2C Intro to real analysis 2020/21

Ex Sheet 6

Solutions and Comments [1] [2]

Q1 *Show that the sequences $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty$ and $(z_n)_{n=1}^\infty$ are eventually strictly monotonic where*

$$x_n = \frac{n^2-3n+7}{n^2+n+1}, \quad y_n = \frac{n^2-7n+1}{n^2-n-5}, \quad z_n = \frac{n^2+n+1}{n^2-3n-7}.$$

In all of these cases it makes sense to consider the difference $a_{n+1} - a_n$ and try and decide whether the resulting expression is eventually positive or negative. Lemma 1.9 can often be used to help us do this.

For $n \in \mathbb{N}$, we have

$$\begin{aligned} x_{n+1} - x_n &= \frac{(n+1)^2 - 3(n+1) + 7}{(n+1)^2 + (n+1) + 1} - \frac{n^2-3n+7}{n^2+n+1} \\ &= \frac{(n^2-n+5)(n^2+n+1) - (n^2+3n+3)(n^2-3n+7)}{(n^2+3n+3)(n^2+n+1)} \\ &= \frac{4n^2-8n-16}{(n^2+3n+3)(n^2+n+1)}. \end{aligned}$$

By Lemma 1.9, there exists $n_1, n_2 \in \mathbb{N}$ such that

$$\begin{aligned} n \geq n_1 &\implies 2n^2 \leq 4n^2 - 8n - 16 \leq 6n^2 \\ n \geq n_2 &\implies \frac{1}{2}n^4 \leq (n^2+3n+3)(n^2+n+1) \leq \frac{3}{2}n^4. \end{aligned}$$

Therefore, for $n \geq \max(n_1, n_2)$, we have

$$x_{n+1} - x_n \geq \frac{2n^2}{3n^4/2} = \frac{4}{3n^2} > 0,$$

so the sequence $(x_n)_{n=1}^\infty$ is eventually strictly increasing.

In the other two calculations, I've omitted the initial algebraic steps.

For $n \in \mathbb{N}$, we have

$$\begin{aligned} y_{n+1} - y_n &= \frac{(n+1)^2 - 7(n+1) + 1}{(n+1)^2 - (n+1) - 5} - \frac{n^2-7n+1}{n^2-n-5} \\ &= 6\frac{n^2-n+5}{n^4-11n^2+25} \end{aligned}$$

By Lemma 1.9, there exists $n_1, n_2 \in \mathbb{N}$ such that

$$\begin{aligned} n \geq n_1 &\implies 3n^2 \leq 6(n^2-n+5) \leq 9n^2 \\ n \geq n_2 &\implies \frac{1}{2}n^4 \leq (n^4-11n^2+25) \leq \frac{3}{2}n^4. \end{aligned}$$

[1] If you've not seriously tried these exercises, please don't look at these solutions and comments, until you have. You'll get the most benefit from reading these comments, when you've first thought hard about them yourself, even if you get really stuck — don't just try for a few minutes and then look at the solutions to work out how to proceed, you don't learn anywhere near as much that way.

[2] Note that I deliberately do not include formal answers for all questions.

Therefore, for $n \geq \max(n_1, n_2)$, we have

$$y_{n+1} - y_n \geq \frac{3n^2}{3n^4/2} = \frac{2}{n^2} > 0,$$

so the sequence $(y_n)_{n=1}^{\infty}$ is eventually strictly increasing.

For $n \in \mathbb{N}$, we have

$$\begin{aligned} z_{n+1} - z_n &= \frac{(n+1)^2 + (n+1) + 1}{(n+1)^2 - 3(n+1) - 7} - \frac{n^2 + n + 1}{n^2 - 3n - 7} \\ &= \frac{-4n^2 - 20n - 12}{n^4 - 4n^3 - 13n^2 + 34n + 63}. \end{aligned}$$

By Lemma 1.9, there exist $n_1, n_2 \in \mathbb{N}$ such that

$$\begin{aligned} n \geq n_1 &\implies 2n^2 \leq 4(n^2 + 5n + 3) \leq 6n^2 \\ n \geq n_2 &\implies \frac{1}{2}n^4 \leq (n^4 - 4n^3 - 13n^2 + 34n + 63) \leq \frac{3}{2}n^4. \end{aligned}$$

Then, for $n \geq \max(n_1, n_2)$, we have

$$z_{n+1} - z_n \leq \frac{-2n^2}{3n^4/2} = -\frac{4}{3n^2} < 0,$$

so the sequence is eventually strictly decreasing.

