2C Revision

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Contents

1 Equivalent statements and inequalities

- If we have a non-empty subset S of $\mathbb{R}$ with an upper bound M, $M = \sup(A)$ if and only if any of the following are true:

  1. $\forall \varepsilon > 0,\ \exists s \in S$ s.t. $s > m - \varepsilon$.
  2. $\forall \varepsilon > 0,\ \exists s \in S$ s.t. $s \geq m - \varepsilon$.
  3. $\forall M' < M,\ \exists s \in S$ s.t. $s > M'$.
  4. $\forall M' < M,\ \exists s \in S$ s.t. $s \geq M'$.

- If we have a non-empty subset S of $\mathbb{R}$ with a lower bound m, $m = \inf(A)$ if and only if any of the following are true:

  1. $\forall \varepsilon > 0,\ \exists s \in S$ s.t. $s < m + \varepsilon$.
  2. $\forall \varepsilon > 0,\ \exists s \in S$ s.t. $s \leq m + \varepsilon$.
  3. $\forall m' > m,\ \exists s \in S$ s.t. $s < m'$.
  4. $\forall m' < m,\ \exists s \in S$ s.t. $s \leq m'$.

- The limit of the sequence $(x_n)_{n=1}^{\infty}$ is L if and only if any of the following are true:

  1. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0,\ |x_n - L| < \varepsilon$.
  2. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0,\ |x_n - L| \leq \varepsilon$.
  3. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n > n_0,\ |x_n - L| < \varepsilon$.
  4. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n > n_0,\ |x_n - L| \leq \varepsilon$.

- The function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $c \in \mathbb{R}$ if and only if any of the following are true:

  1. $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$.
  2. $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $|x - c| < \delta \implies |f(x) - f(c)| \leq \varepsilon$.
  3. $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $|x - c| \leq \delta \implies |f(x) - f(c)| < \varepsilon$.
  4. $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $|x - c| \leq \delta \implies |f(x) - f(c)| \leq \varepsilon$.

2 Uniform continuity

The function

$$f(x) = \frac{1}{x - K}$$

is not uniformly continuous on $\mathbb{R}\backslash\{K\}$.

Proof. Set $\varepsilon = \frac{1}{2}$ and let $\delta > 0$ be arbitrary. For $n \in \mathbb{N}$, consider $c_n = -K + \frac{1}{n}$ and $x_n = -K + \frac{1}{2n}$. Then,

$$|f(x_n) - f(c_n)| = \left|\frac{n}{2} - n\right| = \frac{n}{2} \geq \frac{1}{2} = \varepsilon$$

for $n \in \mathbb{N}$. Since $|x_n - c_n| = \frac{1}{n}$, we can take $n > \frac{1}{\delta}$ to find $|x_n - c_n| < \delta$ but $|g(x_n) - g(c_n)| \geq \varepsilon$. Therefore, the function f isn't uniformly continuous on $\mathbb{R}\backslash\{K\}$. □

The function

$$f(x) = x^3$$

is not uniformly continuous on $\mathbb{R}$.

Proof. Set $\varepsilon = 1$ and let $\delta > 0$ be arbitrary. Set $c = \max(1, \frac{1}{\delta})$ and $x = c + \frac{\delta}{3}$. Then, although $|x - c| = \frac{\delta}{3} < \delta$,

$$\begin{aligned}
|f(x) - f(c)| &= |x^3 - c^3| \\
&= \left(c + \frac{\delta}{3}\right)^3 - c^3 \\
&= c^2\delta + \frac{c\delta^2}{3} + \frac{\delta^3}{27} \\
&\geq c + \frac{1}{3c} + \frac{1}{27c^3} && \delta \geq \frac{1}{c} \\
&\geq 1 = \varepsilon && c \geq 1.
\end{aligned}$$

Therefore, the function f isn't uniformly continuous on $\mathbb{R}$. □

The function

$$f(x) = x^3$$

is uniformly continuous on $(-2, 1]$.

