Algorithmic Foundations 2 - Tutorial Sheet 9

Graphs and Relations

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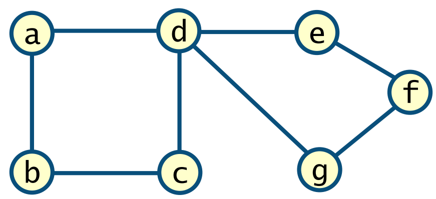
1. Consider the following graph:

$$G = (\{a, b, c, d, e, f, g\}, \{\{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}, \{d, g\}, \{d, e\}, \{f, g\}, \{e, f\}\})$$

   (a) Draw the graph

Solution:



   (b) Is the graph G connected?

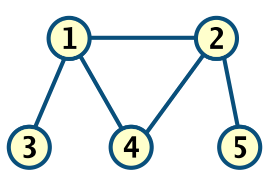
Solution: The graph is connected, i.e. every pair of vertices is joined by a path.

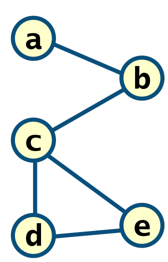
2. How many simple undirected graphs are there with 20 vertices and 60 edges?

Solution: The number of possible edges between 20 vertices is $C(20, 2)$, i.e. the number of 2-combinations from a set of size 20. This yields $20{\cdot}19/2 = 190$ different edges. For a graph to have 60 edges we need to choose 60 out of 190 possible edges i.e. an 60-combination from a set of size 190. The number of graphs therefore equals:

$$C(190, 60) = \frac{190!}{60!{\cdot}130!}$$

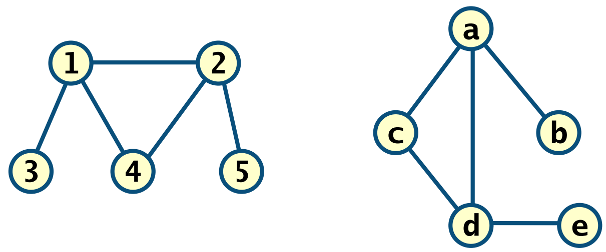
3. Decide whether or not the two graphs below are isomorphic. Explain your answer.





Solution: The graphs are not isomorphic, for example the graph on the left has two vertices with degree 3 (vertices 1 and 2), while the graph on the right has only one vertex with degree 3 (vertex c).

4. Decide whether or not the two graphs below are isomorphic. Explain your answer.



Solution: The graphs are isomorphic as demonstrated by the following bijection:

$$\begin{array}{ccc} 1 & \mapsto & d \\ 2 & \mapsto & a \\ 3 & \mapsto & e \\ 4 & \mapsto & c \\ 5 & \mapsto & b \end{array}$$

5. What is an Euler circuit?

Solution: A Euler circuitis a circuit that contains every edge, where a circuit is a path of length at least 2 that begins and ends with the same vertex.

6. What is a Hamiltonian circuit?

Solution: A Hamiltonian circuit is a circuit that visits each vertex exactly once, where a circuit is a path of length at least 2 that begins and ends with the same vertex.

7. Determine whether each of the following binary relations is

- reflexive;
- symmetric;
- anti-symmetric;
- transitive.

(a) The relation R_1 over $\mathbb{N}\times\mathbb{N}$ where $(a, b) \in R_1$ if and only if $a|b$.

Solution:

- R_1 is reflexive since $a|a$ for any $a \in \mathbb{N}$;
- R_1 is not symmetric since, for example $1|2$ while 2 does not divide 1;
- R_1 is anti-symmetric since, for any $a, b \in \mathbb{N}$, if $a|b$ and $b|a$, then $a{=}b$;

- R_1 is transitive since if $a|b$ and $b|c$ for any $a, b, c \in \mathbb{N}$, then $a|c$ (this was proved in the lectures).

Proof for anti-symmetric case: if $a|b$ and $b|a$ for any $a, b \in \mathbb{N}$, then $a = c_1{\cdot}b$ and $b = c_2{\cdot}a$ for some $c_1, c_2 \in \mathbb{N}$, and hence $a = c_1{\cdot}c_2{\cdot}a$ and $b = c_1{\cdot}c_2{\cdot}b$. Therefore, since $a, b, c_1, c_2 \in \mathbb{N}$, we have either $a = b = 0$ or $c_1 = c_2 = 1$, in either case it follows that $a = b$ as required.