In the last example, notice that we have to switch signs in the set of inequalities for the numerator. Indeed, in order to apply Lemma 1.9. we have to make sure that the leading coefficient of our polynomial is positive. In particular, Lemma 1.9 does *not* imply the existence of $n_1 \in \mathbb{N}$ such that $2n^2 \leq -4(n^2 + 5n + 3) \leq 6n^2$ for $n \geq n_1$.

Q2 *Show that the sequences $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ are eventually strictly monotonic where*

$$x_n = \frac{5^n}{n!}, \quad y_n = \binom{2n}{n} 4^{-n}, \quad z_n = \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}.$$

In each case the nature of these sequences lead me to use the ratio method. Remember that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

For $n \in \mathbb{N}$, we have $x_n > 0$. Then

$$\frac{x_{n+1}}{x_n} = \frac{5^{n+1}}{(n+1)!} \frac{n!}{5^n} = \frac{5}{n+1} < 1,$$

for $n \geq 5$. Therefore $(x_n)_{n=1}^{\infty}$ is eventually strictly decreasing.

For $n \in \mathbb{N}$, we have $y_n > 0$. Now

$$\frac{y_{n+1}}{y_n} = \frac{(2(n+1))!}{((n+1)!)^2 4^{(n+1)}} \frac{(n!)^2 4^n}{(2n)!} = \frac{(2n+2)(2n+1)}{4(n+1)^2} < 1,$$

for all n. Thus $(y_n)_{n=1}^{\infty}$ is strictly decreasing.

For $n \in \mathbb{N}$, we have $z_n > 0$. Now

$$\frac{z_{n+1}}{z_n} = \frac{2n+2}{2n+1} > 1,$$

so the sequence is strictly monotonic.

In the calculation for $(y_n)_{n=1}^{\infty}$, be careful to write $(2(n+1))!$ and note that $(2(n+1))! = (2n+2)! = (2n+2)(2n+1)(2n)!$ for any $n \in \mathbb{N}$.

Q3 *Define a sequence* $(x_n)_{n=1}^{\infty}$ *by*

$$x_1 = 1, \quad x_{n+1} = \frac{2}{5 - 2x_n}.$$

a) *Show that* $1/2 < x_n < 2$ *for all* $n \in \mathbb{N}$*;*

b) *Show that* $(x_n)_{n=1}^{\infty}$ *is decreasing;*

c) *Show* $x_n \to \frac{1}{2}$ *as* $n \to \infty$.

For a), induction comes to mind as the sequence is defined by induction.

We certainly have $1/2 < x_1 < 2$. Suppose moreover that we have shown $1/2 < x_n < 2$ for some $n \in \mathbb{N}$. Then

$$x_{n+1} - \frac{1}{2} = \frac{4 - 5 + 2x_n}{2(5 - 2x_n)} > 0,$$

as $2 > x_n > 1/2$. Thus $x_{n+1} > 1/2$. Also

$$x_{n+1} - 2 = \frac{2 - 10 + 4x_n}{5 - 2x_n} < 0,$$

as $x_n < 2$. Thus $x_{n+1} < 2$. Therefore, by induction $1/2 < x_n < 2$ for all $n \in \mathbb{N}$.

To see that the sequence is decreasing we use the strategy of looking at the difference $x_{n+1} - x_n$.

For $n \in \mathbb{N}$, we have

$$x_{n+1} - x_n = \frac{2 - 5x_n + 2x_n^2}{5 - 2x_n} = \frac{(x_n - 2)(2x_n - 1)}{5 - 2x_n} < 0,$$

as $1/2 < x_n < 2$. Hence $x_{n+1} < x_n$, so the sequence is strictly decreasing.

We have shown in a) that the sequence is bounded and in b) that it is decreasing, so the monotone convergence theorem[3] should come to mind.

[3] When you see a bounded monotonic sequence, you should always be thinking of the monotone convergence theorem.

Since $(x_n)_{n=1}^\infty$ is decreasing by b) and bounded below by the lower bound $1/2$ according to a), it converges to L for some $L \in \mathbb{R}$.

