Algorithmic Foundations 2

Section 10 – Relations

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Relations – Introduction

Relationships may exist between elements of a set or different sets

- the set of activities and the resources to perform the activities
- BA flight numbers and take off times or departure/arrival airport
- students and subjects
- resources and costs of using them
- the set of lecturers and the set of teaching times
- the set of pairs of compatible values
- people and people that they know
- …

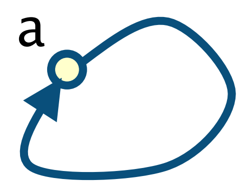
Binary relation – Definition

Let A and B be sets

A binary relation R is a subset of $A \times B$

- recall $A \times B$ is the Cartesian product of A and B
- i.e. R is a set of ordered pairs of the form (a,b) where $a \in A$ and $b \in B$
- put another way R is a subset of $\{ (a,b) \mid a \in A \wedge b \in B \}$
- we say "a is related to b by R" if (a,b) is in the relation R
- and often write aRb for $(a,b) \in R$

Can represent a relation as a digraph



- vertices are the elements of A and B ($V = A \cup B$)
- edges are the elements of the relation R ($E = R$)
- directed graph since ordered pairs
- notice that we can have loops (not a simple graph)

Binary relations – Example

Students: **Alex, Bea, Cath, Don, Eddie, Fiona**

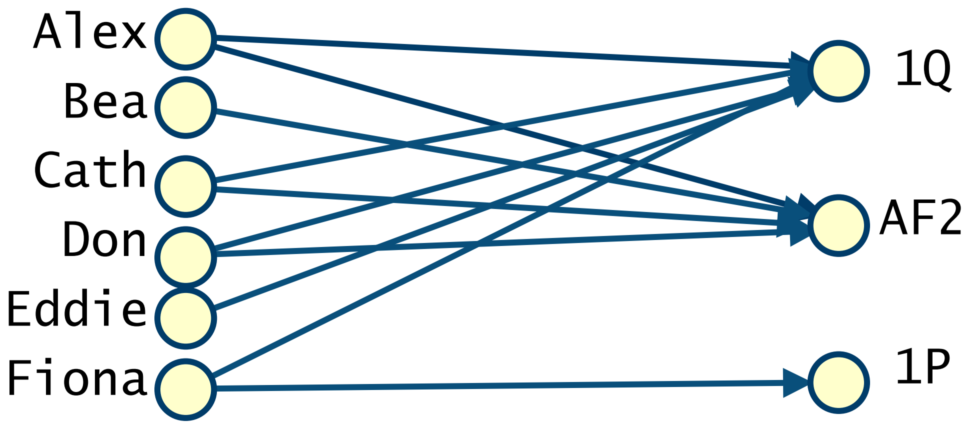
Subjects: **1Q, 1P, AF2**

Let R be the relation of students who passed subjects

R={(Alex,1Q),(Alex,AF2),(Bea,AF2),(Cath,AF2),(Cath,1Q),

(Don,AF2),(Don,1Q),(Fiona,1Q),(Eddie,1Q),(Fiona,1P)}

- order between pairs is insignificant as R is a set
- order within pairs is significant as a set of ordered pairs



notice this is not a function
e.g. Alex related to 1Q and AF2

but functions are relations
(can be expressed as a set of
ordered pairs of domain and
co-domain elements)

Relations – Representation

We can represent a relation as a directed graph

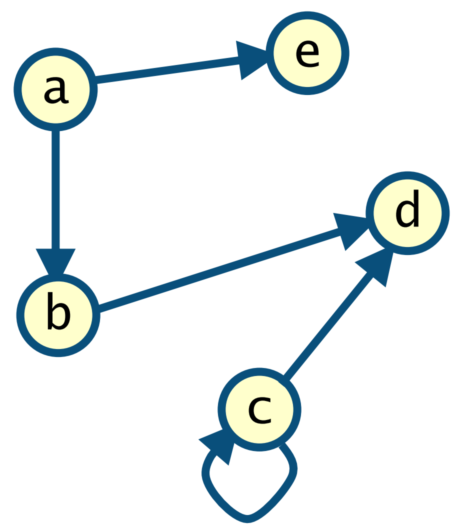
Suppose R is a binary relation over $A \times B$

then we can represent R as a directed graph $G=(A \cup B, E)$ where

- $(a,b) \in E$ if and only if $(a,b) \in R$
- notice that we can have loops (not a simple graph)

