# 2C Week 4 2020/21

Handout

## Sequences

This chapter is devoted to *sequences*. We will start our study of the most central concept in analysis in this context, namely that of *limits*.

A real sequence is an infinite list of real numbers, like

$$1, 1/2, 1/3, 1/4, \ldots.$$

To make this precise we give the following formal definition.

**Definition 3.1.** A *real sequence* is a function $\mathbb{N} \to \mathbb{R}$.

According to this definition[1], we may denote the elements of a real sequence by $a(1), a(2), \ldots$, corresponding to the values of the function $a : \mathbb{N} \to \mathbb{R}$. However, usually we prefer a slightly different notation. We will write $a_n$ for the $n$-th term of a sequence $a$, rather than $a(n)$, and write[2] "let $(a_n)_{n=1}^{\infty}$ be a real sequence". It is equally possible to have sequences which start at other integers. For example, a sequence starting at $n = 0$ will be denoted $(a_n)_{n=0}^{\infty}$, and such a sequence[3] corresponds to a function $\{0, 1, 2, \ldots\} \to \mathbb{R}$. Most often in what follows sequences will be specified by giving a formula describing the $n$-th term, for example: "consider the sequence $(a_n)_{n=1}^{\infty}$ given by $a_n = (-1)^n$ for $n \in \mathbb{N}$." However, it would be a mistake to think that the $n$-th term of a sequence is always given by a nice formula: a sequence is just a list of real numbers, there is no reason why these terms must fit into a nice pattern[4].

Note that the definitions of boundedness (bounded above, and bounded below) from the previous chapter apply to sequences. More precisely, a real sequence $(a_n)_{n=1}^{\infty}$ is *bounded above* if and only if there exists $M \in\ R$ such that for all $n \in \mathbb{N}$, we have $a_n \leq M$. The sequence is *bounded below* if and only if there exists $m \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, we have $m \leq a_n$.

### Limits of sequences

The definition of convergence below is perhaps the most important definition in this course. I expect you to know this definition and to be able to work with it.

**Definition 3.2.** Let $(a_n)_{n=1}^{\infty}$ be a real sequence and let $L \in \mathbb{R}$. We say that $a_n$ *converges to* $L$ as $n$ tends to infinity if and only if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that for $n \in \mathbb{N}$ with $n \geq n_0$, we have $|a_n - L| < \varepsilon$. When $a_n$ converges to $L$ as $n$ tends to infinity, we write[5] "$a_n \to L$ as $n \to \infty$" or $\lim_{n\to\infty} a_n = L$.

We say that the sequence $(a_n)_{n=1}^{\infty}$ *diverges* if it does not converge to any limit.

[1] In this course I am defining the natural numbers to be the set $\{1, 2, \ldots, \}$, i.e. the smallest natural number is 1. Some authors consider 0 as the smallest natural number, but we will not do this.

[2] Note that [ERA], and some other books, use the notation $\{a_n\}_{n=1}^{\infty}$ for a sequence. The downside of this notation is that it leads to potential confusion between sequences and sets. For this reason, I will stick to using round brackets for sequences.

[3] Strictly speaking, this doesn't satisfy the definition of a sequence we've just given - but we'll follow common practice and allow for a wider use of the term *sequence*, including this variant.

[4] For instance, the sequence defined by

$$a_n = \begin{cases} e, & n = 1 \\ \pi, & n = 2 \\ 0, & n = 3, \\ -1, & n \geq 4 \end{cases}$$

is a perfectly valid sequence.

[5] The symbol $\to$ is reserved for this use, and also in specifying the domain and codomain of a function (for example, "$f : X \to Y$"). Please don't use it for any other purposes.

As we will discuss in more detail below, this definition encapsulates our intuitive notion that a sequence $(a_n)_{n=1}^{\infty}$ converges to the limit $L$ if and only if the values of the sequence get closer and closer to $L$ as $n$ gets large. The important point here is that the above definition formalises this is in a precise mathematical way that doesn't use phrases like "closer and closer to $L$", which we couldn't use in a proof.