Once we have proved the sequence converges, we can take limits in the recursion relation $x_{n+1} = \frac{2}{5-2x_n}$.

We also have $x_{n+1} \to L$, so as $x_{n+1} = \frac{2}{5-2x_n}$, by properties of limits we have $L = \frac{2}{5-2L}$. That is, we have $2L^2 - 5L + 2 = 0$. Therefore either $L = 1/2$ or $L = 2$. Since $x_1 = 1$ and the sequence is decreasing, we cannot have $L = 2$. So we must have $L = \frac{1}{2}$.

Q4 *Let $1 \leq a \leq \sqrt{2}$ and define a sequence $(x_n)_{n=1}^\infty$ by*

$$x_1 = a, \quad x_{n+1} = \frac{3x_n + 2}{x_n + 3}.$$

a) *Show that $1 \leq x_n \leq \sqrt{2}$ for all $n \in \mathbb{N}$;*

b) *Show that $(x_n)_{n=1}^\infty$ is increasing;*

c) *Show $x_n \to \sqrt{2}$ as $n \to \infty$.*

This is similar to the previous question. Again, the key thing to remember is the idea of first proving that the limit exists using the monotone convergence theorem, and then taking limits in the recursion formula to find the value of the limit.

a) Certainly $1 \leq x_1 \leq \sqrt{2}$. Suppose inductively that $1 \leq x_n \leq \sqrt{2}$ for some $n \in \mathbb{N}$. Then,

$$x_{n+1} - 1 = \frac{3x_n + 2}{x_n + 3} - 1 = \frac{2x_n - 1}{x_n + 3} \geq 0,$$

as $2x_n - 1 \geq 0$ and $x_n + 3 > 0$. Thus $x_{n+1} \geq 1$. Also

$$x_{n+1} - \sqrt{2} = \frac{3x_n + 2}{x_n + 3} - \sqrt{2} = \frac{(3-\sqrt{2})x_n + 2 - 3\sqrt{2}}{x_n + 3} \leq 0,$$

as $x_n + 3 > 0$ and

$$(3-\sqrt{2})x_n + (2 - 3\sqrt{2}) \leq (3-\sqrt{2})\sqrt{2} + 2 - 3\sqrt{2} = 0.$$

Therefore $x_{n+1} \leq \sqrt{2}$. By induction, we see that $1 \leq x_n \leq \sqrt{2}$ for all $n \in \mathbb{N}$.

b) For $n \in \mathbb{N}$, we have

$$x_{n+1} - x_n = \frac{3x_n + 2}{x_n + 3} - x_n = \frac{3x_n + 2 - x_n^2 - 3x_n}{x_n + 3} = \frac{2 - x_n^2}{x_n + 3} \geq 0,$$

as $1 \leq x_n \leq \sqrt{2}$. Thus $x_n \leq x_{n+1}$, so the sequence $(x_n)_{n=1}^\infty$ is increasing.

c) Since $(x_n)_{n=1}^{\infty}$ is increasing and bounded above, it converges by the monotone convergence theorem to some limit L. By properties of limits and order we must have $1 \leq L \leq \sqrt{2}$. Then, since the sequence $x_{n+1} \to L$, and $L + 3 \neq 0$, properties of limits give $L = \frac{3L+2}{L+3}$, so that $L^2 = 2$. Hence $L = \pm\sqrt{2}$, but since $L \geq 1$, the only possible solution is $L = \sqrt{2}$.

Q5 *By finding suitable subsequences, show that the sequences $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ do not converge, when*

$$x_n = \frac{n(-1)^n}{3n+1}, \quad y_n = \frac{n}{n+1}\cos(n\pi/4), \quad z_n = \frac{(-3)^n}{2^n + 3^n}.$$

We start by looking to see how each of these sequences behave. For the first sequence, the $(-1)^n$ term is -1 when n is odd and $+1$ when n is even. Thus we consider the two subsequences $(x_{2n})_{n=1}^{\infty}$ of even terms and $(x_{2n-1})_{n=1}^{\infty}$ of odd terms of the original sequence. These will converge to different limits.