Example

- $A=\{a,b,c\}$
- $B=\{b,c,d,e\}$
- $R=\{(a,b),(a,e),(b,d),(c,c),(c,d)\}$

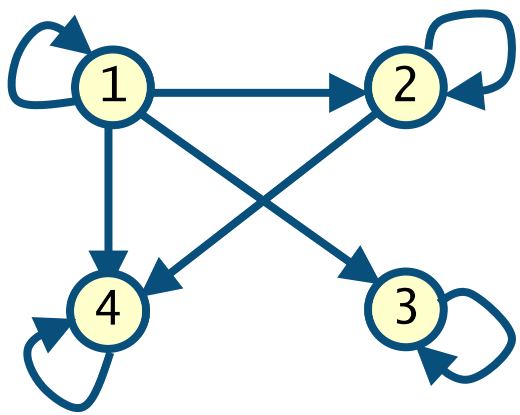


Relation – Divisibility

$A = \{1,2,3,4\}$

R is the relation “a divides b” defined over $A \times A$

$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$



N-ary relations

We can have a relation between **n** sets, i.e. a set of ordered **n-tuples**

A relation between the sets $A_1, A_2, \ldots, A_n$ is a subset of $A_1 \times A_2 \times \ldots \times A_n$

- i.e. an element of the relation is of the form $(a_1, a_2, \ldots, a_n)$

Terminology

- n=1, a unary relation (singletons)
  - e.g. consider a predicate `P(x) : A →` {**true**,**false**}
  - `R = { x | x∈A ∧ P(x)=`**true** `}`
- n=2, a binary relation (pairs)
- n=3, a ternary relation (triples)

  …
- n= …, an n-ary relation (n-tuples)

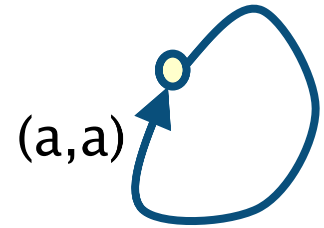
Binary relations – Properties

A binary relation R is…

- **reflexive**: if $a \in A$, then $(a,a) \in R$

  $\forall a{\in}A.\ (a,a){\in}R$

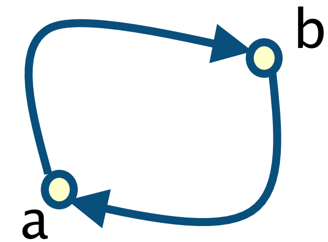
  - i.e. every element is related to itself



- **symmetric**: if (a,b), then $(b,a) \in R$

  $\forall a{\in}A.\ \forall b{\in}A.\ (\ (a,b){\in}R \rightarrow (b,a) \in R\ )$

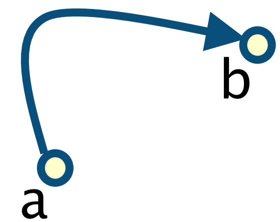
  - i.e. a is related to b if and only if b is related to a



- **anti-symmetric**: if $(a,b) \in R$ and $a \neq b$ then $(b,a) \notin R$

  $\forall a{\in}A.\ \forall b{\in}A.\ (\ (a,b){\in}R \wedge (a{\neq}b) \rightarrow (b,a) \notin R\ )$

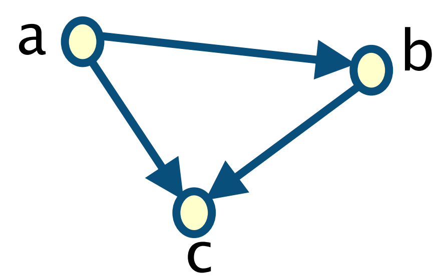
  - i.e. if a is related to b and distinct, then b is not related to a



