2C Intro to real analysis 2020/21

Ex Sheet **2**

Feedback and solutions

Q1 *Let A, B be subsets of $\mathbb{R}$ such that $\sup(A)$ and $\inf(B)$ exist and $\sup(A) < 0$. Define*

$$C = \left\{ \frac{1}{a} + b \mid a \in A, b \in B \right\}.$$

Explain why this makes sense (i.e. why $a \neq 0$ for every $a \in A$), and prove that $\inf(C)$ exists.

The quickest way of answering the question is to use the completeness axiom. More precisely, let us show that C is non-empty and bounded below, then $\inf(C)$ will exist by the completeness axiom[1].

To prove that C is bounded below, we should construct a lower bound for C from bounds for A and B. I'll do this in a way that can be used in my second answer below, by showing that $\frac{1}{\sup(A)} + \inf(B)$ is a lower bound[2] for C. Take care with manipulation of negatives in inequalities[3].

> For $a \in A$ we have $a \leq \sup(A) < 0$, so that $\frac{1}{a}$ is always defined for $a \in A$ and $\frac{1}{a} \geq \frac{1}{\sup(A)}$. For $b \in B$, we have $b \geq \inf(B)$. Adding these last two inequalities, gives $\frac{1}{a} + b \geq \frac{1}{\sup(A)} + \inf(B)$ for $a \in A$ and $b \in B$. Therefore $\frac{1}{\sup(A)} + \inf(B)$ is a lower bound for C.

To complete the argument using the completeness axiom, we need to show that C is non-empty, which will follow once we know that A and B are non-empty. The key point is that the empty set does not have a supremum or infimum, so since $\inf(B)$ exists, B is necessarily non-empty, and similarly as $\sup(A) < 0$, there exists[4] $a \in A$ with $a < 0$. Now let's complete our answer.

> Since $\inf(B)$ exists, the set B is non-empty, so take $b \in B$. Similarly as $\sup(A) < 0$, there exists $a \in A$ with $a < 0$. Then $\frac{1}{a} + b \in C$, so C is non-empty. By the completeness axiom, $\inf(C)$ exists.

[1] One other advantage of doing this here is that it allows me to address some queries about sups and infs of the empty set.

[2] The same argument shows that $\frac{1}{M} + m$ is a lower bound for C, provided $M < 0$ is an upper bound for A and m is a lower bound for B.

[3] $a < b < 0$ implies $-a > -b > 0$ implies $0 < -1/a < -1/b$ implies $0 > 1/a > 1/b$

[4] Let's see why. Let $D \subset \mathbb{R}$ be a set. Recall that $M \in \mathbb{R}$ is an upper bound for D if and only if for all $d \in D$ we have $d \leq M$. It follows that *every* real number $M \in \mathbb{R}$ is an upper bound for the empty set $\emptyset$. Indeed, as there are no elements in the empty set, the statement "$\forall d \in \emptyset, d \leq M$" is true (since there are no values of d to check). Since every real number is an upper bound, it follows that $\emptyset$ cannot have a least upper bound (given any upper bound M, we have that $M - 1 < M$, and $M - 1$ is also an upper bound). Therefore the empty set has no supremum. Likewise it has no infimum. (Some authors write $\sup(\emptyset) = -\infty$ as a notational way of saying this, but we will not adopt this convention). A similar argument shows that as $\sup(A) < 0$, the set A is non-empty. I wouldn't expect you to include all the above as part of an answer using the completeness axiom, but I would expect you to give some indication of why C is non-empty.

Alternatively we could find an explicit form for $\inf(C)$, by proving that $\frac{1}{\sup(A)} + \inf(B)$ is the greatest lower bound[5] for C. To do this, we need to show that $\frac{1}{\sup(A)} + \inf(B)$ is a lower bound for C, and then that it is the greatest lower bound. I've already shown that it's a lower bound in the first box above, so let's concentrate on the last part. I'll do this twice: firstly directly, and then secondly in two steps[6].