We have

$$x_{2n} = \frac{2n}{6n+1} = \frac{2}{6+1/n} \to \frac{1}{3}$$

as $n \to \infty$. Also

$$x_{2n-1} = -\frac{2n-1}{6n-6} = -\frac{2-1/n}{6-5/n} \to -\frac{1}{3}$$

as $n \to \infty$. Thus the subsequences $(x_{2n})_{n=1}^{\infty}$ and $(x_{2n-1})_{n=1}^{\infty}$ converge to different limits, and hence the original sequence $(x_n)_{n=1}^{\infty}$ diverges.

For the sequence $(y_n)_{n=1}^{\infty}$ the $\cos(n\pi/4)$ part repeats every 8 terms. Thus you could look at the subsequences $(y_{8n+1})_{n=1}^{\infty}$ and $(y_{8n+2})_{n=1}^{\infty}$, for instance. Just be careful to choose two subsequences which really do converge to different limits: $y_{8n+2} = 0 = y_{8n+6}$ for all $n \in \mathbb{N}$, so you can't use the subsequences $(y_{8n+2})_{n=1}^{\infty}$ and $(y_{8n+6})_{n=1}^{\infty}$.

Now let's look at the $(z_n)_{n=1}^{\infty}$ sequence. Again we see an alternation between the behaviour of the even terms and the odd terms.

We have

$$z_{2n} = \frac{3^{2n}}{2^{2n} + 3^{2n}} = \frac{1}{(2/3)^{2n} + 1} \to \frac{1}{0+1} = 1,$$

as $n \to \infty$. Also

$$z_{2n-1} = -\frac{3^{2n-1}}{2^{2n-1} + 3^{2n-1}} = -\frac{1}{(2/3)^{2n} + 1} \to -\frac{1}{0+1} = -1,$$

as $n \to \infty$. Since $(z_n)_{n=1}^{\infty}$ has two subsequences converging to different limits it does not converge.

Q6 *Show directly from the definition that* $(x_n)_{n=1}^{\infty}$ *and* $(y_n)_{n=1}^{\infty}$ *are Cauchy, where*

$$x_n = \frac{n-7}{n+7}, \quad y_n = \frac{n}{2n+(-1)^n}.$$

This exercise is similar to showing that certain sequences converge directly from the definition. We let $\varepsilon > 0$ be arbitrary, and then consider $m \geq n$. Our aim is to find some n_0 such that if $n \geq n_0$, then $|x_m - x_n| < \varepsilon$, so we start by looking at the quantity $|x_m - x_n|$. As with sequence convergence, we do not need to find the smallest value of n_0, just some n_0 which works. This means that we can make some approximations to make the calculations easier.

Let $\varepsilon > 0$ be arbitrary. For natural numbers $m \geq n$ we have

$$\begin{aligned} |x_m - x_n| &= \left| \frac{m-7}{m+7} - \frac{n-7}{n+7} \right| \\ &= \left| \frac{(m-7)(n+7)-(n-7)(m+7)}{(m+7)(n+7)} \right| \\ &= \left| \frac{14(m-n)}{(m+7)(n+7)} \right| \leq \frac{14m}{mn} = \frac{14}{n} < \varepsilon, \end{aligned}$$

provided $n > \frac{14}{\varepsilon}$. Thus take $n_0 \in \mathbb{N}$ with $n_0 > \frac{14}{\varepsilon}$. Then for $m \geq n \geq n_0$, we have $|x_n - x_m| < \varepsilon$, so $(x_n)_{n=1}^{\infty}$ is Cauchy.

Here we used the fact that $|14(m-n)| = 14(m-n) \leq 14m$ to estimate the numerator. Our aim in these calculations was to get a final estimate of the form K/n as then we would be able to bound this by ε by taking n large enough.

Now for the second sequence. The idea is the same, we just need to be more careful with the estimates.

Let $\varepsilon > 0$ be arbitrary and take $m, n \in \mathbb{N}$ with $m \geq n$. Then

$$\begin{aligned} |y_m - y_n| &= \left| \frac{m(2n+(-1)^n) - n(2m+(-1)^m)}{(2m+(-1)^m)(2n+(-1)^n)} \right| \\ &= \left| \frac{m(-1)^n - n(-1)^m}{(2m+(-1)^m)(2n+(-1)^n)} \right| \\ &\leq \frac{m+n}{(2m-1)(2n-1)} \\ &\leq \frac{2m}{(2m-1)(2n-1)} \\ &\leq \frac{2m}{mn} = \frac{2}{n} < \varepsilon, \end{aligned}$$

provided $n > 2/\varepsilon$. Thus take $n_0 \in \mathbb{N}$ with $n_0 > 1/\varepsilon$, so that for $m \geq n \geq n_0$, we have $|y_m - y_n| < \varepsilon$.

