2C Week 9 2020/21

HANDOUT

Continuity

In this chapter we study the notion of *continuity* for real functions. Intuitively, continuity of a function f means that one can draw the graph of f without lifting the pen from the paper. Most functions you have seen before are continuous, but there are also some rather basic examples of functions which fail to be continuous.

From a general mathematical perspective, continuity is an important regularity property for functions[1]. From the continuity of a function f a number of nice properties can be deduced, and this plays a role in a wide range of problems in analysis and its applications. We will see some of this below, most notably the intermediate value theorem and the extreme value theorem.

Before entering the discussion of continuity, let us first recall what we mean by a *function*. Although we often simply write a formula like

$$f(x) = \frac{x^3 - 2}{x + 1}$$

and refer to this as a "function", the precise mathematical definition of a function is as follows: If X and Y are sets, then a function from X to Y is an object $f : X \to Y$ which assigns a unique element $f(x) \in Y$ to each $x \in X$. We also refer to X as the *domain* of the function and Y as the *codomain*, and sometimes write $X = \mathrm{dom}(f)$ and $Y = \mathrm{codom}(f)$.

In practice, one often omits the domain and codomain from the definition of a function. For instance, $f(x) = 3 + 2x^2 + \sin(x)$ is a function, tacitly with domain and codomain $\mathbb{R}$. Note that we could also view this as a function with domain, say, $[0, 1]$ and codomain $\mathbb{R}$. In our previous example $f(x) = \frac{x^3-2}{x+1}$ we should similarly choose domain and codomain appropriately[2]. Simply writing down a formula does not specify a function completely, and one should in principle always indicate both the domain and codomain[3].

Let us also point out that a function need not be given by an explicit formula[4]. For instance,

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

is a perfectly valid function $f : \mathbb{R} \to \mathbb{R}$.

By a *real function* we mean a function f whose domain $\mathrm{dom}(f)$ is some subset of $\mathbb{R}$, possibly all of $\mathbb{R}$, and whose codomain is $\mathbb{R}$, or possibly some subset thereof. Thus, for each $x \in \mathrm{dom}(f)$ of a real function f there is a unique element $f(x) \in \mathbb{R}$ associated to x.

[1] Another crucial regularity property is *differentiability*, this will be one of the main topics of the 3H course on Analysis of Differentiation and Integration next year.

[2] Here one could consider the domain $X = \mathbb{R} \setminus \{-1\}$, or any other subset X of $\mathbb{R}$ not containing -1. We have to exclude $x = -1$ from the domain since the expression for $f(-1)$ contains an illegal division by zero. As codomain we can simply take $Y = \mathbb{R}$.

[3] In practice, however, this is often neglected.

[4] We already indicated this in the chapter on sequences. Recall that sequences are particular examples of functions, with domain $\mathbb{N}$ and codomain $\mathbb{R}$.

The definition of continuity

Let us now give the precise definition of continuity.

Definition 5.1. Let $f : \mathrm{dom}(f) \to \mathbb{R}$ be a real function. If $c \in \mathrm{dom}(f)$, we say that f is continuous[5] at c if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathrm{dom}(f)$ with $|x - c| < \delta$ we have $|f(x) - f(c)| < \epsilon$.

We say that f is continuous if f is continuous at all points of its domain $\mathrm{dom}(f)$.

[5] In ERA the definition of continuity at c is slightly different, as it additionally asks for $\mathrm{dom}(f)$ to contain a neighbourhood of c. However, our definition is standard. Ignore those parts of ERA where it discusses the domain containing a neighbourhood!

The definition of continuity at c can also be rewritten using quantifiers. More precisely, the function f is continuous at $c \in \mathrm{dom}(f)$ if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in \mathrm{dom}(f), |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon.$$

As we will see in examples, the value of δ will usually depend on ε. If you make ε smaller you should expect the corresponding value of δ to be smaller, too. The general idea behind the concept of continuity is that *small changes in the argument* x of a continuous function f lead to *small changes in the value* $f(x)$.

Let us consider some examples of continuous functions.

Example 5.2. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 3x + 7$ for $x \in \mathbb{R}$. Show directly from the definition[6] that f is continuous at $c = 2$.

[6] As usual, *directly from the definition* means you shall verify the condition in the definition of continuity, that is Definition 5.1.

