2B Linear Algebra

Ex Sheet 9

1 True/False

a) If an n by n matrix (not necessarily real) has an imaginary eigenvalue it cannot be orthogonally diagonalized.

b) If a real n by n matrix is orthogonally diagonalizable, then all its eigenvalues are real.

c) If the eigenvalues of a real n by n matrix are real, then the matrix is orthogonally diagonalizable.

d) The product of an n by n real matrix with its transpose is orthogonally diagonalizable.

e) A real orthogonal matrix with inverse itself will be orthogonally diagonalizable.

f) Upper triangular real matrices are always orthogonally diagonalizable.

g) If A is orthogonally diagonalized via an orthogonal matrix Q, then A^{2020} is orthognally diagonalized via Q as well.

h) The zero matrix is orthognally diagonalizable.

i) If A and B are real n by n matrices which are orthogonally diagonalizable, then so is AB.

j) $(x_1 - 3x_2)^2$ is a quadratic form

k) $q(x, y) = xy$ is not a quadratic form because it has no x^2 or y^2 terms

l) If $q(\mathbf{x}) = \mathbf{x}^T C\mathbf{x}$ is a quadratic form, and $A = \frac{1}{2}(C + C^T)$, then $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$

[1] **True/False Questions**

Every Exercise Sheet will have a section containing true/false questions. They are designed to test your understanding from lectures. You should look at your lecture notes and/or the textbook to help you answer these questions, but you should not need to write anything to work out the solution.

Solutions to True/False

a) F b) T c) F d) T e) T f) F g) T h) T i) F j) T k) F l) T

Tutorial Exercises

T1 Let

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}.$$

Find an orthogonal matrix Q and a diagonal matrix D such that

$$Q^T A Q = D.$$

Solution

$$\chi_A(t) = \begin{vmatrix} t-4 & -2 \\ -2 & t-7 \end{vmatrix} = (t-4)(t-7) - 4 = t^2 - 11t + 24 = (t-3)(t-8).$$

So 8 and 3 are the eigenvalues of A.

$$(8I - A)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff 2x - y = 0.$$

Therefore

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ is an eigenvector corresponding to } 8.$$

$$(3I - A)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff x + 2y = 0.$$

Therefore

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ is an eigenvector corresponding to } 3.$$

The eigenvectors

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

have lengths $\sqrt{(1^2 + 2^2)} = \sqrt{5}$ and $\sqrt{(2^2 + (-1)^2)} = \sqrt{5}$, respectively. So let

$$Q = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$$

Then Q is orthogonal and $Q^{\mathrm{T}}AQ = \mathrm{diag}(8, 3)$.

T2 Let

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix}.$$

Find an orthogonal matrix Q and a diagonal matrix D such that

$$Q^{\mathrm{T}}AQ = D.$$

Solution

$$\begin{aligned}\chi_A(t) = \begin{vmatrix} t-1 & -1 & -3 \\ -1 & t-3 & -1 \\ -3 & -1 & t-1 \end{vmatrix} &= \begin{vmatrix} t-5 & t-5 & t-5 \\ -1 & t-3 & -1 \\ -3 & -1 & t-1 \end{vmatrix} && \begin{bmatrix} R_1 \to R_1 + R_2 \text{ then} \\ R_1 \to R_1 + R_3 \end{bmatrix} \\ &= (t-5)\begin{vmatrix} 1 & 1 & 1 \\ -1 & t-3 & -1 \\ -3 & -1 & t-1 \end{vmatrix} \\ &= (t-5)\begin{vmatrix} 1 & 0 & 0 \\ -1 & t-2 & 0 \\ -3 & 2 & t+2 \end{vmatrix} && \begin{bmatrix} C_2 \to C_2 - C_1 \\ \text{then} \\ C_3 \to C_3 - C_1 \end{bmatrix} \\ &= (t-5)(t-2)(t+2).\end{aligned}$$

So 5, 2 and -2 are the eigenvalues of A.

$$\begin{aligned}(5I - A)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} &\iff \begin{bmatrix} 4 & -1 & -3 \\ -1 & 2 & -1 \\ -3 & -1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ &\iff \begin{cases} x \quad - z = 0, \\ \quad y - z = 0. \end{cases}\end{aligned}$$

