2C Intro to real analysis 2020/21

Ex Sheet 7

Solutions and Comments [1] [2]

Q1 *Show that each of the following series diverges:*

$$\sum_{n=1}^{\infty} \frac{2n-5}{3n+1}, \quad \sum_{n=1}^{\infty} \frac{3^n-2^n}{3^{n+1}}, \quad \sum_{n=1}^{\infty} \frac{3+\sin(n)}{4+\cos(n)}.$$

In each case the terms of the series do not converge to zero, so the series diverges. In the first two cases we can demonstrate this by showing that the terms converge to some other limit:

We have

$$\frac{2n-5}{3n+1} = \frac{2-5/n}{3+1/n} \to \frac{2-0}{3+0} = \frac{2}{3} \neq 0,$$

as $n \to \infty$. Therefore, the series $\sum_{n=1}^{\infty} \frac{2n-5}{3n+1}$ diverges.

We have

$$\frac{3^n-2^n}{3^{n+1}} = \frac{1-(2/3)^n}{3} \to \frac{1-0}{3} = \frac{1}{3} \neq 0,$$

as $n \to \infty$. Therefore, the series $\sum_{n=1}^{\infty} \frac{3^n-2^n}{3^{n+1}}$ diverges.

In the last case this won't work as the sequence $(a_n)_{n=1}^{\infty}$ given by $a_n = \frac{3+\sin(n)}{4+\cos(n)}$ does not converge to some other value. We need a different strategy of showing that $a_n \not\to 0$, and so we look to bound a_n away from zero. My strategy for doing this is to ask how small is it possible for a_n to be. Remember that $-1 \leq \sin(x) \leq 1$ and $-1 \leq \cos(x) \leq 1$. We use these inequalities below[3]

For $n \in \mathbb{N}$, we have

$$2 \leq 3+\sin(n) \leq 4 \text{ and } 3 \leq 4+\cos(n) \leq 5.$$

Therefore

$$\frac{3+\sin(n)}{4+\cos(n)} \geq \frac{2}{5} > 0,$$

so that

$$\frac{3+\sin(n)}{4+\cos(n)} \not\to 0,$$

as $n \to \infty$. Accordingly, $\sum_{n=1}^{\infty} \frac{3+\sin(n)}{4+\cos(n)}$ diverges.

Q2

a) *Use algebraic and order properties of sequence limits to carefully prove[4] parts b), c) and d) of Theorem 4.6.*

[1] If you've not seriously tried these exercises, please don't look at these solutions and comments, until you have. You'll get the most benefit from reading these comments, when you've first thought hard about them yourself, even if you get really stuck — don't just try for a few minutes and then look at the solutions to work out how to proceed, you don't learn anywhere near as much that way.

[2] Note that I deliberately do not include formal answers for all questions.

[3] being careful to get the inequality for the denominator the right way round: a standard error is to claim that $\frac{1}{4+\cos(n)} \geq \frac{1}{3}$ rather than $\frac{1}{4+\cos(n)} \geq \frac{1}{5}$.

[4] Start by introducing sequences of partial sums for the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$.

b) Give an example of two divergent series $\sum_{n=1}^{\infty} x_n$ *and* $\sum_{n=1}^{\infty} y_n$ *such that* $\sum_{n=1}^{\infty}(x_n + y_n)$ *converges.*

The idea for the proofs in the first part of this question is relatively straightforward - we deduce the respective properties of series limits from the corresponding properties of sequence limits. To write it down correctly, we must first take care to introduce our notation. Make sure you are clear about what needs to be proved in each case.

Let $\sum_{n=1}^{\infty} x_n$ have sum S and $\sum_{n=1}^{\infty} y_n$ have sum T. For $N \in \mathbb{N}$, write $s_N = \sum_{n=1}^{N} x_n$ and $t_N = \sum_{n=1}^{N} y_n$, so that $s_N \to S$ and $t_N \to T$ as $N \to \infty$.

For part b), note that the partial sum of the series $\sum_{n=1}^{\infty}(x_n + y_n)$ is $\sum_{n=1}^{N}(x_n + y_n) = s_N + t_N$. By part b) of Theorem 3.10 we have $s_N + t_N \to S + T$ as $N \to \infty$. Therefore the series $\sum_{n=1}^{\infty}(x_n + y_n)$ converges and has sum $S + T$.

For part c), assume that $x_n \leq y_n$ for all n. Then for each $N \in \mathbb{N}$, we have $s_N \leq t_N$, so that $S \leq T$ by order properties of sequence limits.

