University of Glasgow

EXAMINATION FOR THE DEGREES OF
M.A. AND B.Sc.

Mathematics 2E - Introduction to Real Analysis

An electronic calculator may be used provided that it does not have a facility for either textual storage or display, or for graphical display.

Candidates must attempt all questions.

Question 1 and question 2 are multiple choice questions. Use the response form "2E Degree Exam Multiple Choice Section" to record your answers.

1. (i) Which of the following statements is the negation of the statement

$$\forall p \in \mathbb{Z}, \forall q \in \mathbb{N}, (n \in \mathbb{N} \text{ and } \left(\frac{p}{q}\right)^n = 2) \implies n = 1\,?$$

(A) $n = 1 \implies \forall p \in \mathbb{Z}, \forall q \in \mathbb{N}, (n \in \mathbb{N} \text{ and } (p/q)^n = 2)$.
(B) $\exists p \notin \mathbb{Z}, \exists q \notin \mathbb{N}$ such that $n \notin \mathbb{N}$ and $(p/q)^n \neq 2$ and $n \neq 1$.
(C) $\exists p \in \mathbb{Z}, \exists q \in \mathbb{N}$ such that $n \in \mathbb{N}$ and $(p/q)^n = 2$ and $n \neq 1$.
(D) $n \neq 1 \implies \exists p \in \mathbb{Z}, \exists q \in \mathbb{N}$ such that $(n \notin \mathbb{N}$ or $(p/q)^n \neq 2)$.
(E) None of these statements. **2**

Solution: (C)

(ii) Let $A \subset \mathbb{R}$. Which of the following is equivalent to the statement that A is not bounded below?

(A) $\forall m \in \mathbb{R}, \exists a \in A$ such that $a \leqslant m$.
(B) $\exists m \in \mathbb{R}$ such that $\forall a \in A, a < m$.
(C) $\exists m \in \mathbb{R}, \exists a \in A$ such that $a > m$.
(D) $\forall a \in A, \exists m \in \mathbb{R}$ such that $a < m$.
(E) None of these statements. **2**

Solution: (A)

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(iii) Suppose that A and B are non-empty subsets of $\mathbb{R}$ and suppose that $0 \notin A$. We define the set $C \subset \mathbb{R}$ as

$$C = \left\{ \frac{1}{a} - b \mid a \in A, b \in B \right\}.$$

Which of the following statements implies that $\sup(C)$ exists?

(A) $a > 0$ for all $a \in A$ and B is bounded above.
(B) $a > 0$ for all $a \in A$ and B is bounded below.
(C) $\inf(A) > 0$ and B is bounded.
(D) A is bounded below and B is bounded below.
(E) None of these statements. **2**

Solution: (C)

(iv) Let $(x_n)_{n=1}^{\infty}$ be a real sequence. Which of the following statements guarantees that also $x_n \to L$ as $n \to \infty$?

(A) $\exists \varepsilon > 0$ such that $\forall n_0 \in \mathbb{N}, (n \in \mathbb{N} \text{ and } n \geqslant n_0) \implies |x_n - L| < \varepsilon$.
(B) $\exists \varepsilon > 0, \exists n_0 \in \mathbb{N}$ such that $(n \in \mathbb{N} \text{ and } n \geqslant n_0) \implies |x_n - L| < \varepsilon$.
(C) $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ such that $(n \in \mathbb{N} \text{ and } n \geqslant n_0) \implies |x_n| < \varepsilon$.
(D) $\forall \varepsilon > 0, \forall n_0 \in \mathbb{N}, (n \in \mathbb{N} \text{ and } n \geqslant n_0) \implies |x_n - L| < \varepsilon$.
(E) None of these statements. **2**

Solution: (D)

(v) Let $\varepsilon > 0$ be arbitrary and consider the true statement

$$\exists \delta > 0 \text{ such that } \forall x \in \mathbb{R}, |x-2| < \delta \implies \left|x^2 + 2x - 8\right| < \varepsilon.$$

Which of the following values for δ can be used to demonstrate this?

