Algorithmic Foundations 2 - Tutorial Sheet 7

Counting

1. Suppose that a password is of length between 6 and 8 characters, consists of letters and digits, and is case-sensitive.

   (a) How many distinct passwords are there?

   > **Solution:** There are (26+26+10=)62 possible characters in each position of the password, therefore using both the product and sum rules we have a total of:
   >
   > $$\begin{aligned} 62^6 + 62^7 + 62^8 &= 56,800,235,584 + 3,521,614,606,208 + 218,340,105,584,896 \\ &= 7221,918,520,426,688 \end{aligned}$$

   (b) How many contain at least one letter and at least one digit?

   > **Solution:** We need to exclude all passwords containing only letters and those containing only digits, this gives a total of
   >
   > $$\begin{aligned} &62^6 + 62^7 + 62^8 - (52^6 + 52^7 + 52^8) - (10^6 + 10^7 + 10^8) \\ &= 221,918,520,426,688 - 54,507,570,843,648 - 111,000,000 \\ &= 167,410,838,583,040 \end{aligned}$$

   (c) How long would it take a hacker to test all passwords at 2.8×10^9 tests per second?

   > **Solution:** The seconds required equals $7221,918,520,426,688/(2.8 \times 10^9)$ which is approximately $79,256$ which yields $1,320$ minutes or alternatively 22 hours.

2. How many one-one functions are there from a set with 5 elements to a set with the following number of elements:

   (a) 4

   > **Solution:** 0 since size of domain is larger than size of codomain.

   (b) 5

   > **Solution:** 120 (5 choices for first value in domain; 4 choices for second value in domain,. . . ,1 choice for fifth value in domain, so $5! = 120$ different functions by the product rule).

   (c) 6

   > **Solution:** 720 (6 choices for first value in domain; 5 choices for second value in domain,. . . ,2 choices for fifth value in domain, so $6{\cdot}5{\cdot}4{\cdot}3{\cdot}2 = 720$ different functions by the product rule).

(d) 7

> **Solution:** 2,520 (7 choices for first value in domain; 6 choices for second value in domain,. . . ,3 choices for fifth value in domain, so $7{\cdot}6{\cdot}5{\cdot}4{\cdot}3 = 2520$ different functions by the product rule).

3. How many bit strings of length seven either begin with two 0's or end with three 1's?

> **Solution:** There are $2^5 = 32$ bit strings which begin with two 0's (two choices for each of the five remaining positions). There are $2^4 = 16$ bit strings which end with three 1's (two choices for each of the four remaining positions). There are $2^2 = 4$ bit strings which begin with two 0's and end with three 1's (two choices for each of the two remaining positions). Hence by the Principle of Inclusion-Exclusion, the number of bit strings of length seven which either begin with two 0's or end with three 1's is $32 + 16 - 4 = 44$.

4. How many bit strings of length eight either start with two 1's, or end with two 1's, or have four 1's in the middle four places?

> **Solution:** Let A be the set of bit strings starting with two 1's, B the set ending with two 1's, and C the set with four 1's in the middle four places. Now we have:
>
> $$\begin{aligned} |A| = |B| &= 2^6 = 64 \\ |C| &= 2^4 = 16 \\ |A \cap B| &= 2^4 = 16 \\ |A \cap C| = |B \cap C| &= 2^2 = 4 \\ |A \cap B \cap C| &= 2^0 = 1 \end{aligned}$$
>
> So, applying the Inclusion-Exclusion Principle, we have:
>
> $$\begin{aligned} |A \cup B \cup C| &= |A| + |B \cup C| - |A \cap (B \cup C)| & \\ &= |A| + |B| + |C| - |B \cap C| - |A \cap (B \cup C)| & \text{Inclusion-Exclusion Principle} \\ &= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)| & \text{distributivity} \\ &= |A| + |B| + |C| - |B \cap C| - (|A \cap B| + |A \cap C| - |A \cap B \cap C|) & \text{set counting rule} \\ &= 64 + 64 + 16 - 4 - (16 + 4 - 1) & \text{from above} \\ &= 121 & \end{aligned}$$

5. How many bit strings of length ten have:

   (a) exactly three 0's?

> **Solution:** There are $C(10, 3) = 120$ bit strings, since there are $C(10, 3)$ ways to choose the positions for the three 0's, and that is the only choice to be made (all other positions take value 1).

   (b) the same number of 0's as 1's?