Let me explain why some of the inequalities above hold. The expression $|m(-1)^n - n(-1)^m|$ is either $m+n$ or $m-n$ according to whether m and n are both even or both odd (when the first case occurs) or whether one is even and the other odd (when the second case occurs). Since $m - n \leq m + n$, we see that $m + n$ is an upper bound for $|m(-1)^n - n(-1)^m|$. The estimates on the denominator come from noting that $(2m + (-1)^m) \geq (2m - 1)$ and $(2n + (-1)^n) \geq (2n - 1)$. This justifies the first inequality. The second comes from the assumption that $m \geq n$ and the third as $2m - 1 \geq m$ and $2n - 1 \geq n$ for $m, n \in \mathbb{N}$.

Q7 *Let $(x_n)_{n=1}^{\infty}$ be a real sequence and let $L \in \mathbb{R}$. Suppose that both the subsequences $(x_{2m})_{m=1}^{\infty}$ and $(x_{2m-1})_{m=1}^{\infty}$ converge to L. Prove*[4] *that $x_n \to L$ as $n \to \infty$.*

[4] Start in the usual way by fixing an arbitrary value of $\varepsilon > 0$. Now write down the conclusion you get from $x_{2m} \to L$ and $x_{2m-1} \to L$ for this value of ε. Can you find an n_0 which has the required property?

The start of our answer should come to mind immediately by now.

> Let $\varepsilon > 0$ be arbitrary.

We now need to find some $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $|x_n - L| < \varepsilon$. Since the subsequence $(x_{2m})_{n=1}^{\infty}$ of even terms of $(x_n)_{n=1}^{\infty}$ converges to L, it follows that for sufficiently large *even* n, we will have $|x_n - L| < \varepsilon$. Also, as $x_{2m-1} \to L$ as $m \to \infty$, it follows that sufficiently large *odd* n will also have $|x_n - L| < \varepsilon$. Combining these two statements we see that we will have $|x_n - L| < \varepsilon$ for all sufficiently large n, so we now need to find a way of writing this down.

We start by writing down what we get from $x_{2m} \to L$ and $x_{2m-1} \to L$ as $m \to \infty$.

> As $x_{2m} \to L$, there exists $m_1 \in \mathbb{N}$ such that if $m \geq m_1$, then $|x_{2m} - L| < \varepsilon$. Similarly as $x_{2m-1} \to L$, there exists $m_2 \in \mathbb{N}$ such that if $m \geq m_2$, then $|x_{2m-1} - L| < \varepsilon$.

What do we have so far? Looking at the previous lines, we see that if n is even and $n \geq 2m_1$, then $|x_n - L| < \varepsilon$. Similarly, if n is odd and $n \geq 2m_2 - 1$, then $|x_n - L| < \varepsilon$. This leads us to:

> Take $n_0 = \max(2m_1, 2m_2 - 1)$. Now take $n \in \mathbb{N}$ with $n \geq n_0$. If n is even, we can write $n = 2m$ for some $m \in \mathbb{N}$. As $2m \geq 2m_1$, we have $m \geq m_1$ so $|x_{2m} - L| < \varepsilon$, that is $|x_n - L| < \varepsilon$. If n is odd, we can write $n = 2m - 1$ for some $m \in \mathbb{N}$, and as $2m - 1 \geq 2m_2 - 1$, we have $m \geq m_2$ and hence $|x_{2m-1} - L| < \varepsilon$, that is $|x_n - L| < \varepsilon$. It follows that for any $n \geq n_0$, we have $|x_n - L| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, it follows that $x_n \to L$ as $n \to \infty$.

Q8 *Complete the proofs of the following results stated in the lecture notes. You'll find proofs in the ERA notes, so try and do this without looking at these proofs.*

a) *Proposition.* Every convergent sequence is Cauchy.

Proof. Let $(x_n)_{n=1}^{\infty}$ be a convergent sequence with $x_n \to L$ as $n \to \infty$. Let $\varepsilon > 0$ be arbitrary and find, by definition of convergence, $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, we have $|x_n - L| < \varepsilon/2$. Then ...

