The intermediate value theorem

In this subsection we shall study one of the most central results about continuous functions, namely the intermediate value theorem. Essentially, this theorem says that if we are given a continuous function f defined on an interval $[a, b]$, such that $f(a) \neq f(b)$, then f attains all *intermediate values* between $f(a)$ and $f(b)$.

A special case of this situation is when $f(a) < 0$ and $f(b) > 0$. In this case, the theorem says that there must exist some $c \in (a, b)$ such that $f(c) = 0$. This should be no surprise: if we are able to draw the graph of f without lifting the pen, starting with a negative value of $y = f(x)$ at $x = a$, and ending with a positive value at $x = b$, then somewhere in between we'll have to cross the line $y = 0$.

Here is the precise formulation of the intermediate value theorem.

Theorem 5.16 (Intermediate value theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function and assume that d is a number such that $f(a) < d < f(b)$ or $f(b) < d < f(a)$. Then there exists a point $c \in (a, b)$ such that $f(c) = d$.*

Proof. Consider the case $f(a) < d < f(b)$. We define

$$S = \{x \in [a, b] \mid f(x) \leq d\}.$$

Since $f(a) < d$ by assumption we have $a \in S$, so that S is nonempty. By construction, the set S is bounded above by b. Therefore $c = \sup(S)$ exists. We claim that $f(c) = d$.

In order to show this, choose a sequence[1] $(x_n)_{n=1}^{\infty}$ in S converging to c. Since f is continuous at c, we have $f(x_n) \to f(c)$ as $n \to \infty$, and $f(c) \leq d$ because $f(x_n) \leq d$ for all n. Since $d < f(b)$ we have $c \neq b$. Similarly, choose a sequence $(y_n)_{n=1}^{\infty}$ in $(c, b]$ converging to c. Since $(c, b] \cap S$ is empty, we have $f(y_n) > d$ for all $n \in \mathbb{N}$. Again by continuity, this implies $f(c) \geq d$. Hence we obtain $f(c) = d$ as desired.

The case $f(b) < d < f(a)$ is analogous. $\square$

[1] Why can we find such a sequence?

We also record the following variant of the intermediate value theorem: If $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f(a) \leq d \leq f(b)$ or $f(a) \geq d \geq f(b)$, then there exists $c \in [a, b]$ with $f(c) = d$. Indeed, if $d = f(a)$ we can take $c = a$, if $d = f(b)$ we may take $c = b$. In the case that d lies strictly between $f(a)$ and $f(b)$ we obtain the desired point c using Theorem 5.16.

Let's turn to some applications of the intermediate value theorem. Firstly, the theorem can be used to prove the existence of solutions to certain equations without having to find them explicitly.

Example 5.17. Show that the equation $x^3 - 3x + 1 = 0$ has a solution in the interval $(1, 2)$.

Solution. We consider the function $f(x) = x^3 - 3x + 1$. Due to Theorem **??** we know that f is continuous, and we calculate $f(1) = -1 < 0$ and $f(2) = 3 > 0$. Hence by the intermediate value theorem, there exists a number $c \in (1, 2)$ such that $f(c) = 0$. This means exactly that c solves the equation $x^3 - 3x + 1 = 0$ as desired. $\square$

In a similar spirit, we can revisit our proof of the existence of the real number $\sqrt{2}$ in chapter 2. More precisely, consider the real function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 - 2$. Due to Theorem 5.12, the function f is continuous. Moreover $f(0) = -2 < 0$ and $f(2) = 2 > 0$. By the intermediate value theorem, there exists $c \in (0, 2)$ such that $f(c) = 0$. In other words, the number c thus obtained is positive and satisfies $c^2 = 2$, which is precisely saying that $c = \sqrt{2}$. [2]

Another application of the intermediate value theorem provides the existence of *fixed points* for certain maps.

Example 5.18. Let $f : [a, b] \to [a, b]$ be a continuous function. Then there exists a fixed point for f, that is, a point $c \in [a, b]$ such that $f(c) = c$.

Solution. We consider the function $g(x) = f(x) - x$. Then g is continuous. Moreover, we have $g(a) = f(a) - a \geq 0$ as $a \leq f(a)$, and similarly[3] $g(b) = f(b) - b \leq 0$ as $f(b) \leq b$. Hence by the intermediate value theorem, there exists a number $c \in [a, b]$ such that $g(c) = 0$. This means $f(c) = c$, so c is a fixed point of f. $\square$

Apart from the intermediate value theorem, another central result about continuous functions is the following fact.

