Algorithmic Foundations 2 - Tutorial Sheet 8

Probability (and more Counting)

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1. In roulette there is a wheel with 38 numbers of these 18 are red and 18 are black. The other two numbers are 0 and 00 which are neither red nor black. The probability that when the wheel is spun it lands an a particular number is 1/38.

   (a) What is the probability the wheel lands on a red number?

   > **Solution:** Since 18 numbers are red out of a total of 38, the probability is $18/38 = 9/19$.

   (b) What is the probability the wheel lands on a black number twice in a row?

   > **Solution:** Using the product rule there are $38{\cdot}38 = 1,444$ equally likely outcomes for two spins. Of these, again using the product rule $18{\cdot}18 = 182$ are a pair of black numbers. Therefore the probability is $182/1,444 = 91/722$.

   (c) What is the probability the wheel lands on 0 or 00?

   > **Solution:** There are 2 outcomes out of the 38 equally possible, therefore the probability is $2/38 = 1/19$.

   (d) What is the probability in five spins the wheel neither lands on 0 nor 00?

   > **Solution:** Using the product rule there are $38{\cdot}38{\cdot}38{\cdot}38{\cdot}38 = 79,235,168$ outcomes for five spins. Since 36 outcomes on each spin are neither 0 no 00, there are $36{\cdot}36{\cdot}36{\cdot}36{\cdot}36 = 60,466,176$ outcomes that meet the requirement considered. Therefore the probability is $60,466,176/79,235,168 = 1,889,568/2,476,099$.

   (e) What is the probability the wheel lands on one of the first six integers on one spin, but does not land on any of them on the next spin?

   > **Solution:** In this case the total number of outcomes is $38{\cdot}38 = 1,444$ using the product rule. Using the product rule again there are $6{\cdot}(38-6) = 192$ outcomes that meet the specification. Therefore the probability is $192/1,444 = 48/361$.

2. For each of the following pairs of events determine there probabilities and if they are independent or not when a coin is tossed three times.

   (a) E_1: the first coin comes up `tails`.
   E_2: the second coin comes up `heads`.

   > **Solution:** The total number of outcomes are, using the product rule, $2{\cdot}2{\cdot}2 = 2^3$ and each is equally likely (we will also use this result in the other parts to the question). Using the product rule again we have:
   >
   > - there are $1{\cdot}2{\cdot}2 = 2^2$ outcomes in E_1;
   > - there are $2{\cdot}1{\cdot}2 = 2^2$ outcomes in E_2;

- there are $1{\cdot}1{\cdot}2 = 2$ outcomes in $E_1 \cap E_2$.

These events are independent since $\mathbf{P}[E_1] = \mathbf{P}[E_2] = 2^2/2^3 = 1/2$ and $\mathbf{P}[E_1 \cap E_2] = 2/2^3 = 1/2^2 = (1/2){\cdot}`(1/2)$.

(b) E_3: the first coin comes up `tails`.
E_4: precisely two `heads` in a row.

Solution: Using the product and sum rules:

- there are $1{\cdot}2{\cdot}2 = 2^2$ outcomes in E_3;
- there are $1{\cdot}1{\cdot}1 + 1{\cdot}1{\cdot}1 = 2$ outcomes in E_4;
- there are $1{\cdot}1{\cdot}1 = 1$ outcomes in $E_3 \cap E_4$.

These events are independent since $\mathbf{P}[E_3] = 2^2/2^3 = 1/2$, $\mathbf{P}[E_4] = 2/2^3 = 1/4$ and $\mathbf{P}[E_3 \cap E_4] = (1/2)^3 = 1/8 = (1/2){\cdot}(1/4)$.

(c) E_5: the second coin comes up `tails`.
E_6: precisely two `heads` in a row.

Solution: Using the product rule:

- there are $2{\cdot}1{\cdot}2 = 2^2$ outcomes in E_5;
- there are $21{\cdot}1{\cdot}1 + 1{\cdot}1{\cdot}1 = 2$ outcomes in E_6;
- there are 0 outcomes in $E_5 \cap E_6$.