Solution: There are $C(10, 5) = 252$ bit strings, since there are $C(10, 5)$ ways to choose the positions for the five 0's, and that is the only choice to be made (all other positions take value 1).

(c) at least seven 1's?

Solution: In this case we need to count the number of bit strings with exactly seven, eight, nine and ten 1's. By the same reasoning as above, the number of such strings is $C(10, 7) + C(10, 8) + C(10, 9) + C(10, 10) = 120 + 45 + 10 + 1 = 176$.

(d) at least three 1's?

Solution: It is easier to count the number of bit strings containing fewer than three 1's and then subtract this number from the total number of bit strings. To contain fewer than three 1's, there must be either precisely 0, 1 or 2, yielding the total $C(10, 0) + C(10, 1) + C(10, 2) = 1 + 10 + 45 = 56$. Therefore, since the total number of bit strings is $2^{10} = 1,024$, we have $1,024 - 56 = 968$ bit strings have at least three 1's.

6. How many bit strings contain exactly five 0's and fourteen 1's if every 0 must be immediately followed by two 1's?

Solution: We need two 1's to be to the right of each 0 so we need five copies of 011, this gives ten 1's in the string so we need another four 1's. To calculate the number of strings we need to find the number of ways of rearranging the four 1's and five copies of 011. More precisely, we have 9 objects of which the four 1's and and the five 011's are instinguishable, therefore we have:

$$\frac{9!}{4!5!} = \frac{9{\cdot}8{\cdot}7{\cdot}6}{4{\cdot}3{\cdot}2{\cdot}1} = 9{\cdot}7{\cdot}2 = 126\,.$$

7. A lottery ticket consists of six distinct numbers in the range 1-49.

(a) What are the chances of having exactly four of the winning numbers?

Solution: There are $C(6, 4) = 15$ ways of choosing four of the winning numbers, and $C(43, 2) = 903$ ways of choosing two losing numbers, giving $15{\cdot}903 = 13,545$ ways of having exactly four winning numbers. The odds are $13,545$ in $C(49, 6) = 13,983,816$, that is around 1 in $1,032$.

(b) Exactly three of the winning numbers?

Solution: There are $C(6, 3) = 20$ ways of choosing three of the winning numbers, and $C(43, 3) = 12,341$ ways of choosing three losing numbers. So, there are $20{\times}12341 = 2,468,20$ ways of choosing exactly three winning numbers and three losing numbers (about 1 in 57).

(c) Is it more likely to have at least one of the winning numbers, or none of them?

> **Solution:** There are $C(43,6) = 6,096,454$ ways of choosing no winning numbers. This is less than half $C(49,6) = 13,983,816$, so it is more likely to have at least one winning number than none.

8. In a substitution cipher, a one-to-one mapping is given from the alphabet to itself, and each letter is replaced by its image under this mapping. For example, a simple "rotate 13" cipher maps each letter to the letter in it's position plus 13 (modulo 26); i.e., $a \mapsto n$, $b \mapsto m$, $\ldots$, $m \mapsto z$,$n \mapsto a$, $\ldots$, $z \mapsto m$. This particular cipher encodes the message "hello" as "uryyb".

   (a) How many substitution ciphers are there?

   > **Solution:** This is the number of permutations of the alphabet:
   >
   > $$26! = 403,291,461,126,605,635,584,000,000$$

   (b) If 10^{10} checks per second are performed, how long would it take to test all substitution ciphers?

   > **Solution:** Checking them all would take around 1.3 billion years.

9. A bowl contains 10 red balls and 10 blue balls. A student selects balls at random without looking at them.

   (a) How many balls must be selected to be sure of having at least three balls of the same colour?

   > **Solution:** Let there be two containers (pigeonholes) representing the colours red and blue. We want to calculate the fewest number of objects (balls) needed to ensure that at least one of the containers contains 3 objects. By the Generalised Pigeonhole Principle, we need the smallest n such that $\texttt{ceil}(n/2) = 3$, and hence 5 balls are required.

   (b) How many balls must be selected to be sure of having at least three blue balls?

   > **Solution:** One needs to select 13 balls in order to ensure that there are at least three blue ones. Essentially, since there are a total of 10 red balls, if 12 or fewer balls are selected, then 10 can be red meaning two or less may be blue.
   >
   > Notice that the number of each colour was relevant here but not in part (a).

10. A palindrome is a string that reads the same forwards as backwards; for example,"hannah" and "minim". How many bit strings of length n are palindromes?