Solution. Let $\varepsilon > 0$. Then for $x \in \mathbb{R}$ we have

$$|f(x) - f(2)| = |3x + 7 - 13| = |3x - 6| = 3|x - 2|.$$

Therefore, $|f(x) - f(2)| < \varepsilon$ is equivalent to $3|x - 2| < \varepsilon$, or $|x - 2| < \varepsilon/3$. If we choose $\delta = \varepsilon/3$ we conclude that $|x - 2| < \delta$ implies $|f(x) - f(2)| < \varepsilon$. Thus f is continuous at 2. $\square$

Example 5.3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 + x + 1$ for $x \in \mathbb{R}$. Show that f is continuous at $c = 3$.

Solution. Let $\varepsilon > 0$. For $x \in \mathbb{R}$ we compute

$$|f(x) - f(3)| = |x^2 + x + 1 - 13| = |x^2 + x - 12| = |x - 3||x + 4|.$$

We therefore have to find $\delta > 0$ such that $|x - 3| < \delta$ implies $|x - 3||x + 4| < \varepsilon$. Notice that we can pick the number δ as small as we like. So we can try to find a suitable δ in the interval $(0, 1]$. Now for[7] $|x - 3| < \delta \le 1$ we have

[7] The inequality $\delta \le 1$ I'm using here is just a convenient choice, other choices are possible.

$$-1 < x - 3 < 1 \Rightarrow 6 < x + 4 < 8,$$

which means $|x + 4| < 8$. Inserting this into the relation above yields

$$|x - 3||x + 4| < \delta 8$$

provided[8] $|x - 3| < \delta$ and $\delta \le 1$. Therefore, if we set $\delta = \min(1, \varepsilon/8)$ we obtain $|x - 3| < \delta \Rightarrow |f(x) - f(3)| = |x - 3||x + 4| < \epsilon$. Hence f is continuous at 3. $\square$

[8] What happens if we replace the bound 1 for δ by some other value? What happens e.g. if we consider $\delta \le 2$?

Example 5.4. Let $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ be given by

$$f(x) = \frac{x+1}{x-1}.$$

Show that f is continuous at 2.

Solution. Let $\varepsilon > 0$. For $x \in \mathbb{R}$ we have

$$|f(x) - f(2)| = \left|\frac{x+1}{x-1} - 3\right| = \left|\frac{x+1-3x+3}{x-1}\right| = \frac{2|x-2|}{|x-1|}.$$

Now $|x-2| < 1/2$ implies $-1/2 < x-2 < 1/2$, and hence[9] $1/2 < x-1 < 3/2$. Therefore we obtain

$$\frac{2|x-2|}{|x-1|} < 4|x-2|$$

provided $|x-2| < 1/2$. Taking $\delta = \min(1/2, \varepsilon/4)$, we get that $|x-2| < \delta$ implies $|f(x) - f(2)| < \varepsilon$. □

[9] Here I start from $|x-2| < 1/2$ instead of $|x-2| < 1$, compare with the previous example. Can you see why taking $|x-2| < 1$ would not work?

Example 5.5. Let $f : [0, \infty) \to \mathbb{R}$ be given by $f(x) = \sqrt{x}$. Show directly from the definition that f is continuous.

Solution. Here we have to show continuity at all points in the domain of f. So we start by fixing $c \in [0, \infty)$, and let $\varepsilon > 0$ be arbitrary. For $x \in [0, \infty)$ we have

$$|\sqrt{x} - \sqrt{c}|^2 = |\sqrt{x} - \sqrt{c}||\sqrt{x} - \sqrt{c}| \leq |\sqrt{x} - \sqrt{c}||\sqrt{x} + \sqrt{c}| = |x - c|$$

since both $\sqrt{x}$ and $\sqrt{c}$ are positive[10]. Therefore

$$|f(x) - f(c)| = |\sqrt{x} - \sqrt{c}| \leq |x-c|^{1/2}.$$

Therefore $|f(x) - f(c)| < \varepsilon$ provided $|x-c|^{1/2} < \varepsilon$, or equivalently, $|x - c| < \varepsilon^2$. Hence if we set $\delta = \varepsilon^2$ we conclude that $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$. Thus f is continuous at c. Since $c \in \mathrm{dom}(f)$ was arbitrary this means that f is continuous. □

[10] and hence $|\sqrt{x} - \sqrt{c}| \leq |\sqrt{x} + \sqrt{c}| = \sqrt{x} + \sqrt{c}$.

