EXAMINATION FOR THE DEGREES OF
M.A. AND B.Sc.

Mathematics 2A - Multivariable Calculus

An electronic calculator may be used provided that it does not have a facility for either textual storage or display, or for graphical display.

Candidates must attempt all questions.

1. (i) Let $g(x, y)$ be a twice-differentiable function defined implicitly by the equation $g(\lambda x, \lambda y) = \lambda^n g(x, y)$ with $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$. Show that

$$xg_x + yg_y = ng,$$

by taking a first derivative with respect to λ, and then taking first derivatives with respect to x and y. [HINT: You may find it useful to introduce functions $u(x, \lambda) = \lambda x$ and $v(y, \lambda) = \lambda y$ while differentiating.] **4**

(ii) Find the general solution of the partial differential equation

$$xf_x + yf_y = x \log(x + y)$$

using the change of variables $u = x + y$, $v = x/(x + y)$. [NOTE: The function f is not related to the function g from part (i).] **8**

2. (i) Let $r > 0$ be defined implicitly via $r^2 = x^2 + y^2$. Use implicit differentiation to show that

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \qquad \frac{\partial r}{\partial y} = \frac{y}{r}.$$

For a scalar field $\phi(x, y) = \Phi(r(x, y))$, where $\Phi : \mathbb{R} \to \mathbb{R}$ is a twice-differentiable function of one variable, show that

$$\nabla \phi = \frac{1}{r}\mathbf{x}\Phi'(r)$$

where $\mathbf{x} = (x, y)$. **4**

(ii) State the product rule for

$$\nabla \cdot (f\mathbf{F})$$

where f is a scalar field and $\mathbf{F}$ is a vector field. Using this product rule and the definitions and results from part (i) show that **6**

$$\nabla^2 \phi = \frac{1}{r}\left(r\Phi'\right)'.$$

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3. (i) Let S be the region in the first quadrant enclosed by the hyperbolas $y = 3/x$, $y = 6/x$ and the lines $y = 2x$, $y = 2x + 1$. Find an appropriate change of variables that maps S into a rectangular region R. Sketch the regions S and R. **4**

(ii) Evaluate the double integral

$$\iint_S (y + 2x)^3 dxdy,$$

where S is the region described in part (i). [HINT: You may need to use that $(a + b)^2 = (a - b)^2 + 4ab$.] **7**

(iii) Evaluate the triple integral

$$\int_0^4 dx \int_1^2 dz \int_{\sqrt{x}}^2 \frac{xz^2}{1 + y^5} dy,$$

changing the order of integration if necessary. **7**

4. (i) The curve $\mathcal{C}$ is the curve made from the straight line from the origin to the point $(\sqrt{2}, 0)$, followed by part of the circle centred at the origin with radius $\sqrt{2}$ from $(\sqrt{2}, 0)$ to $(1, 1)$, followed by the straight line from $(1, 1)$ to the origin. Sketch $\mathcal{C}$ and mark the region $\mathcal{D}$ it encloses.

State Green's Theorem. Use Green's Theorem to write the integral

$$\oint_{\mathcal{C}} x^2\, dy - y^2\, dx$$

as a double integral. Evaluate the double integral by changing to polar coordinates. **7**

(ii) Consider the part of the surface $\mathcal{S}$ defined by $z = x^3 - 3xy^2$ that lies inside the cylinder $x^2 + y^2 = 1$. Calculate

$$\iint_{\mathcal{S}} x^2 + y^2\, dS.$$ **6**

(iii) Consider the region $\mathcal{V}$ that is the part of the sphere radius 1 that lies inside the cone $z = \sqrt{3(x^2 + y^2)}$. Use the divergence theorem to calculate

$$\iint_{\partial\mathcal{V}} \mathbf{F} \cdot \mathbf{n}\, dS$$

where $\partial\mathcal{V}$ is the boundary of $\mathcal{V}$ and where $\mathbf{F} = (-x, -y, 3z)$. **7**

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