2B Linear Algebra

Ex Sheet 7

True/False[1]

[1] **True/False Questions**
Every Exercise Sheet will have a section containing true/false questions. They are designed to test your understanding from lectures. You should look at your lecture notes and/or the textbook to help you answer these questions, but you should not need to write anything to work out the solution.

a) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a diagonal matrix.

b) Any matrix $A \in M_{n\times n}(\mathbb{R})$ is similar to itself.

c) Similar matrices have the same eigenvalues.

d) A square matrix is diagonalisable if it is similar to a diagonal matrix.

e) For all diagonalisable matrices A, there is a unique diagonal matrix D and a unique invertible matrix P so that $P^{-1}AP = D$.

f) k eigenvectors corresponding to k distinct eigenvalues are linearly independent.

g) An $n \times n$ matrix with real entries is diagonalisable if and only if it has n distinct real eigenvalues.

h) $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ is diagonalisable.

i) The geometric multiplicity of an eigenvalue λ of the square matrix A is the number of vectors in the λ-eigenspace of A.

j) The sum of the algebraic multiplicities of the eigenvalues of an $n \times n$ real matrix is n.

Solutions to True/False

a) T b) T c) T d) T e) F f) T g) F h) T i) F j) T

Tutorial Exercises

T1 Let $A, B \in M_{n\times n}$.

a) Show that if A and B are similar, then A is invertible if and only if B is invertible.

b) Prove that if A and B are similar and both invertible, then A^{-1} and B^{-1} are similar.

Solution

a) The matrix A is invertible if and only if $\det(A) \neq 0$ and the matrix B is invertible if and only if $\det(B) \neq 0$. Since $\det(A) = \det(B)$ the result follows.

b) We have $B^{-1} = (P^{-1}AP)^{-1} = P^{-1}A^{-1}(P^{-1})^{-1} = P^{-1}A^{-1}P$. Since P is invertible, this means that A^{-1} and B^{-1} are similar.

T2 Consider the matrices

$$A = \begin{bmatrix} 2 & 3 & 2 & 4 \\ -1 & 2 & 1 & 1 \\ 2 & 3 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

By considering determinants, show that A and B are *not* similar.

Solution

Suppose by way of contradiction that A and B are similar. Then $\det(A) = \det(B)$. Now we need only notice that $\det(A) = 0$ (expand along the bottom row), and $\det(B) = 4$ (it's upper triangular, so the determinant is the product of the entries on the main diagonal). These numbers are different, so A and B cannot be similar.

T3 For each of the matrices in T7 and T9 on Exercise Sheet 6:

- Determine whether the matrix is diagonalisable, and if it is diagonalisable, find a diagonal matrix D and an invertible matrix P such that $P^{-1}AP = D$ (replace A by B or C as appropriate).
- Find the algebraic and geometric multiplicities of each of the eigenvalues.

Solution

For T7 on Exercise Sheet 6:

a) Since A has two distinct eigenvalues, A is diagonalisable. Using the answers to T7(a), if

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix} \text{ and } P = \begin{pmatrix} 1 & -\frac{3}{5} \\ 0 & 1 \end{pmatrix}$$

then $P^{-1}BP = D$.

Each of the eigenvalues of A has algebraic multiplicity 1 and geometric multiplicity 1.

b) The only eigenvalue is $\lambda = -2$ and we have

$$E_{-2} = \operatorname{Span}\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$$

which is one-dimensional. Thus $\mathbb{R}^2$ does not have a basis consisting of eigenvectors of B, hence B is not diagonalisable.

The eigenvalue $\lambda = 2$ has algebraic multiplicity 2 and geometric multiplicity 1.

c) Since C has two distinct eigenvalues, C is diagonalisable. Using the answers to T7(c), if

$$D = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } P = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{3} \\ 1 & 1 \end{pmatrix}$$

then $P^{-1}CP = D$.

Each of the eigenvalues of C has algebraic multiplicity 1 and geometric multiplicity 1.

For T9 on Exercise Sheet 6:

a) Since A has three distinct eigenvalues, A is diagonalisable. Using the answers to T9(a), if

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \text{ and } P = \begin{pmatrix} 0 & -2 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

then $P^{-1}AP = D$.