> **Solution:** If n is even, each of the first $n/2$ bits can be chosen in two ways, and the remaining bits are then all fixed. This gives $2^{n/2}$ possibilities.
>
> If n is odd, then each of the first $(n-1)/2+1 = (n+1)/2$ bits can be chosen in two ways, and the remaining bits are then all fixed. This gives $2^{(n+1)/2}$ possibilities.

11. The staff of a Computing Science department comprises fifteen women and ten men. How many ways are there to select a committee of six members of staff so that:

(a) At least one woman must be on the committee?

> **Solution:** There are $C(25, 6) = 177, 100$ ways of choosing the committee if there are no restrictions on membership and there are $C(10, 6) = 210$ ways of choosing the committee if no women serve on the committee. Therefore there are $177, 100-210 = 176, 890$ ways of choosing the committee if at least one woman must serve on the committee.

(b) At least one woman and at least one man must be on the committee?

> **Solution:** Using part (a), there are 177, 100 ways of choosing the committee if there are no restrictions on membership and 210 ways of choosing the committee if no women serve on the committee. Furthermore, there are $C(15, 6) = 5, 005$ ways of choosing the committee if no men serve on the committee. Since the cases where no women and no men do not overlap, there are $177, 100 - (210 + 5, 005) = 17, 1885$ ways of choosing the committee if at least one man and at least one woman must serve on the committee.

(c) The numbers of men and women are equal?

> **Solution:** There are $C(10, 3) = 120$ ways to choose the 3 men, and there are $C(15, 3) = 455$ ways to choose the 3 women. Hence by the product rule, the number of ways of choosing the committee is $120 \cdot 455 = 54, 600$ if the numbers of men and women are to be equal.

(d) There are more women than men?

> **Solution:** There are the following three cases for have more women than men: (i) 6 women and 0 men; (ii) 5 women and 1 man and (iii) 4 women and 2 men. Using the product rule, the number of ways of forming a committee according to cases (i),(ii) and (iii) are $C(15, 6) \cdot C(10, 0) = 5, 005$, $C(15, 5) \cdot C(10, 1) = 30, 030$ and $C(15, 4) \cdot C(10, 2) = 61, 425$ respectively. Since the cases are disjoint, using the sum rule, the total number of ways of choosing the committee such that the number of women is more than the number of men is $5, 005 + 30, 030 + 61, 425 = 96460$.

12. A department teaches three courses, each with two lectures per week. There are six timetable slots set aside in the week for these lectures.

(a) How many different ways can the lectures be distributed among the slots?

> **Solution:** Call the lecture courses A, B and C. Filling the slots involves forming a permutation of the string $AABBCC$, of which there are $6!/(2! \cdot 2! \cdot 2!) = 720/8 = 90$ ways.

(b) At least one of the courses has consecutive slots?

> **Solution:** If the slots for A are consecutive, then we can treat AA as a single symbol (say X) and therefore the number of ways the lectures can be distributed with A consecutive is the number of permutations of $XBBCC$ and there are $5!/(2! \cdot 2!) = 120/4 = 30$ of these. Similarly, there are 30 ways of having B or C consecutive.
>
> However, we need to also consider the cases when two or more courses are consecutive. If two courses are consecutive, then there are $4!/2! = 12$ ways (e.g. permutations

of $XYCC$). If all courses are consecutive, there are $3! = 6$ ways (permutations of XYZ).
Letting S_A, S_B and S_B be the ways of having the lectures A, B and C consecutive respectively, by the Inclusion-Exclusion Principle:

$$\begin{aligned}|S_A \cup S_B \cup S_C| &= |S_A| + |S_B| + |S_C| - |S_A \cap S_B| - |S_A \cap S_B| - |S_B \cap S_C| + |S_A \cap S_B \cap S_C| \\ &= 30 + 30 + 30 - 12 - 12 - 12 + 6 \\ &= 60\end{aligned}$$

Therefore there are 60 allocations in which at least one of the courses occupies consecutive slots.

(c) What is the answer if no course is to have both lectures in consecutive slots?

Solution: Using the answers to parts (a) and (b), the number of solutions in which none of the lectures are in consecutive slots is $90 - 60 = 30$.

13. Consider the letters of the word '*success*'.

(a) How many permutations are there in which the two 'c' characters are not consecutive?

