Inequalities and the modulus function

We use inequalities regularly in mathematical analysis in our arguments. Most often we do not need to get the best possible estimates to make our arguments work, and we can get away with correct but non-optimal inequalities in order to correctly prove statements. For example, in the two proofs given of Example **??** we took the inequality $\frac{3n}{n+1} > 3 - \varepsilon$ and rearranged it to make n the subject, showing that

$$\frac{3n}{n+1} > 3 - \varepsilon \Longleftrightarrow n > \frac{3}{\varepsilon} - 1.$$

However we didn't need to do this, and in some cases it may be essentially impossible to come up with a condition of the form $n > K$ which is *equivalent* to the condition we started with. Here's another proof of Example **??**.

Proof. Let $\varepsilon > 0$ be arbitrary. For $n \in \mathbb{N}$ we have

$$\frac{3n}{n+1} > 3 - \varepsilon \Leftrightarrow \varepsilon(n+1) > 3 \Leftarrow \varepsilon n > 3 \Leftrightarrow n > \frac{3}{\varepsilon}.$$

Therefore take $n \in \mathbb{N}$ with $n > \frac{3}{\varepsilon}$, so that $\frac{3n}{n+1} > 3 - \varepsilon$ holds. □

Basically we've ended up with a "simpler condition" on n which implies that $\frac{3n}{n+1} > 3 - \varepsilon$. Of course, here it was straightforward to rearrange the inequality to make n the subject, so on this occasion we haven't made our life any easier with this third proof. In later examples, however, making the right sort of simplifying estimate will help a lot. The moral of the story is when you're doing this sort of "n-ε argument" we don't need to find the best n which works; any n which works will do.

The modulus function

The modulus function on the real line is defined by

$$|x| = \begin{cases} x, & x \geq 0; \\ -x, & x < 0. \end{cases}$$

We will repeatedly use the following four key properties of the modulus function.

a) For all $x, y \in \mathbb{R}$, $|xy| = |x||y|$;[1]

b) (The triangle inequality). For all $x, y \in \mathbb{R}$, $|x+y| \leq |x| + |y|$;[2]

c) For all $y \in \mathbb{R}$ and $r > 0$, $|y| < r \Longleftrightarrow -r < y < r$;[3]

d) For all $x, a \in \mathbb{R}$ and $r > 0$,

$$|x-a| < r \Longleftrightarrow a - r < x < a + r.$$

[1] This can be proved directly from the definition above by checking all four cases; when x and y are both positive, when x is positive and y is negative, ...

[2] Again this can be proved by checking all four cases.

[3] This follows directly from the definition, and then one obtains (d) by taking $y = x - a$ in (c).

A key piece of intuition is that for $x, a \in \mathbb{R}$, the quantity $|x-a|$ represents **the distance from** x **to** a. We expect distances to obey the inequality[4]

$$\text{distance}(a,c) \leq \text{distance}(a,b) + \text{distance}(b,c).$$

[4] Think of a, b and c being the vertices on a triangle; it's shorter to go from a to c directly compared to traveling from a to c via b.

Taking $x = a-b$ and $y = b-c$ in the triangle inequality, we have

$$|a-c| = |(a-b)+(b-c)| \leq |a-b| + |b-c| \tag{1}$$

so verifying this intuition. It's from this view point that (b) should be thought of as the triangle inequality. The "trick" of adding and subtracting b as in (1) is used a lot; this trick and the inequality (1) needs to be part of your tool kit.

The condition $|x-a| < r$ in (d) says that "the distance between x and a is less than r", i.e. that x is trapped within a band of radius r around a (as described by the inequality $a - r < x < a + r$. We'll move between these two equivalent statements a lot from chapter 2 onwards.