Each of the eigenvalues of A has algebraic multiplicity 1 and geometric multiplicity 1.

b) Since B has three distinct eigenvalues, B is diagonalisable. Using the answers to T9(b), if

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3i & 0 \\ 0 & 0 & -3i \end{pmatrix} \text{ and } P = \begin{pmatrix} 2 & 1+3i & 1-3i \\ -2 & 3i-1 & -3i-1 \\ 1 & -4 & -4 \end{pmatrix}$$

then $P^{-1}AP = D$.

Each of the eigenvalues of B has algebraic multiplicity 1 and geometric multiplicity 1.

c) The only eigenvalue is $\lambda = 2$ and we have

$$E_2 = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

which is one-dimensional. Thus $\mathbb{R}^3$ does not have a basis consisting of eigenvectors of C, hence C is not diagonalisable.

The eigenvalue $\lambda = 2$ has algebraic multiplicity 3 and geometric multiplicity 1.

T4 Let $A, B \in M_{3\times 3}(\mathbb{C})$ and suppose the eigenvalues of both A and B are $1, 2+i$ and 4.

a) Write down a diagonal matrix D to which both A and B are similar.

b) Hence prove that A is similar to B.

Solution

a) Since A and B have the same three distinct eigenvalues, they are both similar to the diagonal matrix

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2+i & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

b) Since A is similar to D, there is an invertible matrix $P \in M_{3\times3}(\mathbb{C})$ such that $P^{-1}AP = D$. Since B is similar to D, there is an invertible matrix $Q \in M_{3\times3}(\mathbb{C})$ such that $Q^{-1}BQ = D$. Now this latter equation means $B = QDQ^{-1}$, so we have

$$B = QDQ^{-1} = QP^{-1}APQ^{-1} = (PQ^{-1})^{-1}A(PQ^{-1}).$$

Therefore A and B are similar matrices.

T5 Construct the matrix A which has eigenvalues 0 and -1, with corresponding eigenspaces

$$E_0 = \mathrm{Span}\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right), \qquad E_{-1} = \mathrm{Span}\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right).$$

Solution

Since A has distinct eigenvalues, we know that it is similar to a diagonal matrix D. Hence, we have $A = PDP^{-1}$ where

$$D = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}.$$

We compute

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 \\ -6 & -3 \end{bmatrix}.$$

T6 Let $A \in M_{n\times n}(\mathbb{R})$ be invertible. Recall that the eigenvalues of an invertible matrix are non-zero.

a) Suppose that $\lambda \in \mathbb{R}$ is an eigenvalue of A. Prove that λ^{-1} is an eigenvalue of A^{-1}.

b) Show that if $D = (d_{ij}) \in M_{n\times n}(\mathbb{R})$ is diagonal, with all diagonal entries $d_{ii} \neq 0$, then D is invertible, with D^{-1} the diagonal matrix with diagonal entries d_{ii}^{-1}.

c) Prove that if A is diagonalisable, then A^{-1} is diagonalisable.

Solution

a) Since λ is an eigenvalue of A, there is a non-zero vector $\boldsymbol{v}$ so that $A\boldsymbol{v} = \lambda\boldsymbol{v}$. Now multiply both sides of this equation by A^{-1} to get

$$A^{-1}A\boldsymbol{v} = A^{-1}(\lambda\boldsymbol{v}) \implies \mathbb{I}_n\boldsymbol{v} = \lambda(A^{-1}\boldsymbol{v}) \implies \boldsymbol{v} = \lambda(A^{-1}\boldsymbol{v}).$$

Since $\lambda \neq 0$, we can then multiply both sides of this last equation by λ^{-1} to get

$$\lambda^{-1}\boldsymbol{v} = 1(A^{-1}\boldsymbol{v}) \implies \lambda^{-1}\boldsymbol{v} = A^{-1}\boldsymbol{v}.$$

Since the vector $\boldsymbol{v}$ is non-zero, this means that λ^{-1} is an eigenvalue of A^{-1}.

b) The determinant of D is the product of its diagonal entries. Since each $d_{ii} \neq 0$, we have that

$\det(D) \neq 0$, so D is invertible. If E is the diagonal matrix with $e_{ii} = d_{ii}^{-1}$ then a direct computation shows that $DE = ED = \mathbb{I}_n$. Hence $D^{-1} = E$ as required.