The definition can be written concisely as a quantified statement: the sequence $(a_n)_{n=1}^{\infty}$ converges to $L \in \mathbb{R}$ if and only if

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, (n \geq n_0 \implies |a_n - L| < \varepsilon).$$

An alternative form is

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall (n \in \mathbb{N} \text{ with } n \geq n_0), |a_n - L| < \varepsilon.$$

You will also often see the $n \in \mathbb{N}$ suppressed in the literature, and in lectures. In these statements, $n$ must be a natural number as it is used as a subscript on $a_n$ to indicate which term of the sequence we are taking, so in using this notation we are implicitly saying that $n$ must be a natural number. Therefore you will see the definition of convergence written as

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \geq n_0, |a_n - L| < \varepsilon,$$

or as[6]

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } n \geq n_0 \implies |a_n - L| < \varepsilon.$$

## How to interpret the definition of convergence?

Let's try and understand how Definition 3.2 works[7]. In general, when I'm trying to understand a quantified statement, I start at the end and work backwards, so here I initially focus on the condition $|a_n - L| < \varepsilon$, and make sure I understand this. Remember that we view the quantity $|a_n - L|$ as the distance from $a_n$ to $L$. Thus the last part "$|a_n - L| < \varepsilon$" of the definition says that "$a_n$ is within distance $\varepsilon$ of $L$"; in this statement $n$ and $\varepsilon$ are variables. The role $\varepsilon$ plays in this definition is that of an allowed error tolerance, we are asking when is it true that the distance between $a_n$ and $L$ lies within this error tolerance.

Working back with the quantified statement, we next look at the statement "$\forall(n \in \mathbb{N}$ with $n \geq n_0), |a_n - L| < \varepsilon$" (or equivalently "$\forall n \in \mathbb{N}, n \geq n_0 \implies |a_n - L| < \varepsilon$") which is a statement with variables $\varepsilon$ and $n_0$. This statement asks that for all $x_n$ for $n \geq n_0$ lie within the error tolerance $\varepsilon$ of $L$, and finally that $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, n \geq n_0 \implies |a_n - L| < \varepsilon$, which asks that there is some index $n_0$, such that past this point in the sequence, the $a_n$ lies within the error tolerance of $L$. We should think of $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, n \geq n_0 \implies |a_n - L| < \varepsilon$ as meaning that "$|a_n - L| < \varepsilon$ holds *eventually*": for all sufficiently large $n$ it's true that $|a_n - L| < \varepsilon$.

With $\varepsilon > 0$ fixed, another way to think about the condition $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, n \geq n_0 \implies |a_n - L| < \varepsilon$ is to imagine that we

[6] I also want to discuss the effects of using $<$ and $\leq$ in this definition, as this is a regular source of confusion. It's possible to change the condition $|a_n - L| < \varepsilon$ to $|a_n - L| \leq \varepsilon$, and also change $n \geq n_0$ to $n > n_0$, without changing the meaning of the definition; making these changes doesn't affect the intuitive interpretation of the definition, and we can in fact give formal proofs that making these changes leads to equivalent statements.

For example, the statement that $a_n \to L$ as $n \to \infty$ is equivalent to

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \geq n_0, |a_n - L| \leq \varepsilon. \tag{1}$$

It is immediate that if $a_n \to L$, then (1) holds (given $\varepsilon > 0$, take $n_0$ such that $n \geq n_0 \implies |a_n - L| < \varepsilon$, so that $n \geq n_0 \implies |a_n - L| \leq \varepsilon$). Conversely, suppose (1) holds, then given $\varepsilon > 0$, note that $0 < \frac{\varepsilon}{2} < \varepsilon$. Taking $\frac{\varepsilon}{2}$ in (1), gives $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, we have $|a_n - L| \leq \frac{\varepsilon}{2} < \varepsilon$, proving that $a_n \to L$ as $n \to \infty$.

In a similar vein, $a_n \to L$ as $n \to \infty$ is equivalent to

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n > n_0, |a_n - L| < \varepsilon.$$

Can you see how to give a formal proof of this equivalence?