(b) The relation R_2 over $S{\times}S$ where $S = \{w, x, y, z\}$ and

$$R_2 = \{(w,w),(w,x),(x,w),(x,x),(x,z),(y,y),(z,y),(z,z)\}.$$

Solution:

- R_2 is reflexive since $(a,a) \in R$ for all $a \in S$;
- R_2 is not symmetric, e.g. $(x,z) \in R$ while $(z,x) \notin R$;
- R_2 is not anti-symmetric, e.g. $(w,x) \in R$ and $(x,w) \in R$;
- R_2 is not transitive, e.g. $(w,x) \in R$ and $(x,z) \in R$ while $(w,z) \notin R$

(c) The relation R_3 over $\mathbb{Z}{\times}\mathbb{Z}$ where $(a,b) \in R_3$ if and only if $a \neq b$.

Solution:

- R_3 is not reflexive since $a = a$ for all $a \in \mathbb{R}$
- R_3 is symmetric since if $a \neq b$ for any $a, b \in \mathbb{Z}$, then $b \neq a$
- R_3 is not anti-symmetric, e.g. $1 \neq 2$ and $2 \neq 1$;
- R_3 is not transitive, e.g. $1 \neq 2$, $2 \neq 1$ and not $1 \neq 1$.

(d) The relation R_4 over $P(X){\times}P(X)$ where $X = \{1,2,3,4\}$ and $(S,T) \in R_4$ if and only if $S \subseteq T$.

Solution:

- R_4 is reflexive since $S \subseteq S$ for any $S \subseteq X$;
- R_4 is not symmetric e.g. $\{1\} \subseteq \{1,2\}$ and not $\{1,2\} \subseteq \{1\}$;
- R_4 is anti-symmetric since if $S \subseteq T$ and and $T \subseteq S$ for any $S, T \subseteq X$, then $S = T$;
- R_4 is transitive since if $S \subseteq T$ and $T \subseteq U$ for any $S, T, U \subseteq X$, then $S \subseteq U$.

(e) The relation R_5 over *People*$\times$*People* where *People* is the set of all people and $(a,b) \in R_5$ if and only if a is younger than b.

Solution:

- R_5 is not reflexive as a person is not younger than them self;

- R_5 is not symmetric as if a is younger than b, then b is not younger than a;
- R_5 is anti-symmetric if a is younger than b and b is younger than a, then $a = b$ (note that this is implication is vacuously true);
- R_5 is transitive since if a is younger than b and b is younger than c, then a is younger than c.

8. Give an example of a relation on a set that is

   (a) symmetric and anti-symmetric

   Solution: For any set A, define a relation R over $A \times A$ by $(a, b) \in R$ if and only if $a = b$, for any $a, b \in A$. Then R is symmetric and anti-symmetric.

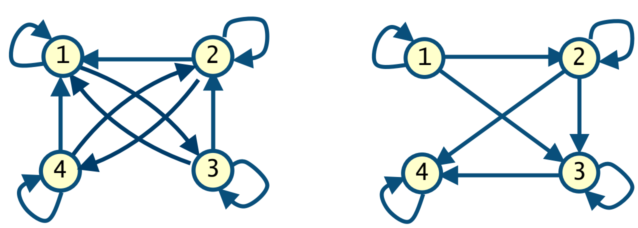
   (b) neither symmetric nor anti-symmetric

   Solution: Define a relation R over $\mathbb{Z} \times \mathbb{Z}$ by $(a, b) \in R$ if and only if $a|b$. Then R is not symmetric, e.g. choose $a = 1$ and $b = 2$. Also R is not anti-symmetric e.g. choose $a = 2$ and $b = -2$.

9. Draw the directed graph for the following relations

$$
\begin{aligned}
R_1 &= \{(1,1),(1,3),(2,1),(2,2),(2,4),(3,1),(3,2),(3,3),(4,1),(4,2),(4,4)\} \\
R_2 &= \{(1,1),(1,2),(1,3),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}
\end{aligned}
$$

Solution:



10. Suppose that the relation R over $A \times A$ is reflexive. Show that R^* is reflexive.

    R^* is the transitive closure of R and is given by $R^* = \cup_{i=1}^{\infty} R^n = R \cup R^2 \cup R^3 \cup R^3 \cup \ldots$.