Finally, for part d), write $r_N = \sum_{n=1}^{N} |x_n|$. Then we have

$$|s_N| = |x_1 + x_2 + \cdots + x_N| \leq |x_1| + |x_2| + \cdots |x_N| = r_N$$

by the triangle inequality. Therefore, by Theorem 3.12 we obtain

$$|S| = \lim_{n\to\infty} |s_N| \leq \lim_{n\to\infty} r_n = \sum_{n=1}^{\infty} |x_n|$$

as claimed.

Let us now turn to part b) of the question. A strategy to get $\sum_{n=1}^{\infty}(x_n + y_n)$ to converge while each of $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ diverges is to arrange for the terms $x_n + y_n$ to cancel. The easiest way to do this is to take $y_n = -x_n$. This leads to:

Take $x_n = 1$ and $y_n = -1$ for all $n \in \mathbb{N}$. Then $x_n + y_n = 0$, so the series $\sum_{n=1}^{\infty}(x_n + y_n)$ converges. Since $x_n \not\to 0$ and $y_n \not\to 0$, the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ both diverge.

Q3 *Use the comparison test to determine which of the series below converge or diverge. Justify your answers.*

a) $\sum_{n=1}^{\infty} \frac{n^2+2n+3}{n^4+n^2+1}$;

b) $\sum_{n=1}^{\infty} \frac{2n+3}{n^2-n+5}$;

c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n-1}}$;

d) $\sum_{n=1}^{\infty} \frac{2^n+5^n}{4^n+7^n}$.

In each of these questions the first thing I do is work out roughly how large the n-th term of the series is, so I know what to compare with and whether I'm expecting convergence or divergence. In most, if not all of these cases, either the limiting or the direct comparison test both work. You may find the limit version more systematic, but sometimes the direct version is quicker[5]. Remember that you need to know the standard convergence results: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$, and $\sum_{n=1}^{\infty} x^n$ converges if and only if $|x| < 1$.

[5] particularly if you can quickly see a relevant inequality.

In a), we note that the n-th term is roughly $1/n^2$, and as $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we expect the original series to converge. I'll use the direct version of the comparison test:

For $n \in \mathbb{N}$, we have

$$0 \leq \frac{n^2+2n+3}{n^4+n^2+1} \leq \frac{n^2+2n^2+3n^2}{n^4} = \frac{6}{n^2}.$$

Since the series $\sum_{n=1}^{\infty} \frac{6}{n^2}$ converges, so too does $\sum_{n=1}^{\infty} \frac{n^2+2n+3}{n^4+n^2+1}$.

In b), we expect the series to diverge by comparing with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. This time I'll use the limit version[6].

[6] though you could use the comparison test directly via the inequality

$$\frac{2n+3}{n^2-n+5} \geq \frac{2n}{n^2} = \frac{2}{n}$$

for $n \geq 5$.

Set $a_n = \frac{2n+3}{n^2-n+5}$ and $b_n = \frac{1}{n}$. Then $a_n > 0$ and $b_n > 0$ for all n, and we have

$$\frac{a_n}{b_n} = \frac{2n^2+3n}{n^2-n+5} = \frac{2+3/n}{1-1/n+5/n^2} \to \frac{2+0}{1-0+0} = 2,$$

as $n \to \infty$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so too does $\sum_{n=1}^{\infty} \frac{2n+3}{n^2-n+5}$ by the limit version of the comparison test.

In c) the n-th term is roughly $\frac{1}{2\sqrt{n}}$. This leads to the following argument:

For $n \geq 1$, we have

$$\frac{1}{\sqrt{n}+\sqrt{n-1}} \geq \frac{1}{2\sqrt{n}} \geq 0,$$

and so $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n-1}}$ diverges as $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}}$ diverges.

Finally, in d) we compare with the geometric series $\sum_{n=1}^{\infty}(5/7)^n$. Again our choice to do this is made by looking at the dominating terms in the numerator and denominator.

Let $a_n = \frac{2^n+5^n}{4^n+7^n}$ and $b_n = \frac{5^n}{7^n}$ so that $a_n > 0$ and $b_n > 0$. We have

$$\frac{a_n}{b_n} = \frac{2^n+5^n}{4^n+7^n}\frac{7^n}{5^n} = \frac{(2/5)^n+1}{(4/7)^n+1} \to \frac{0+1}{0+1} = 1,$$

as $n \to \infty$. By the limit version of the comparison test, $\sum_{n=1}^{\infty} \frac{2^n+5^n}{4^n+7^n}$ converges as the geometric series $\sum_{n=1}^{\infty}(5/7)^n$ converges.

