2467273S

MATHS 2032

1\. i)

a) $\inf(X) = -2$, $\sup(X) = 7$

b) $\inf(Y) \in \emptyset$, $\sup(Y) = 2$

c) $\inf(Z) = 0$, $\sup(Z) = 2$

ii) Set $M = (\sup(A))^2 + 2\sup(A) + \frac{5}{\inf(A)}$. For $a \in A$,

$a \leq \sup(A)$ and $a \geq \inf(A)$ and $\frac{1}{a} \leq \frac{1}{\inf(A)}$. Therefore, $a^2 + 2a + \frac{5}{a} \leq (\sup(A))^2 + 2\sup(A) + \frac{5}{\inf(A)}$

$= M$.

Thus, M is an upper bound for B.

Let $\varepsilon > 0$ be arbitrary. Because $\sup(A)$ exists, $\exists a \in A$ s.t.

$a > \sup(A) - \varepsilon$, and because $\inf(A)$ exists, $\exists a \in A$ s.t.

$a < \inf(A) + \varepsilon$. Then $\frac{1}{a} > \frac{1}{\inf(A)} + \varepsilon$

$a^2 + 2a + \frac{5}{a} > (\cancel{a}\sup(A) - \varepsilon)^2 + 2(\sup(A) - \varepsilon) + 5(\frac{1}{\inf} - \varepsilon)$

Because $\sup(B)$ exis

24672735

MATHS2032

(2) i) Check whether $x_n \to 3$ as $n \to \infty$. Let $\varepsilon > 0$ be arbitrary.

Find $n_0 \in \mathbb{N}$ s.t. $|\frac{2}{n} + 3 - 3| = \frac{2}{n} < \varepsilon$ for all $n \geq n_0$. Then

$$\frac{2}{n} < \varepsilon \Leftrightarrow \frac{2}{\varepsilon} < n$$

for $n \in \mathbb{N}$. Then, if $n_0 > \frac{2}{\varepsilon}$, then $|x_n - 3| < \varepsilon, \forall n \geq n_0$, i.e. x_n converges.

ii) $\lim_{n\to\infty}\left(\frac{n}{1+\frac{1}{n}} - n\right) = \lim_{n\to\infty}\left(\frac{n - n - 1}{1+\frac{1}{n}}\right) = \frac{-1}{1} = -1$

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$$

$$\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = \frac{\lim(x_n)_{n\to\infty}}{\lim_{n\to\infty}(y_n)} \quad \text{if } \lim_{n\to\infty}(y_n) \not\to 0$$

iii) Suppose $\exists L \in \mathbb{R}$ s.t. $y_n \to L$ as $n \to \infty$. Take $\varepsilon = \frac{1}{4} > 0$. ~~Take $n \in \mathbb{N}, n_0 \in \mathbb{N}$ with $n \geq n_0$ and~~ Then, for $n \in \mathbb{N}$, $n_0 \in \mathbb{N}$,

$$n \geq n_0 \Rightarrow |x_n - L| < \frac{1}{4}$$

by definition. Then take $n \geq n_0$ and ~~...~~ $3 | n$ to see that $|0 - L| < \frac{1}{4}$. Then take $n \geq n_0$ and $3 \nmid n$ to see that $|\frac{1}{2} - L| < \frac{1}{4}$. Then, using the triangle inequality,

$$\frac{1}{2} = |0 - \tfrac{1}{2}| = |0 - L + L - \tfrac{1}{2}| \leq |0 - L| + |\tfrac{1}{2} - L| < \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2},$$

which is a contradiction. Thus, y_n does not converge.

24672735

MATHS2032

3. i) Take $y_n = (3^n)^{\frac{1}{n}} = 3$ and $z_n = (2\cdot 3^n)^{\frac{1}{n}} = 2^{\frac{1}{n}} \cdot 3$. Since $y_n \to 3$ as $n \to \infty$, $z_n \to 3$ as $n \to \infty$ (standard limit), and

$$y_n \leq x_n \leq z_n,$$

$x_n \to 3$ as $n \to \infty$; thus, x_n is convergent by the sandwich principle.

ii) If $\frac{x_n - 2}{x_n + 1} \to 0$ as $n \to \infty$, then $(x_n - 2) \to 0$ as $n \to \infty$. Then, by algebraic properties of limits, $\lim\limits_{n\to\infty}(x_n - 2) = 0 \iff \lim\limits_{n\to\infty} x_n - \lim\limits_{n\to\infty} 2 = 0 \iff \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} 2 \iff \lim\limits_{n\to\infty} x_n = 2$.

