2B Linear Algebra

Ex Sheet 6

1 True/False

a) The determinant of $\begin{pmatrix} 1 & 0 & 0 \\ -67 & 2 & 0 \\ -10 & 10 & 1 \end{pmatrix}$ is 2.

b) The determinant of an upper triangular matrix is equal to the product of the entries on the main diagonal.

c) The determinant of an invertible matrix is sometimes zero.

d) For any square matrix A, $\det(A) = \det(A^T)$.

e) Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $B = \begin{pmatrix} 2a_{21} & 2a_{22} & 2a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ two 3×3 matrices. Then $\det(B) = -2\det(A)$.

f) Let E be the elementary matrix corresponding to an ERO that swaps two rows. Then $\det(E) = 1$.

g) Let E be the elementary matrix corresponding to an ERO that multiplies a row by 1000. Then $\det(E) = 1000$.

h) Let A and B be $n \times n$ matrices. Then $\det(AB) = \det(B)\det(A)$.

i) Any non-zero vector $\mathbf{v} \in \mathbb{R}^n$ is an eigenvector of the identity I.

j) If $\mathbf{v}$ is an eigenvector of $A \in M_{n\times n}(\mathbb{R})$ corresponding to λ, then for $\lambda \neq 0$ it follows that $\lambda\mathbf{v}$ is an eigenvector of A corresponding to λ.

k) If $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors of $A \in M_{n\times n}(\mathbb{R})$ corresponding to λ (such that $\mathbf{v} \neq -\mathbf{w}$), then $\mathbf{v} + \mathbf{w}$ is an eigenvector of A corresponding to λ.

l) The characteristic polynomial of the $n \times n$ identity matrix I is λ^n.

m) The characteristic polynomial of $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 3 & 3 \end{pmatrix}$ is $(1-\lambda)(2-\lambda)(3-\lambda)$.

n) If $\det(A + 3I) = 0$ then 3 is an eigenvalue of A.

o) If λ is an eigenvalue of a matrix A then $A\boldsymbol{x} + \lambda\boldsymbol{x} = \mathbf{0}$ for some $\boldsymbol{x} \neq \mathbf{0}$.

p) Suppose a matrix A has eigenvalue 1. Then the 1-eigenspace of A consists of all vectors $\boldsymbol{x}$ so that $A\boldsymbol{x} = \boldsymbol{x}$.

[1] **True/False Questions**

Every Exercise Sheet will have a section containing true/false questions. They are designed to test your understanding from lectures. You should look at your lecture notes and/or the textbook to help you answer these questions, but you should not need to write anything to work out the solution.

q) An eigenvalue can equal 0 but an eigenvector can never equal $\mathbf{0}$.

r) The numbers 0 and 2 are eigenvalues of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

s) The vector $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$.

t) The vector $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ corresponding to the eigenvalue $\lambda = 1$.

u) The vector $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ corresponding to the eigenvalue $\lambda = 2$.

Solutions to True/False

a) T b) T c) F d) T e) T f) F g) T h) T i) T j) T k) T l) F m) T n) F o) F (p) T (q) T (r) T (s) F (t) F (u) T

Tutorial Exercises

T1 Find the determinants of the matrices

$$A = \begin{bmatrix} 3 & 4 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 5 & -10 \\ 2 & 3 & 1 \end{bmatrix}$$

Solution

Using the formula for a 2×2 determinant we have

$$\det A = 3 \times 1 - 4 \times (-1) = 7.$$

Using the definition of determinant of an $n \times n$ matrix, (expanding along the top row), gives

$$\begin{aligned} \det B &= 1 \times \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - 4 \times \begin{vmatrix} 0 & 3 \\ -1 & 2 \end{vmatrix} + 6 \times \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} \\ &= 1(1) - 4(3) + 6(2) \\ &= 1. \end{aligned}$$

Similarly, expanding along the top row of C gives

$$\begin{aligned} \det C &= 1 \times \begin{vmatrix} 5 & -10 \\ 3 & 1 \end{vmatrix} - 2 \times \begin{vmatrix} 5 & -10 \\ 2 & 1 \end{vmatrix} + 3 \times \begin{vmatrix} 5 & 5 \\ 2 & 3 \end{vmatrix} \\ &= 1(35) - 2(25) + 3(5) \\ &= 0. \end{aligned}$$

T2 Consider the following matrices:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a+1 & 1 \\ 1 & 1 & a^2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & 2 \\ 2 & 2 & a \end{bmatrix}.$$

In each case:

a) calculate its determinant; and

b) find the values of a for which it is not invertible.

Solution

For (a) we expand along the top row

$$\begin{aligned} \begin{vmatrix} 1 & 1 & 1 \\ 1 & a+1 & 1 \\ 1 & 1 & a^2 \end{vmatrix} &= 1 \times \begin{bmatrix} a+1 & 1 \\ 1 & a^2 \end{bmatrix} - 1 \times \begin{bmatrix} 1 & 1 \\ 1 & a^2 \end{bmatrix} + 1 \times \begin{bmatrix} 1 & a+1 \\ 1 & 1 \end{bmatrix} \\ &= (a^3 + a^2 - 1) - (a^2 - 1) + (1 - a - 1) \\ &= a^3 - a \\ &= a(a^2 - 1) \\ &= a(a+1)(a-1). \end{aligned}$$

So the matrix is not invertible when $a = -1, 0, +1$.

