2A Multivariable Calculus 2020

Ex Sheet 4

Tutorial Exercises

T1 Evaluate

$$\text{(a)} \int_0^1 dx \int_0^2 3y^2 - 4x\, dy, \quad \text{(b)} \int_0^1 dx \int_0^1 2x + 10y\, dy.$$

Solution

(a) We have

$$\int_0^1 dx \int_0^2 3y^2 - 4x\, dy = \int_0^1 \left[y^3 - 4xy\right]_0^2 dx = 8\int_0^1 1 - x\, dx = 4\left[2x - x^2\right]_0^1 = 4.$$

(b) We have

$$\int_0^1 dx \int_0^1 2x + 10y\, dy = \int_0^1 \left[2xy + 5y^2\right]_0^1 dx = \int_0^1 2x + 5\, dx = \left[x^2 + 5x\right]_0^1 = 6.$$

T2 Evaluate

$$\text{(a)} \int_1^2 dx \int_1^x \frac{1}{x+y}\, dy, \quad \text{(b)} \int_0^{\pi/2} dy \int_y^4 x \sin y\, dx.$$

Solution

(a) The integral is

$$\begin{aligned}\int_1^2 \left[\log|x+y|\right]_1^x dx &= \int_1^2 \log(2x) - \log(x+1)\, dx \\ &= \left[x\log(2x) - (x+1)\log(x+1)\right]_1^2 = 5\log 2 - 3\log 3.\end{aligned}$$

Recall, that to calculate the integral of $\log x$ with respect to x you can express the function as $1 \times \log x$ and then use integration by parts. Please see revision sheet 0 and your 1S/1Y notes.

(b) The integral is

$$\begin{aligned}\int_0^{\pi/2} \left[x^2 \sin y\right]_y^4 dy &= \frac{1}{2}\int_0^{\pi/2} (16 - y^2)\sin y\, dy \\ &= \frac{1}{2}\left[-18\cos y + y^2\cos y - 2y\sin y\right]_0^{\pi/2} = 9 - \frac{\pi}{2}.\end{aligned}$$

T3 Sketch the triangular domain T, bounded by the lines $y = -x$, $y = 0$ and $x = 1$ and illustrate that it is both type I and type II.

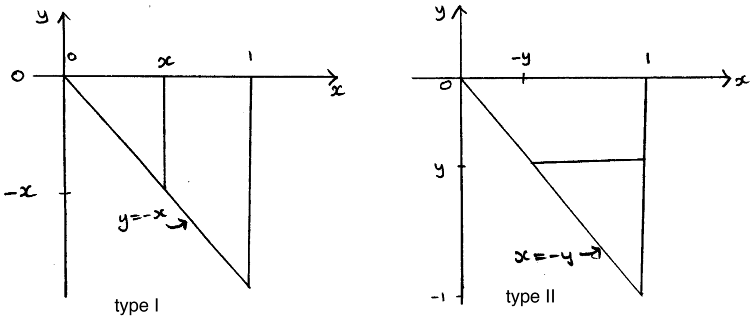
Evaluate the double integral

$$\iint_T x\,dxdy,$$

using (a) the type I formulation of T and (b) the type II formulation of T.[1]

[1] The answers you get to (a) and (b) should, of course, be the same.

Solution



(a) Using the type I formulation the integral is

$$\iint_T x\,dxdy = \int_0^1 \left(\int_{-x}^{0} x\,dy \right)\,dx = \int_0^1 x^2\,dx = \frac{1}{3}.$$

(b) Using the type II formulation the integral is

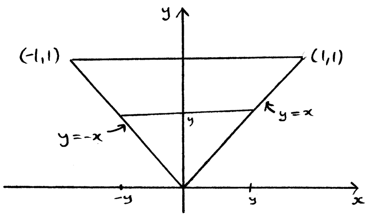
$$\iint_T x\,dxdy = \int_{-1}^{0} \left(\int_{-y}^{1} x\,dx \right)\,dy = \frac{1}{2}\int_{-1}^{0} 1 - y^2\,dy = \frac{1}{3}.$$

T4 Evaluate

$$\iint_D e^{x+y}\,dxdy,$$

where D is the triangle with vertices $(0,0)$, $(1,1)$ and $(-1,1)$.