Example 2.1. Show that[5] for all $x \in \mathbb{R}$,

$$|x-2| \leq 1 \implies \frac{5}{6} \leq \left|\frac{x+4}{x+3}\right| \leq \frac{7}{4}. \tag{2}$$

Hence, find $K \in \mathbb{R}$ such that for all $x \in \mathbb{R}$,

$$|x-2| \leq 1 \implies \left|\frac{x^2+7x+7}{x+3} - 5\right| \leq K|x-2|. \tag{3}$$

[5] At this point it is far from clear why we would want to do this. By the end of the course, you will see that the estimates here form the meat of the proof that the function $f(x) = \frac{x^2+7x+7}{x+3}$ is continuous at $x = 2$.

Let's look at the implication (2) first. We're told that x is within distance 1 of the point 2, and have to obtain an estimate on the size of $\left|\frac{x+4}{x+3}\right|$. Using item (c) above[6], we have

$$\begin{aligned} |x-2| \leq 1 &\Leftrightarrow -1 \leq x-2 \leq 1 \\ &\Leftrightarrow 4 \leq x+3 \leq 6 \\ &\Leftrightarrow 5 \leq x+4 \leq 7. \end{aligned}$$

[6] we could also use item (d). Note too that while both of (c) and (d) are stated for strict inequalities, the corresponding versions also hold when we work with $\leq$ and $\geq$ as here.

In particular, [7]

$$|x-2| \leq 1 \Rightarrow 4 \leq |x+3| \leq 6 \text{ and } 5 \leq |x+4| \leq 7. \tag{4}$$

[7] Would it be correct to use $\Leftrightarrow$ in place of $\Rightarrow$ in (4)?

Now

$$4 \leq |x+3| \leq 6 \Rightarrow \frac{1}{6} \leq \frac{1}{|x+3|} \leq \frac{1}{4},$$

so that[8]

$$|x-2| \leq 1 \Rightarrow \frac{5}{6} \leq \frac{|x+4|}{|x+3|} \leq \frac{7}{4}, \tag{5}$$

as required.

[8] A typical mistake is to incorrectly conclude that

$$|x-2| \leq 1 \Rightarrow \frac{5}{4} \leq \frac{|x+4|}{|x+3|} \leq \frac{7}{6}$$

from (4) by not thinking about what happens when you "cross multiply" inequalities. Whenever you are multiplying and dividing inequalities makes sure what you've done is correct; errors like the one above happen quite frequently.

To make progress with the inequality (3) I'd start by simplifying the expression $\left|\frac{x^2+7x+7}{x+3} - 5\right|$, looking for factors of $|x-2|$. We have

$$\begin{aligned} \left|\frac{x^2+7x+7}{x+3} - 5\right| &= \left|\frac{x^2+7x+7-5x-15}{x+3}\right| \\ &= \left|\frac{(x-2)(x+4)}{x+3}\right| = \frac{|x+4|}{|x+3|}|x-2|. \end{aligned}$$

Thus we can take $K = \frac{7}{4}$, and then using (5), we have

$$|x-2| \le 1 \Rightarrow \left| \frac{x^2+7x+7}{x+3} - 5 \right| \le K|x-2|.$$

A useful estimation lemma

We will often need to estimate polynomials $p(x)$ for large values of x. Intuitively we know that the dominating term is the term of highest degree, and the next lemma gives a method of controlling a polynomial by this term.[9]

Lemma 2.2 (Polynomial estimation lemma). *Let $n \in \mathbb{N}$, and suppose we are given real numbers $a_0, a_1, \cdots, a_n$ with $a_n > 0$. Write*

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Then there exists $N > 0$ such that

$$x \ge N \implies \frac{1}{2}a_n x^n \le p(x) \le \frac{3}{2}a_n x^n.$$

Proof. Let $n, a_n, \cdots, a_0$ be as in the statement of the lemma. Using the triangle inequality, and property (a) of the modulus function, we have

$$\begin{aligned} |p(x) - a_n x^n| &= |a_{n-1}x^{n-1} + \cdots + a_1 x + a_0| \\ &\le |a_{n-1}||x|^{n-1} + \cdots + |a_1||x| + |a_0|. \end{aligned} \tag{6}$$