> ... for $n, m \geq n_0$, we have
>
> $$|x_n - x_m| = |x_n - L + L - x_m| \leq |x_n - L| + |L - x_m| = |x_n - L| + |x_m - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$
>
> Thus $(x_n)_{n=1}^{\infty}$ is Cauchy.

□

b) *Proposition.* Every Cauchy sequence is bounded.

Proof. Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence. Take $\varepsilon = 1$ in the definition of being Cauchy and find $n_0 \in \mathbb{N}$ such that if $n, m \geq n_0$, then $|x_n - x_m| < 1$. In particular, for $n \geq n_0$, we have $|x_n - x_{n_0}| < 1$ so $|x_n| < 1 + |x_{n_0}|$. Then[5] ...

[5] See the proof that convergent sequences are bounded.

> ... take $M = \max(|x_1|, |x_2|, \ldots, |x_{n_0-1}|, |x_{n_0}| + 1)$. Then $|x_n| \leq M$ for all $n \in \mathbb{N}$, and hence $(x_n)_{n=1}^{\infty}$ is bounded.

□

Q9

a) *Let $\alpha \in (0,1)$ and suppose that $(x_n)_{n=1}^{\infty}$ is a sequence of positive reals such that there exists $N \in \mathbb{N}$ with $x_{n+1}/x_n \leq \alpha$ for all $n \geq N$. Use the monotone convergence theorem to prove that x_n converges to some limit L. Use the properties of limits and order to prove that $0 \leq L \leq \alpha L$ and hence that $L = 0$.*

b) *Now let $(y_n)_{n=1}^{\infty}$ be a sequence of positive reals such that*

$$\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \beta$$

for some $\beta \in [0,1)$. Use the previous part to prove[6] *that $\lim_{n\to\infty} y_n = 0$.*

[6] You might want to choose $\alpha \in (\beta, 1)$ and choose $\varepsilon = \alpha - \beta$.

c) *Let $x \in (0,1)$ and $k \in \mathbb{N}$. Prove that $\lim_{n\to\infty} n^k x^n = 0$.*

For a), we are told to use the monotone convergence theorem. Therefore we must show that the sequence satisfies the conditions to use the monotone convergence theorem.

> For $n \geq N$, we have $x_{n+1} \leq \alpha x_n < x_n$ as $\alpha < 1$, so the sequence $(x_n)_{n=1}^{\infty}$ is eventually strictly decreasing. Since $x_n > 0$ for all n, the sequence $(x_n)_{n=1}^{\infty}$ is also bounded below. Hence it converges by the monotone convergence theorem.

As we now have shown the limit exists, we try to compute the value of the limit using properties of limits.

Note that $L \geq 0$ as each $x_n \geq 0$ for all $n \in \mathbb{N}$. Moreover, as $x_{n+1} \leq \alpha x_n$ for all $n \geq N$, we can take limits in this inequality to obtain $L \leq \alpha L$. As $\alpha < 1$, the only solution to these inequalities is $L = 0$.

In b) we aim to reduce to a). The choice of ε below is made such that $\beta + \varepsilon < 1$.

Suppose $(y_n)_{n=1}^{\infty}$ is a sequence of positive terms with

$$\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \beta$$

for some $0 \leq \beta < 1$. Let $\varepsilon = \frac{1-\beta}{2}$ so that $\beta + \varepsilon = \frac{1}{2} + \frac{\beta}{2} < 1$. Then there exists $N \in \mathbb{N}$ such that for $n \geq N$, we have

$$\left|\frac{y_{n+1}}{y_n} - \beta\right| < \varepsilon \implies \beta - \varepsilon < \frac{y_{n+1}}{y_n} < \beta + \varepsilon.$$

Taking $\alpha = \beta + \varepsilon < 1$ in a), we see that $y_n \to 0$ as $n \to \infty$.

In c) we aim to use part b). We start by introducing the notation of the previous part and then check the hypotheses of part b).

