2B Linear Algebra

Ex Sheet 1

True/False

We will start the tutorial by going over these true false questions. Please make sure you've thought about them in advance of the tutorial and so are ready to answer.[1]

a) The vector $\begin{bmatrix}1\\1\end{bmatrix}$ can be written as a linear combination of the vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}4\\0\end{bmatrix}$.

b) A system of linear equations with augmented matrix $[A|\boldsymbol{b}]$ has a unique solution if and only if $\boldsymbol{b}$ is a linear combination of the columns of A.

c) The vector $\begin{bmatrix}1\\1\end{bmatrix}$ is in Span $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}4\\0\end{bmatrix}\right\}$.

d) If S is any non-empty set of vectors then $\mathbf{0}$ is in Span(S).

e) If a vector $\boldsymbol{v}$ is in the span of some set of vectors S, then so is its additive inverse $-\boldsymbol{v}$.

f) The set of vectors $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}4\\0\end{bmatrix}\right\}$ is a spanning set for $\mathbb{R}^2$.

g) For any two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^3$, Span$(\boldsymbol{u}, \boldsymbol{v})$ is a plane through the origin.

h) In $\mathbb{R}^n$, the vectors $(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ are linearly independent.

i) A set of vectors in linearly dependent if at least one vector in the set can be expressed as a linear combination of the others.

j) The zero vector is contained in any linearly independent set of vectors.

k) A set of vectors $S = \{\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}\}$ is linearly independent if and only if there are no solutions to the equation $\lambda_1\boldsymbol{v_1} + \cdots + \lambda_n\boldsymbol{v_n} = \mathbf{0}$ for scalars $\lambda_1, \ldots, \lambda_n$.

l) A set of vectors $S = \{\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}\}$ is linearly independent if $\boldsymbol{v}_1$ is in Span$\{\boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\}$.

m) A homogeneous system of linear equations with augmented matrix $[A|\mathbf{0}]$ has a nontrivial solution if and only if the columns of A are linearly dependent.

n) Any set of 2 vectors in $\mathbb{R}^3$ is linearly dependent.

[1] **True/False Questions**

Every Exercise Sheet will have a section containing true/false questions, with solutions at the end of the sheet. They are designed to test your understanding from lectures. You should look at your lecture notes and/or the textbook to help you answer these questions, but you should not need to write anything to work out the solution.

o) Any set of 4 vectors in $\mathbb{R}^3$ is linearly dependent.

Solutions to True/False

(a) F (b) F (c) F (d) T (e) T (f) F (g) F (h) T (i) T (j) F (k) F (l) F (m) T (n) F (o) T

Tutorial Exercises

Before attempting these questions you should make sure you do all of the questions on Vectors and Systems of Linear Equations from Exercise Sheet 0.

T1 Which of the vectors $(1,2)$ and $(0,0)$ can be written as a linear combination of $\boldsymbol{u} = (1,-1)$ and $\boldsymbol{v} = (2,-1)$?

Solution

Both vectors can be written as a linear combination of $\boldsymbol{u} = (1,-1)$ and $\boldsymbol{v} = (2,-1)$ since

$$(1,2) = -5(1,-1) + 3(2,-1) = -5\boldsymbol{u} + 3\boldsymbol{v}.$$

and

$$(0,0) = 0(1,-1) + 0(2,-1) = 0\boldsymbol{u} + 0\boldsymbol{v}.$$

T2 Which of the vectors $(0,-3,6)$, $(3,-9,-2)$, $(0,0,0)$ and $(7,8,9)$ can be written as a linear combination of $\boldsymbol{u} = (2,1,4)$, $\boldsymbol{v} = (1,-1,3)$ and $\boldsymbol{w} = (3,2,5)$?

Solution

Linear combinations of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are all vectors of the form

$$x(2,1,4) + y(1,-1,3) + z(3,2,5)$$

for some $x, y, z \in \mathbb{R}$, that is, any vector of the form

$$(2x + y + 3z, x - y + 2z, 4x + 3y + 5z).$$

The zero vector $(0,0,0)$ can be written in this form by taking $x = y = z = 0$. As for the rest, rather than do three separate calculations, we just do one: the vector (b_1, b_2, b_3) is of the above form if and only if the following system of equations has a solution:

$$\begin{aligned} 2x + y + 3z &= b_1 \\ x - y + 2z &= b_2 \\ 4x + 3y + 5z &= b_3. \end{aligned}$$

