, *th* , 2017 9:30 am *to* 11:00 am

EXAMINATION FOR THE DEGREES OF
M.A. AND B.Sc.

## Mathematics 2E - Introduction to Real Analysis

*An electronic calculator may be used provided that it does not have a facility for either textual storage or display, or for graphical display.*

*Candidates must attempt all questions.*

1. (i) C

   (ii) C

   (iii) D

   (iv) C

   (v) D 10

2. (i) C

   (ii) B

   (iii) D

   (iv) C

   (v) C 10

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3. (i) $a = 2x + z\sin y \in A$.

$$\begin{aligned} |a| &\leqslant 2|x| + |z||\sin y| && \text{by triangle inequality and properties of modulus} \\ &\leqslant 2|x| + |z| && \text{since } -1 \leqslant \sin y \leqslant y, \forall y \\ &\leqslant 2 + 1 = 3 && \text{for } -1 \leqslant x, z, \leqslant 1 \end{aligned}$$

$|a| \leqslant 3, \forall a \in A \implies A$ is bounded.

(ii) $1 - \frac{n-1}{n+1} = \frac{2}{n+1} > 0, \forall n \in \mathbb{N} \implies 1$ is an upper bound.
$\varepsilon > 0$. Choose $\mathbb{N} \ni n > \frac{2}{\varepsilon}$. Then $1 - \frac{n-1}{n+1} = \frac{2}{n+1} < \frac{2}{n} < \varepsilon \implies \frac{n-1}{n+1} > 1 - \varepsilon$. So $1 - \varepsilon$ is not an upper bound.

4. (i) Let $x_n \to L$ and $x_n \to M$ as $n \to \infty$ for $L \neq M$. Choose $\varepsilon = \frac{|L-M|}{2} > 0$. $\exists n_1, n_2 \in \mathbb{N}$,

$$\begin{aligned} n \geqslant n_1 &\implies |x_n - L| < \varepsilon \\ n \geqslant n_2 &\implies |x_n - M| < \varepsilon. \end{aligned}$$

Let $n = \max(n_1, n_2)$.

$$\begin{aligned} |L - M| &= |L - x_n - (M - x_n)| \\ &\leqslant |x_n - L| + |x_n - M| \\ &< 2\varepsilon = |L - M|. \end{aligned}$$

This contradiction implies $L = M$.

(ii) $\varepsilon > 0$. $|x - 1| < \frac{1}{2} \implies \frac{1}{2} < x < \frac{3}{2}$ and $-\frac{3}{2} < x - 2 < -\frac{1}{2} \implies |x|^{-1} < 2$ and $|x-2|^{-1} < 2$.
Then $|f(x) - f(1)| = |\frac{(x-1)^2}{x(x-2)}| < 2|x-1|$. Let $\delta = \min(\frac{1}{2}, \varepsilon/2)$. Then $|x - 1| < \delta \implies |f(x) - f(1)| < \varepsilon$.

5. (i) A *subsequence* of $(x_n)_1^\infty$ is a sequence $(x_{k_n})_1^\infty$ where $k_n \in \mathbb{N}$ and $k_n < k_{n+1}$.
*Bolzano-Weierstraß*: Every bounded real sequence has a convergent subsequence.

(ii) Choose either even values of $n$ or odd values of $n$. e.g $k_n = 2n$ so $y_n = x_{2n} = 1 - \frac{1}{2n} + \frac{1}{4n^2} \to 1$.
$\varepsilon > 0$. $|y_n - 1| = |-\frac{1}{2n} + \frac{1}{4n^2}| < |\frac{1}{2n}| + |\frac{1}{4n^2}| < 2|\frac{1}{2n}| = \frac{1}{n}$ since $4n^2 > 2n$, $n \in \mathbb{N}$.
Let $\mathbb{N} \ni n_0 > \frac{1}{\varepsilon}$. Then $n \geqslant n_0 \implies |y_n - 1| < \varepsilon$.

6. (i) $a_n = \frac{n^3+3n+1}{n^3+1} = 1 + \frac{3n}{n^3+1} > 1, \forall n \in \mathbb{N}$. Hence $(a_n)_1^\infty$ does not have a zero limit and the sum is divergent. [Alternatively use the algebraic properties of limits to show that $a_n = \frac{1+3n^{-2}+n^{-3}}{1+n^{-3}} \to 1$ for the same conclusion.]

(ii) $\sum_0^\infty (-1)^n a_n$. An alternating series which converges by Leibniz test. Check: $a_n = \frac{n}{n^2+n+1} = \frac{n^{-1}}{1+n^{-1}+n^{-2}} \to 0$, $n \to 0$ by algebraic properties of limits; also

$$\begin{aligned} a_{n+1} - a_n &= \frac{n+1}{n^2+3n+3} - \frac{n}{n^2+n+1} \\ &= \frac{1+n-n^2}{(n^2+3n+3)(n^2+n+1)} \\ &< \frac{2n-n^2}{(n^2+3n+3)(n^2+n+1)} \\ &= \frac{n(2-n)}{(n^2+3n+3)(n^2+n+1)} \\ &< 0 \end{aligned}$$

for $n > 2$. Hence the $a_n$ are decreasing to 0 and the Leibniz test applies.

(iii) Apply the ratio test to a series of positive terms:

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n}\frac{5^n}{5^{n+1}} = (1+\frac{1}{n})\frac{1}{5} \to \frac{1}{5} < 1 \text{ as } n \to \infty.$$

7. (i) Let $f : [a,b] \to \mathbb{R}$ be a continuous function and assume that $d$ is a number such that $f(a) < d < f(b)$ or $f(b) < d < f(a)$. Then there exists a point $c \in (a,b)$ such that $f(c) = d$.

(ii) Consider the function $\tilde{f} : [0,1] \to \mathbb{R}$ defined by $\tilde{f}(x) = 4x(1-x) - x - \frac{1}{4}$. Note that $\tilde{f}(0) = -\frac{1}{4}$, $\tilde{f}(\frac{1}{2}) = \frac{1}{4}$ and $\tilde{f}(1) = -\frac{5}{4}$. Hence by the Intermediate Value Theorem there exist $c_1 \in (0, \frac{1}{2})$ and $c_2 \in (\frac{1}{2}, 1)$ with $\tilde{f}(c_1) = \tilde{f}(c_2) = 0$, that is $f(c_1) = 4c_1(1-c_1)$ and $f(c_2) = 4c_2(1-c_2)$.

8. (i) For $c \in (0, \infty)$, $|f(x) - f(c)| = \frac{|x-c|}{|x-1||c-1|}$.

$$|x-c| < \frac{c-1}{2} \implies \frac{c-1}{2} < x-1 < \frac{3(c-1)}{2} \implies \frac{2}{c-1} > \frac{1}{x-1} > \frac{2}{3(c-1)}.$$

$\varepsilon > 0$. Let $\delta = \min(\frac{c-1}{2}, \frac{(c-1)^2}{2}\varepsilon)$. Then $|x-c| < \delta \implies |f(x) - f(c)| < \frac{2|x-c|}{(c-1)^2} < \varepsilon$.

(ii) Let $\delta > 0$ be arbitary, $\varepsilon = 1$, $c = 1 + \frac{1}{p}$ and $x = 1 + \frac{2}{p}$. Then

$$|f(x) - f(c)| = \frac{|x-c|}{|x-1||c-1|} = \frac{p}{2}.$$

Let $p > \max(2, \frac{1}{\delta})$ and we have $|x-c| < \delta$ but $|f(x) - f(c)| > \varepsilon = 1$.

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