University of Glasgow

Wednesday, 7 December 2016
09.30 am – 11.00 am
(1 hour 30 minutes)

DEGREES OF MSci, MEng, BEng, BSc, MA and MA (Social Sciences)

ALGORITHMIC FOUNDATIONS 2: COMPSCI2003

Answer all questions

This examination paper is worth a total of 60 marks.

The use of calculators is not permitted in this examination.

INSTRUCTIONS TO INVIGILATORS: Please collect all exam question papers and exam answer scripts and retain for school to collect. Candidates must not remove exam question papers.

1. **(a)** Prove that $(p \wedge \neg q) \to q$ and $(p \wedge \neg q) \to \neg p$ are equivalent using laws of logical equivalence. Justify each step. [4]

Solution:

$$
\begin{aligned}
(p \wedge \neg q) \to q &\equiv \neg(p \wedge \neg q) \vee q && \text{implication law}\\
&\equiv (\neg p \vee \neg\neg q) \vee q && \text{de Morgan law}\\
&\equiv (\neg p \vee q) \vee q && \text{double negation law}\\
&\equiv \neg p \vee (q \vee q) && \text{associative law}\\
&\equiv \neg p \vee q && \text{idempotency law}\\
&\equiv (\neg p \vee \neg p) \vee q && \text{idempotency law}\\
&\equiv \neg p \vee (\neg p \vee q) && \text{associative law}\\
&\equiv (\neg p \vee q) \vee \neg p && \text{commutative law}\\
&\equiv (\neg p \vee \neg\neg q) \vee \neg p && \text{double negation law}\\
&\equiv \neg(p \wedge \neg q) \vee \neg p && \text{de Morgan law}\\
&\equiv (p \wedge \neg q) \to \neg p && \text{implication law}
\end{aligned}
$$

(b) Give a simpler form of the two logical statements in **(a)** containing only one logical connective. [1]

Solution: $p \to q$

(c) Prove for any positive integer $n \in \mathbb{Z}^+$: "n is even if and only if $7{\cdot}n+4$ is even". State what proof methods you use. [5]

Solution:
This is the proof of a biconditional and is therefore in two parts.

- We first prove for any positive integer $n \in \mathbb{Z}^+$: "if n is even, then $7{\cdot}n+4$ is even", using a direct proof. Therefore suppose $n \in \mathbb{Z}^+$ is even, that is $n = 2{\cdot}k$ for some positive integer k. It follows that:

  $$7{\cdot}n+4 = 14{\cdot}k+4 = 2{\cdot}(7{\cdot}k+2)$$

  and hence $7{\cdot}n+4$ is even completing this part of the proof.

- Second we prove for any positive integer $n \in \mathbb{Z}^+$: "if $7 \cdot n + 4$ is even, then n is even" using an indirect proof. More precisely, we prove the contrapositive statement for any positive integer $n \in \mathbb{Z}^+$: "if n is not even (i.e. odd), then $7 \cdot n + 4$ is not even (i.e. odd)". Supposing $n \in \mathbb{Z}^+$ is odd, we have $n = 2 \cdot k + 1$, for some non-negative integer k. It follows that:

$$7 \cdot n + 4 = 7 \cdot (2 \cdot k + 1) + 4 = 14 \cdot k + 11 = 2 \cdot (7 \cdot k + 5) + 1$$

and hence $7 \cdot n + 4$ is odd completing the second and final part of the proof.

2. Suppose the domain of discourse P is all people and we have the following predicates:

- $F(x)$: x is friendly;
- $T(x)$: x is tall;
- $A(x)$: x is angry.

Express the following English statements in logical formulae using the above predicates.

(a) Some people are not angry. [2]

Solution:
$\exists x \in P. \neg A(x)$ and $\neg \forall x \in P. A(x)$ are possible solutions.

(b) All tall people are friendly. [2]

Solution:
$\forall x \in P. (T(x) \rightarrow F(x))$ and $\neg \exists x \in P. (T(x) \land \neg F(x))$ are possible solutions.

(c) No friendly people are angry. [2]

Solution: $\neg \exists x \in P. (F(x) \land A(x))$, $\forall x \in P. (F(x) \rightarrow \neg A(x))$ and $\forall x \in P. \neg (F(x) \land A(x))$ are possible solutions.

Express in concise (good) English without variables each of the following logical formulae.