Proof. Let $\varepsilon > 0$ be arbitrary. We note that if $x, c \in (-2, 1]$, then

$$|f(x) - f(c)| = |x^3 - c^3| = |x - c| \cdot |x^2 + cx + c^2|.$$

Moreover, for $x, c \in (-2, 1]$,

$$|x^2 + cx + c^2| \leq |x|^2 + |c||x| + |c|^2 \leq 2^2 + 2 \cdot 2 + 2^2 = 12.$$

So, set $\delta = \frac{\varepsilon}{12}$. Then, for $x, c \in (-2, 1]$, we have

$$|f(x) - f(c)| = |x - c| \cdot |x^2 + cx + c^2| \leq 12|x - c| < \varepsilon,$$

as required. Therefore, the function f is uniformly continuous on $(-2, 1]$. $\square$

3 Sets and bounds

The set

$$S = \left\{ 4xy - \frac{2}{z} \mid x, y, z \in (0, 3) \right\}$$

is bounded above, but not bounded below.

Proof. We first show that S is bounded above. So, let $s \in S$ be arbitrary. We know that there exists an $x, y, z \in (0, 3)$ such that $4xy - \frac{2}{z} = s$. Therefore,

$$s = 4xy - \frac{2}{z} \leq 4xy \leq 4 \cdot 3 \cdot 3 = 36.$$

Since $s \in S$ was arbitrary, we conclude that for all $s \in S$, $s \leq 36$.
Now, we show that S isn't bounded below. For that let $K > 0$ be arbitrary. We show that there exists an $s \in S$ such that $s \leq -K$. We note that for $x = y = 1$ and $z \leq \frac{1}{4}$,

$$4xy - \frac{2}{z} = 4 - \frac{1}{z} - \frac{1}{z} \leq 4 - 4 - \frac{1}{z} = -\frac{1}{z}.$$

In that case,

$$4xy - \frac{2}{z} \leq -\frac{1}{z} \leq -K \iff \frac{1}{z} \geq K \iff \frac{1}{K} \geq z.$$

So, set $z = \min(\frac{1}{4}, \frac{1}{K}) \in (0, 3)$. In that case, $s = 4xy - \frac{2}{z} \in S$ and $s \leq -K$, as required. Since $K > 0$ was arbitrary, we have shown that S isn't bounded below. $\square$

4 Series and rational polynomials

1. *The series* $\sum_{n=1}^{\infty} \frac{5n^2+3}{3n-2}$ *doesn't converge since the sequence diverges- this is because the greatest power of the numerator (2) is higher than the greatest power of denominator (1).*

   Since
   $$\frac{5n^2+3}{3n-2} \geq \frac{5n^2}{3n} = \frac{5}{3}n \geq \frac{5}{3}$$
   for all $n \in \mathbb{N}$, the limit cannot be 0. Therefore, the series $\sum_{n=1}^{\infty} \frac{5n^2+3}{3n-2}$ diverges.

   The series $\sum_{n=1}^{\infty}(-1)^n \frac{5n^2+3}{3n-2}$ *won't pass Leibniz because the limit wasn't 0- it will also diverge.*

   If we let $x_n = (-1)^n \frac{5n^2+3}{3n-2}$, then
   $$x_{2n} = \frac{5(2n)^2+3}{6n-2} = \frac{20n^2+3}{6n-2} \geq \frac{20n^2}{6n} = \frac{10}{3}n \geq \frac{10}{3},$$
   so $x_{2n} \not\to 0$, in which case $x_n \not\to 0$. Therefore, the series $\sum_{n=1}^{\infty} x_n$ diverges.

2. *The series* $\sum_{n=1}^{\infty} \frac{5n+3}{3n-2}$ *doesn't converge since the sequence doesn't converge to 0- this is because the greatest power of the numerator (1) is equal to the greatest power of the denominator (1).*

   Since
   $$\frac{5n+3}{3n-2} = \frac{5+3/n}{3-2/n} \to \frac{5+0}{3-0} = \frac{5}{3} \neq 0,$$
   the limit isn't 0.