[7] It is really important to get to grips with this definition, and to be able to state it without having to look it up.

display the sequence on a computer or graphical calculator, with $\varepsilon$ representing the size of one pixel on the screen, so that if two numbers differ by at most $\varepsilon$, then they are indistinguishable on the screen. From this view point, the statement $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0, |a_n - L| < \varepsilon$ says that there is some value $n_0$ such that from this point onwards, you can't distinguish the values $a_n$ and $L$ on the screen. Of course, one could change the resolution and zoom in: this corresponds to decreasing the value of $\varepsilon$, and convergence of the sequence means that no matter what the pixel size, if you look far enough along the sequence it is impossible to distinguish between values of the sequence and the limit.

## Proving convergence *directly from the definition*

You will sometimes be asked to prove that certain sequences converge *directly from the definition*. This means that you must verify the condition in definition 3.2 directly, without using subsequent results. Results from earlier in the course, like the polynomial estimation lemma 1.9, can be used in such questions. As we have already seen, the definition can be written in the form of a quantified statement, with three quantifiers, so the general methods we discussed for proving such quantified statements in Chapter 1 apply.

When you are asked to prove $a_n \to L$ directly from the definition, you should start by introducing an arbitrary value of $\varepsilon > 0$ to work with by saying "Let $\varepsilon > 0$ be arbitrary." To complete the proof you need to give a value of $n_0$ for which $n \geq n_0 \implies |a_n - L| < \varepsilon$. For this it is best to start by examining the expression $|x_n - L| < \varepsilon$, and to try to simplify it. You may be able to rearrange the inequality $|a_n - L| < \varepsilon$ in the form $n > \dots$, giving you a value of $n_0$. However, it may not be that straightforward to rearrange $|a_n - L| < \varepsilon$ exactly for $n$. Fortunately we can get away with less: it suffices to find *some* value of $n_0$ such that $|a_n - L| < \varepsilon$ when $n \geq n_0$ — you do not need to find the least value of $n_0$ with this property. Note that when verifying the definition of convergence, the value of $n_0$ is allowed to depend on $\varepsilon$, and examples below will show that one should expect it to do so. When $\varepsilon$ gets made smaller, we will typically have to use a larger value of $n_0$, corresponding to looking further down the sequence, in order to verify the condition $n \geq n_0 \implies |a_n - L| < \varepsilon$.

There are many ways you can write a solution, but it is usually best to do some rough work first to examine $|a_n - L|$ and find how large $n$ needs to be to ensure that $|a_n - L| < \varepsilon$, and then start to write down the answer.

Let's start with two simple examples. The first one is really a warm-up exercise.

**Example 3.3.** Let $K \in \mathbb{R}$, and define $x_n = K$ for all $n \in \mathbb{N}$. Show that $x_n \to K$ as $n \to \infty$, directly from the definition.

*Solution.* As explained above, the first step in proving convergence directly from the definition is to give ourselves an arbitrary $\varepsilon > 0$. Next we have to find $n_0 \in \mathbb{N}$ such that $|x_n - K| \leq \varepsilon$ for all $n \geq n_0$.

In this example this is very easy: Taking $n_0 = 1$, for $n \in \mathbb{N}$ with $n \geq n_0$, we have $|x_n - K| = |K - K| = 0 < \varepsilon$. This means $x_n \to K$ as $n \to \infty$. $\square$

Sequences of the form $x_n = K$ for all $n \in \mathbb{N}$ as in the previous example are also called *constant sequences*. Thus, we have proved that constant sequences converge.

**Example 3.4.** Let $(x_n)_{n=1}^{\infty}$ be given by

$$x_n = \frac{1}{n}.$$

Show that $x_n \to 0$ as $n \to \infty$, directly from the definition.

