EXAMINATION FOR THE DEGREES OF
M.A. AND B.Sc.

Mathematics 1

An electronic calculator may be used provided that it is allowed under the School of Mathematics and Statistics Calculator Policy. A copy of this policy has been distributed to the class prior to the exam and is also available via the invigilator.

Candidates must attempt the ALL *of Section A and* ALL *of Section B.*

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Section A Multiple Choice

Attempt ALL *questions from this section. Each question has only ONE correct answer. Enter your answers as well as your student number and name on the provided scanning sheet with a* BLACK BALL POINT PEN. *Return the sheet together with your script book.*

Q1. Suppose a connected planar graph G has 5 vertices and 3 faces. How many edges does G have? **3**

A) 5 edges

B) 6 edges

C) 7 edges **correct**

D) 8 edges

E) None of the above

Q2. Calculate the derivative of $g(x) = \sin^2 x \cos x$. **3**

A) $2 \sin x \cos x$

B) $2 \sin x \cos x - \sin^3 x$

C) $2 \sin x \cos x - \sin x$

D) $2 \sin x \cos^2 x - \sin^3 x$ **correct**

E) None of the above.

Q3 Find all values of x such that $3|x - 6| \leqslant |x - 2|$.

A) $5 \leqslant x \leqslant 8$. **correct**

B) $x \leqslant 5$ and $x \geqslant 8$.

C) $x \geqslant 5$.

D) $x \leqslant 8$.

E) None of the above.

Q4. Let $f(x) = x^2 + 4x + 3$ with domain $[-2, \infty)$. What is the value of $(f^{-1})'(3)$? **3**

A) $1/3$

B) 3

C) 4

D) $1/4$ **correct**

E) None of the above.

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Q5. Which of the following statements is true of the equation $y^3 = x^5$? **3**

A) There are no integer solutions.

B) There is a unique integers solution.

C) There are exactly two integer solutions.

D) There are an infinite number of integer solutions. **correct**

E) None of the above.

Q6. What is the third nonzero term in the Taylor series for $f(x) = 2 \ln x$ centered at $a = 1$? **3**

A) $4x^3$

B) $\frac{1}{3}x^3$

C) $\frac{2}{3}(x-1)^3$ **correct**

D) $\frac{4}{3}(x-1)^3$

E) None of the above.

Q7. Let $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial with real coefficients $a_{n-1}, \ldots, a_1, a_0$ with $a_0 \neq 0$. Which one of the following statements is correct? **3**

A) The polynomial p has n distinct complex roots.

B) If n is odd, then p must have at least one real root. **correct**

C) The roots of p come in complex conjugate pairs, so there must be an even number of roots.

D) One of the roots of $p(x)$ is 0.

E) None of the above.

Q8. Find an equation of the tangent line to the curve defined by $\sinh(xy) = \cos(x+y)$ at the point $(x, y) = (0, \pi/2)$. **3**

A) $y = \frac{\pi}{2} - (1 + \frac{\pi}{2})x$ **correct**

B) $y = \frac{\pi}{2} - x$

C) $y = \frac{\pi}{2} - \frac{x}{2}$

D) $y = \frac{\pi}{2} - \frac{\pi}{2}x$

E) None of the above.

Q9. Identify the remainder r (between 0 and 12) that we get when we divide 6^{82} by 13. **3**

A) $r = 0$.

B) $r = 3$.

C) $r = 6$.

D) $r = 9$.

E) None of the above. **correct (solution is $r = 4$)**

Q10. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be vectors. Which of the following expressions is **not** mathematically meaningful? **3**

A) $|\mathbf{a}|(\mathbf{b} \times \mathbf{c})$

B) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$

C) $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ **correct**

D) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

E) $|\mathbf{a}|(\mathbf{b} \cdot \mathbf{c})$

Section B

Attempt ALL *questions from this section.*

B1. (i) State the Principle of Mathematical Induction. **2**

The Principle of Mathematical Induction: Suppose that for each positive integer n we have a statement $P(n)$. If we prove the following two things:

(a)$P(1)$ is true;

(b)for all n, if $P(n)$ is true, then $P(n+1)$ is also true;

then $P(n)$ is true for all positive integers $n \in \mathbb{N}$.

2 Marks. Bookwork.

(ii) Prove the following formula holds for all $n \in \mathbb{N}$.

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$ **3**

The statement $P(n)$ is that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ is true $\forall n$. To prove $P(1)$ we observe that $1 = \frac{1(1+1)}{2}$. Now assume $P(n)$ is true and consider $P(n+1)$. We have

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \quad \text{by } P(n) \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}, \end{aligned}$$

which is $P(n+1)$. So, the Principle of Mathematical Induction implies that $P(n)$ holds for all n. **3 Marks. Example from class.**

(iii) Compute the rational form of $0.8\overline{594}$. **3**

We have

$$\begin{aligned} 0.8\overline{594} &= \frac{8}{10} + \frac{594}{10^4} + \frac{594}{10^7} + \frac{594}{10^{10}} + \ldots \\ &= \frac{8}{10} + \frac{594}{10^4}\left(1 + \frac{1}{10^3} + \frac{1}{10^6} + \ldots\right) \\ &= \frac{8}{10} + \frac{594}{10^4}\left(\frac{1000}{999}\right) \\ &= \frac{7992 + 594}{9990} = \frac{8586}{9990}. \end{aligned}$$

