.

T2 Which of the following sets of vectors are orthogonal? Which are orthonormal?

a) $\left\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}2\\-1\end{pmatrix}\right\}$

b) $\left\{\begin{pmatrix}1\\3\end{pmatrix},\begin{pmatrix}4\\5\end{pmatrix}\right\}$

c) $\left\{\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix},\begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix}\right\}$

Solution

a) Orthogonal

b) None

c) Orthonormal

T3 Let $\mathbf{x}$ and $\mathbf{y}$ be non-zero vectors in $\mathbb{R}^n$.

a) Prove that $\mathbf{y}=c\mathbf{x}+\mathbf{w}$ for some scalar c and some vector $\mathbf{w}$ orthogonal to $\mathbf{x}$.

b) Show that the scalar c and vector $\mathbf{w}$ in part (a) are unique. I.e. show that if $\mathbf{y} = d\mathbf{x} + \mathbf{v}$ where d is a scalar and $\mathbf{v}$ a vector orthogonal to $\mathbf{x}$ then $d = c$ and $\mathbf{v} = \mathbf{w}$. (Hint: compute $\mathbf{x} \cdot \mathbf{y}$).

Solution

a) We know that once c is chosen $\mathbf{w}$ is forced to be $\mathbf{y} - c\mathbf{x}$. Hence we would also want $\mathbf{w} \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{y} - c\|\mathbf{x}\|^2 = 0$. Set $c = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|^2}$, and $\mathbf{w} = \mathbf{y} - c\mathbf{x}$.

b) Assuming we can also write $\mathbf{y} = d\mathbf{x} + \mathbf{v}$, we have that $\mathbf{x} \cdot \mathbf{y} = c\|\mathbf{x}\|^2$ but also $\mathbf{x} \cdot y = d\|\mathbf{x}\|^2$. Hence $c = d$. This also means $\mathbf{v} = \mathbf{y} - d\mathbf{x} = \mathbf{y} - c\mathbf{x} = \mathbf{w}$.

T4 Let U be the subspace of $\mathbb{R}^4$ with the basis

$$(1, 0, 1, 0), \quad (0, 1, -1, 0), \quad (0, 0, 1, 1).$$

Use the Gram-Schmidt process to find an orthonormal basis for U.

Solution

Let

$$\mathbf{y}_1 = (1, 0, 1, 0), \quad \mathbf{y}_2 = (0, 1, -1, 0), \quad \mathbf{y}_3 = (0, 0, 1, 1).$$

Put $\mathbf{x}_1 = \mathbf{y}_1 = (1, 0, 1, 0)$ to get started. Next put

$$\mathbf{x}_2 = \mathbf{y}_2 - \frac{\mathbf{x}_1 \cdot \mathbf{y}_2}{\mathbf{x}_1 \cdot \mathbf{x}_1}\mathbf{x}_1.$$

Then

$$\mathbf{x}_2 = (0, 1, -1, 0) - \frac{-1}{2}(1, 0, 1, 0) = \frac{1}{2}(1, 2, -1, 0).$$

Next put

$$\mathbf{x}_3 = \mathbf{y}_3 - \frac{\mathbf{x}_1 \cdot \mathbf{y}_3}{\mathbf{x}_1 \cdot \mathbf{x}_1}\mathbf{x}_1 - \frac{\mathbf{x}_2 \cdot \mathbf{y}_3}{\mathbf{x}_2 \cdot \mathbf{x}_2}\mathbf{x}_2.$$

Then

$$\begin{aligned}
\mathbf{x}_3 &= (0, 0, 1, 1) - \frac{1}{2}(1, 0, 1, 0) - \frac{-\frac{1}{2}}{\frac{6}{4}}\frac{1}{2}(1, 2, -1, 0) \\
&= (0, 0, 1, 1) - \frac{1}{2}(1, 0, 1, 0) + \frac{1}{6}(1, 2, -1, 0) \\
&= \frac{1}{6}(-2, 2, 2, 6) \\
&= \frac{1}{3}(-1, 1, 1, 3).
\end{aligned}$$

The vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ have lengths

$$\sqrt{(1^2 + 0^2 + 1^2 + 0^2)} = \sqrt{2}, \qquad \frac{1}{2}\sqrt{(1^2 + 2^2 + (-1)^2 + 0^2)} = \frac{1}{2}\sqrt{6}$$

$$\text{and} \qquad \frac{1}{3}\sqrt{((-1)^2 + 1^2 + 1^2 + 3^2)} = \frac{1}{3}\sqrt{12} = \frac{2}{3}\sqrt{3}.$$

So

$$\frac{1}{\sqrt{2}}(1,0,1,0), \quad \frac{1}{\sqrt{6}}(1,2,-1,0), \quad \frac{1}{2\sqrt{3}}(-1,1,1,3)$$

is an orthonormal basis for U.

T5 Let U be the subspace of $\mathbb{R}^4$ spanned by

$$(1,-1,0,0), \quad (1,1,1,-3), \quad (0,1,-1,0), \quad (0,0,1,-1).$$

Find a basis for U and then use the Gram-Schmidt process to find an orthonormal basis for U.

Solution

The Mathematics-2B course included two methods for finding a basis for U. The first method involves writing the given vectors as the columns of a matrix; the second method involves writing the given vectors as the rows of a matrix.

Method 1

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -3 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -3 & 0 & -1 \end{bmatrix} \qquad [R_2 \to R_2 + R_1]$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & -3 & 0 & -1 \end{bmatrix} \qquad [R_2 \leftrightarrow R_3]$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -3 & 2 \end{bmatrix} \qquad \begin{bmatrix} R_3 \to R_3 - 2R_2 \text{ then} \\ R_4 \to R_4 + 3R_2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad [R_4 \to R_4 + R_3]$$

The last matrix is an echelon matrix with leading entries in columns 1, 2 and 3. So the given vectors corresponding to columns 1, 2 and 3 of the first matrix,

$$\text{i.e.} \quad (1,-1,0,0), \quad (1,1,1,-3), \quad (0,1,-1,0)$$

form a basis of U.

Let

$$\mathbf{y}_1 = (1,-1,0,0), \quad \mathbf{y}_2 = (1,1,1,-3), \quad \mathbf{y}_3 = (0,1,-1,0).$$

Put $\mathbf{x}_1 = \mathbf{y}_1 = (1, -1, 0, 0)$. Next put

$$\mathbf{x}_2 = \mathbf{y}_2 - \frac{\mathbf{x}_1 \cdot \mathbf{y}_2}{\mathbf{x}_1 \cdot \mathbf{x}_1} \mathbf{x}_1.$$

Then

$$\mathbf{x}_2 = (1, 1, 1, -3) - \frac{0}{2}(1, -1, 0, 0) = (1, 1, 1, -3).$$

Next put

$$\mathbf{x}_3 = \mathbf{y}_3 - \frac{\mathbf{x}_1 \cdot \mathbf{y}_3}{\mathbf{x}_1 \cdot \mathbf{x}_1} \mathbf{x}_1 - \frac{\mathbf{x}_2 \cdot \mathbf{y}_3}{\mathbf{x}_2 \cdot \mathbf{x}_2} \mathbf{x}_2.$$

Then

$$\begin{aligned}\mathbf{x}_3 &= (0, 1, -1, 0) - \frac{-1}{2}(1, -1, 0, 0) - \frac{0}{12}(1, 1, 1, -3) \\ &= \frac{1}{2}(1, 1, -2, 0).\end{aligned}$$

The vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ have lengths

$$\sqrt{(1^2 + (-1)^2 + 0^2 + 0^2)} = \sqrt{2}, \qquad \sqrt{(1^2 + 1^2 + 1^2 + (-3)^2)} = \sqrt{12} = 2\sqrt{3}$$

$$\text{and} \qquad \frac{1}{2}\sqrt{(1^2 + 1^2 + (-2)^2 + 0^2)} = \frac{1}{2}\sqrt{6}.$$

So

$$\frac{1}{\sqrt{2}}(1, -1, 0, 0), \quad \frac{1}{2\sqrt{3}}(1, 1, 1, -3), \quad \frac{1}{\sqrt{6}}(1, 1, -2, 0)$$

is an orthonormal basis for U.

Method 2

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & -3 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & -3 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad [R_2 \to R_2 - R_1]$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad [R_2 \leftrightarrow R_3]$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad [R_3 \to R_3 - 2R_2]$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad [R_3 \to \tfrac{1}{3}R_3]$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad [R_4 \to R_4 - R_3]$$

The last matrix is an echelon matrix. So

$$(1, -1, 0, 0), \quad (0, 1, -1, 0), \quad (0, 0, 1, -1)$$

is a basis of U.

Let

$$\mathbf{y}_1 = (1, -1, 0, 0), \quad \mathbf{y}_2 = (0, 1, -1, 0), \quad \mathbf{y}_3 = (0, 0, 1, -1).$$

Put $\mathbf{x}_1 = \mathbf{y}_1 = (1, -1, 0, 0)$. Next put

$$\mathbf{x}_2 = \mathbf{y}_2 - \frac{\mathbf{x}_1 \cdot \mathbf{y}_2}{\mathbf{x}_1 \cdot \mathbf{x}_1}\mathbf{x}_1.$$

Then

$$\mathbf{x}_2 = (0, 1, -1, 0) - \frac{-1}{2}(1, -1, 0, 0) = \frac{1}{2}(1, 1, -2, 0).$$

Next put

$$\mathbf{x}_3 = \mathbf{y}_3 - \frac{\mathbf{x}_1 \cdot \mathbf{y}_3}{\mathbf{x}_1 \cdot \mathbf{x}_1}\mathbf{x}_1 - \frac{\mathbf{x}_2 \cdot \mathbf{y}_3}{\mathbf{x}_2 \cdot \mathbf{x}_2}\mathbf{x}_2.$$

Then

$$\begin{aligned} \mathbf{x}_3 &= (0, 0, 1, -1) - \frac{0}{2}(1, -1, 0, 0) - \frac{-1}{\frac{6}{4}}\frac{1}{2}(1, 1, -2, 0) \\ &= (0, 0, 1, -1) + \frac{1}{3}(1, 1, -2, 0) \\ &= \frac{1}{3}(1, 1, 1, -3). \end{aligned}$$

The vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ have lengths

$$\sqrt{(1^2 + (-1)^2 + 0^2 + 0^2)} = \sqrt{2}, \qquad \frac{1}{2}\sqrt{(1^2 + 1^2 + (-2)^2 + 0^2)} = \frac{1}{2}\sqrt{6}$$

$$\text{and} \qquad \frac{1}{3}\sqrt{(1^2 + 1^2 + 1^2 + (-3)^2)} = \frac{1}{3}\sqrt{12} = \frac{2}{3}\sqrt{3}.$$

So

$$\frac{1}{\sqrt{2}}(1, -1, 0, 0), \quad \frac{1}{\sqrt{6}}(1, 1, -2, 0), \quad \frac{1}{2\sqrt{3}}(1, 1, 1, -3)$$

is an orthonormal basis for U. (It is just a rearrangement of the first orthonormal basis we found.)