The next example shows that continuous functions can exhibit a certain kind of singular behaviour, in the sense that they need not be *smooth*[11].

[11] That is, there can be "corners" in the graph of f.

Example 5.6. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = |x|$. Show that f is continuous[12].

[12] Intuitively, the function f "changes its slope" drastically at zero. It will be shown in the 3H course on Analysis of Differentiation and integration that f fails to be *differentiable* at 0.

Solution. Again we have to prove continuity at c for every $c \in \mathrm{dom}(f) = \mathbb{R}$. We consider first the case $c > 0$, and let $\varepsilon > 0$ be arbitary. In this case, for $|x - c| < c$ the number x is positive as well, and hence

$$|f(x) - f(c)| = |x - c|.$$

Therefore, if we take $\delta = \min(\varepsilon, c)$, then[13] $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$.

In a similar way on proves[14] continuity at c for $c < 0$.

The most interesting part is to show that f is continuous at 0. Let $\varepsilon > 0$. We have

$$|f(x) - f(0)| = ||x| - 0| = |x|,$$

[13] In a similar way as before, we take $\delta = \min(\varepsilon, c)$, and not simply $\delta = \varepsilon$ in order to ensure that our previous calculation $|f(x) - f(c)| = |x - c|$ is valid.

[14] Try to write down the details!

so if we set $\delta = \varepsilon$ then $|x - 0| < \delta$ implies $|f(x) - f(0)| < \epsilon$ as required.

Summarising, we have shown that f is continuous at c for all $c \in \operatorname{dom}(f)$, and hence f is continuous. □

Let us now discuss an example where continuity fails. We say that a real function f is *discontinuous* at $c \in \operatorname{dom}(f)$ if f fails to be continuous at c.

Example 5.7. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Show that f is discontinuous at 0.

Solution. Let[15] $\varepsilon = 1/2$. Then for any $\delta > 0$ consider $x = -\delta/2$. We have $|x - 0| = |-\delta/2| = \delta/2 < \delta$ but $|f(x) - f(0)| = |0 - 1| = 1 > \epsilon$. Therefore, f is not continuous at 0. □

[15] Can you see why I chose ε in this way? Which other choices would also work in this argument?

We can even produce real functions which are discontinuous *everywhere*. The following example is an exercise.

Example 5.8. Let $\chi_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ be defined by

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Then $\chi_{\mathbb{Q}}$ is not continuous at any $c \in \mathbb{R}$.

Intuitively, it should be clear that $\chi_{\mathbb{Q}}$ must be discontinuous: We cannot draw the graph of $\chi_{\mathbb{Q}}$ without lifting the pen from the paper; in fact, it is hard to draw the graph of f at all!

Properties of continuous functions

In this section we collect some results which will help us to check continuity, and prove that a large variety of familiar functions are continuous.

The first result which we'll present allows us to reduce matters to the study of certain sequences and their limits, that is, things we have already discussed in detail before.

Theorem 5.9 (Sequential characterisation of continuity)**.** *Let $f : \operatorname{dom}(f) \to \mathbb{R}$ be a real function and $c \in \operatorname{dom}(f)$. Then f is continuous at c if and only for every sequence $(x_n)_{n=1}^{\infty}$ in $\operatorname{dom}(f)$ with $x_n \to c$ as $n \to \infty$ we have $f(x_n) \to f(c)$ as $n \to \infty$.*

Proof. Assume first that f is continuous at c, and let $x_n \to c$ be a sequence in[16] $\operatorname{dom}(f)$ converging to c. Moreover let $\varepsilon > 0$ be arbitrary. Then, by the definition of continuity, there exists $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$. Since $x_n \to c$ and $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that $|x_n - c| < \delta$ for all $n \geq n_0$. In particular,

[16] We say that a sequence $(x_n)_{n=1}^{\infty}$ is *in* $\operatorname{dom}(f)$ if $x_n \in \operatorname{dom}(f)$ for all $n \in \mathbb{N}$, that is, if all its terms are contained in $\operatorname{dom}(f)$.

this implies $|f(x_n) - f(c)| < \varepsilon$ for $n \geq n_0$. That is, $f(x_n) \to f(c)$ as $n \to \infty$.

