2C Intro to real analysis 2020/21

Ex Sheet 9

Solutions and Comments [1] [2]

Q1 *Show* directly from the definition *that the functions below are continuous at the stated points.*

a) $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = 3x + 4$, *at* 5.

b) $g : \mathbb{R} \to \mathbb{R}$ *defined by* $g(x) = x^2 + 4x + 2$, *at* 2.

c) $h : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ *defined by* $h(x) = \frac{x+5}{x-1}$ *at* -2.

d) $u : \mathbb{R} \setminus \{-2\} \to \mathbb{R}$ *defined by* $u(x) = \frac{x^2+2x+5}{x+2}$ *at* 3.

e) $v : \mathbb{R} \setminus \{-2\} \to \mathbb{R}$ *defined by* $v(x) = \frac{2x-3}{x+2}$ *at* -1.

Remember that the definition of a function f being continuous at $c \in \mathrm{dom}(f)$ is

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in \mathrm{dom}(f), |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon.$$

Since we have to prove a statement which begins "$\forall \varepsilon > 0$", the first line of our answer should always be "Let $\varepsilon > 0$ be arbitrary". Once we've fixed ε we have to find a value of δ for which the implication $|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$ holds. To do this, start by trying to simplify and understand $|f(x) - f(c)|$.

In the first part of the question, looking at $f(x) = 3x + 4$ gives us everything we need as $|f(x) - f(5)| = |3x + 4 - 19| = |3x - 15|$. The point here is that $|x - 5|$ comes out as a factor, so that $|f(x) - f(5)| < \varepsilon$ if and only if $|x - 5| < \varepsilon/3$. This leads us to the required value of δ.

> Let $\varepsilon > 0$ be arbitrary. We have
>
> $$|f(x) - f(5)| = |3x + 4 - 19| = 3|x - 5|,$$
>
> so take $\delta = \varepsilon/3 > 0$. Then $|x - 5| < \delta \implies |f(x) - f(5)| < \varepsilon$. Therefore f is continuous at 5.

In part $b)$ things are a bit more involved. We again simplify $|g(x) - g(2)| = |x^2 + 4x + 2 - 14| = |x^2 + 4x - 12|$. Now we can identify a factor of $|x - 2|$ and see that $|g(x) - g(2)| = |x - 2||x + 6|$. The next step is to handle the $|x - 6|$ factor — the $|x - 2|$ factor is already under control, we will have $|x - 2| < \delta$ and we get to choose δ. So we look to control $|x + 6|$, which is the distance from x to the number -6, assuming that x is close to 2. Notice that when x is close to 2 we should expect $|x + 6|$ to be around 8. The way we deal with this in practice is to consider some fixed bound on $|x - 2|$, say $|x - 2| < 1$, and see what we can say about $|x + 6|$ under this hypothesis.

[1] If you've not seriously tried these exercises, please don't look at these solutions and comments, until you have. You'll get the most benefit from reading these comments, when you've first thought hard about them yourself, even if you get really stuck — don't just try for a few minutes and then look at the solutions to work out how to proceed, you don't learn anywhere near as much that way.

[2] Note that I deliberately do not include formal answers for all questions.

Let $\varepsilon > 0$ be arbitrary. We have

$$|g(x) - g(2)| = |x^2 + 4x - 12| = |x-2||x+6|.$$

We have

$$\begin{aligned} |x-2| < 1 &\implies -1 < x-2 < 1 \\ &\implies 7 < x+6 < 9 \\ &\implies |x+6| < 9. \end{aligned}$$

Take $\delta = \min(1, \varepsilon/9)$. Then, for $|x-2| < \delta$, we have

$$|g(x) - g(2)| = |x-2||x+6| < 9|x-2| < \varepsilon,$$

so g is continuous at 2.

The line starting "$|x-2| < 1$" is an assumption that x is within distance 1 of the point $+2$. Assuming this, we then see where $x+6$ is, finding that it's between 7 and 9. Thus assuming x is "close enough" to 2 (meaning $|x-2| < 1$), we get $|x+6| < 9$. We use this fact in our final value of δ. Make sure you understand why we take $\delta = \min(1, \varepsilon/9)$. We want to use both the conditions $|x-2| < 1$ and $|x-2| < \varepsilon/9$, so we take[3] $|x-2| < \min(1, \varepsilon/9)$.