Perform EROs on the augmented matrix

$$\begin{pmatrix} 2 & 1 & 3 & b_1 \\ 1 & -1 & 2 & b_2 \\ 4 & 3 & 5 & b_3 \end{pmatrix}$$

to obtain the augmented matrix in echelon form (this result isn't unique, yours may differ)

$$\begin{pmatrix} 1 & -1 & 2 & b_1 \\ 0 & 1 & -1 & b_3 - 2b_1 \\ 0 & 0 & 1 & \frac{7}{2}b_1 - b_2 - \frac{3}{2}b_3 \end{pmatrix}$$

This system is consistent, that is, we can solve the system by back substitution for any b_1, b_2, b_3. This means that the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ span the whole of $\mathbb{R}^3$. In particular, each of the vectors $(0, -3, 6)$, $(3, -9, -2)$, $(0, 0, 0)$ and $(7, 8, 9)$ can be written as a linear combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

T3 Give a geometric description of the span of the following pairs of vectors in $\mathbb{R}^3$. Is the span a plane or a line?

a) $\boldsymbol{v}_1 = [3, -4, 5]$, $\boldsymbol{v}_2 = [4, 2, 7]$

b) $\boldsymbol{v}_1 = [3, 5, 2]$, $\boldsymbol{v}_2 = [6, 10, 4]$

Solution

a) We have

$$\mathrm{Span}(\boldsymbol{v}_1, \boldsymbol{v}_2) = \{\lambda_1[3, -4, 5] + \lambda_2[4, 2, 7] : \lambda_1, \lambda_2 \in \mathbb{R}\}.$$

As $\boldsymbol{v}_1 = [3, -4, 5]$ is not a scalar multiple of $\boldsymbol{v}_2 = [4, 2, 7]$, the span is the plane through the origin spanned by $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

b) We have

$$\mathrm{Span}(\boldsymbol{v}_1, \boldsymbol{v}_2) = \{\lambda_1[3, 5, 2] + \lambda_2[6, 10, 4] : \lambda_1, \lambda_2 \in \mathbb{R}\}.$$

Since $[6, 10, 4] = 2[3, 5, 2]$, the span is the line through the origin in the direction of $[3, 5, 2]$.

T4 Which of the following sets are spanning sets for $\mathbb{R}^3$?

a) $S_1 = \{(2, 1, 0), (1, 1, -1)\}$;

b) $S_2 = \{(1, 0, 1), (2, 1, 0), (1, 1, -1)\}$;

c) $S_3 = \{(1, 0, 1), (2, 1, 0), (1, 1, -1), (3, -1, 3)\}$;

d) $S_4 = \{(2, 1, 0), (1, 1, -1), (3, -1, 3)\}$.

Solution

We examine which of these sets span $\mathbb{R}^3$.

a) $S_1 = \{(2, 1, 0), (1, 1, -1)\}$.

If this set spans $\mathbb{R}^3$ then any $(x,y,z) \in \mathbb{R}^3$ van be written as

$$(x,y,z) = \lambda(2,1,0) + \mu(1,1,-1)$$

for some $\lambda, \mu \in \mathbb{R}$. This vector equation is equivalent to a system of three equations that we can represent in the augmented matrix

$$\begin{pmatrix} 2 & 1 & x \\ 1 & 1 & y \\ 0 & -1 & z \end{pmatrix}.$$

Simple EROs show that the *reduced* row echelon form of this matrix (remember that the reduced row echelon form of a matrix is unique — you should get this answer!) is

$$\begin{pmatrix} 1 & 0 & x-y \\ 0 & 1 & 2y-x \\ 0 & 0 & z+2y-x \end{pmatrix}.$$

The final line shows that the system is consistent only when the equation $z + 2y - x = 0$ holds. In other words, we cannot find $\lambda, \mu \in \mathbb{R}$ for every $(x,y,z) \in \mathbb{R}^3$. For example, if $(x,y,z) = (1,1,1)$ then the equation becomes $2 = 0$ which is not true. Therefore $(1,1,1)$ does not lie in the span of the set $S_1 = \{(2,1,0),(1,1,-1)\}$, so S_1 does not span $\mathbb{R}^3$.

b) $S_2 = \{(1,0,1),(2,1,0),(1,1,-1)\}$.