Therefore

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ is an eigenvector corresponding to 5.}$$

$$\begin{aligned}(2I - A)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} &\iff \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -1 \\ -3 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ &\iff \begin{cases} x \quad - \; z = 0, \\ \quad y + 2z = 0. \end{cases}\end{aligned}$$

Therefore

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \text{ is an eigenvector corresponding to 2.}$$

$$((-2)I - A)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} -3 & -1 & -3 \\ -1 & -5 & -1 \\ -3 & -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\iff \begin{cases} x \quad + z = 0, \\ \quad y \quad\; = 0. \end{cases}$$

Therefore

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ is an eigenvector corresponding to } -2.$$

The eigenvectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

have lengths

$$\sqrt{(1^2+1^2+1^2)} = \sqrt{3}, \qquad \sqrt{(1^2+(-2)^2+1^2)} = \sqrt{6}$$
$$\text{and} \qquad \sqrt{(1^2+0^2+(-1)^2)} = \sqrt{2},$$

respectively. So let

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then Q is orthogonal and $Q^{\mathrm{T}}AQ = \operatorname{diag}(5, 2, -2)$.

T3 Let

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Find an orthogonal matrix Q and a diagonal matrix D such that

$$Q^{\mathrm{T}}AQ = D.$$

Solution

$$\chi_A(t) = \begin{vmatrix} t & -1 & -1 \\ -1 & t & -1 \\ -1 & -1 & t \end{vmatrix} = \begin{vmatrix} t-2 & t-2 & t-2 \\ -1 & t & -1 \\ -1 & -1 & t \end{vmatrix} \qquad \begin{bmatrix} R_1 \to R_1 + R_2 \text{ then} \\ R_1 \to R_1 + R_3 \end{bmatrix}$$

$$= (t-2) \begin{vmatrix} 1 & 1 & 1 \\ -1 & t & -1 \\ -1 & -1 & t \end{vmatrix}$$

$$= (t-2) \begin{vmatrix} 1 & 0 & 0 \\ -1 & t+1 & 0 \\ -1 & 0 & t+1 \end{vmatrix} \qquad \begin{bmatrix} C_2 \to C_2 - C_1 \text{ then} \\ C_3 \to C_3 - C_1 \end{bmatrix}$$

$$= (t-2)(t+1)^2.$$

So 2 and -1 are the eigenvalues of A.

$$(2I - A) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{cases} x \quad - z = 0, \\ \quad y - z = 0. \end{cases}$$

Therefore

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ is an eigenvector corresponding to 2.}$$

$$((-1)I - A) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\iff x + y + z = 0.$$

Therefore

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \text{ is an eigenvector corresponding to } -1.$$

We must find a second eigenvector corresponding to -1 which is orthogonal to the first one, i.e. a column $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that

$$x + y + z = 0 \qquad \text{and} \qquad x - y = 0.$$

Therefore

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \text{ is a suitable second eigenvector corresponding to } -1.$$

The eigenvectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

have lengths

$$\sqrt{(1^2+1^2+1^2)} = \sqrt{3}, \qquad \sqrt{(1^2+(-1)^2+0^2)} = \sqrt{2}$$

$$\text{and} \qquad \sqrt{(1^2+1^2+(-2)^2)} = \sqrt{6},$$

respectively. So let

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}.$$

Then Q is orthogonal and $Q^{\mathrm{T}}AQ = \mathrm{diag}(2, -1, -1)$.

T4 Let

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}.$$

Find an orthogonal matrix Q and a diagonal matrix D such that

$$Q^{\mathrm{T}}AQ = D.$$

Solution

$$\chi_A(t) = \begin{vmatrix} t-1 & 1 & 1 & 1 \\ 1 & t-1 & 1 & 1 \\ 1 & 1 & t-1 & 1 \\ 1 & 1 & 1 & t-1 \end{vmatrix} = \begin{vmatrix} t+2 & t+2 & t+2 & t+2 \\ 1 & t-1 & 1 & 1 \\ 1 & 1 & t-1 & 1 \\ 1 & 1 & 1 & t-1 \end{vmatrix}$$

$$\begin{bmatrix} R_1 \to R_1 + R_2 \text{ then} \\ R_1 \to R_1 + R_3 \text{ then} \\ R_1 \to R_1 + R_4 \end{bmatrix}$$

$$= (t+2)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & t-1 & 1 & 1 \\ 1 & 1 & t-1 & 1 \\ 1 & 1 & 1 & t-1 \end{vmatrix}$$

$$= (t+2)\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & t-2 & 0 & 0 \\ 1 & 0 & t-2 & 0 \\ 1 & 0 & 0 & t-2 \end{vmatrix} \quad \begin{bmatrix} C_2 \to C_2 - C_1 \text{ then} \\ C_3 \to C_3 - C_1 \text{ then} \\ C_4 \to C_4 - C_1 \end{bmatrix}$$

$$= (t+2)(t-2)^3.$$

So 2 and -2 are the eigenvalues of A.

