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Q1/ Let $\varepsilon > 0$ be arbitrary. Then

$$|f(x) - f(2)| = \left|\frac{x^2-x-2}{2x-3} - 8\right| = \left|\frac{x^2-15+26x}{2x-3}\right|$$

$$\Rightarrow |f(x) - f(2)| = \left|\frac{(x-2)(x-13)}{2x-13}\right|$$

Let $|x-2| < \delta$, since δ can be any positive real number, we say $\delta \in [0,1]$. Then

$$-\delta < x-2 < \delta \iff -\delta - 11 < x - 13 < \delta - 11$$
$$\iff -2\delta < 2x-4 < 2\delta \iff -2\delta+1 < 2x-3 < 2\delta+1$$

Taking the boundaries for both expressions, we see $|x-13| < 12$ and $\left|\frac{1}{2x-3}\right| < 1$, therefore

$$\frac{|x-13|}{|2x-3|} < 12 \iff \frac{|x-2||x-13|}{|2x-3|} < 12 \cdot \delta < \varepsilon$$

Hence, let $\delta = \min(1, \varepsilon/12)$. We see that $|x-2| < \delta \Rightarrow |f(x) - f(2)| < \varepsilon$, as required.

Q2 a) If f is continuous at 0, then, for any $\varepsilon > 0$, there exists $\delta > 0$ s.t.

$$|x| < \delta \Rightarrow \begin{cases} |x \sin \frac{1}{x}| < \varepsilon & \text{for } x \neq 0 \\ 0 < \varepsilon & \text{for } x = 0 \end{cases}$$

Note that $|f(x) - f(0)| = |f(x)| \leq |x| < \delta \quad \forall x \in dom(f)$. Then we let $\delta = \varepsilon$, and we see that f is continuous at 0, as required.

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b) Let us assume g is continuous at 0. Then, by Theorem 5.9, the sequential characterisation of continuity, any sequence $(x_n)_{n=1}^{\infty}$ in $dom(f)$ with $x_n \to c$ as $n \to \infty \Rightarrow f(x_n) \to f(c)$ as $n \to \infty$.

Let us define $(x_n)_{n=1}^{\infty}$ as $x_n = \frac{2}{\pi} \cdot \frac{1}{1+4n}$, so clearly $x_n \to 0$ as $n \to \infty$.

Since g is continuous, then $g(x_n) \to g(0) = 0$. Observe that

$$g(x_n) = \begin{cases} \left|\sin\left(\frac{1}{\frac{2}{\pi} \cdot \frac{1}{1+4n}}\right)\right| & x_n \neq 0 \\ 0 & x_n = 0 \end{cases}$$

$$\Leftrightarrow \quad g(x_n) = \begin{cases} \left|\sin\left(\frac{\pi}{2} + 2\pi n\right)\right| & x_n \neq 0 \\ 0 & x_n = 0 \end{cases}$$

Then we have $g(x_n) = \sin\left(\frac{\pi}{2}\right) = 1 \quad \forall x_n \in \mathbb{R} \setminus \{0\}$. This implies that $g(x_n) \to 1$ as $n \to \infty$. But this is a contradiction, since we said that for g to be continuous, $g(x_n) \to 0$ as $n \to \infty$. Therefore, g is not continuous at 0.

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a) We are concerned with the value of $f(0)$. We see that for $x=y=0$ $(x,y\in\mathbb{R})$ we have

$$f(x+y) = f(x)+f(y)$$
$$\iff f(0+0) = f(0)+f(0) \iff f(0) = f(0)+f(0)$$

Let $c = f(0)$, then we have $c = 2c$. We see this can only be true if $c=0$ for $c\in\mathbb{R}$.
So $f(0)=0$.

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b) We are concerned with the value of $f(q)$. Let us start with $f(n)$ for $n\in\mathbb{N}$.
We can easily see

$$f(n) = f(\tfrac{n}{2})+f(\tfrac{n}{2}) = f(\tfrac{n}{3})+f(\tfrac{n}{3})+f(\tfrac{n}{3}).$$
$$\Rightarrow f(n) = \sum^{k} f(\tfrac{n}{k}) = k\cdot f(\tfrac{n}{k})$$

Now, let $k=n$, so we get $f(n) = n\cdot f(\frac{n}{n}) = n\,f(1)$.

Let us extend our notation to $f(\frac{n}{m})$ for $n,m\in\mathbb{N}$. Then we have, from our previous result,

$$f(\tfrac{n}{m}) = f(\tfrac{1}{m}\cdot n) = n\cdot f(\tfrac{1}{m})$$

~~The~~ To discover the value of $f(\frac{1}{m})$, observe that

$$f(1) = f(m\cdot\tfrac{1}{m}) = m\cdot f(\tfrac{1}{m})$$
$$\iff f(\tfrac{1}{m}) = \tfrac{1}{m}\cdot f(1)$$

Therefore

$$f(\tfrac{n}{m}) = n\cdot f(\tfrac{1}{m}) = n\cdot\tfrac{1}{m}\cdot f(1) = \tfrac{n}{m}f(1).$$

To extend to the set of rational numbers, observe that $f((-1)\cdot\frac{n}{m}) = \frac{n}{m}f(-1)$ from our previous result. Then we're left with the task of discovering the value of $f(-1)$. Using the definition of f, we ~~let~~ $x=1$ and $y=-1$, ~~then~~ and observe

$$f(1-1) = f(0) = f(1)+f(-1)$$
$$\iff f(-1) = f(0)-f(1) = -f(1)$$

So we have

$$f((-1)\tfrac{n}{m}) = \tfrac{n}{m}f(-1) = -\tfrac{n}{m}f(1)$$

So that $\forall q\in\mathbb{Q}$ we have $f(q) = q\,f(1)$, as required.

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c) Suppose that there exists a sequence $(q_n)_{n=1}^{\infty}$ with $q_n\in\mathbb{Q}$ $\forall n\in\mathbb{N}$ such that $q_n\to x$ as $n\to\infty$.
From above, we know $f(q_n) = q_n f(1)$ $\forall q_n\in\mathbb{Q}$. Since f is continuous, we have that $q_n\to x \Rightarrow f(q_n)\to f(x)$ since the sequence $(q_n)_{n=1}^{\infty}$ and its converging value are in $\mathrm{dom}(f)$.
Therefore, if f is continuous then $f(x) = x\,f(1)$, as required.