2A: Tutorial 8

School of Mathematics and Statistics

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Semester 1 2020–21

INSTRUCTIONS

Access your tutorial lecture

TU11,TU12,TU13,TU14,TU15: November 16th, 09:00-10:00

TU16,TU17,TU18,TU19,TU20: November 16th, 15:00-16:00

using the Zoom plug-in in the Moodle page of the course.

Please mute yourself when you are in the main room of the meeting.

Be ready to work in groups using Microsoft Whiteboard. Having a method of writing on screen, as mouse, tablet or similar is useful. Screen sharing is allowed!

From the Moodle forum 1

Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ be a function and let $\mathbf{d} \in \mathbb{R}^n$ be a vector. How to compute the directional derivative in the direction of $\mathbf{d}$?

- Compute $\nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$;
- Normalise the vector $\mathbf{d}$: compute $\mathbf{u} = \frac{\mathbf{d}}{|\mathbf{d}|}$.

(recall that if $\mathbf{d} = (d_1, \ldots, d_n)$, then $|\mathbf{d}| = \sqrt{d_1^2 + \ldots d_n^2}$.)

The directional derivative is the dot product of the two vectors you just computed:

$$\frac{\partial f}{\partial \mathbf{u}} = \nabla f \cdot \mathbf{u}\,.$$

From the Moodle forum 2

Let $P \in D$. The rate of change of f at P in the direction of $\mathbf{u}$ is

$$\frac{\partial f}{\partial \mathbf{u}}(P)\,.$$

What is the maximal rate of change at P?

It is $|\nabla f(P)|$.

In which direction does it occur?

In the direction of the vector $\nabla f(P)$ (if you are asked for a unit vector direction, then you need to normalise $\nabla f(P)$ as in the previous slide).

Multiple Choice Question 1

Ex Sheet 7, T8(c)

Find the rate of change of $f(x, y, z) = \sin xy + \log yz$ at $(x, y, z) = (\pi, 1, 2)$ in the direction $\mathbf{d} = (0, 1, 2)$.

(A) $(y \cos xy, x \cos xy + 1/y, 1/z)$ **(B)** $\frac{2-\pi}{\sqrt{5}}$

(C) $2 - \pi$ **(D)** $(-1, -\pi + 1, 1/2)$

Multiple Choice Question 1

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(C) $2 - \pi$ **(D)** $(-1, -\pi + 1, 1/2)$

The unit vector in the direction of $\mathbf{d}$ is $\mathbf{u} = (0, 1, 2)/\sqrt{5}$ and

$$\nabla f(x, y, z) = (y \cos xy, x \cos xy + 1/y, 1/z)\,.$$

Hence, $\nabla f(\pi, 1, 2) = (-1, -\pi + 1, 1/2)$. Therefore the rate of change at $(\pi, 1, 2)$ in the direction $\mathbf{d}$ (or in the unit vector direction $\mathbf{u}$) is

$$\nabla f(\pi, 1, 2) \cdot \mathbf{u} = \frac{1}{\sqrt{5}}(-1, -\pi + 1, 1/2) \cdot (0, 1, 2) = \frac{2 - \pi}{\sqrt{5}}.$$

Multiple Choice Question 2

Unseen question

Compute the gradient of

$$f(x, y, z) = xyz.$$

(A) $x + y + z$

(B) (x, y, z)

(C) (yz, xz, xy)

(D) $xy + xz + yz$

Multiple Choice Question 2

Unseen question

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$$f(x, y, z) = xyz.$$

(A) $x + y + z$

(B) (x, y, z)

(C) (yz, xz, xy)

(D) $xy + xz + yz$

ANSWER: **(C)** The gradient is

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

and so in this case

$$\nabla f = (yz, xz, xy)\,.$$

Multiple Choice Question 3

Ex Sheet 8, T3 (related)

Which one of the following vector fields is incompressible?

(A) $(\sin xy, \cos yz, \sin xz)$

(B) $(y^2 z, 2xyz, xy^2)$

(C) $(x, y, z)/\sqrt{x^2 + y^2 + z^2}$

(D) (yz, xz, xy)

Multiple Choice Question 3

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(A) $(\sin xy, \cos yz, \sin xz)$ **(B)** $(y^2 z, 2xyz, xy^2)$

(C) $(x, y, z)/\sqrt{x^2 + y^2 + z^2}$ **(D)** (yz, xz, xy)

ANSWER: **(D)** A vector field is incompressible if its divergence is zero. We can check the divergence in each case:

(A)

$$\frac{\partial}{\partial x}(\sin xy) + \frac{\partial}{\partial y}(\cos yz) + \frac{\partial}{\partial z}(\sin xz) = y\cos xy - z\sin yz + x\cos xz;$$

(B)

$$\frac{\partial}{\partial x}(y^2 z) + \frac{\partial}{\partial y}(2xyz) + \frac{\partial}{\partial z}(xy^2) = 0 + 2xz + 0 = 2xz;$$

Multiple Choice Question 3

Ex Sheet 8, T3 (related)

Which one of the following vector fields is incompressible?