$$((-2)I - A) = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} -4 & 0 & 0 & 4 \\ 0 & -4 & 0 & 4 \\ 0 & 0 & -4 & 4 \\ 1 & 1 & 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} R_1 \to R_1 - R_4 \text{ then} \\ R_2 \to R_2 - R_4 \text{ then} \\ R_3 \to R_3 - R_4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} R_1 \to -\frac{1}{4}R_1 \text{ then} \\ R_2 \to -\frac{1}{4}R_2 \text{ then} \\ R_3 \to -\frac{1}{4}R_3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} R_4 \to R_4 - R_1 \text{ then} \\ R_4 \to R_4 - R_2 \text{ then} \\ R_4 \to R_4 - R_3 \end{bmatrix}$$

So

$$((-2)I - A)\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{cases} w \quad\; - z = 0, \\ \quad x \; - z = 0, \\ \quad\;\; y - z = 0. \end{cases}$$

Therefore

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ is an eigenvector corresponding to } -2.$$

$$(2I - A)\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\iff w + x + y + z = 0.$$

Therefore

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \text{ is an eigenvector corresponding to 2.}$$

We must find a second eigenvector corresponding to 2 which is orthogonal to the first one, i.e. a column $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$ such that

$$w + x + y + z = 0 \qquad \text{and} \qquad w - x = 0.$$

Therefore

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \text{ is a suitable second eigenvector corresponding to 2.}$$

Next we must find a third eigenvector corresponding to 2 which is orthogonal to the first two, i.e. a column $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$ such that

$$w + x + y + z = 0, \qquad w - x = 0 \qquad \text{and} \qquad y - z = 0.$$

Therefore

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \text{ is a suitable third eigenvector corresponding to 2.}$$

The eigenvectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

have lengths

$$\sqrt{(1^2+1^2+1^2+1^2)} = 2, \qquad \sqrt{(1^2+(-1)^2+0^2+0^2)} = \sqrt{2},$$

$$\sqrt{(0^2+0^2+1^2+(-1)^2)} = \sqrt{2} \qquad \text{and} \qquad \sqrt{(1^2+1^2+(-1)^2+(-1)^2)} = 2,$$

respectively. So let

$$Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix}.$$

Then Q is orthogonal and $Q^{\mathrm{T}}AQ = \operatorname{diag}(-2, 2, 2, 2)$.

T5 Find the eigenvalues of the orthogonal matrix

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Solution

$$\begin{aligned} \chi_Q(t) = \begin{vmatrix} t-\cos\theta & \sin\theta \\ -\sin\theta & t-\cos\theta \end{vmatrix} &= (t-\cos\theta)^2 + \sin^2\theta \\ &= (t-\cos\theta)^2 - i^2\sin^2\theta \\ &= (t-\cos\theta - i\sin\theta)(t-\cos\theta + i\sin\theta) \\ &= \left(t-e^{i\theta}\right)\left(t-e^{-i\theta}\right). \end{aligned}$$

So $e^{i\theta}$ and $e^{-i\theta}$ are the eigenvalues of Q. (Recall that the matrix transformation determined by Q is rotation through θ radians.)

T6 Let α be an imaginary eigenvalue of an orthogonal matrix Q and let $\mathbf{x}$ be a corresponding eigenvector. Prove that $\mathbf{x}^{\mathrm{T}}\mathbf{x} = 0$.

