2A Multivariable Calculus 2020

Tutorial Exercises

T1 Find grad f at the point P for

(a) $f = x^2 + y^2 - 3yz,\ P(1,2,1),$ (b) (b) $f = e^x \log(yz),\ P(0,2,3),$

(c) $f = \cos(yz)\log(xz),\ P(1,0,3).$

Solution

(a) grad $f = (2x, 2y - 3z, -3y) = (2,1,-6)$ at $P(1,2,1)$.
(b) grad $f = (\log(yz), e^x/y, e^x, z) = (\log 6, \frac{1}{2}, \frac{1}{3})$ at $P(0,2,3)$.
(c) grad $f = (\cos(yz)/x, -z\sin(yz)\log(xz), -y\sin(yz)\log(xz) + \cos(yz)/z) = (1, 0, \frac{1}{3})$ at $P(1,0,3)$.

T2 Find the directional derivative of xyz^2 at the point $(1,5,1)$ in the direction of the vector $(1,-1,2)$.

Solution

The unit vector in the direction of $(1,-1,2)$ is $\mathbf{n} = (1,-1,2)/\sqrt{6}$ and $\nabla\phi = (yz^2, xz^2, 2xyz)(1,5,1) = (5,1,10)$ at $P(1,5,1)$. Therefore the directional derivative is

$$\frac{\partial \phi}{\partial n} = \nabla\phi(1,5,1)\cdot\mathbf{n} = \frac{1}{\sqrt{6}}(1,-1,2)\cdot(5,1,10) = \frac{5-1+20}{\sqrt{6}} = \frac{24}{\sqrt{6}} = 4\sqrt{6}.$$

T3 Let f be a scalar field, $\mathbf{u}$ a unit vector and let θ be the angle between $\mathbf{u}$ and ∇f evaluated at some point P.

a) Show that the directional derivative of f at P in the direction of vector $\mathbf{u}$ is $|\nabla f|\cos\theta$.

b) Deduce that the directional derivative of f at P in the direction of $\mathbf{u}$ is a maximum when $\mathbf{u}$ has the same direction as ∇f. When is this directional derivative a minimum?

c) In what directions from the point $P(1,3,2)$ is the directional derivative of $f = xyz - y^2 z$ a maximum and a minimum respectively? Find these directional derivatives.

d) The temperature at a point $P(x,y,z)$ in space is given by $T = x^2 + y^2 - z$. In what direction should an insect at $P(1,1,2)$ move so that it warms up as rapidly as possible?

Solution

(a) The directional derivative is

$$\nabla F \cdot \mathbf{u} = |\nabla F||\mathbf{u}| \cos\theta = |\nabla F| \cos\theta,$$

since $\mathbf{u}$ is a unit vector.
(b) This directional derivative is a maximum when $\cos\theta$ is a maximum, that is when $\cos\theta = 1$. Hence the maximum occurs when the angle between $\mathbf{u}$ and ∇f (at P)is zero, i.e., they have the same direction. The minimum occurs when $\cos\theta = -1$ which means that $\mathbf{u}$ and ∇f have opposite directions.
(c) We have $\operatorname{grad} f = (yz, xz - 2yz, xy - y^2) = (6, -10, -6)$ at $P(1, 3, 2)$ and so the maximum directional derivative occurs in the direction of $(3, -5, -3)$ and the minimum in the direction of $(-3, 5, 3)$. The unit vector having the same directions are

$$\pm\frac{(3, -5, -3)}{\sqrt{43}}.$$

Hence the maximum/minimum directional derivative are

$$\pm\frac{1}{\sqrt{43}}(6, -10, -6) \cdot (3, -5, -3) = \pm\frac{86}{\sqrt{43}} = \pm 2\sqrt{43}.$$

(d) We have $\nabla T = (2x, 2y, -1) = (2, 2, -1)$ at $P(1, 1, 2)$. Hence the temperature in the surroundings increases most rapidly in the direction $(2, 2, -1)$. This is the direction the insect should move in.

T4 A scalar field f is called harmonic if the Laplacian of the scalar field is zero. Show the following scalar fields are harmonic.

(a) $u(x, y, z) = e^{(x+y)} \cos(\sqrt{2}z)$, (b) $v(x, y) = x^2 - y^2$.