Solution



The type II formulation is simpler and we get

$$\iint_D e^{x+y}\,dxdy = \int_0^1 \left(\int_{-y}^{y} e^{x+y}\,dx \right)\,dy = \int_0^1 \left[e^{x+y}\right]_{-y}^{y}\,dy = \int_0^1 e^{2y} - 1\,dy$$
$$= \frac{1}{2}(e^2 - 3).$$

T5 Evaluate

$$\int\int x^2 + 2y \; dxdy$$

over the rectangle with vertices at $(0,0)$, $(2,0)$, $(2,3)$ and $(0,3)$.

Solution

We have

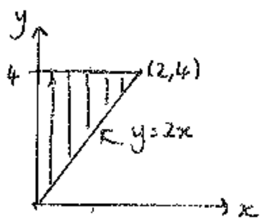
$$\int_0^2 dx \int_0^3 x^2 + 2y \, dy = \int_0^2 [x^2 y + y^2]_0^3 \, dx = \int_0^2 3x^2 + 9 \, dx = [x^3 + 9x]_0^2 = 26.$$

T6 Evaluate

$$\int\int xy \; dxdy$$

over the triangle enclosed by the lines $y = 2x$, $y = 4$ and the y-axis.

Solution



We have

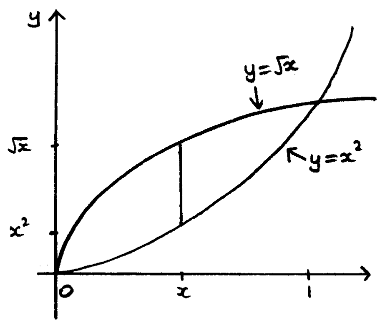
$$\int_0^2 dx \int_{2x}^4 xy \, dy = \int_0^2 x[\frac{1}{2}y^2]_{2x}^4 \, dx = \int_0^2 8x - 2x^3 \, dx = [4x^2 - \frac{2x^4}{4}]_0^2 = 8.$$

T7 Evaluate

$$\iint_D xy \, dxdy,$$

where D is the finite region bounded by the curves $y = x^2$ and $x = y^2$.

Solution



Using the type I formulation,

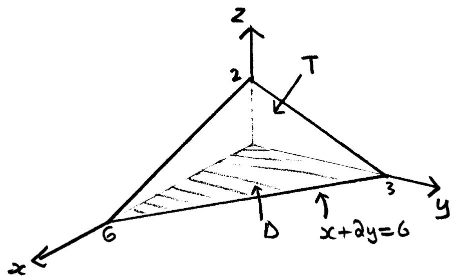
$$\iint_D xy\,dxdy = \int_0^1 \left(\int_{x^2}^{\sqrt{x}} xy\,dy \right) dx = \frac{1}{2}\int_0^1 \left[xy^2 \right]_{x^2}^{\sqrt{x}} dx = \frac{1}{2}\int_0^1 x^2 - x^5\,dx$$
$$= \frac{1}{12}.$$

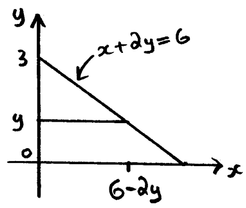
T8 Sketch the tetrahedron T formed by the plane $x + 2y + 3z = 6$ and the xy-, xz- and yz-planes. Show that the volume of T is

$$V = \frac{1}{3}\iint_D 6 - x - 2y\,dxdy,$$

where D is the finite region bounded by $x = 0$, $y = 0$ and $x + 2y = 6$. Hence evaluate V.

