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Assessed Coursework

Course Name	Algorithmic Foundations 2			
Coursework Number	2			
Deadline	Time:	16:30	Date:	23/11/2020
% Contribution to final course mark	10%			
Solo or Group ✓	Solo	✓	Group	
Anticipated Hours	10			
Submission Instructions	Use the latex template and submit the generated pdf through moodle (do not submit the source latex file). Failure to follow the submission instructions will lead to a penalty of 2 bands.			
Please Note: This Coursework cannot be Re-Assessed				

Code of Assessment Rules for Coursework Submission

Deadlines for the submission of coursework which is to be formally assessed will be published in course documentation, and work which is submitted later than the deadline will be subject to penalty as set out below.

The primary grade and secondary band awarded for coursework which is submitted after the published deadline will be calculated as follows:

(i) in respect of work submitted not more than five working days after the deadline
  a. the work will be assessed in the usual way;
  b. the primary grade and secondary band so determined will then be reduced by two secondary bands for each working day (or part of a working day) the work was submitted late.

(ii) work submitted more than five working days after the deadline will be awarded Grade H.

Penalties for late submission of coursework will not be imposed if good cause is established for the late submission. You should submit documents supporting good cause via MyCampus.

Penalty for non-adherence to Submission Instructions is 2 bands

Algorithmic Foundations 2

Assessed Exercise 2

Notes for guidance

1. This is the second of the two assessed exercises. It is worth 10% of your final grade for this module. Your answers must be the result of your own individual efforts.
2. Please use the latex template and submit your the generated pdf via moodle (do not submit the latex source file).
3. Please ensure you have filled out your tutorial group, name and student id.
4. **Failure to follow the submission instructions will lead to a penalty for non-adherence to submission instructions of 2 bands**.
5. As stated on the cover sheet deadline for completing this assessed exercise is **16:30 Monday November 23, 2020**.
6. You must complete the online "Declaration of Originality" form on the LTC website.
7. The exercise is marked out of 30 using the included marking scheme. Credit will be given for partial answers.

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1. (a) Show that if n divides m where n and m are positive integers greater than 1, then $a \equiv b \pmod{m}$ implies $a \equiv b \pmod{n}$ for any positive integers a and b. [4]

Solution:
By definition of congruence, $a \equiv b \pmod{m} \Rightarrow m|(a-b)$, which can be expressed as

$$a - b = mk$$

for some positive integer k. Since n divides m, m can be substituted into the equation as

$$a - b = npk$$

for some positive integer p. Hence,

$$n|(a-b)$$
$$\Rightarrow a \equiv b \pmod{n},$$

as required.

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(b) Show that $a{\cdot}c \equiv b{\cdot}c \pmod{m}$ with a, b, c and m integers with $m{\geq}2$ does not imply $a \equiv b \pmod{m}$. [3]

Solution: For $a = 1, b = 2, m = 3$ and $c = 0$,

$$1 \cdot 0 \equiv 2 \cdot 0 \pmod{3}$$

because

$$0 \equiv 0 \pmod{3}.$$

However,

$$1 \not\equiv 2 \pmod{3}.$$

Thus, $a{\cdot}c \equiv b{\cdot}c \pmod{m}$ with a, b, c and m integers with $m{\geq}2$ does not imply $a \equiv b \pmod{m}$, as required.

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(c) Using the Euclidean Algorithm, find $\texttt{gcd}(3084, 1424)$. Show your working. [3]

Solution: By the Euclidean Algorithm:

$$\begin{aligned} 3084 &= 1424{\cdot}2 + 236 \\ 1424 &= 236{\cdot}6 + 8 \\ 236 &= 8{\cdot}29 + 4 \\ 8 &= 4{\cdot}2 + 0. \end{aligned}$$

Thus, $\texttt{gcd}(3084, 1424) = 4$.

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2. A company has a contract to cover the four walls, ceiling, and floor of a factory building with fire-retardant material. The building is rectangular and of width 280m, length 336m and height 168m. Square panels can be manufactured in any size of whole metres. For safety reasons, the building must be covered in complete panels (i.e. panels cannot be cut). What is the minimum number of equally sized square panels that are required to line the interior of the building? Explain your answer. [5]

Solution:

To find the smallest number of panels required, one must find the biggest (greatest) possible size of each panel in order to cover everything on each wall. Thus, one must find the greatest common divisor of all three dimensions (280 x 336 x 168). Firstly, find the greatest common divisor of the two largest dimensions. By the Euclidean Algorithm:

$$\begin{aligned} 336 &= 280 \cdot 1 + 56 \\ 280 &= 56 \cdot 5 + 0 \end{aligned}$$

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Thus, $\texttt{gcd}(336, 280) = 56$.