[5] Can you see why we expect this to be $\inf(C)$? If not, ask yourself how you can make a number of the form $\frac{1}{a} + b$ small. To do this, you'd want to make $b \in B$ small, and then take $a \in A$ with $a < 0$ large (i.e not very negative), so that $\frac{1}{a}$ is small (i.e very negative). This leads us to predict that $\inf(C) = \frac{1}{\sup(A)} + \inf(B)$.

[6] A standard error is to manipulate fractions in a wishful thinking kind of way. It is perhaps tempting to take $a \in A$ with $a > \sup(A) - \frac{2}{\varepsilon}$, and then claim that this a would satisfy $\frac{1}{a} < \frac{1}{\sup(A)} - \frac{\varepsilon}{2}$. This is not the case:

$$\frac{1}{\sup(A) - \frac{2}{\varepsilon}} \neq \frac{1}{\sup(A)} - \frac{\varepsilon}{2},$$

remember that in general $\frac{1}{b+c} \neq \frac{1}{b} + \frac{1}{c}$. Instead, if you want to find $a \in A$ with $\frac{1}{a} < \frac{1}{\sup(A)} + \frac{\varepsilon}{2}$, rearrange this inequality, and find out what condition a needs to satisfy; this is what leads me to the answer on the left.

Let $\varepsilon > 0$ satisfy $\varepsilon < -\frac{2}{\sup(A)}$. As $\sup(A) < 0$ and $0 < 2 + \varepsilon \sup(A) < 2$ note that

$$\frac{2\sup(A)}{2 + \varepsilon \sup(A)} < \sup(A).$$

Therefore we can take $a \in A$ with $a > \frac{2\sup(A)}{2+\varepsilon\sup(A)}$ and $b \in B$ with $b < \inf(B) + \frac{\varepsilon}{2}$. Then

$$\begin{aligned}\frac{1}{a} + b &< \left(\frac{2 + \varepsilon\sup(A)}{2\sup(A)}\right) + \left(\inf(B) + \frac{\varepsilon}{2}\right)\\ &= \left(\frac{1}{\sup(A)} + \frac{\varepsilon}{2}\right) + \left(\inf(B) + \frac{\varepsilon}{2}\right)\\ &= \frac{1}{\sup(A)} + \inf(B) + \varepsilon.\end{aligned}$$

Thus there exists $c \in C$ with $c < \frac{1}{\sup(A)} + \inf(B) + \varepsilon$. It follows that for any $\varepsilon > 0$ there exists $c \in C$ with $c < \frac{1}{\sup(A)} + \inf(B) + \varepsilon$ (note that if the condition $\exists c \in C$ with $c < \frac{1}{\sup(A)} + \inf(B) + \varepsilon$ holds for all ε with $0 < \varepsilon < -\frac{2}{\sup(A)}$, then it holds for all $\varepsilon > 0$, as given c with $c < \frac{1}{\sup(A)} + \inf(B) + \varepsilon$, then $c < \frac{1}{\sup(A)} + \inf(B) + \varepsilon'$ for all $\varepsilon' > \varepsilon$). As such $\frac{1}{\sup(A)} + \inf(B)$ is the greatest lower bound for C.

The key point in the argument above is that I knew I wanted to arrange to find $a \in A$ with

$$\frac{1}{a} < \frac{1}{\sup(A)} + \frac{\varepsilon}{2}.$$

Therefore I started my rough work, which I'm not showing you this time, by rearranging this inequality to find what condition on a I needed in order to ensure this. Then I needed to justify, using the definition of the supremum of A, why I could achieve this condition. I also noted that I needed to multiply inequalities, so it was necessary to take care with signs of terms to ensure the directions of the inequalities were correct.

An alternative approach would be to proceed in steps, and define a new set $D = \{\frac{1}{a} \mid a \in A\}$, prove that $\inf(D) = \frac{1}{\sup(A)}$, and use this to show that $\inf C = \frac{1}{\sup(A)} + \inf(B)$. Here are the details, divided into the two parts.