   The series $\sum_{n=1}^{\infty}(-1)^n \frac{5n+3}{3n-2}$ *won't pass Leibniz because the limit wasn't 0- it will also diverge.*

   If we let $x_n = (-1)^n \frac{5n+3}{3n-2}$, then
   $$x_{2n} = \frac{5(2n)+3}{3(2n)-2} = \frac{10n+3}{6n-2} = \frac{10+3/n}{6-2/n} \to \frac{10+0}{6-0} = \frac{5}{3} \neq 0,$$
   so $x_{2n} \not\to 0$, in which case $x_n \not\to 0$. Therefore, the series $\sum_{n=1}^{\infty} x_n$ diverges.

3. *The series* $\sum_{n=1}^{\infty} \frac{5n+3}{3n^2-2}$ *doesn't converge since it can only be compared with a multiple of the Harmonic series- this is because the denominator is only 1 power higher than the numerator. So, we want a smaller multiple of the harmonic series to compare it to.*

   Since
   $$\frac{5n+3}{3n^2-2} \geq \frac{5n}{3n^2} = \frac{5}{3n} \geq 0,$$
   and $\sum_{n=1}^{\infty} \frac{5}{3n}$ diverges, the series must also diverge by the comparison test.

The series $\sum_{n=1}^{\infty}(-1)^n \frac{5n+3}{3n^2-2}$ *will pass Leibniz because the limit is* 0*- it is conditionally convergent.*

Denote as $(x_n)_{n=1}^{\infty}$ the sequence $x_n = \frac{5n+3}{3n^2-2}$. Then, $x_n \geq 0$ for all $n \in \mathbb{N}$. Also,

$$x_n = \frac{5n+3}{3n^2-2} = \frac{5/n + 3/n^2}{3 - 2/n^2} \to \frac{0+0}{3-0} = 0.$$

Moreover, since (x_n) is decreasing (*Should show that* $x_{n+1} - x_n \leq 0$), we can apply the Leibniz test to conclude that $\sum_{n=1}^{\infty}(-1)^n x_n$ converges.

4. *The series* $\sum_{n=1}^{\infty} \frac{5n+3}{3n^3-2}$ *converges since the denominator has the higher power and the difference of the highest powers in the numerator and the denominator is strictly greater than* 1*- we should use the comparison test to prove this.*

   Since
$$0 \leq \frac{5n+3}{3n^3-2} \leq \frac{5n+3n}{3n^3-2n^3} = \frac{8n}{n^3} = \frac{8}{n^2},$$
   and $\sum_{n=1}^{\infty} \frac{8}{n^2}$ converges, the series must also converge by the comparison test.

   The series $\sum_{n=1}^{\infty}(-1)^n \frac{5n+3}{3n^3-2}$ *will pass the Leibniz because the limit is* 0*, although we don't need it since the sum is absolutely convergent.*

   Since the series $\sum_{n=1}^{\infty} \frac{5n+3}{3n^3-2}$ converges, the series $\sum_{n=1}^{\infty}(-1)^n \frac{5n+3}{3n^3-2}$ must also converge.

5 More on series

1. *Manipulating series: If we have the two series* $\sum_{n=1}^{\infty} \frac{5\sqrt{n}+3}{2n^2\sqrt{n}-1}$ *and* $\sum_{n=1}^{\infty} \frac{5\sqrt{n}-3}{n^2\sqrt{n}+1}$*, then they both converge since the degree of the denominator* (3.5) *is higher than the degree of the numerator* (0.5) *by more than* 1*. We should also use the comparison test, but the proof goes very differently in the two cases because of the signs of the scalars at the end.*

   We note that
   $$0 \leq \frac{5\sqrt{n}+3}{2n^2\sqrt{n}-1} \leq \frac{5\sqrt{n}}{2n^2\sqrt{n}} = \frac{5}{2n^2}$$
   for all $n \in \mathbb{N}$. Moreover, since the series $\sum_{n=1}^{\infty} \frac{5}{2n^2}$ converges, we can apply the comparison test and conclude that the series $\frac{5\sqrt{n}+3}{2n^2\sqrt{n}-1}$ converges.

