EXAMINATION FOR THE DEGREES OF
M.A. AND B.Sc.

Mathematics 2E - Introduction to Real Analysis

An electronic calculator may be used provided that it does not have a facility for either textual storage or display, or for graphical display.

Candidates must attempt all questions.

Question 1 and question 2 are multiple choice questions. Use the response form "2E Degree Exam Multiple Choice Section" to record your answers.

1. (i) Let x, y be real numbers, and let P be the statement

$$x < y \implies x^3 < y^3.$$

Which one of the following statements is equivalent to the negation of P?

(A) $x^3 \geqslant y^3 \implies x < y$.
(B) $x \geqslant y \implies x^3 < y^3$.
(C) $x < y \implies x^3 \geqslant y^3$.
(D) $x < y$ and $x^3 \geqslant y^3$.
(E) None of these statements. **2**

(D)

(ii) Let $A \subset \mathbb{R}$ be a non-empty subset and let m be a lower bound for A. Which one of the following statements is equivalent to $m = \inf(A)$?

(A) For every $m' \geqslant m$ there exists $a \in A$ with $a > m'$.
(B) For every $m' > m$ there exists $a \in A$ with $a < m'$.
(C) For every $m' \geqslant m$ there exists $a \in A$ with $a < m'$.
(D) For every $m' < m$ there exists $a \in A$ with $a > m'$.
(E) None of these statements. **2**

(B)

(iii) Let A and B be nonempty bounded subsets of $\mathbb{R}$ and let

$$C = \{6a - 3b \mid a \in A, b \in B\}.$$

Which one of the following quantities is the infimum of C?

(A) $6\inf(A) - 3\inf(B)$.
(B) $6\inf(A) + 3\inf(B)$.
(C) $6\inf(A) - 3\sup(B)$.
(D) $6\inf(A) + 3\sup(B)$.
(E) None of these quantities. **2**

(C)

(iv) Let $(x_n)_{n=1}^{\infty}$ be a real sequence and let $L \in \mathbb{R}$. Which one of the following statements is equivalent to the negation of $x_n \to L$ as $n \to \infty$?

(A) $\exists M \neq L$ such that $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ such that $n \geqslant n_0 \implies |x_n - M| < \varepsilon$.
(B) $\exists \varepsilon > 0$ such that $\forall n_0 \in \mathbb{N}, \exists n \geqslant n_0$ such that $|x_n - L| \geqslant \varepsilon$.
(C) $\exists \varepsilon > 0$ such that $\forall n_0 \in \mathbb{N}, \exists n < n_0$ such that $|x_n - L| < \varepsilon$.
(D) $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ such that $n \geqslant n_0 \implies |x_n - L| \geqslant \varepsilon$.
(E) None of these statements. **2**

(B)

(v) Let $(x_n)_{n=1}^{\infty}$ be a sequence such that $x_n > 0$ for all $n \in \mathbb{N}$. Which one of the following statements is a consequence of the statement that $\sum_{n=1}^{\infty} x_n$ is convergent?

(A) $\forall \varepsilon > 0, \forall N \in \mathbb{N}, x_N < \varepsilon$.
(B) $\frac{x_{n+1}}{x_n} \to 0$ as $n \to \infty$.
(C) $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \geqslant N, \sum_{j=1}^{n} x_j < \varepsilon$.
(D) $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $\sum_{n=1}^{N} x_n < \varepsilon$.
(E) None of these statements. **2**

(E)

2. (i) Let $\varepsilon > 0$ be arbitrary. The statement

$$\exists \delta > 0 \text{ such that } \forall x \in \mathbb{R}, |x - 2| < \delta \Rightarrow |x^2 - 4| < \epsilon$$

is correct. Which one of the following values of δ demonstrates this?

(A) $\delta = \epsilon/5$.
(B) $\delta = \min(1, \epsilon/5)$.
(C) $\delta = \frac{\epsilon}{|x+2|}$.
(D) $\delta = \min(1, \epsilon)$.
(E) None of these values. **2**

(B)

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Which one of the following statements is equivalent to f not being bounded below?