- **transitive**: if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$

  $\forall a{\in}A.\ \forall b{\in}A.\ \forall c{\in}A.\ (\ ((a,b){\in}R \wedge (b,c) \in R) \rightarrow (a,c){\in}R\ )$

  - i.e. if a is related to b and b related to c, then a is related to c



Binary relations – Properties

A relation R over $A \times A$ is called an equivalence relation if it is reflexive, symmetric and transitive

- elements related by an equivalence relation are said to be equivalent

Example: Let R be a relation on the set of people, such that $(x,y) \in R$ if x and y are the same age in years

- R is reflexive: you are the same age as yourself
- R is symmetric: if x is same age as y, then y is same age as x
- R is transitive: if x is the same age as y and y is the same age as z, then x is the same age as z

Combining relations – Using set operations

Assuming two relations R_1 and R_2 over $A \times B$

- then each is a subset of $A \times B$
- can therefore combine R_1 and R_2 using set theoretic operations
- e.g. union, intersection, set difference

Composing relations

Analogous to the composition of functions

Let R be a relation over $A \times B$ and S be a relation over $B \times C$

Composition of R and S (denoted $S \circ R$) is the relation over $A \times C$ such that $(a,c) \in S \circ R$ if and only if

there exists $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$

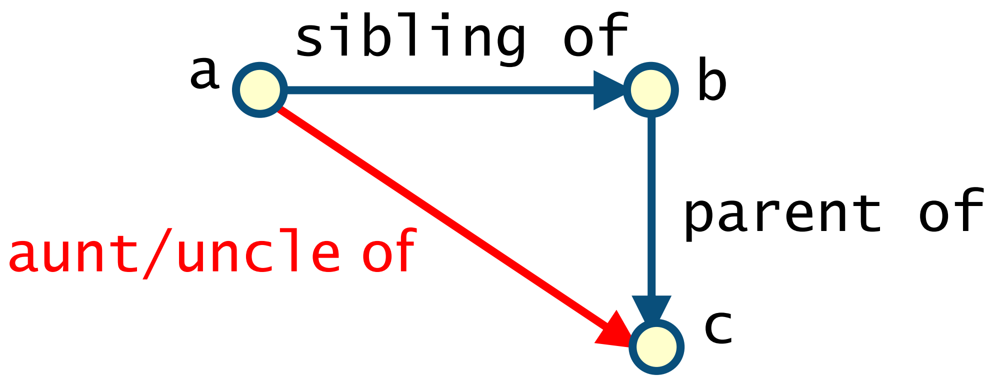
Composing relations – Example

Let R be a relation on people such that (a,b) is "a is a sibling of b"

Let S be a relation on people such that (a,b) is "a is a parent of b"

What is $S \circ R$?

- by definition $(a,c) \in S \circ R$ if there exists b such that $(a,b) \in R$ and $(b,c) \in S$
  - i.e. $(a,c) \in S \circ R$ if there is a b which has a as a sibling and is c's parent
- therefore $(a,c) \in S \circ R$ is "a is an aunt/uncle of c"



Composing relations – Example

Let R be a relation on people such that (a,b) is "a is a sibling of b"

Let S be a relation on people such that (a,b) is "a is a parent of b"