   We note that
   $$0 \leq \frac{5\sqrt{n}-3}{n^2\sqrt{n}+1} \leq \frac{5\sqrt{n}-3\sqrt{n}}{n^2\sqrt{n}+n^2\sqrt{n}} \leq \frac{2\sqrt{n}}{2n^2\sqrt{n}} = \frac{1}{n^2}$$
   for all $n \in \mathbb{N}$. Moreover, since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we can apply the comparison test and conclude that the series $\sum_{n=1}^{\infty} \frac{5\sqrt{n}-3}{n^2\sqrt{n}+1}$ converges.

2. *Negative series: If a series has a negative terms no matter how far you go into the sequence, then it will (very likely) be convergent. We'd establish this by checking for absolute convergence- very likely using comparison test.*

   We note that
   $$0 \leq \left| \frac{(-1)^n - 2}{5n^2+3n+1} \right| = \frac{|(-1)^n - 2|}{5n^2+3n+1} \leq \frac{1+2}{5n^2+3n+1} \leq \frac{3}{5n^2}$$
   for all $n \in \mathbb{N}$. Moreover, since the series $\sum_{n=1}^{\infty} \frac{3}{5n^2}$ converges, we find that the series $\sum_{n=1}^{\infty} \frac{(-1)^n-2}{5n^2+3n+1}$ is absolutely convergent, and therefore convergent.

3. *Trig in the numerator without* π*: For a series like* $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^3+5^n}$*, we would expect for absolute convergence- very likely using comparison test.* We note that
   $$0 \leq \left| \frac{\cos(n)}{n^3+5^n} \right| \leq \frac{1}{n^3+5^n} \leq \frac{1}{n^3}$$
   for all $n \in \mathbb{N}$. Moreover, since the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, we find that the series $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^3+5^n}$ is absolutely convergent, and therefore convergent.

   Sometimes, it might always be positive so we don't need to take absolute value. Still, we show convergence by the comparison test. We note that
   $$0 \leq \frac{\cos(n)+3}{n^3+5^n} \leq \frac{4}{n^3+5^n} \leq \frac{1}{n^3}$$

for all $n \in \mathbb{N}$. Moreover, since the series $\sum_{n=1}^{\infty} \frac{4}{n^3}$ converges, we find that the series $\sum_{n=1}^{\infty} \frac{\cos(n)+3}{n^3+5^n}$ is absolutely convergent, and therefore convergent.

4. *Ratio test with comparison test: It is possible for the numerator and the denominator to have something that is ideal for the ratio test (raising to the n-th power or factorials). We need to apply the comparison test to select the right expressions from the numerator and the denominator before applying the ratio test.*

   If there is something at the numerator, it is pretty easy to get rid of, like in the series $\sum_{n=1}^{\infty} \frac{5^n-7}{3n!}$ We find that

$$0 \leq \frac{5^n - 7}{3n!} \leq \frac{5^n}{3n!}$$

   for all $n \geq 2$ (*Might not always be $n \in \mathbb{N}$!*). Now, if we let $x_n = \frac{5^n}{3n!}$, we find that

$$\frac{x_{n+1}}{x_n} = \frac{5^{n+1}}{3(n+1)!} \cdot \frac{3n!}{5^n} = \frac{5}{n+1} \leq \frac{5}{6}$$

   for $n \geq 6$. Therefore, the series $\sum_{n=1}^{\infty} x_n$ converges. In that case, the series $\sum_{n=1}^{\infty} \frac{5^n-7}{3n!}$ also converges.