Additional Remark

What would happen if we applied the Gram-Schmidt process without first finding a basis for U? In order to investigate this, let

$$\mathbf{y}_1 = (1, -1, 0, 0), \quad \mathbf{y}_2 = (1, 1, 1, -3), \quad \mathbf{y}_3 = (0, 1, -1, 0), \quad \mathbf{y}_4 = (0, 0, 1, -1).$$

As before, put $\mathbf{x}_1 = \mathbf{y}_1 = (1, -1, 0, 0)$. Next put

$$\mathbf{x}_2 = \mathbf{y}_2 - \frac{\mathbf{x}_1 \cdot \mathbf{y}_2}{\mathbf{x}_1 \cdot \mathbf{x}_1}\,\mathbf{x}_1.$$

Then

$$\mathbf{x}_2 = (1, 1, 1, -3) - \frac{0}{2}(1, -1, 0, 0) = (1, 1, 1, -3).$$

Next put

$$\mathbf{x}_3 = \mathbf{y}_3 - \frac{\mathbf{x}_1 \cdot \mathbf{y}_3}{\mathbf{x}_1 \cdot \mathbf{x}_1}\,\mathbf{x}_1 - \frac{\mathbf{x}_2 \cdot \mathbf{y}_3}{\mathbf{x}_2 \cdot \mathbf{x}_2}\,\mathbf{x}_2.$$

Then

$$\begin{aligned}\mathbf{x}_3 &= (0, 1, -1, 0) - \frac{-1}{2}(1, -1, 0, 0) - \frac{0}{12}(1, 1, 1, -3)\\ &= \frac{1}{2}(1, 1, -2, 0).\end{aligned}$$

Next put

$$\mathbf{x}_4 = \mathbf{y}_4 - \frac{\mathbf{x}_1 \cdot \mathbf{y}_4}{\mathbf{x}_1 \cdot \mathbf{x}_1}\,\mathbf{x}_1 - \frac{\mathbf{x}_2 \cdot \mathbf{y}_4}{\mathbf{x}_2 \cdot \mathbf{x}_2}\,\mathbf{x}_2 - \frac{\mathbf{x}_3 \cdot \mathbf{y}_4}{\mathbf{x}_3 \cdot \mathbf{x}_3}\,\mathbf{x}_3.$$

Then

$$\begin{aligned}\mathbf{x}_4 &= (0, 0, 1, -1) - \frac{0}{2}(1, -1, 0, 0) - \frac{4}{12}(1, 1, 1, -3) - \frac{-1}{\frac{6}{4}}\,\frac{1}{2}(1, 1, -2, 0)\\ &= (0, 0, 1, -1) - \frac{1}{3}(1, 1, 1, -3) + \frac{1}{3}(1, 1, -2, 0)\\ &= (0, 0, 0, 0).\end{aligned}$$

So the Gram-Schmidt process fails to extend the orthogonal list of non-zero vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ we have already produced. The process failed because $\mathbf{y}_4$ is in the subspace of U spanned by $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, contrary to the requirements for the choice of the vector $\mathbf{y}_4$ to continue the process. (It is impossible to extend the list $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ since it is already a basis of U.) We found that

$$\mathbf{y}_4 = \frac{\mathbf{x}_1 \cdot \mathbf{y}_4}{\mathbf{x}_1 \cdot \mathbf{x}_1}\,\mathbf{x}_1 + \frac{\mathbf{x}_2 \cdot \mathbf{y}_4}{\mathbf{x}_2 \cdot \mathbf{x}_2}\,\mathbf{x}_2 + \frac{\mathbf{x}_3 \cdot \mathbf{y}_4}{\mathbf{x}_3 \cdot \mathbf{x}_3}\,\mathbf{x}_3.$$

T6 Which of the following matrices are orthogonal. For those that are, find their inverses.

a) $\begin{pmatrix} 0 & 0 & -1 \\ 0 & \sqrt{5} & 0 \\ \pi & 0 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

c) Let $a = \sqrt{4 + \sin^2(\theta)}$, and $b = \sqrt{\cos^2(\theta)\sin^2(\theta) + 1}$ and consider $\begin{pmatrix} \frac{2}{a} & \frac{\cos(\theta)\sin(\theta)}{b} \\ \frac{\sin(2\theta)}{a} & \frac{-1}{b} \end{pmatrix}$

Solution

a) Not orthogonal

b) Orthogonal, and the transpose is the inverse

c) Orthogonal, and the transpose is the inverse

T7 Let Q be an orthogonal $n \times n$ matrix. What is $\text{col}(A)$?

Solution

Since Q is orthogonal, the columns are orthonormal and hence form a basis for $\mathbb{R}^n$. Thus $\text{col}(A) = \mathbb{R}^n$.

T8 Let Q be an orthogonal $n \times n$ matrix. Show that for any n by n matrix A, there exists an n by n matrix B such that $A = BQ$.

Solution

We have $Q^TQ = I$. Let $B = AQ^T$. Then $BQ = AQ^TQ = AI = A$.

T9 Let U be a subspace of $\mathbb{R}^n$. The orthogonal complement of U is the set $U^\perp$ of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in U, i.e.

$$U^\perp = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \cdot \mathbf{u} = 0 \text{ for all } \mathbf{u} \in U\}.$$

Prove that $U^\perp$ is a subspace of $\mathbb{R}^n$ and $U^\perp \cap U = \{0\}$.

Solution

First $\mathbf{0} \in U^\perp$ since, for all $\mathbf{u} \in U$,

$$\mathbf{0} \cdot \mathbf{u} = 0.$$

Now suppose that $\mathbf{v}, \mathbf{w} \in U^\perp$ and α is a scalar. Then, for all $\mathbf{u} \in U$,

$$\mathbf{v} \cdot \mathbf{u} = \mathbf{w} \cdot \mathbf{u} = 0.$$

Therefore

$$(\alpha\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \alpha\mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u} = (\alpha \times 0) + 0 = 0.$$

So $\alpha\mathbf{v} + \mathbf{w} \in U^\perp$. Hence $U^\perp$ is a subspace of V.

Now suppose that $\mathbf{u} \in U \cap U^\perp$. Then

$$\mathbf{u} \cdot \mathbf{u} = 0.$$

Therefore

$$\mathbf{u} = \mathbf{0}.$$

Hence

$$U \cap U^\perp = \{\mathbf{0}\}.$$

T10 Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be an orthonormal basis for $\mathbb{R}^n$. Show that, for all vectors $\mathbf{v} \in \mathbb{R}^n$,

$$\mathbf{v} = \sum_{j=1}^{n} (\mathbf{v} \cdot \mathbf{e}_j)\mathbf{e}_j.$$

Solution

Let $\mathbf{v} \in V$. Since $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ is a basis for V, we can find scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\mathbf{v} = \sum_{j=1}^{n} \alpha_j \mathbf{e}_j.$$

Then, for $k = 1, 2, \ldots, n$,

$$\begin{aligned} \mathbf{v} \cdot \mathbf{e}_k &= \Big(\sum_{j=1}^{n} \alpha_j \mathbf{e}_j\Big) \cdot \mathbf{e}_k \\ &= \sum_{j=1}^{n} \alpha_j \mathbf{e}_j \cdot \mathbf{e}_k \\ &= \sum_{j=1}^{n} \alpha_j (\mathbf{e}_j \cdot \mathbf{e}_k) \\ &= \alpha_k. && [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \text{ is an orthonormal list}] \end{aligned}$$

Hence

$$\mathbf{v} = \sum_{j=1}^{n} (\mathbf{v} \cdot \mathbf{e}_j)\mathbf{e}_j.$$

Suppose that we were given an orthogonal basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ instead of an orthonormal one. Then, we would have obtained

$$\mathbf{v} = \sum_{j=1}^{n} \frac{\mathbf{v} \cdot \mathbf{e}_j}{\mathbf{e}_j \cdot \mathbf{e}_j} \mathbf{e}_j.$$

T11 Let A and B be orthogonal $n \times n$ matrices such that $A^T B + B^T A = -I$. Show that $A + B$ is orthogonal.

Solution

$(A+B)^T(A+B) = A^T A + B^T B + A^T B + B^T A = I + I - I = I.$

T12 Let U be a subspace of $\mathbb{R}^n$, and $U^\perp$ the orthogonal complement (as defined in T9). Let $\mathbf{x}$ be some vector in $\mathbb{R}^n$.

a) Use Gram-Schmidt to show that there is a basis $\{\mathbf{f}_1, \mathbf{f}_2, \ldots \mathbf{f}_n\}$ for $\mathbb{R}^n$ such that $\{\mathbf{f}_1, \ldots, \mathbf{f}_m\}$ belong to U and $\{\mathbf{f}_{m+1}, \ldots, \mathbf{f}_n\}$ belong to $U^\perp$.

b) Prove that there is a vector $\mathbf{y} \in U$ and $\mathbf{z} \in U^{\perp}$ such that $\mathbf{x} = \mathbf{y} + \mathbf{z}$.

c) Prove that this choice of $\mathbf{y}$ and $\mathbf{z}$ is unique.