[3] I'd encourage you to think about what happens if you change the 1 in the first formula $|x-2| < 1$ of the calculation. There is no particular reason why the value 1 was chosen in my answer — for this question any value would work. If you instead start with $|x-2| < 2$, what is the resulting value of δ you get?

Part $c)$ is very similar to $b)$, though we should be careful with the directions of the inequalities when we are trying to control the denominator of a fraction.

Let $\varepsilon > 0$ be arbitrary. We have

$$|h(x) - h(-2)| = \left|\frac{x+5}{x-1} - \frac{3}{-3}\right| = \left|\frac{2x+4}{x-1}\right| = \frac{2|x+2|}{|x-1|}.$$

Now

$$\begin{aligned} |x+2| < 1 \implies -1 < x+2 < 1 &\implies -4 < x-1 < -2 \\ &\implies \frac{1}{4} < \frac{1}{|x-1|} < \frac{1}{2}. \end{aligned}$$

Take $\delta = \min(1, \varepsilon)$, so that

$$|x+2| < \delta \implies |h(x) - h(-2)| = \frac{2|x+2|}{|x-1|} < |x+2| < \varepsilon,$$

and hence h is continuous at -2.

Part $d)$ is more of the same. Again be careful with your inequalities: we get $\frac{|x+1|}{|x+2|} \leq \frac{5}{4}$ in the following calculation, not the standard error $\frac{|x+1|}{|x+2|} \leq \frac{5}{6}$.

d) Let $\varepsilon > 0$ be arbitrary. We have

$$|u(x) - u(3)| = \left|\frac{x^2+2x+5}{x+2} - 4\right| = \left|\frac{x^2-2x-3}{x+2}\right| = \frac{|x-3||x+1|}{|x+2|}.$$

Now

$$|x-3| < 1 \implies -1 < x-3 < 1 \implies 4 < x+2 < 6 \text{ and } 3 < x+1 < 5$$
$$\implies \frac{|x+1|}{|x+2|} \leq \frac{5}{4}.$$

Take $\delta = \min(1, \frac{4\varepsilon}{5})$. Then,

$$|x-3| < \delta \implies |u(x) - u(3)| \leq \frac{5}{4}|x-3| < \varepsilon,$$

so u is continuous at 3.

In part *e*), there is an additional point to take care of. First, let's do the standard simplification.

e) Let $\varepsilon > 0$ be arbitrary. We have

$$|v(x) - v(-1)| = \left|\frac{2x-3}{x+2} + 5\right| = \left|\frac{7x+7}{x+2}\right| = 7\frac{|x+1|}{|x+2|}.$$

If we carry on as normal, that is, by considering what happens when $|x+1| < 1$, something goes wrong. We have

$$|x+1| < 1 \implies -1 < x+1 < 1 \implies 0 < x+2 < 2.$$

We learn that $\frac{1}{|x+2|} > \frac{1}{2}$, but don't get an upper bound on $\frac{1}{|x+2|}$ from this inequality, as we can't divide by 0. The problem here is by only asking for x to be within distance 1 of the point -1, we don't manage to keep $x+2$ away from zero. The solution is to ask for x to be closer[4] to -1, say $|x+1| < 1/2$. This leads to the rest of the solution:

[4] in fact $|x+1| < a$ for any $a < 1$ works.

Now

$$|x+1| < \frac{1}{2} \implies -\frac{1}{2} < x+1 < \frac{1}{2}$$
$$\implies \frac{1}{2} < x+2 < \frac{3}{2} \implies \frac{1}{|x+2|} < 2.$$

Take $\delta = \min(\frac{1}{2}, \frac{\varepsilon}{14})$. Then

$$|x+1| < \delta \implies |v(x) - v(-1)| < 7 \times 2|x+1| < \varepsilon,$$

so v is continuous at -1.

