The Fibonacci numbers $f_0, f_1, f_2, \ldots$ and Lucas numbers $l_0, l_1, l_2, \ldots$ are defined by the equations:

- $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1}+f_{n-2}$ for all $n \geq 2$;
- $l_0 = 2$, $l_1 = 1$ and $l_n = l_{n-1}+l_{n-2}$ for all $n \geq 2$

respectively. Prove that $f_n+f_{n+2} = l_{n+1}$ for all $n \geq 1$.

Let $P(n)$ be the proposition $f_n + f_{n+2} = l_{n+1}$ for all $n \geq 1$. We use the second principle of mathematical induction.

Base cases: $P(1)$ and $P(2)$ hold since:

$$\begin{aligned} f_1 + f_3 = 1 + 2 &= l_1 + l_0 = l_2 \\ f_2 + f_4 = 1 + 3 = 3 + 1 &= l_2 + l_1 = l_3 \end{aligned}$$

Inductive step: Suppose $n \geq 2$ and $P(k)$ is true for all $1{\leq}k{\leq}n$, Now, by definition of the Fibonacci numbers we have:

$$\begin{aligned} f_{n+1} + f_{n+3} &= (f_{n-1} + f_n) + (f_{n+1} + f_{n+2}) \\ &= (f_{n-1} + f_{n+1}) + (f_n + f_{n+2}) && \text{rearranging} \\ &= l_n + l_{n+1} && \text{by induction (using } P(n{-}1) \text{ and } P(n)) \\ &= l_{n+2} && \text{by definition of the Lucas numbers} \end{aligned}$$

and hence $P(n{+}1)$ holds.

Therefore by the principle of induction we have proved that $P(n)$ holds for all $n \geq 1$.

The set of bit strings $\mathbb{B}^*$ are be defined recursively by:

- $\varepsilon \in \mathbb{B}^*$ (where ε is the empty string);
- if $w \in \mathbb{B}^*$ and $x \in \{0, 1\}$, then $wx \in \mathbb{B}^*$.

We can define concatenation of two bit strings denoted ++, recursively as follows:

- if $w \in \mathbb{B}^*$, then $w{+}{+}\varepsilon = w$;
- if $w, v \in \mathbb{B}^*$ and $x \in \{0, 1\}$, then $w{+}{+}(vx) = (w{+}{+}v)x$.

Give a recursive definition of the function $\texttt{ones} : \mathbb{B}^* \to \mathbb{N}$ which counts the number of ones in a bit string. The function $\texttt{ones} : \mathbb{B}^* \to \mathbb{N}$ is defined as follows. For any $v \in \mathbb{B}^*$:

$$\texttt{ones}(v) = \begin{cases} 0 & \text{if } v = \varepsilon \\ \texttt{ones}(w) & \text{if } v = wx \text{ and } x{=}0 \\ 1 + \texttt{ones}(w) & \text{if } v = wx \text{ and } x{=}1 \end{cases}$$

The use structural induction to prove that $\texttt{ones}(w{+}{+}v) = \texttt{ones}(w) + \texttt{ones}(v)$ for all $w, v \in \mathbb{B}^*$.

We will prove $\texttt{ones}(w{+}{+}v) = \texttt{ones}(w) + \texttt{ones}(v)$ for all $w, v \in \mathbb{B}^*$ by induction on the structure of v.

Base cases: in this case we have $v = \varepsilon$, and hence by definition of concatenation:

$$\begin{aligned} \texttt{ones}(w{+}{+}v) &= \texttt{ones}(w) \\ &= \texttt{ones}(w) + 0 && \text{rearranging} \\ &= \texttt{ones}(w) + \texttt{ones}(v) && \text{by definition of } \texttt{ones} \text{ and since } v{=}\varepsilon. \end{aligned}$$

Induction step: in this case we have $v = v'x$ for some $v' \in \mathbb{B}^*$ and $x \in \{0, 1\}$. We have two cases to consider.