For (b), expanding along the top row

$$\begin{aligned} \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 2 \\ 2 & 2 & a \end{vmatrix} . &= 1 \times \begin{bmatrix} a & 2 \\ 2 & a \end{bmatrix} - 1 \times \begin{bmatrix} 1 & 2 \\ 2 & a \end{bmatrix} + 1 \times \begin{bmatrix} 1 & a \\ 2 & 2 \end{bmatrix} \\ &= (a^2 - 4) - (a - 4) + (2 - 2a) \\ &= a^2 - 3a + 2 \\ &= (a-1)(a-2). \end{aligned}$$

So the matrix is not invertible when $a = 1$ or 2.

T3 Find the determinants of the matrices B, C, D, E and F, given that the matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

satisfies $\det(A) = 2$.

$$B = \begin{pmatrix} 3a & 3b & 3c \\ d & e & f \\ g & h & i \end{pmatrix}, \quad C = \begin{pmatrix} a+d+g & b+e+h & c+f+i \\ d & e & f \\ g & h & i \end{pmatrix},$$

$$D = \begin{pmatrix} a & b & c \\ g & h & i \\ d & e & f \end{pmatrix}, \quad E = \begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix},$$

$$F = \begin{pmatrix} 2a & 2b & 2c \\ -d & -e & -f \\ g+4a & h+4b & i+4c \end{pmatrix}.$$

Solution

- $\det(B) = 3\det(A) = 6$ since multiplying a row by $\lambda = 3$ multiplies the determinant by $\lambda = 3$.
- $\det(C) = \det(A) = 2$ since adding multiples of one row to another does not change the determinant.
- $\det(D) = -\det(A) = -2$ since swapping two rows multiplies the determinant by -1.
- $\det(E) = -\det(D) = -(-2) = 2$ since E is obtained from D by swapping two rows, and swapping two rows multiplies the determinant by -1. Alternatively, E is obtained by A by carrying out two row-swaps, and so $\det(E) = -(-\det(A)) = 2$.
- $\det(F) = 2(-1)\det(A) = -4$ since we are multiplying row 1 by 2 and row 2 by -1, hence we must multiply the determinant by $2(-1)$. Adding 4 times row 1 to row 4 does not change the determinant.

T4 Solve each of the following equations. Here, $|A|$ means $\det(A)$.

(a) $\begin{vmatrix} x+5 & 2 & -1 \\ 3 & x & x+2 \\ 24 & 8 & -3 \end{vmatrix} = 0,$ (b) $\begin{vmatrix} 1 & 3 & -2 \\ 3 & x+5 & -4 \\ 0 & 4 & x+6 \end{vmatrix} = 0.$

Solution

We solve these equations by expanding the determinants and solving the polynomial equations in x. For (a), expanding along the top row

$$\begin{aligned} \begin{vmatrix} x+5 & 2 & -1 \\ 3 & x & x+2 \\ 24 & 8 & -3 \end{vmatrix} &= (x+5)\begin{vmatrix} x & x+2 \\ 8 & -3 \end{vmatrix} - 2\begin{vmatrix} 3 & x+2 \\ 24 & -3 \end{vmatrix} - 1\begin{vmatrix} 3 & x \\ 24 & 8 \end{vmatrix} \\ &= (x+5)(-3x-8x-16) - 2(-9-24x-48) - (24-24x) \\ &= (x+5)(-11x-16) + 72x + 90 \\ &= -11x^2 + x + 10. \end{aligned}$$

We use the quadratic formula to solve the equation $-11x^2 + x + 10 = 0$, giving

$$x = \frac{-1 \pm \sqrt{1+440}}{(-22)}$$

Since $\sqrt{441} = 21$ we have

$$x = \frac{20}{-22} = -\frac{10}{11}, \quad \text{or} \quad x = \frac{-22}{-22} = 1.$$

For (b) we illustrate a different approach using the elementary column operations $C_2 \longrightarrow C_2 - 3C_1$

and $C_3 \longrightarrow C_3 + 2C_1$ to introduce zeros along the top row.

$$\begin{vmatrix} 1 & 0 & 0 \\ 3 & x-4 & 2 \\ 0 & 4 & x+6 \end{vmatrix} = 0.$$

Then expanding the determinant along the top row we have

$$\begin{aligned} 1 \times \big[(x-4)(x+6) - 8\big] &= 0 \\ x^2 + 2x - 32 &= 0. \end{aligned}$$

Hence

$$x = \frac{-2 \pm \sqrt{4+128}}{2} = -1 \pm \sqrt{33}.$$

T5 Let $A \in M_{n\times n}(\mathbb{R})$.

a) Suppose $A^2 = I$. Find all possible values of $\det(A)$. Must A be invertible?

b) Suppose $A^2 = A$. Find all possible values of $\det(A)$. Must A be invertible?

c) Suppose $AA^T = I$. Find all possible values of $\det(A)$. Must A be invertible?

d) Suppose $A^k = O$ for some positive integer k, where O is the zero matrix. Find all possible values of $\det(A)$. Can A be invertible?