These events are dependent since $\mathbf{P}[E_5] = 2^2/2^3 = 1/2$ and $\mathbf{P}[E_6] = 2/2^3 = 1/4$ $\mathbf{P}[E_3 \cap E_4] = (1/2)^3 = 0 \neq (1/2){\cdot}(1/4)$.

3. What probabilities should be assigned to the outcomes of a biased coin if the probability of heads equals four times the probability of tails.

Solution: The requirement is $\mathbf{P}[\texttt{heads}] = 4{\cdot}\mathbf{P}[\texttt{tails}]$. From the first and third axiom of probability it follows that $\mathbf{P}[\texttt{heads}] + \mathbf{P}[\texttt{tail}] = 1$. Solving these simultaneous equations we have $\mathbf{P}[\texttt{heads}] = 1/5$ and $\mathbf{P}[\texttt{tails}] = 4/5$.

4. What is the conditional probability that a randomly generated bit string of length four contain at least two consecutive `0`'s, given that the first bit is a `1`?

Solution: Let A be the event that the string contain at least two consecutive `0`'s and B the event that the first bit is a `1`. There are 2^4 outcomes in total, the number satisfying B is 2^3 and therefore $\mathbf{P}[B] = 1/2$, and the number satisfying $A \cap B$ is 3 (the bit strings `1000`, `1100`, `1001`) and therefore $\mathbf{P}[A \cap B] = 3/16$. By the definition of conditional probability:

$$\mathbf{P}[A \mid B] = \frac{\mathbf{P}[A \cap B]}{\mathbf{P}[B]} = \frac{3/16}{1/2} = 3/8\,.$$

5. A *Bernoulli trial* is an experiment which can have only two possible outcomes (denoted *success* and *failure*).

   Find each of the probabilities when n independent Bernoulli trials are carried out, each with a probability of success equal to p.

   (a) The probability of no successes.

   > **Solution:** Using the fact that the experiments are independent, the probability equals $(1{-}p)^n$.

   (b) The probability of at least one success.

   > **Solution:** Using the result from part (a), the probability of at least one success equals $1-(1{-}p)^n$.

   (c) The probability of at most one success.

   > **Solution:** At most one success means one of the n is a success and all others are failures or all are failures. In the first case, since there are n trials, there are n different positions the success can be in, and therefore, using fact the experiments are independent, the probability equals $n{\cdot}(p{\cdot}(1{-}p)^{n-1})$. In the second case this is just probability $(1{-}p)^n$. Since these two events are mutually exclusive we an add the probabilities to give the result:
   >
   > $$(1{-}p)^n + n{\cdot}(p{\cdot}(1{-}p)^{n-1})$$

   (d) The probability of precisely two successes.

   > **Solution:** The number of outcomes with precisely two successes is all the permutations of two `s`'s, and $n{-}2$ `f`'s where the `s`'s and the `f`'s are indistinguishable. Furthermore, since the experiments are independent, each of these outcomes has probability $p^2{\cdot}(1{-}p)^{n-2}$. Finally, since the outcomes are mutually exclusive, using axiom 3 the probability equals:
   >
   > $$n!/((n{-}2)!{\cdot}2!)\cdot p^2{\cdot}(1{-}p)^{n-2} = (n{\cdot}(n{-}1))/2\cdot p^2{\cdot}(1{-}p)^{n-2}\,.$$

   (e) The probability of at least two successes.

   > **Solution:** Here it is easier to compute the complement and then subtract this probability from 1. The complement is at most one success, i.e. part (c), and therefore the probability of at least two successes equals:
   >
   > $$1-\left((1{-}p)^n + n{\cdot}(p{\cdot}(1{-}p)^{n-1})\right).$$

6. Suppose there are two boxes of balls, the first box contains two white balls and three blue balls, while the second contains four white and one blue ball.

   Suppose you choose a box at random and then select a ball from that box at random, what is the probability that a ball from the first box was chosen, given you selected a blue ball.