Fix $k \in \mathbb{N}$ and $x \in (0,1)$ and let $y_n = n^k x^n$. Then

$$\begin{aligned} \frac{y_{n+1}}{y_n} &= \frac{(n+1)^k}{n^k}\frac{x^{n+1}}{x^n} \\ &= \frac{n^k + kn^{k-1} + \frac{k(k-1)}{2}n^{k-2} + \cdots + kn + 1}{n^k}x \\ &= \frac{1 + k/n + \frac{k(k-1)}{2n^2} + \cdots + k/n^{k-1} + 1/n^k}{1}x \to x \end{aligned}$$

as $n \to \infty$. Since $0 < x < 1$, we have $y_n \to 0$ as $n \to \infty$ by b).

Q10 *This question is harder than the others; it will not feature on the exam.*
Let $a_1 > b_1 > 0$. For $n \geq 1$, define

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \frac{2a_n b_n}{a_n + b_n}.$$

a) *Prove[7] that $a_n > a_{n+1} > b_{n+1} > b_n$ for all $n \in \mathbb{N}$.*

b) *Use the monotone convergence theorem to show that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ both converge.*

c) *Write $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$. Use one of the recurrence relations to show $a = b$.*

[7] I'd recommend first proving $a_n > b_n$ for all $n \in \mathbb{N}$ and then proving the remaining two inequalities.

d) *Find a formula for a in terms of a_1 and b_1.*

e) *How does this exercise give another proof that $\sqrt{2}$ exists?*

a) Following the hint, we first prove that $a_n > b_n$ for all $n \in \mathbb{N}$.

We prove that $a_n > b_n > 0$ for all $n \in \mathbb{N}$ by induction. We are given that $a_1 > b_1 > 0$, so assume inductively that $a_n > b_n > 0$ for some $n \in \mathbb{N}$. Then $b_{n+1} > 0$ by construction, and

$$(a_{n+1} - b_{n+1}) = \frac{1}{a_n + b_n}\left(\frac{(a_n + b_n)^2}{2} - 2a_n b_n\right)$$
$$= \frac{1}{a_n + b_n}\left(\frac{a_n^2 - 2a_n b_n + b_n^2}{2}\right) = \frac{(a_n - b_n)^2}{2(a_n + b_n)} > 0,$$

so that $a_{n+1} > b_{n+1}$. Therefore $a_{n+1} > b_{n+1} > 0$ for all $n \in \mathbb{N}$.

With this, the outer inequalities become easier to check:

For each $n \in \mathbb{N}$, we have

$$a_{n+1} = \frac{a_n + b_n}{2} < \frac{a_n + a_n}{2} = a_n.$$

Also,

$$b_{n+1} = \frac{2a_n b_n}{a_n + b_n} = \frac{a_n b_n}{a_{n+1}} > b_n$$

as $a_n > a_{n+1} > 0$.

b) From the inequalities

$$a_n > a_{n+1} > b_{n+1} > b_n$$

we see that the sequence $(a_n)_{n=1}^{\infty}$ is strictly decreasing and $(b_n)_{n=1}^{\infty}$ is strictly increasing. Since $(a_n)_{n=1}^{\infty}$ is bounded below by b_1, and $(b_n)_{n=1}^{\infty}$ is bounded above by a_1, both sequences converge by the monotone convergence theorem.

Note that b_m is a lower bound for $(a_n)_{n=1}^{\infty}$ for each $m \in \mathbb{N}$, so you didn't have to choose b_1 as your lower bound in the previous answer[8].

[8] Though one should avoid writing that a_n is bounded below by b_n, because the lower bound for the terms a_n of the sequence $(a_n)_{n=1}^{\infty}$ is supposed to be independent of n.

c) Let $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$. Then $a_{n+1} \to a$ as $n \to \infty$, so taking limits in the formula $a_{n+1} = \frac{a_n + b_n}{2}$ gives $a = \frac{a+b}{2}$, so that $a = b$.

d) Note that $a_{n+1}b_{n+1} = a_n b_n$ for all $n \in \mathbb{N}$. Applying this repeatedly we see that $a_n b_n = a_1 b_1$ for all n, so taking limits, we have $a^2 = a_1 b_1$, so that (as $a \geq b_1 > 0$ by properties of limits), a is the unique positive square root of $a_1 b_1$.

e) Taking $a_1 = 2$ and $b_1 = 1$, we obtain sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ which converge to a positive number $a \in \mathbb{R}$ satisfying $a^2 = 2$. That is, we obtain $a = \sqrt{2}$ in this way.