It is instructive to see what happens if we try to solve the problem by making the distance from x to -1 bigger, and say consider $|x+1| <$

2. We have

$$|x+1| < 2 \implies -2 < x+1 < 2 \implies -1 < x+2 < 3.$$

A standard error here is to say that this gives $\frac{1}{|x+2|} < 1$, but this isn't the case. The condition $-1 < x+2 < 3$ still allows $x+2 = 0$, and so we can't get an upper bound on $\frac{1}{|x+2|}$. It's important to make sure $x+2$ is kept away from 0.

Q2 *Show* directly from the definition *that the functions below are continuous.*

a) $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = 3 - 4x$.

b) $g : \mathbb{R} \to \mathbb{R}$ *defined by* $g(x) = x^2 + 2x + 3$.

c) $h : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *defined by* $h(x) = \frac{1}{x}$.

These questions are very similar to the previous question, but remember what the definition of continuity is. A function f is continuous if it is continuous at c for every point c in the domain of f. This means we should start our answer by first fixing an arbitrary point c of the domain, then fixing an arbitrary $\varepsilon > 0$. Then look to simplify $|f(x) - f(c)|$ and identify a factor of $x - c$, and proceed as in question 1. We also need to work with conditions like $|x - c| < 1$, and for questions like 1*e*), we might need to use a condition like $|x - c| < |c|/2$, when the distance we require x to be from c depends on the value of c.

a) Let $c \in \mathbb{R}$ be arbitrary and let $\varepsilon > 0$ be arbitrary. We have

$$|f(x) - f(c)| = 4|c - x| = 4|x - c| < \varepsilon,$$

provided $|x - c| < \varepsilon/4$. Taking $\delta = \varepsilon/4$, we see that f is continuous at c. Since c was arbitrary, f is continuous.

In our answer to *b*), we see that the value of δ we get depends on c as well as ε. This is the usual state of affairs.

b) Let $c \in \mathbb{R}$ be arbitrary and let $\varepsilon > 0$ be arbitrary. We have

$$|g(x) - g(c)| = |x^2 + 2x + 3 - (c^2 + 2c + 3)| = |(x - c)(x + c + 2)|.$$

Now

$$|x - c| < 1 \implies |x| < |c| + 1,$$

so for $|x - c| < 1$, we have

$$|x + c + 2| < |x| + |c| + 2 < 2|c| + 3.$$

Take $\delta = \min(1, \frac{\varepsilon}{2|c|+3})$. For $|x - c| < \delta$, we have

$$|g(x) - g(c)| < (2|c| + 3)|x - c| < \varepsilon,$$

so g is continuous at c. Since c was arbitrary, g is continuous.

In c), you need to think carefully about how to bound $\frac{1}{|x|}$ from above. Compare this with the answer to part e) of question 1.

> c) Let $c \neq 0$ be arbitrary and let $\varepsilon > 0$ be arbitrary. For $x \neq 0$, we have
> $$|h(x) - h(c)| = \left|\frac{1}{x} - \frac{1}{c}\right| = \frac{|x-c|}{|x||c|}.$$
> For $|x - c| < |c|/2$, we have $|x| > |c|/2$, so that
> $$\frac{1}{|x||c|} \leq \frac{2}{|c|^2}.$$
> Take $\delta = \min(\frac{|c|}{2}, \frac{\varepsilon|c|^2}{2})$, so that
> $$|x - c| < \delta \implies |h(x) - h(c)| < \frac{2|x-c|}{|c|^2} < \varepsilon.$$
> Since $c \in \mathbb{R} \setminus \{0\}$ was arbitrary, h is continuous.

Q3 *Let $c \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ have the property that there exists $\mu > 0$ and $K > 0$ such that for $x \in \mathbb{R}$,*

$$|x - c| < \mu \implies |f(x)| \leq K.$$

Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = (x - c)f(x)$. Prove directly from the definition that g is continuous at c.
Then look at your answers to question 1 again and explain how they fit into the framework of this question.

> Let $\varepsilon > 0$ be arbitrary and suppose μ and K are given with the properties of the question. Define $\delta = \min(\mu, \varepsilon/K)$. Then for $|x - c| < \delta$, we have
> $$|g(x) - g(c)| = |x - c||f(x)| \leq K|x - c| < \varepsilon,$$
> so g is continuous at c.