Solution

a) Since $A^2 = I$, we have $\det(A^2) = \det(I)$ and so $\det(A)\det(A) = 1$. Put $d = \det(A)$ then $d^2 = 1$. Thus the possible values of $\det(A)$ are 1 and -1. Since both these values are non-zero, A must be invertible.

b) Since $A^2 = A$, we have $\det(A^2) = \det(A)$ and so $\det(A)\det(A) = \det(A)$. Put $d = \det(A)$ then $d^2 = d$ and so $d(d-1) = 0$, thus $d = 0$ or $d = 1$. Thus the possible values of $\det(A)$ are 0 and 1. Since $\det(A)$ could be zero, the matrix A does not have to be invertible.

c) Since $AA^T = I$, we have $\det(AA^T) = \det(I)$ and so $\det(A)\det(A^T) = \det(A)\det(A) = 1$. Put $d = \det(A)$ then $d^2 = 1$ and so $d = \pm 1$. Thus the possible values of $\det(A)$ are -1 and 1. Since $\det(A)$ cannot be zero, the matrix A must be invertible.

d) Since $A^k = O$, we have $\det(A^k) = \det(O)$ and so $\det(A)^k = 0$. Thus $\det(A) = 0$. The matrix A is never invertible.

T6 Show that the vector $v = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}$ is an eigenvector of the matrix

$$A = \begin{pmatrix} 2 & -2 & 2 & 2 \\ -2 & 2 & 2 & 2 \\ 2 & 2 & 2 & -2 \\ 2 & 2 & -2 & 2 \end{pmatrix}$$

and find the corresponding eigenvalue.[2]

[2] Hint: compute Av.

Solution

We have

$$Av = \begin{pmatrix} 2 & -2 & 2 & 2 \\ -2 & 2 & 2 & 2 \\ 2 & 2 & 2 & -2 \\ 2 & 2 & -2 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \\ 12 \\ 12 \end{pmatrix} = 4 \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix} = 4v.$$

Thus v is an eigenvector of A with corresponding eigenvalue 4.

T7 Find the eigenvalues over $\mathbb{R}$ of each of the following matrices, and give bases for each of the corresponding eigenspaces.

(a) $A = \begin{pmatrix} 1 & 3 \\ 0 & -4 \end{pmatrix}$, (b) $B = \begin{pmatrix} 1 & -9 \\ 1 & -5 \end{pmatrix}$, (c) $C = \begin{pmatrix} 2 & 1 \\ -6 & -3 \end{pmatrix}$

Solution

a) We have

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 3 \\ 0 & -4-\lambda \end{pmatrix} = (1-\lambda)(-4-\lambda)$$

so the eigenvalues of A are $\lambda = 1, -4$.

Consider $\lambda = 1$: We need to find the null space of the matrix $A - 1I = A + I$. The augmented matrix $(A - 1I|\mathbf{0})$ is

$$\begin{pmatrix} 0 & 3 & 0 \\ 0 & -5 & 0 \end{pmatrix}.$$

We perform EROs on the augmented matrix to get it into reduced row echelon form

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the general solution to $(A - 1I)\boldsymbol{x} = \mathbf{0}$ is $x_2 = 0$, $x_1 = t$ with $t \in \mathbb{R}$. So

$$\text{null}(A - 1I) = \left\{ \begin{pmatrix} t \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

Hence the 1-eigenspace of A is

$$E_1 = \text{Span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$$

and so E_1 has basis $\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$.

Consider $\lambda = -4$: We need to find the null space of the matrix $A - (-4)I = A + 4I$. The augmented matrix $(A - (-4)I|\mathbf{0})$ is

$$\left(\begin{array}{cc|c}5 & 3 & 0\\ 0 & 0 & 0\end{array}\right).$$

We perform EROs on the augmented matrix to get it into reduced row echelon form

$$\left(\begin{array}{cc|c}1 & \frac{3}{5} & 0\\ 0 & 0 & 0\end{array}\right).$$

Thus the general solution to $(A - (-4)I)\boldsymbol{x} = \mathbf{0}$ is $x_2 = t$, $x_1 = -\frac{3}{5}t$ with $t \in \mathbb{R}$. So

$$\text{null}(A - (-4)I) = \left\{\begin{pmatrix}-\frac{3}{5}t\\ t\end{pmatrix} : t \in \mathbb{R}\right\} = \text{Span}\left\{\begin{pmatrix}-\frac{3}{5}\\ 1\end{pmatrix}\right\}.$$

Hence the (-4)-eigenspace of A is

$$E_{-4} = \text{Span}\left\{\begin{pmatrix}-\frac{3}{5}\\ 1\end{pmatrix}\right\}$$

and so E_{-4} has basis $\left\{\begin{pmatrix}-\frac{3}{5}\\ 1\end{pmatrix}\right\}$.

b) We have

$$\begin{aligned}\det(B - \lambda I) &= \det\begin{pmatrix}1-\lambda & -9\\ 1 & -5-\lambda\end{pmatrix}\\ &= (1-\lambda)(-5-\lambda) + 9\\ &= \lambda^2 + 4\lambda - 5 + 9\\ &= \lambda^2 + 4\lambda + 4\\ &= (\lambda+2)^2\end{aligned}$$

so B has only one eigenvalue $\lambda = -2$ (repeated twice).