*Solution.* Again, the first step is to let $\varepsilon > 0$ be arbitrary. Our task it to find $n_0 \in \mathbb{N}$ such that $|x_n - 0| = \frac{1}{n} < \epsilon$ for all $n \geq n_0$. Now we have

$$\frac{1}{n} < \varepsilon \iff \frac{1}{\varepsilon} < n$$

for $n \in \mathbb{N}$, so if we take[8] $n_0 \in \mathbb{N}$ such that $n_0 > \frac{1}{\varepsilon}$, then we get $|x_n - 0| < \epsilon$ for all $n \geq n_0$. That is, we have shown $x_n \to 0$ as $n \to \infty$. $\square$

[8] It's worth being clear that here, and elsewhere where we will use expressions like "Take $n_0 > \frac{14}{\varepsilon}$", this is justified by Archimedes' axiom, which we proved in Section 2 as a consequence of the completeness axiom. Indeed, we can find $n_0$ as claimed from the fact that the natural numbers are not bounded above. More precisely, for $\varepsilon > 0$, the expression $\frac{1}{\varepsilon}$ is not an upper bound for $\mathbb{N}$, and so there exists $n_0 \in \mathbb{N}$ with $n_0 > \frac{1}{\varepsilon}$. I don't expect you to justify this step in this way in your answers: you may assume that such an $n_0$ can be found without further comment, but it's worth being aware of what's going on here.

This example is good to keep in mind, it displays very clearly the idea behind the notion of convergence. Intuitively, the numbers $x_n = \frac{1}{n}$ get closer and closer to 0 as $n$ gets large, so that we should expect the sequence $(x_n)_{n=1}^{\infty}$ to converge to 0. Indeed, that's exactly what we have just shown.

Let's have a look at more examples.

**Example 3.5.** Let $(x_n)_{n=1}^{\infty}$ be given by

$$x_n = \frac{3n-2}{n+4}.$$

Show that $x_n \to 3$ as $n \to \infty$, directly from the definition.

*Solution.* Let $\varepsilon > 0$. For $n \in \mathbb{N}$, we have[9]

$$\left|\frac{3n-2}{n+4} - 3\right| = \left|\frac{3n-2-3(n+4)}{n+4}\right| = \frac{14}{n+4} < \frac{14}{n} < \varepsilon, \qquad (2)$$

provided $n > \frac{14}{\varepsilon}$. Take $n_0 \in \mathbb{N}$ with $n_0 \geq \frac{14}{\varepsilon}$. Then for $n \in \mathbb{N}$ with $n \geq n_0$, we have $|x_n - 3| < \varepsilon$, so that $x_n \to 3$ as $n \to \infty$. $\square$

[9] Note that in equation (2), the last inequalities $\frac{14}{n} < \varepsilon$ is not always true: this only holds under the condition that $n > \frac{14}{\varepsilon}$, hence the sentence continues with the condition "provided $n > \frac{14}{\varepsilon}$".

The step $\frac{14}{n+4} < \frac{14}{n}$ is not necessary in this answer; it's included as it is often a good idea to estimate complicated expressions in $n$, by simpler ones which you can still make small. Instead you may want to give a solution starting: "Let $\varepsilon > 0$ be arbitrary. Then for $n \in \mathbb{N}$, we have

$$\left|\frac{3n-2}{n+4} - 3\right| = \left|\frac{3n-2-3(n+4)}{n+4}\right| = \frac{14}{n+4} < \varepsilon,$$

provided $n > \frac{14}{\varepsilon} - 4$. Take $n_0 \in \mathbb{N}$ with $n_0 > \frac{14}{\varepsilon} - 4 \ldots$".

As another alternative, you could have done the calculation $|x_n - L| < \varepsilon \iff \frac{14}{\varepsilon} - 4 < n$ as rough work, and then started your solution with: "Let $\varepsilon > 0$ be arbitrary. Take $n_0 \in \mathbb{N}$ with $n_0 > \frac{14}{\varepsilon} - 4$, and $n \in \mathbb{N}$ with $n \geq n_0$. Then ...".

For more complicated expressions, it may be convenient to use Lemma 1.9 to get the required estimate on $n$.

**Example 3.6.** Let $(y_n)_{n=1}^{\infty}$ be the sequence given by

$$y_n = \frac{3n^3 + 4n - 5}{5n^3 - 3n^2 + 4}$$

Show that $y_n \to \frac{3}{5}$ as $n \to \infty$.

