2B Linear Algebra

Ex Sheet **3**

True/False

[1]

a) If A, B and C are $n \times n$ matrices and $AB = AC$ then $B = C$.

b) If A and B are $n \times n$ matrices and $AB = I$ then $B = A^{-1}$.

c) If A is an invertible matrix then the inverse of A can be written as $\frac{1}{A}$.

d) If A is invertible then the system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution.

e) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then A is invertible if and only if $ad - bc > 0$.

f) If A and B are invertible then AB is invertible and $(AB)^{-1} = A^{-1}B^{-1}$.

g) If A is invertible then A^T is invertible, and the inverse of A^T is the transpose of A^{-1}.

h) Performing an elementary row operation is equivalent to left multiplication by an elementary matrix.

i) Every invertible matrix A can be expressed uniquely as a product of elementary matrices.

j) Let A be a non-invertible $n \times n$ matrix. Then the matrix obtained by swapping rows one and two of A is also non-invertible.

k) Let A be an invertible matrix. Then the matrix obtained by adding twice the first row to the second may not be invertible.

l) Let A be a non-invertible $n \times n$ matrix. Then the linear system $A\boldsymbol{x} = \boldsymbol{0}$ has infinitely many solutions.

m) For matrices A, B and C, if A is row equivalent to B and B is row equivalent to C then A is row equivalent to C.

n) Every elementary matrix is invertible.

o) The product of two $n \times n$ elementary matrices must also be an elementary matrix.

p) If E_1 and E_2 are both $n \times n$ elementary matrices then $E_1E_2 = E_2E_1$.

q) A square matrix A is invertible if and only if its reduced row echelon form is the identity.

r) Any line in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.

[1] **True/False Questions**

Every Exercise Sheet will have a section containing true/false questions. They are designed to test your understanding from lectures. You should look at your lecture notes and/or the textbook to help you answer these questions, but you should not need to write anything to work out the solution.

s) The set of solutions to the equation $3v - 10w + 3x + 2y - 5z = 0$ is a subspace of $\mathbb{R}^5$.

t) The set $\{(a, b, c, 0, 0) : a, b, c \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^5$.

u) Let A be an $m \times n$ matrix over $\mathbb{R}$. Then the row space of A is the subspace of $\mathbb{R}^n$ given by the solutions to $Av = \mathbf{0}$.

v) Let A be an $m \times n$ matrix over $\mathbb{R}$. Then the null space of A is a subspace of $\mathbb{R}^n$.

Solutions to True/False

(a) F (b) T (c) F (d) T (e) F (f) F (g) T (h) T (i) F (j) T (k) F (l) T (m) T (n) T (o) F (p) F (q) T (r) F (s) T (t) T (u) F (v) T

Tutorial Exercises

T1 Let

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \; B = \begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix}.$$

Work out $A^{-1}B$.

Solution

First we calculate the inverse of A to be

$$\frac{1}{-2}\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$$

Then the required product is

$$A^{-1}B = \begin{bmatrix} -\frac{9}{2} & 5 \\ \frac{7}{2} & -2 \end{bmatrix}.$$

T2 Use Theorem 3.8 to determine whether the matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is invertible, and if it is invertible finds its inverse.

Solution

The matrix has determinant $\cos^2\theta + \sin^2\theta = 1 \neq 0$ so it is invertible (no matter what the value of θ). Its inverse is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

T3 Let A and P be $n \times n$ matrices; P being invertible. Simplify each of the following expressions:

a) $(P^{-1}AP)^2$, b) $(P^{-1}AP)^3$.

Solution

a) $(P^{-1}AP)^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A(PP^{-1})AP = P^{-1}A(I)AP = P^{-1}A^2P$.

b) $(P^{-1}AP)^3 = (P^{-1}AP)^2(P^{-1}AP) = P^{-1}A^2(PP^{-1})AP = P^{-1}A^2(I)AP = P^{-1}A^3P$.