Consider $t = -2$: We need to find the null space of the matrix $B - (-2)I = B + 2I$. The augmented matrix $(B - (-2)I|\mathbf{0})$ is

$$\left(\begin{array}{cc|c}3 & -9 & 0\\ 1 & -3 & 0\end{array}\right).$$

We perform EROs on the augmented matrix to get it into reduced row echelon form

$$\left(\begin{array}{cc|c}1 & -3 & 0\\ 0 & 0 & 0\end{array}\right).$$

Thus the general solution to $(B-(-2)I)\boldsymbol{x} = \mathbf{0}$ is $x_2 = t$, $x_1 = 3t$ with $t \in \mathbb{R}$. So

$$\operatorname{null}(B-(-2)I) = \left\{ \begin{pmatrix} 3t \\ t \end{pmatrix} : t \in \mathbb{R} \right\} = \operatorname{Span}\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}.$$

Hence the (-2)-eigenspace of B is

$$E_{-2} = \operatorname{Span}\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$$

and so E_{-2} has basis $\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$.

c) We have

$$\begin{aligned} \det(C-\lambda I) &= \det \begin{pmatrix} 2-\lambda & 1 \\ -6 & -3-\lambda \end{pmatrix} \\ &= (2-\lambda)(-3-\lambda)+6 \\ &= \lambda^2+\lambda \\ &= \lambda(\lambda+1) \end{aligned}$$

so the eigenvalues of C are $\lambda = 0, -1$.

Consider $\lambda = 0$: We need to find the null space of the matrix $C - 0I = C$. The augmented matrix $(C-0I|\mathbf{0})$ is

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ -6 & -3 & 0 \end{array} \right).$$

We perform EROs on the augmented matrix to get it into reduced row echelon form

$$\left(\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array} \right).$$

Thus the general solution to $(C-0I)\boldsymbol{x} = \mathbf{0}$ is $x_2 = t$, $x_1 = -\frac{1}{2}t$ with $t \in \mathbb{R}$. So

$$\operatorname{null}(C-0I) = \left\{ \begin{pmatrix} -\frac{1}{2}t \\ t \end{pmatrix} : t \in \mathbb{R} \right\} = \operatorname{Span}\left\{ \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} \right\}.$$

Hence the 0-eigenspace of C is

$$E_0 = \operatorname{Span}\left\{ \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} \right\}$$

and so E_0 has basis $\left\{ \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} \right\}$.

Consider $\lambda = -1$: We need to find the null space of the matrix $C-(-1)I = C+I$. The augmented matrix $(C-(-1)I|\mathbf{0})$ is

$$\left(\begin{array}{cc|c} 3 & 1 & 0 \\ -6 & -2 & 0 \end{array} \right).$$

We perform EROs on the augmented matrix to get it into reduced row echelon form

$$\begin{pmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the general solution to $(C-(-1)I)\boldsymbol{x}=\boldsymbol{0}$ is $x_2=t$, $x_1=-\frac{1}{3}t$ with $t\in\mathbb{R}$. So

$$\text{null}(C-(-1)I)=\left\{\begin{pmatrix} -\frac{1}{3}t \\ t \end{pmatrix} : t\in\mathbb{R}\right\}=\text{Span}\left\{\begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix}\right\}.$$

Hence the (-1)-eigenspace of C is

$$E_{-1}=\text{Span}\left\{\begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix}\right\}$$

and so E_{-1} has basis $\left\{\begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix}\right\}$.

T8 Find the eigenvalues over $\mathbb{C}$ of the following matrix A, and give bases for each of the corresponding eigenspaces.

$$A=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Solution

We have

$$\det(A-\lambda I)=\det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}=\lambda^2+1$$

so the eigenvalues of A over $\mathbb{C}$ are $\lambda=i,-i$. (Note that a matrix with all entries real may have complex eigenvalues.)

Consider $\lambda=i$: We need to find the null space of the matrix $A-iI$. The augmented matrix $(A-iI|\boldsymbol{0})$ is

$$\begin{pmatrix} -i & -1 & 0 \\ 1 & -i & 0 \end{pmatrix}.$$

We perform EROs on the augmented matrix to get it into reduced row echelon form, as follows:

$$\begin{pmatrix} -i & -1 & 0 \\ 1 & -i & 0 \end{pmatrix}\xrightarrow{R_1\to iR_1}\begin{pmatrix} 1 & -i & 0 \\ 1 & -i & 0 \end{pmatrix}\xrightarrow{R_2\to R_2-R_1}\begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus the general solution to $(A-iI)\boldsymbol{x}=\boldsymbol{0}$ is $x_2=t$, $x_1=it$ with $t\in\mathbb{C}$ (the scalars are now $\mathbb{C}$). So

$$\text{null}(A-iI)=\left\{\begin{pmatrix} it \\ t \end{pmatrix} : t\in\mathbb{C}\right\}=\text{Span}\left\{\begin{pmatrix} i \\ 1 \end{pmatrix}\right\}.$$

Hence the i-eigenspace of A is

$$E_i=\text{Span}\left\{\begin{pmatrix} i \\ 1 \end{pmatrix}\right\}$$

and so E_i has basis $\left\{\begin{pmatrix} i \\ 1 \end{pmatrix}\right\}$.