Solution

a) If U is $\{\mathbf{0}\}$ or $\mathbb{R}^n$ (then $U^{\perp} = \mathbb{R}^n$ or $\{\mathbf{0}\}$ respectively), the problem is easy. Therefore we can assume that we are not is such a case and so we may assume U has a basis $\{\mathbf{f}_1, \mathbf{f}_2, \ldots \mathbf{f}_m\}$, where $1 < m < n$. Using Gram-Schmidt we may produce orthogonal vectors $\mathbf{f}_{m+1}, \mathbf{f}_{m+2} \ldots, \mathbf{f}_n$ such that these are orthogonal to all the f_i's, $1 \leq i \leq m$, and hence belong to $U^{\perp}$, and such that $\{\mathbf{f}_1, \mathbf{f}_2, \ldots \mathbf{f}_m, \mathbf{f}_{m+1}, f_{m+2} \ldots, \mathbf{f}_n\}$ forms a basis for $\mathbb{R}^n$.

b) Given our basis, we have $\mathbf{x} = \alpha_1 \mathbf{f}_1 + \ldots \alpha_n \mathbf{f}_n$ for some scalars $\alpha_1, \ldots, \alpha_n$. Let $\mathbf{y} = \alpha_1 \mathbf{f}_1 + \ldots + \alpha_m \mathbf{f}_m$, and $\mathbf{z} = \alpha_{m+1} \mathbf{f}_{m+1} + \ldots \alpha_n \mathbf{f}_n$, then $\mathbf{x} = \mathbf{y} + \mathbf{z}$ where $\mathbf{y}$ and $\mathbf{z}$ have the required properties.

c) Assume there was another way of writing $\mathbf{x} = \overline{\mathbf{y}} + \overline{\mathbf{z}}$ where $\overline{\mathbf{y}} \in U$ and $\overline{\mathbf{z}} \in U^{\perp}$. Then $\mathbf{0} = \mathbf{x} - \mathbf{x} = (\mathbf{y} + \mathbf{z}) - (\overline{\mathbf{y}} + \overline{\mathbf{z}}) = (\mathbf{y} - \overline{\mathbf{y}}) + (\mathbf{z} - \overline{\mathbf{z}})$. Let $\mathbf{Y} = \mathbf{y} - \overline{\mathbf{y}} \in U$ and $\mathbf{Z} = \mathbf{z} - \overline{\mathbf{z}} \in U^{\perp}$. Then $\mathbf{0} = \mathbf{Y} + \mathbf{Z}$ and so $\mathbf{0} \cdot \mathbf{Y} = 0 = \mathbf{Y} \cdot \mathbf{Y} + \mathbf{Z} \cdot \mathbf{Y} = \|\mathbf{Y}\|^2$. Hence $\mathbf{Y} = \mathbf{0}$, and thus $\mathbf{Z} = \mathbf{0}$. This implies $\mathbf{y} = \overline{\mathbf{y}}$ and $\mathbf{z} = \overline{\mathbf{z}}$.

T13 A permutation f on the set $\{1, 2, \ldots, n\}$ is a bijection

$$f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}.$$

Let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$. Assume there is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ which satisfies $T(\mathbf{e}_i) = \mathbf{e}_{f(i)}, 1 \leq i \leq n$, and let A be the corresponding matrix associated with the transformation.

a) Describe, in general, what the matrix A looks like. (Hint: come up with simple permutations to get a feel for what the matrix is).

b) Hence show that A is orthogonal.

Solution

a) By considering how T acts on the standard unit vectors, we see that it shuffles them around (and the way the shuffle is performed depends on what the permutation is). Hence A looks like the identity matrix but with the columns shuffled around.

b) The columns of A are the standard unit vectors, hence orthogonal.

2B Linear Algebra

Ex Sheet 9

1 True/False

a) If an n by n matrix (not necessarily real) has an imaginary eigenvalue it cannot be orthogonally diagonalized.

b) If a real n by n matrix is orthogonally diagonalizable, then all its eigenvalues are real.

c) If the eigenvalues of a real n by n matrix are real, then the matrix is orthogonally diagonalizable.

d) The product of an n by n real matrix with its transpose is orthogonally diagonalizable.

e) A real orthogonal matrix with inverse itself will be orthogonally diagonalizable.

f) Upper triangular real matrices are always orthogonally diagonalizable.

g) If A is orthogonally diagonalized via an orthogonal matrix Q, then A^{2020} is orthognally diagonalized via Q as well.

h) The zero matrix is orthognally diagonalizable.

i) If A and B are real n by n matrices which are orthogonally diagonalizable, then so is AB.

j) $(x_1 - 3x_2)^2$ is a quadratic form

k) $q(x, y) = xy$ is not a quadratic form because it has no x^2 or y^2 terms

l) If $q(\mathbf{x}) = \mathbf{x}^T C\mathbf{x}$ is a quadratic form, and $A = \frac{1}{2}(C + C^T)$, then $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$

[1] **True/False Questions**

Every Exercise Sheet will have a section containing true/false questions. They are designed to test your understanding from lectures. You should look at your lecture notes and/or the textbook to help you answer these questions, but you should not need to write anything to work out the solution.

Solutions to True/False

a) F b) T c) F d) T e) T f) F g) T h) T i) F j) T k) F l) T

Tutorial Exercises

T1 Let

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}.$$

Find an orthogonal matrix Q and a diagonal matrix D such that

$$Q^{\mathrm{T}} A Q = D.$$

Solution

$$\chi_A(t) = \begin{vmatrix} t-4 & -2 \\ -2 & t-7 \end{vmatrix} = (t-4)(t-7) - 4 = t^2 - 11t + 24 = (t-3)(t-8).$$

So 8 and 3 are the eigenvalues of A.

$$(8I - A)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff 2x - y = 0.$$

Therefore

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ is an eigenvector corresponding to 8.}$$

$$(3I - A)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff x + 2y = 0.$$

Therefore

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ is an eigenvector corresponding to 3.}$$

The eigenvectors

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

have lengths $\sqrt{(1^2 + 2^2)} = \sqrt{5}$ and $\sqrt{(2^2 + (-1)^2)} = \sqrt{5}$, respectively. So let

$$Q = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$$

Then Q is orthogonal and $Q^{\mathrm{T}}AQ = \mathrm{diag}(8, 3)$.

T2 Let

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix}.$$

Find an orthogonal matrix Q and a diagonal matrix D such that

$$Q^{\mathrm{T}}AQ = D.$$

Solution

$$\chi_A(t) = \begin{vmatrix} t-1 & -1 & -3 \\ -1 & t-3 & -1 \\ -3 & -1 & t-1 \end{vmatrix} = \begin{vmatrix} t-5 & t-5 & t-5 \\ -1 & t-3 & -1 \\ -3 & -1 & t-1 \end{vmatrix} \qquad \begin{bmatrix} R_1 \to R_1 + R_2 \text{ then} \\ R_1 \to R_1 + R_3 \end{bmatrix}$$

$$= (t-5)\begin{vmatrix} 1 & 1 & 1 \\ -1 & t-3 & -1 \\ -3 & -1 & t-1 \end{vmatrix}$$

$$= (t-5)\begin{vmatrix} 1 & 0 & 0 \\ -1 & t-2 & 0 \\ -3 & 2 & t+2 \end{vmatrix} \qquad \begin{bmatrix} C_2 \to C_2 - C_1 \\ \text{then} \\ C_3 \to C_3 - C_1 \end{bmatrix}$$

$$= (t-5)(t-2)(t+2).$$

So 5, 2 and -2 are the eigenvalues of A.

$$(5I - A)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 4 & -1 & -3 \\ -1 & 2 & -1 \\ -3 & -1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{cases} x \quad - z = 0, \\ \quad y - z = 0. \end{cases}$$

Therefore

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ is an eigenvector corresponding to 5.}$$

$$(2I - A)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -1 \\ -3 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{cases} x \quad - \quad z = 0, \\ \quad y + 2z = 0. \end{cases}$$

Therefore

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \text{ is an eigenvector corresponding to 2.}$$

$$((-2)I - A)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} -3 & -1 & -3 \\ -1 & -5 & -1 \\ -3 & -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\iff \begin{cases} x \quad + z = 0, \\ \quad y \qquad = 0. \end{cases}$$

Therefore

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ is an eigenvector corresponding to } -2.$$

The eigenvectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

have lengths

$$\sqrt{(1^2 + 1^2 + 1^2)} = \sqrt{3}, \qquad \sqrt{(1^2 + (-2)^2 + 1^2)} = \sqrt{6}$$
$$\text{and} \qquad \sqrt{(1^2 + 0^2 + (-1)^2)} = \sqrt{2},$$

respectively. So let

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then Q is orthogonal and $Q^{\mathrm{T}}AQ = \mathrm{diag}(5, 2, -2)$.

T3 Let

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Find an orthogonal matrix Q and a diagonal matrix D such that

$$Q^{\mathrm{T}}AQ = D.$$

Solution

$$\chi_A(t) = \begin{vmatrix} t & -1 & -1 \\ -1 & t & -1 \\ -1 & -1 & t \end{vmatrix} = \begin{vmatrix} t-2 & t-2 & t-2 \\ -1 & t & -1 \\ -1 & -1 & t \end{vmatrix} \qquad \begin{bmatrix} R_1 \to R_1 + R_2 \text{ then} \\ R_1 \to R_1 + R_3 \end{bmatrix}$$

$$= (t-2)\begin{vmatrix} 1 & 1 & 1 \\ -1 & t & -1 \\ -1 & -1 & t \end{vmatrix}$$

$$= (t-2)\begin{vmatrix} 1 & 0 & 0 \\ -1 & t+1 & 0 \\ -1 & 0 & t+1 \end{vmatrix} \qquad \begin{bmatrix} C_2 \to C_2 - C_1 \text{ then} \\ C_3 \to C_3 - C_1 \end{bmatrix}$$

$$= (t-2)(t+1)^2.$$

So 2 and -1 are the eigenvalues of A.

$$(2I - A)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{cases} x \quad - z = 0, \\ \quad y - z = 0. \end{cases}$$

Therefore

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ is an eigenvector corresponding to 2.}$$

$$((-1)I - A)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\iff x + y + z = 0.$$

Therefore

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \text{ is an eigenvector corresponding to } -1.$$

We must find a second eigenvector corresponding to -1 which is orthogonal to the first one, i.e. a column $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that

$$x + y + z = 0 \qquad \text{and} \qquad x - y = 0.$$

Therefore

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \text{ is a suitable second eigenvector corresponding to } -1.$$

The eigenvectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

have lengths

$$\sqrt{(1^2 + 1^2 + 1^2)} = \sqrt{3}, \qquad \sqrt{(1^2 + (-1)^2 + 0^2)} = \sqrt{2}$$

$$\text{and} \qquad \sqrt{(1^2 + 1^2 + (-2)^2)} = \sqrt{6},$$

respectively. So let

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}.$$

Then Q is orthogonal and $Q^{\mathrm{T}}AQ = \operatorname{diag}(2, -1, -1)$.