T4 Find the inverse of the given matrix if it exists[2], or justify why it does not exist. For 2×2 matrices you can use either Theorem 3.8 or the Gauss–Jordan method.

[2] and check your answer!

$$A = \begin{pmatrix} -3 & 6 \\ 4 & 5 \end{pmatrix}; \quad B = \begin{pmatrix} 6 & -4 \\ -3 & 2 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 4 & 4 \\ 1 & 2 & 4 \\ 1 & 3 & 2 \end{pmatrix}.$$

Solution

The Gauss–Jordan method is written out for the 2×2 matrices here.

Let's first consider A: the Gauss–Jordan method tells us that we need to find the reduced row echelon form of the matrix

$$(A \mid \mathbb{I}_2) = \begin{pmatrix} -3 & 6 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{pmatrix}$$

Some calculation shows that the reduced row echelon form is

$$\begin{pmatrix} 1 & 0 & -\frac{5}{39} & \frac{2}{13} \\ 0 & 1 & \frac{4}{39} & \frac{1}{13} \end{pmatrix}$$

This is of the form $(\mathbb{I}_2 \mid X)$ and so

$$X = \begin{pmatrix} -\frac{5}{39} & \frac{2}{13} \\ \frac{4}{39} & \frac{1}{13} \end{pmatrix}$$

is the inverse of A.

Now let's consider B: we need to find the reduced row echelon form of the matrix

$$(B \mid \mathbb{I}_2) = \begin{pmatrix} 6 & -4 & 1 & 0 \\ -3 & 2 & 0 & 1 \end{pmatrix}$$

Again some calculation shows that the reduced row echelon form is

$$\begin{pmatrix} 1 & -\frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

This is not of the form $(\mathbb{I}_2 \mid Y)$ and so B is not invertible.

Finally we consider C: we need to find the reduced row echelon form of the matrix

$$(C \mid \mathbb{I}_3) = \begin{pmatrix} 1 & 4 & 4 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{pmatrix}$$

Again some calculation shows that this is the matrix

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 2 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \end{array}\right)$$

This is of the form $(\mathbb{I}_3 \mid Z)$ and so

$$Z = \begin{pmatrix} -2 & 1 & 2 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \end{pmatrix}$$

is the inverse of C.

T5 For which real values of a and b is the matrix $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ invertible? For these values, find A^{-1}.

Solution

We have $\det(A) = a^2 + b^2$. Since a and b are real numbers, $a^2 + b^2 \geq 0$ with $a^2 + b^2 = 0$ if and only if $a = b = 0$. So A is invertible unless a and b both equal 0. For all other real values of a and b, $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \frac{1}{a^2+b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

T6 Let A be as in T4 and let $\boldsymbol{b} = \begin{pmatrix} 39 \\ 13 \end{pmatrix}$. Use A^{-1} to solve the system $A\boldsymbol{x} = \boldsymbol{b}$.

Solution

The unique solution to this system is

$$A^{-1}\boldsymbol{b} = \begin{pmatrix} -\frac{5}{39} & \frac{2}{13} \\ \frac{4}{39} & \frac{1}{13} \end{pmatrix} \begin{pmatrix} 39 \\ 13 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}.$$

T7 In the context of 3×3 matrices, write down the elementary matrix E corresponding to the ERO

$$R_2 \to R_2 + 5R_3.$$

Verify that pre-multiplication by E has the same effect on a 3×3 matrix as the ERO

$$R_2 \to R_2 + 5R_3.$$

Solution

The elementary matrix corresponding to the ERO $R_2 \to R_2 + 5R_3$ is

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$

For $a, b, c, d, e, f, g, h, i \in \mathbb{R}$, let F be the matrix

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Then

$$EF = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d+5g & e+5h & f+5i \\ g & h & i \end{bmatrix},$$

which is the same as performing the ERO $R_2 \to R_2 + 5R_3$ on F.