Q4 *Use the ratio test to determine which of the series below converge or diverge. Justify your answers.*

a) $\sum_{n=1}^{\infty} \frac{5^n}{n^3}$;

b) $\sum_{n=1}^{\infty} \frac{5^n}{n!}$;

c) $\sum_{n=1}^{\infty} \binom{2n}{n} \frac{1}{2^n}$.

In each case, start by introducing your notation - no a_n is mentioned in the question, so you must introduce this symbol if you plan to use it. Also take care to note that $a_n > 0$ as this was a hypothesis in the ratio test. Make sure you give enough information in the calculation of any ratio limits so that the person reading your work can see how this limit was computed[7]. It's also a good idea to make sure your answer shows your knowledge of the ratio test: in all the answers below, the last sentence shows that I know what the critical threshold in the limit version of the ratio test is.

[7] You don't necessarily need to give as much information as in a question about computing sequence limits, so I tend not to refer to standard limits in these questions, but it should be clear what was done.

a) Let $a_n = \frac{5^n}{n^3}$ and note that $a_n > 0$ for all n. Then

$$\frac{a_{n+1}}{a_n} = \frac{5^{n+1}}{(n+1)^3}\frac{n^3}{5^n} = \frac{5}{(1+1/n)^3} \to 5,$$

as $n \to \infty$. Since $5 > 1$, the series $\sum_{n=1}^{\infty} \frac{5^n}{n^3}$ diverges by the ratio test.

b) Let $a_n = \frac{5^n}{n!}$ and note that $a_n > 0$ for all n. Then

$$\frac{a_{n+1}}{a_n} = \frac{5^{n+1}}{(n+1)!}\frac{n!}{5^n} = \frac{5}{n+1} \to 0,$$

as $n \to \infty$. Since $0 < 1$, the series $\sum_{n=1}^{\infty} \frac{5^n}{n!}$ converges by the ratio test.

c) Set $a_n = \binom{2n}{n}\frac{1}{2^n}$ and note that $a_n > 0$ for all n. Since $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$, we have

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(2(n+1))!}{((n+1)!)^2 2^{n+1}}\frac{(n!)^2 2^n}{(2n)!}\\ &= \frac{(2n+2)(2n+1)}{2(n+1)^2} = \frac{2n+1}{n+1} = \frac{2+1/n}{1+1/n} \to 2\end{aligned}$$

as $n \to \infty$. Since $2 > 1$, the series $\sum_{n=1}^{\infty} \binom{2n}{n}\frac{1}{2^n}$ diverges by the ratio test.

In this last answer be careful with brackets. A standard error is to not substitute $n+1$ correctly into the computation of a_{n+1} obtaining $\frac{(2n+1)!}{((n+1)!)^2 2^{n+1}}$ rather than the correct $\frac{(2n+2)!}{((n+1)!)^2 2^{n+1}}$.

Q5 *Use the ratio test to determine which of the series below converge or diverge. Justify your answers.*

a) $\sum_{n=1}^{\infty} \frac{n+1}{5^{2n+1}}$;

b) $\sum_{n=1}^{\infty} \frac{n!}{5^{n^2}}$;

c) $\sum_{n=1}^{\infty} \frac{3\times 8\times 13\times\cdots\times(5n-2)}{5\times 9\times 13\times\cdots\times(4n+1)} \left(\frac{3}{4}\right)^n$.

This is similar to the previous question.

a) Set $a_n = \frac{n+1}{5^{2n+1}}$ and note that $a_n > 0$. Then

$$\frac{a_{n+1}}{a_n} = \frac{n+2}{5^{2n+3}} \frac{5^{2n+1}}{n+1} = \frac{n+2}{25(n+1)} \to \frac{1}{25},$$

as $n \to \infty$. As $\frac{1}{25} < 1$, the series $\sum_{n=1}^{\infty} \frac{n+1}{5^{2n+1}}$ converges by the ratio test.

