FB1:

a) Use (i) the Midpoint Rule and (ii) Simpson's Rule to approximate the integral

$$\int_2^4 \frac{\mathrm{d}x}{\sqrt{x^2+1}},$$

using n = 10 grid intervals. Round your answer to 6 decimal places.

b) Compute the integral exactly using antiderivatives and decide which method from part (a) is closest to the true value.

a)

i) By dividing the interval [2, 4] into 10 grid intervals, the width Δx of each interval would be

$$\Delta x = (4-2)/10 = 0.2.$$

The intervals would then be [2, 2.2], [2.2, 2.4], ..., [3.8, 4]. The Midpoint Rule states that

$$\int_2^4 \frac{\mathrm{d}x}{\sqrt{x^2+1}} \approx M_{10} = \sum_{i=1}^{10} \frac{1}{\sqrt{\overline{x_i}^2+1}} \Delta x,$$

for some positive integer *i*, where $\overline{x_i} = \frac{1}{2}(x_{i-1}+x_i)$. Thus, the list of midpoints $\bar{x}$ would be $2.1, 2.3, 2.5, \dots, 3.9$. Therefore, approximation using the Midpoint Rule would be

$$\sum_{i=1}^{10} \frac{1}{\sqrt{(1.9+0.2i)^2+1}} \cdot 0.2 \approx 0.650874.$$

ii) Simpson's Rule states that

$$\int_2^4 \frac{\mathrm{d}x}{\sqrt{x^2+1}} \approx S_{10} = \frac{\Delta x}{3}(y_0+4y_1+2y_2+4y_3+\cdots+2y_{n-2}+4y_{n-1}+y_n),$$

where $y_0, y_1, \dots, y_n$ are the values of the function (in this case $\frac{1}{\sqrt{x^2+1}}$) for each x in 2, 2.2, ..., 4.

When substituting Δx for 0.2 as found in part (a)(i) and y values calculated by inserting each x in the above expression, the approximation by Simpson's Rule becomes

$$S_{10} \approx 0.651078.$$

b) Since the antiderivative of $\frac{1}{\sqrt{x^2+1}}$ is

$$\int \frac{1}{\sqrt{x^2+1}} = \sinh^{-1}(x) + C,$$

the definite integral of the function in the interval (2, 4) is

$$\int_2^4 \frac{\mathrm{d}x}{\sqrt{x^2+1}} = \sinh^{-1}(4) - \sinh^{-1}(2) \approx 0.651077.$$

Calculating the absolute error made by the Midpoint Rule $|E_M|$,

$$|E_M| = 0.651077 - 0.650874 = 0.000203,$$

while the absolute error made by Simpson's Rule $|E_S|$ is

$$|E_S| = 0.651078 - 0.651077 = 0.000001.$$

Since $|E_S| < |E_M|$, the result obtained by Simpson's Rule is the closest to the true value.

FB2:

Let S be a set, and let A and $\{B_i \mid i \in N\}$ be subsets of S. Is it true that

$$A \cup \left(\bigcap_{i \in \mathbb{N}} \mathrm{B_i} \right) = \bigcap_{i \in \mathbb{N}} (A \cup \mathrm{B}_i) \,?$$

If so provide a proof, and if not provide a counterexample.

Suppose an arbitrary element $x \in A \cup (\bigcap_{i \in \mathbb{N}} \mathrm{B_i})$. Then *x* is at least either in A or in $\mathrm{B_i}$ for all *i* (or both). Hence, $x \in A \cup B_i$ for all *i*, meaning that $x \in \bigcap_{i \in \mathbb{N}} (A \cup \mathrm{B}_i)$.

Going the other way, suppose an arbitrary element $y \in \bigcap_{i \in \mathbb{N}} (A \cup \mathrm{B}_i)$. Then $y \in A \cup \mathrm{B}_i$ for all *i* (in other words, if $y \notin A \cup B_k$ for some integer *k* in $1 \leq k \leq i$, y will not be in the intersection of all instances of $A \cup \mathrm{B}_i$). Hence, $y \in A$ or $y \in B_i$ for all *i*. Thus, $y \in A \cup (\bigcap_{i \in \mathbb{N}} \mathrm{B_i})$.

Since all elements from each side of the equation are in the other, the statement

$$A \cup \left(\bigcap_{i \in \mathbb{N}} \mathrm{B_i} \right) = \bigcap_{i \in \mathbb{N}} (A \cup \mathrm{B}_i)$$

is proven true.

FB3:

Consider the relation $\sim$ on $\mathbb{R} \times \mathbb{R}$ defined as follows for all $(x, y), (a, b) \in \mathbb{R} \times \mathbb{R}$.

$$(x, y) \sim (a, b) \text{ if and only if } x - a = y - b.$$

Show that $\sim$ is an equivalence relation and describe geometrically how this relation partitions the plane.

To check for **reflexivity**, let an arbitrary $(x, y) \in \mathbb{R}^2$. Since

$$x - x = y - y = 0,$$

the relation is reflexive because $(x, y) \sim (x, y)$.

To check for **symmetry**, the condition of the relation can be multiplied by (-1) to get

$$a - x = b - y,$$

which is the condition for the relation of $(a, b) \sim (x, y)$; thus, the relation is symmetric.

To check for **transitivity**, let an arbitrary $(\alpha, \beta) \in \mathbb{R}^2$. If $(x, y) \sim (a, b)$ and assuming $(a, b) \sim (\alpha, \beta)$,

$$x - a = y - b \text{ and } a - \alpha = b - \beta.$$

Rearranging both equations, one gets

$$x - y = a - b \text{ and } a - b = \alpha - \beta.$$

By comparing both equations, one can see that

$$x - y = \alpha - \beta,$$

which, rearranged, is

$$x - \alpha = y - \beta,$$

the condition for the relation $(x, y) \sim (\alpha, \beta)$; thus, the relation is transitive.

When the condition for the relation $(x, y) \sim (a, b)$ is rearranged to make a function y, the function is

$$y = x + (b - a).$$

This arrangement shows that the relation partitions the plane into linear equations with gradient 1 and y-axis intersection of $(0, b - a)$.