Theorem 5.19. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is bounded.*

Proof. Suppose f is not bounded above. Then for any $n \in N$, there exists $x_n \in [a, b]$ such that $f(x_n) > n$. By Bolzano-Weierstrass, the sequence $(x_n)_{n=1}^{\infty}$ has a subsequence $(x_{k_n})_{n=1}^{\infty}$ converging to some $c \in [a, b]$. Therefore, by continuity of f at c, we obtain that $f(x_{k_n}) \to f(c)$ as $n \to \infty$. This means in particular that $(f(x_{k_n}))_{n=1}^{\infty}$ is bounded, since we know that convergent sequences are bounded[4]. If M is an upper bound for this sequence, then we have $f(x_{k_n}) \leq M$ for all n. This contradicts the fact that

$$f(x_{k_n}) > k_n \geq n > M$$

for $n \in \mathbb{N}$ with $n > M$. Hence our assumption was wrong, which means that f is bounded above.

In a similar way one shows that f is bounded below. $\square$

The extreme value theorem

As a consequence of Theorem 5.19 we obtain the *extreme value theorem*.

[2] In chapter 2 we already gave a proof for the existence of $\sqrt{2}$, based on the completeness axiom. We now have another proof of this result, which is shorter and less technical than the first one. If you trace through the ingredients in the second proof, you'll find however that eventually we're again relying completeness; things are just organised in a different way, using more advanced ideas like the concept of continuity.

[3] Here we are using $a \leq f(x) \leq b$ for any $x \in [a, b]$, which holds by assumption.

[4] Again we are referring to known results on sequences, compare chapter 3.

Theorem 5.20 (Extreme value theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ be continuous. Then there exist $u, v \in [a,b]$ such that $f(u) \leq f(x) \leq f(v)$ for all $x \in [a,b]$.*

Proof. Let us first show that there exists $v \in [a,b]$ such that $f(x) \leq f(v)$ for all $x \in [a,b]$. We consider the set

$$S = f([a,b]) = \{f(x) \mid x \in [a,b]\}.$$

Clearly S is nonempty, and according to Theorem 5.19 it is bounded. Therefore $M = \sup(S)$ exists. We claim that there exists a $v \in [a,b]$ with $f(v) = M$.

Suppose this is not the case. Then we have $f(x) < M$ for all $x \in [a,b]$. Therefore, we can define a function $g : [a,b] \to \mathbb{R}$ by

$$g(x) = \frac{1}{M - f(x)}.$$

The function g is continuous according to Theorem **??**, and therefore bounded by Theorem 5.19. Let $K > 0$ be an upper bound for g, so that $g(x) \leq K$ for all $x \in [a,b]$. This means

$$\frac{1}{M - f(x)} \leq K \iff f(x) \leq M - \frac{1}{K}$$

for all $x \in [a,b]$. Thus $M - \frac{1}{K}$ is an upper bound for f, which contradicts the fact that M is the least upper bound of f.

This contradiction establishes that there exists some $v \in [a,b]$ such that $f(v) = M$. In particular, we have $f(x) \leq f(v)$ for all $x \in [a,b]$ by the fact that M is an upper bound for f.

The existence of u is proved in a similar way. □

Combining the intermediate value theorem and the extreme value theorem, it follows that the image $f([a,b]) = \{f(x) \mid x \in [a,b]\}$ of any bounded interval $[a,b]$ under a continuous function $f : \mathbb{R} \to \mathbb{R}$ is again some bounded interval.

Example 5.21. Let $f : [a,b] \to \mathbb{R}$ be continuous. Show that

$$f([a,b]) = [f(u), f(v)]$$

for some $u, v \in [a,b]$.

Proof. By the extreme value theorem, there exist $u, v \in [a,b]$ such that $f(u) \leq f(x) \leq f(v)$ for all $x \in [a,b]$. Hence

$$f([a,b]) \subset [f(u), f(v)].$$

If $u = v$ then the set $[f(u), f(v)]$ consists of a single point, and the previous inclusion is clearly an equality.

If $u < v$, then the intermediate value theorem, applied to the continuous function $f : [u,v] \to \mathbb{R}$, shows that for any $d \in [f(u), f(v)]$ there exists $c \in [a,b]$ with $f(c) = d$. Hence we have $f([a,b]) = [f(u), f(v)]$.