Solution

$Q\mathbf{x} = \alpha\mathbf{x}$. So

$$\mathbf{x}^{\mathrm{T}} Q^{\mathrm{T}} Q\mathbf{x} = \mathbf{x}^{\mathrm{T}} I\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{x}$$

and

$$\mathbf{x}^{\mathrm{T}} Q^{\mathrm{T}} Q\mathbf{x} = (Q\mathbf{x})^{\mathrm{T}} Q\mathbf{x} = (\alpha\mathbf{x})^{\mathrm{T}}(\alpha\mathbf{x}) = \alpha^2\mathbf{x}^{\mathrm{T}}\mathbf{x}.$$

Therefore

$$\begin{aligned} \alpha^2\mathbf{x}^{\mathrm{T}}\mathbf{x} &= \mathbf{x}^{\mathrm{T}}\mathbf{x}, \\ \text{i.e.}\quad (\alpha^2 - 1)\mathbf{x}^{\mathrm{T}}\mathbf{x} &= 0, \\ \text{i.e.}\quad (\alpha - 1)(\alpha + 1)\mathbf{x}^{\mathrm{T}}\mathbf{x} &= 0. \end{aligned}$$

But $\alpha \neq \pm 1$. Hence $\mathbf{x}^{\mathrm{T}}\mathbf{x} = 0$.

(Note that $\mathbf{x}$ must be a non-zero complex column matrix and we know that $\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x} > 0$.)

T7 Let Q be an $n \times n$ orthogonal matrix, where n is an odd positive integer. Show that 1 or -1 is an eigenvalue of Q.

Solution

$\chi_Q(t)$ is a real polynomial. So its imaginary roots occur as pairs of complex conjugates. (See the Level-1 courses.) Since $\chi_Q(t)$ has an odd degree, it must therefore have a real root. But 1 and -1 are the only real numbers that can be eigenvalues of an orthogonal matrix. Hence 1 or -1 is an eigenvalue of Q.

T8 Let A be a real skew-symmetric matrix, i.e. a real matrix such that $A^{\mathrm{T}} = -A$. Prove that the eigenvalues of A have the form $i\alpha$ for some real number α.

Solution

Let λ be an eigenvalue of the real skew-symmetric matrix A and let $\mathbf{x}$ be an eigenvector of A corresponding to λ. Then

$$\overline{\mathbf{x}}^{\mathrm{T}} A\mathbf{x} = \overline{\mathbf{x}}^{\mathrm{T}}(\lambda\mathbf{x}) = \lambda\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x}$$

and, since $\overline{A}^{\mathrm{T}} = A^{\mathrm{T}} = -A$,

$$\overline{\mathbf{x}}^{\mathrm{T}} A = -\overline{\mathbf{x}}^{\mathrm{T}}\overline{A}^{\mathrm{T}} = -\left(\overline{A}\,\overline{\mathbf{x}}\right)^{\mathrm{T}} = -\overline{(A\mathbf{x})}^{\mathrm{T}} = -\overline{(\lambda\mathbf{x})}^{\mathrm{T}} = -\left(\overline{\lambda}\,\overline{\mathbf{x}}\right)^{\mathrm{T}} = -\overline{\lambda}\,\overline{\mathbf{x}}^{\mathrm{T}}.$$

So

$$\overline{\mathbf{x}}^{\mathrm{T}} A\mathbf{x} = -\overline{\lambda}\,\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x}.$$

Therefore

$$\begin{aligned} \lambda\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x} &= -\overline{\lambda}\,\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x}, \\ \text{i.e.}\quad (\lambda + \overline{\lambda})\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x} &= 0, \\ \text{i.e.}\quad 2\,\mathrm{Re}(\lambda)\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{x} &= 0. \end{aligned}$$

But $\bar{\mathbf{x}}^{\mathrm{T}}\mathbf{x} \neq 0$ since $\mathbf{x} \neq \mathbf{0}$. So $\mathrm{Re}(\lambda) = 0$, i.e. $\lambda = i\alpha$ for some real number α.

T9 Find the eigenvalues of the skew-symmetric matrices

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -2 \\ -1 & 2 & 0 \end{bmatrix}.$$

Solution

$$\chi_A(t) = \begin{vmatrix} t & -2 \\ 2 & t \end{vmatrix} = t^2 + 4 = (t-2i)(t+2i).$$

So $2i$ and $-2i$ are the eigenvalues of A.

$$\begin{aligned} \chi_B(t) = \begin{vmatrix} t & -2 & -1 \\ 2 & t & 2 \\ 1 & -2 & t \end{vmatrix} &= t \begin{vmatrix} t & 2 \\ -2 & t \end{vmatrix} + 2 \begin{vmatrix} 2 & 2 \\ 1 & t \end{vmatrix} - 1 \begin{vmatrix} 2 & t \\ 1 & -2 \end{vmatrix} \\ &= t(t^2+4) + 2(2t-2) - (-4-t) \\ &= t(t^2+9) \\ &= t(t-3i)(t+3i). \end{aligned}$$

So 0, $3i$ and $-3i$ are the eigenvalues of B.