(A) $\forall K \in \mathbb{R}, \exists x \in \mathbb{R}$ such that $f(x) < K$.
(B) $\exists K \in \mathbb{R}$ such that $\forall x \in \mathbb{R}, f(x) < K$.
(C) $\exists x \in \mathbb{R}$ such that $\forall K \in \mathbb{R}, f(x) < K$.
(D) $\forall x \in \mathbb{R}, \exists K \in \mathbb{R}$ such that $f(x) < K$.
(E) None of these statements. **2**

(A)

(iii) Let $\sum_{n=1}^{\infty} x_n$ be an absolutely convergent series. Which one of the following statements is correct?

(A) $\sum_{n=1}^{\infty}(-1)^n x_n$ is absolutely convergent.
(B) $\sum_{n=1}^{\infty}(-1)^n x_n$ is conditionally convergent.
(C) $\sum_{n=1}^{\infty}(-1)^n x_n$ is divergent.
(D) $\sum_{n=1}^{\infty}(-1)^n x_n$ is absolutely convergent, but not convergent.
(E) None of these statements. **2**

(A)

(iv) Let $f : \mathbb{R} \to \mathbb{R}$ be a real function and $c \in \mathbb{R}$. Which one of the following statements is equivalent to f not being continuous at c?

(A) $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x \in \mathbb{R}, |x - c| < \delta \not\Rightarrow |f(x) - f(c)| < \varepsilon$.
(B) $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x \in \mathbb{R}, |x - c| \geqslant \delta \Rightarrow |f(x) - f(c)| < \varepsilon$.
(C) $\exists \varepsilon > 0$ such that $\forall \delta > 0, |x - c| \geqslant \delta \Rightarrow |f(x) - f(c)| < \varepsilon$.
(D) $\exists \varepsilon > 0$ such that $\forall \delta > 0, \exists x \in \mathbb{R}$ such that $|x-c| \leqslant \delta$ and $|f(x)-f(c)| > \varepsilon$.
(E) None of these statements. **2**

(D)

(v) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Which one of the following statements is a valid deduction from the extremal value theorem?

(A) $\forall d \in [a, b], \forall x \in [a, b], f(x) \geqslant f(d)$.
(B) $\forall x \in [a, b], \exists d \in [a, b]$ such that $f(d) < f(x)$.
(C) $\exists d \in [a, b]$ such that $\inf\{f(x) \mid x \in [a, b]\} = f(d)$.
(D) $\exists d \in (a, b)$ such that $\forall x \in [a, b], f(d) \leqslant f(x)$.
(E) None of these statements. **2**

(C)

3. (i) Show that the set

$$A = \{3x - 2y + \frac{1}{2z} \mid x, y, z \in (2, 4)\}$$

is bounded. **3**

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(ii) Show that

$$\inf\left\{\frac{2n+5}{7n+7} \mid n \in \mathbb{N}\right\} = \frac{2}{7}.$$ **3**

(i) For $2 < x, y, z < 4$ *we have* $6 < 3x < 12$, $-8 < -2y < -4$ *and* $\frac{1}{8} < \frac{1}{2z} < \frac{1}{4}$. *Therefore*

$$-\frac{15}{8} = 6 - 8 + \frac{1}{8} < 3x - 2y + \frac{1}{2z} < 12 - 4 + \frac{1}{4} = \frac{33}{4}$$

for all $x, y, z \in (2, 4)$.
That is, $-\frac{15}{8}$ *is a lower bound for* A*, and* $\frac{33}{4}$ *is an upper bound for* A.
(ii) We have

$$\frac{2n+5}{7n+7} \geqslant \frac{2n+2}{7n+7} = \frac{2}{7}$$

for all $n \in \mathbb{N}$*, so that* $\frac{2}{7}$ *is a lower bound.*
Now let $\varepsilon > 0$. *We have to show that there exists* $n \in \mathbb{N}$ *such that* $\frac{2n+5}{7n+7} < \frac{2}{7} + \varepsilon$. *For this we write*