T8 Consider the matrices

$$A = \begin{pmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 4 & 7 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 0 & -5 & 25 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 \\ -5 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}.$$

a) Find elementary matrices E_1 and E_2 such that $E_1 A = B$ and $E_2 B = A$.

b) Write the matrix C as a product of elementary matrices.

c) Is it possible to repeat part (b) for the matrix D? Justify your answer.

Solution

a) We first find an elementary row operation (ERO) which takes the matrix A to the matrix B:

$$\begin{pmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 4 & 7 & 9 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} \begin{pmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 0 & -5 & 25 \end{pmatrix}$$

Clearly to get back to the matrix A from the matrix B we need to reverse this ERO:

$$\begin{pmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 0 & -5 & 25 \end{pmatrix} \xrightarrow{R_3 \to R_3 + 2R_1} \begin{pmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 4 & 7 & 9 \end{pmatrix}$$

Now recall that doing an ERO to a matrix is the same thing as left multiplying by the *elementary matrix* corresponding to the ERO. What is the elementary matrix E_1 corresponding to the ERO

$R_3 \to R_3 - 2R_1$? This is just the matrix we get by applying this ERO to the 3×3 identity matrix $\mathbb{I}_3$, in other words

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

Therefore $E_1 A = B$ (since B is obtained from A by doing the ERO corresponding to E_1). Likewise the elementary matrix E_2 corresponding to the ERO $R_3 \to R_3 + 2R_1$ is

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

Therefore $E_2 B = A$ (since A is obtained from B by doing the ERO corresponding to E_2). Since the two EROs reverse the effect of each other (i.e. doing one and then the other one gets you back to where you started) the matrices E_1 and E_2 are inverse to one another, i.e. $E_1 E_2 = \mathbb{I}_3 = E_2 E_1$.

b) We can write the matrix C as a product of elementary matrices if and only if we can find a sequence of EROs which make C row equivalent to $\mathbb{I}_2$, i.e. if and only if the reduced row echelon form of C is $\mathbb{I}_2$. We have

$$\begin{pmatrix} 1 & 0 \\ -5 & 2 \end{pmatrix} \xrightarrow{R_2 \to R_2 + 5R_1} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \xrightarrow{R_2 \to \frac{1}{2} R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The EROs $R_2 \to R_2 + 5R_1$ and $R_2 \to \frac{1}{2} R_2$ correspond to elementary matrices E_1 and E_2 given by

$$E_1 = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

respectively. Therefore $E_2 E_1 C = \mathbb{I}_2$ (note the order), and so

$$C = E_1^{-1} E_2^{-1} \mathbb{I}_2 = E_1^{-1} E_2^{-1}.$$

By Theorem 3.11 the matrices E_1^{-1} and E_2^{-1} are elementary, and it is easy to see that

$$E_1^{-1} = \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix} \quad \text{and} \quad E_2^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

c) By Theorem 3.12, D can be written as a product of elementary matrices if and only if D is invertible. But D is not invertible, since $\det(D) = 0$. So we cannot repeat part (b) for D.

T9 Suppose A and B are $n \times n$ matrices such that $(AB)^2 = I$. Prove that A is invertible and find A^{-1} in terms of A and B. Show also that $(BA)^2 = I$.

Solution

In this and subsequent questions we have to know what 'invertible' means.

Advice: *realising that definitions are central to understanding mathematics is often the key to doing well in mathematics at Level 2 and beyond. It's impossible to get started on many questions without knowing the definitions, but on the plus side, it can become easy to solve such questions when you get the hang of it. This* *word of advice could be the difference between a summer holiday and a resit, so please take note!*

The equation $(AB)^2 = I$ can be expanded as $ABAB = I$, hence $A(BAB) = I$. Therefore by Theorem 3.13, A is invertible and $A^{-1} = BAB$. Now $(BA)^2 = BABA = (BAB)A = A^{-1}A = I$ as required.

T10 Let A and B be $n \times n$ matrices. Prove that if A and B are invertible then AB is invertible.