c) If A is diagonalisable there exists an invertible matrix P and a diagonal matrix D so that $P^{-1}AP = D$. The diagonal entries of D are the eigenvalues of A, so the diagonal entries of D are non-zero as A is invertible. Hence D is invertible and D^{-1} is diagonal, by (ii). So we may take the inverse of both sides of the equation $P^{-1}AP = D$ to get

$$(P^{-1}AP)^{-1} = D^{-1} \implies P^{-1}A^{-1}(P^{-1})^{-1} = D^{-1}.$$

Put $Q = P^{-1}$ then we have $QA^{-1}Q^{-1} = D^{-1}$, with D^{-1} diagonal, hence A^{-1} is diagonalisable.

T7 Let A, B and C be $n \times n$ matrices and suppose that A is similar to B, with $P^{-1}AP = B$, and B is similar to C, with $Q^{-1}BQ = C$, where P and Q are $n \times n$ invertible matrices. Prove that A is similar to C.

Solution

Since P and Q are invertible, PQ is invertible. Now

$$(PQ)^{-1}A(PQ) = Q^{-1}P^{-1}APQ = Q^{-1}BQ = C$$

and so A and C are similar.

T8 Let A and B be $n \times n$ matrices and suppose that A is similar to B, with $AP = PB$ for an $n \times n$ invertible matrix P.

a) Recall that row-equivalent matrices have the same row space. Use this to show that $\operatorname{rank}(B) = \operatorname{rank}(PB)$ and that $\operatorname{rank}((AP)^T) = \operatorname{rank}(A^T)$.

b) Deduce that $\operatorname{rank}(A) = \operatorname{rank}(B)$.

Solution

a) Since P is invertible, P is a product of elementary matrices. So PB is row-equivalent to B, hence $\operatorname{row}(PB) = \operatorname{row}(B)$ and thus $\operatorname{rank}(PB) = \operatorname{rank}(B)$.
Now $(AP)^T = P^TA^T$ and P^T is invertible since P is invertible. Thus by the same argument $\operatorname{rank}((AP)^T) = \operatorname{rank}(A^T)$.

b) Using a) and the equation $AP = PB$, as well as results about rank, we have

$$\operatorname{rank}(A) = \operatorname{rank}(A^T) = \operatorname{rank}((AP)^T) = \operatorname{rank}(AP) = \operatorname{rank}(PB) = \operatorname{rank}(B)$$

and so $\operatorname{rank}(A) = \operatorname{rank}(B)$ as required.

In the remaining exercises you will prove the Diagonalisation Theorem, Theorem 4.27. The notation is as follows. Suppose $A = (a_{ij}) \in M_{n\times n}(\mathbb{R})$ has eigenvalues $\lambda_1, \ldots, \lambda_k$. For $1 \leq i \leq k$ let

m_i = the algebraic multiplicity of the eigenvalue λ_i

and

d_i = the geometric multiplicity of the eigenvalue λ_i.

The aim is to show that A is diagonalisable if and only, for each $1 \leq i \leq k$, we have $d_i = m_i$.

T9

a) Prove by induction on $n \geq 2$ if $B = (b_{ij}) \in M_{n\times n}(\mathbb{R})$ then $\det(B)$ is a polynomial in the entries of B of degree at most n. This means that for each monomial term of $\det(B)$, the sum of the powers of the b_{ij} appearing in that monomial is at most n.

b) Using (a), prove by induction on $n \geq 2$ that

$$\det(A - tI) = (a_{11} - t)(a_{22} - t)\cdots(a_{nn} - t) + g(t)$$

where $g(t)$ is a polynomial in the variable t with coefficients in $\mathbb{R}$ and degree strictly less than n.

c) Conclude that $\det(A - tI)$ has degree equal to n, hence

$$m_1 + m_2 + \cdots + m_k = n.$$

Solution

a) In the case $n = 2$ we have

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

so $\det(B) = b_{11}b_{22} - b_{12}b_{21}$. Thus $\det(B)$ is a polynomial in the entries of B of degree at most 2. Now assume that for any $k \times k$ real matrix B, $\det(B)$ is a polynomial in the entries of B of degree at most k.