If this set spans $\mathbb{R}^3$ then any $(x,y,z) \in \mathbb{R}^3$ van be written as

$$(x,y,z) = \lambda(1,0,1) + \mu(2,1,0) + \nu(1,1,-1)$$

for some $\lambda, \mu, \nu \in \mathbb{R}$. As in (i) we consider the augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 & x \\ 0 & 1 & 1 & y \\ 1 & 0 & -1 & z \end{bmatrix}$$

which has reduced row echelon form

$$\begin{bmatrix} 1 & 0 & -1 & x-2y \\ 0 & 1 & 1 & y \\ 0 & 0 & 0 & z+2y-x \end{bmatrix}.$$

As in the previous example we require $z + 2y - x = 0$ which does not hold for all $(x,y,z) \in \mathbb{R}^3$. Hence S_2 does not span $\mathbb{R}^3$.

c) $S_3 = \{(1,0,1),(2,1,0),(1,1,-1),(3,-1,3)\}$.

If this set spans $\mathbb{R}^3$ then any $(x,y,z) \in \mathbb{R}^3$ van be written as

$$(x,y,z) = \lambda(1,0,1) + \mu(2,1,0) + \nu(1,1,-1) + \delta(3,-1,3)$$

for some $\lambda, \mu, \nu, \delta \in \mathbb{R}$. As above we consider the augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 & 3 & x \\ 0 & 1 & 1 & -1 & y \\ 1 & 0 & -1 & 3 & z \end{bmatrix}$$

and find that it has reduced row echelon form

$$\left[\begin{array}{cccc|c}1 & 0 & -1 & 0 & -\frac{3}{2}x+3y+\frac{5}{2}z \\ 0 & 1 & 1 & 0 & \frac{1}{2}x-\frac{1}{2}z \\ 0 & 0 & 0 & 1 & \frac{1}{2}x-y-\frac{1}{2}z\end{array}\right].$$

In this case the equations always have a real solution in $\lambda, \mu, \nu, \delta$ given by

$$\begin{aligned}\lambda &= -\frac{3}{2}x+3y+\frac{5}{2}z+\nu \\ \mu &= \frac{1}{2}x-\frac{1}{2}z+\nu \\ \delta &= \frac{1}{2}x-y-\frac{1}{2}z.\end{aligned}$$

So given any $(x,y,z) \in \mathbb{R}$ we can let ν to be any real number and calculate λ, μ, δ from the equations. So in fact, there are infinitely many solutions to the original equations. So S_3 does span $\mathbb{R}^3$.

d) $S_4 = \{(2,1,0),(1,1,-1),(3,-1,3)\}$.
If this set spans $\mathbb{R}^3$ then any $(x,y,z) \in \mathbb{R}^3$ van be written as

$$(x,y,z) = \lambda(2,1,0)+\mu(1,1,-1)+\nu(3,-1,3)$$

for some $\lambda, \mu, \nu, \in \mathbb{R}$. We consider the augmented matrix

$$\left[\begin{array}{ccc|c}2 & 1 & 3 & x \\ 1 & 1 & -1 & y \\ 0 & -1 & 3 & z\end{array}\right]$$

which has reduced row echelon form

$$\left[\begin{array}{ccc|c}1 & 0 & 0 & -x+3y+2z \\ 0 & 1 & 0 & \frac{3}{2}x-3y-\frac{5}{2}z \\ 0 & 0 & 1 & \frac{x}{2}-y-\frac{z}{2}\end{array}\right].$$

In this case the equations always have a real solution in $\lambda, \mu, \nu, \delta$ given by

$$\begin{aligned}\lambda &= -x+3y+2z \\ \mu &= \frac{3}{2}x-3y-\frac{5}{2}z \\ \nu &= \frac{x}{2}-y-\frac{z}{2}.\end{aligned}$$

So given any $(x,y,z) \in \mathbb{R}$ we calculate λ, μ, δ from the equations above. This time the values will be unique and there is exactly one solution to the original equations. So, S_4 does span $\mathbb{R}^3$.

T5 Prove that $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are all in $\text{Span}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$.

Solution

The key point is that the coefficient 0 is allowed in the definition of span. Since

$$\boldsymbol{u} = 1\boldsymbol{u}+0\boldsymbol{v}+0\boldsymbol{w}, \quad \boldsymbol{v} = 0\boldsymbol{u}+1\boldsymbol{v}+0\boldsymbol{w}, \quad \boldsymbol{w} = 0\boldsymbol{u}+0\boldsymbol{v}+1\boldsymbol{w}$$

the result follows.

