Q1: Define a sequence $(x_n)_{n=1}^{\infty}$ by $x_1 = 2$ and $x_{n+1} = \frac{1}{3}x_n^2 - \frac{1}{3}x_n + 1$ for $n \in \mathbb{N}$.

a) Show that $1 < x_n < 3$ for all $n \in \mathbb{N}$.

Since the sequence is defined recursively, the pair of inequalities can be proven inductively. Firstly, the base case holds for x_1 by $1 < 2 < 3$. Secondly, assume that $1 < x_n < 3$ holds. Then

$$x_{n+1} - 1 = \frac{1}{3}x_n^2 - \frac{1}{3}x_n = \frac{1}{3}x_n(x_n - 1) > 0$$

and

$$x_{n+1} - 3 = \frac{1}{3}x_n^2 - \frac{1}{3}x_n - 2 = \frac{1}{3}(x_n - 3)(x_n + 2) < 0$$

as $1 < x_n < 3$ (per the inductive hypothesis). Thus, $1 < x_{n+1} < 3$, proving the statement $1 < x_n < 3$ for all $n \in \mathbb{N}$ by induction.

b) Show that $(x_n)_{n=1}^{\infty}$ is decreasing.

By looking at the difference between consecutive terms, we have

$$x_{n+1} - x_n = \frac{1}{3}x_n^2 - \frac{4}{3}x_n + 1 = \frac{1}{3}(x_n^2 - 4 + 3) = \frac{1}{3}(x_n - 1)(x_n - 3) < 0$$

as $1 < x_n < 3$. Therefore, $(x_n)_{n=1}^{\infty}$ is decreasing.

c) Prove that $(x_n)_{n=1}^{\infty}$ converges and find its limit.

Since $(x_n)_{n=1}^{\infty}$ is decreasing and bounded below, it converges by the monotone convergence theorem to some number $L \in \mathbb{R}$. As $1 < x_n \leq x_1 < 3$ for all *n*, we have $1 \leq L \leq x_1 < 3$ by the order properties of limits. Additionally, as $x_{n+1} \to L$ as $n \to \infty$, taking limits in the recursion formula gives

$$\begin{aligned} L &= \frac{1}{3}L^2 - \frac{1}{3}L + 1 \\ &\Rightarrow \frac{1}{3}L^2 - \frac{4}{3}L + 1 = 0 \\ &\Rightarrow \frac{1}{3}(L^2 - 4L + 3) = 0 \\ &\Rightarrow \frac{1}{3}(L - 1)(L - 3) = 0. \end{aligned}$$

Thus, $L = 1$ or $L = 3$. Since $L < 3$ by the previous statement, $L = 1$, as required.

Q2: Determine whether the series

$$\sum_{n=1}^{\infty} \frac{n!}{n^n + 3^n}$$

converges or diverges.

Let $x_n = \frac{n!}{n^n+3^n}$. Since $n! > 0$ and $n^n + 3^n > 0$, $x_n > 0$. Now let $y_n = \frac{n!}{n^n} > x_n$. By the limit ratio test,