Let $B = (b_{ij})$ be a $(k+1) \times (k+1)$ real matrix. Then

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1,k+1} \\ b_{21} & b_{22} & \cdots & b_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k+1,1} & b_{k+1,2} & \cdots & b_{k+1,k+1} \end{pmatrix}.$$

We calculate $\det(B)$ by expanding along the first row. We have

$$\det(B) = \sum_{j=1}^{k+1} (-1)^{j+1} b_{1j} \det(B_{1j}).$$

Now each cofactor B_{1j} is a $k \times k$ matrix, so by the inductive assumption, $\det(B_{1j})$ is a polynomial in the entries of B of degree at most k. Hence each summand $(-1)^{j+1} b_{1j} \det(B_{1j})$ has degree at most $k+1$, and thus $\det(B)$ has degree at most $k+1$ as required.

b) In the case $n = 2$ we have

$$\det(A - tI) = \begin{vmatrix} a_{11} - t & a_{12} \\ a_{21} & a_{22} - t \end{vmatrix} = (a_{11} - t)(a_{22} - t) - a_{12}a_{21}$$

Let $g(t) = -a_{12}a_{21}$. That is, $g(t)$ is the constant polynomial $-a_{12}a_{21} \in \mathbb{R}$. Then we have $\det(A - tI) = (a_{11} - t)(a_{22} - t) + g(t)$ with $g(t)$ a real polynomial of degree 0. Since $0 < 2$, this proves the statement in the case $n = 2$.

Now assume that if A is $k \times k$ then

$$\det(A - tI) = (a_{11} - t)(a_{22} - t) \cdots (a_{kk} - t) + g(t)$$

where $g(t)$ is a polynomial in the variable t with coefficients in $\mathbb{R}$ and degree less than k.

Let A be $(k+1) \times (k+1)$. Then

$$\det(A - tI) = \det \begin{pmatrix} a_{11} - t & a_{12} & \cdots & a_{1,k+1} \\ a_{21} & a_{22} - t & \cdots & a_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+1,1} & a_{k+1,2} & \cdots & a_{k+1,k+1} - t \end{pmatrix}.$$

We compute this determinant by expanding along the top row. This gives

$$\det(A - tI) = (a_{11} - t)\det((A - tI)_{11}) + \sum_{j=2}^{k+1} (-1)^{j+1} a_{1j} \det((A - tI)_{1j}).$$

Consider the term $(a_{11} - t)\det((A - tI)_{11})$. The cofactor $(A - tI)_{11} = (A - tI_{k+1})_{11}$ is equal to $(A_{11} - tI_k)$, and so

$$\det((A - tI_{k+1})_{11}) = \det(A_{11} - tI_k).$$

This is the characteristic polynomial of A_{11}, so by inductive assumption, since A_{11} is $k \times k$ we have

$$\det(A - tI_k) = (a_{22} - t) \cdots (a_{k+1,k+1} - t) + g(t)$$

where $g(t)$ is a polynomial in the variable t with coefficients in $\mathbb{R}$ and degree less than k. Thus

$$(a_{11} - t)\det((A - tI_{k+1})_{11}) = (a_{11} - t)(a_{22} - t) \cdots (a_{k+1,k+1} - t) + (a_{11} - t)g(t).$$

Since $g(t)$ has degree less than k, the term $(a_{11} - t)g(t)$ has degree less than $(k+1)$.

It now suffices to show that

$$\sum_{j=2}^{k+1} (-1)^{j+1} a_{1j} \det((A - tI)_{1j})$$

has degree less than $(k+1)$. For this, it suffices to show that for each $2 \leq j \leq n$, the polynomial in t given by $\det((A - tI)_{1j})$ has degree at most k. By part (a), since $(A - tI)_{1j}$ is a $k \times k$ matrix, $\det(A - tI)_{1j})$ is a polynomial in the entries of $(A - tI)_{1j}$ of degree at most k. This completes the proof.

c) By part (b), we have

$$\det(A - tI) = (a_{11} - t)(a_{22} - t) \cdots (a_{nn} - t) + g(t)$$

where $g(t)$ has degree less than n. Since the expression $(a_{11} - t)(a_{22} - t) \cdots (a_{nn} - t)$ is a polynomial in t of degree equal to n (the coefficient of t^n is $(-1)^n \neq 0$), it follows that $\det(A - tI)$ has degree n.

