University of Glasgow

EXAMINATION FOR THE DEGREES OF
M.A. AND B.Sc.

Mathematics 2A — Multivariable Calculus

An electronic calculator may be used provided that it is allowed under the School of Mathematics and Statistics Calculator Policy. A copy of this policy has been distributed to the class prior to the exam and is also available via the invigilator.

Candidates must attempt all questions.

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1. Let $f(x,y) = \tan^{-1}\left(\dfrac{y}{x}\right)$, $x \neq 0$. Compute the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ and show that
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0\,.$$
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2. If $f(x,y) = F(v(x,y), w(x,y))$, write down the chain rule for the calculation of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.
Find the general solution $f(x,y)$ of the partial differential equation
$$x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} = 2x^2$$
by using the change of variables $v = xy$ and $w = \dfrac{x}{y}$. **8**

3. Let $\mathbf{c} \in \mathbb{R}^3$ be a constant vector and $\mathbf{r} = (x, y, z)$. Compute
$$\nabla(\mathbf{c} \cdot \mathbf{r}) \qquad \text{and} \qquad \operatorname{curl}(\mathbf{c} \times \mathbf{r})\,.$$
Using the identity
$$\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \operatorname{curl}(\mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot \operatorname{curl}(\mathbf{G})$$
show that
$$\operatorname{div}\left((\mathbf{c} \times \mathbf{r}) \times \nabla(\mathbf{c} \cdot \mathbf{r})\right) \geqslant 0\,.$$
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4. Sketch the region of integration for
$$\int_0^1 \left(\int_{\sqrt{x}}^1 e^{x/y}\, dy\right)\, dx.$$
Rewrite the integral by changing the order of integration, hence evaluate the integral. **8**

5. By making a change of variable, compute
$$\iint_D xy^3\, dx dy\,,$$
where D is the region that lies between the curves $xy = 1$, $xy = e$, $xy^2 = 1$ and $xy^2 = 2$. **8**

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6. Compute the integral

$$\iint_S y^2 dS\,,$$

where S is the part of the sphere $x^2+y^2+z^2=4$ with $x^2+y^2 \leqslant 1$ and above the xy-plane.
HINT: It may be useful to use the parametrisation of a sphere of radius R given by

$$\mathbf{r}(\theta,\phi) = (R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi)$$

and to find appropriate limits for the variables θ and ϕ. **8**

7. State and use Green's theorem to compute

$$\oint (2019 + 2x^2 y \sinh x^2)\,dx + (x\cosh x^2 + \tan^{-1} y)\,dy\,,$$

where the line integral is taken along the path consisting of the line joining $(0,1)$ to $(1,0)$, followed by the line joining $(1,0)$ to $(1,2)$, followed by the line joining $(1,2)$ to $(0,1)$. **8**

8. State and use the divergence theorem to compute

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS\,,$$

where $\mathbf{F} = (xz\sin(yz), yz + \cos(yz), x^y + y^x)$ and S is the surface of the solid bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$. **8**

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