Solution: Let A_i be the event choose the ith box and B a blue ball is chosen. We want to find $\mathbf{P}[A_1 \mid B]$. Clearly $\mathbf{P}[A_i] = 1/2$ for $1 \le i \le 2$, $\mathbf{P}[B \mid A_1] = 3/5$ and $\mathbf{P}[B \mid A_2] = 1/5$. Now since $\mathbf{P}[A_1] + \mathbf{P}[A_2] = 1$, using Bayes' law we have:

$$\mathbf{P}[A_1 \mid B] = \frac{\mathbf{P}[B \mid A_1]\mathbf{P}[A_1]}{\mathbf{P}[B \mid A_1]\mathbf{P}[A_1] + \mathbf{P}[B \mid A_2]\mathbf{P}[A_2]} = \frac{3/10}{3/10 + 1/10} = \frac{3/10}{4/10} = \frac{3}{4}\,.$$

7. Given three cards where:
   - the first is red on each side;
   - the second is green on each side;
   - and the third is red on one side and green on the other.

   If we draw one card at random and look at one side only, what is the probability that given the side we are looking at is green that the other side is also green?

Solution: This is a conditional probability, let G_1 be the side we are looking at is green and G_2 the other side is green. There are 6 sides and 3 are green, therefore the probability the other side of a card we are looking at is green equals 1/2. Alternatively, if you think you look at a side with equal chance, then you can use the law of total probability:

$$\begin{aligned}
&\mathbf{P}[G_2 \mid \mathit{card}_1] + \mathbf{P}[G_2 \mid \mathit{card}_2] + \mathbf{P}[G_2 \mid \mathit{card}_3] \\
&= \mathbf{P}[G_2 \cup \mathit{card}_1]\mathbf{P}[\mathit{card}_1] + \mathbf{P}[G_2 \cup \mathit{card}_2]\mathbf{P}[\mathit{card}_1] + \mathbf{P}[G_2 \cup \mathit{card}_3]\mathbf{P}[\mathit{card}_1] \\
&= 0{\cdot}(1/3) + 1{\cdot}(1/3) + (1/2){\cdot}(1/3) = 1/3 + 1/6 = 1/2\,.
\end{aligned}$$

The probability we choose a card which is green on both side ($G_1 \cap G_2$) equals 1/3, i.e. the probability you choose the second card. Therefore by definition of conditional probability we have:

$$\mathbf{P}[G_1 \mid G_2] = \frac{\mathbf{P}[G_1 \cap G_2]}{\mathbf{P}[G_2]} = \frac{1/3}{1/2} = \frac{2}{3}\,.$$

8. Suppose a test for opioids use has a 2% false positive rate and a 5% false negative rate. (More precisely, 2% of people who have not taken opioids test positive and 5% of people who have taken opioids test negative.) Assume that 1% of people have taken opioids.

   (a) Find the probability that someone who tests negative for opioids has not taken opioids.

   Solution: Let O and $\neg O$ be the events have and have not taken opioids, while P and N the events testing positive and negative. Clearly we have $\mathbf{P}[O] + \mathbf{P}[\neg O] = 0.01 + 0.99 = 1$. Therefore using Bayes' theorem we have:

$$\begin{aligned}
\mathbf{P}[\neg O \mid N] &= \frac{\mathbf{P}[N \mid \neg O]\mathbf{P}[\neg O]}{\mathbf{P}[N \mid \neg O]\mathbf{P}[\neg O] + \mathbf{P}[N \mid O]\mathbf{P}[O]} \\
&= \frac{0.98{\cdot}0.99}{0.98{\cdot}0.99 + 0.05{\cdot}0.01} = \frac{0.9702}{0.9707} = 0.9995\,.
\end{aligned}$$

   (b) Find the probability that someone who tests positive for opioids has actually taken opioids.

