Algorithmic Foundations 2 - Tutorial Sheet 6

Induction and Recursive Definitions

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1. Use the principle of mathematical induction to show $\sum_{i=1}^{n} i{\cdot}(i!) = (n{+}1)! - 1$ for all $n \in \mathbb{N}$.
2. Use the principle of mathematical induction to show $3^n < n!$ for all $n{>}6$.
3. Use the principle of mathematical induction to show $n^3 > n^2 + 3$ for all $n \geq 2$.
4. Suppose that
   - $a_1 = 2$;
   - $a_2 = 9$;
   - $a_n = 2{\cdot}a_{n-1} + 3{\cdot}a_{n-2}$ for $n \geq 3$.

   Use (the second principle of) mathematical induction to show $a_n \leq 3^n$ for all $n \in \mathbb{Z}^+$.
5. Use the principle of mathematical induction to show a function f defined by specifying $f(0)$ and a rule for obtaining $f(n{+}1)$ from $f(n)$ (for each $n \geq 0$) is well-defined.
6. Find $f(i)$ for $i = 1, 2, 3, 4$ given $f(n)$ is defined recursively by $f(0) = 3$ and for each $n \geq 0$:
   (a) $f(n{+}1) = -2{\cdot}f(n)$;
   (b) $f(n{+}1) = 3{\cdot}f(n) + 7$;
   (c) $f(n{+}1) = f(n)^2 - 2{\cdot}f(n) - 2$;
   (d) $f(n{+}1) = 3{\cdot}f(n)/3$.
7. Give a recursive definition for each of the following non-recursive definitions:
   (a) $g_1(n) = 4{\cdot}7^n$ for all $n \geq 0$;
   (b) $g_2(n) = 3{\cdot}n + 5$ for all $n \geq 0$;
   (c) $g_3(n) = n!$ for all $n \geq 1$;
   (d) $g_4(n) = n^2$ for all $n \geq 0$.
8. Give recursive definitions of the functions max and min, so that $\max(a_1, a_2, \ldots, a_n)$ and $\min(a_1, a_2, \ldots, a_n)$ are the maximum and minimum of the n real numbers $a_1, a_2, \ldots, a_n$ respectively.
9. Give a recursive definition of the following sets:
   (a) the odd positive integers;
   (b) the positive integer powers of 3;
   (c) the polynomials with integer coefficients.
10. Give recursive definitions with initial condition(s) for each of the following sets:
    (a) $\{0.1, 0.01, 0.001, \ldots\}$
    (b) the set of positive integers congruent to $4 \pmod 7$
    (c) the set of integers not divisible by 3
11. Assume that we have a list l, and are given the functions:
    - `head`(l) which returns the first element of a non-empty list;

- `tail`(l) which returns the tail of a non-empty list;
- `isEmpty`(l) returns `true` if the list is empty and `false` otherwise.

For example if l equals $\langle 5, 3, 4, 2, 7, 8, 3, 4\rangle$, then `head`($l$) would deliver 5, `tail`($l$) would deliver $\langle 3, 4, 2, 7, 8, 3, 4\rangle$, and `isEmpty`($l$) would deliver `false`.

Using the above functions, in a pseudo code of your choice:

(a) write a recursive function `length`(l) that returns the length of the list l as an integer.

For example, `length`($\langle 1, 5, 2, 9, 8, 3, 2\rangle$) would return 7.

(b) write a recursive function `sum`(l), that returns the summation of the elements in a list.

For example, `sum`($\langle 1, 5, 2, 3\rangle$) returns $1 + 5 + 2 + 3 = 11$.

(c) write a recursive function `present`(e, l), that delivers `true` if e appears in the list l and `false` otherwise.

For example, `present`($6, \langle 1, 5, 2, 3\rangle$) returns `false` and `present`($4, \langle 1, 2, 3, 1, 2, 4, 2\rangle$) returns `true`.

(d) write a recursive function `remove`(e, l) that removes all occurrences of e from the list l.

For example, `remove`($5, \langle 1, 5, 2, 3, 5\rangle$) returns $\langle 1, 2, 3\rangle$.

Difficult/challenging questions.

12. Show that the set S defined by:

- $5 \in S$;
- if $s \in S$ and $t \in S$, then $s + t \in S$

is the set of positive integers divisible by 5.

13. Prove that

$$\sum_{j=0}^{n} \left(-\frac{1}{2}\right)^{j} = \frac{2^{n+1} + (-1)^n}{3 \cdot 2^n}$$

for all $n \in \mathbb{N}$.