Since the eigenvalues of A are the roots of $\det(A-tI)$, the polynomial $\det(A-tI)$ factors as

$$\det(A-tI) = (-1)^n (t-\lambda_1)^{m_1} (t-\lambda_2)^{m_2} \cdots (t-\lambda_k)^{m_k}.$$

As $\det(A-tI)$ has degree n, the sum of the m_i must equal n, that is,

$$m_1 + m_2 + \cdots + m_k = n.$$

T10 Let $\lambda = \lambda_i$ be an eigenvalue of A, let $m = m_i$ be the algebraic multiplicity of λ and let $d = d_i$ be the geometric multiplicity of λ. Let $S : \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_d$ be an ordered basis for the λ-eigenspace E_λ.

a) Let U be the $n \times d$ matrix which has $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_d$ as its columns. Explain why $AU = \lambda U$.

b) Let Q be any invertible matrix in $M_{n\times n}(\mathbb{R})$ which has $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_d$ as its first d columns. Then we can write Q as a "partitioned matrix" $Q = (U \mid V)$, where V is $n \times (n-d)$. By considering the product $Q^{-1}Q$, prove that if Q^{-1} is the partitioned matrix

$$Q^{-1} = \begin{pmatrix} C \\ D \end{pmatrix}$$

where C is $d \times n$ and D is $(n-d) \times n$, then the following equations hold:

$$CU = \mathbb{I}_d \quad CV = \mathbb{O}_{d,n-d} \quad DU = \mathbb{O}_{n-d,d} \quad CU = \mathbb{I}_{n-d}.$$

Here, $\mathbb{O}_{k,l}$ is the $k \times l$ matrix with all entries 0.

c) Hence prove that

$$\det(Q^{-1}AQ - tI) = (\lambda - t)^d \det(DAV - tI).$$

d) Conclude that $d \le m$. That is, for each eigenvalue of A, the geometric multiplicity is less than or equal to the algebraic multiplicity.

Solution

a) Since $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_d$ are all eigenvectors of A with corresponding eigenvector λ, we have $A\boldsymbol{v}_i = \lambda \boldsymbol{v}_i$ for each $1 \le i \le d$. Thus AU is the $n \times d$ matrix with i^{th} column $A\boldsymbol{v}_i$, and so $AU = \lambda U$.

b) We have

$$Q^{-1}Q = \begin{pmatrix} C \\ D \end{pmatrix} (U \mid V) = \begin{pmatrix} CU & CV \\ DU & DV \end{pmatrix}$$

But also $Q^{-1}Q = \mathbb{I}_n$ which we partition as

$$Q^{-1}Q = \mathbb{I}_n = \begin{pmatrix} \mathbb{I}_d & \mathbb{O}_{d,n-d} \\ \mathbb{O}_{n-d,d} & \mathbb{I}_{n-d} \end{pmatrix}.$$

By considering the sizes of the products CU, CV, DU and DV we obtain the required equations

$$CU = \mathbb{I}_d \quad CV = \mathbb{O}_{d,n-d} \quad DU = \mathbb{O}_{n-d,d} \quad DV = \mathbb{I}_{n-d}.$$

c) We first compute $Q^{-1}AQ$:

$$Q^{-1}AQ = \begin{pmatrix} C \\ D \end{pmatrix} A(U \mid V) = \begin{pmatrix} CAU & CAV \\ DAU & DAV \end{pmatrix}$$

Now $AU = \lambda U$ by part (a), and $CU = \mathbb{I}_d$ and $DU = \mathbb{O}_{n-d,d}$ by part (b). So

$$Q^{-1}AQ = \begin{pmatrix} C\lambda U & CAV \\ D\lambda U & DAV \end{pmatrix} = \begin{pmatrix} \lambda CU & CAV \\ \lambda DU & DAV \end{pmatrix} = \begin{pmatrix} \lambda \mathbb{I}_d & CAV \\ \lambda \mathbb{O}_{n-d,d} & DAV \end{pmatrix} = \begin{pmatrix} \lambda \mathbb{I}_d & CAV \\ \mathbb{O}_{n-d,d} & DAV \end{pmatrix}.$$

Hence

$$\det(Q^{-1}AQ - tI) = \det \begin{pmatrix} (\lambda - t)\mathbb{I}_d & CAV \\ \mathbb{O}_{n-d,d} & DAV - t\mathbb{I}_{n-d} \end{pmatrix} = (\lambda - t)^d \det(DAV - t\mathbb{I}_{n-d})$$

as required.

d) Since A and $Q^{-1}AQ$ are similar matrices, they have the same characteristic polynomial. Therefore by part (c),

$$\det(A - tI) = \det(Q^{-1}AQ - tI) = (\lambda - t)^d \det(DAV - tI).$$

Thus the algebraic multiplicity of λ is at least d, and so $d \leq m$ as required.