Define $D = \{\frac{1}{a} \mid a \in A\}$. For $a \in A$, we have $a \leq \sup(A) < 0$, so that $\frac{1}{a} \geq \frac{1}{\sup(A)}$. Therefore $\frac{1}{\sup(A)}$ is a lower bound for D. Note that as A is non-empty, it must contain some element $a \in A$ with $a \leq \sup(A) < 0$. Therefore $\frac{1}{a} \in D$, and any lower bound for D, say m, must have $m < 0$. Given a lower bound $m < 0$ for D, we have that $\frac{1}{a} \geq m$ for all $a \in A$. Therefore $\frac{1}{m} \geq a$ for all $a \in A$, so that $\frac{1}{m} \geq \sup(A)$. Therefore $\frac{1}{\sup(A)} \geq m$ for all lower bounds m of D. This shows that $\inf(D) = \frac{1}{\sup(A)}$.

Now $C = \{b + d \mid b \in B, d \in D\}$. For $b \in B$ and $d \in D$, we have $b \geq \inf(B)$ and $d \geq \inf(D)$, so that $b + d \geq \inf(B) + \inf(D)$. Therefore $\inf(B) + \inf(D)$ is a lower bound for C. Given $\varepsilon > 0$, there exist $b \in B$ and $d \in D$ with $b < \inf(B) + \frac{\varepsilon}{2}$ and $d < \inf(D) + \frac{\varepsilon}{2}$. Therefore $b + d < \inf(B) + \inf(D) + \varepsilon$, and $\inf(B) + \inf(D)$ is the greatest lower bound for C. Combining this with the previous step, we have $\inf(C) = \frac{1}{\sup(A)} + \inf(B)$.

Q2 *Show* directly from the definition[7] *that*

$$\lim_{n\to\infty} \frac{4n^4 + 5n^3 + 1}{2n^4 - n^2 + 3} = 2.$$

[7] you may use Lemma 1.9 if it helps.

This question is pretty similar to one I did in lectures, so I'll get stuck right in with an answer[8].

[8] Remember what you're trying to show, namely: for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that for $n \geq n_0$, we have $\left|\frac{4n^4+5n^3+1}{2n^4-n^2+3} - 2\right| < \varepsilon$. You must start by introducing the arbitrary value of ε before you use it.

Let $\varepsilon > 0$ be arbitrary. For $n \in \mathbb{N}$, we have

$$\frac{4n^4 + 5n^3 + 1}{2n^4 - n^2 + 3} - 2 = \frac{5n^3 + 2n^2 - 5}{2n^4 - n^2 + 3}.$$

By Lemma 1.9, there exists $n_1, n_2 \in \mathbb{N}$, such that

$$n \geq n_1 \implies \frac{1}{2}5n^3 \leq 5n^3 + 2n^2 - 5 \leq \frac{3}{2}5n^3$$
$$n \geq n_2 \implies \frac{1}{2}2n^4 \leq 2n^4 - n^2 + 3 \leq \frac{3}{2}2n^4.$$

For $n \geq \max(n_1, n_2)$, we have

$$\left|\frac{4n^4 + 5n^3 + 1}{2n^4 - n^2 + 3} - 2\right| = \left|\frac{5n^3 + 2n^2 - 5}{2n^4 - n^2 + 3}\right|$$
$$= \frac{5n^3 + 2n^2 - 5}{2n^4 - n^2 + 3} \leq \frac{\frac{3}{2}5n^3}{\frac{1}{2}2n^4} = \frac{15}{2n}.$$

Therefore take $n_0 > \max(n_1, n_2, \frac{15}{2\varepsilon})$, then for $n \geq n_0$ we have

$$\left|\frac{4n^4 + 5n^3 + 1}{2n^4 - n^2 + 3} - 2\right| < \varepsilon,$$

and so

$$\lim_{n\to\infty} \frac{4n^4 + 5n^3 + 1}{2n^4 - n^2 + 3} = 2.$$

Q3 *In this question we use n-th roots. For $x > 0$ and $n \in \mathbb{N}$, the quantity $x^{1/n}$ is defined to be the unique positive real number*[9] *which has $(x^{1/n})^n = x$. In this question you may use the fact that if $0 < x \le y$ and $n \in \mathbb{N}$, then $x^{1/n} \le y^{1/n}$.*

a) For $n \in \mathbb{N}$ use the binomial expansion for $\left(1+\frac{2}{n}\right)^n$ to show that $\left(1+\frac{2}{n}\right)^n \ge 3$, and deduce that $1 \le 3^{1/n} \le 1+\frac{2}{n}$.