Consider $\lambda = -i$: We need to find the null space of the matrix $A - (-i)I = A + iI$. The augmented matrix $(A - (-i)I|\mathbf{0})$ is

$$\left(\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right).$$

We perform EROs on the augmented matrix to get it into reduced row echelon form, as follows:

$$\left(\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right) \xrightarrow{R_1 \to (-i)R_1} \left(\begin{array}{cc|c} 1 & i & 0 \\ 1 & i & 0 \end{array}\right) \xrightarrow{R_2 \to R_2 - R_1} \left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Thus the general solution to $(A - (-i)I)\boldsymbol{x} = \mathbf{0}$ is $x_2 = t$, $x_1 = -it$ with $t \in \mathbb{C}$ (the scalars are now $\mathbb{C}$). So

$$\text{null}(A - (-i)I) = \left\{\begin{pmatrix} -it \\ t \end{pmatrix} : t \in \mathbb{C}\right\} = \text{Span}\left\{\begin{pmatrix} -i \\ 1 \end{pmatrix}\right\}.$$

Hence the $(-i)$-eigenspace of A is

$$E_{-i} = \text{Span}\left\{\begin{pmatrix} -i \\ 1 \end{pmatrix}\right\}$$

and so E_{-i} has basis $\left\{\begin{pmatrix} -i \\ 1 \end{pmatrix}\right\}$.

T9 Suppose $A \in M_{7\times 7}(\mathbb{R})$ has characteristic polynomial

$$(1-\lambda)^2(3+\lambda)(17+\lambda)(9-\lambda)^3$$

Write down the eigenvalues of A.

Solution

The eigenvalues of A are the roots of the characteristic polynomial, that is, $\lambda = 1, -3, -17, 9$.

T10 Find the characteristic polynomial and the eigenvalues of each of the following matrices, then find a basis for each of the corresponding eigenspaces.

(a) $A = \begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix}$, (b) $B = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 2 \\ -2 & -2 & 0 \end{pmatrix}$, (c) $C = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

Solution

a) The characteristic polynomial is

$$\begin{aligned}\det(A-\lambda I) &= \det\begin{pmatrix}-1-\lambda & 2 & 2\\ 2 & 2-\lambda & 2\\ -3 & -6 & -6-\lambda\end{pmatrix}\\ &= (-1-\lambda)[(2-\lambda)(-6-\lambda)+12]-2[2(-6-\lambda)+6]+2[-12+3(2-\lambda)]\\ &= -\lambda^3-5\lambda^2-6\lambda\\ &= -\lambda(\lambda^2+5\lambda+6)\\ &= -\lambda(\lambda+2)(\lambda+3),\end{aligned}$$

so the eigenvalues are $\lambda = 0, -2, -3$.

Consider $t = 0$: We need to find the null space of the matrix $A - 0I = A$. The augmented matrix $(A - 0I|\mathbf{0})$ is just the matrix $(A|\mathbf{0})$:

$$\begin{pmatrix}-1 & 2 & 2 & 0\\ 2 & 2 & 2 & 0\\ -3 & -6 & -6 & 0\end{pmatrix}.$$

We perform EROs on this augmented matrix to get it into reduced row echelon form

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Thus the general solution to $(A-0I)\boldsymbol{x} = \mathbf{0}$ is $x_1 = 0$, $x_2 = -t$, $x_3 = t$ with $t \in \mathbb{R}$. So

$$\text{null}(A-0I) = \left\{\begin{pmatrix}0\\ -t\\ t\end{pmatrix} : t \in \mathbb{R}\right\} = \text{Span}\left\{\begin{pmatrix}0\\ -1\\ 1\end{pmatrix}\right\}.$$

Hence the 0-eigenspace of A is

$$E_0 = \text{Span}\left\{\begin{pmatrix}0\\ -1\\ 1\end{pmatrix}\right\}$$

and so E_0 has basis $\left\{\begin{pmatrix}0\\ -1\\ 1\end{pmatrix}\right\}$.

Consider $\lambda = -2$: This time we jump straight to the matrix $(A - \lambda I)$ with $\lambda = -2$ and the column of zeros.

$$\begin{pmatrix}1 & 2 & 2 & 0\\ 2 & 4 & 2 & 0\\ -3 & -6 & -4 & 0\end{pmatrix}$$

which has reduced row echelon form

$$\begin{pmatrix}1 & 2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Thus the general solution to $(A-(-2)I)\boldsymbol{x}=\mathbf{0}$ is $x_1=-2t$, $x_2=t$, $x_3=0$ with $t\in\mathbb{R}$. So

$$\operatorname{null}(A-(-2)I)=\left\{\begin{pmatrix}-2t\\t\\0\end{pmatrix}: t\in\mathbb{R}\right\}=\operatorname{Span}\left\{\begin{pmatrix}-2\\1\\0\end{pmatrix}\right\}.$$

Hence the (-2)-eigenspace of A is

$$E_{-2}=\operatorname{Span}\left\{\begin{pmatrix}-2\\1\\0\end{pmatrix}\right\}$$

and so E_{-2} has basis $\left\{\begin{pmatrix}-2\\1\\0\end{pmatrix}\right\}$.

Consider $\lambda=-3$**:** This time we jump straight to the matrix $(A-\lambda I)$ with $\lambda=-3$ and the column of zeros.

$$\begin{pmatrix}2&2&2&0\\2&5&2&0\\-3&-6&-3&0\end{pmatrix}$$

which has reduced row echelon form

$$\begin{pmatrix}1&0&1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}.$$

Thus the general solution to $(A-(-3)I)\boldsymbol{x}=\mathbf{0}$ is $x_1=-t$, $x_2=0$, $x_3=t$ with $t\in\mathbb{R}$. So

$$\operatorname{null}(A-(-3)I)=\left\{\begin{pmatrix}-t\\0\\t\end{pmatrix}: t\in\mathbb{R}\right\}=\operatorname{Span}\left\{\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}.$$

Hence the (-3)-eigenspace of A is

$$E_{-3}=\operatorname{Span}\left\{\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$$

and so E_{-3} has basis $\left\{\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$.

b) The characteristic polynomial is

$$\begin{aligned}\det(B-\lambda I)&=\det\begin{pmatrix}-\lambda&1&2\\-1&-\lambda&2\\-2&-2&-\lambda\end{pmatrix}\\&=-\lambda(\lambda^2+4)-1(\lambda+4)+2(2-2\lambda)\\&=-\lambda^3-4\lambda-\lambda-4+4-4\lambda\\&=-\lambda^3-9\lambda\\&=-\lambda(\lambda^2+9).\end{aligned}$$

The eigenvalues are the roots of the characteristic polynomial and so are $\lambda = 0, \pm 3i$.