Solution

By definition, since A and B are invertible there exist matrices M and N such that

$$AM = \mathbb{I}_n = MA, \tag{1}$$
$$BN = \mathbb{I}_n = NB. \tag{2}$$

To show that the product AB is invertible, we must find a matrix P such that

$$(AB)P = \mathbb{I}_n = P(AB).$$

[Aside: How do we choose P? The *only* information we've got is from the equations (1) and (2) above, and here you just have to guess and check. You might guess that P equals M, or N, or MN or NM or,.... so just try to compute $(AB)P$ for each of these guesses, and you'll see that only one of them is the identity matrix.] Define $P = NM$, that is, $P = B^{-1}A^{-1}$. Then

$$\begin{aligned}
(AB)P &= A(BN)M && \text{since matrix multiplication is associative} \\
&= A\mathbb{I}_n M && \text{by (2)} \\
&= AM && \\
&= \mathbb{I}_n && \text{by (1).}
\end{aligned}$$

You can now check similarly that $P(AB) = \mathbb{I}_n$, so P is indeed the inverse of AB, that is,

$$(AB)^{-1} = B^{-1}A^{-1}.$$

T11 Prove by induction on $k \geq 2$ that the product of k invertible matrices, all of size $n \times n$, is an invertible matrix.

Solution

We've just proved the result for two matrices, so suppose by induction that the analogous formula holds for a product of $k - 1$ matrices:

$$(A_1 A_2 \cdots A_{k-1})^{-1} = A_{k-1}^{-1} \cdots A_2^{-1} A_1^{-1}. \tag{3}$$

If A_k is also invertible, then A_k^{-1} exists and we can define

$$P := A_k^{-1} A_{k-1}^{-1} \cdots A_2^{-1} A_1^{-1}. \tag{4}$$

Now, just as for two matrices, we compute that

$$\begin{aligned}(A_1A_2\cdots A_k)P &= (A_1A_2\cdots A_k)(A_k^{-1}A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}) \\ &= A_1A_2\cdots A_{k-1}(A_k\cdot A_k^{-1})A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1} && \text{as matrix mult is associative} \\ &= A_1A_2\cdots A_{k-1}\mathbb{I}_nA_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1} && \text{as } A_k^{-1} \text{ is inverse of } A_k \\ &= (A_1A_2\cdots A_{k-1})(A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}) \\ &= \mathbb{I}_n && \text{by (3).}\end{aligned}$$

Again, one shows similarly that $PA_1A_2\cdots A_k = \mathbb{I}_n$, so the matrix P defined in (4) is the inverse of the matrix $A_1A_2\cdots A_k$. This completes the proof.

T12 Let A and B be $n\times n$ matrices. Prove that if AB is invertible then both A and B are invertible.

Solution

Since AB is invertible there is an $n\times n$ matrix C so that $(AB)C = I = C(AB)$. Therefore $A(BC) = I$ and $(CA)B = I$. Hence by Theorem 3.13, both A and B are invertible (with $A^{-1} = BC$ and $B^{-1} = CA$).

T13 For which (if any) elementary matrices E is it true that $E^{-1} = E$? Justify your answer.

Solution

If E corresponds to the row operation of multiplying row i by λ then E^{-1} corresponds to the row operation of multiplying row i by $\frac{1}{\lambda}$. So E is equal to E^{-1} if and only if $\lambda = \frac{1}{\lambda}$, equivalently $\lambda^2 = 1$, equivalently $\lambda = \pm 1$. Note that if $\lambda = 1$ then the row operation does nothing, that is, $E = I$, but the equation $E = E^{-1}$ still holds since $I = I^{-1}$. The other case is when $\lambda = -1$, and so there is a nontrivial case in which $E = E^{-1}$ for this type of row operation.

If E is the elementary matrix corresponding to swapping row i and row j, then $E^2 = I$ since doing this swap twice results in the identity matrix. Hence $E^{-1} = E$ for all such row operations.

