Use a direct proof to show that the sum of two even integers is even.

Proof. If a and b are even numbers, then $a = 2{\cdot}k$ and $b = 2{\cdot}l$ for some integers k and l. Therefore we have:

$$a + b = 2{\cdot}k + b = 2{\cdot}l = 2{\cdot}(k + 1)$$

and hence $a + b$ is even as requried.

Use an indirect proof to show that if $x + y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$.

Proof. The proof is of the form $\forall x \in \mathbb{R}. \forall y \in \mathbb{R}. (P(x, y) \rightarrow Q(x, y))$ where $P(x, y) = (x + y \geq 2)$ and $Q(x, y) = (x \geq 1) \vee (y \geq 1)$. Since we are using an indirect proof we will consider arbitrary $x, y \in \mathbb{R}$ and show if $\neg Q(x, y)$ holds, then $\neg P(x, y)$ holds. Now

$$\begin{aligned}
\neg Q(x, y) = \;& \neg((x \geq 1) \vee (y \geq 1)) && \\
\equiv \;& \neg(x \geq 1) \wedge \neg(y \geq 1) && \text{using de Morgan Law} \\
\equiv \;& (x < 1) \wedge (y < 1) && \text{rearranging}
\end{aligned}$$

Therefore since $\neg Q(x, y)$ holds it follows that:

$$x + y < 1 + 1 = 2$$

and hence $\neg P(x, y)$ holds as required.

Show that if n is an integer and $n^3 + 5$ is odd, then n is even using an indirect proof.

Proof. This is an indirect proof, there we assume n is odd and try and show $n^3 + 5$ is even. Since n is odd we have $n = 2{\cdot}k + 1$ for some integer k and therefore:

$$\begin{aligned}
n^3 + 5 = \;& (2{\cdot}k + 1)^3 + 5 && \\
= \;& (2{\cdot}k + 1)^2(2{\cdot}k + 1) + 5 && \text{rearranging} \\
= \;& (4{\cdot}k^2 + 4{\cdot}k + 1)(2{\cdot}k + 1) + 5 && \text{rearranging} \\
= \;& 8{\cdot}k^3 + 8{\cdot}k^2 + 2{\cdot}k + 4{\cdot}k^2 + 4{\cdot}4 + 1 + 5 && \text{rearranging} \\
= \;& 8{\cdot}k^3 + 12{\cdot}k^2 + 6{\cdot}4 + 6 && \text{rearranging} \\
= \;& 2{\cdot}(4{\cdot}k^3 + 6{\cdot}k^2 + 3{\cdot}4 + 3) && \text{rearranging}
\end{aligned}$$

Therefore $n^3 + 5$ is odd as required.

Prove that if n is an integer, these four statements are equivalent:

(a) n is even;

(b) $n + 1$ is odd;

(c) $3n + 1$ is odd;

(d) $3n$ is even.

Proof. Here we show (a) $\rightarrow$ (b), (b) $\rightarrow$ (c), (c) $\rightarrow$ (d) and (d) $\rightarrow$ (a).

(a) $\rightarrow$ **(b)** If n is even, then $n = 2{\cdot}k$ for some integer k and hence $n + 1 = 2{\cdot}k + 1$ and is odd as required.

(b) $\rightarrow$ **(c)** If $n+1$ is odd, then $n+1 = 2{\cdot}k+1$ for some integer k. Therefore it follows $n = 2{\cdot}k$ by subtracting one 1 from both sides. Using this fact we have $3{\cdot}n+1 = 3{\cdot}(2{\cdot}k)+1$ which rearranging equals $2{\cdot}(3{\cdot}k)+1$, and hence $3{\cdot}n + 1$ is even as required.

(c) $\rightarrow$ **(d)** If $3{\cdot}n+1$ is odd, then $n3{\cdot}+1 = 2{\cdot}k+1$ for some integer k. Therefore it follows $3{\cdot}n = 2{\cdot}k$ by subtracting one 1 from both sides as required.

(d) $\rightarrow$ **(a)** Here we will use an indirect proof, so we assume n is odd, and hence $n = 2{\cdot}k+1$ for some integer k. It follows that:

$$\begin{aligned} 3{\cdot}n &= 3{\cdot}(2{\cdot}k+1) & \\ &= 6{\cdot}k+3 & \text{rearranging} \\ &= 6{\cdot}k+2+1 & \text{rearranging} \\ &= 2{\cdot}(3{\cdot}k+1)+1 & \text{rearranging} \end{aligned}$$

Therefore $3{\cdot}n$ is odd as required.

This completes the proof.

Show that $\sqrt{n}$ is irrational if n is a positive integer that is not a perfect square (an integer n is a perfect square if $n = k^2$ for some integer k).

Proof. Suppose for a contradiction that $\sqrt{n}$ is rational. Then $\sqrt{n} = a/b$ for some positive integers a and b, so that $a = b\sqrt{n}$, which implies that $a^2 = n{\cdot}b^2$.