Let's look at this in the context of question 1 part b). Here we have $c = 2$ and $g(x) - g(c) = (x - c)(x + 6)$. Looking at the solution above, it corresponds to taking $\mu = 1$ and $K = 9$, in the abstract calculation above.

Can you describe the answers to the other parts of question 1 in this way?

Q4 *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by setting $f(x)$ to be the greatest integer n with $n \leq x$ (the number $f(x)$ defined this way is known as the* integer part *of x, and is often written $\lfloor x \rfloor$). Show that the function f is continuous at c for $c \in \mathbb{R} \setminus \mathbb{Z}$ and discontinuous at c for $c \in \mathbb{Z}$.*

This is fairly intuitive, but a little hard to write down. If c is not an integer, then we find a small interval around c which doesn't contain any integers. The way I do this is by looking at the distance from c to the nearest integer, and taking that to be the δ giving us the interval.

Let $c \in \mathbb{R} \setminus \mathbb{Z}$ be arbitrary and let $\varepsilon > 0$ be arbitrary. Let $n = f(c) = \lfloor c \rfloor$ so that $n < c < n+1$. Define $\delta = \min(n+1-c, c-n)$. Then

$$\begin{aligned} |x-c| < \delta &\implies c-\delta < x < c+\delta \\ &\implies n < x < n+1 \\ &\implies f(x) = n \\ &\implies |f(x)-f(c)| = 0 < \varepsilon, \end{aligned}$$

and so f is continuous at c.

To show that f is not continuous at the integers, it's quickest to use the sequential characterisation of continuity.

Let $c \in \mathbb{Z}$ be arbitrary. For $n \in \mathbb{N}$ define $x_n = c - 1/n$ so that $f(x_n) = c-1$ for all n. Since $x_n \to c$, but $f(x_n) = c-1 \not\to c = f(c)$, it follows that f is not continuous at c.

Q5 *Let* $\chi_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Show directly from the definition[5] *that* $\chi_{\mathbb{Q}}$ *is not continuous at any* $c \in \mathbb{R}$.

[5] You may use the fact that any open interval $(a, b) \subset \mathbb{R}$ contains both a rational and an irrational number, see question 7 below.

Start by fixing an arbitrary value of $c \in \mathbb{R}$. We will show, directly from the definition, that $\chi_{\mathbb{Q}}$ is not continuous at c. To do this, we must know what this statement means, so we formally negate the definition of continuity at c. Recall that f is continuous at c if and only if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in \operatorname{dom}(f), |x-c| < \delta \implies |f(x)-f(c)| < \varepsilon.$$

Remember how to negate these formal definitions from the beginning of the course (particularly the implication sign at the end). We get that f is *not* continuous at c if and only if

$$\exists \varepsilon > 0 \text{ s.t. } \forall \delta > 0, \exists x \in \operatorname{dom}(f) \text{ s.t. } |x-c| < \delta \text{ and } |f(x)-f(c)| \geq \varepsilon.$$

This gives us the framework for a solution. Our answer must start by telling the reader what the value of ε should be. Then we should take an arbitrary value of $\delta > 0$, and explain why there is an x satisfying the two properties.

So what value of ε should we take? The function $\chi_{\mathbb{Q}}$ takes the two values 0 and 1, and at a point c where $\chi_{\mathbb{Q}}(c)$ is 1, we can find nearby points where it is 0, and vice versa. This leads us to take $\varepsilon = 1$ as we will be able to arrange for $|\chi_{\mathbb{Q}}(x) - \chi_{\mathbb{Q}}(c)| = 1$. We could also take ε to be any number satisfying $0 < \varepsilon \leq 1$, but you will not be able to get it to work with $\varepsilon > 1$. Here's my answer.

> Let $c \in \mathbb{R}$ be arbitrary and take $\varepsilon = 1$. Let $\delta > 0$ be arbitrary. If c is rational, then take a number $x \in (c-\delta, c+\delta)$ which is irrational, whereas if c is irrational, take some $x \in (c-\delta, c+\delta)$ which is rational. (These choices can be made by the facts which we are allowed to use in the question). Then $|x - c| < \delta$, but $|\chi_{\mathbb{Q}}(x) - \chi_{\mathbb{Q}}(c)| = 1 \geq \varepsilon$, as exactly one of $\chi_{\mathbb{Q}}(x)$ and $\chi_{\mathbb{Q}}(c)$ is 1 and the other value is 0. Therefore $\chi_{\mathbb{Q}}$ is not continuous at c. Since c was arbitrary, $\chi_{\mathbb{Q}}$ is not continuous at any $c \in \mathbb{R}$.