*Solution.* Let $\varepsilon > 0$ be arbitrary. For $n \in \mathbb{N}$, we have

$$\left| \frac{3n^3 + 4n - 5}{5n^3 - 3n^2 + 4} - \frac{3}{5} \right| = \left| \frac{9n^2 + 20n - 37}{5(5n^3 - 3n^2 + 4)} \right|.$$

By[10] Lemma 1.9 there exist $n_1, n_2 \in \mathbb{N}$ such that

$$n \geq n_1 \implies \frac{1}{2} 9n^2 \leq 9n^2 + 20n - 37 \leq \frac{3}{2} 9n^2$$
$$n \geq n_2 \implies \frac{1}{2} 25n^3 \leq 5(5n^3 - 3n^2 + 4) \leq \frac{3}{2} 25n^3.$$

In particular, when $n \geq \max(n_1, n_2)$, we have

$$\left| \frac{9n^2 + 20n - 37}{5(5n^3 - 3n^2 + 4)} \right| = \frac{9n^2 + 20n - 37}{5(5n^3 - 3n^2 + 4)} \leq \frac{\frac{3}{2} 9n^2}{\frac{1}{2} 25n^3} = \frac{27}{25n} < \varepsilon,$$

provided $n$ also satisfies $n > \frac{27}{25\varepsilon}$. Therefore[11] take $n_0 \in \mathbb{N}$ with $n_0 > \max(n_1, n_2, \frac{27}{25\varepsilon})$. For $n \in \mathbb{N}$ with $n \geq n_0$, we have $\left| y_n - \frac{3}{5} \right| < \varepsilon$, so $y_n \to \frac{3}{5}$ as $n \to \infty$. $\square$

You don't need to use Lemma 1.9 to answer this question, another approach would be to estimate

$$\left| \frac{9n^2 + 20n - 37}{5(5n^3 - 3n^2 + 4)} \right|$$

directly[12].

One may ask "why doesn't this work when you change $\frac{3}{5}$ to some other number?", i.e. why can't we use Lemma 1.9 to prove that $y_n$ converges to any real number we like? This is a very good question, and exactly the sort of question you should be asking yourself when you read an argument in order to make sure you really understand what's going on. The best way to answer this question is to try and see what happens. Here if we replace $\frac{3}{5}$ by, say, 1, and perform the initial calculation, we get

$$\left| \frac{3n^3 + 4n - 5}{5n^3 - 3n^2 + 4} - 1 \right| = \left| \frac{2n^3 - 3n^2 - 4n + 9}{5n^3 - 3n^2 + 4} \right|.$$

Now when we use Lemma 1.9, for sufficiently large $n \in \mathbb{N}$ we have

$$\left| \frac{3n^3 + 4n - 5}{5n^3 - 3n^2 + 4} - 1 \right| \leq \frac{\frac{3}{2} 2n^3}{\frac{1}{2} 5n^3} = \frac{6}{5}$$

[10] In your answers to the feedback exercises I expect you to say "By Lemma 1.9" or "By the polynomial estimation lemma" if you use it, however in the exam I do not expect you to know the numbers of results in the course, so when you use a non-named result say something like "by a lemma from lectures". If a result you want to use has a name, such as the monotone convergence theorem below, you should use that name.

[11] This is the "max trick" we saw earlier. We take $n_0 > \max(n_1, n_2, \frac{27}{25\varepsilon})$ as in order to conclude that $|y_n - \frac{3}{5}| < \varepsilon$, we need three conditions to hold: $n \geq n_1, n \geq n_2$ and $n > \frac{27}{25\varepsilon}$. We must package these three conditions in the form $n \geq n_0$ for some suitable $n_0$, so the easiest option is to take $n_0 > \max(n_1, n_2, \frac{27}{25\varepsilon})$.

[12] For example, I note that for $n \geq 2$, we have $9n^2 + 20n - 37 > 0$ so $|9n^2 + 20n - 37| = 9n^2 + 20n - 37$, and then $9n^2 + 20n - 37 \leq 9n^2 + 20n^2 = 29n^2$. Similarly, we have $5n^3 - 3n^3 + 4 > 0$ for all $n \in \mathbb{N}$, so $|5(5n^3 - 3n^2 + 4)| = 5(5n^3 - 3n^2 + 4)$. Then $5(5n^3 - 3n^2 + 4) \geq 5(5n^3 - 3n^3) = 10n^3$. Therefore, for $n \geq 2$, we have

$$\left| \frac{9n^2 + 20n - 37}{5(5n^3 - 3n^2 + 4)} \right| \leq \frac{29n^2}{10n^3} = \frac{29}{10n}.$$

Then if $\varepsilon > 0$ is specified, we can take $n_0 \in \mathbb{N}$ with $n_0 > \max(2, \frac{29}{10\varepsilon})$ in order to verify that $y_n \to \frac{3}{5}$ as $n \to \infty$.