If E is the elementary matrix corresponding to the row operation $R_i \to R_i + \lambda R_j$ then E^{-1} is the elementary matrix corresponding to the row operation $R_i \to R_i - \lambda R_j$. Thus $E = E^{-1}$ if and only if $\lambda = 0$. Note that if $\lambda = 0$ then the row operation does nothing, that is, $E = I$, but the equation $E = E^{-1}$ still holds since $I = I^{-1}$.

T14 For each of the following subsets of $\mathbb{R}^2$, sketch the region of the plane described by the set U, then determine whether the set U contains $\mathbf{0}$, whether U is closed under vector addition and whether U is closed under scalar multiplication, giving reasons for your answers (either proofs or a counterexample). Hence determine whether U is a subspace of $\mathbb{R}^2$.

a) $U = \{(x,y)\in\mathbb{R}^2 : y = 0\}$

b) $U = \{(x,y)\in\mathbb{R}^2 : y = x\}$

c) $U = \{(x, y) \in \mathbb{R}^2 : x = y + 1\}$

d) $U = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0\}$

e) $U = \{(x, y) \in \mathbb{R}^2 : xy \geq 0\}$

Solution

For the sketches, ask your tutor or lecturer.

a) The set U contains $\mathbf{0} = (0, 0)$ since it contains all points $(x, 0)$ with $x \in \mathbb{R}$. It is closed under vector addition since if $(x, 0), (x', 0) \in U$ then $(x, 0) + (x', 0) = (x + x', 0 + 0) = (x + x', 0)$ is also in U. It is closed under scalar multiplication since if $(x, 0) \in U$ and $c \in \mathbb{R}$ then $c(x, 0) = (cx, c0) = (cx, 0)$ is in U. Therefore U is a subspace of $\mathbb{R}^2$. (Geometrically, it is the x-axis.)

b) The set U contains $\mathbf{0} = (0, 0)$ since it contains all points (x, x) with $x \in \mathbb{R}$. It is closed under vector addition since if $(x, x), (x', x') \in U$ then $(x, x) + (x', x') = (x + x', x + x')$ is also in U. It is closed under scalar multiplication since if $(x, x) \in U$ and $c \in \mathbb{R}$ then $c(x, x) = (cx, cx)$ is in U. Therefore U is a subspace of $\mathbb{R}^2$. (Geometrically, it is the line through the origin $y = x$.)

c) The set U does not contain $\mathbf{0}$ since $0 \neq 0 + 1$. It is not closed under vector addition since the points $(2, 1)$ and $(3, 2)$ both lie in U, but $(2, 1) + (3, 2) = (5, 3)$ is not in U because $5 \neq 3 + 1$. It is not closed under scalar multiplication because $(2, 1) \in U$ but $2(2, 1) = (4, 2)$ is not in U since $4 \neq 2 + 1$. Thus U is not a subspace; in fact it does not satisfy any of the conditions required for a subspace. (Geometrically, U is the line $y = x - 1$, which does not pass through the origin.)

d) The set U contains $\mathbf{0} = (0, 0)$ since it contains all points (x, y) with $x \geq 0$ and $y \geq 0$. It is closed under vector addition since if $(x, y), (x', y') \in U$ then $x, y, x', y' \geq 0$ so $x + y \geq 0$ and $x' + y' \geq 0$. Thus $(x, y) + (x', y') = (x + x', y + y')$ is also in U. It is not closed under scalar multiplication since $(1, 1)$ is in U but $-1(1, 1) = (-1, -1)$ is not in U. Therefore U is not a subspace of $\mathbb{R}^2$. (Geometrically, it is the first quadrant.)

e) The set U contains $\mathbf{0} = (0, 0)$ since it contains all points (x, y) with $xy \geq 0$, and $0 \cdot 0 = 0$. It is not closed under vector addition since $(2, 2), (-1, -3) \in U$ but $(2, 2) + (-1, -3) = (2 - 1, 2 - 3) = (1, -1)$ is not in U. It is closed under scalar multiplication since if $(x, y) \in U$ and $c \in \mathbb{R}$ then as $xy \geq 0$ and $c^2 \geq 0$ we have $(cx)(cy) = c^2 xy \geq 0$, hence $c(x, y) = (cx, cy) \in U$. Therefore U is not a subspace. (Geometrically, it is the first and third quadrants.)