T11 For $1 \leq i \leq k$, let

$$S_i : \boldsymbol{v}_{i1}, \boldsymbol{v}_{i2}, \ldots, \boldsymbol{v}_{id_i}$$

be an ordered basis for the λ_i-eigenspace $\mathrm{Eig}_{\lambda_i}(A)$.

a) Prove that

$$S : \boldsymbol{v}_{11}, \boldsymbol{v}_{12}, \ldots, \boldsymbol{v}_{1d_1}, \boldsymbol{v}_{21}, \boldsymbol{v}_{22}, \ldots, \boldsymbol{v}_{2d_2}, \ldots, \boldsymbol{v}_{k1}, \boldsymbol{v}_{k2}, \ldots, \boldsymbol{v}_{kd_k}$$

obtained by taking the union of the S_i is linearly independent. [Hint: remember that eigenspaces are subspaces, and use Theorem 4.20.]

b) Hence prove that $\mathbb{R}^n$ has a basis consisting of eigenvectors of A if and only if $d_1 + d_2 + \cdots + d_k = n$.

Solution

a) Suppose that

$$\lambda_{11}\boldsymbol{v}_{11} + \lambda_{12}\boldsymbol{v}_{12} + \cdots + \lambda_{1d_1}\boldsymbol{v}_{1d_1} + \lambda_{21}\boldsymbol{v}_{21} + \lambda_{22}\boldsymbol{v}_{22} + \cdots + \lambda_{2d_2}\boldsymbol{v}_{2d_2} + \cdots + \lambda_{k1}\boldsymbol{v}_{k1} + \lambda_{k2}\boldsymbol{v}_{k2} + \cdots + \lambda_{kd_k}\boldsymbol{v}_{kd_k} = \boldsymbol{0}$$

where each $\lambda_{ij} \in \mathbb{R}$. Let

$$\begin{aligned} \boldsymbol{v}_1 &= \lambda_{11}\boldsymbol{v}_{11} + \lambda_{12}\boldsymbol{v}_{12} + \cdots + \lambda_{1d_1}\boldsymbol{v}_{1d_1} \\ \boldsymbol{v}_2 &= \lambda_{21}\boldsymbol{v}_{21} + \lambda_{22}\boldsymbol{v}_{22} + \cdots + \lambda_{2d_2}\boldsymbol{v}_{2d_2} \\ \vdots &\ \vdots\ \vdots \\ \boldsymbol{v}_k &= \lambda_{k1}\boldsymbol{v}_{k1} + \lambda_{k2}\boldsymbol{v}_{k2} + \cdots + \lambda_{kd_k}\boldsymbol{v}_{kd_k}. \end{aligned}$$

Then

$$v_1 + v_2 + \cdots + v_k = \mathbf{0}. \tag{1}$$

Now for each $1 \leq i \leq k$, we have that $v_i \in \mathrm{Eig}_{\lambda_i}(A)$, since eigenspaces are subspaces. So either $v_i = \mathbf{0}$ or v_i is an eigenvector of A. If there is some $v_i \neq \mathbf{0}$ then the collection $\{v_i \mid v_i \neq \mathbf{0}\}$ is a collection of eigenvectors of A corresponding to distinct eigenvalues. By Theorem 4.20, this collection is linearly independent. However Equation (1) above gives a linear dependence between these eigenvectors, a contradiction. Therefore each $v_i = \mathbf{0}$. Now as each S_i is a basis, and thus linearly independent, the k equations above defining the v_i mean that $\lambda_{ij} = 0$ for all i, j. Hence S is linearly independent.

b) Suppose $d_1 + d_2 + \cdots + d_k = n$. Then the set S is a linearly independent set in $\mathbb{R}^n$ containing n vectors, hence S is a basis. So $\mathbb{R}^n$ has a basis consisting of eigenvectors of A.