T4 Let

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}.$$

Find an orthogonal matrix Q and a diagonal matrix D such that

$$Q^{\mathrm{T}}AQ = D.$$

Solution

$$\chi_A(t) = \begin{vmatrix} t-1 & 1 & 1 & 1 \\ 1 & t-1 & 1 & 1 \\ 1 & 1 & t-1 & 1 \\ 1 & 1 & 1 & t-1 \end{vmatrix} = \begin{vmatrix} t+2 & t+2 & t+2 & t+2 \\ 1 & t-1 & 1 & 1 \\ 1 & 1 & t-1 & 1 \\ 1 & 1 & 1 & t-1 \end{vmatrix}$$

$$\begin{bmatrix} R_1 \to R_1 + R_2 \text{ then} \\ R_1 \to R_1 + R_3 \text{ then} \\ R_1 \to R_1 + R_4 \end{bmatrix}$$

$$= (t+2)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & t-1 & 1 & 1 \\ 1 & 1 & t-1 & 1 \\ 1 & 1 & 1 & t-1 \end{vmatrix}$$

$$= (t+2)\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & t-2 & 0 & 0 \\ 1 & 0 & t-2 & 0 \\ 1 & 0 & 0 & t-2 \end{vmatrix} \quad \begin{bmatrix} C_2 \to C_2 - C_1 \text{ then} \\ C_3 \to C_3 - C_1 \text{ then} \\ C_4 \to C_4 - C_1 \end{bmatrix}$$

$$= (t+2)(t-2)^3.$$

So 2 and -2 are the eigenvalues of A.

$$\big((-2)I - A\big) = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} -4 & 0 & 0 & 4 \\ 0 & -4 & 0 & 4 \\ 0 & 0 & -4 & 4 \\ 1 & 1 & 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} R_1 \to R_1 - R_4 \text{ then} \\ R_2 \to R_2 - R_4 \text{ then} \\ R_3 \to R_3 - R_4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} R_1 \to -\tfrac{1}{4}R_1 \text{ then} \\ R_2 \to -\tfrac{1}{4}R_2 \text{ then} \\ R_3 \to -\tfrac{1}{4}R_3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} R_4 \to R_4 - R_1 \text{ then} \\ R_4 \to R_4 - R_2 \text{ then} \\ R_4 \to R_4 - R_3 \end{bmatrix}$$

So

$$((-2)I - A)\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{cases} w \quad\quad - z = 0, \\ \quad x \quad - z = 0, \\ \quad\quad y - z = 0. \end{cases}$$

Therefore

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ is an eigenvector corresponding to } -2.$$

$$(2I - A)\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\iff w + x + y + z = 0.$$

Therefore

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \text{ is an eigenvector corresponding to 2.}$$

We must find a second eigenvector corresponding to 2 which is orthogonal to the first one, i.e. a column $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$ such that

$$w + x + y + z = 0 \qquad \text{and} \qquad w - x = 0.$$

Therefore

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \text{ is a suitable second eigenvector corresponding to 2.}$$

Next we must find a third eigenvector corresponding to 2 which is orthogonal to the first two, i.e. a column $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$ such that

$$w + x + y + z = 0, \qquad w - x = 0 \qquad \text{and} \qquad y - z = 0.$$

Therefore

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \text{ is a suitable third eigenvector corresponding to 2.}$$

The eigenvectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

have lengths

$$\sqrt{(1^2+1^2+1^2+1^2)} = 2, \qquad \sqrt{(1^2+(-1)^2+0^2+0^2)} = \sqrt{2},$$

$$\sqrt{(0^2+0^2+1^2+(-1)^2)} = \sqrt{2} \qquad \text{and} \qquad \sqrt{(1^2+1^2+(-1)^2+(-1)^2)} = 2,$$

respectively. So let

$$Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix}.$$

Then Q is orthogonal and $Q^{\mathrm{T}}AQ = \operatorname{diag}(-2, 2, 2, 2)$.

T5 Find the eigenvalues of the orthogonal matrix

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Solution

$$\begin{aligned} \chi_Q(t) = \begin{vmatrix} t-\cos\theta & \sin\theta \\ -\sin\theta & t-\cos\theta \end{vmatrix} &= (t-\cos\theta)^2 + \sin^2\theta \\ &= (t-\cos\theta)^2 - i^2\sin^2\theta \\ &= (t-\cos\theta - i\sin\theta)(t-\cos\theta + i\sin\theta) \\ &= \left(t-e^{i\theta}\right)\left(t-e^{-i\theta}\right). \end{aligned}$$

So $e^{i\theta}$ and $e^{-i\theta}$ are the eigenvalues of Q. (Recall that the matrix transformation determined by Q is rotation through θ radians.)

T6 Let α be an imaginary eigenvalue of an orthogonal matrix Q and let $\mathbf{x}$ be a corresponding eigenvector. Prove that $\mathbf{x}^{\mathrm{T}}\mathbf{x} = 0$.

Solution

$Q\mathbf{x} = \alpha\mathbf{x}$. So

$$\mathbf{x}^{\mathrm{T}} Q^{\mathrm{T}} Q\mathbf{x} = \mathbf{x}^{\mathrm{T}} I\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{x}$$

and

$$\mathbf{x}^{\mathrm{T}} Q^{\mathrm{T}} Q\mathbf{x} = (Q\mathbf{x})^{\mathrm{T}} Q\mathbf{x} = (\alpha\mathbf{x})^{\mathrm{T}}(\alpha\mathbf{x}) = \alpha^2\mathbf{x}^{\mathrm{T}}\mathbf{x}.$$

Therefore

$$\begin{aligned} \alpha^2\mathbf{x}^{\mathrm{T}}\mathbf{x} &= \mathbf{x}^{\mathrm{T}}\mathbf{x}, \\ \text{i.e.} \quad (\alpha^2 - 1)\mathbf{x}^{\mathrm{T}}\mathbf{x} &= 0, \\ \text{i.e.} \quad (\alpha - 1)(\alpha + 1)\mathbf{x}^{\mathrm{T}}\mathbf{x} &= 0. \end{aligned}$$

But $\alpha \neq \pm 1$. Hence $\mathbf{x}^{\mathrm{T}}\mathbf{x} = 0$.

(Note that $\mathbf{x}$ must be a non-zero complex column matrix and we know that $\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x} > 0$.)

T7 Let Q be an $n \times n$ orthogonal matrix, where n is an odd positive integer. Show that 1 or -1 is an eigenvalue of Q.

Solution

$\chi_Q(t)$ is a real polynomial. So its imaginary roots occur as pairs of complex conjugates. (See the Level-1 courses.) Since $\chi_Q(t)$ has an odd degree, it must therefore have a real root. But 1 and -1 are the only real numbers that can be eigenvalues of an orthogonal matrix. Hence 1 or -1 is an eigenvalue of Q.

T8 Let A be a real skew-symmetric matrix, i.e. a real matrix such that $A^{\mathrm{T}} = -A$. Prove that the eigenvalues of A have the form $i\alpha$ for some real number α.

Solution

Let λ be an eigenvalue of the real skew-symmetric matrix A and let $\mathbf{x}$ be an eigenvector of A corresponding to λ. Then

$$\overline{\mathbf{x}}^{\mathrm{T}} A\mathbf{x} = \overline{\mathbf{x}}^{\mathrm{T}}(\lambda\mathbf{x}) = \lambda\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x}$$

and, since $\overline{A}^{\mathrm{T}} = A^{\mathrm{T}} = -A$,

$$\overline{\mathbf{x}}^{\mathrm{T}} A = -\overline{\mathbf{x}}^{\mathrm{T}}\overline{A}^{\mathrm{T}} = -\left(\overline{A}\,\overline{\mathbf{x}}\right)^{\mathrm{T}} = -\overline{(A\mathbf{x})}^{\mathrm{T}} = -\overline{(\lambda\mathbf{x})}^{\mathrm{T}} = -\left(\overline{\lambda}\,\overline{\mathbf{x}}\right)^{\mathrm{T}} = -\overline{\lambda}\,\overline{\mathbf{x}}^{\mathrm{T}}.$$

So

$$\overline{\mathbf{x}}^{\mathrm{T}} A\mathbf{x} = -\overline{\lambda}\,\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x}.$$

Therefore

$$\begin{aligned} \lambda\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x} &= -\overline{\lambda}\,\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x}, \\ \text{i.e.} \quad (\lambda + \overline{\lambda})\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x} &= 0, \\ \text{i.e.} \quad 2\,\mathrm{Re}(\lambda)\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x} &= 0. \end{aligned}$$

But $\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x} \neq 0$ since $\mathbf{x} \neq \mathbf{0}$. So $\mathrm{Re}(\lambda) = 0$, i.e. $\lambda = i\alpha$ for some real number α.

T9 Find the eigenvalues of the skew-symmetric matrices

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -2 \\ -1 & 2 & 0 \end{bmatrix}.$$

Solution

$$\chi_A(t) = \begin{vmatrix} t & -2 \\ 2 & t \end{vmatrix} = t^2 + 4 = (t-2i)(t+2i).$$

So $2i$ and $-2i$ are the eigenvalues of A.

$$\begin{aligned} \chi_B(t) = \begin{vmatrix} t & -2 & -1 \\ 2 & t & 2 \\ 1 & -2 & t \end{vmatrix} &= t\begin{vmatrix} t & 2 \\ -2 & t \end{vmatrix} + 2\begin{vmatrix} 2 & 2 \\ 1 & t \end{vmatrix} - 1\begin{vmatrix} 2 & t \\ 1 & -2 \end{vmatrix} \\ &= t(t^2+4) + 2(2t-2) - (-4-t) \\ &= t(t^2+9) \\ &= t(t-3i)(t+3i). \end{aligned}$$

So 0, $3i$ and $-3i$ are the eigenvalues of B.