In the case $v < u$ one argues similarly, by considering the continuous function $f : [v,u] \to \mathbb{R}$. □

Recall that a map $f : X \to Y$ is a bijection if and only if f is injective and surjective. This is equivalent to the existence of an inverse map $g : Y \to X$ to f. The inverse is uniquely determined by the requirements

$$f \circ g = \mathrm{id}_Y, \qquad g \circ f = \mathrm{id}_X$$

where id_X and id_Y are the identity maps of X and Y, respectively[5].

[5] Following standard notation, we'll write $f^{-1} : Y \to X$ for the inverse map of f in the sequel.

Lemma 5.22. *Let $f : [a,b] \to [c,d]$ be a continuous bijection. Then $f(a) = c$ and $f(b) = d$ or $f(a) = d$ and $f(b) = c$.*

Proof. According to Example 5.21, surjectivity of f implies that there exist $u, v \in [a,b]$ with $f(u) = c$ and $f(v) = d$, respectively.

Assume first $u \leq v$ and consider $f([u,v]) \subset [c,d]$. By the Intermediate Value Theorem, for any $y \in [f(u), f(v)] = [c,d]$ we find $x \in [u,v]$ such that $f(x) = y$. In other words, we have $f([u,v]) = [c,d]$. In particular, if $a < u$ then $y = f(a) \in [c,d]$ is equal to $f(x)$ for some $x \in [u,v]$. That is, $f(a) = f(x)$ and $a < x$, in particular, $a \neq x$. This contradicts injectivity of f. Similarly, if $v < b$ we get $y = f(b)$ equals $f(x)$ for $x \in [u,v]$, which contradicts injectivity of f again. We conclude $a = u$ and $b = v$.

The case $u \geq v$ is dealt with in a similar way, in this case one obtains $a = v$ and $b = u$.

$\square$

Theorem 5.23. *Let $f : [a,b] \to [c,d]$ be a bijective continuous map. Then*

a) f and f^{-1} are both strictly increasing or strictly decreasing;

b) The inverse map f^{-1} is continuous.

Proof. Consider first part a). We'll consider the case that $f(a) < f(b)$. The according to Lemma 5.22 we have $f(a) = c$ and $f(b) = d$. Now let $x, y \in [a,b]$ with $x < y$. If $f(x) \geq f(y)$ then $f([a,x]) \cap f([y,b])$ is nonempty by the intermediate value theorem, contradicting injectivity of f. Hence $f(x) < f(y)$, which means that f is strictly increasing. The case $f(a) > f(b)$ is analogous.

Now assume that f is strictly increasing but f^{-1} is not. Then there exists $x, y \in [c,d]$ such that $x < y$ but $f^{-1}(x) \geq f^{-1}(y)$. Applying the strictly increasing function f to $f^{-1}(x)$ and $f^{-1}(y)$ yields $x = f(f^{-1}(x)) \geq f(f^{-1}(y)) = y$, which is a contradiction. Again, the case of strictly decreasing functions is analogous.

For part b) let us consider the case that f is strictly increasing. Moreover let $(y_n)_{n=1}^{\infty}$ be a monotonic sequence in $[c,d]$ converging to s. Then $(f^{-1}(y_n))_{n=1}^{\infty}$ is a monotonic sequence in $[a,b]$, and by the monotone convergence theorem we have $(f^{-1}(y_n))_{n=1}^{\infty} \to r$ for some $r \in [a,b]$. Since f is continuous we obtain $y_n = f(f^{-1}(y_n)) \to f(r)$. Moreover we have $y_n \to s$, which means $s = f(r)$ since f is bijective. Hence $f^{-1}(s) = r$.

Now let $\varepsilon > 0$ be arbitrary. Considering $y_n = s - \frac{1}{n}$ we obtain $n_1 \in \mathbb{N}$ such that $f^{-1}(s - \frac{1}{n_1}) \geq r - \varepsilon$. Similarly, considering $y_n =$

$s + \frac{1}{n}$ we obtain $n_2 \in \mathbb{N}$ such that $f^{-1}(s + \frac{1}{n_2}) \leq r + \varepsilon$. If we pick $\delta = \min(\frac{1}{n_1}, \frac{1}{n_2})$, then using monotonicity of f^{-1} we get $f^{-1}((s - \delta, s + \delta)) \subset (r - \varepsilon, r + \varepsilon)$. That is, for $x \in [c, d]$ with $|x - s| < \delta$ we get $|f^{-1}(x) - f^{-1}(s)| < \varepsilon$. In other words, f^{-1} is continuous at s. Since s was arbitrary, this shows that f^{-1} is continuous. $\square$