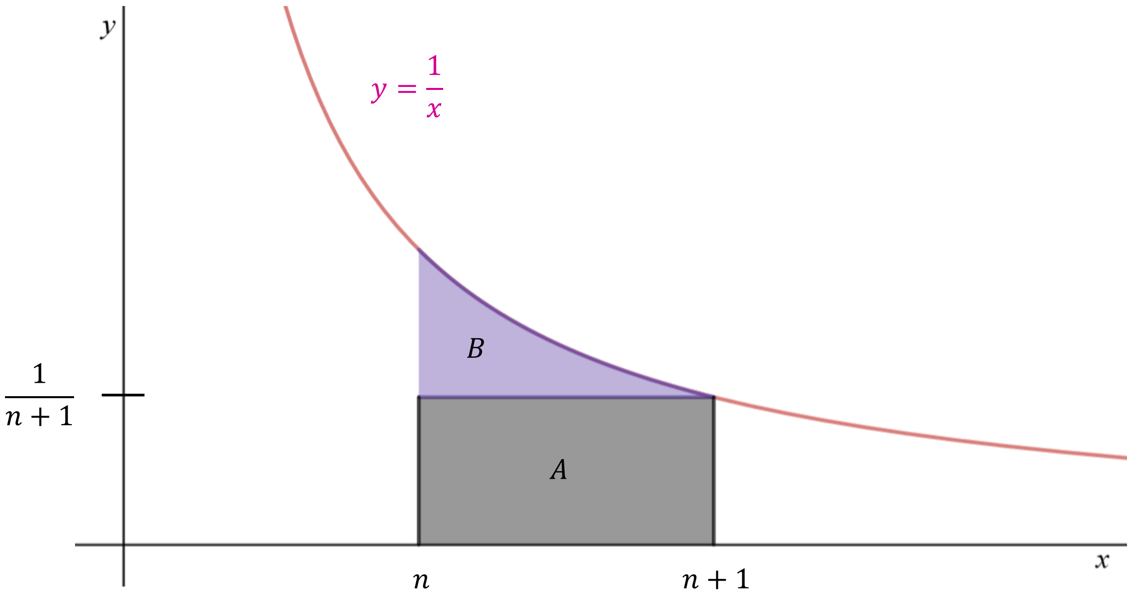
$$\frac{y_{n+1}}{y_n} = \frac{(n+1)n!\,n^n}{(n+1)n^n n!} = \left(\frac{n}{n+1}\right)^n \to \frac{1}{e} < 1$$

as $n \to \infty$ by the definition of e. Thus, y_n converges. By the comparison test, since $0 < x_n < y_n$ is true and y_n converges, $\sum_{n=1}^{\infty} \frac{n!}{n^n+3^n}$ also converges.

Q3:

a) Explain why $\frac{1}{n+1} \leq \int_n^{n+1} \frac{1}{x} dx$ for each $n \in \mathbb{N}$.

Let us look at the RHS of the inequality with a geometric point of view. Looking at graph A, the function $y = \frac{1}{x}$ is strictly decreasing and continuous in the first quadrant, and the integral is a computation of the area below the graph between the values n and $n + 1$. Since the function is decreasing, the area can be divided into two parts – a rectangle A with height $\frac{1}{n+1}$ and width $n + 1 - n = 1$ and a triangle-like area B on top of the rectangle.



Graph A

Then, by Graph A, we have

$$\int_n^{n+1} \frac{1}{x} dx = A + B = \frac{1}{n+1} + B \geq \frac{1}{n+1},$$

as required.

b) By writing the natural logarithm function for some x as $\ln x$, define a sequence $(t_n)_{n=1}^{\infty}$ by $t_n = \left(\sum_{r=1}^{n} \frac{1}{r}\right) - \ln n$. Show that this sequence is decreasing and that $0 \leq t_n \leq 1$ for all n.

By looking at the difference between consecutive terms of the sequence, we have

$$t_{n+1} - t_n = \left(\sum_{r=1}^{n} \frac{1}{r}\right) + \frac{1}{n+1} - \ln(n+1) - \left(\sum_{r=1}^{n} \frac{1}{r}\right) + \ln n$$
$$= \frac{1}{n+1} + \ln n - \ln(n+1).$$

Then, by properties of integration from Q3(a) we have

$$\frac{1}{n+1} \leq \int_n^{n+1} \frac{1}{x} dx = [\ln x]_n^{n+1} = \ln(n+1) - \ln n$$
$$\Rightarrow \frac{1}{n+1} + \ln n - \ln(n+1) \leq 0.$$

Thus, $t_{n+1} - t_n \leq 0$, which means that t_n is decreasing.

Since

$$t_1 = \sum_{r=1}^{1} \frac{1}{r} - \ln 1 = 1$$

and t_n is decreasing, $t_n \leq 1$.

The first term in the definition of the sequence, $\sum_{r=1}^{n} \frac{1}{r}$, can be seen as a left Riemann sum, which is an overestimation of the sequence t_n since it is decreasing. Thus,

$$\sum_{r=1}^{n} \frac{1}{r} > \int_n^{n} \frac{1}{x} dx = \ln n$$
$$\Rightarrow \sum_{r=1}^{n} \frac{1}{r} - \ln n \geq 0.$$

Thus, $0 \leq t_n \leq 1$, as required.

c) Why does $\lim_{n\to\infty} t_n$ exist?

Since t_n is decreasing and bounded below, $\lim_{n\to\infty} t_n$ exists by the monotone convergence theorem, as required.