T10 Let A be an $n \times n$ real skew-symmetric matrix, where n is an odd positive integer. Show that 0 is an eigenvalue of A. Could 0 be an eigenvalue of A if n were even?

Solution

$\chi_A(t)$ is a real polynomial. So its imaginary roots occur as pairs of complex conjugates. (See the Level-1 courses.) Since $\chi_A(t)$ has an odd degree, it must therefore have a real root. This must be 0 because 0 is the only real number that can be an eigenvalue of a real skew-symmetric matrix. Hence 0 is an eigenvalue of A.

Alternatively, observe that

$$\det A = \det(A^{\mathrm{T}}) = \det(-A) = (-1)^n \det A = -\det A$$

since n is odd. Therefore $\det A = 0$. Since $\det A$ is the product of the eigenvalues of A, 0 must be an eigenvalue of A. (To see this, observe that $\det(-A)$ is the constant term of the characteristic polynomial $\chi_A(t)$.)

An $n \times n$ real skew-symmetric matrix with n even could have 0 as an eigenvalue; $O_{2,2}$ is the only 2×2 real skew-symmetric matrix which has 0 as an eigenvalue.

T11 Let λ, μ be distinct eigenvalues of a real skew-symmetric matrix A and let $\mathbf{x}$, $\mathbf{y}$ be eigenvectors of A corresponding to λ, μ, respectively. Prove that $\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{y} = 0$.

Solution

$A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \mu\mathbf{y}$. Then

$$\overline{\mathbf{x}}^{\mathrm{T}}A\mathbf{y} = \overline{\mathbf{x}}^{\mathrm{T}}(\mu\mathbf{y}) = \mu\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{y}$$

and, since $\overline{A}^{\mathrm{T}} = A^{\mathrm{T}} = -A$,

$$\overline{\mathbf{x}}^{\mathrm{T}}A = -\overline{\mathbf{x}}^{\mathrm{T}}\overline{A}^{\mathrm{T}} = -\left(\overline{A}\,\overline{\mathbf{x}}\right)^{\mathrm{T}} = -\overline{(A\mathbf{x})}^{\mathrm{T}} = -\overline{(\lambda\mathbf{x})}^{\mathrm{T}} = -\left(\overline{\lambda}\,\overline{\mathbf{x}}\right)^{\mathrm{T}} = -\overline{\lambda}\,\overline{\mathbf{x}}^{\mathrm{T}} = \lambda\overline{\mathbf{x}}^{\mathrm{T}}$$

because $\overline{\lambda} = -\lambda$ by T9. So

$$\overline{\mathbf{x}}^{\mathrm{T}}A\mathbf{y} = \lambda\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{y}.$$

Therefore

$$\lambda\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{y} = \mu\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{y},$$

$$\text{i.e.}\quad (\lambda-\mu)\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{y} = 0.$$

But $\lambda \neq \mu$. Therefore $\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{y} = 0$.

T12 Let

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 10 \end{bmatrix}.$$

Find an invertible matrix P such that $P^{\mathrm{T}}AP = I$. Deduce that $A = B^{\mathrm{T}}B$ for some invertible matrix B.

Solution

Let $q = \mathbf{x}^{\mathrm{T}}A\mathbf{x}$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Then

$$\begin{aligned} q &= 2x_1^2 + 10x_2^2 + 8x_1x_2 \\ &= 2\left[x_1^2 + 4x_1x_2\right] + 10x_2^2 \\ &= 2\left[(x_1+2x_2)^2 - 4x_2^2\right] + 10x_2^2 \\ &= 2(x_1+2x_2)^2 + 2x_2^2 \\ &= \left(\sqrt{2}\,x_1 + 2\sqrt{2}\,x_2\right)^2 + \left(\sqrt{2}\,x_2\right)^2 \\ &= y_1^2 + y_2^2, \end{aligned}$$

where

$$\begin{aligned} y_1 &= \sqrt{2}\,x_1 + 2\sqrt{2}\,x_2, \\ y_2 &= \sqrt{2}\,x_2. \end{aligned}$$

Then

$$\begin{aligned} x_2 &= \frac{1}{\sqrt{2}} y_2, \\ x_1 &= \frac{1}{\sqrt{2}} (y_1 - 2\sqrt{2}\, x_2) = \frac{1}{\sqrt{2}} y_1 - \frac{2}{\sqrt{2}} y_2 \end{aligned}$$

and so

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{2}} y_1 - \frac{2}{\sqrt{2}} y_2, \\ x_2 &= \phantom{\frac{1}{\sqrt{2}} y_1 -{}} \frac{1}{\sqrt{2}} y_2. \end{aligned}$$

Let

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

Then P is invertible and

$$P^{\mathrm{T}} A P = \mathrm{diag}(1, 1) = I.$$

Therefore

$$A = \left(P^{\mathrm{T}}\right)^{-1} I P^{-1} = \left(P^{-1}\right)^{\mathrm{T}} P^{-1} = B^{\mathrm{T}} B,$$

where

$$B = P^{-1} = \sqrt{2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

T13 Let A be a real symmetric matrix whose eigenvalues are all positive. Show that $A = B^{\mathrm{T}} B$ for some invertible matrix B.

Solution

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A (including any repetitions). We can find an orthogonal matrix Q such that

$$Q^{\mathrm{T}} A Q = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = D^2,$$

where

$$D = \mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n}).$$

Then

$$A = Q D^2 Q^{\mathrm{T}} = (QD)(QD)^{\mathrm{T}} = B^{\mathrm{T}} B,$$

where $B = (QD)^{\mathrm{T}}$, and B is invertible because both Q and D are invertible.

Alternatively, let q be the quadratic form defined by

$$q = \mathbf{x}^{\mathrm{T}} A \mathbf{x},$$

as in the previous example. Use the matrices Q and D introduced above. Under the nonsingular change of variables $\mathbf{x} = Q\mathbf{y}$,

$$\begin{aligned} q = (Q\mathbf{y})^{\mathrm{T}} A (Q\mathbf{y}) &= \mathbf{y}^{\mathrm{T}} Q^{\mathrm{T}} A Q \mathbf{y} \\ &= \mathbf{y}^{\mathrm{T}} \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \mathbf{y} \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2, \end{aligned}$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

So

$$\begin{aligned} q &= \left(\sqrt{\lambda_1}\, y_1\right)^2 + \left(\sqrt{\lambda_2}\, y_2\right)^2 + \cdots + \left(\sqrt{\lambda_n}\, y_n\right)^2 \\ &= z_1^2 + z_2^2 + \cdots + z_n^2, \end{aligned}$$

where

$$\begin{aligned} z_1 &= \sqrt{\lambda_1}\, y_1, \\ z_2 &= \qquad \sqrt{\lambda_2}\, y_2, \\ \vdots & \qquad\qquad\qquad \ddots \\ z_n &= \qquad\qquad\qquad \sqrt{\lambda_n}\, y_n, \end{aligned}$$

$$\text{i.e.} \quad \mathbf{z} = D\mathbf{y},$$

where

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}.$$

Then

$$\mathbf{y} = D^{-1}\mathbf{z}.$$

Now let $P = QD^{-1}$. Then $\mathbf{x} = Q\mathbf{y} = P\mathbf{z}$ is a nonsingular change of variables such that

$$P^{\mathrm{T}}AP = \operatorname{diag}(1, 1, \ldots, 1) = I.$$

Therefore

$$A = \left(P^{\mathrm{T}}\right)^{-1} I P^{-1} = \left(P^{-1}\right)^{\mathrm{T}} P^{-1} = B^{\mathrm{T}}B,$$

where

$$B = P^{-1}.$$

Observe that

$$B = \left(QD^{-1}\right)^{-1} = DQ^{-1} = D^{\mathrm{T}}Q^{\mathrm{T}} = (QD)^{\mathrm{T}},$$

as in the first method.

T14 Let A and B be real $n \times n$ matrices. We say that A is congruent to B if $P^{\mathrm{T}}AP = B$ for some invertible matrix P. Deduce that, in this case, B is also congruent to A. So we can simply say that A and B are congruent.

Solution

Suppose that A is congruent to B. Then

$$P^{\mathrm{T}}AP = B$$

for some invertible matrix P. So P^{-1} is also invertible and

$$\left(P^{-1}\right)^{\mathrm{T}}BP^{-1} = \left(P^{\mathrm{T}}\right)^{-1}BP^{-1} = A.$$

Therefore B is congruent to A.

T15 Suppose that the quadratic form q in n variables can be defined by both

$$q(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$$

and

$$q(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}x_ix_j,$$

where a_{ij} and b_{ij} $(i, j = 1, 2, \ldots, n)$ are real numbers such that $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$. By choosing suitable values for the variables, show that $a_{ij} = b_{ij}$ for $i, j = 1, 2, \ldots, n$. (Hint: first show that $a_{11} = b_{11}$.)

Solution

Put $x_1 = 1$ and $x_i = 0$ for $i = 2, \ldots, n$. Then $q = a_{11}$ and $q = b_{11}$. Therefore $a_{11} = b_{11}$. Similarly, $a_{ii} = b_{ii}$ for $i = 2, \ldots, n$.

Next, provided $n > 1$, put $x_1 = x_2 = 1$ and $x_i = 0$ for $i = 3, \ldots, n$. Then $q = a_{11} + a_{22} + 2a_{12}$ and $q = b_{11} + b_{22} + 2b_{12}$. Therefore

$$a_{11} + a_{22} + 2a_{12} = b_{11} + b_{22} + 2b_{12}.$$

By the first part, $2a_{12} = 2b_{12}$, i.e. $a_{12} = b_{12}$. Similarly $a_{ij} = b_{ij}$ for all other relevant values of i and j with $i < j$.