   If there is something in the denominator, we need to select the one that will give convergence with the ratio test, if possible. For example, for the series $\sum_{n=1}^{\infty} \frac{3^{2n}}{5^n+n!}$, we find that the series $\sum_{n=1}^{\infty} \frac{3^{2n}}{5^n}$ diverges since the ratio is $\frac{3^2}{5} > 1$. On the other hand, the series $\sum_{n=1}^{\infty} \frac{3^{2n}}{n!}$ converges since the ratio has limit 0. We note that

$$0 \leq \frac{3^{2n}}{5^n + n!} \leq \frac{3^{2n}}{n!}$$

   for $n \in \mathbb{N}$. Now, denote as $(x_n)_{n=1}^{\infty}$ the sequence $x_n = \frac{9^n}{n!}$. Since

$$\frac{x_{n+1}}{x_n} = \frac{9^{n+1}}{(n+1)!} \cdot \frac{n!}{9^n} = \frac{9}{n+1} \leq \frac{9}{10}$$

   for $n \geq 10$, we can apply the ratio test and conclude that the series $\sum_{n=1}^{\infty} x_n$ converges. In that case, the series $\sum_{n=1}^{\infty} \frac{3^{2n}}{5^n+n!}$ converges as well by the comparison test.

   If there is no term in the denominator that can be used to prove convergence, then the series diverges. To prove this, we have to establish the smallest series in the denominator and use the comparison test on that for divergence. The example here is the series $\sum_{n=1}^{\infty} \frac{n!}{2n!+3^{2n}}$, where both the denominators will give divergence in isolation. Also, a factorial will always eventually win any power.

We note that for $n \geq 22$, $2n! \geq 3^{2n}$. In that case, $2n! + 3^{2n} \leq 2n! + 2n!$, or

$$\frac{n!}{2n! + 3^{2n}} \geq \frac{n!}{2n! + 2n!} \geq 0.$$

Moreover, since $\sum_{n=1}^{\infty} \frac{n!}{4n!} = \frac{1}{4}$ diverges, the series $\sum_{n=1}^{\infty} \frac{n!}{2n!+2n!}$ also diverges by the comparison test.

6 Infimum and supremum

Let
$$S = \left\{ \frac{2n-3}{3n+2} \mid n \in \mathbb{N} \right\}.$$

Then, $\sup(S) = \frac{2}{3}$.

Proof. We first show that $\frac{2}{3}$ is an upper bound for S. We note that

$$\frac{2n-3}{3n+2} - \frac{2}{3} = \frac{6n-9-(6n+4)}{3(3n+2)} = \frac{-13}{3(3n+2)} \leq 0$$

for all $n \in \mathbb{N}$. So, $\frac{2}{3}$ is an upper bound for S.
Now, we show that $\frac{2}{3}$ is the least upper bound for S. So, let $\varepsilon > 0$ be arbitrary. We find that for some $n \in \mathbb{N}$,

$$\begin{aligned}
\frac{2n-3}{3n+2} > \frac{2}{3} - \varepsilon &\iff \frac{2n-3}{3n+2} - \frac{2}{3} > -\varepsilon \\
&\iff \frac{-13}{3(3n+2)} > -\varepsilon \\
&\iff \frac{13}{9n+6} < \varepsilon \\
&\iff \frac{13}{\varepsilon} < 9n+6 \\
&\iff \frac{13}{\varepsilon} - 6 < 9n \\
&\iff \frac{13}{9\varepsilon} - \frac{2}{3} < n.
\end{aligned}$$

So, select $n \in \mathbb{N}$ with $n > \frac{13}{9\varepsilon} - \frac{2}{3}$. In that case, we have $\frac{2n-3}{3n+2} > \frac{2}{3} - \varepsilon$. Therefore, we have found an element $s \in S$ such that $s > \frac{2}{3} - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this implies that $\sup(S) = \frac{2}{3}$. $\square$

Let
$$S = \left\{ \frac{2n+3}{3n-2} \mid n \in \mathbb{N} \right\}.$$
Then, $\inf(S) = \frac{2}{3}$.