Now by the Fundamental Theorem of Arithmetic any number can be expressed as the product of prime factors. It therefore follows that:

- when expressing the square of any number as the product of primes each power is even and, in particular when expressing a^2 and b^2 as the product of primes each power is even;
- since n is not a perfect square, expressing n as a product of powers of primes at least one of these prime factors must be raised to an odd power.

Thus when expressing $n{\cdot}b^2$ as the product of primes at least one prime is raised to an odd power, which contradicts the fact that $a^2 = n{\cdot}b^2$. Hence n cannot be rational and must be irrational.

Use a proof by cases to show that $\min(a, \min(b, c)) = \min(\min(a, b), c)$ whenever a, b, and c are real numbers.

Proof. There are three cases, depending on which of the three numbers is least.

- If $a \leq b, c$, then clearly $a \leq \min(b, c)$, and hence the left-hand side equals a. On the other hand, for the right-hand side we have $\min(a, b) = \min(a, c) = a$, and therefore the right hand side also equals a.
- if $b \leq a, c$, then similar reasoning shows us that both sides equal b.
- if $c \leq b, c$, then again similar reasoning shows us that both sides equal c.

The sum of the first n odd integers equals n^2.

Theorem. $\forall n \in \mathbb{Z}^+ .\, P(n)$ where $P(n) : \sum\limits_{i=1}^{n}(2{\cdot}i-1) = n^2$.

Base case: If $n = 1$, then $\sum_{i=1}^{1}(2{\cdot}i{-}1) = 1 = 1^2$ as required.

Inductive step: Suppose that $P(n)$ holds for some $n \geq 1$. Considering $n{+}1$ we have:

$$\begin{aligned} \sum_{i=1}^{n+1}(2{\cdot}i{-}1) &= \left(\sum_{i=1}^{n}(2{\cdot}i{-}1)\right) + 2{\cdot}(n+1){-}1 & \\ &= n^2 + 2{\cdot}(n+1){-}1 & \text{by induction} \\ &= n^2 + 2{\cdot}n + 1 & \text{rearranging} \\ &= (n+1)^2 & \text{rearranging} \end{aligned}$$

and hence $P(n{+}1)$ holds.

Therefore by the principle of induction we have proved that $P(n)$ holds for all $n \geq 1$.

Theorem. $\forall n \in \mathbb{Z}^+ .\, P(n)$ where $P(n) : \sum_{i=1}^{n} i = \frac{n \cdot (n+1)}{2}$

Base case: If $n = 1$, then $\sum_{i=1}^{1} i = 1 = (1)(2)/2$ as required.

Inductive step: Suppose that $P(n)$ holds for some $n \geq 1$. Considering $n{+}1$ we have:

$$\begin{aligned}
\sum_{i=1}^{n+1} i &= \left(\sum_{i=1}^{n} i\right) + (n+1) \\
&= \frac{n \cdot (n+1)}{2} + (n+1) && \text{by induction} \\
&= \frac{(n+1)}{2}(n+2) && \text{rearranging} \\
&= \frac{(n+1)((n+1)+1)}{2} && \text{rearranging}
\end{aligned}$$

and hence $P(n{+}1)$ holds.

Therefore by the principle of induction we have proved that $P(n)$ holds for all $n \geq 1$.

Theorem. For any $a, r \in \mathbb{Z}$ such that $r \neq 1$ and $n \in \mathbb{N}$: $\sum_{i=0}^{n} a \cdot r^i = \frac{a \cdot (r^{n+1} - 1)}{(r-1)}$.

Let $P(n)$ be the proposition $\sum_{i=0}^{n} a \cdot r^i = \frac{a \cdot (r^{n+1} - 1)}{(r-1)}$ and consider any

Base case: If $n = 0$, then for $a, r \in \mathbb{Z}$ such that $r \neq 1$

$$\sum_{i=1}^{1} a \cdot r^i = a \cdot r^0 = a \cdot 1 = \frac{a \cdot (r^1 - 1)}{(r-1)}$$

as required.

Inductive step: Suppose that $P(n)$ holds for some $n \geq 1$. Considering $n{+}1$ we have:

$$\begin{aligned}
\sum_{i=1}^{n+1} a \cdot r^i &= \left(\sum_{i=1}^{n} a \cdot r^i\right) + a \cdot r^{n+1} \\
&= \frac{a \cdot (r^{n+1} - 1)}{(r-1)} + a \cdot r^{n+1} && \text{by induction} \\
&= \frac{a}{(r-1)}(r^{n+1} - 1 + (r-1) \cdot r^{n+1}) && \text{rearranging} \\
&= \frac{a}{(r-1)}(r^{n+1} - 1 + r^{n+1} - r^{n+1}) && \text{rearranging} \\
&= \frac{a}{(r-1)}(r^{n+2} - 1) && \text{rearranging} \\
&= \frac{a}{(r-1)}(r^{(n+1)+1} - 1) && \text{rearranging}
\end{aligned}$$

and hence $P(n{+}1)$ holds.

Therefore by the principle of induction we have proved that $P(n)$ holds for all $n \geq 1$.