FB1: Find the volume of the solid generated by rotating the region bounded by the curves $y = e^x$, $y = cos\left(\frac{\pi x^2}{2}\right)$ and $x = 1$ about the y-axis.

When all three curves are drawn, the graph can be seen in Figure A:

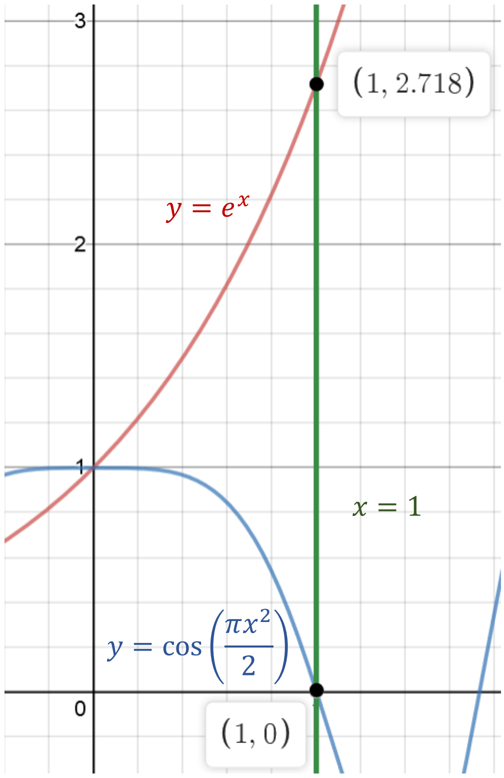

Figure A

As can be seen from the graph, the region is bounded within the points $(1,0)$, $(1, e)$, and $(0,1)$. To find the volume of the solid, let us divide the *y* interval of the region into [0,1] and [1,e]. Then the volume of the solid becomes a sum of two different volumes of the two disks, for which the formula for calculating the areas S is

$$S = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2.$$

To find the volumes of both disks, the curves must be rearranged in terms of *y*, which gives us the following curves:

$$x = 1,$$

$$x = \ln(y),$$

$$x = \sqrt{\frac{2\, arccos\, y}{\pi}},$$

where x is positive because the region is in the positive side.

Thus, the volume of the lower disk is

$$\begin{aligned} V_1 &= \int_0^1 \pi\left(1 - \frac{2\, arccos\, y}{\pi}\right) \mathrm{dy} \\ &= \pi\left(\int_0^1 \mathrm{dy} - 2\int_0^1 arccos\, y\, \mathrm{dy}\right) \\ &= \pi\left(1 - 2\left(\left[1 \cdot arccos(1) - \sqrt{1-1^2}\right] - \left[0 \cdot arccos(0) - \sqrt{1-0^2}\right]\right)\right) \\ &= \pi - 2, \end{aligned}$$

and the volume of the upper disk is

$$\begin{aligned} V_2 &= \int_1^e \pi(1 - \ln^2(y))\, \mathrm{dy} \\ &= \pi\left(\int_1^e \mathrm{dy} - \int_1^e \ln^2(y)\, \mathrm{dy}\right) \\ &= \pi(e - 1 - [e\ln^2(e) - 2e\ln(e) + 2e] - [\ln^2(1) - 2\ln(1) + 2]) \\ &= \pi. \end{aligned}$$

Hence, the volume of the whole solid of revolution is the sum of the volumes of both disks:

$$V = V_1 + V_2 = 2\pi - 2.$$

FB2: Decide whether the set A of positive integers divisible by 17 and B the set of positive integers divisible by 11 are in bijection.

The set A can be written in the notation $\{a|a = 17s, s \in \mathbb{N}\}$, and the set B can be written as $\{a|a = 11t, t \in \mathbb{N}\}$. As can be seen, a function $f: A \to B$ can be defined to be

$$\mathrm{f(a)} = \frac{11\mathrm{a}}{17}.$$

Since *f* has an inverse function

$$\mathrm{f}^{-1}(\mathrm{a}) = \frac{17\mathrm{a}}{11}$$

because it is injective (given by the fact it is a linear function) and surjective (given by the fact that the inverse function is defined for all a ∈ A), the function *f* is a bijection. Thus, the two sets A and B are in bijection.

FB3: Suppose that α, β are disjoint cycles in the symmetric group S_n for $n \geq 9$.

a) Let α by a cycle of length 3 and β a cycle of length 9. What is the order of $\alpha\beta$. Is the permutation $\alpha\beta$ even or odd?

b) Show that for every positive integer *n* we have

$$(\alpha\beta)^n = \alpha^n\beta^n.$$

a)

The order of $\alpha\beta$ is 9 because it is the least common multiple of 3 and 9. Using the formula

$$sgn(g) = (-1)^{r_1-1}(-1)^{r_2-1} \dots (-1)^{r_k-1}$$

to find the parity of the permutation, $r_1 = 9$ and $r_2 = 3$, which gives

$$sgn(g) = (-1)^8(-1)^2 = 1$$

and proves that the permutation is even.

b)

By the definition of exponentiating permutations,

$$\begin{aligned}(\alpha\beta)^n &= \underbrace{(\alpha\beta) \cdot (\alpha\beta) \cdot \dots \cdot (\alpha\beta)}_{n} \\ &= \underbrace{\alpha \cdot \alpha \cdot \dots \cdot \alpha}_{n} \cdot \underbrace{\beta \cdot \beta \cdot \dots \beta}_{n} \\ &= \alpha^n\beta^n.\end{aligned}$$

This rearrangement can be done because α and β are disjoint, which means that they commute.