2B 2020 – Homework 3 Solutions

Q1 We are given $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that
$$T(x,y,z) = (2x+z, -y, 3z).$$

(i) Consider the vectors $(x,y,z), (x',y',z') \in \mathbb{R}^3$ and consider $c \in \mathbb{R}$. We have
$$\begin{aligned} T((x,y,z)+(x',y',z')) &= T(x+x', y+y', z+z') \\ &= (2(x+x')+(z+z'), -(y+y'), 3(z+z')) \\ &= ((2x+z)+(2x'+z'), -y-y', 3z+3z') \\ &= (2x+z, -y, 3z) + (2x'+z', -y', 3z') \\ &= T(x,y,z) + T(x',y',z'). \end{aligned}$$

We also have:
$$\begin{aligned} T(c(x,y,z)) &= T(cx, cy, cz) \\ &= (2cx+cz, -cy, 3cz) \\ &= (c(2x+z), -cy, c(3z)) \\ &= c(2x+z, -y, 3z) \\ &= cT(x,y,z). \end{aligned}$$

We hence conclude that, as $T(\underline{x}+\underline{x}') = T(\underline{x}) + T(\underline{x}')$ and $T(c\underline{x}) = cT(\underline{x})$ $\forall\, \underline{x}, \underline{x}' \in \mathbb{R}^3$ and $c \in \mathbb{R}$, T is a linear transformation.

(ii)
$$[T] = \begin{pmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(iii) Yes, $(-2,3,6) \in \operatorname{range}(T)$, indeed $T(-2,-3,2) = (-2,3,6)$ (i.e. $x=-2$, $y=-3$, $z=2$).

Q2 WE HAVE

$$\mathcal{B}=\left\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right\},\quad \mathcal{C}=\left\{\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$$

(i) WE HAVE:

$$(\underline{v}=)\begin{pmatrix}3\\2\\3\end{pmatrix}=2\begin{pmatrix}1\\0\\1\end{pmatrix}+\begin{pmatrix}0\\1\\1\end{pmatrix}+\begin{pmatrix}1\\1\\0\end{pmatrix}\Rightarrow[\underline{v}]_{\mathcal{B}}=\begin{pmatrix}2\\1\\1\end{pmatrix}$$

(ii) TO CONSTRUCT THE CHANGE OF BASIS MATRIX $P_{\mathcal{C}\leftarrow\mathcal{B}}$ WE FIND THE COORDINATES OF THE BASIS VECTORS IN $\mathcal{B}$ WITH RESPECT TO THE ORDERED BASIS $\mathcal{C}$. WE HAVE:

$$\begin{pmatrix}1\\0\\1\end{pmatrix}=-\begin{pmatrix}0\\1\\1\end{pmatrix}+\begin{pmatrix}0\\0\\1\end{pmatrix}+\begin{pmatrix}1\\1\\1\end{pmatrix}\Rightarrow\left[\begin{pmatrix}1\\0\\1\end{pmatrix}\right]_{\mathcal{C}}=\begin{pmatrix}-1\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}0\\1\\1\end{pmatrix}+0\begin{pmatrix}0\\0\\1\end{pmatrix}+0\begin{pmatrix}1\\1\\1\end{pmatrix}\Rightarrow\left[\begin{pmatrix}0\\1\\1\end{pmatrix}\right]_{\mathcal{C}}=\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1\\1\\0\end{pmatrix}=0\begin{pmatrix}0\\1\\1\end{pmatrix}-\begin{pmatrix}0\\0\\1\end{pmatrix}+\begin{pmatrix}1\\1\\1\end{pmatrix}\Rightarrow\left[\begin{pmatrix}1\\1\\0\end{pmatrix}\right]_{\mathcal{C}}=\begin{pmatrix}0\\-1\\1\end{pmatrix}$$

AND SO THE CHANGE OF BASIS MATRIX IS:

$$P_{\mathcal{C}\leftarrow\mathcal{B}}=\begin{pmatrix}-1&1&0\\1&0&-1\\1&0&1\end{pmatrix}$$

Q2 (iii) WE HAVE

$$[\underline{v}]_{\mathcal{E}} = P_{\mathcal{E} \leftarrow \mathcal{B}} [\underline{v}]_{\mathcal{B}}$$

AND SO USING OUR SOLUTIONS TO PARTS (i) AND (ii) WE HAVE:

$$[\underline{v}]_{\mathcal{E}} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\Longrightarrow [\underline{v}]_{\mathcal{E}} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

Q3) WE ARE GIVEN $CD = DC$ WITH D INVERTIBLE AND WE ARE TOLD $D\underline{x}$ IS AN EIGENVECTOR OF C WITH EIGENVALUE λ. HENCE

$$C(D\underline{x}) = \lambda(D\underline{x})$$
$$\Rightarrow DC\underline{x} = \lambda D\underline{x} \quad (\text{AS } CD = DC)$$
$$\Rightarrow D^{-1}DC\underline{x} = D^{-1}\lambda D\underline{x} \quad (D^{-1} \text{ EXISTS AS } \det(D) \neq 0)$$
$$\Rightarrow \mathbb{I}C\underline{x} = \lambda \mathbb{I}\underline{x}$$
$$\Rightarrow C\underline{x} = \lambda\underline{x}$$

AND SO $\underline{x}$ IS AN EIGENVECTOR OF C WITH CORRESPONDING EIGENVALUE λ.