In the next part the ratio gives us a slightly more complicated limit to compute. We can see that $\frac{n+1}{5^{2n+1}}$ should converge to zero, but we need to justify this. In my answer below, I make a very crude estimate and then use the sandwich principle.

b) Set $a_n = \frac{n!}{5^{n^2}}$ and note $a_n > 0$ for all n. Then

$$0 \leq \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{5^{(n+1)^2}} \frac{5^{n^2}}{n!} = \frac{(n+1)}{5^{2n+1}} \leq \frac{5^n}{5^{2n+1}} = \frac{1}{5^{n+1}} \to 0,$$

as $n+1 \leq 5^n$ for all $n \geq 1$. By the sandwich principle $\frac{a_{n+1}}{a_n} \to 0$, as $n \to \infty$, so the series $\sum_{n=1}^{\infty} a_n$ converges by the ratio test.

In the final part, a typical mistake comes from the computation of a_{n+1} and the ratio $\frac{a_{n+1}}{a_n}$. Make sure you can see why this ratio is not

$$\frac{(5n-1)(5n)(5n+1)(5n+2)(5n+3)}{(4n+2)(4n+3)(4n+4)(4n+5)} \frac{3}{4}.$$

c) Let $a_n = \frac{3\times 8\times 13\times\cdots\times(5n-2)}{5\times 9\times 13\times\cdots\times(4n+1)} \left(\frac{3}{4}\right)^n$ and note that $a_n > 0$ for all n. Then

$$a_{n+1} = \frac{3\times 8\times 13\times\cdots\times(5n-2)(5n+3)}{5\times 9\times 13\times\cdots\times(4n+1)(4n+5)} \left(\frac{3}{4}\right)^{n+1},$$

so that

$$\frac{a_{n+1}}{a_n} = \frac{5n+3}{4n+5}\frac{3}{4} \to \frac{15}{16}.$$

Since $15/16 < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges by the ratio test.

Q6 *Let $\sum_{n=1}^{\infty} x_n$ be a series, and assume that $x_n \geq 0$ for all $n \in \mathbb{N}$. Assume moreover that $\sqrt[n]{x_n} \to L$ as $n \to \infty$ for some $L \in \mathbb{R}$ with $0 \leq L < 1$.*

a) Show[8] that there exists $0 \leq q < 1$ and $n_0 \in \mathbb{N}$ such that $x_n < q^n < 1$ for all $n \geq n_0$.

b) Use the comparison test to prove that $\sum_{n=1}^{\infty} x_n$ converges.

[8] You may try to use $q = L + \frac{1-L}{2}$.

For part a), first recall that $\sqrt[n]{x_n} \to L$ as $n \to \infty$ means that for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|\sqrt[n]{x_n} - L| < \varepsilon$ for all $n \geq n_0$. Since $L < 1$ we may apply this for $\varepsilon = \frac{1-L}{2}$, and hence obtain

$$-\frac{1-L}{2} < \sqrt[n]{x_n} - L < \frac{1-L}{2},$$

in particular

$$\sqrt[n]{x_n} < L + \frac{1-L}{2}$$

for all $n \geq n_0$. If we set $q = L + \frac{1-L}{2}$, then $q < 1$ and the previous relation implies

$$x_n < q^n < 1$$

as desired.

For part b) it suffices to apply the comparison test, using the geometric series $\sum_{n=1}^{\infty} q^n$.

Q7 This exercise is more difficult than the others. It will not feature on the exam.

The aim of this question is to prove for any real number $p > 1$ that $\sum_{n=1}^{\infty} n^{-p}$ converges[9]. For $p \geq 2$, the claim is a consequence of proposition 4.9 in the lectures notes and the comparison test, but for $1 < p < 2$ more work is required.

[9] In this exercise, you may use laws of indices freely, we have not defined n^{-p} rigorously when p is irrational!

a) Let $(a_n)_{n=1}^{\infty}$ be a decreasing sequence of positive terms. For $m \in \mathbb{N}$, explain[10] why

$$S_{2^m-1} = \sum_{n=1}^{2^m-1} a_n \leq \sum_{r=1}^{m} 2^{r-1} a_{2^{r-1}}.$$

[10] Write out what this inequality says for $m = 1, m = 2, m = 3$ before trying to explain the general case.

b) Let $(a_n)_{n=1}^{\infty}$ be a decreasing sequence of positive terms, and suppose that the series $\sum_{r=1}^{\infty} 2^{r-1} a_{2^{r-1}}$ converges. Prove that $\sum_{n=1}^{\infty} a_n$ converges.

c) Now let $p > 1$ be a real number and set $a_n = 1/n^p$. By applying the previous part, show $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

d) Let $p > 1$. Does $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converge?

I'll not provide solutions to this exercise, it's really for those of you who want an extra challenge.