2A: Tutorial 2

School of Mathematics and Statistics

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Semester 1 2020–21

Instructions

Access your tutorial lecture

TU11,TU12,TU13,TU14,TU15: October 5th, 09:00-10:00

TU16,TU17,TU18,TU19,TU20: October 5th, 15:00-16:00

using the Zoom plug-in in the Moodle page of the course.

Please mute yourself when you are in the main room of the meeting.

Be ready to work in groups using Microsoft Whiteboard as instructed by your tutor. Having a method of writing on screen, as mouse, tablet or similar is useful.

Multiple Choice Question 1

Unseen question, implicit differentiation.

Find $\partial p/\partial r$ when $p = p(q, r)$ is implicitly defined by the equation

$$\ln(p + r) = pq.$$

(A) $\dfrac{1}{pq + qr}$

(B) 0

(C) $\dfrac{1}{pq + qr - 1}$

(D) $\dfrac{p + r}{pq + qr - 1}$

Multiple Choice Question 1

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Find $\partial p/\partial r$ when $p = p(q, r)$ is implicitly defined by the equation

$$\ln(p + r) = pq.$$

(A) $\dfrac{1}{pq + qr}$ **(B)** 0

(C) $\dfrac{1}{pq + qr - 1}$ **(D)** $\dfrac{p + r}{pq + qr - 1}$

ANSWER: **(C)** The expression is equivalent to $p + r = e^{pq}$. Differentiate it with respect to r to get

$$\frac{\partial p}{\partial r} + 1 = q\frac{\partial p}{\partial r}e^{pq} \Rightarrow \left(qe^{pq} - 1\right)\frac{\partial p}{\partial r} = 1\,.$$

Then divide both sides by $qe^{pq} - 1 = pq + qr - 1$.

Multiple Choice Question 2

Ex Sheet 1, T5(a)

The set of points $(x, y, z) \in \mathbb{R}^3$ that satisfy $x^2 + y^2 = z^2$ is best described as

(A) A sphere

(B) A cone

(C) A double cone

(D) An ellipsoid

Multiple Choice Question 2

Ex Sheet 1, T5(a)

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ANSWER: **(C)** The $x = 0$ cross-section gives $z = \pm y$ and the $y = 0$ cross-section gives $z = \pm x$. Both equations describe *perpendicular lines*. Level curves are circles of radius $|z|$. This is a double cone.

Tutorial questions

Ex Sheet 1, T4

Complete the square in each of the following expressions

$$x^2 + y^2 + z^2 + 2 = 2(x + y + z), \quad z = \sqrt{2x + 2y - x^2 - y^2 - 1},$$

and describe the surfaces they represent.

Ex Sheet 2, T4(a)

Let $z = z(x, y)$. Use the chain rule to find z_x and z_y when

$$z = \tan^{-1} r, \qquad \text{where } r^2 = x^2 + y^2.$$

Ex Sheet 2, T9

Let $\phi(x, y) = f(r)$ where $r^2 = x^2 + y^2$. Show that

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$

Tutorial questions

Ex Sheet 1, T4

Complete the square in each of the following expressions

$$x^2 + y^2 + z^2 + 2 = 2(x + y + z)\,, \quad z = \sqrt{2x + 2y - x^2 - y^2 - 1}\,,$$

and describe the surfaces they represent.

We have $x^2 - 2x + y^2 - 2y + z^2 - 2z + 2 = 0$ and so completing the square gives

$$(x - 1)^2 + (y - 1)^2 + (z - 1)^2 = 1\,,$$

the sphere with centre $(1, 1, 1)$ and radius 1.

In the second expression, $z \geq 0$ and

$$(x - 1)^2 + (y - 1)^2 + z^2 = 1\,.$$

So we have the upper hemisphere with centre $(1, 1, 0)$ and radius 1.

EX SHEET 2, T4(A)

Let $z = z(x, y)$. Use the chain rule to find z_x and z_y when

$$z = \tan^{-1} r, \qquad \text{where } r^2 = x^2 + y^2.$$

Using implicit differentiation (as in Ex 1.6 in the lecture notes) we have $r_x = x/r$ and $r_y = y/r$. The chain rule for $z(x, y) = f(r(x, y))$ is

$$z_x = r_x f'(r), \qquad z_y = r_y f'(r).$$

In this case $f(r) = \tan^{-1} r$ so $f'(r) = 1/(1 + r^2)$ and

$$z_x = \frac{x}{r\,(1 + r^2)}, \qquad z_y = \frac{y}{r\,(1 + r^2)}.$$

Tutorial questions

Ex Sheet 2, T9

Let $\phi(x, y) = f(r)$ where $r^2 = x^2 + y^2$. Show that

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$

Again, implicit differentiation gives $r_x = x/r$ and $r_y = y/r$. The first derivatives are

$$\frac{\partial \phi}{\partial x} = r_x f'(r) = \frac{x}{r} f'(r), \qquad \frac{\partial \phi}{\partial x} = r_y f'(r) = \frac{y}{r} f'(r).$$

Tutorial questions

Using the product and chain rules again for the second derivatives

$$\phi_{xx} = \frac{1}{r} f'(r) + x \frac{\partial}{\partial x}\left(\frac{f'(r)}{r}\right) = \frac{1}{r} f'(r) + x \frac{x}{r}\frac{d}{dr}\left(\frac{f'(r)}{r}\right)$$
$$= \frac{1}{r} f'(r) + \frac{x^2}{r}\left(\frac{f''(r)}{r} - \frac{f'(r)}{r^2}\right)$$

and similarly for ϕ_{yy}. Adding gives

$$\phi_{xx} + \phi_{yy} = \frac{2}{r} f'(r) + \left(x^2 + y^2\right)\left(\frac{f''(r)}{r^2} - \frac{f'(r)}{r^3}\right)$$
$$= f''(r) + \frac{f'(r)}{r}.$$