T10 Let A be an $n \times n$ real skew-symmetric matrix, where n is an odd positive integer. Show that 0 is an eigenvalue of A. Could 0 be an eigenvalue of A if n were even?

Solution

$\chi_A(t)$ is a real polynomial. So its imaginary roots occur as pairs of complex conjugates. (See the Level-1 courses.) Since $\chi_A(t)$ has an odd degree, it must therefore have a real root. This must be 0 because 0 is the only real number that can be an eigenvalue of a real skew-symmetric matrix. Hence 0 is an eigenvalue of A.

Alternatively, observe that

$$\det A = \det(A^{\mathrm{T}}) = \det(-A) = (-1)^n \det A = -\det A$$

since n is odd. Therefore $\det A = 0$. Since $\det A$ is the product of the eigenvalues of A, 0 must be an eigenvalue of A. (To see this, observe that $\det(-A)$ is the constant term of the characteristic polynomial $\chi_A(t)$.)

An $n \times n$ real skew-symmetric matrix with n even could have 0 as an eigenvalue; $O_{2,2}$ is the only 2×2 real skew-symmetric matrix which has 0 as an eigenvalue.

T11 Let λ, μ be distinct eigenvalues of a real skew-symmetric matrix A and let $\mathbf{x}$, $\mathbf{y}$ be eigenvectors of A corresponding to λ, μ, respectively. Prove that $\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{y} = 0$.

Solution

$A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \mu\mathbf{y}$. Then

$$\overline{\mathbf{x}}^{\mathrm{T}}A\mathbf{y} = \overline{\mathbf{x}}^{\mathrm{T}}(\mu\mathbf{y}) = \mu\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{y}$$

and, since $\overline{A}^{\mathrm{T}} = A^{\mathrm{T}} = -A$,

$$\overline{\mathbf{x}}^{\mathrm{T}}A = -\overline{\mathbf{x}}^{\mathrm{T}}\overline{A}^{\mathrm{T}} = -\left(\overline{A}\,\overline{\mathbf{x}}\right)^{\mathrm{T}} = -\overline{(A\mathbf{x})}^{\mathrm{T}} = -\overline{(\lambda\mathbf{x})}^{\mathrm{T}} = -\left(\overline{\lambda}\,\overline{\mathbf{x}}\right)^{\mathrm{T}} = -\overline{\lambda}\,\overline{\mathbf{x}}^{\mathrm{T}} = \lambda\overline{\mathbf{x}}^{\mathrm{T}}$$

because $\overline{\lambda} = -\lambda$ by T9. So

$$\overline{\mathbf{x}}^{\mathrm{T}}A\mathbf{y} = \lambda\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{y}.$$

Therefore

$$\begin{aligned} \lambda\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{y} &= \mu\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{y}, \\ \text{i.e.} \quad (\lambda-\mu)\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{y} &= 0. \end{aligned}$$

But $\lambda \neq \mu$. Therefore $\overline{\mathbf{x}}^{\mathrm{T}}\mathbf{y} = 0$.

T12 Let

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 10 \end{bmatrix}.$$

Find an invertible matrix P such that $P^{\mathrm{T}}AP = I$. Deduce that $A = B^{\mathrm{T}}B$ for some invertible matrix B.

Solution

Let $q = \mathbf{x}^{\mathrm{T}}A\mathbf{x}$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Then

$$\begin{aligned} q &= 2x_1^2 + 10x_2^2 + 8x_1x_2 \\ &= 2\left[x_1^2 + 4x_1x_2\right] + 10x_2^2 \\ &= 2\left[(x_1+2x_2)^2 - 4x_2^2\right] + 10x_2^2 \\ &= 2(x_1+2x_2)^2 + 2x_2^2 \\ &= \left(\sqrt{2}\,x_1 + 2\sqrt{2}\,x_2\right)^2 + \left(\sqrt{2}\,x_2\right)^2 \\ &= y_1^2 + y_2^2, \end{aligned}$$

where

$$\begin{aligned} y_1 &= \sqrt{2}\,x_1 + 2\sqrt{2}\,x_2, \\ y_2 &= \sqrt{2}\,x_2. \end{aligned}$$