Note that for $x \ge 1$, we have $|x|^r \le x^{n-1}$ for $r = 0, 1, \ldots, n-1$. This gives

$$x \ge 1 \implies$$
$$|a_{n-1}||x|^{n-1} + \cdots + |a_1||x| + |a_0| \le x^{n-1}\left(|a_{n-1}| + \cdots + |a_1| + |a_0|\right).$$

Setting $K = |a_{n-1}| + \cdots + |a_1| + |a_0|$ we can rewrite this as

$$x \ge 1 \implies |a_{n-1}||x|^{n-1} + \cdots + |a_1||x| + |a_0| \le Kx^{n-1}. \tag{7}$$

Then, for $x > 0$, we have

$$Kx^{n-1} \le \frac{a_n x^n}{2} \iff \frac{2K}{a_n} \le x. \tag{8}$$

Take[10] $N = \max(1, \frac{2K}{a_n})$. For $x \ge N$, we have $x \ge 1$ and $x \ge \frac{2K}{a_n}$ so by (6), (7) and (8), we have

$$|p(x) - a_n x^n| \le \frac{1}{2}a_n x^n \Leftrightarrow \frac{1}{2}a_n x^n \le p(x) \le \frac{3}{2}a_n x^n,$$

where the last equivalence is property (d) of the modulus function. $\square$

Note that we can assume that the $N > 0$ given in the lemma is a natural number if we wish (by replacing the N by a larger value which lies in $\mathbb{N}$).

Example 2.3. Show that there exist $K, N > 0$ such that for $x \in \mathbb{R}$,

$$x \ge N \implies \frac{3x^2 - 4x + 8}{5x + 6} \ge Kx. \tag{9}$$

[9] When you're reading the proof of the lemma, there's a number of things you should try and think about in order to understand what is going on. Where is the hypothesis that $a_n > 0$ used — it's not explicitly mentioned? How would you state (and prove) a version of the lemma for a polynomial with $a_n < 0$? How would you prove a version of the lemma which replaced the conclusion (2.2) with

$$x \ge N \implies \frac{3}{4}a_n x^n \le p(x) \le \frac{5}{4}a_n x^n?$$

(This is perfectly possible, and the choice to use $\frac{1}{2}$ and $\frac{3}{2}$ in the lemma is just that — a choice). Could you prove the stronger statement that: $\forall \varepsilon > 0, \exists N > 0$ such that

$$x \ge N \implies$$
$$(1-\varepsilon)\, a_n x^n \le p(x) \le (1+\varepsilon)\, a_n x^n?$$

[10] This is a really important idea, which you'll see repeatedly in this course. We want two conditions to hold, namely $x \ge 1$ and $x \ge \frac{2K}{a_n}$, and we need to package this in one condition of the form $x \ge N$. Taking N to be the maximum of 1 and $\frac{2K}{a_n}$ achieves this. Notice that given a finite list of real numbers $b_1, \ldots, b_m$, the number $\max(b_1, \ldots, b_m)$ is defined to be the largest of these numbers.

Solution. By Lemma 2.2, there exist $N_1, N_2 > 0$ such that[11]

$$x \geq N_1 \implies \frac{1}{2}3x^2 \leq 3x^2 - 4x + 8 \leq \frac{3}{2}3x^2$$
$$x \geq N_2 \implies \frac{1}{2}5x \leq 5x + 6 \leq \frac{3}{2}5x.$$

Then define $N = \max(N_1, N_2)$,[12] so that

$$x \geq N \implies \frac{1}{2}3x^2 \leq 3x^2 - 4x + 8 \leq \frac{3}{2}3x^2 \text{ and } \frac{1}{2}5x \leq 5x + 6 \leq \frac{3}{2}5x.$$

You should note that the last condition $\frac{1}{2}5x \leq 5x + 6 \leq \frac{3}{2}5x$ ensures that $x \geq 0$, so we can deduce that $\frac{2}{15x} \leq \frac{1}{5x+6} \leq \frac{2}{5x}$. Therefore

$$x \geq N \implies \frac{3x^2 - 4x + 8}{5x + 6} \geq \frac{\frac{1}{2}3x^2}{\frac{3}{2}5x} = \frac{1}{5}x.$$

Hence the implication (9) holds with this value of N and $K = 1/5$. □

Using the estimation lemma 2.2 isn't the only approach here. You could also examine the inequality directly, arguing as follows.