Proof. We first show that $\frac{2}{3}$ is a lower bound for S. We note that

$$\frac{2n+3}{3n-2} - \frac{2}{3} = \frac{6n+9-(6n-4)}{3(3n-2)} = \frac{13}{3(3n-2)} \geq 0$$

for all $n \in \mathbb{N}$. So, $\frac{2}{3}$ is a lower bound for S.
Now, we show that $\frac{2}{3}$ is the greatest lower bound for S. So, let $\varepsilon > 0$ be arbitrary. We find that for some $n \in \mathbb{N}$,

$$\begin{aligned}
\frac{2n+3}{3n-2} < \frac{2}{3} + \varepsilon &\iff \frac{2n+3}{3n-2} - \frac{2}{3} < \varepsilon \\
&\iff \frac{13}{9n-6} < \varepsilon \\
&\iff \frac{13}{\varepsilon} < 9n-6 \\
&\iff \frac{13}{\varepsilon} + 6 < 9n \\
&\iff \frac{13}{9\varepsilon} + \frac{2}{3} < n.
\end{aligned}$$

So, select $n \in \mathbb{N}$ with $n > \frac{13}{9\varepsilon} + \frac{2}{3}$. In that case, we have $\frac{2n+3}{3n-2} < \frac{2}{3} + \varepsilon$. Therefore, we have found an element $s \in S$ such that $s < \frac{2}{3} + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this implies that $\inf(S) = \frac{2}{3}$. $\square$

7 Continuity of $\frac{1}{x-K}$

The function

$$f(x) = \frac{1}{x+1}$$

is continuous on $(-1, \infty)$.

Proof. Let $c \in (-1, \infty)$ be arbitrary and let $\varepsilon > 0$ be arbitrary. If we have $\delta \leq \frac{c+1}{2}$, then for $|x - c| < \delta$, we find that

$$\begin{aligned} |x - c| < \frac{c+1}{2} &\implies \frac{-c-1}{2} < x - c < \frac{c+1}{2} \\ &\implies \frac{c-1}{2} < x < \frac{3c+1}{2} \\ &\implies \frac{c+1}{2} < x + 1 < \frac{3c+3}{2}. \end{aligned}$$

In that case,

$$|f(x) - f(c)| = \left| \frac{1}{x+1} - \frac{1}{c+1} \right| = \frac{|x-c|}{(x+1)(c+1)} \leq \frac{2|x-c|}{(c+1)(c+1)}.$$

So, set $\delta = \min(\frac{c+1}{2}, \frac{(c+1)^2}{2}\varepsilon)$. In that case, if $|x - c| < \delta$, then $|x - c| < \frac{c+1}{2}$ and $\frac{2|x-c|}{(c+1)^2} < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, f is continuous at c. Moreover, since $c \in (-1, \infty)$ was arbitrary, we conclude that f is continuous on $(-1, \infty)$. $\square$

8 Series from a positive sequence

Let $(a_n)_{n=1}^{\infty}$ be a sequence with $a_n > 0$ for all $n \in \mathbb{N}$.
Then, if $\sum_{n=1}^{\infty} a_n$ converges,

- $a_n \to 0$ as $n \to \infty$.

  This is true. If a series converges, the sequence must converge to 0.

- for all $\varepsilon > 0$, there exists a $n_0 \in \mathbb{N}$ such that if $n > n_0$, $a_n < \varepsilon$.

  This is true. The limit is 0, *and* $|a_n| = a_n$ *since* $a_n > 0$.

- $\frac{a_{n+1}}{a_n} \to 1$ as $n \to \infty$.

  This is false. We can only say the ratio is less than or equal to 1, not equal to 1.

- $\frac{a_{n+1}}{a_n} \to L < 1$ as $n \to \infty$.

  This is false. We can only say the ratio is less than or equal to 1, not equal to 1.