Then

$$x_2 = \frac{1}{\sqrt{2}} y_2,$$
$$x_1 = \frac{1}{\sqrt{2}}(y_1 - 2\sqrt{2}\,x_2) = \frac{1}{\sqrt{2}} y_1 - \frac{2}{\sqrt{2}} y_2$$

and so

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{2}} y_1 - \frac{2}{\sqrt{2}} y_2, \\ x_2 &= \qquad\qquad \frac{1}{\sqrt{2}} y_2. \end{aligned}$$

Let

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

Then P is invertible and

$$P^{\mathrm{T}} A P = \operatorname{diag}(1, 1) = I.$$

Therefore

$$A = \left(P^{\mathrm{T}}\right)^{-1} I P^{-1} = \left(P^{-1}\right)^{\mathrm{T}} P^{-1} = B^{\mathrm{T}} B,$$

where

$$B = P^{-1} = \sqrt{2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

T13 Let A be a real symmetric matrix whose eigenvalues are all positive. Show that $A = B^{\mathrm{T}} B$ for some invertible matrix B.

Solution

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A (including any repetitions). We can find an orthogonal matrix Q such that

$$Q^{\mathrm{T}} A Q = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = D^2,$$

where

$$D = \operatorname{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n}).$$

Then

$$A = Q D^2 Q^{\mathrm{T}} = (QD)(QD)^{\mathrm{T}} = B^{\mathrm{T}} B,$$

where $B = (QD)^{\mathrm{T}}$, and B is invertible because both Q and D are invertible.

Alternatively, let q be the quadratic form defined by

$$q = \mathbf{x}^{\mathrm{T}} A \mathbf{x},$$

as in the previous example. Use the matrices Q and D introduced above. Under the nonsingular change of variables $\mathbf{x} = Q\mathbf{y}$,

$$\begin{aligned} q = (Q\mathbf{y})^{\mathrm{T}} A (Q\mathbf{y}) &= \mathbf{y}^{\mathrm{T}} Q^{\mathrm{T}} A Q \mathbf{y} \\ &= \mathbf{y}^{\mathrm{T}} \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \mathbf{y} \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2, \end{aligned}$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

So

$$\begin{aligned} q &= \left(\sqrt{\lambda_1}\, y_1\right)^2 + \left(\sqrt{\lambda_2}\, y_2\right)^2 + \cdots + \left(\sqrt{\lambda_n}\, y_n\right)^2 \\ &= z_1^2 + z_2^2 + \cdots + z_n^2, \end{aligned}$$

where

$$\begin{aligned} z_1 &= \sqrt{\lambda_1}\, y_1, \\ z_2 &= \qquad \sqrt{\lambda_2}\, y_2, \\ &\vdots \qquad\qquad \ddots \\ z_n &= \qquad\qquad\qquad \sqrt{\lambda_n}\, y_n, \end{aligned}$$

$$\text{i.e.} \quad \mathbf{z} = D\mathbf{y},$$

where

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}.$$

Then

$$\mathbf{y} = D^{-1}\mathbf{z}.$$

Now let $P = QD^{-1}$. Then $\mathbf{x} = Q\mathbf{y} = P\mathbf{z}$ is a nonsingular change of variables such that

$$P^{\mathrm{T}}AP = \mathrm{diag}(1, 1, \ldots, 1) = I.$$

Therefore

$$A = \left(P^{\mathrm{T}}\right)^{-1} I P^{-1} = \left(P^{-1}\right)^{\mathrm{T}} P^{-1} = B^{\mathrm{T}}B,$$

where

$$B = P^{-1}.$$

Observe that

$$B = \left(QD^{-1}\right)^{-1} = DQ^{-1} = D^{\mathrm{T}}Q^{\mathrm{T}} = (QD)^{\mathrm{T}},$$

as in the first method.

T14 Let A and B be real $n \times n$ matrices. We say that A is congruent to B if $P^{\mathrm{T}}AP = B$ for some invertible matrix P. Deduce that, in this case, B is also congruent to A. So we can simply say that A and B are congruent.