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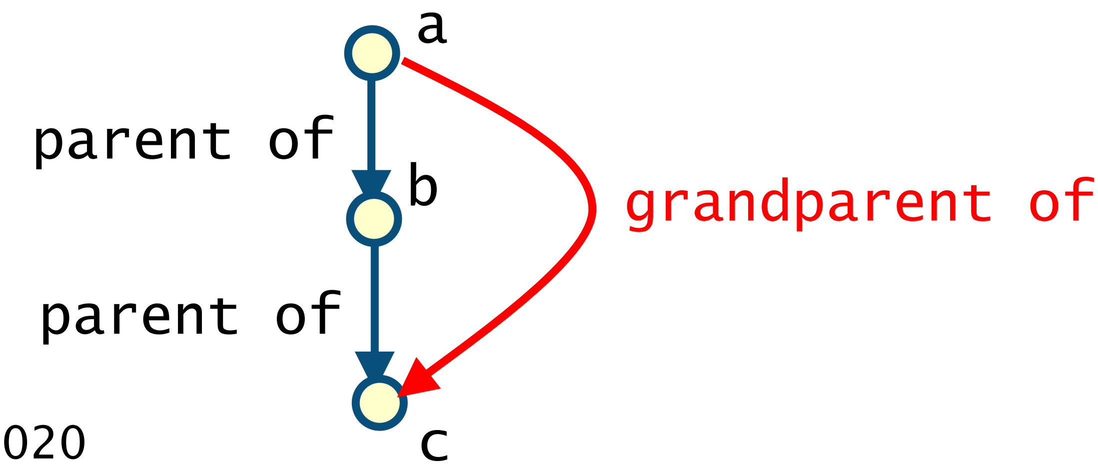
- by definition $(a,c) \in S \circ S$ if there exists b such that $(a,b) \in S$ and $(b,c) \in S$
    - i.e. $(a,c) \in S \circ S$ if there is a b which has a as a parent and is c's parent
- therefore $(a,c) \in S \circ S$ is "a is an grandfather/grandmother of c"



Closures

If we have a relation R, then the closure of R with respect to some property P is given by the relation S where S is R union the minimum number of tuples that ensures property P holds

Property P could be reflexivity, symmetry or transitivity

Reflexive and symmetric closure

To obtain the reflexive closure we set $S = R \cup \Delta$

- where Δ is the diagonal relation $\Delta=\{(a,a) \mid a \in A\}$
- this is the minimum we need to add to R to make it reflexive

What is the reflexive closure of less than “$<$” on the reals/integers?

- answer: less than or equal to “$\leq$”

To obtain the symmetric closure we set $S = R \cup R^{-1}$

- where R^{-1} is the inverse of relation R, i.e. $R^{-1}=\{(b,a) \mid (a,b) \in R\}$
- this is the minimum we need to add to R to make it symmetric

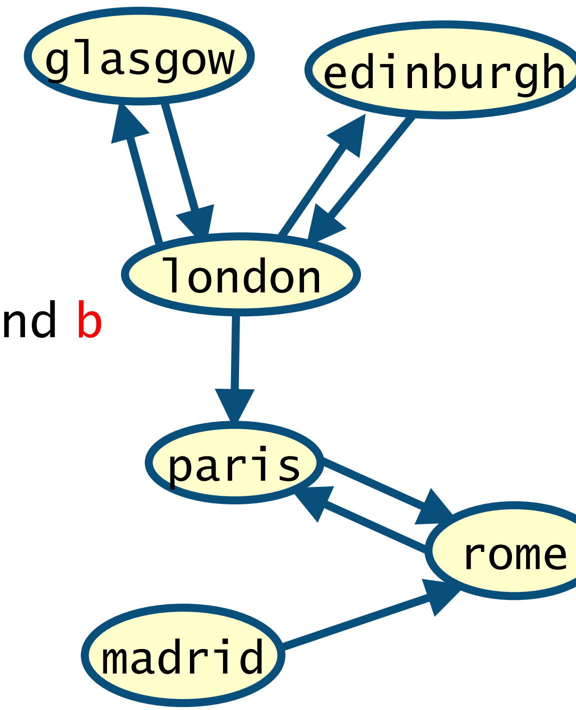
What is the symmetric closure of less than “$<$” on the reals/integers?