Solution





The volume of T is the volume under the the surface $z = \frac{1}{3}(6 - x - 2y)$ and so

$$V = \iint_D z\,dxdy = \frac{1}{3}\iint_D 6 - x - 2y\,dxdy,$$

where, as illustrated, D is the finite region bounded by $x = 0$, $y = 0$ and $x + 2y = 6$. Thus

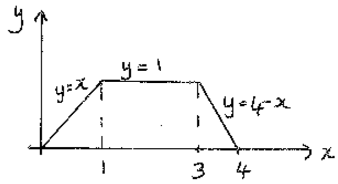
$$V = \frac{1}{3}\int_0^3 \left(\int_0^{6-2y} 6 - x - 2y\,dx \right) dy = \frac{1}{3}\int_0^3 \left[6x - \tfrac{1}{2}x^2 - 2xy \right]_0^{2(3-y)} dy$$
$$= \frac{1}{3}\int_0^3 12(3-y) - 2(3-y)^2 - 4y(3-y)\,dy = \int_0^3 6 - 4y + \tfrac{2}{3}y^2\,dy = 6.$$

T9 Evaluate

$$\int\int x\,dx\,dy$$

over the trapezium with vertices at $(0,0)$, $(4,0)$, $(3,1)$ and $(1,1)$.

Solution



To avoid splitting up the domain, we treat it as type II and integrate with respect to x first.

$$\int_0^1 dy \int_y^{4-y} x\,dx = \int_0^1 [\frac{x^2}{2}]_y^{4-y}\,dy$$
$$= \frac{1}{2}\int_0^1 (4-y)^2 - y^2\,dy = \frac{1}{2}\int_0^1 16-8y\,dy = \frac{1}{2}\big[16y-4y^2\big]_0^1 = 6.$$

With the other order of integration we would have bee to split the domain into 3 pieces (see diagram).

T10 Evaluate

$$\int\int e^{-(x+y)}\,dx\,dy$$

over the region given by the inequalities $y \geq 0$, $y \leq 1$ and $y \leq x$.

Solution

We have,

$$\int_0^1 dy \int_y^{\infty} e^{-x}e^{-y}\,dx = \int_0^1 e^{-y}\big[-e^{-x}\big]_y^{\infty}\,dy$$
$$= \int_0^1 e^{-y}e^{-y}\,dy = \int_0^1 e^{-2y}\,dy = [-\frac{1}{2}e^{-2y}]_0^1 = \frac{1}{2}(1-e^{-2}).$$

T11 Find the volume of the given solid

a) Bounded by the cylinder $y^2+z^2=4$ and the planes $x=2y$, $x=0$, $z=0$ in the first octant.

b) Bounded by the cylinders $x^2+y^2=r^2$ and $y^2+z^2=r^2$.

Solution

a) We observe the solid bounded by the cylinder $y^2+z^2=4$ and the planes $x=2y$, $x=0$, $z=0$ in the first octant lies under the surface $z=\sqrt{4-y^2}$ and above the triangle $x/2 \leq y \leq 2$ and $0 \leq x \leq 4$, hence

$$\int_0^4\int_{x/2}^2 \sqrt{4-y^2}\,dy\,dx = \int_0^2\int_0^{2y}\sqrt{4-y^2}\,dx\,dy = 16/3.$$

b) Using symmetry we find the volume in the first octant and multiply the answer by 8 to get the total volume of the solid. We observe that the solid bounded by the cylinders $x^2+y^2=r^2$ and $y^2+z^2=r^2$ lies under the surface $z=\sqrt{r^2-y^2}$ and above the quarter circle $0 \leq y \leq r$ and

$0 \le x \le \sqrt{r^2 - y^2}$, hence the total volume of the solid is

$$8 \int_0^r \int_0^{\sqrt{r^2-y^2}} \sqrt{r^2 - y^2}\, dx\, dy = (16/3)r^3.$$

T12 Use geometry or symmetry, or both, to evaluate the double integral

$$\iint_D (2 + x^2y^3 - y^2 \sin x)\, dA,$$

where $D = \{(x,y) | |x| + |y| \le 1\}$.