- If x=0, then

$$
\begin{aligned}
\texttt{ones}(w{+}{+}v) &= \texttt{ones}(w{+}{+}(v'x)) \\
&= \texttt{ones}((w{+}{+}v')x) && \text{by definition of concatenation} \\
&= \texttt{ones}(w{+}{+}v') && \text{by definition of } \texttt{ones} \text{ \& since } x{=}0 \\
&= \texttt{ones}(w) + \texttt{ones}(v') && \text{by the induction hypothesis} \\
&= \texttt{ones}(w) + \texttt{ones}(v'x) && \text{by definition of } \texttt{ones} \text{ \& since } x{=}0 \\
&= \texttt{ones}(w) + \texttt{ones}(v) && \text{by construction.}
\end{aligned}
$$

- If x=1, then

$$
\begin{aligned}
\texttt{ones}(w{+}{+}v) &= \texttt{ones}(w{+}{+}(v'x)) \\
&= \texttt{ones}((w{+}{+}v')x) && \text{by definition of concatenation} \\
&= 1 + \texttt{ones}(w{+}{+}v') && \text{by definition of } \texttt{ones} \text{ \& since } x{=}0 \\
&= 1 + \texttt{ones}(w) + \texttt{ones}(v') && \text{by the induction hypothesis} \\
&= \texttt{ones}(w) + (1 + \texttt{ones}(v')) && \text{rearranging} \\
&= \texttt{ones}(w) + \texttt{ones}(v'x) && \text{by definition of } \texttt{ones} \text{ \& since } x{=}1 \\
&= \texttt{ones}(w) + \texttt{ones}(v) && \text{by construction.}
\end{aligned}
$$

Since these are the other cases to consider the inductive step holds.

Therefore by the principle of structural induction we have proved that

$$\texttt{ones}(w{+}{+}v) = \texttt{ones}(w) + \texttt{ones}(v)$$

for all $w, v \in \mathbb{B}^*$.

Non-empty proper binary trees over X (where X is a data set):

- base case: if $x \in X$, then $\texttt{node}(\texttt{nil}, \texttt{nil}, x)$ is a tree over X;
- inductive step: if t_1 and t_2 are non-empty proper binary trees over X and $x \in X$, then $\texttt{node}(t_1, t_2, x)$ is a tree over X.

Define recursive functions for the number of nodes $\texttt{n}(t)$ and height $\texttt{h}(t)$ of a complete non-empty binary tree.

The height of a true is the length of the longest path from the root and the length of a path is the number of edges in the path.

The functions $\texttt{n}$ and $\texttt{h}$ are defined as follows. For any tree t:

$$\texttt{n}(t) = \begin{cases} 1 & \text{if } t = \texttt{node}(\texttt{nil}, \texttt{nil}, x) \\ 1 + \texttt{n}(t_1) + \texttt{n}(t_2) & \text{if } t = \texttt{node}(t_1, t_2, x) \end{cases}$$

$$\texttt{h}(t) = \begin{cases} 0 & \text{if } t = \texttt{node}(\texttt{nil}, \texttt{nil}, x) \\ 1 + \max\{\texttt{n}(t_1), \texttt{n}(t_2)\} & \text{if } t = \texttt{node}(t_1, t_2, x) \end{cases}$$

Use structural induction to show $\texttt{n}(t) \geq 2{\cdot}\texttt{h}(t){+}1$ for any complete non-empty binary tree t.

Base case: In the base case we have that $t = \varepsilon$ and by definition of $\texttt{n}$:

$$
\begin{aligned}
\texttt{n}(\varepsilon) &= 1 \\
&= 2{\cdot}0 + 1 && \text{rearranging} \\
&= 2{\cdot}\texttt{h}(\varepsilon) + 1 && \text{by definition of } \texttt{h}
\end{aligned}
$$

as required.

Inductive step: Noe assume $\mathtt{n}(t_i) \geq 2{\cdot}\mathtt{h}(t_i){+}1$ for $i = 1, 2$ and consider an arbitrary $x \in X$. By definition of $\mathtt{n}$:

$$
\begin{aligned}
\mathtt{n}(\mathtt{node}(t_1, t_2, x)) &= 1 + \mathtt{n}(t_1) + \mathtt{n}(t_2) && \\
&\geq 1 + 2{\cdot}\mathtt{h}(t_1){+}1 + 2{\cdot}\mathtt{h}(t_2){+}1 && \text{by the inductive hypothesis} \\
&= 1 + 2{\cdot}(1 + \mathtt{h}(t_1) + 2{\cdot}\mathtt{h}(t_2)) && \text{rearranging} \\
&\geq 1 + 2{\cdot}(1 + \max\{\mathtt{h}(t_1) + 2{\cdot}\mathtt{h}(t_2)\}) && \text{since } l + m \geq \max\{l, m\} \text{ for any } l, m \in \mathbb{N} \\
&\geq 1 + 2{\cdot}\mathtt{h}(\mathtt{node}(t_1, t_2, x)) && \text{by definition of } \mathtt{h}
\end{aligned}
$$

and therefore the the inductive step holds.

Therefore by the principle of structural induction we have proved that $\mathtt{n}(t) \geq 2{\cdot}\mathtt{h}(t){+}1$ for any complete non-empty binary tree t.