- answer: not equal to “$\neq$”

Transitive closure

Add minimum number of tuples to R to give a transitive relation S

- i.e. if (a,b)∈S and (b,c)∈S, then (a,c)∈S



Example: flights between various cities

- set of elements: cities
- relation: (a,b)∈R if there is a flight between a and b
- suppose we define the relation S such that (a,b)∈S if there is a trip from a to b
  - i.e. can flight from a to b allowing for transfers
- S is actually the transitive closure of R
- we have (a,b)∈S if there is a path from a to b

Computing the transitive closure of R reduces to finding all (a,b) such that there is a path from a to b in the digraph representing R

Partial orders

A relation R over $S \times S$ is a partial order on S if it is

- reflexive
- anti-symmetric (if $(s,t) \in R$ and $s \neq t$ then $(t,s) \notin R$)
- transitive

Standard convention is to use $\sqsubseteq$ to represent partial orders

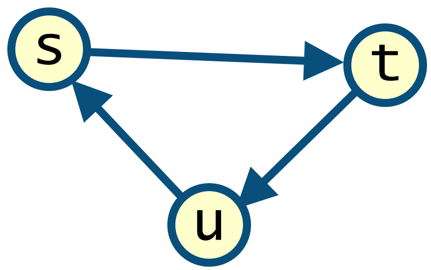
A set S with a partial order $\sqsubseteq$ on S is called a partially ordered set or a poset and is denoted $(S,\sqsubseteq)$

- it is a partial ordering because pairs of elements may be incomparable

Partial orders

A partially ordered set $(S,\sqsubseteq)$ cannot have cycles

For example suppose $(S,\sqsubseteq)$ has the cycle $(s,t),(t,u),(u,s)$



$\sqsubseteq$ is reflexive, anti-symmetric and transitive, therefore

- since $s \sqsubseteq t$ and $t \sqsubseteq u$ consequently $s \sqsubseteq u$ (by transitivity)
- hence $s \sqsubseteq u$ and $u \sqsubseteq s$ which would mean $\sqsubseteq$ is not anti-symmetric
- contradiction, therefore $(S,\sqsubseteq)$ does not have cycle $(s,t),(t,u),(u,s)$
- (this generalises to cycles of any length)

Example – Lexicographic ordering

Used for ordering sets constructed as

- products, strings and words (requires ordering on the original sets)

Example: if we have partially ordered sets $(S_1,\sqsubseteq_1)$ and $(S_2,\sqsubseteq_2)$, then we can construct the partially ordered set $(S_1 \times S_2,\sqsubseteq)$ where

- $(s_1,s_2) \sqsubset (t_1,t_2)$ if $s_1 \sqsubset_1 t_1$ or $s_1 = t_1$ and $s_2 \sqsubset_2 t_2$

For more general product spaces…

- $(s_1,s_2,\ldots,s_n) \sqsubset (t_1,t_2,\ldots,t_n)$ if $s_1 \sqsubset_1 t_1$ or there exists $i>0$ such that $s_j = t_j$ for all $j \leq i$ and $s_{i+1} \sqsubset_{i+1} t_{i+1}$

When strings are of different lengths

- $(s_1,s_2,\ldots,s_m) \sqsubset (t_1,t_2,\ldots,t_n)$ if $(s_1,s_2,\ldots,s_t) \sqsubset (t_1,t_2,\ldots,t_t)$ where $t=\min(m,n)$ or $m<n$ and $(s_1,s_2,\ldots,s_m) = (t_1,t_2,\ldots,t_m)$
  - i.e. first string is shorter and is a prefix of the second

Hasse diagram

A poset can be drawn as a digraph

- it has loops at nodes (reflexive)
- it has directed asymmetric edges
- it has transitive edges