(A) $(\sin xy, \cos yz, \sin xz)$ **(B)** $(y^2 z, 2xyz, xy^2)$

(C) $(x, y, z)/\sqrt{x^2 + y^2 + z^2}$ **(D)** (yz, xz, xy)

ANSWER: **(D)** A vector field is incompressible if its divergence is zero. We can check the divergence in each case:

(C) using the nabla identity for $\operatorname{div}(f\mathbf{F})$ and that $\operatorname{div}\mathbf{r} = 3$ and $\nabla f(r) = f'(r)\mathbf{r}/r$ we have

$$\operatorname{div}\left(\frac{\mathbf{r}}{r}\right) = \frac{3}{r} + \mathbf{r} \cdot \left(-\frac{1}{r^3}\mathbf{r}\right) = \frac{2}{r};$$

(D)

$$\frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0 + 0 + 0 = 0\,.$$

Tutorial questions

Ex Sheet 8, T5

For the vector field $\mathbf{F} = (F_1, F_2, F_3)$ in Cartesian components, calculate

$$\nabla(\operatorname{div}\mathbf{F}) - \operatorname{curl}(\operatorname{curl}\mathbf{F}).$$

Ex Sheet 8, T9(c),(d)

Let $\mathbf{r} = (x, y, z)$, $r = |\mathbf{r}|$, and let $\mathbf{a}$ be a constant vector. Calculate

$$\operatorname{div}\left((\mathbf{a}\cdot\mathbf{r})\,\mathbf{r}\right) \quad \text{and} \quad \operatorname{curl}\left((\mathbf{a}\cdot\mathbf{r})\,\mathbf{r}\right).$$

EX SHEET 8, T5

For the vector field $\mathbf{F} = (F_1, F_2, F_3)$ in Cartesian components, calculate

$$\nabla(\operatorname{div}\mathbf{F}) - \operatorname{curl}(\operatorname{curl}\mathbf{F}).$$

In components, we have $\operatorname{div}\mathbf{F} = F_{1,x} + F_{2,y} + F_{3,z}$ and

$$\nabla(\operatorname{div}\mathbf{F}) = (F_{1,xx} + F_{2,yx} + F_{3,zx}, F_{1,xy} + F_{2,yy} + F_{3,zy}, F_{1,xz} + F_{2,yz} + F_{3,zz}) \; .$$

Moreover

$$\operatorname{curl}\mathbf{F} = (F_{3,y} - F_{2,z}, F_{1,z} - F_{3,x}, F_{2,x} - F_{1,y})$$

so

$$\begin{aligned}&\operatorname{curl}(\operatorname{curl}\mathbf{F}) = (F_{2,xy} - F_{1,yy} - F_{1,zz} + F_{3,xz})\,\mathbf{i}\\ &+ (F_{3,yz} - F_{2,zz} - F_{2,xx} + F_{1,yx})\,\mathbf{j} + (F_{1,xz} - F_{3,xx} - F_{3,yy} + F_{2,yz})\,\mathbf{k}\,.\end{aligned}$$

Then

$$\nabla(\operatorname{div}\mathbf{F}) - \operatorname{curl}(\operatorname{curl}\mathbf{F}) = (\Delta F_1, \Delta F_2, \Delta F_3)\,.$$

Ex Sheet 8, T9(c),(d)

Let $\mathbf{r} = (x, y, z)$, $r = |\mathbf{r}|$, and let $\mathbf{a}$ be a constant vector. Calculate

$$\operatorname{div}\left((\mathbf{a}\cdot\mathbf{r})\,\mathbf{r}\right) \quad \text{and} \quad \operatorname{curl}\left((\mathbf{a}\cdot\mathbf{r})\,\mathbf{r}\right).$$

For the divergence we use the nabla identity

$$\operatorname{div}(f\mathbf{F}) = \nabla f\cdot\mathbf{F} + f\operatorname{div}\mathbf{F}$$

and note that $\nabla(\mathbf{a}\cdot\mathbf{r}) = \mathbf{a}$ and $\operatorname{div}\mathbf{r} = 3$ so

$$\operatorname{div}\left((\mathbf{a}\cdot\mathbf{r})\,\mathbf{r}\right) = \mathbf{a}\cdot\mathbf{r} + 3\mathbf{a}\cdot\mathbf{r} = 4\mathbf{a}\cdot\mathbf{r}.$$

Ex Sheet 8, T9(c),(d)

Let $\mathbf{r} = (x, y, z)$, $r = |\mathbf{r}|$, and let $\mathbf{a}$ be a constant vector. Calculate

$$\operatorname{div}\left((\mathbf{a}\cdot\mathbf{r})\,\mathbf{r}\right) \quad \text{and} \quad \operatorname{curl}\left((\mathbf{a}\cdot\mathbf{r})\,\mathbf{r}\right).$$

For the curl we use the nabla identity

$$\operatorname{curl}(f\mathbf{F}) = \nabla f \times \mathbf{F} + f \operatorname{curl}\mathbf{F}$$

and note (in addition to the results above) that $\operatorname{curl}\mathbf{r} = \mathbf{0}$ so

$$\operatorname{curl}\left((\mathbf{a}\cdot\mathbf{r})\,\mathbf{r}\right) = \mathbf{a}\times\mathbf{r}.$$