(A) $\delta < \min(-(x-2), x-2)$.
(B) $\delta < \min(1, \varepsilon/7)$.
(C) $\delta < \varepsilon/11$.
(D) $\delta < 8 - 2\varepsilon - \varepsilon^2$.
(E) None of these statements. **2**

Solution: (B) To see this, note that for $\delta < 1$ we have that

$$|x-2| < \delta \implies -1 < x - 2 < 1 \implies 5 < x + 4 < 7 \implies |x+4| < 7.$$

So for $\delta < 1$ we have that

$$|x^2 + 2x - 8| = |x+4||x-2| < 7|x-2| < \varepsilon$$

under the additional assumption that $|x-2| < \varepsilon/7$.

2. (i) Let $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ be real sequences for which $x_n \leqslant y_n$ for all $n \in \mathbb{N}$. Which of the following is guaranteed to be true?

(A) If one of the sequences converges then so must the other.
(B) If both $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ converge, then they have the same limit.
(C) If $x_n > 0$ for all $n \in \mathbb{N}$ then $y_n \to L$ as $n \to \infty$ for some $L \geqslant 0$.
(D) If $y_n \to 0$ as $n \to \infty$ then also $x_n \to 0$ as $n \to \infty$.
(E) None of these statements. **2**

Solution: (E)

(ii) Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be continuous and suppose that $g(x) > 0$ for all $x \in \mathbb{R}$. Let $I = (0, 1)$ be the open unit interval and let $h\colon I \to \mathbb{R}$ be the function defined by $h(x) = f(x)/g(x)$ for $x \in I$. Which of the following statements is true?

(A) The function h must attain its maximum.
(B) The function h is not necessarily uniformly continuous.
(C) The set $\{h(x) \mid x \in I\}$ must be bounded.
(D) All of the above statements are true.
(E) None of these statements. **2**

Solution: (C) There are simple counter-examples to (A). We know that h extends to a continuous function on the closed interval $[0, 1]$, so by a result from lectures the function is uniformly continuous, which obviously implies that h is uniformly continuous. This rules out (B). Since h extends to $[0, 1]$, a result from lectures shows that h must be bounded.

(iii) Let $\varepsilon > 0$ be arbitrary. Which of the following conditions on $n_0 \in \mathbb{N}$ ensure that the statement

$$(n \in \mathbb{N} \text{ and } n \geqslant n_0) \implies \frac{2n}{5n^2 - 3n - 1} < \varepsilon$$

is true?

(A) $n_0 > \max(1/(2\varepsilon), 3)$.
(B) $n_0 > 2/(5\varepsilon)$.
(C) $n_0 > 2\varepsilon$.
(D) $n_0 < 2\varepsilon$.
(E) None of these conditions. **2**

Solution: (A) For $n \in \mathbb{N}$ with $n > 3$ we have that $n^2 \geqslant 4n$ and hence

$$\frac{2n}{5n^2 - 3n - 1} < \frac{2n}{5n^2 - 3n - n} = \frac{2n}{5n^2 - 4n} \leqslant \frac{2n}{5n^2 - n^2} = \frac{1}{2n}.$$

Thus for $n_0 > \max(1/(2\varepsilon), 3)$, the above is $< \varepsilon$ for $n \geqslant n_0$.

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(iv) Which of the following series is conditionally convergent?

(A) $\sum_{n=1}^{\infty}(-1)^n$.
(B) $\sum_{n=1}^{\infty} 1/n$.
(C) $\sum_{n=1}^{\infty}(-1)^n/(2n)$.
(D) $\sum_{n=1}^{\infty}(-1/2)^n$.
(E) None of these series. **2**

Solution: **(C)** This series converges by the Leibniz test. But the series of absolute values is 1/2 times the harmonic series, which diverges by a result from lectures.

(v) Select a statement below which is false for some real sequence $(x_n)_{n=1}^{\infty}$ and function $f\colon \mathbb{R} \to \mathbb{R}$ or, if the four statements (A–D) must all be true, select (E).