Solution

(a)

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = e^{(x+y)} \cos(\sqrt{2}z) + e^{(x+y)} \cos(\sqrt{2}z) + (-(\sqrt{2})^2 e^{(x+y)} \cos(\sqrt{2}z)) = 0.$$

(b)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + (-2) = 0$$

T5 Find the divergence of the vector fields

(a) $\mathbf{F} = (3xyz^2, 2xy^3, -x^2yz)$, (b) $\mathbf{G} = (e^{xz}, x^2 + y^2, yz)$,

at an arbitrary point and at $P(1, 1, 1)$.

Solution

(a) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = 3yz^2 + 6xy^2 - x^2y = 8$ at $(1,1,1)$. (b) $\operatorname{div}\mathbf{G} = \nabla\cdot\mathbf{G} = ze^{xz} + 2y + y = ze^{xz} + 3y = e + 3$ at $(1,1,1)$.

T6 Find $\operatorname{grad} f$ for

(a) $f = x\sin(y)$, (b) (b) $f = x\log(x+3z)$, (c) $f = \sqrt{zy}\cot(x+y)$.

Solution

(a) $\operatorname{grad} f = (\sin(y), x\cos(y), 0)$.
(b) $\operatorname{grad} f = (\log(x+3z) + \frac{x}{x+3z}, 0, \frac{3x}{x+3z})$.
(c) $\operatorname{grad} f = (-\sqrt{zy}\csc^2(x+y), -\sqrt{zy}\csc^2(x+y) + \sqrt{z/(4y)}\cot(x+y), \sqrt{y/(4z)}\cot(x+y))$.

T7 Give two examples from the natural world of (i) a scalar field, (ii) a vector field.

Solution

Scalar fields include height above sea-level or temperature or air pressure as a function of location on the surface of the earth. Vector fields include velocity as a function of position in fluid flow, gravitational force or magnetic force as a function of location in space.

T8 Find the directional derivative of

a) $f = e^{2x-y+z}$ at $P(1,1,-1)$ in the direction $\mathbf{d} = (-1,-3,-5)$;

b) $f = x^3 + 3xy - 3yz + z^3$ at $P(1,2,1)$ in the direction $\mathbf{d} = (1,4,3)$;

c) $f = \sin xy + \log yz$ at $P(\pi,1,2)$ in the direction $\mathbf{d} = (0,1,2)$.

Solution

(a) The unit vector in the direction of $\mathbf{d}$ is $\hat{\mathbf{d}} = \mathbf{d}/\sqrt{35}$ and $\nabla f = e^{2x-y+z}(2,-1,1) = (2,-1,1)$ at $P(1,1,-1)$. Therefore the directional derivative is

$$\nabla f(1,1,-1)\cdot\hat{\mathbf{d}} = \frac{1}{\sqrt{35}}(2,-1,1)\cdot(-1,-3,-5) = \frac{-2+3-5}{\sqrt{35}} = -\frac{4}{\sqrt{35}}.$$

(b) $\hat{\mathbf{d}} = \mathbf{d}/\sqrt{26}$ and $\nabla f = (3x^2+3y, 3x-3z, -3y+3z^2) = (9,0,-3)$ at $P(1,2,1)$. Therefore

$$\nabla f(1,2,1)\cdot\hat{\mathbf{d}} = \frac{1}{\sqrt{26}}(9,0,-3)\cdot(1,4,3) = 0.$$

(c) $\hat{\mathbf{d}} = \mathbf{d}/\sqrt{5}$ and $\nabla f = (y\cos xy, x\cos xy + 1/y, 1/z) = (-1, -\pi+1, \frac{1}{2})$ at $P(\pi, 1, 2)$. Therefore

$$\nabla f(\pi, 1, 2) \cdot \hat{\mathbf{d}} = \frac{1}{\sqrt{5}}(-1, -\pi+1, \tfrac{1}{2}) \cdot (0, 1, 2) = \frac{2-\pi}{\sqrt{5}}.$$

T9 Find the directional derivative of $xy + 3yz$ at the point (0, 3, - 2) in the direction of each of the vectors

$$\text{(a) } (2, 2, -1), \quad \text{(b) } (1, 0, 1), \quad \text{(c) } (4, -7, -4).$$

What are the maximum and minimum values of the directional derivative at $(0, 3, -2)$ and in which directions do they occur?