$$\begin{aligned} \frac{2n+5}{7n+7} < \frac{2}{7} + \varepsilon &\Leftrightarrow 2n + 5 < 2n + 2 + \varepsilon(7n+7) \\ &\Leftrightarrow 3 - 7\varepsilon < 7\varepsilon n \\ &\Leftrightarrow n > \frac{3}{7\varepsilon} - 1, \end{aligned}$$

so that for $n > \frac{3}{7\varepsilon} - 1$ *we get the claim.*

4. Prove the following statements *directly from the definition.*

(i) Every convergent real sequence is bounded. **4**

(ii) The function $f : (0, \infty) \to \mathbb{R}$ given by

$$f(x) = \frac{x^2 - 1}{x + 2}$$

is continuous at $x = 1$. **4**

(i) Suppose that $x_n \to L$ *as* $n \to \infty$. *Taking* $\varepsilon = 1$ *in the definition of convergence there exists* $n_0 \in \mathbb{N}$ *such that for* $n \geqslant n_0$*, we have* $|x_n - L| < 1$. *In particular, for* $n \geqslant n_0$ *we get*

$$|x_n| = |x_n - L + L| \leqslant |x_n - L| + |L| < 1 + |L|.$$

Define $M = \max(|x_1|, |x_2|, \ldots, |x_{n_0-1}|, |L|+1) > 0$. *Then for all* $n \in \mathbb{N}$, $|x_n| \leqslant M$*, so that* $(x_n)_{n=1}^{\infty}$ *is bounded.*
(ii) Let $\varepsilon > 0$ *be arbitrary. Note that* $f(1) = 0$ *and compute*

$$|f(x) - f(1)| = \frac{|x+1|}{|x+2|}|x - 1|.$$

We have

$$|x-1|<1 \implies -1<x-1<1 \implies 1<x+1<3 \implies 1<|x+1|<3$$

and

$$|x-1|<1 \implies -1<x-1<1 \implies 2<x+2<4 \implies \frac{1}{4}<\frac{1}{|x+2|}<\frac{1}{2}.$$

Therefore take $\delta=\min(1,\frac{2}{3}\varepsilon)>0$. *Then for* $|x-1|<\delta$, *we have*

$$|f(x)-f(1)|=\frac{|x+1|}{|x+2|}<\frac{3}{2}|x-1|<\varepsilon,$$

so that f *is continuous at* 1.

5. (i) State the sandwich principle. **3**

   (ii) Calculate
$$\lim_{n\to\infty}\frac{4n^3+2n^2}{3n^4+(-1)^n},$$
   stating clearly all properties of limits used. **4**

(i) Let $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}$ *and* $(z_n)_{n=1}^{\infty}$ *be sequences and suppose that* $x_n\to L$ *and* $z_n\to L$ *as* $n\to\infty$. *If there exists* $N\in\mathbb{N}$ *such that*

$$n\geqslant N \implies x_n\leqslant y_n\leqslant z_n,$$

then $\lim_{n\to\infty}y_n=L$. *(Full marks also if* $x_n\leqslant y_n\leqslant z_n$ *is required for all* $n\in\mathbb{N}$.*)*
(ii) Write $y_n=\frac{4n^3+2n^2}{3n^4+(-1)^n}$ *and* $z_n=\frac{4n^3+2n^2}{3n^4-1}$, *and let* $x_n=0$ *for all* n. *Then*

$$0=x_n\leqslant y_n=\frac{4n^3+2n^2}{3n^4+(-1)^n}\leqslant\frac{4n^3+2n^2}{3n^4-1}=z_n$$

for all $n\in N$. *By algebraic properties of limits, we obtain*

$$z_n=\frac{4n^3+2n^2}{3n^4-1}=\frac{4\frac{1}{n}+2\frac{1}{n^2}}{3-\frac{1}{n^4}}\to\frac{0+0}{3-0}=0,$$

as $n\to\infty$. *Hence, by the sandwich principle, we have* $y_n\to 0$ *as* $n\to\infty$.

