2B 2020 – Homework 4. Solutions

Q1) Let

$$y_1 = (0,1,1,0), \quad y_2 = (0,2,0,1), \quad y_3 = (1,1,0,2)$$

Setting $\underline{x}_1 = y_1 = (0,1,1,0)$ we then have:

$$\begin{aligned}
\underline{x}_2 &= y_2 - \frac{y_2 \cdot x_1}{x_1 \cdot x_1} x_1 \\
&= (0,2,0,1) - \frac{(0,2,0,1)\cdot(0,1,1,0)}{(0,1,1,0)\cdot(0,1,1,0)}(0,1,1,0) \\
&= (0,2,0,1) - \frac{2}{2}(0,1,1,0) \\
&= (0,1,-1,1).
\end{aligned}$$

We proceed by setting:

$$\begin{aligned}
\underline{x}_3 &= y_3 - \frac{y_3 \cdot x_1}{x_1 \cdot x_1} x_1 - \frac{y_3 \cdot x_2}{x_2 \cdot x_2} x_2 \\
&= (1,1,0,2) - \frac{(1,1,0,2)\cdot(0,1,1,0)}{(0,1,1,0)\cdot(0,1,1,0)}(0,1,1,0) - \frac{(1,1,0,2)\cdot(0,1,-1,1)}{(0,1,-1,1)\cdot(0,1,-1,1)}(0,1,-1,1) \\
&= (1,1,0,2) - \frac{1}{2}(0,1,1,0) - \frac{3}{3}(0,1,-1,1) \\
&= (1, -\tfrac{1}{2}, \tfrac{1}{2}, 1). \\
&= \tfrac{1}{2}(2,-1,1,2)
\end{aligned}$$

Hence an orthogonal basis for U is

$$\{\underline{x}_1, \underline{x}_2, \underline{x}_3\} = \{(0,1,1,0), (0,1,-1,1), (1,-\tfrac{1}{2},\tfrac{1}{2},1)\}.$$

The vectors are indeed orthogonal as:

$$\begin{aligned}
\underline{x}_1 \cdot \underline{x}_2 &= 1 - 1 = 0, \\
\underline{x}_1 \cdot \underline{x}_3 &= -\tfrac{1}{2} + \tfrac{1}{2} = 0, \\
\underline{x}_2 \cdot \underline{x}_3 &= -\tfrac{1}{2} - \tfrac{1}{2} + 1 = 0.
\end{aligned}$$

Q2 (i) WE HAVE

$$A = \begin{pmatrix} 1 & 0 \\ 4 & 2 \end{pmatrix}$$

WHICH HAS CHARACTERISTIC POLYNOMIAL $p_A(\lambda) = (\lambda - 1)(\lambda - 2)$ SO THE EIGENVALUES OF A ARE $\lambda_1 = 1$, $\lambda_2 = 2$. TO FIND THE E_1 - EIGENSPACE WE CALCULATE:

$$(A - \mathbb{I})\underline{x} = \underline{0}$$

WHERE $\underline{x}_1 = (x_1, y_1) \in \mathbb{R}^2$. THE CORRESPONDING AUGMENTED MATRIX SYSTEM IS:

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 4 & 1 & 0 \end{array}\right)$$

SO $4x_1 + y_1 = 0$ GIVING $y_1 = -4x_1$ AND SO

$$E_1 = \text{Span}\left\{\begin{pmatrix} 1 \\ -4 \end{pmatrix}\right\}.$$

TO FIND THE E_2 - EIGENSPACE WE CALCULATE

$$(A - 2\mathbb{I})\underline{x}_2 = \underline{0}$$

WHERE $\underline{x}_2 = (x_2, y_2) \in \mathbb{R}^2$. THE CORRESPONDING AUGMENTED SYSTEM IS:

$$\left(\begin{array}{cc|c} -1 & 0 & 0 \\ 4 & 0 & 0 \end{array}\right)$$

AND SO $x_2 = 0$ AND WE HAVE:

$$E_2 = \text{Span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}.$$

HENCE A IS DIAGONALISABLE WITH $P^{-1}AP = D$ WHERE

$$P = \begin{pmatrix} 0 & 1 \\ 1 & -4 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

(ii) WE HAVE

$$P^{-1}AP = D$$
$$\Longrightarrow A = PDP^{-1}$$
$$\Longrightarrow A^4 = (PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1})$$
$$\Longrightarrow A^4 = PD^4P^{-1}$$

HENCE,

$$A^4 = \begin{pmatrix} 0 & 1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} 2^4 & 0 \\ 0 & 1^4 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} 64 & 16 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 60 & 16 \end{pmatrix}$$

Q3 WE FIRST SHOW THAT IF A IS ORTHOGONAL THEN A IS SYMMETRIC.
ASSUME A IS ORTHOGONAL, SO $A^{-1} = A^T$. GIVEN $A^2 = I_n$
APPLY A^{-1} TO BOTH SIDES GIVES $A^{-1}A^2 = A^{-1}I_n = A^{-1} = A^T$.
SO $A^{-1}A^2 = A = A^T$.
HENCE A IS SYMMETRIC.

WE NOW PROVE THAT IF A IS SYMMETRIC THEN A IS ORTHOGONAL.
ASSUME $A = A^T$. APPLYING A TO BOTH SIDES GIVES
$A^2 = AA^T = I$, SINCE $A^2 = I$,
SIMILARLY,
$A^2 = A^T A = I$.
SO $A^T A = A A^T = I$ SO A IS INVERTIBLE
AND $A^{-1} = A^T$ HENCE A IS ORTHOGONAL.

MATHS: /15
SOLVE: /5
TOTAL: /20