(d) $\exists x \in P. (T(x) \land \neg A(x))$ [2]

Solution:
Some people are tall but not angry. Other solutions are possible.

(e) $\forall x \in P.(T(x) \rightarrow (F(x) \lor \neg A(x))$ [2]

Solution:
If a person is tall, then they are either friendly or not angry. Other solutions are possible.

3. **(a)** Give a recursive definition with initial conditions for the set $\{0.5, 0.05, 0.005, \dots\}$. [1]

> **Solution:**
> $0.5 \in S$ and if $x \in S$, then $x/10 \in S$

(b) Give a recursive definition for the set of integers not divisible by 4. [1]

> **Solution:**
> $1 \in S$, $2 \in S$, $3 \in S$ and if $x \in S$, then $x{+}4 \in S$ and $x{-}4 \in S$

A non-empty proper binary tree over X (where X is a data set) can be inductively defined as follows:

- **base case:** if $x \in X$, then $\texttt{node}(\texttt{nil}, \texttt{nil}, x)$ is a non-empty complete binary tree;
- **inductive step:** if t_1 and t_2 are non-empty complete binary trees and $x \in X$, then $\texttt{node}(t_1, t_2, x)$ is a non-empty complete binary tree.

(c) Give a recursive function `nn` which returns the number of nodes in a non-empty proper binary tree. [2]

> **Solution:**
>
> $$\texttt{nn}((t_1, t_2, x)) = \begin{cases} 1 & \text{if } t_1 = t_2 = \texttt{nil} \\ 1 + \texttt{nn}(t_1) + \texttt{nn}(t_2) & \text{otherwise} \end{cases}$$

(d) Give a recursive function `ne` which returns the number of edges in a non-empty proper binary tree. [2]

> **Solution:**
>
> $$\texttt{ne}((t_1, t_2, x)) = \begin{cases} 0 & \text{if } t_1 = t_2 = \texttt{nil} \\ 2 + \texttt{ne}(t_1) + \texttt{ne}(t_2) & \text{otherwise} \end{cases}$$

(e) Using the above functions, prove using induction that the number of nodes in a non-empty proper binary tree equals 1 plus the number of edges (i.e. $\texttt{nn}(t) = 1 + \texttt{ne}(t)$ for all non-empty proper binary trees t). Justify each step. [4]

Solution:
The proof is by induction on the binary tree t.

- **base case:** If $t = \mathtt{node}(\mathtt{nil}, \mathtt{nil}, x)$ for some $x \in X$, then by definition we have $\mathtt{nn}(t) = 1 = 1 + 0 = 1 + \mathtt{ne}(t)$ as required.
- **inductive step:** Assume that the hypothesis holds for the non-empty proper binary trees t_1 and t_2 and $x \in X$. Now considering the non-empty proper binary tree $t = \mathtt{node}(t_1, t_2, x)$, by definition of $\mathtt{nn}$ we have:

$$\begin{aligned}
\mathtt{nn}(t) &= 1 + \mathtt{nn}(t_1) + \mathtt{nn}(t_2) && \\
&= 1 + (1 + \mathtt{ne}(t_1)) + (1 + \mathtt{nn}(t_2)) && \text{by the induction hypothesis} \\
&= 1 + (2 + \mathtt{ne}(t_1) + \mathtt{ne}(t_2)) && \text{rearranging} \\
&= 1 + \mathtt{ne}(t) && \text{by definition of } \mathtt{ne}
\end{aligned}$$

  as required.

Therefore we have proved by induction that $\mathtt{nn}(t) = 1 + \mathtt{ne}(t)$ for all non-empty proper binary trees t.

4. **(a)** A car number plate has two forms:

(i) three digits followed by three letters;

(ii) four letters followed by two digits.

How many car number plates are there (you can leave your answer in powers of 10 and 26). Explain your answer. [2]

Solution:
Using the product rule there are $10{\cdot}10{\cdot}10{\cdot}26{\cdot}26{\cdot}26$ combinations of three digits followed by three letters and $26{\cdot}26{\cdot}26{\cdot}26{\cdot}10{\cdot}10$ combinations of four letters followed by two digits, since these are disjoint using the sum rule the total number of number plates equals:

$$10^3{\cdot}26^3 + 26^4{\cdot}10^2 = 10^2{\cdot}26^3{\cdot}(10+26) = 10^2{\cdot}26^3{\cdot}36\,.$$

(b) How may ways are there to choose 4 pieces of fruit from a bowl that contains 6 bananas, 5 apples, 7 pears, and 4 oranges? Explain your answer. [3]

Solution:
We can solve this using the "stars and bars" approach or noticing the fact this is a case of combinations with repetitions.