Solution. For $x \geq 4$, we have $4x \leq x^2$ so that

$$3x^2 - 4x + 8 \geq 3x^2 - x^2 = 2x^2.$$

Similarly, for $x \geq 1$, we have

$$5x + 6 \leq 11x.$$

Therefore, when $x \geq 4$, we have

$$\frac{3x^2 - 4x + 8}{5x + 6} \geq \frac{2x^2}{11x} = \frac{2}{11}x,$$

and so we can take $N = 4$ and $K = \frac{2}{11}$. □

Order, bounds, and axioms for the real numbers

The ordered field axioms

What are the real numbers anyway? We have been working with them for some time, but without having discussed what it means to be a real number, and exactly what properties they have. In this course we will take an axiomatic approach, describing precisely the properties, or rules, we require our real numbers to obey — in principle all other properties are deducible from these rules[13]. In our work on the foundations of analysis, we will focus on the key differences between the rational numbers and the real numbers.

Let us start with our axiomatic description of $\mathbb{R}$ by discussing the *field axioms*.

Axiom 2.4. The real numbers $\mathbb{R}$ are a set equipped with an addition $+$, and multiplication $\cdot$, and form a *field* under these operations, that is, the following nine axioms are satisfied[14].

[11] Note that we should use different symbols N_1 and N_2 for the two applications of the lemma in the first instance.

[12] Another application of the "max trick" — see side note 10.

[13] We will not, however, show how to construct the real numbers, i.e. demonstrate that there is a set which obeys these rules. This can be done, though it is relatively time consuming. If you're interested I'd recommend reading Rudin's account of the construction of the reals from the rationals via Dedekind cuts in Chapter 1 of his book. We will also not discuss uniqueness: to what extent is the real number system unique, or are there multiple constructions of systems obeying our axioms which behave differently? The short answer is that the real numbers are unique, but to make this precise would be too big a diversion.

[14] I do not expect you to remember these axioms in this course.

a) $\forall a, b \in \mathbb{R}, a + b = b + a$;

b) $\forall a, b, c \in \mathbb{R}, (a + b) + c = a + (b + c)$;

c) $\exists 0 \in \mathbb{R}$ s.t. $\forall a \in \mathbb{R}, 0 + a = a$;

d) $\forall a \in \mathbb{R}, \exists(-a) \in \mathbb{R}$ s.t. $a + (-a) = 0$;

e) $\forall a, b \in \mathbb{R}, a \cdot b = b \cdot a$;

f) $\forall a, b, c \in \mathbb{R}, a \cdot (b \cdot c) = (a \cdot b) \cdot c$;

g) $\exists 1 \in \mathbb{R}$ s.t. $(1 \neq 0$ and $(\forall a \in A, a \cdot 1 = a))$;

h) $\forall a \in \mathbb{R}$ with $a \neq 0, \exists a^{-1} \in \mathbb{R}$ s.t. $a \cdot a^{-1} = 1$;

i) $\forall a, b, c \in \mathbb{R}, a \cdot (b + c) = (a \cdot b) + (a \cdot c)$;

What this essentially means is that we can add, subtract, multiply[15] and divide[16] real numbers in the usual way.