Consider $t = 0$: We need to consider the matrix $(B - \lambda I)$ with $\lambda = 0$ and a column of zeros:

$$\begin{pmatrix} 0 & 1 & 2 & 0 \\ -1 & 0 & 2 & 0 \\ -2 & -2 & 0 & 0 \end{pmatrix}.$$

which has reduced row echelon matrix

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus the general solution to $(B - 0I)\boldsymbol{x} = \boldsymbol{0}$ is $x_1 = 2t$, $x_2 = -2t$, $x_3 = t$ with $t \in \mathbb{C}$. So

$$\text{null}(B - 0I) = \left\{ \begin{pmatrix} 2t \\ -2t \\ t \end{pmatrix} : t \in \mathbb{C} \right\} = \text{Span}\left\{ \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

Hence the 0-eigenspace of B is

$$E_0 = \text{Span}\left\{ \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}$$

and so E_0 has basis $\left\{ \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}$.

Consider $t = 3i$: We need to consider the matrix $(B - \lambda I)$ with $\lambda = 3i$ and a column of zeros:

$$\begin{pmatrix} -3i & 1 & 2 & 0 \\ -1 & -3i & 2 & 0 \\ -2 & -2 & -3i & 0 \end{pmatrix}.$$

We swap R_1 and R_2 to give

$$\begin{pmatrix} -1 & -3i & 2 & 0 \\ -3i & 1 & 2 & 0 \\ -2 & -2 & -3i & 0 \end{pmatrix}$$

and multiply R_1 by -1 to give

$$\begin{pmatrix} 1 & 3i & -2 & 0 \\ -3i & 1 & 2 & 0 \\ -2 & -2 & -3i & 0 \end{pmatrix}.$$

Then we use $R_2 \longrightarrow R_2 + 3iR_1$ and $R_3 \longrightarrow R_3 + 2R_1$ to give

$$\begin{pmatrix} 1 & 3i & -2 & 0 \\ 0 & -8 & 2 - 6i & 0 \\ 0 & -2 + 6i & -3i - 4 & 0 \end{pmatrix}.$$

Using $R_2 \longrightarrow -\frac{1}{8}R_2$ gives

$$\begin{pmatrix} 1 & 3i & -2 & 0 \\ 0 & 1 & \frac{3i-1}{4} & 0 \\ 0 & -2 + 6i & -3i - 4 & 0 \end{pmatrix}.$$

Using $R_1 \longrightarrow R_1 - 3iR_2$ and $R_3 \longrightarrow R_3 - (-2+6i)R_2$ we have that the reduced row echelon form is

$$\begin{pmatrix} 1 & 0 & \frac{1+3i}{4} & 0 \\ 0 & 1 & \frac{3i-1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus the general solution to $(B - 3iI)\boldsymbol{x} = \mathbf{0}$ is $x_1 = -\frac{1+3i}{4}t$, $x_2 = -\frac{3i-1}{4}t$, $x_3 = t$ with $t \in \mathbb{C}$. So

$$\operatorname{null}(B-3iI) = \left\{ \begin{pmatrix} -\frac{1+3i}{4}t \\ -\frac{3i-1}{4}t \\ t \end{pmatrix} : t \in \mathbb{C} \right\} = \operatorname{Span}\left\{ \begin{pmatrix} -\frac{1+3i}{4} \\ -\frac{3i-1}{4} \\ 1 \end{pmatrix} \right\}.$$

Hence the $3i$-eigenspace of B is

$$E_{3i} = \operatorname{Span}\left\{ \begin{pmatrix} -\frac{1+3i}{4} \\ -\frac{3i-1}{4} \\ 1 \end{pmatrix} \right\}$$

and so E_{3i} has basis $\left\{ \begin{pmatrix} -\frac{1+3i}{4} \\ -\frac{3i-1}{4} \\ 1 \end{pmatrix} \right\}$. Another correct answer, which avoids fractions, would be that a basis for E_{3i} is $\left\{ \begin{pmatrix} 1+3i \\ 3i-1 \\ -4 \end{pmatrix} \right\}$. (Multiply through by -4.)