3 Marks. Easy question. Similar to question in book.

B2. (i) State the Mean Value Theorem. **2**

Let f be a function that satisfies the following hypotheses:

- f is continuous on the closed interval $[a, b]$
- f is differentiable on the open interval (a, b)

Then there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

2 Marks. Bookwork.

(ii) Find the derivative of $f(x) = \frac{x}{x+4}$. **2**

Using the Quotient Rule we find

$$f'(x) = \frac{(x+4) - x}{(x+4)^2} = \frac{4}{(x+4)^2}.$$

2 Marks. Unseen, straightforward example.

(iii) Find all numbers, c, that satisfy the conclusion of the Mean Value Theorem for $f(x)$ defined on the interval $[-1, 8]$. **4**

$f(x)$ is continuous on $[-1, 8]$ and differentiable on $(-1, 8)$ and so it satisfies the hypotheses of the Mean Value Theorem. Therefore $\exists c \in (-1, 8)$ such that

$$f'(c) = \frac{f(8) - f(-1)}{8 - (-1)} = \frac{\frac{2}{3} - (\frac{-1}{3})}{9} = \frac{1}{9}.$$

Therefore, using the result from part (ii), we have

$$\frac{4}{(c+4)^2} = \frac{1}{9} \Rightarrow (c+4)^2 = 36 \Rightarrow c = \pm 6 - 4 \Rightarrow c = 2 \text{ or } c = -10.$$

Then $2 \in (-1, 8)$ but $-10 \notin (-1, 8)$ so the only valid value is $c = 2$.
4 Marks. Unseen example.

B3. (i) State the relationship between the cross product of two vectors $\mathbf{u}$ and $\mathbf{v}$ and the angle, θ, between them. **2**

$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$.
2 Marks. Bookwork.

(ii) Let $\mathbf{a} = \langle 1, 2, 0 \rangle$ and $\mathbf{b} = \langle 0, 4, 1 \rangle$ and show that $\langle 2, -1, 4 \rangle$ is orthogonal to both vectors. **2**

The cross product, $\mathbf{a} \times \mathbf{b}$, produces a vector that is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. We find

$$\mathbf{a} \times \mathbf{b} = \langle 2-0, 0-1, 4-0 \rangle = \langle 2, -1, 4 \rangle,$$

so $\langle 2-0, 0-1, 4-0 \rangle$ as required.

Alternatively, we could use the dot product twice to show that $\langle 2, -1, 4 \rangle$ is orthogonal to $\mathbf{a}$ and, separately, orthogonal to $\mathbf{b}$. Two vectors are orthogonal if their dot product vanishes. We find $\mathbf{a} \cdot \langle 2, -1, 4 \rangle = \langle 1, 2, 0 \rangle \cdot \langle 2, -1, 4 \rangle = 2 - 2 + 0 = 0$ and $\mathbf{b} \cdot \langle 2, -1, 4 \rangle = \langle 0, 4, 1 \rangle \cdot \langle 2, -1, 4 \rangle = 0 - 4 + 4 = 0$, as expected.

2 Marks. Unseen, straightforward example.

(iii) Find the (scalar) equation of the plane that contains the vectors $\mathbf{a}$ and $\mathbf{b}$ and the point $(1, 1, 1)$. **3**

If both $\mathbf{a}$ and $\mathbf{b}$ are contained in the plane then a normal vector to the plane will be a vector orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. From part (ii) we have such a vector so we let the normal vector to the plane be $\mathbf{n} = \langle 2, -1, 4 \rangle$. Then the vector equation of the plane is

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0,$$

where $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r_0}$ is the position vector of point in the plane. So let $\mathbf{r_0} = \langle 1, 1, 1 \rangle$ and then

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = \langle 2, -1, 4 \rangle \cdot \langle x-1, y-1, z-1 \rangle = 2(x-1) - (y-1) + 4(z-1) = 2x - y + 4z - 5.$$

The equation of the plane is then given by

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0 &\Rightarrow 2(x-1) - (y-1) + 4(z-1) = 2x - y + 4z - 5 = 0 \\ &\Rightarrow 2x - y + 4z = 5. \end{aligned}$$

3 Marks. Unseen example.

B4. (i) State Fermat's Little Theorem. **2**

Fermat's Little Theorem: Let p be a prime number, and let a be an integer that is not divisible by p. Then

$$a^{p-1} \equiv 1 (\operatorname{mod} p).$$

2 Marks. Bookwork.

(ii) Prove by contradiction or otherwise that there are infinitely many prime numbers. **5**

Assume there are only finitely many prime numbers, so that we can list all the primes in order as $2 = p_1 < p_2 < p_3 < \ldots < p_n$. Now define a positive integer by

$$N = p_1 p_2 \cdots p_n + 1.$$

By the Fundamental Theorem of Arithmetic, N is equal to a product of prime numbers, and must be divisible by some prime on the original list, so $\exists\ 1 \leqslant i \leqslant n$ so that a is a natural number and

$$a = \frac{N}{p_i} = \frac{p_1 p_2 p_3 \cdots p_n + 1}{p_i} = (p_1 p_2 p_3 \cdots p_{i-1} p_{i+1} \cdots p_n) + \frac{1}{p_i},$$

which is impossible since $p_i \geqslant 2$, contradicting our hypothesis that there are finitely many primes. **5 Marks. Bookwork.**

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