T16 Write down the matrix of each of the following quadratic forms:

(i) $q(x_1, x_2, x_3) = 2x_1^2 + 3x_2^2 - 4x_3^2 + 10x_1x_2 + 16x_2x_3$,

(ii) $q(x_1, x_2, x_3) = x_1x_2 + x_1x_3 - x_2x_3$,

(iii) $q(x_1, x_2, x_3, x_4) = x_1^2 - 2x_2x_3$.

Solution

$$\textbf{(i)} \begin{bmatrix} 2 & 5 & 0 \\ 5 & 3 & 8 \\ 0 & 8 & -4 \end{bmatrix} \qquad \textbf{(ii)} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}. \qquad \textbf{(iii)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

T17 Write down the formula for the quadratic form $q(x_1, x_2, x_3)$ which has the matrix

$$\begin{bmatrix} 2 & -1 & 2 \\ -1 & 0 & 4 \\ 2 & 4 & -5 \end{bmatrix}.$$

Solution

$$q(x_1, x_2, x_3) = 2x_1^2 - 5x_3^2 - 2x_1x_2 + 4x_1x_3 + 8x_2x_3.$$

T18 For each of the following quadratic forms, let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

and find a nonsingular change of variables $\mathbf{x} = P\mathbf{y}$ such that

$$q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$$

for some non-zero real numbers $\lambda_1, \lambda_2, \lambda_3$.

(i) $q(x_1, x_2, x_3) = x_1^2 + 3x_2^2 + 9x_3^2 + 2x_1x_2 - 8x_2x_3$.

(ii) $q(x_1, x_2, x_3) = x_1^2 - 2x_1x_2 - 2x_2x_3$.

(iii) $q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^2 - 6x_1x_3 + 4x_2x_3$.

Solution

(i)

$$\begin{aligned}
q &= x_1^2 + 3x_2^2 + 9x_3^2 + 2x_1x_2 - 8x_2x_3 \\
&= \left[x_1^2 + 2x_1x_2\right] + 3x_2^2 + 9x_3^2 - 8x_2x_3 \\
&= \left[(x_1+x_2)^2 - x_2^2\right] + 3x_2^2 + 9x_3^2 - 8x_2x_3 \\
&= (x_1+x_2)^2 + 2x_2^2 + 9x_3^2 - 8x_2x_3 \\
&= (x_1+x_2)^2 + 2\left[x_2^2 - 4x_2x_3\right] + 9x_3^2 \\
&= (x_1+x_2)^2 + 2\left[(x_2-2x_3)^2 - 4x_3^2\right] + 9x_3^2 \\
&= (x_1+x_2)^2 + 2(x_2-2x_3)^2 + x_3^2 \\
&= y_1^2 + 2y_2^2 + y_3^2,
\end{aligned}$$

where

$$\begin{aligned}
y_1 &= x_1 + x_2, \\
y_2 &= \qquad x_2 - 2x_3, \\
y_3 &= \qquad\qquad x_3.
\end{aligned}$$

Then

$$\begin{aligned}
x_3 &= y_3, \\
x_2 &= y_2 + 2x_3 = y_2 + 2y_3, \\
x_1 &= y_1 - x_2 = y_1 - y_2 - 2y_3
\end{aligned}$$

and so

$$\begin{aligned}
x_1 &= y_1 - y_2 - 2y_3, \\
x_2 &= \qquad y_2 + 2y_3, \\
x_3 &= \qquad\qquad y_3,
\end{aligned}$$

$$\text{i.e.}\quad \mathbf{x} = P\mathbf{y},$$

where

$$P = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix},$$

which is invertible.

(ii)

$$\begin{aligned}
q &= x_1^2 - 2x_1x_2 - 2x_2x_3 \\
&= \left[x_1^2 - 2x_1x_2\right] - 2x_2x_3 \\
&= \left[(x_1 - x_2)^2 - x_2^2\right] - 2x_2x_3 \\
&= (x_1 - x_2)^2 - x_2^2 - 2x_2x_3 \\
&= (x_1 - x_2)^2 - \left[x_2^2 + 2x_2x_3\right] \\
&= (x_1 - x_2)^2 - \left[(x_2 + x_3)^2 - x_3^2\right] \\
&= (x_1 - x_2)^2 - (x_2 + x_3)^2 + x_3^2 \\
&= y_1^2 - y_2^2 + y_3^2,
\end{aligned}$$

where

$$\begin{aligned}
y_1 &= x_1 - x_2, \\
y_2 &= \qquad x_2 + x_3, \\
y_3 &= \qquad\qquad x_3.
\end{aligned}$$

Then

$$\begin{aligned}
x_3 &= y_3, \\
x_2 &= y_2 - x_3 = y_2 - y_3, \\
x_1 &= y_1 + x_2 = y_1 + y_2 - y_3
\end{aligned}$$

and so

$$\begin{aligned}
x_1 &= y_1 + y_2 - y_3, \\
x_2 &= \qquad y_2 - y_3, \\
x_3 &= \qquad\qquad y_3,
\end{aligned}$$

$$\text{i.e.} \quad \mathbf{x} = P\mathbf{y},$$

where

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix},$$

which is invertible.

(iii)

$$\begin{aligned}
q &= x_1^2 + 2x_2^2 + 3x_3^2 - 6x_1x_3 + 4x_2x_3 \\
&= \left[x_1^2 - 6x_1x_3\right] + 2x_2^2 + 3x_3^2 + 4x_2x_3 \\
&= \left[(x_1 - 3x_3)^2 - 9x_3^2\right] + 2x_2^2 + 3x_3^2 + 4x_2x_3 \\
&= (x_1 - 3x_3)^2 + 2x_2^2 - 6x_3^2 + 4x_2x_3 \\
&= (x_1 - 3x_3)^2 + 2\left[x_2^2 + 2x_2x_3\right] - 6x_3^2 \\
&= (x_1 - 3x_3)^2 + 2\left[(x_2 + x_3)^2 - x_3^2\right] - 6x_3^2 \\
&= (x_1 - 3x_3)^2 + 2(x_2 + x_3)^2 - 8x_3^2 \\
&= y_1^2 + 2y_2^2 - 8y_3^2,
\end{aligned}$$

where

$$\begin{aligned}
y_1 &= x_1 \qquad - 3x_3, \\
y_2 &= \qquad x_2 + \ x_3, \\
y_3 &= \qquad\qquad\ \ x_3.
\end{aligned}$$

Then

$$\begin{aligned}
x_3 &= y_3, \\
x_2 &= y_2 - x_3 = y_2 - y_3, \\
x_1 &= y_1 + 3x_3 = y_1 + 3y_3
\end{aligned}$$

and so

$$\begin{aligned}
x_1 &= y_1 \qquad + 3y_3, \\
x_2 &= \qquad y_2 - \ y_3, \\
x_3 &= \qquad\qquad\ \ y_3,
\end{aligned}$$

$$\text{i.e.} \quad \mathbf{x} = P\mathbf{y},$$

where

$$P = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix},$$

which is invertible.

2B Linear Algebra

Ex Sheet 10

1 True/False

a) The polynomial $q(x) = 0$ is a quadratic form.

b) The polynomial $q(x_1, x_2) = x_1 - x_2$ is a quadratic form.

c) Any quadratic form can be represented by an upper triangular matrix.

d) The determinant of the matrix representing a quadratic form in n variables is $a_{11}a_{22}\ldots a_{nn}$.

e) Two non-zero quadratic forms in one variable are equivalent if and only if one is a scalar multiple of the other.

f) Two non-zero quadratic forms in one variable are equivalent if and only if one is a positive scalar multiple of the other.

g) If A is an invertible matrix in $M_{n\times n}(\mathbb{C})$, then the conjugate transpose A^* is invertible.

h) A symmetric matrix in $M_{n\times n}(\mathbb{C})$ is automatically Hermitian.

i) A symmetric matrix in $M_{n\times n}(\mathbb{R})$ is automatically Hermitian.

j) The diagonal entries of any Hermitian matrix in $M_{n\times n}(\mathbb{C})$ must be real- valued.

k) For any $A \in M_{n\times n}(\mathbb{C})$, $B = A + A^*$ is Hermitian.

l) The sum of two unitary matrices in $M_{n\times n}(\mathbb{C})$ is again a unitary matrix.

m) If $A \in M_{n\times n}(\mathbb{C})$ is both Hermitain and unitary, then $A^2 = \mathbb{I}$.

n) In a unitary matrix, the sum of the entries of the first row must add up to 1.

o) In a unitary matrix, the sum of the squares of the entries of the first row must add up to 1.

p) A diagonal matrix that is unitary must have all of its entries be either 1 or -1.

q) If $A \in M_{n\times n}(\mathbb{C})$ is Hermitian, then A^2 is positive-definite.

r) If $U \in M_{n\times n}(\mathbb{C})$ is unitary, then $U\mathbf{v} \cdot U\mathbf{w} = \mathbf{v} \cdot \mathbf{w}$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$.

[1] **True/False Questions**

Every Exercise Sheet will have a section containing true/false questions. They are designed to test your understanding from lectures. You should look at your lecture notes and/or the textbook to help you answer these questions, but you should not need to write anything to work out the solution.

Solutions to True/False

a) T b) F c) T d) T e) F f) T g) T h) F i) T j) T k)T l) F m) T n)F o)T p)T q) F r) T

Tutorial Exercises

T1 (i) Find the rank and signature of each of the following quadratic forms:

(a) $q(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 5x_3^2 - 2x_1x_2$,

(b) $q(x_1, x_2) = 5x_1^2 + 5x_2^2 - 4x_1x_2$,

(c) $q(x_1, x_2) = -(x_1 - x_2)^2$,

(d) $q(x_1, x_2, x_3) = x_1x_2$.

(ii) State whether each of the forms is positive-definite, positive-semidefinite, negative-definite, negative-semidefinite or indefinite.

