EXAMINATION FOR THE DEGREES OF
M.A. AND B.Sc.

Mathematics 2E - Introduction to Real Analysis

An electronic calculator may be used provided that it does not have a facility for either textual storage or display, or for graphical display.

Candidates must attempt all questions.

Question 1 and question 2 are multiple choice questions. Use the response form "2E Degree Exam Multiple Choice Section" to record your answers.

1. (i) Consider the statement

$$P : \exists x > 0, x^2 < 0.$$

Which of the following is equivalent to the negation of P.

(A) $\exists x > 0, x^2 \geqslant 0$
(B) $\exists x < 0, x^2 > 0$
(C) $\forall x > 0, x^2 \geqslant 0$
(D) $\forall x \leqslant 0, x^2 \geqslant 0$
(E) None of the above. **2**

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $c \in \mathbb{R}$. Let Q be the statement

$$Q : \forall \varepsilon > 0, \exists \delta > 0, \forall (x \in \mathbb{R}, |x - c| < \delta), |f(x) - f(c)| < \varepsilon.$$

Which of the following statements is the negation of Q?

(A) $\exists \varepsilon > 0, \forall \delta > 0, \exists x \in \mathbb{R}, (|x - c| \geqslant \delta)$ and $|f(x) - f(c)| \geqslant \varepsilon$.
(B) $\exists \varepsilon \leqslant 0, \forall \delta \leqslant 0, \exists x \in \mathbb{R}, (|x - c| \geqslant \delta)$ and $|f(x) - f(c)| \geqslant \varepsilon$.
(C) $\exists \varepsilon > 0, \forall \delta > 0, \exists x \in \mathbb{R}, (|x - c| < \delta)$ and $|f(x) - f(c)| \geqslant \varepsilon$.
(D) $\forall \varepsilon > 0, \exists \delta > 0, \forall (x \in \mathbb{R}, |x - c| < \delta), |f(x) - f(c)| \geqslant \varepsilon$.
(E) None of the above. **2**

(iii) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Which of the following statements is equivalent to the statement that f is not bounded above?

(A) $\exists x \in \mathbb{R}, \forall M \in \mathbb{R}, f(x) > M$.
(B) $\exists M \in \mathbb{R}, \forall x \in \mathbb{R}, f(x) > M$.

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(C) $\forall M \in \mathbb{R}, \forall x \in \mathbb{R}, f(x) > M.$
(D) $\forall M \in \mathbb{R}, \exists x \in \mathbb{R}, f(x) > M.$
(E) None of the above. **2**

(iv) Let $\varepsilon > 0$ be arbitrary. Which of the following conditions on $n_0 \in \mathbb{N}$ ensure that the implication

$$(n \in \mathbb{N} \text{ and } n \geqslant n_0) \implies \left| \frac{4n}{6n+4} - \frac{2}{3} \right| < \varepsilon$$

is true?

(A) $n_0 > \frac{1}{3\varepsilon}$
(B) $n_0 < \frac{1}{3\varepsilon}$
(C) $n_0 > \frac{4}{9\varepsilon}$
(D) $n_0 > \frac{4}{9}\varepsilon$
(E) None of the above. **2**

(v) Which of the following statements is correct?

(A) If $f, g : \mathbb{R} \to \mathbb{R}$ are functions and $f + g$ is continuous then both f and g are continuous
(B) If $f, g : \mathbb{R} \to \mathbb{R}$ are functions and $f + g$ is continuous then one of f or g is continuous.
(C) If $f, g : \mathbb{R} \to \mathbb{R}$ are functions not continous at $x = c$ then $f + g$ is not continuous at $x = c$.
(D) If $f, g : \mathbb{R} \to \mathbb{R}$ are functions and both f and $f + g$ are continuous then g is continuous
(E) None of the above. **2**

2. (i) Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n \geqslant 0$ for all $n \in \mathbb{N}$. Which of the following statements is equivalent to the statement that $\sum_1^{\infty} a_n$ converges?

