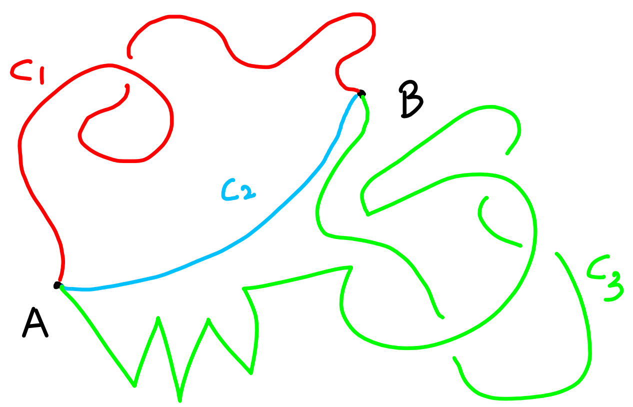
# Mathematics 2A - Lecture 19

## • Conservative / Path independent vector fields

**Definition**: a vector field $\underline{F} = (F_1, \dots, F_m) : D \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is called conservative if there exists a certain scalar field $f : D \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}$ such that

$$\left( \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_m} \right) = \nabla f = \underline{F} = (F_1, \dots, F_m)$$

$f$ is called a potential
(it is not unique : $f + c$ , $c \in \mathbb{R}$, is also a potential for $\underline{F}$ )




In general

$$\int_{C_1} \underline{F} \cdot d\underline{x} \neq \int_{C_2} \underline{F} \cdot d\underline{x} \neq \int_{C_3} \underline{F} \cdot d\underline{x}$$

**Definition**: a vector field $\underline{F}$ is called path independent if

$$\boxed{\int_{C_1} \underline{F} \cdot d\underline{x} = \int_{C_2} \underline{F} \cdot d\underline{x}}$$

for every path $C_1, C_2$ from A to B.

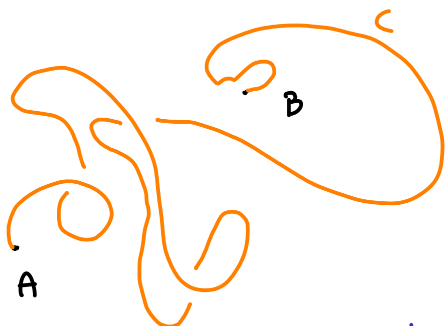
## · Conservative vector fields are path independent

**Recall:** suppose $F(x) = f'(x)$. Then

$$\int_a^b F(x)\,dx = \int_a^b f'(x)\,dx = f(b) - f(a)$$

**Theorem:** a conservative vector field $\underline{F}$ is path independent.

**proof:**




we need to check that $\int_C \underline{F} \cdot d\underline{x}$ does not depend on the choice of the curve $C$ connecting $A$ and $B$.

Let's compute:

$$\sum_{i=1}^{m} \frac{\partial f}{\partial r_i} \cdot r_i' = \frac{d}{dt}\big(f(\underline{r}(t))\big)$$

$$\int_C \underline{F} \cdot d\underline{x} = \int_{\cancel{c}\; a}^{b} \underline{F}(\underline{r}(t)) \cdot \frac{d\underline{r}}{dt}\,dt = \int_{\cancel{c}\; a}^{b} \overbrace{\nabla f(\underline{r}(t)) \cdot \frac{d\underline{r}}{dt}}^{\parallel}\,dt$$

choose a parametrisation of $C$
$\underline{r}(t) = (r_1(t), \dots, r_m(t))$, $t \in [a,b]$
s.t. $\underline{r}(a) = A$, $\underline{r}(b) = B$

$\underline{F}$ conservative
$\Downarrow$
$\underline{F} = \nabla f$

$$= \int_a^b \frac{d}{dt}\Big(f(\underline{r}(t))\Big)\,dt = f(\underline{r}(b)) - f(\underline{r}(a))$$

by the chain rule

$$= f(B) - f(A)$$

and this number does not depend on $C$.

. Sometimes path independent vector fields are also conservative

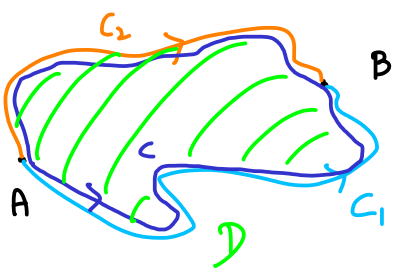
**Recall**: if $C$ is an oriented curve, then $-C$ is the same curve with opposite orientation and

$$\int_C \underline{F}\cdot d\underline{x} = -\int_{-C} \underline{F}\cdot d\underline{x}$$

**Theorem**: if $\underline{F}: \mathbb{R}^m \to \mathbb{R}^n$ everywhere defined and continuous.
If $\underline{F}$ is path independent, then it is also conservative.

This help us to check whether a vector field is conservative!

