2A: Tutorial 10

School of Mathematics and Statistics

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Semester 1 2020–21

Instructions

Access your tutorial lecture

TU11,TU12,TU13,TU14,TU15: November 30th, 09:00-10:00

TU16,TU17,TU18,TU19,TU20: November 30th, 15:00-16:00

using the Zoom plug-in in the Moodle page of the course.

Please mute yourself when you are in the main room of the meeting.

Be ready to work in groups. Having a method of writing on screen, as mouse, tablet or similar is useful. Screen sharing is allowed!

From the Moodle forum: surface integrals

Let $S \subset \mathbb{R}^3$ be a surface and let $f : S \to \mathbb{R}$ be a function. The following number

$$\iint_S f(x, y, z)\, dS$$

is called a surface integral.

If f is the density of snow at each point of the surface of the mountain, then the surface integral measures the total amount of snow covering the mountain.

A special case: for $f = 1$ we get

$$Area(S) = \iint_S dS\,.$$

How to compute surface integrals?

- Choose a parametrization of the surface S: this is a vector-valued function

$$\mathbf{r} : D \subset \mathbb{R}^2 \to \mathbb{R}^3 \,,$$

where $\mathbf{r}(u, v) = (r_1(u, v), r_2(u, v), r_3(u, v))$ describes all the points of S as (u, v) varies in D.

- Compute $\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right|$: the length of the cross product of the vectors $\frac{\partial \mathbf{r}}{\partial u} = (\frac{\partial r_1}{\partial u}, \frac{\partial r_2}{\partial u}, \frac{\partial r_3}{\partial u})$ and $\frac{\partial \mathbf{r}}{\partial v} = (\frac{\partial r_1}{\partial v}, \frac{\partial r_2}{\partial v}, \frac{\partial r_3}{\partial v})$.

Then

$$\iint_S f(x, y, z)\, dS = \iint_D f(r_1, r_2, r_3) \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| \, dudv \,.$$

Useful parametrizations

- S is part of a sphere of radius R centred in the origin: then a parametrization is

$$\mathbf{r}(u, v) = (R \sin u \cos v, R \sin u \sin v, R \cos u)$$

and

$$\left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| = R^2 \sin u \,.$$

- S is part of the graph of a function $z = z(x, y)$: then a parametrization is

$$\mathbf{r}(x, y) = (x, y, z(x, y))$$

and

$$\left| \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \right| = \sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2} \,.$$

Multiple Choice Question 1

Unseen question

Given the parameterisation of the hemisphere H (given by $x^2 + y^2 + z^2 = 1$ in $z \geq 0$),

$$\mathbf{r}(u, v) = (\sin u \cos v, \sin u \sin v, \cos u)$$

which expression can be used to evaluate

$$\iint_H f(z)\, dS.$$

(A) $2\pi \displaystyle\int_0^{\pi/2} f(\sin u) \cos u \, du$

(B) $\dfrac{\pi}{2} \displaystyle\int_0^{2\pi} f(\sin u) \cos u \, du$

(C) $2\pi \displaystyle\int_0^{\pi/2} f(\cos u) \, du$

(D) $\dfrac{\pi}{2} \displaystyle\int_0^{\pi/2} f(\cos u) \, du$

(E) $2\pi \displaystyle\int_0^{\pi/2} f(\cos u) \sin u \, du$

(F) $\dfrac{\pi}{2} \displaystyle\int_0^{2\pi} f(\cos u) \sin u \, du$

Multiple Choice Question 1

Unseen question

(A) $2\pi \int_0^{\pi/2} f(\sin u) \cos u \, du$

(B) $\frac{\pi}{2} \int_0^{2\pi} f(\sin u) \cos u \, du$

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(E) $2\pi \int_0^{\pi/2} f(\cos u) \sin u \, du$