T15 Determine which of the following subsets are subspaces of $\mathbb{R}^4$.

$$\begin{aligned}
W_1 &= \{(w, x, y, z) \in \mathbb{R}^4 : x = y,\ w = 2z\}, \\
W_2 &= \{(w, x, y, z) \in \mathbb{R}^4 : w + x - y = 0\}, \\
W_3 &= \{(w, x, y, z) \in \mathbb{R}^4 : y = 1\}, \\
W_4 &= \{(w, x, y, z) \in \mathbb{R}^4 : y = x^2\}, \\
W_5 &= \{(w, x, y, z) \in \mathbb{R}^4 : wy = xz\}.
\end{aligned}$$

In each case, justify fully your conclusion.

Solution

- There are two approaches. One is to notice that $W_1 = \{(w,x,y,z) \in \mathbb{R}^4 : x = y,\ w = 2z\}$ can be written as the set of solutions to a homogeneous system of linear equations (which ones?) and hence by Theorem 3.21 is a subspace.

  Alternatively, we can check the definition of a subspace. First, the zero vector of $\mathbb{R}^4$ is $(0,0,0,0)$ and $(0,0,0,0) \in W_1$ since $0 = 0$ and $0 = 2 \cdot 0$.

  Now suppose that $(w_1,x_1,y_1,z_1), (w_2,x_2,y_2,z_2) \in W_1$ and $\lambda \in \mathbb{R}$ is arbitrary. Then

$$\begin{aligned} x_1 &= y_1 \quad \text{and} \quad w_1 = 2z_1 \quad \text{since} \quad (w_1,x_1,y_1,z_1) \in W_1 \quad \text{and} \\ x_2 &= y_2 \quad \text{and} \quad w_2 = 2z_2 \quad \text{since} \quad (w_2,x_2,y_2,z_2) \in W_1. \end{aligned}$$

  We first consider the sum

$$(w_1,x_1,y_1,z_1) + (w_2,x_2,y_2,z_2) = (w_1+w_2, x_1+x_2, y_1+y_2, z_1+z_2).$$

  We need to see whether this is also an element of W_1. Since $x_1 = y_1$ and $x_2 = y_2$ we have

$$x_1 + x_2 = y_1 + y_2$$

  and since $w_1 = 2z_1$ and $w_2 = 2z_2$ we have

$$w_1 + w_2 = 2z_1 + 2z_2 = 2(z_1 + z_2).$$

  Therefore the vector $(w_1+w_2, x_1+x_2, y_1+y_2, z_1+z_2)$ is in W_1.

  We now consider the scalar multiple

$$\lambda(w_1,x_1,y_1,z_1) = (\lambda w_1, \lambda x_1, \lambda y_1, \lambda z_1).$$

  We need to see whether this is also an element of W_1. Since $x_1 = y_1$ we have

$$\lambda x_1 = \lambda y_1$$

  and since $w_1 = 2z_1$ we have

$$\lambda w_1 = \lambda(2z_1) = 2(\lambda z_1).$$

  Therefore the vector $(\lambda w_1, \lambda x_1, \lambda y_1, \lambda z_1)$ is in W_1. We conclude that W_1 is a subspace of $\mathbb{R}^4$.

- Again, one approach is to notice that $W_2 = \{(w,x,y,z) \in \mathbb{R}^4 : w + x - y = 0\}$ can be written as the set of solutions to a homogeneous system of linear equations (which ones?) and hence by Theorem 3.21 is a subspace.

  Alternatively, one checks the definition of subspace. First, the zero vector $(0,0,0,0)$ is in W_2 since $0 + 0 - 0 = 0$.