Solution: Using the notation from the first part and applying Bayes' law:

$$\begin{aligned}\mathbf{P}[O \mid P] &= \frac{\mathbf{P}[P \mid O]\mathbf{P}[O]}{\mathbf{P}[P \mid O]\mathbf{P}[O] + \mathbf{P}[P \mid \neg O]\mathbf{P}[\neg O]} \\ &= \frac{0.95 \cdot 0.01}{0.95 \cdot 0.01 + 0.02 \cdot 0.99} = \frac{0.0095}{0.0293} = 0.3242\,.\end{aligned}$$

9. A box contains 3 yellow balls and 5 red balls. A ball is chosen at random from the box, then replaced in the box along with two other balls of the same colour.
   (a) If a second ball is now chosen at random from the box, what is the probability that it will be red?

Solution: The second ball is dependent on the choice of the first so let R_i and Y_i be the event the ith ball is red and yellow respectively. We have:

$$\begin{aligned}\mathbf{P}[R_2] &= \mathbf{P}[R_2 \cap R_1] + \mathbf{P}[R_2 \cap Y_1] \\ &= (5/8) \cdot (7/10) + (3/8) \cdot (5/10) = 50/80 = 5/8\,.\end{aligned}$$

   (b) Given that the second ball is red, what is the probability that the first ball was yellow?

Solution: Using Bayes' law we have:

$$\mathbf{P}[Y_1 \mid R_2] = \frac{\mathbf{P}[R_2 \mid Y_1]\mathbf{P}[Y_1]}{\mathbf{P}[R_2]} = \frac{\mathbf{P}[R_2 \cap Y_1]}{\mathbf{P}[R_2]} = \frac{(3/8) \cdot (5/10)}{5/8} = 3/10\,.$$

10. An *octahedral die* has eight faces that are number 1 to 8. What are the expected value and the variance when a fair octahedral die is rolled.

Solution: The expected value is given by

$$\mathbf{E}[X] = \sum_{i=1}^{8} (i \cdot (1/8)) = (1/8) \cdot \left(\sum_{i=1}^{8} i\right) = (1/8) \cdot (8 \cdot 9/2) = 9/2$$

On the other hand the variance is given by:

$$\begin{aligned}\mathbf{V}[X] &= \sum_{i=1}^{8} ((i - 9/2)^2 \cdot (1/8)) \\ &= (1/8) \cdot \left(\sum_{i=1}^{8} (i - 9/2)^2\right) \\ &= (1/8) \cdot \left(\sum_{i=1}^{8} (i^2 - 9 \cdot i + 81/4\right) \\ &= (1/8) \cdot \left(\sum_{i=1}^{8} i^2\right) - 9 \cdot \left(\sum_{i=1}^{8} i\right) + 81/4 \cdot \left(\sum_{i=1}^{8} 1\right) \\ &= (1/8) \cdot ((8 \cdot 9 \cdot 17)/6 - 9 \cdot ((8 \cdot 9)/2) + 8 \cdot (81/4)) \\ &= 5.25\end{aligned}$$

where final steps follow from the following results you can/have proved by induction. For any $n \in \mathbb{Z}$:

$$\sum_{i=1}^{n} i^2 = \frac{n\cdot(n+1)\cdot(2\cdot n+1)}{6}\,, \qquad \sum_{i=1}^{n} i = \frac{n\cdot(n+1)}{2} \qquad \text{and} \qquad \sum_{i=1}^{n} 1 = n\,.$$

Difficult/challenging questions.

11. Both undergraduates and postgraduates can use the university cafeteria. Each diner can choose between buying a meal or bringing a packed lunch. (Everyone has exactly one meal each, no more and no less). The cafeteria offers a daily choice between a hot meal or a cold meal. A survey of undergraduate diners finds that 40% of them bring their own food. Overall, only 25% of the diners bring their own food. Postgraduates make up one fifth of the diners in the cafeteria.

   (a) What is the probability that a diner is an undergraduate and buys a meal?

   Solution: From the assumptions it follows that the probability an undergraduate buy lunch is 0.6 and the probability a student is an undergraduate is 0.8. Therefore we have:

$$\begin{aligned} \mathbf{P}[\mathit{buys} \cap \mathit{undergrad}] &= \mathbf{P}[\mathit{buys} \mid \mathit{undergrad}] \cdot \mathbf{P}[\mathit{undergrad}] \\ &= 0.6\cdot 0.8 = 0.48\,. \end{aligned}$$

   (b) What is the probability that someone that buys a meal is a postgraduate?