Conversely, assume that $f(x_n) \to f(c)$ as $n \to \infty$ for every sequence $(x_n)_{n=1}^{\infty}$ in $\mathrm{dom}(f)$ satisfying $x_n \to c$. Moreover suppose that f is not continuous at c. Then there exists an $\epsilon > 0$ such that for any $\delta > 0$ we find $x \in \mathrm{dom}(f)$ with $|x - c| < \delta$ and $|f(x) - f(c)| \geq \epsilon$. In particular, for each $n \in \mathbb{N}$ we can choose $x_n \in \mathrm{dom}(f)$ such that $|x - c| < 1/n$ and $|f(x_n) - f(c)| \geq \epsilon$. Then $x_n \to c$ as $n \to \infty$, so our assumption implies $f(x_n) \to f(c)$ as $n \to \infty$. This is a contradiction to $|f(x_n) - f(c)| \geq \epsilon$ for all $n \in \mathbb{N}$. □

We can rephrase Theorem 5.9 by saying that

$$f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n)$$

whenever $(x_n)_{n=1}^{\infty}$ is a convergent sequence in $\mathrm{dom}(f)$, provided f is continuous.

Let us revisit one of our previous examples using Theorem 5.9.

Example 5.10. Use the sequential characterisation of continuity to show that $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

is discontinuous at 0.

Solution. Consider $x_n = -\frac{1}{n}$. Then $x_n \to 0$, but since $f(x_n) = 0$ for all n we have $f(x_n) \to 0$, and this differs from $f(0) = 1$. By the sequential characterisation of continuity, the function f is not continuous at 0. □

We now come to algebraic properties of continuous functions. You should compare the following theorem and its proof with algebraic properties of sequence limits in chapter 3.

Theorem 5.11 (Algebraic properties of continuity)**.** *Let $f : \mathrm{dom}(f) \to \mathbb{R}$ and $g : \mathrm{dom}(g) \to \mathbb{R}$ be real functions and $\lambda \in \mathbb{R}$. Moreover let $c \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ and assume that f and g are continuous at c. Then*

a) $f + g$ is continuous at c;

b) λf is continuous at c;

c) fg is continuous at c;

d) If $g(c) \neq 0$, then f/g is continuous at c.

Let us first clarify what the functions in these statements are. If f and g are functions, then $f + g$ and fg are defined[17] by

$$(f+g)(x) = f(x) + g(x), \qquad (fg)(x) = f(x)g(x),$$

respectively. Moreover, we define the function λf by

$$(\lambda f)(x) = \lambda f(x)$$

[17] Here we tacitly pick the domains of definition appropriately: Both $f + g$ and fg are defined for $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$, so it is natural to set $\mathrm{dom}(f + g) = \mathrm{dom}(f) \cap \mathrm{dom}(g) = \mathrm{dom}(fg)$. For λf we do not need to restrict the domain, we can simply take $\mathrm{dom}(\lambda f) = \mathrm{dom}(f)$. Notice however that in principle we could also choose smaller domains for all these functions.

for any $\lambda \in \mathbb{R}$. Finally, f/g is defined by $(f/g)(x) = f(x)/g(x)$ for $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ such that $g(x) \neq 0$.

I'm going to prove the first and last statement of the theorem, and encourage you to prove the other parts.

Proof of a). Let $\epsilon > 0$. By assumption, there exists $\delta_1, \delta_2 > 0$ such that[18]

$$|x - c| < \delta_1 \implies |f(x) - f(c)| < \frac{\varepsilon}{2}$$
$$|x - c| < \delta_2 \implies |g(x) - g(c)| < \frac{\varepsilon}{2}.$$

Take $\delta = \min(\delta_1, \delta_2)$. Then, for $|x - c| < \delta$, we have

$$\begin{aligned} |(f(x) + g(x)) - (f(c) + g(c))| &= |(f(x) - f(c)) + (g(x) - g(c))| \\ &\leq |f(x) - f(c)| + |g(x) - g(c)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

using the triangle inequality. Since $\varepsilon > 0$ was arbitrary, $f + g$ is continuous at c. □

[18] I'm going to use $\frac{\varepsilon}{2}$ in the definition of continuity for f and g. Can you see why I want to take $\frac{\varepsilon}{2}$ in this definition?