b) Use the previous part[10] *to show,* directly from the definition, *that $\lim_{n\to\infty} 3^{1/n} = 1$.*

[9] We can argue in a similar fashion to Theorem 2.13 to show that such a real number exists (though I do not expect you to do this in your answer). To see that $x^{1/n}$ is unique (which I also do not expect you to do as part of your answer) note that if $y, z > 0$ have $y^n = z^n$, then $0 = y^n - z^n = (y-z)(y^{n-1} + y^{n-2}z + y^{n-3}z^2 + \cdots + yz^{n-2} + z^{n-1})$. Since $(y^{n-1} + y^{n-2}z + y^{n-3}z^2 + \cdots + yz^{n-2} + z^{n-1}) > 0$, we must have $y = z$.

[10] As with any question which is divided into parts, this question is designed so that if you can't do part a), you should still be able to do part b).

For the first part you probably want to experiment a bit, to try and find out why something like this is true[11]. Let's look at the first few cases. When $n = 1$, we have $(1+2)^1 = 3 \ge 3$. For $n = 2$, we have

$$\left(1+\frac{2}{2}\right)^2 = 1 + 2\frac{2}{2} + \frac{4}{4} \ge 3.$$

For $n = 3$, we have

$$\left(1+\frac{2}{3}\right)^3 = 1 + 3\frac{2}{3} + 3\frac{4}{9} + \frac{8}{27} \ge 3.$$

[11] This time, there hasn't been a similar example in lectures. This is deliberate, as one of the objectives of this course is for you to be able to produce your own mathematical arguments.

In each case, you can see that the first two terms of the binomial expansion sum to 3, and as the other terms are positive, this gives the result below[12].

[12] It's easier to see this if you write it out as above, compared with writing $\left(1+\frac{2}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{2^k}{n^k}$.

a) For $n \in \mathbb{N}$, using the binomial expansion, we have

$$\begin{aligned}\left(1+\frac{2}{n}\right)^n &= 1 + n\frac{2}{n} + \binom{n}{2}\frac{4}{n^2} + \binom{n}{3}\frac{8}{n^3} + \cdots + \frac{2^n}{n^n} \\ &\ge 1 + 2 + 0 + \cdots + 0 = 3.\end{aligned}$$

For $n \in \mathbb{N}$, we have $1 \le 3^{1/n}$. Taking n-th roots of the inequality above gives $3^{1/n} \le 1 + \frac{2}{n}$. Therefore

$$1 \le 3^{1/n} \le 1 + \frac{2}{n},$$

for all $n \in \mathbb{N}$.

Now we want to use this result to prove that $3^{1/n} \to 1$ directly from the definition[13]. We will let $\varepsilon > 0$ be arbitrary, and we need to find n_0 such that for $n \ge n_0$ we have $|3^{1/n} - 1| < \varepsilon$. What does the first part of the question tell us about $|3^{1/n} - 1|$?

[13] If you get stuck, the first thing to do is to write down the definition of convergence, and be sure exactly what you have to prove.

b) Let $\varepsilon > 0$ be arbitrary. For $n \in \mathbb{N}$, since

$$1 \le 3^{1/n} \le 1 + \frac{2}{n},$$

we have

$$|3^{1/n} - 1| \le \frac{2}{n}.$$

Take $n_0 \in \mathbb{N}$, with $n_0 > \frac{2}{\varepsilon}$. Then, for $n \geq n_0$, we have

$$|3^{1/n} - 1| \leq \frac{2}{n} \leq \frac{2}{n_0} < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $3^{1/n} \to 1$ as $n \to \infty$.