(F) $\frac{\pi}{2} \int_0^{2\pi} f(\cos u) \sin u \, du$

ANSWER: **(E)** The hemisphere is the image of the region $(u, v) \in [0, \pi/2] \times [0, 2\pi]$. Compute (or use the previous slides)

$$\frac{\partial \mathbf{r}}{\partial u} = (\cos u \cos v, \cos u \sin v, -\sin u)\,, \quad \frac{\partial \mathbf{r}}{\partial v} = (-\sin u \sin v, \sin u \cos v, 0)$$

so $|\mathbf{r}_u \times \mathbf{r}_v| = \sin u$. Note that $f(z(u, v)) = f(\cos u)$. Hence we obtain

$$\iint_H f(z)\, dS = \int_0^{2\pi} dv \int_0^{\pi/2} f(\cos u) \sin u \, du = 2\pi \int_0^{\pi/2} f(\cos u) \sin u \, du.$$

Multiple Choice Question 2

Related to 2A MD Exam 2016-17, Q4(ii)

Consider the surface S given by $z = x^3 - 3xy^2$, for $(x, y) \in D$. Which expression can be used to compute

$$\iint_S x^2 + y^2 \, dS.$$

(A) $\iint_D (x^2 + y^2)\sqrt{1 + 9(x^2 - y^2)}\, dxdy$

(B) $\iint_D x^2 + y^2 \, dxdy$

(C) $\iint_D (x^2 + y^2)\sqrt{1 + 9(x^2 + y^2)}\, dxdy$

(D) $\iint_D \sqrt{1 + 9(x^2 + y^2)}\, dxdy$

MULTIPLE CHOICE QUESTION 2

ANSWER: **(C)** A parametrization of S is $\mathbf{r}(x, y) = (x, y, z(x, y))$, where $z(x, y) = x^3 - 3xy^2$ and $(x, y) \in D$.

Compute $\frac{\partial z}{\partial x} = 3x^2 - 3y^2$ and $\frac{\partial z}{\partial y} = -6xy$ and use the results in the previous slides to get

$$\left|\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y}\right| = \sqrt{1 + \left(3x^2 - 3y^2\right)^2 + (-6xy)^2}$$
$$= \sqrt{1 + 9\left(x^2 + y^2\right)^2}$$

and

$$\iint_S x^2 + y^2\, dS = \iint_D (x^2 + y^2)\sqrt{1 + 9\left(x^2 + y^2\right)^2}\, dxdy.$$

Multiple Choice Question 3

Ex Sheet 9, T3(c) related

Which one is not a parametric description of the parabola $y^2 = x$ for $y \in [1, 2]$?

(A) $\mathbf{r}(t) = (t^{1/2}, t^{1/4})$, $t \in [1, 16]$

(B) $\mathbf{r}(t) = (t^2, t)$, $t \in [1, 4]$

(C) $\mathbf{r}(t) = (t, t^{1/2})$, $t \in [1, 4]$

(D) $\mathbf{r}(t) = (t^4, t^2)$, $t \in [1, \sqrt{2}]$

Multiple Choice Question 3

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(A) $\mathbf{r}(t) = (t^{1/2}, t^{1/4})$, $t \in [1, 16]$ **(B)** $\mathbf{r}(t) = (t^2, t)$, $t \in [1, 4]$

(C) $\mathbf{r}(t) = (t, t^{1/2})$, $t \in [1, 4]$ **(D)** $\mathbf{r}(t) = (t^4, t^2)$, $t \in [1, \sqrt{2}]$

ANSWER: **(B)** All parametrizations satisfy the equation $y^2 = x$. Hence, the only question is whether the range of t given gives the part of parabola we're interested in (starting point is $(1, 1)$, endpoint is $(4, 2)$) and that the direction is correct. In the case **(B)** we get $\mathbf{r}(4) = (16, 4)$, so the endpoint does not match with what we want.

Ex Sheet 9, T14(c)

Given that

$$\mathbf{H} = \left(\frac{2xz}{1+x^2+y^2}, \frac{2yz}{1+x^2+y^2}, \log\left(1+x^2+y^2\right) \right)$$

is conservative, find a potential.

2A MD Exam 2016-17, Q4(ii)

Consider the surface S given by $z = x^3 - 3xy^2$, inside the cylinder $x^2 + y^2 = 1$. Calculate

$$\iint_S x^2 + y^2 \, dS.$$

Ex Sheet 9, T14(c)

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is conservative, find a potential.