Solution

$D = \{(x,y) | |x| + |y| \le 1\}$ is a square with corners at $(0,1)$, $(1,0)$, $(-1,0)$ and $(0,-1)$. We then look at the symmetry of the integrand and notice that x^2y^3 is symmetrical about the y-axis and a rotation by π about the x-axis, so the integral of this term must be zero. Similarly the $y^2 \sin x$ is symmetrical about the x-axis and a rotation by π about the y-axis, so the integral of this term is zero also. This leaves $\iint_D 2\, dA$. The symmetry of the domain means this integral is

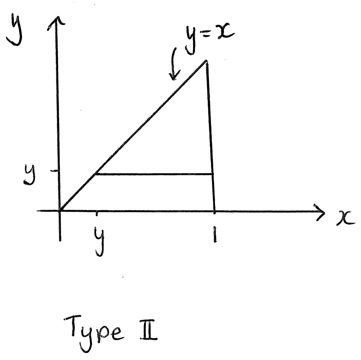
$$4 \int_0^1 \int_0^{1-x} 2\, dy\, dx = 4.$$

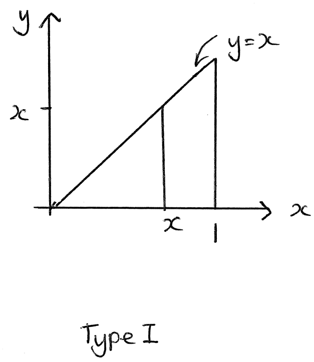
T13 By reversing the order of integration, evaluate

(a) $\displaystyle\int_0^1 dy \int_y^1 \sinh(x^2)\, dx$, (b) $\displaystyle\int_1^e dx \int_{\log x}^1 \frac{e^{-y^2}}{x}\, dy.$

Solution

(a) Sketching the two formulations of the integral we get

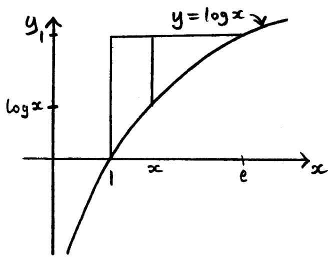


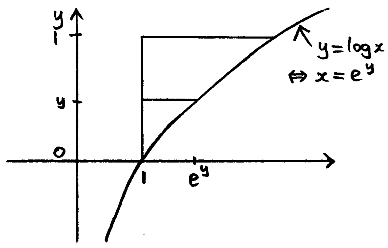


So the integral is

$$\int_0^1 \left(\int_0^x \sinh(x^2)\, dy \right) dx = \int_0^1 x \sinh(x^2)\, dx = \frac{1}{2} \left[\cosh(x^2)\right]_0^1 = \frac{1}{2}(\cosh 1 - 1).$$

(b)



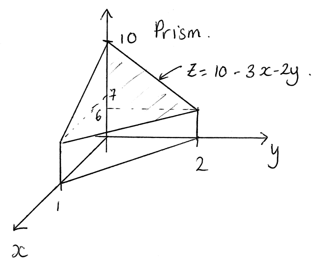


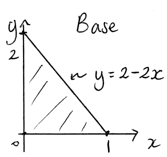
The integral is

$$\int_0^1 \left(\int_1^{e^y} \frac{e^{-y^2}}{x} \, dx \right) dy = \int_0^1 y e^{-y^2} \, dy = -\frac{1}{2} \left[e^{-y^2} \right]_0^1 = \frac{1}{2}(1 - e^{-1}).$$

T14 Find the volume of the prism whose base is the triangle with vertices at $(0,0,0)$, $(1,0,0)$ and $(0,2,0)$, which has sides parallel to the z-axis and the top of which is the plane $3x + 2y + z = 10$.