6. For each of the series below, determine whether they converge or diverge. Justify your answers, clearly referring to any results or tests you use from the course. Answers without a justification will receive zero marks.

   (i)
$$\sum_{n=1}^{\infty}(-1)^n\frac{2n^2+1}{n^2}$$ **3**

(ii)
$$\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$$ **3**

(iii)
$$\sum_{n=1}^{\infty} \frac{5^n}{3^n + 3n!}$$ **3**

(i) Setting $a_n = (-1)^n \frac{2n^2+1}{n^2}$ *we have*

$$a_{2n} = \frac{8n^2+1}{4n^2} = \frac{8+\frac{1}{n^2}}{4} \to 2 \neq 0$$

as $n \to \infty$. *Hence the sequence* a_n *has a subsequence which does not converge to* 0*, and therefore does not converge to zero. It follows that the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*
(ii) Let $a_n = \frac{\cos(n)}{n^2}$ *and note* $0 \leqslant |a_n| \leqslant \frac{1}{n^2}$ *for all* $n \in \mathbb{N}$. *By the comparison test, the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent. Therefore it is convergent.*
(iii) Let $a_n = \frac{5^n}{3^n+3n!}$ *and note that for* $n \in \mathbb{N}$,

$$0 \leqslant a_n \leqslant \frac{5^n}{3n!}.$$

By the limit version of the ratio test, the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{5^n}{3n!}$ *is convergent because*
$$\frac{b_{n+1}}{b_n} = \frac{5}{n+1} \to 0$$
as $n \to \infty$. *By the comparison test, therefore* $\sum_{n=1}^{\infty} a_n$ *is convergent.*

7. (i) State the intermediate value theorem. **2**

(ii) Suppose that $f : [0,1] \to [0,1]$ is a continuous function. Show that there exists $c \in [0,1]$ such that $f(c) = 1 - c^2$. **3**

(i) Let $f : [a,b] \to \mathbb{R}$ *be a continuous function, and assume that* $d \in \mathbb{R}$ *satisfies* $f(a) < d < f(b)$ *or* $f(b) < d < f(a)$. *Then there exists a point* $c \in (a,b)$ *such that* $f(c) = d$.
(ii) Consider $g : [0,1] \to \mathbb{R}$ *given by* $g(x) = f(x) - (1 - x^2)$. *Then* g *is continuous. Moreover we have*

$$g(0) = f(0) - 1 \leqslant 0, \quad g(1) = f(1) - 0 \geqslant 0$$

Hence, by the intermediate value theorem there exists $c \in [0,1]$ *such that* $g(c) = 0$. *This means* $0 = f(c) - (1 - c^2)$, *or equivalently,* $f(c) = 1 - c^2$.

8. (i) Let $(x_n)_{n=1}^{\infty}$ be a real sequence, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and assume $\lim_{n\to\infty} x_n = L$. Show that

$$\lim_{n\to\infty} f(x_n) = f(L).$$ **3**

(ii) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ and a convergent real sequence $(x_n)_{n=1}^{\infty}$ such that $(f(x_n))_{n=1}^{\infty}$ does not converge. **2**

(i) Let $\varepsilon > 0$*. By definition of continuity, there exists* $\delta > 0$ *such that* $|x - L| < \delta$ *implies* $|f(x) - f(L)| < \epsilon$*. By definition of convergence, there exists* $n_0 \in \mathbb{N}$ *such that* $|x_n - L| < \delta$ *for all* $n \geqslant n_0$*. Hence for* $n \geqslant n_0$ *we have*

$$|f(x_n) - f(L)| < \varepsilon,$$

which means $\lim_{n\to\infty} f(x_n) = f(L)$*.*
(ii) Take

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leqslant 0 \end{cases}$$

and consider $x_n = \frac{1}{n}$*. Then* $x_n \to 0$ *as* $n \to \infty$*, but the constant sequence* $f(x_n) = 1$ *does not converge to* $f(0) = 0$*.*

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