Suppose $\mathbb{R}^n$ has a basis consisting of eigenvectors of A. Now by F5 we have $d_i \leq m_i$ for each i, and by T10 we have $m_1 + m_2 + \cdots + m_k = n$. So $d_1 + d_2 + \cdots + d_k \leq n$. If $d_1 + d_2 + \cdots + d_k < n$ then the set S does not span $\mathbb{R}^n$, since it contains fewer than n vectors. However every eigenvector in the basis of eigenvectors belongs to $\mathrm{Eig}_{\lambda_i}(A)$ for some i, and so must be in the span of S_i. This is a contradiction. Hence $d_1 + d_2 + \cdots + d_k = n$.

T12 It follows from Theorem 4.23 that A is diagonalisable if and only if $\mathbb{R}^n$ has a basis consisting of eigenvectors of A. Use this together with results above to prove that A is diagonalisable if and only if $d_i = m_i$ for each $1 \leq i \leq k$.

Solution

Suppose A is diagonalisable. Then it follows from Theorem 4.23 that $\mathbb{R}^n$ has a basis consisting of eigenvectors of A. By T11, this means $d_1 + d_2 + \cdots + d_k = n$. Now by T11, we have $m_1 + m_2 + \cdots + m_k = n$. Thus

$$0 = (m_1 + m_2 + \cdots + m_k) - (d_1 + d_2 + \cdots + d_k) = (m_1 - d_1) + (m_2 - d_2) + \cdots (m_k - d_k).$$

By F5, $d_i \leq m_i$ for each i, hence $m_i - d_i \geq 0$ for each i. Thus we must have $m_i - d_i = 0$ for all i. That is, the geometric and algebraic multiplicities of each eigenvalue are equal.

Suppose that $d_i = m_i$ for each i. Then using F4 it follows that

$$m_1 + m_2 + \cdots + m_k = n = d_1 + d_2 + \cdots + d_k.$$

Hence by F6, $\mathbb{R}^n$ has a basis consisting of eigenvectors, and so by Theorem 4.23, A is diagonalisable.

T13 Let

$$A = \begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix}.$$

Find an expression for A^n, where n is a positive integer, in a form that displays the entries of the matrix explicitly.

Solution

In **T9** exercise sheet 6, we found that

$$P^{-1}AP = D,$$

where

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad D = \operatorname{diag}(4, 3).$$

Hence

$$A = PDP^{-1}.$$

So, for any positive integer n,

$$\begin{aligned} A^n &= PD^nP^{-1} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4^n & 0 \\ 0 & 3^n \end{bmatrix} \frac{1}{1} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2(4^n) & 3^n \\ 4^n & 3^n \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2(4^n) - 3^n & -2(4^n) + 2(3^n) \\ 4^n - 3^n & -4^n + 2(3^n) \end{bmatrix}. \end{aligned}$$

T14 Consider the matrix

$$A(x) = \begin{bmatrix} (x-2) & 2 \\ -1 & (x+1) \end{bmatrix},$$

where $x \in \mathbb{R}$. Find an invertible matrix P and a diagonal matrix $D(x)$ (which depends on x) such that $A(x) = P^{-1}D(x)P$. Calculate $A(0)^8 + A(1)^9$.

Solution

The eigenvalues λ satisfy the quadratic equation $\lambda^2 + (1-2x)\lambda + x^2 - x = 0$. This has solutions $\lambda_1 = x$ and $\lambda_2 = x - 1$.

Solving $A(x)\mathbf{y} = x\mathbf{y}$ yields the eigenvector $\mathbf{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Solving $A(x)\mathbf{y} = (x-1)\mathbf{y}$ yields the eigenvector $\mathbf{y} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Hence $D(x) = \begin{pmatrix} x & 0 \\ 0 & x-1 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$.

Now

$$\begin{aligned}
A(0)^8 + A(1)^9 &= P^{-1}D(0)^8P + P^{-1}D(1)^9P \\
&= P^{-1}(D(0)^8 + D(1)^9)P \\
&= P^{-1}\left(\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}^8 + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^9\right)P \\
&= P^{-1}IP \\
&= I.
\end{aligned}$$