a) Let $\boldsymbol{u} = [1, -1, 5]$, $\boldsymbol{v} = [2, 3, -1]$ and $\boldsymbol{w} = [-3, -7, 7]$ be vectors in $\mathbb{R}^3$. Are **u**, **v** and **w** linearly independent? Justify your answer.

In order for the vectors to be linearly independent, the only solution to the equation

$$p\boldsymbol{u} + q\boldsymbol{v} + r\boldsymbol{w} = \boldsymbol{0}$$

has to be $p = q = r = 0$ for scalars $p, q, r \in \mathbb{R}$. The above equation then can be rewritten as a system of linear equations

$$\begin{aligned} &p + 2q - 3r = 0, \\ &-p + 3q - 7r = 0, \\ &5p - q + 7r = 0. \end{aligned}$$

By $R_1 + R_2$ and $R_3 - 5R_1$ it is reduced to

$$\begin{aligned} &5q - 10r = 0, \\ &-11q + 22r = 0, \end{aligned}$$

where both equations reduce to

$$q - 2r = 0;$$

Since both equations are the same, there are infinitely many solutions to the system. The scalars can then be expressed as

$$\begin{aligned} &r = t, \\ &q = 2t, \\ &p = 3r - 2q = 3t - 4t = -t. \end{aligned}$$

Thus, the three vectors are linearly dependent, as required.

b) Consider the 2×2 matrices

$$A_1 = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$$

Show that the matrix

$$B = \begin{pmatrix} -3 & 8 \\ 4 & -7 \end{pmatrix}$$

belongs to Span(A_1, A_2).

To prove that B is in the span, B can be expressed as a linear combination of A_1 and A_2 as

$$\begin{aligned} &B = xA_1 + yA_2 \\ &\Rightarrow \begin{pmatrix} -3 & 8 \\ 4 & -7 \end{pmatrix} = x\begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix} + y\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}, \end{aligned}$$

which can be written as a system of linear equations

$$\begin{aligned}
x + 3y &= -3,\\
4x + 2y &= 8,\\
2x + y &= 4,\\
-x + 2y &= -7.
\end{aligned}$$

By $R_1 + R_4$, we get

$$\begin{aligned}
&5y = -10\\
&\Rightarrow y = -2.
\end{aligned}$$

By R_1, we get

$$x = -3 - 3y = -3 + 6 = 3.$$

To check for consistency,

$$\begin{aligned}
&R_2: 12 - 4 = 8,\\
&R_3: 6 - 2 = 4,\\
&R_4: -3 - 4 = -7.
\end{aligned}$$

Thus, B belongs to Span(A_1, A_2), as required.

c) Let A and B be invertible 2×2 matrices given by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

Let $\lambda \in R$ be a scalar. Prove that $(A^{-1} + \lambda B^{-1})^T = (A^T)^{-1} + \lambda(B^T)^{-1}$. Suppose A and B are both symmetric, how does this result simplify further.

The transposes of both matrices are

$$A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad B^T = \begin{pmatrix} e & g \\ f & h \end{pmatrix}.$$

The inverses of both matrices are

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, B^{-1} = \frac{1}{eh - fg}\begin{pmatrix} h & -g \\ -f & e \end{pmatrix}$$

The LHS can be simplified step-by-step as

$$\begin{aligned}
(A^{-1} + \lambda B^{-1})^T &= \left(\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} + \frac{\lambda}{eh - fg}\begin{pmatrix} h & -g \\ -f & e \end{pmatrix}\right)^T\\
&= \begin{pmatrix} d \cdot \det(A) + \lambda h \cdot \det(B) & -b \cdot \det(A) - \lambda g \cdot \det(B) \\ -c \cdot \det(A) - \lambda f \cdot \det(B) & a \cdot \det(A) + \lambda e \cdot \det(B) \end{pmatrix}^T\\
&= \begin{pmatrix} d \cdot \det(A) + \lambda h \cdot \det(B) & -c \cdot \det(A) - \lambda f \cdot \det(B) \\ -b \cdot \det(A) - \lambda g \cdot \det(B) & a \cdot \det(A) + \lambda e \cdot \det(B) \end{pmatrix}.
\end{aligned}$$

The RHS can be simplified step-by-step as

$$
\begin{aligned}
&(A^T)^{-1} + \lambda(B^T)^{-1} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} + \lambda \begin{pmatrix} e & g \\ f & h \end{pmatrix}^{-1} \\
&= \frac{1}{ad - bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix} + \frac{\lambda}{eh - fg}\begin{pmatrix} h & -g \\ -f & e \end{pmatrix} \\
&= \begin{pmatrix} d \cdot \det(A) + \lambda h \cdot \det(B) & -c \cdot \det(A) - \lambda f \cdot \det(B) \\ -b \cdot \det(A) - \lambda g \cdot \det(B) & a \cdot \det(A) + \lambda e \cdot \det(B) \end{pmatrix}.
\end{aligned}
$$

As can be seen, the LHS and RHS are equal; thus, $(A^{-1} + \lambda B^{-1})^T = (A^T)^{-1} + \lambda(B^T)^{-1}$. If A and B are both symmetric, then $b = c$ and $f = g$. Therefore,

$$
\begin{aligned}
&(A^{-1} + \lambda B^{-1})^T = (A^T)^{-1} + \lambda(B^T)^{-1} = A^{-1} + \lambda B^{-1} \\
&= \begin{pmatrix} d \cdot \det(A) + \lambda h \cdot \det(B) & -b \cdot \det(A) - \lambda f \cdot \det(B) \\ -b \cdot \det(A) - \lambda f \cdot \det(B) & a \cdot \det(A) + \lambda e \cdot \det(B) \end{pmatrix},
\end{aligned}
$$

where

$$
\det(A) = \frac{1}{ad - b^2} \text{ and } \det(B) = \frac{1}{eh - f^2}.
$$