Solution

a) The quadratic form can be written as $\mathbf{x}^t A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^3$ and

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

The eigenvalues are of A are $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = 5$, as these are all positive the quadratic form has (i) rank 3, signature 3, (ii) is positive definite .

b) The quadratic form can be written as $\mathbf{x}^t A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^2$ and

$$A = \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}.$$

The eigenvalues are of A are $\lambda_1 = 3$, $\lambda_2 = 7$, as these are both positive the quadratic form has (i) rank 2, signature 2, (ii) is positive definite.

c) The quadratic form can be written as $\mathbf{x}^t A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^2$ and

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The eigenvalues are of A are $\lambda_1 = 0$, $\lambda_2 = -2$, hence the quadratic form has (i) rank 1, signature -1 (ii) is negative semi-definite.

d) The quadratic form can be written as $\mathbf{x}^t A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^2$ and

$$A = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}.$$

The eigenvalues are of A are $\lambda_1 = 1/2$, $\lambda_2 = -1/2$, hence the quadratic form has (i) rank 2, signature 0, (ii) is indefinite.

T2 Identify the conic section represented by the equation:

a) $2x^2 - 2y^2 = 20$

b) $5x^2 + 3y^2 - 15 = 0$

c) $x^2 - 3 = -y^2$

d) $4y^2 - x^2 = 20$

Solution

(a) Hyperbola, (b) Ellipse (c) Circle (d) Hyperbola

T3 Identify the conic section represented by the equation by rotating axes to place the conic in standard position. Find an equation of the conic in the rotated coordinates, and find the angle of rotation.

a) $x^2 - 4xy - 2y^2 + 8 = 0$

b) $5x^2 + 4xy + 5y^2 = 9$

c) $2x^2 - 12xy - 3y^2 - 7 = 0$

Solution

a) Rewriting the quadratic as $-x^2 + 4xy + 2y^2 = 8$, the left hand side can be written as $\mathbf{x}^T A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^2$ and

$$A = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = 3$, $\lambda_2 = -2$. Applying orthogonal diagonalisation, the equation of the conic in the rotated coordinates is an hyperbola opening left and right

$$3(x')^2 - 2(y')^2 = 8,$$

where the rotated coordinates are $\mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$ and satisfy $Q\mathbf{x}' = \mathbf{x}$ where Q is the orthogonal matrix

$$Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Hence, the change of coordinates corresponds to a rotation (counterclockwise) by the angle θ where $\theta = \tan^{-1} 1 = \pi/4$ or 45^o. The hyperbola intersects the x'-axis at $\pm\sqrt{8/3}$.

b) The left hand side of the quadratic can be written as $\mathbf{x}^T A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^2$ and

$$A = \begin{bmatrix} 2 & -6 \\ -6 & -3 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = -7$, $\lambda_2 = 3$. Applying orthogonal diagonalisation, the equation of the conic in the rotated coordinates is the ellipse

$$7(x')^2 + 3(y')^2 = 8,$$

where the rotated coordinates are $\mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$ and satisfy $Q\mathbf{x}' = \mathbf{x}$ where Q is the orthogonal matrix

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Hence, the change of coordinates corresponds to a rotation (counterclockwise) by the angle θ where $\theta = \tan^{-1} 1 = \pi/4$ or 45^o. The ellipse intersects the x'-axis at $\pm\sqrt{8/7}$ and the y'-axis at $\pm\sqrt{8/3}$.

c) Rewriting the quadratic as $2x^2 - 12xy - 3y^2 = 7$, the left hand side can be written as $\mathbf{x}^T A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^2$ and

$$A = \begin{bmatrix} 2 & -6 \\ -6 & -3 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = -7$, $\lambda_2 = 6$. Applying orthogonal diagonalisation, the equation of the conic in the rotated coordinates the hyperbola opening up and down

$$-7(x')^2 + 6(y')^2 = 7,$$

where the rotated coordinates are $\mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$ and satisfy $Q\mathbf{x}' = \mathbf{x}$ where Q is the orthogonal matrix

$$Q = \frac{1}{\sqrt{13}} \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Hence, the change of coordinates corresponds to a rotation (counterclockwise) by the angle θ where $\theta = \tan^{-1}(3/2)$, or 56.3^o. The hyperbola intersects the y'-axis at $\pm\sqrt{7/6}$.

T4 Let a, b, c be real numbers. (i) Prove that the quadratic form

$$q(x, y) = ax^2 + by^2 + 2cxy$$

is positive-definite if and only if $a > 0$ and $ab > c^2$. Find similar necessary and sufficient conditions for $q(x, y)$ to be (ii) negative-definite, (iii) indefinite. Try to find necessary and sufficient conditions for $q(x, y)$ to be (iv) positive-semidefinite, (v) negative-semidefinite.

Solution

(i) First suppose that $a \neq 0$. Then

$$\begin{aligned} q(x, y) &= ax^2 + by^2 + 2cxy \\ &= a\left[x^2 + 2\frac{c}{a}xy\right] + by^2 \end{aligned}$$

$$= a\left[\left(x + \frac{c}{a}y\right)^2 - \frac{c^2}{a^2}y^2\right] + by^2$$
$$= a\left(x + \frac{c}{a}y\right)^2 + \frac{ab - c^2}{a}y^2.$$

Therefore, provided $a \neq 0$,

$$\begin{aligned} q \text{ is positive-definite} &\iff a > 0 \text{ and } \frac{ab - c^2}{a} > 0 \\ &\iff a > 0 \text{ and } ab - c^2 > 0 \\ &\iff a > 0 \text{ and } ab > c^2. \end{aligned}$$

Now suppose that $a = 0$ and $b \neq 0$. Then

$$q(x, y) = by^2 + 2cxy = b\left(y + \frac{c}{b}x\right)^2 - \frac{c^2}{b}x^2.$$

In this case, q cannot be positive-definite because if $b > 0$ then $-\frac{c^2}{b} \leqslant 0$.

Finally, suppose that $a = b = 0$. Then

$$q(x, y) = 2cxy = \frac{c}{2}(x + y)^2 - \frac{c}{2}(x - y)^2.$$

In this case, q cannot be positive-definite because if $\frac{c}{2} > 0$ then $-\frac{c}{2} < 0$.

Hence, from the three cases,

$$q \text{ is positive-definite} \iff a > 0 \text{ and } ab > c^2.$$

(Note that, from $ab > c^2$, it follows that $ab > 0$ and so a, b have the same sign. This means that

$$\begin{aligned} q \text{ is positive-definite} &\iff b > 0 \text{ and } ab > c^2 \\ &\iff a + b > 0 \text{ and } ab > c^2. \end{aligned}$$

The symmetry of the latter condition is attractive. But we have focused on the first of our conditions because of its relevance for Chapter 5.)

(ii) Similarly, provided $a \neq 0$,

$$\begin{aligned} q \text{ is negative-definite} &\iff a < 0 \text{ and } \frac{ab - c^2}{a} < 0 \\ &\iff a < 0 \text{ and } ab - c^2 > 0 \\ &\iff a < 0 \text{ and } ab > c^2. \end{aligned}$$

Furthermore, q cannot be negative-definite in the second or third cases because if $b < 0$ then $-\frac{c^2}{b} \geqslant 0$, and if $\frac{c}{2} < 0$ then $-\frac{c}{2} > 0$.

Hence, from the three cases,

$$q \text{ is negative-definite} \iff a < 0 \text{ and } ab > c^2.$$

(Note that, from $ab > c^2$, it follows that $ab > 0$ and so a, b have the same sign. This means that

$$\begin{aligned} q \text{ is negative-definite} &\iff b < 0 \text{ and } ab > c^2 \\ &\iff a + b < 0 \text{ and } ab > c^2. \end{aligned}$$

Once again, we have focused on the first of our conditions because of its relevance for Chapter 5.)

(iii) Provided $a \neq 0$,

$$\begin{aligned} q \text{ is indefinite} &\iff a \text{ and } \frac{ab - c^2}{a} \text{ have opposite signs} \\ &\iff ab - c^2 < 0 \\ &\iff ab < c^2. \end{aligned}$$

In the case $a = 0$ and $b \neq 0$,

$$\begin{aligned} q \text{ is indefinite} &\iff b \text{ and } -\frac{c^2}{b} \text{ have opposite signs} \\ &\iff -c^2 < 0 \\ &\iff 0 < c^2 \\ &\iff ab < c^2. \end{aligned}$$

In the case $a = 0$ and $b = 0$,

$$\begin{aligned} q \text{ is indefinite} &\iff \frac{c}{2} \text{ and } -\frac{c}{2} \text{ have opposite signs} \\ &\iff c \neq 0 \\ &\iff 0 < c^2 \\ &\iff ab < c^2. \end{aligned}$$

Hence, from the three cases,

$$q \text{ is indefinite} \iff ab < c^2.$$

It is already clear that if $ab = c^2$ then q must be positive-semidefinite or negative-semidefinite.

(iv) Provided $a \neq 0$,

$$\begin{aligned} q \text{ is positive-semidefinite} &\iff a \geqslant 0 \text{ and } \frac{ab - c^2}{a} \geqslant 0 \\ &\iff a \geqslant 0 \text{ and } ab - c^2 \geqslant 0 \\ &\iff a \geqslant 0 \text{ and } ab \geqslant c^2 \\ &\iff a + b \geqslant 0 \text{ and } ab \geqslant c^2. \end{aligned}$$

(Here, since $a \neq 0$,

$$a \geqslant 0 \iff a > 0.$$

Also, from $ab \geqslant c^2$, it follows that $ab \geqslant 0$, i.e. a, b cannot have opposite signs.)

In the case $a = 0$ and $b \neq 0$,

$$\begin{aligned} q \text{ is positive-semidefinite} &\iff b \geqslant 0 \text{ and } -\frac{c^2}{b} \geqslant 0 \\ &\iff b \geqslant 0 \text{ and } -c^2 \geqslant 0 \\ &\iff a + b \geqslant 0 \text{ and } ab \geqslant c^2. \end{aligned}$$

(Here, since $a = 0$ and $b \neq 0$, we have

$$b \geqslant 0 \iff b > 0,$$

$a + b = b$ and $ab = 0$.)