   Solution: First using the definition of conditional probabilities:

$$\mathbf{P}[\mathit{postgrad} \mid \mathit{buys}] = \frac{\mathbf{P}[\mathit{postgrad} \cup \mathit{buys}]}{\mathbf{P}[\mathit{buys}]}\,.$$

   However we do not currently have a value for $\mathbf{P}[\mathit{postgrad} \cap \mathit{buys}]$. Using the law of total probability:

$$\mathbf{P}[\mathit{buys}] = \mathbf{P}[\mathit{postgrad} \cap \mathit{buys}] + \mathbf{P}[\mathit{undergrad} \cap \mathit{buys}]$$

   So rearranging we have

$$\begin{aligned} \mathbf{P}[\mathit{postgrad} \cap \mathit{buys}] &= \mathbf{P}[\mathit{buys}] - \mathbf{P}[\mathit{undergrad} \cap \mathit{buys}] \\ &= 0.75 - 0.48 \qquad \text{(using the answer to part (a) and the assumptions)} \\ &= 0.27 \end{aligned}$$

   and hence using the expression above for the conditional probability:

$$\mathbf{P}[\mathit{postgrad} \mid \mathit{buys}] = \frac{0.27}{0.75} = 0.36$$

12. The *Birthday Problem* asks what is the minimum number of people who need to be in a room so the probability at least two people have the same birthday is greater than 1/2.

    Find this number under the assumption that the birthdays of the people in the room are independent, each birthday is equally likely and the number of days in a year is 366.

    Solution: Here it is easier to calculate the complement probability, i.e. that all people have different birthdays and then find the actual probability by subtracting this probability from 1. We can consider one person at a time, the first can have any birthday, then the second must have a different birthday (i.e. from the remaining 365 days), then the third again must have a different birthday (i.e. from the remaining 364) and so on (so the nth must have a birthday from the remaining $366-(n-1)$ days). Using the fact that the birthdays of people are independent yields the probability p_n for at least two out of n people having the same birthday:

    $$p_n = 1 - \frac{366}{366}\cdot\frac{365}{366}\cdots\frac{366-(n-2)}{366}\cdot\frac{366-(n-1)}{366}\,.$$

    After some trial and error we find $p_{22} < 0.5$ and $p_{23} > 0.5$.

13. A space probe near Neptune communicates with Earth using bit strings. Suppose that in its transmissions it sends `1`'s one-third of the time and `0`'s the remaining two thirds. When a `0` is sent, the probability that a `0` is received is 0.9 (and a `1` is received with probability 0.1). On the other hand, when a `1` is sent, the probability that a `1` is received is 0.8 (and a `0` is received with probability 0.2).

    (a) What is the probability that a `0` is received.

    Solution: Let T_i and R_i be the events of transmitting and receiving a `i` respectively. Using the law of total probability:

    $$\mathbf{P}[R_0] = \mathbf{P}[R_0 \mid T_0]\mathbf{P}[T_0] + \mathbf{P}[R_0 \mid T_1]\mathbf{P}[T_1] = 9/10\cdot 2/3 + 2/10\cdot 1/3 = 20/30 = 2/3\,.$$

    (b) What is the probability that a `0` was transmitted given that a `0` was received.

    Solution: Using the notation from the first part and Bayes' law:

    $$\begin{aligned}\mathbf{P}[T_0 \mid R_0] &= \frac{\mathbf{P}[R_0 \mid T_0]\mathbf{P}[T_0]}{\mathbf{P}[R_0 \mid T_0]\mathbf{P}[T_0] + \mathbf{P}[R_0 \mid T_1]\mathbf{P}[T_1]} \\ &= \frac{(9/10)\cdot(2/3)}{(9/10)\cdot(2/3) + (2/10)\cdot(1/3)} = \frac{3/5}{18/30 + 2/30} = \frac{3/5}{20/30} = \frac{9}{10}\,.\end{aligned}$$