Consider $\lambda = -3i$: We could proceed as in the $\lambda = +3i$ case. However, it is quicker to note that any eigenvector for this case will be a complex conjugate of an eigenvector for the one above. Indeed, from the working above we know that

$$\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 2 \\ -2 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1+3i \\ 3i-1 \\ -4 \end{pmatrix} = 3i \begin{pmatrix} 1+3i \\ 3i-1 \\ -4 \end{pmatrix}.$$

If we take the complex conjugate of this equation we have

$$\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 2 \\ -2 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1-3i \\ -3i-1 \\ -4 \end{pmatrix} = -3i \begin{pmatrix} 1-3i \\ -3i-1 \\ -4 \end{pmatrix}.$$

Thus a basis for the $(-3i)$-eigenspace is $\left\{ \begin{pmatrix} 1-3i \\ -3i-1 \\ -4 \end{pmatrix} \right\}$. (You should fill in the details.)

c) The characteristic polynomial is

$$\begin{aligned} \det(C - \lambda I) &= \det \begin{pmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{pmatrix} \\ &= (2-\lambda)[(2-\lambda)(2-\lambda) - 0] \\ &= (2-\lambda)^3. \end{aligned}$$

So there is only one eigenvalue, $\lambda = 2$, repeated three times.

For the eigenspace, we need to consider the matrix $(C - \lambda I)$ with $\lambda = 2$ and a column of zeros.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix is already in reduced row echelon form. Thus the general solution to $(C - 2I)\boldsymbol{x} = \mathbf{0}$ is $x_1 = t$, $x_2 = 0$, $x_3 = 0$ with $t \in \mathbb{R}$. So

$$\text{null}(C - 2I) = \left\{ \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Hence the 2-eigenspace of C is

$$E_2 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

and so E_2 has basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

T11 Show that the eigenvalues of $A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ are a and d. Assuming that $a \neq d$, find a basis for the corresponding eigenspaces.[3]

[3] Hint: Consider the cases $c = 0$ and $c \neq 0$ separately.

Solution

We have

$$\det(A - \lambda I) = \det \begin{pmatrix} a - \lambda & 0 \\ c & d - \lambda \end{pmatrix} = (a - \lambda)(d - \lambda).$$

This has solutions $\lambda = a$ and $\lambda = d$, so the eigenvalues of A are a and d.

Suppose first that $c = 0$, so that $A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. Then

$$(A - aI|\mathbf{0}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & d - a & 0 \end{pmatrix}$$

which, since $d - a \neq 0$, has reduced row echelon form

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence the a-eigenspace has basis $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.

We also have

$$(A - dI|\mathbf{0}) = \begin{pmatrix} a - d & 0 & 0 \\ c & 0 & 0 \end{pmatrix}$$

which, since $a - d \neq 0$, has reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence the d-eigenspace has basis $\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. Note that this does not depend upon the value of c!

Now suppose that $c \neq 0$. Then

$$(A - aI|\mathbf{0}) = \begin{pmatrix} 0 & 0 & 0 \\ c & d-a & 0 \end{pmatrix}$$

which, since $c \neq 0$, has reduced row echelon form

$$\begin{pmatrix} 1 & \frac{d-a}{c} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence the a-eigenspace has basis $\left\{ \begin{pmatrix} -\frac{d-a}{c} \\ 1 \end{pmatrix} \right\}$.

By the same calculation as in the case $c = 0$, the d-eigenspace again has basis $\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

T12 Let A be an $n \times n$ matrix with real entries. Show that A is invertible if and only if 0 is *not* an eigenvalue of A.

Solution

a) We have $A\boldsymbol{v} = \mathbf{0} = 0\boldsymbol{v}$. Since $\boldsymbol{v}$ is non-zero, this implies that 0 is an eigenvalue of A.

b) Many solutions are possible. Here are some.

The matrix A is invertible if and only if the unique solution to $A\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$. But 0 is an eigenvalue of A if and only if there is a nontrivial vector $\boldsymbol{x}$ so that $A\boldsymbol{x} = 0\boldsymbol{x} = \mathbf{0}$. Therefore A is invertible if and only if 0 is not an eigenvalue of A.

Alternatively, suppose that A is invertible. Assume by way of contradiction that 0 is an eigenvalue of A, with corresponding eigenvector $\boldsymbol{v}$. Then

$$\begin{aligned} \boldsymbol{v} &= I\boldsymbol{v} && \text{since multiplying by } I \text{ changes nothing} \\ &= (A^{-1}A)\boldsymbol{v} && \text{since } A^{-1}A = I \\ &= A^{-1}(A\boldsymbol{v}) && \text{since matrix multiplication is associative} \\ &= A^{-1}\mathbf{0} && \text{since } A\boldsymbol{v} = 0\boldsymbol{v} = \mathbf{0} \\ &= \mathbf{0}. \end{aligned}$$

But since $\boldsymbol{v}$ is an eigenvector we have $\boldsymbol{v} \neq \mathbf{0}$, a contradiction. Therefore 0 is not an eigenvalue of A. A similar computation can be used to establish the converse.

T13 Let A be an $n \times n$ matrix with real entries, and let $\boldsymbol{v}$ be an eigenvector for A with corresponding eigenvalue λ.

a) Show that $\boldsymbol{v}$ is also an eigenvector for the matrix A^2 with corresponding eigenvalue λ^2.

b) Show tutorial that $\boldsymbol{v}$ is an eigenvector for A^3 with corresponding eigenvalue λ^3.

c) Generalise this!