Using the stars and bars approach, we have 4 regions separated by bars $-|-|-|-$ where each region corresponds to a type of fruit. The stars correspond to the quantity of the item chosen, i.e. $-|*|**|*$ would correspond to one item of class two, two items of class three, and one item of class four. There are 7! ways to permute the 4 stars and 3 bars. But the stars are indistinguishable as are the bars, therefore we over-count the stars by 4! and the bars by 3!. Therefore there are $7!/(3!{\cdot}4!)$ ways, i.e. 35 ways.

On the other hand, noticing that this is an r-combination from a set of n elements where n=r=4, we have the total number of ways equals:

$$C(n{+}r{-}1,r) = C(4{+}4{-}1,4) = C(7,4) = \frac{7!}{3!{\cdot}4!} = \frac{7{\cdot}6{\cdot}5}{3{\cdot}2{\cdot}1} = 7{\cdot}5 = 35$$

(c) How many 8 bit strings are there, that start with a 0 or end with 111 (you can leave you answer in powers of 2)? Explain your answer. [3]

Solution:
There are 2^7 bit strings of the form $0*******$ and 2^5 bit strings of the form $*****111$. But both of these include the strings $0****111$, i.e. 2^4 strings. Therefore there are $2^7+2^5-2^4$ strings that begin with one 0's or end with three 1's.

For completeness we can compute the actual number as

$$2^7+2^5-2^4 = 2^4{\cdot}(2^3+2-1) = 16{\cdot}(8+1) = 16{\cdot}9 = 144$$

(d) How many students must be in a class to guarantee that at least 4 were born on the same day of the week? Explain your answer. [2]

Solution:
We have 7 containers, i.e. the days of the week, and we need to know the minimum number objects n, i.e. the students, such that there are 4 objects in one container. Using the pigeon hole principle we are looking for the smallest n such that $\lceil n/7 \rceil \geq 4$.

Since $21/7 = 3$ it follows that we need at least 22 students.

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5. Suppose that you have three urns that you cannot see into. Urn_1 has 9 green balls and 1 red. Urn_2 has 5 green and 5 blue. Urn_3 has 2 green, 4 red, and 4 blue.

(a) If you select a ball from Urn_1, what is the probability you select a green ball? [1]

> **Solution:** Let G_i be the event you select a green ball from urn i. Clearly we have $\mathbf{P}[G_1] = 9/10$.

(b) If you select a ball from each urn what is the probability you select three green balls? [2]

> **Solution:** Let G_i be the event you select a green ball from urn i. Since the events G_1, G_2 and G_3 are independent we have:
>
> $$P[G_1 \cap G_2 \cap G_3] = P[G_1] \cdot P[G_2] \cdot P[G_3] = \frac{9}{10} \cdot \frac{1}{2} \cdot \frac{2}{10} = \frac{9}{100}$$

(c) If you select a ball from Urn_1 and a ball from Urn_3, what is the probability you select at least one red ball? [3]

> **Solution:** Let R_i be the event you select a red ball from urn i.
>
> $$\begin{aligned} P[R_1 \cup R_3] &= P[R_1 \cap \neg R_3] + P[\neg R_1 \cap R_3] + P[R_1 \cap R_3] \\ &= \frac{1}{10} \cdot \frac{6}{10} + \frac{9}{10} \cdot \frac{4}{10} + \frac{1}{10} \cdot \frac{4}{10} \\ &= \frac{6 + 36 + 4}{100} \\ &= \frac{46}{100} \\ &= \frac{23}{50} \end{aligned}$$

(d) Suppose you randomly select an urn and then randomly select a ball from it. Given the ball you drew was green, what is the probability that it came from Urn_1? [4]