- $a_n - a_{n+1} \to 0$ as $n \to \infty$. *This is true. We know* $a_n \to 0$ *so* $a_{n+1} \to 0$ *as well, and the algebraic properties of limits tells us that* $a_n - a_{n+1} \to 0$.

- $(a_n)_{n=1}^{\infty}$ is decreasing.

  This is false. There are non-decreasing sequences that result in a convergent series.

- $(a_n)_{n=1}^{\infty}$ is bounded.

  This is true. If the series converges, the sequence is convergent and therefore also bounded.

- $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

  This is false. The series is absolutely convergent and nothing else.

- $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

  This is true.

- $\sum_{n=1}^{\infty} (-1)^n a_n$ is conditionally convergent.

  This is false because we know the series is absolutely convergent.

- $\sum_{n=1}^{\infty} (-1)^n a_n$ is absolutely convergent.

  This is true.

- $\sum_{n=1}^{\infty} n a_n$ is convergent.

  This is false, e.g. let $a_n = \frac{1}{n^2}$*- although* $\sum_{n=1}^{\infty} a_n$ *converges,* $\sum_{n=1}^{\infty} n a_n$ *doesn't converge.*

- $\sum_{n=1}^{\infty} \frac{a_n}{n}$ is convergent.

  This is true. We can apply the comparison test here since $0 \leq \frac{a_n}{n} \leq a_n$ *for all* $n \in \mathbb{N}$.

Also, the following imply that $\sum_{n=1}^{\infty} a_n$ converges

- the sequence $a_n \to 0$ as $n \to \infty$.

  This is false- it is a necessary but not a sufficient condition for the convergence of the series.

- the sequence $(a_n)_{n=1}^{\infty}$ converges to 0 and is monotonic decreasing.

  This is false- it is a necessary but not a sufficient condition for the convergence of the series. Being monotonic decreasing makes no difference.

- the sequence $(a_n)_{n=1}^{\infty}$ is Cauchy.

  This is false- we don't even know whether the sequence converges to 0.

- the series $\sum_{n=1}^{\infty} a_n$ is bounded below.

  This is false- we already knew $\sum_{n=1}^{\infty} a_n \geq 0$, *so this adds nothing.*

- the series $\sum_{n=1}^{\infty} a_n$ is bounded above.

  This is true. Because $a_n > 0$, *the series* $\sum_{n=1}^{\infty}$ *is an increasing sequence. If it is bounded above, then we can apply the monotone convergence theorem to prove that the series is convergent.*

- $\frac{a_{n+1}}{a_n} \to 1$ as $n \to \infty$.

  This is false. If the ratio is 1, *we can't say anything about the convergence of the series.*

- $\frac{a_{n+1}}{a_n} \to L < 1$ as $n \to \infty$.

  This is true. If the ratio is less than 1, *then the ratio test tells us the series converges.*

- $\frac{a_n}{a_{n+1}} \to L < 1$ as $n \to \infty$.

  This is false. If the ratio $\frac{a_n}{a_{n+1}} \to L < 1$, *then the ratio* $\frac{a_{n+1}}{a_n} \to \frac{1}{L} > 1$ *(if L isn't zero- if zero, it diverges automatically), and therefore the series diverges by the ratio test.*

- $a_n - a_{n+1} \to 0$ as $n \to \infty$.

  This is false. This doesn't ensure anything, not even that the series converges to 0.

- the series $\sum_{n=1}^{\infty} na_n$ converges. *This is true. This follows since* $0 \leq a_n \leq na_n$, *so we can apply the comparison test.*

- the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

  This is false. For example, let $a_n = \frac{1}{n^2}$.

- the series $\sum_{n=1}^{\infty} \cos(n) a_n$ is conditionally convergent.

  This is false. By the definition of conditional convergence, we actually find that the series $\sum_{n=1}^{\infty} a_n$ diverges.

- the series $\sum_{n=1}^{\infty} \cos(n) a_n$ is absolutely convergent.

  This is true. If the series is absolutely convergent, then the series $\sum_{n=1}^{\infty} a_n$ must converge.