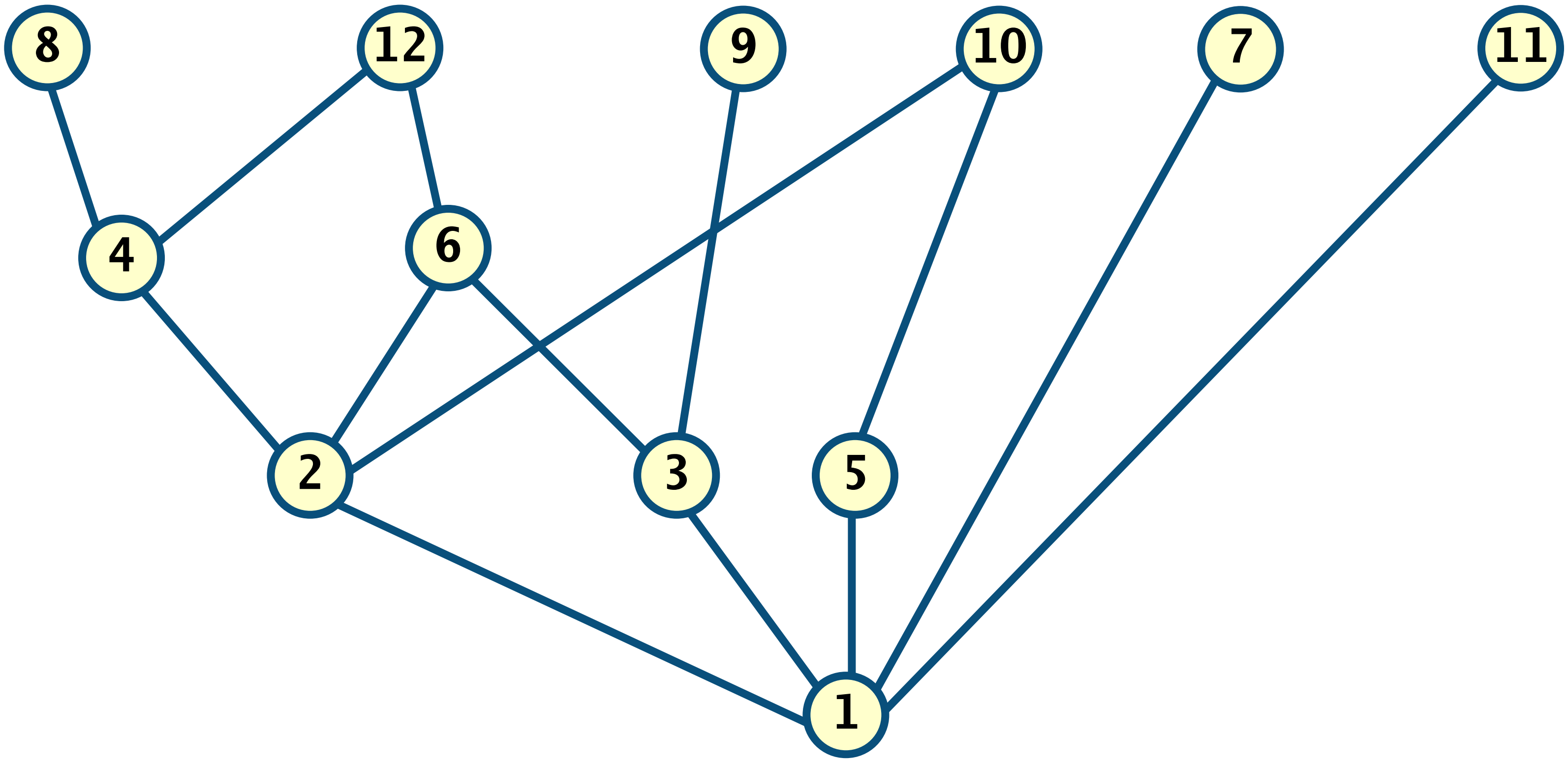
Draw this removing all redundant information: a Hasse diagram

- remove all loops `(x,x)`
- remove all transitive edges (if `(x,y)` and `(y,z)`, then remove `(x,z)`)
- remove all directions (draw pointing upwards)