4. i) ~~Since~~ Set $x_n = \frac{7^n}{n!}$. Since $x_n > 0, \forall n$, one can use the ratio test:

$$\frac{x_{n+1}}{x_n} = \frac{7 \cdot \cancel{7^n} \cdot \cancel{n!}}{(n+1)\cancel{n!} \cdot \cancel{7^n}} = \frac{7}{n+1} \to 0 < 1.$$ Thus, $\sum\limits_{n=1}^{\infty} x_n$ converges by the limit version of the ratio test.

ii) ~~Set~~ Since $\sum\limits_{n=1}^{\infty} \frac{2}{3^n - \cos(2n)} \leq \sum\limits_{n=1}^{\infty} \frac{10}{3^n}$, which converges, $\sum\limits_{n=1}^{\infty} \frac{2}{3^n - \cos(2n)}$ converges by the comparison test. (Since $0 \leq \cos(2n) \leq 1$)

iii) Since $\sum\limits_{n=1}^{\infty} \frac{n^3 - n}{3n^4 + 3n^2 + 2} \geq \sum\limits_{n=1}^{\infty} \frac{n^3}{4n^4} = \sum\limits_{n=1}^{\infty} \frac{1}{4n}$, which diverges (harmonic series), $\sum\limits_{n=1}^{\infty} \frac{n^3 - 3}{3n^4 + 3n^2 + 2}$ also diverges by the comparison test.

24672273
MATHS2032

(5) i) ~~Define a sequence~~ Define a series $\sum_{i=1}^{\infty} s_i$, where $s_{2i} = -\frac{1}{i}$ and $s_{2i+1} = \frac{1}{i}$ (not the harmonic series for the terms to cancel out). The series is alternating and converges to 0 by construction. However, the series is not absolutely convergent as, ~~then the~~ with the absolute value, the underlying sequence is $\frac{1}{n}$, making the series the harmonic series (if without alternating).

ii) If $\sum_{n=1}^{\infty} a_n$ converges, $a_n \to 0$ as $n \to \infty$. Then, by definition of convergence, $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0$, $|a_n - 0| = |a_n| < 1$.

Then $a_n^2 \leq a_n$ for all $n \geq n_0$. Then, by the comparison test,

$$0 \leq \sum_{n=1}^{\infty} a_n^2 \leq \sum_{n=1}^{\infty} a_n,$$

$\sum_{n=1}^{\infty} a_n^2$ converges.

24672735

MATHS2032

~~(5)~~

(6) i) Let $c \in \mathbb{R}$ be arbitrary and let $\varepsilon > 0$ be arbitrary. Then

$$|f(x) - f(c)| = |4x - 7 - 4c + 7| = 4|x - c| < \varepsilon,$$

provided $|x - c| < \frac{\varepsilon}{4}$. Taking $\delta = \frac{\varepsilon}{4}$, f is continuous at c.
Since c is arbitrary, f is continuous.

ii) Let $\varepsilon > 0$ and take $\delta = \varepsilon$. Given $x \in \mathbb{R}$ with $|x - 0| < \delta$, either

$x = 0$ when $|f(x) - f(0)| = 0 < \varepsilon$, or $x \neq 0$ when

$$|f(x) - f(0)| = |x \sin(\tfrac{1}{x})| \leq |x| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, f is continuous at 0 because $|x| < \delta$ implies $|f(x) - f(0)| < \varepsilon$.

iii) Let $g(x) = f(x) - (1 - x)$. Since f is continuous, and the function $x \mapsto 1 - x$ is continuous, then g is continuous. Since $f(-1), f(1) \in [0, 2]$, we have

$$g(-1) = f(-1) - (1 + 1) = f(-1) - 2 \leq 0,$$
$$g(1) = f(1) - 0 = f(1) \geq 0.$$

Therefore, by the intermediate value theorem, $\exists x \in [-1, 1]$ with $g(x) = 0$. Thus, $f(x) = 1 - x$.