Solution

Suppose that A is congruent to B. Then

$$P^{\mathrm{T}}AP = B$$

for some invertible matrix P. So P^{-1} is also invertible and

$$\left(P^{-1}\right)^{\mathrm{T}}BP^{-1} = \left(P^{\mathrm{T}}\right)^{-1}BP^{-1} = A.$$

Therefore B is congruent to A.

T15 Suppose that the quadratic form q in n variables can be defined by both

$$q(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$$

and

$$q(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}x_ix_j,$$

where a_{ij} and b_{ij} $(i, j = 1, 2, \ldots, n)$ are real numbers such that $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$. By choosing suitable values for the variables, show that $a_{ij} = b_{ij}$ for $i, j = 1, 2, \ldots, n$. (Hint: first show that $a_{11} = b_{11}$.)

Solution

Put $x_1 = 1$ and $x_i = 0$ for $i = 2, \ldots, n$. Then $q = a_{11}$ and $q = b_{11}$. Therefore $a_{11} = b_{11}$. Similarly, $a_{ii} = b_{ii}$ for $i = 2, \ldots, n$.

Next, provided $n > 1$, put $x_1 = x_2 = 1$ and $x_i = 0$ for $i = 3, \ldots, n$. Then $q = a_{11} + a_{22} + 2a_{12}$ and $q = b_{11} + b_{22} + 2b_{12}$. Therefore

$$a_{11} + a_{22} + 2a_{12} = b_{11} + b_{22} + 2b_{12}.$$

By the first part, $2a_{12} = 2b_{12}$, i.e. $a_{12} = b_{12}$. Similarly $a_{ij} = b_{ij}$ for all other relevant values of i and j with $i < j$.

T16 Write down the matrix of each of the following quadratic forms:

(i) $q(x_1, x_2, x_3) = 2x_1^2 + 3x_2^2 - 4x_3^2 + 10x_1x_2 + 16x_2x_3$,

(ii) $q(x_1, x_2, x_3) = x_1x_2 + x_1x_3 - x_2x_3$,

(iii) $q(x_1, x_2, x_3, x_4) = x_1^2 - 2x_2x_3$.

Solution

(i) $\begin{bmatrix} 2 & 5 & 0 \\ 5 & 3 & 8 \\ 0 & 8 & -4 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$. **(iii)** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

T17 Write down the formula for the quadratic form $q(x_1, x_2, x_3)$ which has the matrix

$$\begin{bmatrix} 2 & -1 & 2 \\ -1 & 0 & 4 \\ 2 & 4 & -5 \end{bmatrix}.$$

Solution

$$q(x_1, x_2, x_3) = 2x_1^2 - 5x_3^2 - 2x_1x_2 + 4x_1x_3 + 8x_2x_3.$$

T18 For each of the following quadratic forms, let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

and find a nonsingular change of variables $\mathbf{x} = P\mathbf{y}$ such that

$$q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$$

for some non-zero real numbers $\lambda_1, \lambda_2, \lambda_3$.

(i) $q(x_1, x_2, x_3) = x_1^2 + 3x_2^2 + 9x_3^2 + 2x_1x_2 - 8x_2x_3$.

(ii) $q(x_1, x_2, x_3) = x_1^2 - 2x_1x_2 - 2x_2x_3$.

(iii) $q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^2 - 6x_1x_3 + 4x_2x_3$.

Solution

(i)

$$\begin{aligned}
q &= x_1^2 + 3x_2^2 + 9x_3^2 + 2x_1x_2 - 8x_2x_3 \\
&= \left[x_1^2 + 2x_1x_2\right] + 3x_2^2 + 9x_3^2 - 8x_2x_3 \\
&= \left[(x_1 + x_2)^2 - x_2^2\right] + 3x_2^2 + 9x_3^2 - 8x_2x_3 \\
&= (x_1 + x_2)^2 + 2x_2^2 + 9x_3^2 - 8x_2x_3 \\
&= (x_1 + x_2)^2 + 2\left[x_2^2 - 4x_2x_3\right] + 9x_3^2 \\
&= (x_1 + x_2)^2 + 2\left[(x_2 - 2x_3)^2 - 4x_3^2\right] + 9x_3^2 \\
&= (x_1 + x_2)^2 + 2(x_2 - 2x_3)^2 + x_3^2 \\
&= y_1^2 + 2y_2^2 + y_3^2,
\end{aligned}$$

where

$$\begin{aligned}
y_1 &= x_1 + x_2, \\
y_2 &= \qquad x_2 - 2x_3, \\
y_3 &= \qquad\qquad x_3.
\end{aligned}$$