Secondly, since $\texttt{gcd}(336, 280)$ divides 168, $\texttt{gcd}(336, 280, 168) = 56$. Then the largest possible size for square panels is 56×56 m^2. Thus, the number of panels is

$$\frac{2 \cdot (280 \cdot 336 + 336 \cdot 168 + 168 \cdot 280)}{56^2} = 126.$$

Therefore, the minimum number of equally sized panels is 126.

3. Prove that the least significant digit of the square of an even integer is either 0, 4, or 6. [5]

Hint: considering splitting into cases where integers are of the form $a \cdot k + b$ or $-(a \cdot k + b)$ for $k \in \mathbb{N}$ where a and b are fixed for a given case, b varies over the cases and the least significant digit of the integer depends on only b.

Note: the *least significant digit* of an integer is the digit farthest to the right in an integer. For example, the least significant digits of 1007 and 26 are 7 and 6 respectively.

Solution: Because $(a \cdot k + b)^2 = (-(a \cdot k + b))^2$ for $a, b, k \in \mathbb{R}$, it is sufficient to look only at the squares of positive even integers. Even integers can then be expressed in cases as $10a + b$ for $a \in \mathbb{Z}$, where:

- $b = 0$. Then $(10a + 0)^2 = 100a^2 + 0 \Rightarrow$ remainder is 0;
- $b = 2$. Then $(10a + 2)^2 = 100a^2 + 40a + 4 \Rightarrow$ remainder is 4;
- $b = 4$. Then $(10a + 4)^2 = 100a^2 + 80a + 16 \Rightarrow$ remainder is 16, the least significant digit of which is 6;
- $b = 6$. Then $(10a + 6)^2 = 100a^2 + 120a + 36 \Rightarrow$ remainder is 36, the least significant digit of which is 6;
- $b = 8$. Then $(10a + 8)^2 = 100a^2 + 160a + 64 \Rightarrow$ remainder is 64, the least significant digit of which is 4.

All non-remainder terms in the expressions for squares of even integers have least significant digit 0 because they are multiples of 10; thus, it is sufficient to look only at the remainders and their least significant digits. Therefore, the least significant digit of the square of an even integer is either 0, 4, or 6.

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4. Use mathematical induction to show that for any $n \in \mathbb{N}$, if $n \geq 2$, then [5]

$$\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2 \cdot n}\,.$$

Solution: Let $P(n)$ be the proposition $\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2 \cdot n}$.

Base case: $P(2)$ holds since $\prod_{i=2}^{2} \left(1 - \frac{1}{i^2}\right) = 1 - \frac{1}{2^2} = \frac{3}{4} = \frac{2+1}{2 \cdot 2}$.

Inductive step: We now assume $P(n)$ is true for $n \geq 2, n \in \mathbb{N}$. Considering $n + 1$,

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we have:

$$
\begin{aligned}
\prod_{i=2}^{n+1}\left(1-\frac{1}{i^2}\right) &= \left(\prod_{i=2}^{n} 1-\frac{1}{i^2}\right)\cdot\left(1-\frac{1}{(n+1)^2}\right) \\
&= \frac{n+1}{2\cdot n}\cdot\left(1-\frac{1}{(n+1)^2}\right) && \text{by inductive hypothesis} \\
&= \frac{(n+1)n(n+2)}{2n(n+1)^2} \\
&= \frac{n+2}{2(n+1)} \\
&= \frac{(n+1)+1}{2(n+1)}
\end{aligned}
$$

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Hence, $P(n+1)$ holds. Therefore, by the principle of induction, we have proved that $P(n)$ holds for all $n \geq 2, n \in \mathbb{N}$.

5. Use mathematical induction to show that 2 divides $n^2 - n$ for all $n \in \mathbb{N}$. [5]

Solution:

Let $P(n)$ be the hypothesis $2|(n^2 - n)$.

Base case: $P(0)$ holds since $0^2 - 0 = 0$, which is divisible by 2.

Inductive step: Assume $P(n)$ is true for all $n \in \mathbb{N}$. Then $n^2 - n$ can be rewritten as

$$n^2 - n = 2k$$

for some integer k. Considering $n+1$, we have:

$$
\begin{aligned}
(n+1)^2 - (n+1) &= n^2 + 2n + 1 - n - 1 \\
&= n^2 - n + 2n \\
&= 2k + 2n && \text{by inductive hypothesis} \\
&= 2(k+n)
\end{aligned}
$$

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Since $2|2(k+n), P(n)$ holds. Therefore, by the principle of induction, we have proved that $P(n)$ holds for all $n \in \mathbb{N}$.

Index of comments

5.1 base case includes n=0 and n=1