Although we were required to prove the discontinuity of $\chi_{\mathbb{Q}}$ directly from the definition, let me also give an answer to the question using the sequential characterisation of continuity: Given $c \in \mathbb{R}$, the idea is to construct two sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ both converging to c, but with each x_n being rational and each y_n irrational.

> Let $c \in \mathbb{R}$ be arbitrary. For $n \in \mathbb{N}$, choose (using the facts we are allowed to assume), a rational number $x_n \in (c - 1/n, c + 1/n)$ and an irrational number $y_n \in (c - 1/n, c + 1/n)$. Since
>
> $$c - \frac{1}{n} < x_n < c + \frac{1}{n}, \quad c - \frac{1}{n} < y_n < c + \frac{1}{n},$$
>
> for all n, the sandwich principle shows that $x_n \to c$ and $y_n \to c$. As
>
> $$\chi_{\mathbb{Q}}(x_n) = 0, \quad \chi_{\mathbb{Q}}(y_n) = 1,$$
>
> for all n, it follows that $(\chi_{\mathbb{Q}}(x_n))_{n=1}^{\infty}$ and $(\chi_{\mathbb{Q}}(y_n))_{n=1}^{\infty}$ can not both converge to $\chi_{\mathbb{Q}}(c)$. Hence $\chi_{\mathbb{Q}}$ is not continuous at c by the sequential characterisation of continuity.

Q6 *Let $\chi_{\mathbb{Q}}$ be as above, and let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = f(x)\chi_{\mathbb{Q}}(x)$.*

a) For $c \in \mathbb{R}$ show that g is continuous at c if and only if $f(c) = 0$. You may use the fact that given any $c \in \mathbb{R}$, there exists a sequence $(q_n)_{n=1}^{\infty}$ of rational numbers converging to c and a sequence $(r_n)_{n=1}^{\infty}$ of irrational numbers converging to c, see question 7 below.

b) Construct a function $g : \mathbb{R} \to \mathbb{R}$ which is continuous only at $1, 2$ and 3.

We first use the continuity of f to see that g is continuous at c when $f(c) = 0$.

Suppose $f(c) = 0$ and let $\varepsilon > 0$ be arbitrary. By continuity of f there exists $\delta > 0$ such that

$$|x - c| < \delta \implies |f(x) - 0| < \varepsilon.$$

Then, as $|\chi_{\mathbb{Q}}(x)| \leq 1$, we see that $|x - c| < \delta$ implies

$$|g(x) - g(c)| = |g(x) - 0| = |f(x)||\chi_{\mathbb{Q}}(x)| \leq |f(x)| < \varepsilon,$$

so g is continuous at c.

For the reverse direction, we use the sequential characterisation of continuity.

Suppose $f(c) \neq 0$. By the stated fact, we can find sequences $(r_n)_{n=1}^{\infty}$ of irrational numbers and $(q_n)_{n=1}^{\infty}$ of rational numbers so that $r_n \to c$ and $q_n \to c$ as $n \to \infty$. By the sequential characterisation of continuity, $f(r_n) \to f(c)$ and $f(q_n) \to f(c)$. Then $g(r_n) \to 0$ and $g(q_n) \to f(c)$. Since $f(c) \neq 0$, it follows that at g is not continuous at c.

For the final part you simply need to construct a continuous function f which is only zero at $1, 2$ and 3, then apply part *a*), to get a corresponding function g. An easy choice for such a function is the polynomial $f(x) = (x - 1)(x - 2)(x - 3)$.

Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \chi_{\mathbb{Q}}(x)(x - 1)(x - 2)(x - 3)$. By the previous part, this is continuous at c precisely when $c = 1, 2, 3$.