T6 Determine whether the following sets of vectors are linearly independent, giving reasons for your answer. For any sets that are linearly dependent, find a nontrivial linear combination of the vectors in that set which equals $\mathbf{0}$. (Such a linear combination is called a dependence relationship.)

a) In $\mathbb{R}^2$:
$$\boldsymbol{u} = [1,1], \quad \boldsymbol{v} = [-2,-2]$$

b) In $\mathbb{R}^2$:
$$\boldsymbol{u} = [1,1], \quad \boldsymbol{v} = [-2,2]$$

c) In $\mathbb{R}^2$:
$$\boldsymbol{u} = [1,1], \quad \boldsymbol{v} = [-2,1], \quad \boldsymbol{w} = [4,1]$$

d) In $\mathbb{R}^3$:
$$\boldsymbol{u} = [1,0,1], \quad \boldsymbol{v} = [-2,-2,1], \quad \boldsymbol{w} = [1,2,-3]$$

e) In $\mathbb{R}^3$:
$$\boldsymbol{u} = [1,0,1], \quad \boldsymbol{v} = [-2,-2,1], \quad \boldsymbol{w} = [-1,-2,2]$$

f) In $\mathbb{R}^7$:
$$\boldsymbol{u} = [1,2,3,4,5,6,7], \boldsymbol{v} = [8,6,7,5,3,0,9], \boldsymbol{w} = [0,0,0,0,0,0,0]$$

Solution

a) These vectors are not linearly independent since they are scalar multiples of each other. Since $\boldsymbol{v} = -2\boldsymbol{u}$ a dependence relationship is $2\boldsymbol{u} + \boldsymbol{v} = \mathbf{0}$.

b) These vectors are linearly independent since they are not scalar multiples of each other.

c) These vectors are not linearly independent since one of them can be written as a linear combination of the others, for instance
$$\boldsymbol{w} = [4,1] = 2[1,1] - [-2,1] = 2\boldsymbol{u} - \boldsymbol{v}.$$
Alternatively, no set of 3 vectors in $\mathbb{R}^2$ is linearly independent. Using the previous equation, a dependence relationship is $2\boldsymbol{u} - \boldsymbol{v} - \boldsymbol{w} = \mathbf{0}$.

d) This set of vectors is linearly independent. We want to solve the equation $a\boldsymbol{u} + b\boldsymbol{v} + c\boldsymbol{w} = \mathbf{0}$. The corresponding system of linear equations has augmented matrix
$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 1 & 1 & -3 & 0 \end{pmatrix}.$$

After applying EROs we obtain a row echelon form

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix}.$$

This system has a unique solution $a = b = c = 0$, hence the vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly independent.

e) These vectors are not linearly independent since one of them can be written as a linear combination of the others, for instance

$$\boldsymbol{w} = [-1, -2, 2] = [1, 0, 1] + [-2, -2, 1] = \boldsymbol{u} + \boldsymbol{v}.$$

Using this equation, a dependence relationship is $\boldsymbol{u} + \boldsymbol{v} - \boldsymbol{w} = \mathbf{0}$.

f) These vectors are not linearly independent since this set contains the zero vector. A dependence relationship is

$$0\boldsymbol{u} + 0\boldsymbol{v} + 1\boldsymbol{w} = \mathbf{0}.$$

T7

a) Draw diagrams to illustrate properties (a), (d) and (e) of Theorem 1.1.

b) Give an algebraic proof of properties (d) and (e) of Theorem 1.1.

Solution

a) Ask your tutor or lecturer.

b) For (d), let $\boldsymbol{u} \in \mathbb{R}^n$. We want to show that $\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}$. For this, let $\boldsymbol{u} = [u_1, u_2, \ldots, u_n]$. Then $-\boldsymbol{u} = [-u_1, -u_2, \ldots, -u_n]$, so

$$\begin{aligned} \boldsymbol{u} + (-\boldsymbol{u}) &= [u_1, u_2, \ldots, u_n] + [-u_1, -u_2, \ldots, -u_n] \\ &= [u_1 + (-u_1), u_2 + (-u_2), \ldots, u_n + (-u_n)] \\ &= [u_1 - u_1, u_2 - u_2, \ldots, u_n - u_n] \\ &= [0, 0, \ldots, 0] \\ &= \mathbf{0} \end{aligned}$$

as required.