Solution: By construction $R \subseteq R^*$, and hence for any $a \in A$, if $(a, a) \in R$, then $(a, a) \in R^*$. The result then follows from the fact that R is reflexive.

11. If a relation R over $A \times A$ is irreflexive, then is the relation R^2 necessarily irreflexive?

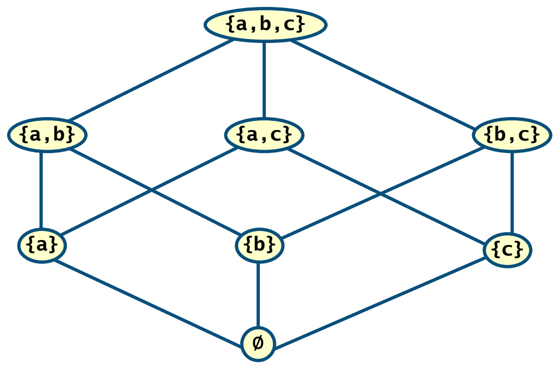
Solution: The answer is no, for example if $A = \{a, b\}$ and $R = \{(a, b), (b, a)\}$, then R is irreflexive while R^2 equals $\{(a, a), (b, b)\}$ and is therefore reflexive.

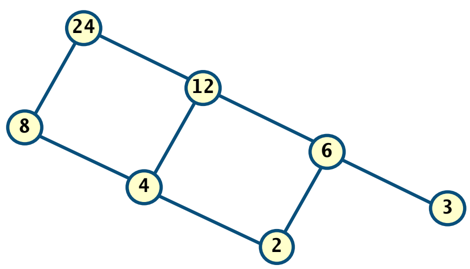
12. Consider the partially ordered sets:

- $(P(S), \subseteq)$ where $S = \{a, b, c\}$;
- $(\{2, 3, 4, 6, 8, 12, 24\}, |)$, i.e. where the relation is the divides relation.

(a) Draw a Hass diagram for each of the partially ordered sets.

Solution:





(b) State both the maximal and minimal elements of each partially ordered set and the greatest and/or least elements when they exist.

Solution: In the first case there is a single maximal element (the set $\{a, b, c\}$) and a single minimal element (the emptyset), these are also the greatest and least elements, respectively, of this partially ordered set.

For the second partially ordered set, 2 and 3 are both minimal, while 24 is maximal. This partially ordered set has no least element, while 24 is the greatest element.

Difficult/challenging questions.

13. What is the minimum number of edges required to produce a connected undirected graph?

Solution: The minimum number of edges equals $n-1$ where n is the number of vertices.

We first show that given n vertices $V = \{v_1, \ldots, v_n\}$ we can construct a connected graph with $n-1$ edges. Considering the graph $G = (V, E)$ where

$$E = \{\{v_i, v_{i+1}\} \mid 1 \leq i \leq n-1\}$$

we have that G has $n-1$ edges. Now for any distinct vertices v_i and v_j, without loss of generality we can assume $i < j$ and we can construct the path between v_i and v_j as follows:

$$\{v_i, v_{i+1}\}, \{v_{i+1}, v_{i+2}\}, \ldots, \{v_{j-1}, v_j\}$$

Therefore, since v_i and v_j were arbitrary, the graph is connected.

Next we show that we cannot construct a connected graph with n vertices and $n-2$ edges. We start with the edgeless graph G, and add edges till the graph is connected.

- First, pick any two vertices of G, label them v_1 and v_2 for convenience, and use one edge to connect them, labelling that edge e_1.
- Second, pick any other vertex, label it v_3, and use one edge to connect it to either v_1 or v_2, labelling that edge e_2.
- Third, pick any other vertex, label it v_4, and use one edge to connect it to v_1, v_2 or v_3, labelling that edge e_3.
- Continue in this way, until we pick a vertex, label it v_{n-1}, and use one edge to connect it to either v_1, v_2, $\ldots$, v_{n-2} labelling that edge e_{n-2}.