Solution: There are seven letters, which suggests $7! = 5,040$ permutations. However, these 5,040 permutations include identical rearrangments of letters 's' and 'c'. There are $3! = 6$ ways of rearranging the three 's' characters, and $2! = 2$ ways of rearranging 'c' characters. So, altogether there are $5040/(6!{\cdot}2!) = 420$ distinct permutations.
We can find the number of permutations in which the two 'c' characters are consecutive by considering the two 'c' characters together as a single character. That gives $6!/3! = 120$ distinct permutations in which two 'c' characters are consecutive, and therefore $420 - 120 = 300$ permutations in which the two 'c' characters are not consecutive.

(b) How many of the permutations in which the two 'c' characters are not consecutive and also do not have not all three of the 's' characters consecutive?

Solution: Let A be the set of permutations in which the two 'c' characters are consecutive, and B the set in which the three 's' characters are consecutive. Imagine the 'c' or 's' characters are "glued together" (so that choosing the first 'c' is actually choosing 'cc', and choosing the first 's' is actually choosing 'sss'). We thus have $|A| = 6!/3! = 120$ distinct permutations involving 'cc', $|B| = 5!/2! = 60$ distinct permutations involving 'sss', and $|A \cap B| = 4! = 24$.
By the Inclusion-Exclusion Principle, $|A \cup B| = |A| + |B| - |A \cap B| = 120 + 60 - 24 = 156$. However, we want the complement of this set, and hence the answer is $420 - 156 = 264$ (using the fact that from (a) there are 420 permutations of '*success*').

14. (a) How many sets are there containing five letters of the alphabet?

Solution: The number of sets is $C(26, 5) = 65,780$.

(b) How many bags (sets in which duplicates are permitted) are there containing five letters of the alphabet?

Solution: Calculating the number of bags requires counting combinations with repetition. You can think of combinations with repetition as a loop which at each iteration either selects a letter or moves on to consider the next letter. Zero or more selections can occur between each move. If we write 's' for "select" and '.' for "move", each execution of the loop is a sequence of the form "$s.s.ss..s$....................." (there are 25 '.' because the loop starts on the letter 'a', and only has to advance 25 times to get to 'z'). This is the same as saying there are 5+25 items of which you get to choose five; i.e. $C(30, 5) = 142, 506$.

Difficult/challenging questions.

15. There are 51 houses on a particular block of a street. Each house in this block has an address between 100 and 199. Show that at least two houses in this block have addresses that are consecutive integers.

Solution: Let $a_1, a_2, \ldots, a_{51}$ denote the house numbers in increasing order, so that $100 \leq a_i < a_j \leq 199$ for $1 \leq i < j \leq 51$. Now let $b_1, b_2, \ldots, b_{51}$ be an increasing sequence of integers such that $b_i = a_i + 1$ for $1 \leq i \leq 51$. Then the sequence

$$a_1, a_2, \ldots, a_{51}, b_1, b_2, \ldots, b_{51}$$

contains 102 values in the range $[100, 200]$. However, the range $[100, 200]$ contains only 101 integers. Hence by the pigeonhole principle, at least two of the values in the sequence are equal. Since by construction $a_i \neq a_j$ for $1 \leq i < j \leq 51$ and $b_i \neq b_j$ for $1 \leq i < j \leq 51$, it follows that $a_i = b_j$ for some i and j, and hence by definition of b_j we have $a_i = a_j + 1$. Thus there are two houses with consecutive numbers.

16. Prove that, at a party where there are at least two people, there are two people who each know the same number of other people there. (Assume that if person x knows person y, then y knows x.)

Solution: Let $K(x)$ denote the number of other people that person x knows. Therefore $0 \leq K(x) \leq n-1$. Now it is impossible for both 0 and $n-1$ to be in the domain of K. For, if somebody knows everybody, then it cannot be the case that somebody knows nobody (recall that if x knows y then y knows x). Hence the range of K has at most $n - 1$ elements. Since the domain of K has n elements, then the pigeonhole principle tells us that at least two elements in the domain map to the same integer in the codomain. That is, there are two people at the party who each know the same number of other people there.

17. Suppose there are 12 students in a tutorial group. In how many different ways can the 12 students be split into six pairs?

Solution: There are $12! = 479,001,600$ permutations of 12 students. Once permuted, the pairs can be read off, student1 with student2, student3 with student4, etc. For a given permutation, we need to know the number of ways it can be rearranged so the pairing outcome is the same. First, each pair can be reversed, giving $2^6 = 64$ possibilities. Next, each of the six pairings can be permuted in $6! = 720$ different ways. Each of these two rearrangements is independent, giving $64 \cdot 6!$.

The overall answer is therefore $12!/(64 \cdot 6!) = 10,395$.