(A) $a_n \to 0$ as $n \to \infty$
(B) $\frac{a_{n+1}}{a_n} \to L$ for some $0 \leqslant L < 1$ in $\mathbb{R}$
(C) There exists some $K \in \mathbb{R}$ such that $\sum_{n=1}^{N} a_n \leqslant K$ for all $N \in \mathbb{N}$
(D) $(a_n)_1^{\infty}$ is bounded above
(E) None of the above. **2**

(ii) Which of the following statements is equivalent to the statement that the sequence $(x_n)_1^{\infty}$ does *not* converge to 0 as $n \to \infty$?

(A) $\exists \varepsilon > 0, \exists n_0, |x_{n_0}| \geqslant \varepsilon.$
(B) $\exists \varepsilon > 0, \forall n_0 \in \mathbb{N}, \exists n \geqslant n_0, |x_n| \geqslant \varepsilon.$
(C) $\exists L \neq 0, \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, n \geqslant n_0 \implies |x_n - L| < \varepsilon.$
(D) $\exists \varepsilon > 0, \forall n_0 \in \mathbb{N}, n \geqslant n_0 \implies |x_n| \geqslant \varepsilon.$
(E) None of the above. **2**

(iii) Let $(x_n)_1^\infty$ be a real sequence. Which of the following statements is equivalent to the statement that $\sum_{n=1}^\infty x_n$ is absolutely convergent?

(A) $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geqslant N, |x_n| < \varepsilon$

(B) $\forall \varepsilon > 0, \forall N \in \mathbb{N}, \exists n \geqslant N, \sum_{r=1}^n |x_r| < \varepsilon$

(C) $|\sum_{r=1}^n x_r| \to 0$ as $n \to \infty$.

(D) $\exists S > 0, \forall n \in \mathbb{N}, \sum_{r=1}^n |x_r| < S$

(E) None of the above **2**

(iv) Let $f, g : (a, b) \to (a, b)$ be bounded, nowhere zero functions. Which of the following statements is false?

(A) $f + g$ is bounded on (a, b)

(B) fg is bounded on (a, b)

(C) f/g is bounded on (a, b)

(D) $f \circ g$ is bounded on (a, b)

(E) None of these statements **2**

(v) Let A be the set $\{n + \frac{1}{m} | n, m \in \mathbb{N}\}$. Which of the following statements is true?

(A) A is bounded above

(B) A is bounded below and $\inf A \in A$

(C) A is bounded below and $\inf A = 1$

(D) A is bounded below and $\inf A = 2$

(E) None of the above **2**

3. (i) Show that the set

$$A = \{2x + z \sin y | x, y, z \in (-1, 1)\}$$

is bounded. **3**

(ii) Show that

$$\sup\{\frac{n-1}{n+1} | n \in \mathbb{N}\} = 1.$$

3

4. Prove the following statements *directly from the definition.*

(i) The limit of a convergent sequence is unique. **4**

(ii) The function $f : (0, 2) \to \mathbb{R}$ given by

$$f(x) = \frac{1}{x(x-2)}$$

is continuous at $x = 1$. **4**

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5. (i) Let $(x_n)_1^\infty$ be a sequence. Give the definition of a *subsequence* and state the Bolzano-Weierstraß theorem. **3**

(ii) Find a subsequence of the sequence

$$x_n = (-1)^n - (-1)^n/n + 1/n^2, \quad n \in \mathbb{N}$$

which is convergent. State the limit of your a subsequence and prove from the limit definition that your limit is correct. **4**

6. For each of the series below, determine whether they converge or diverge. Justify your answers clearly, referring to any results or tests you use from the course. An answer with no justification will receive no marks.

(i)

$$\sum_{n=1}^{\infty} \frac{n^3 + 3n + 1}{n^3 + 1},$$

3

(ii)

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 + n + 1},$$

3

(iii)

$$\sum_{n=1}^{\infty} \frac{n}{5^n}.$$

3

7. (i) State the Intermediate Value Theorem. **2**

(ii) The function $f : [0, 1] \to \mathbb{R}$ is defined by $f(x) = 4x(1 - x)$. Use the Intermediate Value Theorem to show that there exist *two* distinct values of $c \in [0, 1]$ such that $f(c) = c + \frac{1}{4}$. **3**

8. This question concerns the function $f : (1, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x-1}$.

(i) Show *from the definition* that f is continuous throughout its domain. **3**

(ii) Show that f is not uniformly continuous on $(1, \infty)$. **2**

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