Solution





We have,

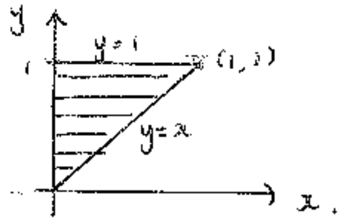
$$\begin{aligned} \text{Volume} = \int\int_{\mathrm{T}} 10 - 3x - 2y \, dxdy &= \int_0^1 dx \int_0^{2-2x} 10 - 3x - 2y \, dy \\ &= \int_0^1 \left[10y - 3xy - y^2 \right]_0^{2-2x} dx \\ &= \int_0^1 10(2-2x) - 3x(2-2x) - (2-2x)^2 \, dx \\ &= 2 \int_0^1 x^2 - 9x + 8 \, dx = 2\left[\frac{x^3}{3} - \frac{9x^2}{2} + 8x \right]_0^1 = \frac{23}{3}. \end{aligned}$$

T15 By changing the order of integration, evaluate the following integrals

$$\text{(a)} \int_0^1 dx \int_x^1 \frac{x}{1+y^3} \, dy, \quad \text{(b)} \int_0^1 dx \int_{x^2}^1 x^3 \sqrt{y^3 + 15} \, dy,$$
$$\text{(c)} \int_0^2 dx \int_{x^3}^8 \frac{x^2}{(1+y^2)^2} \, dy.$$

Solution

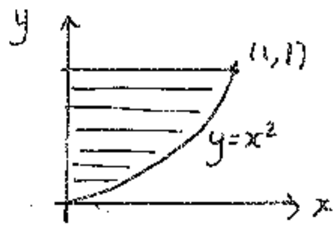
(a)



We cannot easily integrate $1/(1+y^3)$ with respect to y, so we change the order of the integration.

$$\begin{aligned}\int_0^1 dy \int_0^y \frac{1}{1+y^3}\,dx &= \int_0^1 \frac{1}{1+y^3}\Big[\frac{x^2}{2}\Big]_0^y\,dy = \frac{1}{2}\int_0^1 \frac{y^2}{1+y^3}\,dy \\ &= \frac{1}{2}\int_1^2 \frac{1}{3u}\,du \text{ (where } u = 1+y^3) \\ &= \frac{1}{6}[\log u]_1^2 = \frac{1}{6}[\log(1+y^3)]_0^1 = \frac{1}{6}\log 2.\end{aligned}$$

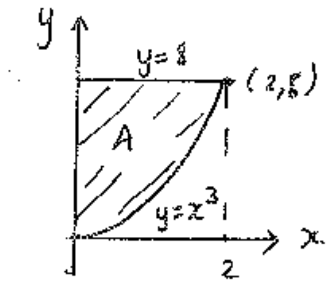
(b)



We have,

$$\begin{aligned}\int_0^1 dy \int_0^{\sqrt{y}} x^3\sqrt{y^3+15}\,dx &= \int_0^1 \sqrt{y^3+15}\Big[\frac{x^4}{4}\Big]_0^{\sqrt{y}}\,dy \\ &= \frac{1}{4}\int_0^1 y^2\sqrt{y^3+15}\,dy \\ &= \frac{1}{4}\int_{15}^{16} \frac{\sqrt{u}}{3}\,du \text{ (where } u = 1+y^3,\ \frac{1}{3}du = y^2dy.) \\ &= \frac{1}{12}\Big[\frac{u^{3/2}}{3/2}\Big]_{15}^{16} = \frac{2}{36}[16^{3/2} - 15^{3/2}] \\ &= \frac{1}{18}[64 - 15\sqrt{15}].\end{aligned}$$

(c)



We have,

$$\int_0^8 dy \int_0^{y^{1/3}} \frac{x^2}{(1+y^2)^2}\, dx = \int_0^8 \frac{1}{(1+y^2)^2} \big[\frac{x^3}{3}\big]_0^{y^{1/3}}\, dy$$
$$= \frac{1}{3}\int_0^8 \frac{y}{(1+y^2)^2}\, dy$$
$$= \frac{1}{3}\int_1^{65} \frac{\frac{1}{2}}{u^2}\, du \text{ (where } u = 1+y^2,\ \frac{1}{2}du = ydy.)$$
$$= \frac{1}{6}\big[-\frac{1}{u}\big]_1^{65} = \frac{1}{6}\cdot\frac{64}{65} = \frac{32}{195}.$$