For example, the axioms imply the usual cancellation operations: if $a, b, x \in \mathbb{R}$ satisfy $a + x = b + x$, then $a = b$. To prove this formally, we would argue as follows. For $a, b, x \in \mathbb{R}$,

$$\begin{aligned}
& a + x = b + x \\
\implies & (a + x) + (-x) = (b + x) + (-x) \\
\implies & a + (x + (-x)) = b + (x + (-x)) \\
\implies & a + 0 = b + 0 \\
\implies & a = b,
\end{aligned}$$

using axiom (d) for the existence of $(-x)$ for the first implication; (b) for the second; (d) for the third; and (c) for the last implication.

It is possible to use the algebraic axioms above to prove a number of simple consequences about the behaviour of the algebraic operations in a field. You should have seen some of these in 2B or 2F, and you can find more examples in Section 1.1 of [ERA].

We have seen a number of examples of fields in the mathematics studied so far: notably the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$. We will need extra axioms to distinguish $\mathbb{R}$ from these other examples.

The next step for this is to introduce the order relation $<$ on $\mathbb{R}$ and specify its behaviour.

Axiom 2.5 (Order axioms for $\mathbb{R}$)**.** There is a relation $>$ on $\mathbb{R}$ satisfying

j) For all $a \in \mathbb{R}$, exactly one of the statements $a = 0, a > 0$ and $0 > a$ is true;

k) For all $a, b \in \mathbb{R}$, $b > a \iff b - a > 0$;

l) For all $a, b \in \mathbb{R}$ with $a > 0$ and $b > 0$, we have $a + b > 0$ and $ab > 0$.

[15] We will usually omit the $\cdot$ sign and write ab instead of $a \cdot b$ for the product of two real numbers.

[16] provided that the denominator is non-zero, of course.

As usual, we also write $a < b$ instead of $b > a$, and $a \geq b$ for $a > b$ or $a = b$. Similarly, we also write $a \leq b$ for $b \geq a$. Notice that the order axiom only introduces the relation $>$, so that in our axiomatic approach we have to explain these other symbols before we can use them [17].

[17] Try to rewrite the conditions in the order axiom in terms of $<, \geq, \leq$.

The thing to take away so far is that the usual arithmetic rules for manipulating addition, subtraction, multiplication and division, and the order structure can be axiomatised[18].

[18] As already mentioned above, I do not expect you to know these axioms by heart, though of course I do expect you to be able to manipulate inequalities accurately.

The rational numbers $\mathbb{Q}$ also have an order $<$, and they also obey the order axioms above — in particular, the 12 axioms so far do not suffice[19] to distinguish $\mathbb{R}$ from $\mathbb{Q}$. Among other things, this means it is not possible to prove there exists a real number x with $x^2 = 2$ from the ordered field axioms, since we know (from 1R and 1X) that there does not exist a rational number x with $x^2 = 2$. Therefore we should not talk of the square root function $\sqrt{\cdot}$ at present — we have not defined it yet[20]. Likewise, many other familiar mathematical functions, such as $\exp, \log, \sin, \cos$, have not yet been defined from our point of view — we will formally define these using power series in the 3H course Analysis of Differentiation and Integration. At the moment, the only operations that we have defined are the basic arithmetic operations of addition, subtraction, multiplication[21] and division. Nevertheless, functions like exp and log may occasionally appear in the exercises — when this happens you should feel free to use any familiar properties of these functions unless otherwise stated.

[19] They suffice to distinguish $\mathbb{R}$ from $\mathbb{C}$: there is no order $<$ on $\mathbb{C}$ satisfying axioms (i), (j) and (k). Note that I really do mean that no possible order $<$ can be constructed, and I invite you to prove this as an extra exercise on sheet 2.

[20] By the end of this chapter, we will see one possible way of defining $\sqrt{\cdot}$.

[21] and hence taking powers of the form a^n where a is real and $n \in \mathbb{N}$.

Bounds

In order to state the final axiom that is needed to describe the real numbers, we will have to investigate bounded sets. Let us make some precise definitions.