Solution

$\text{grad}(xy + 3yz) = (y, x + 3z, 3y) = (3, -6, 9)$ at $(0, 3, -2)$.
(a) The unit vector is $\mathbf{n} = (2, 2, -1)/3$. Therefore the directional derivative is

$$\frac{\partial\phi}{\partial n} = \nabla\phi(0, 3, -2) \cdot \mathbf{n} = (3, -6, 9) \cdot \frac{1}{3}(2, 2, -1) = \frac{6 - 12 - 9}{3} = -5.$$

(b) The unit vector is $\mathbf{n} = (1, 0, 1)/\sqrt{2}$. Therefore the directional derivative is

$$\frac{\partial\phi}{\partial n} = \nabla\phi(0, 3, -2) \cdot \mathbf{n} = (3, -6, 9) \cdot \frac{1}{\sqrt{2}}(1, 0, 1) = \frac{3 + 9}{\sqrt{2}} = 6\sqrt{2}.$$

(c) The unit vector is $\mathbf{n} = (4, -7, -4)/9$. Therefore the directional derivative is

$$\frac{\partial\phi}{\partial n} = \nabla\phi(0, 3, -2) \cdot \mathbf{n} = (3, -6, 9) \cdot \frac{1}{9}(4, -7, -4) = \frac{12 + 42 - 36}{9} = 2.$$

We want the unit vectors $\mathbf{n}$ such that at $(0, 3, -2)$ $\frac{\partial\phi}{\partial n}$ has its maximum and minimum values. Using the definition of dot product, $\frac{\partial\phi}{\partial n}(0, 3, -2) = \nabla\phi(0, 3, -2) \cdot \mathbf{n} = |\nabla\phi(0, 3, -2)||\mathbf{n}|\cos\theta = 1.\sqrt{9 + 36 + 81}\cos\theta = \sqrt{126}\cos\theta$. The angle θ is between the two vectors $(3, -6, 9)$ and $\mathbf{n}$ So the maximum occurs when $\cos\theta = 1$, i.e. $\theta = 0$, which is when $\mathbf{n}$ is parallel to $\text{grad}\,\phi$. The maximum value of the directional derivative is therefore $\sqrt{126}$.

Similarly, the minimum occurs when $\cos\theta = -1$, i.e. $\theta = \pi$, which is when $\mathbf{n}$ lies in the direction $(-3, 6, -9)$, which is the direction opposite to $\text{grad}\,\phi$. The minimum value of the directional derivative is therefore $-\sqrt{126}$.

T10 The temperature at the point (x, y, z) is given by

$$T(x, y, z) = (x + 3y)z^2.$$

Find the direction in which you should move from the point $(2, 2, 1)$ in order to achieve (a) the most rapid increase in temperature, (b) the most rapid decrease in temperature.

Solution

grad $T = (z^2, 3z^2, 2(x+3y)z) = (1, 3, 16)$ at $(2, 2, 1)$.
$\frac{\partial\phi}{\partial n}(1,3,16) = \nabla\phi(1,3,16)\cdot\mathbf{n} = |\nabla\phi(1,3,16)||\mathbf{n}|\cos\theta = 1.\sqrt{266}\cos\theta = \sqrt{266}\cos\theta$. The angle θ is between the two vectors $(1,3,16)$ and $\mathbf{n}$ So the maximum (the fastest increase of ϕ) occurs when $\cos\theta = 1$, i.e. $\theta = 0$, which is when $\mathbf{n}$ is parallel to grad ϕ, so is in the direction $(1,3,16)$.

Similarly, the minimum (the fastest decrease of ϕ) occurs when $\cos\theta = -1$, i.e. $\theta = \pi$, which is when $\mathbf{n}$ lies in the direction $(-1,-3,-16)$, which is the direction opposite to grad ϕ.

T11 Calculate the divergence of the vector field $\mathbf{F}$ and state whether the vector field is incompressible.

a) $\mathbf{F} = (z\ln(x), yz/x, z^2/x)$,

b) $\mathbf{F} = y^2\mathbf{i} + z^2\mathbf{j} + x^2\mathbf{k}$,

c) $\mathbf{F} = x^2\sin(y)(\mathbf{i} - \mathbf{j} + \mathbf{k})$.

Solution

(a) div $\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{z}{x} + \frac{z}{x} + \frac{2z}{x} = \frac{4z}{x}$. $\mathbf{F}$ is NOT incompressible.
(b) div $\mathbf{F} = \nabla\cdot\mathbf{F} = 0 + 0 + 0 = 0$. $\mathbf{F}$ is incompressible.
(c) div $\mathbf{F} = \nabla\cdot\mathbf{F} = 2x - x^2\cos y + 0 = 2x - x^2\cos y$. $\mathbf{F}$ is NOT incompressible.