The key thing that made this proof work was the use of the triangle inequality, in a similar way as we have already seen a few times before. The triangle inequality enables me to control the distance from $(f + g)(x) = f(x) + g(x)$ to $(f + g)(c) = f(c) + g(c)$ in terms to the two distances I understand, namely $|f(x) - f(c)|$ and $|g(x) - g(c)|$. I had found this inequality before embarking on writing my proof.

In the above proof, I verified directly the definition of continuity for the function $f + g$. For part *d*), I will use Theorem 5.9 in order to reduce the claim to a known result about sequence limits.

Before we embark on this, we notice that $g(x)$ is nonzero for x sufficiently close to c, so that $f(x)/g(x)$ makes sense[19].

In particular, if $(x_n)_{n=1}^{\infty}$ is a sequence in $\mathrm{dom}(g)$ converging to c, then the terms $g(x_n)$ will be nonzero except for a finite number of indices $n \in \mathbb{N}$.

With this in mind, let us now write down a proof.

[19] Indeed, consider $\varepsilon_0 = |g(c)|$. By continuity of g at c there exists $\delta_0 > 0$ such that $|x - c| < \delta_0$ implies $|g(x) - g(c)| < \varepsilon_0$. This means $-|g(c)| < g(x) - g(c) < |g(c)|$. If $g(c) > 0$ we conclude $0 < g(c) - |g(c)| < g(x)$, and if $g(c) < 0$ we have $g(x) < |g(c)| + g(c) = 0$. Hence in both cases $g(x) \neq 0$ for all x with $|x - c| < \delta_0$.

Proof of d). Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathrm{dom}(f) \cap \mathrm{dom}(g)$ converging to c. Since f and g are continuous, we have $f(x_n) \to f(c)$ and $g(x_n) \to g(c)$ as $n \to \infty$ due to Theorem 5.9. Using algebraic properties of sequence limits [20], this implies $(f/g)(x_n) = f(x_n)/g(x_n) \to f(c)/g(c) = (f/g)(c)$ as $n \to \infty$. Using again Theorem 5.9, this means that f/g is continuous at c. □

[20] More precisely, part *d*) of Theorem 3.10 in chapter 3.

In this proof we have made use of results from earlier on, and that's why our argument becomes quite short and simple. Trying to write down a direct proof as for part *a*) above would actually be rather cumbersome[21].

Using Theorem 5.11, we can prove that polynomial functions and rational functions are always continuous[22].

[21] This is a key idea to keep in mind when thinking about how to construct proofs: can I use known results to reduce the argument to something simpler? Working this way will often save time and effort.

[22] You should compare this to the proofs *directly from the definition* in examples 5.2, 5.3 and 5.4 of the previous subsection.

Theorem 5.12. *Let*

$$g(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

be a polynomial with real coefficients $a_0, \ldots, a_n$. *Then the function* $g : \mathbb{R} \to \mathbb{R}$ *mapping* x *to* $g(x)$ *is continuous.*

More generally, let

$$f(x) = \frac{g(x)}{h(x)}$$

be a rational function where $g(x), h(x)$ *are polynomials with real coefficients, and let* $N = \{x \in \mathbb{R} \mid h(x) = 0\}$ *be the set of zeros of* $h(x)$. *Then the function* $f : \mathbb{R} \setminus N \to \mathbb{R}$ *mapping* x *to* $f(x)$ *is continuous.*

Proof. Notice first that the identity function $x \mapsto x$ is continuous[23]. Using parts $b)$ and $c)$ of Theorem 5.11 we see that $x \mapsto a_j x^n$ is continuous on $\mathbb{R}$ for any $a_j \in R$. Then part $a)$ of Theorem 5.11 implies that $g : \mathbb{R} \to \mathbb{R}, g(x) = \sum_{j=0}^{n} a_j x^j$ is continuous.