For (e), let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ and let $c \in \mathbb{R}$. We want to show that $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$. For this, let $\boldsymbol{u} = [u_1, u_2, \ldots, u_n]$ and $\boldsymbol{v} = [v_1, v_2, \ldots, v_n]$. Then

$$\begin{aligned} c(\boldsymbol{u} + \boldsymbol{v}) &= c([u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n]) \\ &= c[u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n] \\ &= [c(u_1 + v_1), c(u_2 + v_2), \ldots, c(u_n + v_n)]. \end{aligned}$$

On the other hand

$$\begin{aligned} c\boldsymbol{u}+c\boldsymbol{v} &= c[u_1,u_2,\ldots,u_n]+c[v_1,v_2,\ldots,v_n] \\ &= [cu_1,cu_2,\ldots,cu_n]+[cv_1,cv_2,\ldots,cv_n] \\ &= [cu_1+cv_1,cu_2+cv_2,\ldots,cu_n+cv_n] \\ &= [c(u_1+v_1),c(u_2+v_2),\ldots,c(u_n+v_n)]. \end{aligned}$$

Since both $c(\boldsymbol{u}+\boldsymbol{v})$ and $c\boldsymbol{u}+c\boldsymbol{v}$ are equal to $[c(u_1+v_1),c(u_2+v_2),\ldots,c(u_n+v_n)]$, we have that $c(\boldsymbol{u}+\boldsymbol{v})=c\boldsymbol{u}+c\boldsymbol{v}$ as required.

General comment: sometimes it is easier to get two expressions to equal the same thing by manipulating both of them, rather than trying to work on just one of them to obtain the other.

T8 Suppose $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ are vectors in $\mathbb{R}^n$, and let $S=\{\boldsymbol{u}_1,\boldsymbol{u}_2\}$ and $T=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\}$. Prove that $\mathrm{Span}(S)\subseteq\mathrm{Span}(T)$. That is, prove that $\mathrm{Span}(S)$ is a subset of $\mathrm{Span}(T)$. [Hint: What you need to show is that every vector in $\mathrm{Span}(S)$ is also contained in $\mathrm{Span}(T)$. So your proof should probably start with: "Let $\boldsymbol{v}$ be a vector in $\mathrm{Span}(S)$."]

Solution

Let $\boldsymbol{v}$ be a vector in $\mathrm{Span}(S)$. Then by definition of span, there are scalars λ_1,λ_2 so that $\boldsymbol{v}=\lambda_1\boldsymbol{u}_1+\lambda_2\boldsymbol{u}_2$. Hence

$$\boldsymbol{v}=\lambda_1\boldsymbol{u}_1+\lambda_2\boldsymbol{u}_2+0\boldsymbol{u}_3$$

and so $\boldsymbol{v}$ is in $\mathrm{Span}(T)$ as well. Therefore $\mathrm{Span}(S)\subseteq\mathrm{Span}(T)$.

T9 Suppose that vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly independent. Are the vectors $\boldsymbol{u}+\boldsymbol{v}$, $\boldsymbol{v}+\boldsymbol{w}$ and $\boldsymbol{u}+\boldsymbol{w}$ linearly independent? What about the vectors $\boldsymbol{u}-\boldsymbol{v}$, $\boldsymbol{v}-\boldsymbol{w}$ and $\boldsymbol{u}-\boldsymbol{w}$? Justify your answers.

Solution

Suppose that

$$a(\boldsymbol{u}+\boldsymbol{v})+b(\boldsymbol{v}+\boldsymbol{w})+c(\boldsymbol{u}+\boldsymbol{w})=\boldsymbol{0}$$

where a,b,c are scalars. This rearranges to give

$$(a+c)\boldsymbol{u}+(a+b)\boldsymbol{v}+(b+c)\boldsymbol{w}=\boldsymbol{0}.$$

Since $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly independent, the only solution to this equation is $a+c=0$, $a+b=0$ and $b+c=0$. Check that the unique solution to this system is $a=0$, $b=0$ and $c=0$. Thus the vectors $\boldsymbol{u}+\boldsymbol{v}$, $\boldsymbol{v}+\boldsymbol{w}$ and $\boldsymbol{u}+\boldsymbol{w}$ are linearly independent.

Now suppose that

$$a(\boldsymbol{u}-\boldsymbol{v})+b(\boldsymbol{v}-\boldsymbol{w})+c(\boldsymbol{u}-\boldsymbol{w})=\boldsymbol{0}$$

where a,b,c are scalars. This rearranges to give

$$(a+c)\boldsymbol{u}+(-a+b)\boldsymbol{v}+(-b-c)\boldsymbol{w}=\boldsymbol{0}.$$

Since $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly independent, the only solution to this equation is $a + c = 0$, $-a + b = 0$ and $-b - c = 0$. However this system has infinitely many solutions and in particular has nonzero solutions, for example $a = 1$, $b = 1$ and $c = -1$. Therefore the vectors $\boldsymbol{u} - \boldsymbol{v}$, $\boldsymbol{v} - \boldsymbol{w}$ and $\boldsymbol{u} - \boldsymbol{w}$ are not linearly independent.