(A) If $(x_n)_{n=1}^{\infty}$ converges and f is continuous, then $(f(x_n))_{n=1}^{\infty}$ converges.
(B) If $(x_n)_{n=1}^{\infty}$ diverges but $(f(x_n))_{n=1}^{\infty}$ converges, then f is not continuous.
(C) If $(x_n)_{n=1}^{\infty}$ is bounded and f is continuous, then $(f(x_n))_{n=1}^{\infty}$ is bounded.
(D) If $(x_n)_{n=1}^{\infty}$ is convergent and f is continuous, then $(f(x_n)-x_n)_{n=1}^{\infty}$ converges.
(E) All of the above statements are true. **2**

Solution: **(B)**

3. (i) Let $A \subset \mathbb{R}$ be non-empty and bounded above. Define the supremum of A. **2**

Solution: *The supremum of A, denoted $\sup(A)$, is the unique least upper bound of A, which means that:*

- *the supremum is an upper bound for A, that is $\sup(A) \geqslant a$ for all $a \in A$;*
- *the supremum is the least such upper bound for A, which means that for any other upper bound $M \in \mathbb{R}$ for A, we have that $\sup(A) \leqslant M$.*

(ii) Show that

$$\sup\left\{\frac{n+1}{3n+7} \mid n \in \mathbb{N}\right\} = \frac{1}{3}.$$

4

Solution: *We denote $A = \{\frac{n+1}{3n+7} \mid n \in \mathbb{N}\}$. For any $n \in \mathbb{N}$ we have that*

$$\frac{n+1}{3n+7} = \frac{n+1}{3(n+1)+1} \leqslant \frac{n+1}{3(n+1)} = 1/3.$$

It follows that $1/3$ is an upper bound for A.
Let $\varepsilon > 0$ be arbitrary and set $n > 4/(9\varepsilon)$. Then

$$\frac{1}{3} - \frac{n+1}{3n+7} = \frac{(3n+7)-(3n+3)}{9n+21} = \frac{4}{9n+21} < \frac{4}{9n} < \varepsilon.$$

Thus we have found an element $a \in A$ for which $1/3 - a < \varepsilon$, equivalently $a > 1/3 - \varepsilon$. By a result from lectures, $1/3$ must be the least upper bound for A.

4. (i) Prove directly from the definition of convergence that $x_n \to 1/2$ as $n \to \infty$, where

$$x_n = \frac{n^3 - n}{2n^3 + 3}.$$

3

Solution: *Let $\varepsilon > 0$ be arbitrary. For $n \in \mathbb{N}$ with $n \geqslant 3$,*

$$\left|x_n - \frac{1}{2}\right| = \left|\frac{n^3-n}{2n^3+3} - \frac{1}{2}\right| = \left|\frac{-2n-3}{4n^3+6}\right| = \frac{2n+3}{4n^3+6} < \frac{2n+n}{4n^3} = \frac{3}{4n^2} < \frac{1}{n^2}.$$

Let $n_0 > \max(2, 1/\sqrt{\varepsilon})$. If $n \in \mathbb{N}$ with $n \geqslant n_0$ then $n \geqslant 3$ and $n^2 \geqslant n_0^2 > 1/\varepsilon$ and thus $1/n^2 < \varepsilon$. So by the above, $|x_n - 1/2| < \varepsilon$ for any $n \in \mathbb{N}$ with $n \geqslant n_0$. Hence $x_n \to 1/2$ as $n \to \infty$.

(ii) Using whichever techniques from the course that you wish, prove that $(y_n)_{n=1}^{\infty}$ does not converge, where

$$y_n = \frac{(-1)^n(n^3 - n)}{2n^3 + 3}.$$

2

Solution: *For n even we have that $y_n = x_n$, where x_n is as above. For n odd we have $y_n = -x_n$. Since subsequences of convergent subsequences converge to the same limit, the subsequence of even terms of $(y_n)_{n=1}^{\infty}$ converges to $1/2$ by the above. By algebraic properties of limits, the subsequence of odd terms converges to $-1/2$. But then $(y_n)_{n=1}^{\infty}$ has two subsequences which converge to different limits and thus must diverge.*

5. For each of the series below, determine whether they converge or diverge. Justify your answers clearly, referring to any results or tests you use from the course. Any answer with no justification will receive no marks.