Solution: Let U_i be the event you select a urn i. Using Bayes' law we have:

$$
\begin{aligned}
\mathbf{P}[U_1 \mid G] &= \frac{\mathbf{P}[G \mid U_1]\mathbf{P}[U_1]}{\mathbf{P}[G \mid U_1]\mathbf{P}[U_1] + \mathbf{P}[G \mid U_2]\mathbf{P}[U_2] + \mathbf{P}[G \mid U_3]\mathbf{P}[U_3]} \\
&= \frac{9/10 \cdot 1/3}{9/10 \cdot 1/3 + 1/2 \cdot 1/3 + 2/10 \cdot 1/3} \\
&= \frac{9/10}{9/10 + 1/2 + 2/10} \\
&= \frac{9/10}{16/10} \\
&= \frac{9}{16}
\end{aligned}
$$

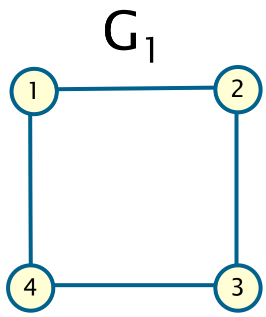
6. **(a)** Define what we mean when we say that two graphs are isomorphic. [3]

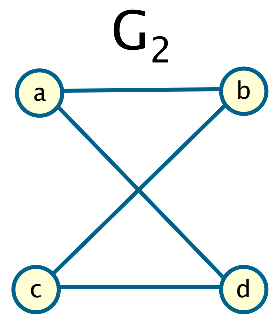
Solution:
Graph $G = (V_1, E_1)$ is isomorphic to graph $H = (V_2, E_2)$ if there is a bijective function $f : V_1 \to V_2$ such that $\{v, w\} \in E_1$ if and only if $\{f(v), f(w)\} \in E_2$.

Alternatively, f is a bijection between the vertex sets such that for any part of vertices v and w we have that v and w are adjacent if and only if $f(v)$ and $f(w)$ are adjacent.

(b) Draw two isomorphic graphs (e.g. with four vertices) one in which the edges cross and one in which the edges do not cross. Explain how the graphs are isomorphic. [3]

Solution: Two possible graphs are:

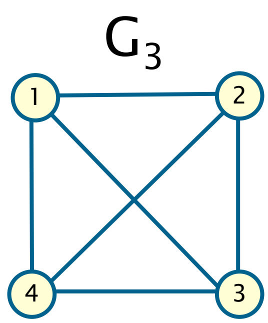


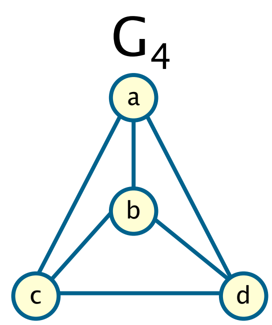


and the isomorphism is given by the following bijection between the vertices of the graphs:

$$\begin{array}{ccc} 1 & \rightarrow & a \\ 2 & \rightarrow & b \\ 4 & \rightarrow & d \\ 3 & \rightarrow & c \end{array}$$

An alternative solution is the following two graphs:





and the isomorphism is given by the following bijection between the vertices of the graphs:

$$\begin{array}{ccc} 1 & \rightarrow & a \\ 2 & \rightarrow & b \\ 3 & \rightarrow & c \\ 4 & \rightarrow & d \end{array}$$

Define what it means for a relation R over a set A to be:

(c) symmetric; [1]

Solution:
A relation $R \subseteq A \times A$ is symmetric if for any $a, b \in A$ we have $(a,b) \in R$ if and only if

$(b,a) \in R$.

Alternatively, in english we have a is related to b if and only if b is related to a.

(d) anti-symmetric. [1]

Solution:
A relation $R \subseteq A \times A$ is anti-symmetric if for any $a,b \in A$ such that $a \neq b$ we have if $(a,b) \in R$, then $(b,a) \notin R$.

Alternatively, in english we have if a and b are distinct and a is related to b, then b is not related to a.

Give an example of a relation on a set that is:

(e) both symmetric and anti-symmetric; [1]

Solution:
For any set A, define a relation R over $A \times A$ by the set $\{(a,a) \,|\, a \in A\}$, then R is symmetric and anti-symmetric.

(f) neither symmetric nor anti-symmetric. [1]

Solution:
Consider the relation "divides" over the integers ($\mathbb{Z} \times \mathbb{Z}$), then:

- this relation is not symmetric, since for example 2 divides 4 while 4 does not divide 2
- this relation is not anti-symmetric, since for example 2 divides -2 and -2 divide 2.

Note that there may be other solutions.