Hasse diagram

Consider $S=\{1,2,3,4,5,6,7,8,9,10,11,12\}$ and $\sqsubseteq\ =\ |$

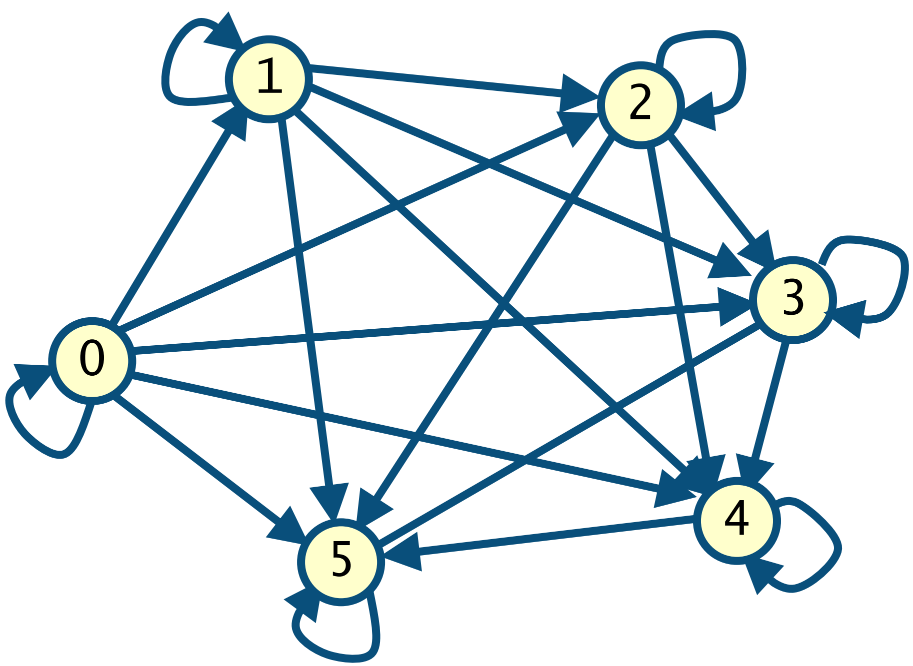
- i.e. $s|t$ if s divides t

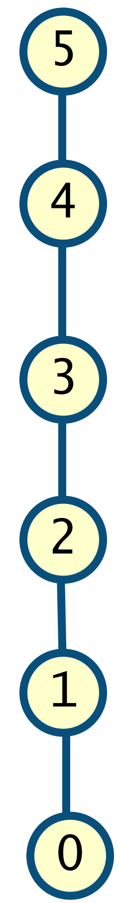


Hasse diagram

Draw Hasse diagram for **({0,1,2,3,4,5},≤)**

- consider its digraph
- remove loops
- remove transitive edges
- remove direction (point upwards)





Minimal, maximal, greatest & least elements

Consider a partially ordered set $(S, \sqsubseteq)$

- s is a maximal element if $\neg\exists t \in S.\ (s \sqsubset t)$
- s is a minimal element if $\neg\exists t \in S.\ (t \sqsubset s)$

Maximal elements are at the top of the Hasse diagram

Minimal elements are at the bottom of the Hasse diagram

Greatest/least elements may (or may not) exist

- s is the greatest element if $\forall t \in S.\ (t \sqsubseteq s)$
- s is the least element if $\forall t \in S.\ (s \sqsubseteq t)$

Minimal, maximal, greatest & least elements

A **lattice** is a partially ordered set such that every pair of elements has both a **least upper bound** (**`lub`**) and **greatest lower bound** (**`glb`**)

- called the "join" ($s \vee t$) and the "meet" ($s \wedge t$)

Examples

- $(\mathbb{Z},\leq)$ is a lattice: **`max`**`(s,t)` and **`min`**`(s,t)` are the `lub` and `glb` of `s` and `t`
- $(\mathbb{Z}^+,|)$ is a lattice: **`lcm`**`(s,t)` and **`gcd`**`(s,t)` are the `lub` and `glb` of `s` and `t`
- $(P(S),\subseteq)$ is lattice: $s \cup t$ and $s \cap t$ are the lub and glb of `s` and `t`