Definition 2.6. Let $A \subseteq \mathbb{R}$. We say that $M \in \mathbb{R}$ is *an upper bound for* A if and only if for all $a \in A$, we have $a \leq M$. Define A to be *bounded above* if and only if there exists an upper bound for A.

Similarly, $m \in \mathbb{R}$ is said to be a *lower bound* for $A \subseteq \mathbb{R}$ if and only if for all $a \in A$, we have $m \leq a$. Say that A is *bounded below* if and only if there exists a lower bound for A.

Symbolically, the set $A \subseteq \mathbb{R}$ is bounded above if and only if

$$\exists M \in \mathbb{R} \text{ s.t. } \forall a \in A, a \leq M,$$

and A is bounded below if and only if

$$\exists m \in \mathbb{R} \text{ s.t. } \forall a \in A, m \leq a.$$

Definition 2.7. Let $A \subseteq \mathbb{R}$. We say that A is *bounded* if and only if A is bounded above and bounded below.

It is often useful to reformulate boundedness using the modulus function, as in the following lemma[22].

[22] I'm quite happy for you to use this lemma without comment in your work, so if you want to prove a set A is bounded, feel free to either show it is both bounded above, and bounded below, or show there exists $K > 0$ such that for all $x \in A$, we have $|x| \leq K$.

Lemma 2.8. *Let $A \subseteq \mathbb{R}$. Then A is bounded if and only if,*

$$\exists K > 0 \text{ s.t. } \forall a \in A, |a| \leq K.$$

Proof. Suppose first that A is bounded. Then there exists $M \in \mathbb{R}$ and $m \in \mathbb{R}$ such that for all $x \in A$, we have $m \leq x \leq M$. Let us define $K = \max(|M|, |m|) + 1$, so that $K > 0$ and for $x \in A$, we have $|x| \leq K$ as desired[23].

Conversely, assume there exists $K > 0$ such that for all $x \in A$, we have $|x| \leq K$. Then $-K \leq x \leq K$, so A is bounded above by K and below by $-K$, that is, A is bounded. □

[23] The reason I added 1 to $\max(|M|, |m|)$ was to ensure that $K > 0$ — if I'd just taken $K = \max(|M|, |m|)$, then when $M = m = 0$ (which could happen when $A = \{0\}$) we would have $K = 0$, and this doesn't satisfy all the conditions. There are other ways of solving this problem, for example, by taking $K = \max(|M|, |m|, 1)$.

Let us look at some examples.

Example Show that the set $P = \{4\sin(x) - \cos(3y) \mid x, y \in \mathbb{R}\}$ is bounded and $Q = \{\frac{1}{x-1} \mid x > 1\}$ is not bounded.

Proof that P is bounded. For $x, y \in \mathbb{R}$, we have[24]

$$\begin{aligned} |4\sin(x) - \cos(3y)| &\leq |4\sin(x)| + |-\cos(3y)| \\ &\leq 4|\sin(x)| + |\cos(3y)| \leq 5. \end{aligned}$$

Therefore P is bounded by Lemma 2.8. □

[24] Be careful in performing the inequalities below: in particular note how we use the triangle inequality to estimate the modulus $|4\sin(x) - \cos(3y)|$.

For Q it is worth getting a feel for what is going on. Firstly, we can see that all the elements of Q are positive, so Q is certainly bounded below. Therefore, we should be trying to show that Q is not bounded above. If you're not sure how to do this, write down the formal negation of the statement that Q is bounded above, so you know what to do. That is, Q is not bounded above if and only if

$$\forall M \in \mathbb{R}, \exists q \in Q \text{ s.t. } q > M.$$

This tells us we should start our proof by taking an arbitrary value of $M \in \mathbb{R}$, and showing that this is not an upper bound[25] for Q.

[25] In particular, an answer consisting of a waffly explanation that Q contains very large values, so isn't bounded above, won't get any credit in this course.