Solution

a) By the definition of eigenvector and eigenvalue, $\boldsymbol{v} \neq \mathbf{0}$ and $A\boldsymbol{v} = \lambda\boldsymbol{v}$. Then we have

$$A^2\boldsymbol{v} = A(A\boldsymbol{v}) = A(\lambda\boldsymbol{v}) = \lambda A\boldsymbol{v} = \lambda(A\boldsymbol{v}) = \lambda(\lambda\boldsymbol{v}) = \lambda^2\boldsymbol{v}.$$

We know that $\boldsymbol{v} \neq \mathbf{0}$, so the equation $A^2\boldsymbol{v} = \lambda^2\boldsymbol{v}$ shows that $\boldsymbol{v}$ is an eigenvector of A^2 with corresponding eigenvalue λ^2.

b) Consider

$$\begin{aligned} A^3\boldsymbol{v} &= A(A^2\boldsymbol{v}) = A(\lambda^2\boldsymbol{v}) && \text{by part (i) above} \\ &= \lambda^2 A\boldsymbol{v} = \lambda^2(A\boldsymbol{v}) \\ &= \lambda^2(\lambda\boldsymbol{v}) \\ &= \lambda^3\boldsymbol{v}. \end{aligned}$$

It is still true that $\boldsymbol{v} \neq \mathbf{0}$, but now $A^3\boldsymbol{v} = \lambda^3\boldsymbol{v}$ shows that $\boldsymbol{v}$ is an eigenvector of A^3 with corresponding eigenvalue λ^3.

c) The generalisation is that *$\boldsymbol{v}$ is an eigenvector of the matrix A^n with corresponding eigenvalue λ^n* for all positive integers n. We will prove this by induction. Let $P(n)$ be the statement in italics above. We know that $P(1)$ is true as this was the starting point. (We also showed $P(2)$ and $P(3)$ are true above, although this isn't required for the induction.) Assume by induction that $P(k)$ is true for some $k \geq 1$. Then

$$\begin{aligned} A^{k+1}\boldsymbol{v} = A(A^k\boldsymbol{v}) &= A(\lambda^k\boldsymbol{v}) && \text{by the inductive hypothesis } P(k) \\ &= \lambda^k A\boldsymbol{v} \\ &= \lambda^k(\lambda\boldsymbol{v}) && \text{since } \lambda \text{ is an eigenvalue of } A \text{ with eigenvector } \boldsymbol{v} \\ &= \lambda^{k+1}\boldsymbol{v}. \end{aligned}$$

Since $\boldsymbol{v} \neq \mathbf{0}$ and $A^{k+1}\boldsymbol{v} = \lambda^{k+1}\boldsymbol{v}$, we deduce that $\boldsymbol{v}$ is an eigenvector of A^{k+1} corresponding to eigenvalue λ^{k+1}. Hence $P(k+1)$ is true. It follows by induction that $P(n)$ is true for all $n \geq 1$.

T14 Let A and B be $n \times n$ matrices. Prove the following statements using determinants.

a) AB is invertible if and only if both A and B are invertible.

b) If A is invertible then A^{-1} is invertible.

Solution

a) If A and B are invertible then $\det(A) \neq 0$ and $\det(B) \neq 0$. Thus $\det(A)\det(B) \neq 0$. But $\det(A)\det(B) = \det(AB)$, hence $\det(AB) \neq 0$ and so AB is invertible.

If AB is invertible then $\det(AB) \neq 0$. Now $\det(AB) = \det(A)\det(B)$, so neither $\det(A)$ nor $\det(B)$ can equal 0. Hence A and B are both invertible.

b) If A is invertible then $\det(A) \neq 0$, and $\det(A^{-1}) = \frac{1}{\det(A)}$. Since $\det(A) \neq 0$, we have $\frac{1}{\det(A)} \neq 0$, hence $\det(A^{-1}) \neq 0$, and so A^{-1} is invertible.

T15

a) Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$. Show that

$$\det(A - \lambda I) = \lambda^2 - (\operatorname{tr}(A))\lambda + \det A.$$

b) Consider an arbitrary 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

By expanding $\det(A - \lambda I)$ along the first row, verify that $\det(A - \lambda I)$ is a polynomial of degree 3 in λ in which the coefficient of λ^3 is -1, the coefficient of λ^2 is $\operatorname{tr} A$, and the constant term is $\det A$.

Solution

a) We have

$$\begin{aligned}\det(A - \lambda I) &= \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \\ &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + (ad - bc) \\ &= \lambda^2 - (\operatorname{tr}(A))\lambda + \det(A).\end{aligned}$$

b) We have

$$\det(A - \lambda I) = \det \begin{pmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{pmatrix}.$$

Expanding along the top row gives

$$\begin{aligned}\det(A - \lambda I) = (a_{11} - \lambda)&[(a_{22}-\lambda)(a_{33}-\lambda) - a_{23}a_{32}] \\ &- a_{12}[a_{21}(a_{33}-\lambda) - a_{23}a_{31}] + a_{13}[a_{21}a_{32} - a_{31}(a_{22}-\lambda)]\end{aligned}$$

Expanding the brackets and collecting together the terms with the same powers of λ we have

$$\begin{aligned}\det(A-\lambda I) = -\lambda^3 + \lambda^2(a_{11}+a_{22}+a_{33})\\ -\lambda(a_{11}a_{22}+a_{11}a_{33}+a_{22}a_{33}-a_{13}a_{31}-a_{23}a_{32}-a_{12}a_{21})\\ +(a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33})\end{aligned}$$

We now have a polynomial of degree 3 where the coefficient of λ^3 is -1. We can check from the matrix A that

$$\operatorname{tr}(A) = a_{11}+a_{22}+a_{33}$$

and so the coefficient of λ^2 is $\operatorname{tr}(A)$. Compute directly that the formula for the determinant is

$$\det(A) = a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}$$

and so the constant term is $\det(A)$ as required.