  Now suppose that $(w_1,x_1,y_1,z_1), (w_2,x_2,y_2,z_2) \in W_2$ and $\lambda \in \mathbb{R}$ is arbitrary. Then

$$w_1 + x_1 - y_1 = 0 \quad \text{and} \quad w_2 + x_2 - y_2 = 0.$$

  We consider first the sum

$$(w_1,x_1,y_1,z_1) + (w_2,x_2,y_2,z_2) = (w_1+w_2, x_1+x_2, y_1+y_2, z_1+z_2).$$

We need to see if this is also an element of W_2. We have

$$\begin{aligned}(w_1+w_2)+(x_1+x_2)-(y_1+y_2) = &= (w_1+x_1-y_1)+(w_2+x_2-y_2) \\ &= 0+0=0.\end{aligned}$$

So we have that $(w_1+w_2, x_1+x_2, y_1+y_2, z_1+z_2)$ is in W_2.
We now consider the scalar multiple

$$\lambda(w_1, x_1, y_1, z_1) = (\lambda w_1, \lambda x_1, \lambda y_1, \lambda z_1).$$

We need to see whether this is also an element of W_2. We have

$$\lambda w_1 + \lambda x_1 - \lambda y_1 = \lambda(w_1+x_1-y_1) = \lambda \cdot 0 = 0.$$

Therefore the vector $(\lambda w_1, \lambda x_1, \lambda y_1, \lambda z_1)$ is in W_2. We conclude that W_2 is a subspace of $\mathbb{R}^4$.

- $W_3 = \{(w,x,y,z) \in \mathbb{R}^4 : y = 1\}$.
  This time the zero vector $(0,0,0,0)$ is not in W_3, so W_3 is not a subspace.
- $W_4 = \{(w,x,y,z) \in \mathbb{R}^4 : y = x^2\}$.
  This is not a subspace of $\mathbb{R}_4$. Consider $(0,2,4,0) \in W_4$ and $2 \in \mathbb{R}$. Then

  $$2(0,2,4,0) = (0,4,8,0) \notin W_4 \qquad (\text{since } 8 \neq 4^2).$$

  Hence W_4 is not closed under scalar multiplication and so is not a subspace.
- $W_5 = \{(w,x,y,z) \in \mathbb{R}^4 : wy = xz\}$.
  This is not a subspace of $\mathbb{R}^4$. Consider

  $$\begin{aligned}(1,1,1,1) &\in W_5 & (\text{since } 1\times 1 = 1 \times 1) \\ (1,-1,1,-1) &\in W_5 & (\text{since } 1 \times 1 = (-1)\times(-1)).\end{aligned}$$

  Then

  $$(1,1,1,1)+(1,-1,1,-1) = (2,0,2,0) \notin W_5 \qquad (\text{since } 2\times 2 = 0 \times 0).$$

  Hence W_5 is not closed under vector addition and so is not a subspace.

T16 In $\mathbb{R}^6$, let

$$\boldsymbol{u} = (3, 200, 456, -188, 87, 900000000) \quad \text{and} \quad \boldsymbol{v} = (1,2,4,8,16,32)$$

Use a theorem from lectures to explain why $\operatorname{Span}(\boldsymbol{u}, \boldsymbol{v})$ is a subspace of $\mathbb{R}^6$.

Solution

By Theorem 3.19, the span of any collection of vectors in $\mathbb{R}^6$ is a subspace of $\mathbb{R}^6$. (So the particular values of $\boldsymbol{u}$ and $\boldsymbol{v}$ don't matter.)

T17 Show that the vector $\boldsymbol{v} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ is in the null space of

$$A = \begin{pmatrix} 1 & 267584 & 3 \\ 4 & 3765417 & 12 \end{pmatrix}$$

by carrying out a matrix multiplication.

Solution

Check that $A\boldsymbol{v} = \mathbf{0}$. Therefore $\boldsymbol{v} \in \text{null}(A)$.