All the approximations above are made so that I end up with the estimate below, which takes the same form (that is, $|y_n - \frac{3}{5}| \leq \frac{K}{n}$ for some constant $K > 0$) as in the solution above.

which you can't make smaller than an arbitrary value of $\varepsilon > 0$. This also happens with

$$\left|\frac{3n^3 + 4n - 5}{5n^3 - 3n^2 + 4} - L\right|$$

for any other value[13] $L \in \mathbb{R}$ with $L \neq \frac{3}{5}$.

[13] Note that $\frac{3}{5}$ is the only value of $L$ for which when you write

$$\frac{3n^3 + 4n - 5}{5n^3 - 3n^2 + 4} - L = \frac{p(n)}{5n^3 - 3n^2 + 4}$$

for some polynomial $p(n)$, you get cancellation of the $n^3$ term of $p(n)$, so that $p(n)$ has a lower degree than $5n^3 - 3n^2 + 4$. This was vital to our method.

The key thing that I want you to learn here is that when you come up with a question like this, don't be afraid to try an example and see if you can work out what's going on.

## Proving that sequences do *not* converge directly from the definition

We can also use Definition 3.2 to check that a sequence does *not* converge.

**Example 3.7.** Show that the sequence $(z_n)_{n=1}^{\infty}$ given by $z_n = (-1)^n$, does not converge to any limit, directly from the definition.

*Solution.* Suppose that there exists $L \in \mathbb{R}$ with $z_n \to L$ as $n \to \infty$. Taking $\varepsilon = 1 > 0$, in the definition of convergence, there exists $n_0 \in \mathbb{N}$, such that for $n \in \mathbb{N}$ with $n \geq n_0$, we have $|z_n - L| < \varepsilon$. In particular, for $n \geq n_0$, note that $|z_n - z_{n+1}| = 2$, as one of $z_n$ and $z_{n+1}$ is 1, while the other is $-1$. Then[14]

$$\begin{aligned} 2 = |z_n - z_{n+1}| &= |z_n - L + L - z_{n+1}| \\ &\leq |z_n - L| + |L - z_{n+1}| < \varepsilon + \varepsilon = 2. \end{aligned}$$

[14] A key use of the triangle inequality is coming up: we estimate the distance from $z_n$ to $z_{n+1}$ as being at most the distance from $z_n$ to $L$ plus the distance from $L$ to $z_{n+1}$.

This is a contradiction. Therefore the sequence $(z_n)_{n=1}^{\infty}$ does not converge to $L$. Since $L \in \mathbb{R}$ was arbitrary, the sequence does not converge to any limit. $\square$

To understand how this answer was constructed, let's have a think about this sequence. We see directly from the formula that the sequence $(z_n)_{n=1}^{\infty}$ alternates between 1 and $-1$, more precisely it takes the form $-1, +1, -1, +1, \ldots$. If it converges to $L$, then, fixing a value of $\varepsilon > 0$, for sufficiently large $n$, the values $z_n$ must lie in the $\varepsilon$-error band around $L$. In particular both $+1$ and $-1$ must lie[15] in the interval $(L - \varepsilon, L + \varepsilon)$. This interval has total width $2\varepsilon$. Since 1 and $-1$ are distance 2 apart, we will get a contradiction when $2\varepsilon \leq 2$, leading to the choice $\varepsilon = 1$ in the answer above[16].

[15] You might find it useful to draw a picture of this.

[16] In particular, any value of $\varepsilon$ with $0 < \varepsilon \leq 1$ would work in the above proof, but values of $\varepsilon$ with $\varepsilon > 1$ would not.