Then

$$\begin{aligned}
x_3 &= y_3, \\
x_2 &= y_2 + 2x_3 = y_2 + 2y_3, \\
x_1 &= y_1 - x_2 = y_1 - y_2 - 2y_3
\end{aligned}$$

and so

$$\begin{aligned}
x_1 &= y_1 - y_2 - 2y_3, \\
x_2 &= \qquad y_2 + 2y_3, \\
x_3 &= \qquad\qquad y_3,
\end{aligned}$$

$$\text{i.e.} \quad \mathbf{x} = P\mathbf{y},$$

where

$$P = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix},$$

which is invertible.

(ii)

$$\begin{aligned}
q &= x_1^2 - 2x_1x_2 - 2x_2x_3 \\
&= \left[x_1^2 - 2x_1x_2\right] - 2x_2x_3 \\
&= \left[(x_1 - x_2)^2 - x_2^2\right] - 2x_2x_3 \\
&= (x_1 - x_2)^2 - x_2^2 - 2x_2x_3 \\
&= (x_1 - x_2)^2 - \left[x_2^2 + 2x_2x_3\right] \\
&= (x_1 - x_2)^2 - \left[(x_2 + x_3)^2 - x_3^2\right] \\
&= (x_1 - x_2)^2 - (x_2 + x_3)^2 + x_3^2 \\
&= y_1^2 - y_2^2 + y_3^2,
\end{aligned}$$

where

$$\begin{aligned}
y_1 &= x_1 - x_2, \\
y_2 &= \phantom{x_1 - {}} x_2 + x_3, \\
y_3 &= \phantom{x_1 - x_2 + {}} x_3.
\end{aligned}$$

Then

$$\begin{aligned}
x_3 &= y_3, \\
x_2 &= y_2 - x_3 = y_2 - y_3, \\
x_1 &= y_1 + x_2 = y_1 + y_2 - y_3
\end{aligned}$$

and so

$$\begin{aligned}
x_1 &= y_1 + y_2 - y_3, \\
x_2 &= \phantom{y_1 + {}} y_2 - y_3, \\
x_3 &= \phantom{y_1 + y_2 - {}} y_3,
\end{aligned}$$

$$\text{i.e.} \quad \mathbf{x} = P\mathbf{y},$$

where

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix},$$

which is invertible.

(iii)

$$
\begin{aligned}
q &= x_1^2 + 2x_2^2 + 3x_3^2 - 6x_1x_3 + 4x_2x_3 \\
&= \left[x_1^2 - 6x_1x_3\right] + 2x_2^2 + 3x_3^2 + 4x_2x_3 \\
&= \left[(x_1 - 3x_3)^2 - 9x_3^2\right] + 2x_2^2 + 3x_3^2 + 4x_2x_3 \\
&= (x_1 - 3x_3)^2 + 2x_2^2 - 6x_3^2 + 4x_2x_3 \\
&= (x_1 - 3x_3)^2 + 2\left[x_2^2 + 2x_2x_3\right] - 6x_3^2 \\
&= (x_1 - 3x_3)^2 + 2\left[(x_2 + x_3)^2 - x_3^2\right] - 6x_3^2 \\
&= (x_1 - 3x_3)^2 + 2(x_2 + x_3)^2 - 8x_3^2 \\
&= y_1^2 + 2y_2^2 - 8y_3^2,
\end{aligned}
$$

where

$$
\begin{aligned}
y_1 &= x_1 \qquad - 3x_3, \\
y_2 &= \qquad x_2 + \; x_3, \\
y_3 &= \qquad\qquad\; x_3.
\end{aligned}
$$

Then

$$
\begin{aligned}
x_3 &= y_3, \\
x_2 &= y_2 - x_3 = y_2 - y_3, \\
x_1 &= y_1 + 3x_3 = y_1 + 3y_3
\end{aligned}
$$

and so

$$
\begin{aligned}
x_1 &= y_1 \qquad + 3y_3, \\
x_2 &= \qquad y_2 - \; y_3, \\
x_3 &= \qquad\qquad\; y_3,
\end{aligned}
$$

$$\text{i.e.} \quad \mathbf{x} = P\mathbf{y},$$

where

$$
P = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix},
$$

which is invertible.