Proof that Q is not bounded. Let $M \in \mathbb{R}$ be arbitrary. If $M \leq 0$, then as $1 = \frac{1}{2-1} \in Q$, we see that M is not an upper bound for Q. Suppose then that $M > 0$. Take $x = \frac{1}{2M} + 1$, so that[26] $x > 1$. Then $\frac{1}{x-1} = 2M > M$. Since $2M \in Q$, it follows that M is not an upper bound for Q. Since $M \in \mathbb{R}$ was arbitrary, Q is not bounded above, and so not bounded. □

[26] How did I decide to make this choice? I knew that I wanted to have $x > 1$ with $\frac{1}{x-1} > M$, so I rearranged this last inequality: $\frac{1}{x-1} > M \Leftrightarrow \frac{1}{M} + 1 > x$ (provided $x - 1 > 0$). Therefore I need to choose some x satisfying $1 < x < 1 + \frac{1}{M}$, leading me to take the average of the two end points of this inequality, namely $x = 1 + \frac{1}{2M}$.

We apply all these boundedness definitions also to *functions*. Given a set X and a function $f : X \to \mathbb{R}$, we say that f is *bounded* (bounded above or bounded below) if and only if the set $f(X)$ is bounded (bounded above or bounded below). Here

$$f(X) = \{f(x) \mid x \in X\}$$

is the range of f — the set of all values taken by f, compare 2F.

Example Let $f : \mathbb{N} \to \mathbb{R}$ be given by

$$f(n) = \frac{2n^2 - 7n + 1}{3n^2 - 2n - 5}.$$

Show that f is bounded above.

We can approach this example in a number of different ways. Firstly, let us use the polynomial estimation lemma (Lemma 1.9) to control the values of $f(n)$ in terms of the leading terms of the denominator and numerator when n is large.

Solution. By Lemma 1.9, there exist[27] $N_1, N_2 \in \mathbb{N}$ such that

$$n \geq N_1 \implies \frac{1}{2}2n^2 \leq 2n^2 - 7n + 1 \leq \frac{3}{2}2n^2;$$
$$n \geq N_2 \implies \frac{1}{2}3n^2 \leq 3n^2 - 2n - 5 \leq \frac{3}{2}3n^2.$$

Define $N = \max(N_1, N_2)$ so that for $n \geq N$, we have

$$f(n) \leq \frac{\frac{3}{2}2n^2}{\frac{1}{2}3n^2} = 2.$$

Define[28] $M = \max(f(1), f(2), \ldots, f(N-1), 2)$. Then, for any $n \in \mathbb{N}$, we have $f(n) \leq M$, so that f is bounded above[29]. □

We could also proceed directly, by thinking about how to get upper bounds for the numerator and denominator.

Another Solution. For $n \in \mathbb{N}$, we have $2n^2 - 7n + 1 \leq 2n^2$ (as $-7n + 1 \leq 0$). We also have[30] $2n + 5 \leq n^2$ when $n \geq 4$. Therefore, for $n \geq 4$, we have $3n^2 - 2n - 5 \geq 3n^2 - n^2 = 2n^2$. This gives

$$f(n) \leq \max(f(1), f(2), f(3), 1),$$

and so f is bounded above[31]. □

[27] Recall the remark following lemma 1.9, that says that we can take the real number N in that lemma to be a natural number.

[28] The point is that since the domain of f is $\mathbb{N}$, there are only finitely many values $n \in \mathbb{N}$ with $n \leq N$, so we can list $f(1), \ldots, f(N-1)$ and take the largest number in this list.

[29] To see this, note that if $n \leq N-1$, we have $f(n) \leq \max(f(1), \ldots, f(N-1)) \leq M$, while if $n \geq N$, then $f(n) \leq 2 \leq M$.

[30] Make sure that you check this.

[31] The advantage of this method, is that we could compute $f(1), f(2), f(3)$ to find an explicit upper bound, unlike the previous solution, which only shows that an upper bound exists, but doesn't give a method for finding one.