Now consider the rational function $f(x) = \frac{g(x)}{h(x)}$. Using part $d)$ of Theorem 5.11 we see that $f(x)$ is continuous on its maximal domain of definition[24] $\mathbb{R} \setminus N$. Notice here that the expression $\frac{g(x)}{h(x)}$ is only defined when the denominator $h(x)$ is nonzero. $\square$

[23] Formally, we consider $id : \mathbb{R} \to \mathbb{R}$ defined by $id(x) = x$. Given $\epsilon > 0$ choose $\delta = \epsilon$. Then obviously $|x - c| < \delta$ implies $|id(x) - c| = |x - c| < \delta = \epsilon$.

[24] Recall that if X is a set and $Y \subset X$ is any subset, then $X \setminus Y$ is defined by $X \setminus Y = \{x \in X \mid x \notin Y\}$.

Next we discuss how continuity behaves with respect to the composition of functions. Assume that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions. Recall that the composition $g \circ f : \mathbb{R} \to \mathbb{R}$ is the real function defined by

$$(g \circ f)(x) = g(f(x)).$$

More generally, if f and g are not necessarily defined on all of $\mathbb{R}$, the previous formula defines a function $g \circ f : \mathrm{dom}(f) \to \mathbb{R}$ provided the image[25] $\mathrm{im}(f)$ of f satisfies $\mathrm{im}(f) \subset \mathrm{dom}(g)$.

[25] The image of g is defined by $\mathrm{im}(g) = \{g(x) : x \in \mathrm{dom}(g)\}$.

Theorem 5.13. *Let* f, g *be real functions and assume that* f *is continuous at* c *and* g *is continuous at* $f(c)$. *Then* $g \circ f$ *is continuous at* c.

Proof. Let $\varepsilon > 0$ be arbitrary. Since g is continuous at $f(c)$, there exists $\eta > 0$ such that $|y - f(c)| < \eta$ implies $|g(y) - g \circ f(c)| < \varepsilon$. Since f is continuous at c, there exists $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \eta$. This means that for $|x - c| < \delta$ we have $|(g \circ f)(x) - (g \circ f)(c)| < \varepsilon$. Hence $g \circ f$ is continuous at c. $\square$

Combining Theorem 5.13 with the results above we obtain further examples of continuous functions.

Example 5.14. Let $f : \mathbb{R} \setminus \{5\} \to \mathbb{R}$ be the function

$$f(x) = \sqrt{\frac{1 + x^2 + x^6}{x^2 - 10x + 25}}$$

Show that f is continuous.

Solution. First of all notice that the denominator of the above expression can be written as $x^2 - 10x + 25 = (x-5)^2$, so that the function f is well-defined[26] on its domain $\mathrm{dom}(f) = \mathbb{R} \setminus \{5\}$.

[26] The term "well-defined" means that our definition makes sense, in particular, that it does not involve any invalid operations like dividing by zero.

We observe that f can be written as the composition $f = g \circ h$ of the rational function $h : \mathbb{R} \setminus \{5\} \to \mathbb{R}$ given by $h(x) = \frac{1+x^2+x^6}{x^2-10x+25}$ with the square root function $g : [0,\infty) \to \mathbb{R}$ givenby $g(x) = \sqrt{x}$. Here we use that $h(x) \geq 0$ for all $x \in \mathrm{dom}(f)$, so that $\mathrm{im}(h) \subset \mathrm{dom}(g)$.

Now h is continuous according to Theorem 5.12, and we have shown in Example 5.5 that g is continuous. Hence Theorem 5.13 implies that f is continuous. $\square$

Let us discuss another application of Theorem 5.13. If $f, g : \mathbb{R} \to \mathbb{R}$ are functions, then we define new functions $\max(f,g), \min(f,g) : \mathbb{R} \to \mathbb{R}$ by

$$\max(f,g)(x) = \max(f(x), g(x)), \quad \min(f,g)(x) = \min(f(x), g(x)),$$

respectively.

Example 5.15. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Show that $\max(f,g)$ and $\min(f,g)$ are continuous.

Solution. We have

$$\max(f,g)(x) = \frac{1}{2}(f(x) + g(x) + |f(x) - g(x)|),$$
$$\min(f,g)(x) = \frac{1}{2}(f(x) + g(x) - |f(x) - g(x)|).$$

Since $x \mapsto |f(x) - g(x)|$ is the composition of the continuous functions $f - g$ and absolute value $x \mapsto |x|$, we see using algebraic properties that $\max(f,g)$ is continuous.

The argument for $\min(f,g)$ is analogous. $\square$