This is the last of our edges, and we still have not connected the last vertex.

14. Prove that an undirected graph with more than $(n-1)\cdot(n-2)/2$ edges is connected.

Solution: Here we consider the dual problem and find the maximum number of edges allowed for a graph to be disconnected and show this equals $(n-1)\cdot(n-2)/2$.

Therefore, consider the highest number of edges a graph can have without being connected. It must have two connected components, and, to maximize the number of edges, they must be size $n-1$ and 1. To maximize the edges, the large component must be a complete graph (there can be no edges in the other graph as it only has one vertex), which will have $C(n-1, 2) = (n-1)(n-2)/2$ edges.

15. Prove that a relation R over $A \times A$ is transitive if and only if R^n is a subset of R for all $n \in \mathbb{Z}^+$.

Solution: This is an if and only if so we need to prove both directions.

First we show if $R^n \subseteq R$ for all $n \in \mathbb{Z}^+$, then R is transitive. Consider any $(a, b) \in R$ and $(b, c) \in R$, since (a, b) and (b, c) are arbitrary elements of R it is sufficient to show $(a, c) \in R$. Now by definition of R^2 we have $(a, c) \in R^2$ and by the hypothesis we have $R^2 \subseteq R$, and hence $(a, c) \in R$ as required.

Second we show if R is transitive, then $R^n \subseteq R$ for all $n \in \mathbb{Z}^+$. We need to show this holds for all positive integers n so prove by induction on n.

Base case: if $n = 1$, then trivially $R^1 = R \subseteq R$ as required.

Inductive step: we assume $R^n \subseteq R$ and consider any $(a, c) \in R^{n+1}$. Since (a, c) is arbitrary, it is sufficient to prove $(a, c) \in R$. Now, by definition we have $R^{n+1} = R^n \circ R$, and therefore there exists $b \in A$ such that $(a, b) \in R^n$ and $(b, c) \in R$. By the induction hypothesis we have $(a, b) \in R$, i.e. since $R^n \subseteq R$ and $(a, b) \in R^n$, and hence by transitivity of R we have $(a, c) \in R$ as required.

Therefore by the principle of induction we have proved that if R is transitive, then $R^n \subseteq R$ for all $n \in \mathbb{Z}^+$.

16. Let R be a relation that is reflexive and transitive. Show that $R^n = R$ for all $n \geq 1$.

Solution: From results presented in the lectures, since R is transitive we have $R^n \subseteq R$ for all $n \geq 1$. Thus it remains to prove that $R \subseteq R^n$ for all $n \geq 1$. The proof is by mathematical induction on $n \in \mathbb{N}$. Clearly the base case holds with $n = 1$. Now assume that $R \subseteq R^n$ for some $n \in \mathbb{N}$, and consider any $(a, b) \in R$. Since R is reflexive, $(b, b) \in R$. Hence by induction hypothesis, $(b, b) \in R^n$. Thus by definition of the composition operator on relations, $(a, b) \in R^{n+1}$, since $(a, b) \in R$ was arbitrary we have $R \subseteq R^{n+1}$ as required.

17. Let R be a symmetric relation. Show that R^n is symmetric for all $n \in \mathbb{Z}^+$.

Solution: The proof is by induction on $n \in \mathbb{Z}^+$. The proof relies on first showing that for any relation S and $n \in \mathbb{Z}^+$ we have $S^{n+1} = S \circ S^{n+1}$ which follows from the fact that $\circ$ is associative.

Base case. The base holds as $R^1 = R$ and since R is symmetric.

Inductive step. Suppose R^n is symmetric and consider any $(a, c) \in R^{n+1}$, by definition of R^{n+1} there exists b such that $(a, b) \in R^n$ and $(b, c) \in R$. By by the hypothesis R is symmetric and by the inductive hypothesis we have R^n is symmetric. Therefore we have $(c, b) \in R$ and $(b, a) \in R^n$, and hence since $R^{n+1} = R \circ R^{n+1}$ we have $(c, a) \in R^{n+1}$. Since $(a, c) \in R^{n+1}$ was arbitrary it follows that R^{n+1} is symmetric.

Therefore by the principle of induction we have proved that R^n is symmetric for all $n \in \mathbb{Z}^+$.