(i)

$$\sum_{n=1}^{\infty} \frac{2 \cdot 9^n}{\binom{2n}{n}}.$$

3

Solution: *Let $a_n = \frac{2 \cdot 9^n}{\binom{2n}{n}}$. Then*

$$\frac{a_{n+1}}{a_n} = \frac{2 \cdot 9^{n+1} \cdot \binom{2n}{n}}{\binom{2(n+1)}{n+1} \cdot 2 \cdot 9^n} = \frac{9 \cdot (2n)!(n+1)!(n+1)!}{(2n+2)!n!n!} = \frac{9(n+1)}{4n+2}$$

So it is easily seen (for example, using algebraic properties of limits) that $a_{n+1}/a_n \to 9/4 > 1$ as $n \to \infty$. Since each term $a_n > 0$, it follows from the limit version of the ratio test that $\sum a_n$ diverges.

(ii)

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2 - 5}{n^3 + 2}.$$

3

Solution: *Let $b_n = \frac{n^2-5}{n^3+2}$. It is easily proven that $b_n \to 0$ as $n \to \infty$. Moreover, $b_n > 0$ for $n \geqslant 3$. We have that*

$$b_n - b_{n+1} = \frac{n^2 - 5}{n^3 + 2} - \frac{(n+1)^2 - 5}{(n+1)^3 + 2} =$$
$$\frac{(n^2 - 5)((n+1)^3 + 2) - (n^3 + 2)((n+1)^2 - 5)}{(n^3 + 2)((n+1)^3 + 2)}$$

The numerator of the above quantity is given by $p(n) - q(n)$, where $p(n) = (n^2 - 5)((n+1)^3 + 2)$ and $q(n) = (n^3 + 2)((n+1)^2 - 5)$. The first two leading terms of the polynomial $p(n)$ are $n^5 + 3n^4$ and the two leading terms of $q(n)$ are $n^5 + 2n^4$. So the numerator is a polynomial with leading term n^4. By the Polynomial Estimation Lemma, there exists some $N \in \mathbb{N}$ for which the numerator is positive for all $n \geqslant N$. Since the denominator is positive for all n, we see that $b_n - b_{n+1} \geqslant 0$ for all $n \geqslant N$. So $(b_n)_{n=1}^{\infty}$ is eventually decreasing. The series $\sum b_n$ then converges by the Leibniz Test.

(iii)

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1}.$$

3

Solution:
Let $c_n = \frac{1}{2^n+1}$ and $d_n = \frac{1}{2^n}$. We have that the geometric series d_n converges. Since $0 \leqslant c_n \leqslant d_n$ for all $n \in \mathbb{N}$, we also have that $\sum c_n$ converges by the comparison test.

6. (i) State the Sandwich Principle. **2**

Solution: *Let $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ be real sequences and suppose that $x_n \to L$ and $z_n \to L$ as $n \to \infty$. The Sandwich Principle states that if there exists $N \in \mathbb{N}$ for which $x_n \leqslant y_n \leqslant z_n$ for all $n \geqslant N$, then $y_n \to L$ as $n \to \infty$.*

(ii) Prove that $(\sqrt{n+3} - \sqrt{n+1}) \to 0$ as $n \to \infty$. **2**

Solution: *For $n \in \mathbb{N}$ we have*

$$0 \leqslant \sqrt{n+3} - \sqrt{n+1} = (\sqrt{n+3} - \sqrt{n+1})\frac{\sqrt{n+3} + \sqrt{n+1}}{\sqrt{n+3} + \sqrt{n+1}} = \frac{(n+3) - (n+1)}{\sqrt{n+3} + \sqrt{n+1}} \leqslant \frac{2}{\sqrt{n}}.$$

Since the constant sequence at 0 and the sequence $2/\sqrt{n}$ both converge to 0, the given sequence converges to 0 too by the Sandwich Principle.