We must find ϕ such that $\mathbf{H} = \nabla\phi$. This can be done by solving the following partial differential equations

$$\frac{\partial \phi}{\partial x} = \frac{2xz}{1+x^2+y^2}, \qquad \frac{\partial \phi}{\partial y} = \frac{2yz}{1+x^2+y^2}, \qquad \frac{\partial \phi}{\partial z} = \log\left(1+x^2+y^2\right).$$

The last equation has solution

$$\phi = z\log\left(1+x^2+y^2\right) + A(x,y).$$

Substitute in the second equation to give an equation for $A(x, y)$,

$$\frac{\partial \phi}{\partial y} = \frac{2yz}{1+x^2+y^2} + \frac{\partial A}{\partial y} = \frac{2yz}{1+x^2+y^2}.$$

So $A_y = 0$ and therefore $A(x, y) = B(x)$ (a function of x alone).

Substitute in the first equation to see that $B'(x) = 0$ so $B(x) = C$ a constant.

Therefore a potential for **H** is

$$\phi = z \log\left(1 + x^2 + y^2\right) + C.$$

(C is arbitrary so you can set $C = 0$ with no harm. But if you are sitting the exam, then letting C to be your lucky number is a wise choice.)

2A MD Exam 2016-17, Q4(ii)

Consider the surface S given by $z = x^3 - 3xy^2$, inside the cylinder $x^2 + y^2 = 1$. Calculate

$$\iint_S x^2 + y^2 \, dS.$$

The projection of S on the xy-plane is the unit disc $x^2 + y^2 \leq 1$, call this D. Then by MCQ2

$$\iint_S x^2 + y^2 \, dS = \iint_D \left(x^2 + y^2\right) \sqrt{1 + 9\left(x^2 + y^2\right)^2} \, dxdy.$$

Both the integrand and domain of integration suggest switching to polar coordinates. We obtain

$$\int_0^{2\pi} \left(\int_0^1 r^3 \sqrt{1 + 9r^4} \, dr \right) d\theta = 2\pi \left[\frac{1}{54} \left(1 + 9r^4\right)^{3/2} \right]_0^1 = \frac{\pi}{27} \left(10^{3/2} - 1 \right).$$

Bonus picture

$$A = \frac{1}{2}\left|\begin{matrix} x_0 & x_1 & x_2 & \cdots & x_{n-1} & x_0 \\ y_0 & y_1 & y_2 & \cdots & y_{n-1} & y_0 \end{matrix}\right|$$

Bonus question

Unseen question: the shoelace formula

Use Green's Theorem to show that the area of a region D is

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{x}$$

where $\mathbf{F}(x, y) = (-y, x)/2$ and ∂D is the anticlockwise boundary of D. Use this to show that the area of a polygon with vertices (x_i, y_i) for $i = 0, \ldots (n-1)$, is

$$\text{Area}(D) = \frac{1}{2} \sum_{i=0}^{n-1} \begin{vmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{vmatrix}$$

where $(x_n, y_n) = (x_0, y_0)$. *HINT: parametrise each straight side of the polygon.*

Bonus question

With $\mathbf{F} = (-y, x)/2$ we have, using Green's theorem

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{x} = \frac{1}{2} \int_{\partial D} x\, dy - y\, dx = \frac{1}{2} \int_D \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\ dxdy = \int_D 1\, dxdy$$

which is the area of D. Now parametrise the line from (x_i, y_i) to (x_{i+1}, y_{i+1}) using

$$\mathbf{r}(t) = (x_i + t(x_{i+1} - x_i), y_i + t(y_{i+1} - y_i))\ , \qquad t \in [0, 1]\,.$$

The integral on this edge gives

$$\frac{1}{2} \int_0^1 -(y_i + t(y_{i+1} - y_i))(x_{i+1} - x_i) + (x_i + t(x_{i+1} - x_i))(y_{i+1} - y_i)\ dt$$
$$= \frac{1}{4} \left[(y_{i+1} - y_i)(x_{i+1} + x_i) - (x_{i+1} - x_i)(y_i + y_{i+1})\right]$$

$= \frac{1}{2}(y_{i+1} x_i - x_{i+1} y_i) = \frac{1}{2} \begin{vmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{vmatrix}$. Now sum over i to obtain the result.