Q7 *In this question we justify the facts about rational and irrational numbers used above. This exercise will not feature on the exam.*

a) Let a, b be real numbers with $a < b$. Use the fact that $\mathbb{N}$ is not bounded above to prove that there exists a rational number q with $a < q < b$.

b) Let a, b be real numbers with $a < b$. Use the previous part to find rational numbers q_1, q_2 with $a < q_1 < q_2 < b$, and then show that

$$r = \frac{1}{\sqrt{2}}q_1 + (1 - \frac{1}{\sqrt{2}})q_2 = q_1 + (1 - \frac{1}{\sqrt{2}})(q_2 - q_1)$$

is an irrational number with $a < r < b$.

c) Let $c \in \mathbb{R}$. Use the previous parts and the sandwich principle to find sequences $(q_n)_{n=1}^{\infty}$ of rational numbers and $(r_n)_{n=1}^{\infty}$ of irrational numbers with $q_n \to c$ and $r_n \to c$ as $n \to \infty$.

For part *a*), we may assume without loss of generality that a and b have the same sign — if a is negative and b is positive we can simply take $q = 0$.

Let us first consider the case that a and b are both positive, so that $0 < a < b$. Then $\delta = b - a > 0$, and since $\mathbb{N}$ is unbounded we find some $n \in \mathbb{N}$ such that $n > \frac{1}{\delta}$. This means $n\delta > 1$. Now let $m \in \mathbb{N}$ be the smallest natural number[6] with $m > na$.

We claim that $nb > m$. Suppose that this is not the case. Then

$$m \geq nb = n(b-a) + na = n\delta + na > 1 + na$$

so that $m - 1 > na$, contradicting the minimality of m.

Hence our assumption $nb \leq m$ was wrong, and we have $nb > m > na$. Dividing by n yields

$$b > \frac{m}{n} > a,$$

and since $\frac{m}{n} \in \mathbb{Q}$ this yields the claim.

In order to treat the case $a < b < 0$ it suffices to use the previous argument to find $q \in \mathbb{Q}$ such that $0 < -b < q < -a$. Then the rational number $-q$ clearly satisfies $a < -q < b$.

For part b) let $a < b$ be given. Then part a) allows us to find a rational number q_1 such that $a < q_1 < b$. Applying part a) again with a replaced by q_1 we find a rational number q_2 such that $q_1 < q_2 < b$.

Now consider

$$r = \frac{1}{\sqrt{2}}q_1 + (1 - \frac{1}{\sqrt{2}})q_2 = q_1 + (1 - \frac{1}{\sqrt{2}})(q_2 - q_1).$$

Since $\sqrt{2} > 1$ we have $(1 - \frac{1}{\sqrt{2}})(q_2 - q_1) > 0$, which means $r > q_1$. Similarly,

$$r < \frac{1}{\sqrt{2}}q_2 + (1 - \frac{1}{\sqrt{2}})q_2 = q_2.$$

That is, we have $q_1 < r < q_2$.

It remains to show that r is irrational. Suppose this is not the case. If r is rational, the same holds for $r - q_2$. Now consider $r - q_2 = \frac{1}{\sqrt{2}}(q_1 - q_2)$. Multiplying this equality by $\sqrt{2}$ and dividing by $r - q_2$ implies that $\sqrt{2} = \frac{q_1 - q_2}{r - q_2}$ is rational, which we know is not true. Hence r is indeed irrational.

Let us finally consider part c). Assume first that c is rational. Then we can simply take $q_n = c$ for all n. Moreover consider the sequence $c + \frac{1}{n}$. Using part b) we find an irrational number r_n such that $c < r_n < c + \frac{1}{n}$. Then we have $0 \leq r_n - c \leq \frac{1}{n}$, and hence $r_n - c \to 0$ as $n \to \infty$ by the sandwich principle. This means $r_n \to c$ as $n \to \infty$.

Now assume that c is irrational. In this case we can take $r_n = c$ for all n. Moreover by part a) we find a rational number q_n such that $c < q_n < c + \frac{1}{n}$ for all n. Using the sandwich principle in the same way as before we obtain $q_n \to c$ as $n \to \infty$.

[6] Using the unboundedness of $\mathbb{N}$ we see that the set

$$\{x \in \mathbb{R} \mid x \in \mathbb{N} \text{ and } x > na\}$$

is nonempty, and clearly bounded below. So the infimum of this set exists.