Start with $m=2$: let $C = C_1 \cup (-C_2)$. Then




$$\int_C \underset{\substack{\shortparallel \\ (P,Q)}}{\underline{F}}\cdot d\underline{x} = \int_{C_1} \underline{F}\cdot d\underline{x} + \int_{-C_2} \underline{F}\cdot d\underline{x}$$

$$= \int_{C_1} \underline{F}\cdot d\underline{x} - \int_{C_2} \underline{F}\cdot dx \qquad (A)$$

By Green Theorem: $$\int_C \underline{F}\cdot d\underline{x} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \qquad (B)$$

Combining (A) and (B) we get:

$$\int_{C_1} \underline{F}\cdot d\underline{x} - \int_{C_2} \underline{F}\cdot d\underline{x} = \iint_D \underbrace{\cancel{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}}_{=0} dx\,dy$$

$$\boxed{\text{If } \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \implies \underline{F} \text{ is path independent}}$$

Summarising: if $\underline{F} = (P, Q)$ is everywhere defined and continuos in $\mathbb{R}^2$. Then $\underline{F}$ is conservative if and only if

$$\boxed{\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}}$$

For $n=3$: Let $\underline{F}: \mathbb{R}^3 \to \mathbb{R}^3$ everywhere defined and continuos. Then, $\underline{F}$ is conservative if and only if

$$\boxed{\operatorname{curl} \underline{F} = 0}$$

irrotational vector field

. If $\underline{F}$ is conservative, how to find the potential $f$?

$$\underline{F} = \nabla f \iff (F_1, F_2, \dots, F_n) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right)$$

We are left to solve a system of partial differential equations

$$\begin{cases} \dfrac{\partial f}{\partial x_1} = F_1 \\ \dfrac{\partial f}{\partial x_2} = F_2 \\ \vdots \\ \dfrac{\partial f}{\partial x_n} = F_n \end{cases}$$

**Example 4.6** Vector fields **V** and **W** are defined by

$$\mathbf{V} = (2x - 3y + z, -3x - y + 4z, 4y + z)$$

$$\mathbf{W} = (2x - 4y - 5z, -4x + 2y, -5x + 6z)\ .$$

One of these is conservative while the other is not. Determine which is conservative and denote it by **F**. Find a potential function $\phi$ for **F** and evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}\ ,$$

where $C$ is the curve from $A = (1,0,0)$ to $B = (0,0,1)$ in which the plane $x+z = 1$ cuts the hemisphere given by $x^2 + y^2 + z^2 = 1$, $y \geq 0$.

Recall:

$$\operatorname{curl}(\underline{F}) = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$\underline{V}$ and $\underline{W}$ are everywhere defined in $\mathbb{R}^3$, hence they are conservative if their curl is the zero vector.

Let's check this:

- $\operatorname{curl}(\underline{V}) = (4-4,\ 1-0,\ -3-(-3)) = (0,1,0) \neq (0,0,0)$

  $\Longrightarrow \underline{V}$ not conservative

- $\operatorname{curl}(\underline{W}) = (0,\ -5-(-5),\ -4-(-4)) = (0,0,0) \Longrightarrow \underline{W}$ conservative

Now we have $\underline{F} = \underline{W} = \nabla f$, for a scalar field $f$, which is found solving:

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2x - 4y - 5z & (1) \\[2ex] \dfrac{\partial f}{\partial y} = -4x + 2y & (2) \\[2ex] \dfrac{\partial f}{\partial z} = -5x + 6z & (3) \end{cases}$$

Solve (1) and plug the solution into (2) and solve, and keep going until you solve all equations.

The solution to (1) is $f = x^2 - 4xy - 5xz + A(y,z)$ (A)

$$\Longrightarrow \frac{\partial f}{\partial y} = -4x + \frac{\partial A}{\partial y} \qquad (2')$$

Combining 2 and 2' we get

$$-4x + \frac{\partial A}{\partial y} = -4x + 2y \Longrightarrow \frac{\partial A}{\partial y} = 2y$$

$$\Longrightarrow A(y,z) = y^2 + B(z) \qquad \text{B}$$

From A and B we get:

$$f = x^2 - 4xy - 5xz + y^2 + B(z) \qquad \text{A'}$$

$$\Longrightarrow \frac{\partial f}{\partial z} = -5x + B'(z) \qquad 3'$$

Combining 3 and 3' we get

$$-5x + B'(z) = -5x + 6z \Longrightarrow B'(z) = 6z$$

$$\Longrightarrow B(z) = 3z^2 + c \quad , \quad c \in \mathbb{R}.$$

C

From A' and C we get:

$$\boxed{f = x^2 - 4xy - 5xz + y^2 + 3z^2 + c}$$

is the potential function.

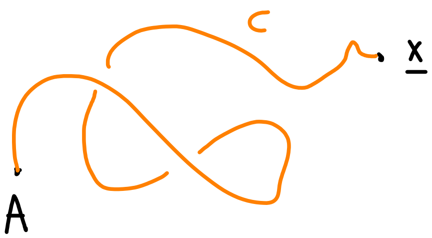
Since $\underline{F}$ is conservative, it is also path independent:

$$\int_C \underline{F} \cdot d\underline{x} = f(\underset{(0,0,1)}{\underset{\shortparallel}{B}}) - f(\underset{(1,0,0)}{\underset{\shortparallel}{A}}) = 3 + \cancel{c} - 1 - \cancel{c} = 2$$

## • Another way to compute the potential function f

( for a conservative vector field $\underline{F}$ )

Choose a point $A \in \mathbb{R}^n$ and choose some path from A to $\underline{x} = (x_1, \dots, x_n)$




Define $f(\underline{x}) = \int_C \underline{F}(\underline{y}) \cdot d\underline{y}$ use a different variable

Then $\nabla f(\underline{x}) = \underline{F}(\underline{x})$.

<u>Recall</u>: if $F(x)$ given, then $f(x) = \int_a^x F(y)\,dy$ has the property that $f'(x) = F(x)$