In the case $a = 0$ and $b = 0$,

$$\begin{aligned} q \text{ is positive-semidefinite} &\iff \frac{c}{2} \geqslant 0 \text{ and } -\frac{c}{2} \geqslant 0 \\ &\iff c \geqslant 0 \text{ and } c \leqslant 0 \\ &\iff c = 0 \\ &\iff a + b \geqslant 0 \text{ and } ab \geqslant c^2. \end{aligned}$$

Hence, from the three cases,

$$q \text{ is positive-semidefinite} \iff a + b \geqslant 0 \text{ and } ab \geqslant c^2.$$

(v) Provided $a \neq 0$,

$$\begin{aligned} q \text{ is negative-semidefinite} &\iff a \leqslant 0 \text{ and } \frac{ab - c^2}{a} \leqslant 0 \\ &\iff a \leqslant 0 \text{ and } ab - c^2 \geqslant 0 \\ &\iff a \leqslant 0 \text{ and } ab \geqslant c^2 \\ &\iff a + b \leqslant 0 \text{ and } ab \geqslant c^2. \end{aligned}$$

(Here, since $a \neq 0$,

$$a \leqslant 0 \iff a < 0.$$

Also, from $ab \geqslant c^2$, it follows that $ab \geqslant 0$, i.e. a, b cannot have opposite signs.)

In the case $a = 0$ and $b \neq 0$,

$$\begin{aligned} q \text{ is negative-semidefinite} &\iff b \leqslant 0 \text{ and } -\frac{c^2}{b} \leqslant 0 \\ &\iff b \leqslant 0 \text{ and } -c^2 \geqslant 0 \\ &\iff a + b \leqslant 0 \text{ and } ab \geqslant c^2. \end{aligned}$$

(Here, since $a = 0$ and $b \neq 0$, we have

$$b \geqslant 0 \iff b > 0,$$

$a + b = b$ and $ab = 0$.)

In the case $a = 0$ and $b = 0$,

$$\begin{aligned} q \text{ is negative-semidefinite} &\iff \frac{c}{2} \geqslant 0 \text{ and } -\frac{c}{2} \geqslant 0 \\ &\iff c \geqslant 0 \text{ and } c \leqslant 0 \\ &\iff c = 0 \\ &\iff a + b \leqslant 0 \text{ and } ab \geqslant c^2. \end{aligned}$$

Hence, from the three cases,

$$q \text{ is negative-semidefinite} \iff a + b \leqslant 0 \text{ and } ab \geqslant c^2.$$

T5 For each of the following matrices A, find A^*:

a)

$$A = \begin{bmatrix} 2+i & 1 \\ 2 & -3i \\ 0 & 1+5i \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 2i & 1-i & -1+i \\ 4 & 5-7i & -i \end{bmatrix}.$$

Solution

a)

$$A = \begin{bmatrix} 2-i & 2 & 0 \\ 1 & 3i & 1-5i \end{bmatrix}$$

b)

$$A = \begin{bmatrix} -2i & 4 \\ 1+i & 5+7i \\ -1-i & i \end{bmatrix}.$$

T6 For each of the following matrices A, substitute numbers for the ?s so that A is Hermitian:

a)

$$A = \begin{bmatrix} 3 & ? & ? \\ 3+2i & -2 & ? \\ 7 & 1-5i & 6 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 2 & 0 & 3+5i \\ ? & -4 & -i \\ ? & ? & 6 \end{bmatrix}.$$

Solution

a)

$$A = \begin{bmatrix} 3 & 3-2i & 7 \\ 3+2i & -2 & 1+5i \\ 7 & 1-5i & 6 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 2 & 0 & 3+5i \\ 0 & -4 & -i \\ 3-5i & i & 6 \end{bmatrix}.$$

T7 For each of the following matrices A, show that A is not Hermitian for any choice of the ?s:

a)

$$A = \begin{bmatrix} 2 & ? & 3-7i \\ ? & 1+i & 2i \\ 3+7i & ? & 0 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 2 & 4+i & ? \\ -4+i & -1 & ? \\ ? & ? & ? \end{bmatrix}$$

c)

$$A = \begin{bmatrix} 1 & 1+i & ? \\ 1+i & 7 & ? \\ 6-2i & ? & 0 \end{bmatrix}$$

d)

$$A = \begin{bmatrix} 1 & ? & 3+5i \\ ? & 3 & 1-i \\ 3-5i & ? & 2+i \end{bmatrix}.$$

Solution

a) Non-real entry on the diagonal.

b) $(1,2)$- and $(2,1)$-entries are not conjugate.

c) $(1,2)$- and $(2,1)$-entries are not conjugate.

d) Non-real entry on the diagonal.

T8 For each of the following matrices A, show that A is unitary and find A^{-1}:

a)

$$A = \begin{bmatrix} \frac{4}{5} & \frac{3i}{5} \\ -\frac{3i}{5} & -\frac{4}{5} \end{bmatrix}$$

b)

$$A = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

Solution

In each case, it is enough to show that $A^*A = I$, and then A is unitary with $A^{-1} = A^*$.

a)

$$A^{-1} = \begin{bmatrix} \frac{4}{5} & \frac{3i}{5} \\ -\frac{3i}{5} & -\frac{4}{5} \end{bmatrix} = A.$$

b)

$$A^{-1} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \\ -\frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

T9 For each of the following Hermitian matrices A, find a unitary matrix P that diagonalises A, and determine $P^{-1}AP$:

a)

$$A = \begin{bmatrix} 9 & 12i \\ -12i & 16 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 0 & 3+i \\ 3-i & -3 \end{bmatrix}$$

c)

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & -1+i \\ 0 & -1-i & 0 \end{bmatrix}$$

d)

$$A = \begin{bmatrix} 2 & \frac{1}{\sqrt{2}}i & -\frac{1}{\sqrt{2}}i \\ -\frac{1}{\sqrt{2}}i & 2 & 0 \\ \frac{1}{\sqrt{2}}i & 0 & 2 \end{bmatrix}.$$

Solution

a) $P = \begin{bmatrix} \frac{3i}{5} & -\frac{4i}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$ gives $P^{-1}AP = \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix}$.

b) $P = \frac{1}{\sqrt{14}} \begin{bmatrix} 2 & 3+i \\ -3+i & 2 \end{bmatrix}$ gives $P^{-1}AP = \begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}$.

c) $P = \frac{1}{\sqrt{6}} \begin{bmatrix} 0 & 0 & 1 \\ 2 & -1+i & 0 \\ 1+i & 2 & 0 \end{bmatrix}$ gives $P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

d) $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{i}{2} & \frac{1}{\sqrt{2}} & -\frac{i}{2} \\ -\frac{i}{2} & \frac{1}{\sqrt{2}} & \frac{i}{2} \end{bmatrix}$ gives $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

T10 Let A be any $n \times n$ matrix with complex entries, and define the matrices B and C by

$$B = \frac{1}{2}(A + A^*), \qquad C = \frac{1}{2i}(A - A^*).$$

a) Show that B and C are Hermitian.

b) Show that $A = B + iC$ and $A^* = B - iC$.

c) What condition must B and C satisfy for A to be normal?

Solution

a) $B^* = \frac{1}{2}(A^* + A) = B$ and $C^* = \frac{-1}{2i}(A^* - A) = C$.

b) Substitute.

c) $AA^* = B^2 + C^2 + (CB - BC)i$ and $A^*A = B^2 + C^2 + (BC - CB)i$, so A is normal if and only if $CB - BC = BC - CB$, i.e. $BC = CB$, i.e. B and C commute.

T11 Let A be an $n \times n$ matrix with complex entries, and let $\mathbf{u}$ and $\mathbf{v}$ be column vectors in $\mathbb{C}^n$. Prove that

$$A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^*\mathbf{v} \qquad \text{and} \qquad \mathbf{u} \cdot A\mathbf{v} = A^*\mathbf{u} \cdot \mathbf{v}.$$

Solution

$A\mathbf{u} \cdot \mathbf{v} = (A\mathbf{u})^*\mathbf{v} = \mathbf{u}^*A^*\mathbf{v} = \mathbf{u} \cdot A^*\mathbf{v}$ and $\mathbf{u} \cdot A\mathbf{v} = \mathbf{u}^*A\mathbf{v} = (A^*\mathbf{u})^*\mathbf{v} = A^*\mathbf{u} \cdot \mathbf{v}$.

T12 Prove that the eigenvalues of a unitary matrix have modulus 1.

Solution

Let λ be an eigenvalue of a unitary matrix A. Choose an eigenvector $\mathbf{u}$.

First solution: putting $\mathbf{v} = A\mathbf{u}$ in the first equation of the last question gives $\|A\mathbf{u}\|^2 = \|\mathbf{u}\|^2$ (since $A^*A = I$). But $A\mathbf{u} = \lambda\mathbf{u}$, so $|\lambda|^2\|\mathbf{u}\|^2 = \|\mathbf{u}\|^2$. Since $\mathbf{u} \neq \mathbf{0}$, this implies that $|\lambda| = 1$.

Second solution: we have

$$(A\mathbf{u}) \cdot (A\mathbf{u}) = (\lambda\mathbf{u}) \cdot (\lambda\mathbf{u}) = \bar{\lambda}\lambda(\mathbf{u} \cdot \mathbf{u}) = |\lambda|^2\|\mathbf{u}\|^2,$$

but also

$$(A\mathbf{u}) \cdot (A\mathbf{u}) = (A\mathbf{u})^*A\mathbf{u} = \mathbf{u}^*A^*A\mathbf{u} = \mathbf{u}^*\mathbf{u} = \|\mathbf{u}\|^2,$$

so $|\lambda|^2\|\mathbf{u}\|^2 = \|\mathbf{u}\|^2$. Since $\mathbf{u} \neq \mathbf{0}$, this implies that $|\lambda| = 1$.

T13 Let $\mathbf{u}$ be a nonzero column vector in $\mathbb{C}^n$. Prove that the matrix $P = \mathbf{u}\mathbf{u}^*$ is Hermitian.

Solution

$P^* = (\mathbf{u}\mathbf{u}^*)^* = \mathbf{u}^{**}\mathbf{u}^* = \mathbf{u}\mathbf{u}^* = P$.

T14 Let A be an invertible matrix. Prove that A^* is invertible and that $(A^*)^{-1} = (A^{-1})^*$.

Solution

$(A^{-1})^*A^* = (AA^{-1})^* = I^* = I$ and $A^*(A^{-1})^* = (A^{-1}A)^* = I^* = I$, so by definition of inverse matrix, A^* is invertible with inverse $(A^{-1})^*$.