(iii) Let a, b, $c \in \mathbb{N}$. Prove that the sequence $x_n = (a^n + b^n + c^n)^{1/n}$ converges to $\max(a, b, c)$. **3**

Solution: *Let* $\ell = \max(a, b, c)$. *For* $n \in \mathbb{N}$ *we have*

$$\ell = (\ell^n)^{1/n} \leqslant (a^n + b^n + c^n)^{1/n}$$

and

$$(a^n + b^n + c^n)^{1/n} \leqslant (\ell^n + \ell^n + \ell^n)^{1/n} = (3 \cdot \ell^n)^{1/n} = 3^{1/n} \cdot \ell.$$

We have the standard limit $x^{1/n} \to 1$ *as* $n \to \infty$ *for any* $x > 0$. *Hence* $3^{1/n} \cdot \ell \to \ell$ *as* $n \to \infty$. *It follows that* $(a^n + b^n + c^n)^{1/n} \to \ell$ *as* $n \to \infty$ *by the Sandwich Principle.*

7. (i) State the Intermediate Value Theorem. **2**

Solution: *Let* $f\colon [a, b] \to \mathbb{R}$ *be continuous and suppose that* $d \in \mathbb{R}$ *satisfies*

$$f(a) < d < f(b) \text{ or } f(a) > d > f(b).$$

Then there exists $c \in (a, b)$ *with* $f(c) = d$.

(ii) Suppose that $f\colon \mathbb{R} \to [0, 1]$ is continuous. Prove that there exists some $c \in \mathbb{R}$ for which $f(c) = (c - 2)(c + 1)$. **4**

Solution: *Define* $g(x) = f(x) - (x - 2)(x + 1)$. *By results from lectures, continuity of* g *follows from that of* f.
Since the range of f *belongs to* $[0, 1]$, *we have that*

$$g(x) \leqslant 1 - (x - 2)(x + 1) \text{ and } g(x) \geqslant -(x - 2)(x + 1)$$

for all $x \in \mathbb{R}$. *Notice that at* $x = 2$ *(or* $x = -1$*) we have that* $(x-2)(x+1) = 0$, *so the second equation shows that* $g(2) \geqslant 0$. *Clearly for sufficiently large* x *we have that* $g(x)$ *is negative; for example, we have that* $g(3) \leqslant 1 - (1 \times 4) = -3$. *So on the endpoints of the interval* $[2, 3]$, *we have that* $g(2) \geqslant 0$ *and* $g(3) < 0$. *Either* $g(2) = 0$ *or it is strictly positive, in which case the Intermediate Value Theorem implies that there exists some* $c \in (2, 3)$ *for which* $g(c) = 0$. *In either case there exists* $c \in [2, 3]$ *for which* $g(c) = 0$. *For such a value of* c,

$$0 = g(c) = f(c) - (c - 2)(c + 1) \implies f(c) = (c - 2)(c + 1),$$

as desired.

8. (i) Let $f\colon \mathbb{R} \to \mathbb{R}$. State, in terms of ε and δ, the definition of f being continuous at a point $c \in \mathbb{R}$. **2**

Solution: *We call $f \colon \mathbb{R} \to \mathbb{R}$ continuous at $c \in \mathbb{R}$ if*

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } (|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon).$$

(ii) Consider the function $f \colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1/2 & \text{for } x < 1; \\ x^2/2 & \text{for } 1 \leqslant x < 2; \\ x - 1 & \text{for } x \geqslant 2. \end{cases}$$

Is the function f continuous? Provide a detailed proof of your claim. **5**

Solution: *The function is discontinuous since it is not continuous at* 2. *To see that f is not continuous at 2, let $\varepsilon = 1/2$. For $3/2 < x < 2$ we have*

$$|f(x) - f(2)| = |x^2/2 - 1| = x^2/2 - 1,$$

since $x^2/2 > ((3/2)^2)/2 = 9/8 > 1$. Suppose that $x > 7/4$. Then $x^2 > 49/16 > 48/16 = 3$ and hence $x^2/2 - 1 > 3/2 - 1 = 1/2$. So given arbitrary $\delta > 0$, choose $x < 2$ with $x > \max(2 - \delta, 7/4)$. Then $|x - 2| = 2 - x < \delta$ and by the above $|f(x) - f(2)| > 1/2 = \varepsilon$, so f is not continuous at $c = 2$.

END]