2A Multivariable Calculus 2020

Ex Sheet 1

Tutorial Exercises

T1 State the type of surface given by each of the following equations in *three dimensional space*.

(a) $4x + 5y - 2z = 20$, (b) $x^2 + y^2 = 1$, (c) $x^2 + y^2 + z^2 - 2x = 10$,

(d) $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$, (e) $25x^2 + 4y^2 + z^2 = 100$, (f) $x^2 + y^2 + z^2 = 16$.

Solution

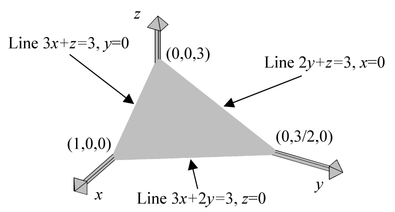
(a) Plane, (b) Cylinder, (c) Sphere centre $(1, 0, 0)$, radius $\sqrt{11}$, as seen by completing the square in the x terms, $(x^2 - 2x + 1) + y^2 + z^2 = 10 + 1$ giving $(x-1)^2 + y^2 + z^2 = 11$, (d) Ellipsoid, (e) Ellipsoid – Divide the equation by 100 to reduce to standard form:

$$\frac{x^2}{4} + \frac{y^2}{25} + \frac{z^2}{100} = 1$$

(f) Sphere.

T2 Sketch the part of the plane $3x + 2y + z = 3$, that lies in the first octant ($= \{(x, y, z) : x \geq 0, y \geq 0, z \geq 0\}$.)

Solution



T3 Match the graphs in Figure 1 with its corresponding contour-maps (cross-sections) from Figure 2. Give reasons for your choices.

Solution

A-(c), B-(d) , C-(a), D-(b).

T4 Complete the square in each of the following expressions

(a) $x^2 + y^2 + z^2 + 2 = 2(x + y + z)$, (b) $z = \sqrt{2x + 2y - x^2 - y^2 - 1}$.

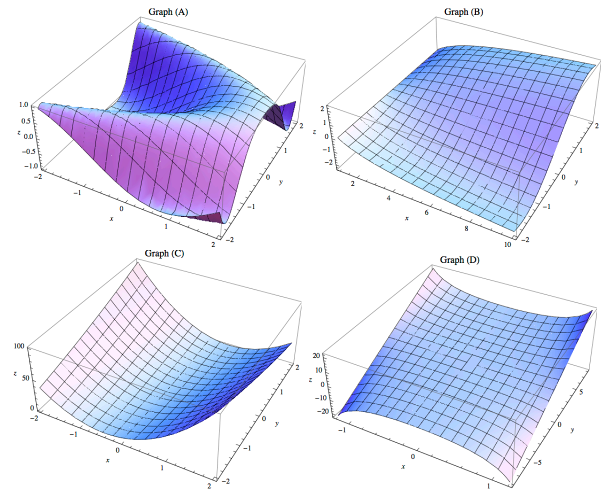


Figure 1: Cross sections of 4 graphs (See question T3).

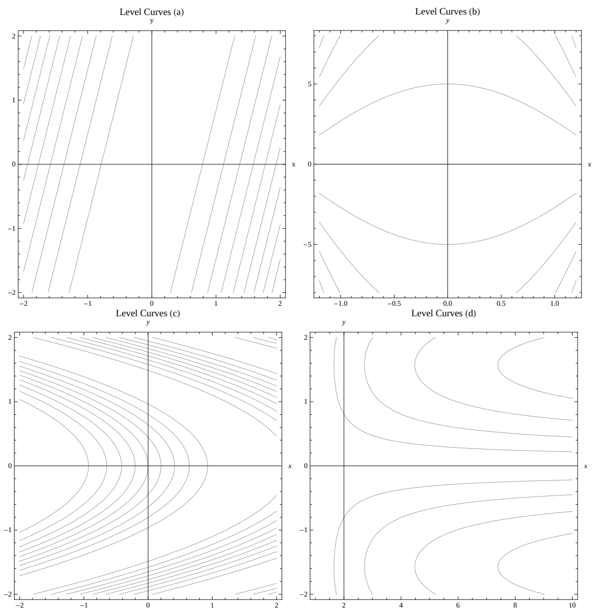
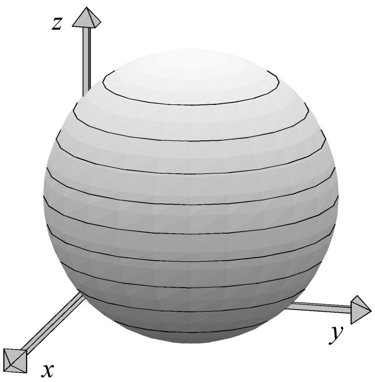


Figure 2: Cross sections of 4 graphs (See question T3).

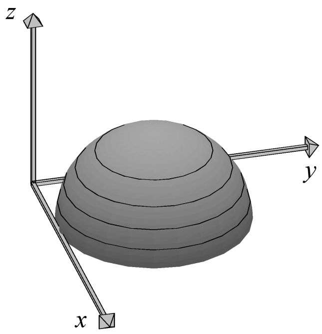
and hence describe and sketch the surfaces they represent.

Solution

(a) We have $x^2 - 2x + y^2 - 2y + z^2 - 2z + 2 = 0$ and so completing the square gives $(x-1)^2 + (y-1)^2 + (z-1)^2 = 1$, the sphere with centre $(1, 1, 1)$ and radius 1.
(b) We have $z \geq 0$ and $(x-1)^2 + (y-1)^2 + z^2 = 1$, the upper hemisphere with centre $(1, 1, 0)$ and radius 1.



Sphere, centre (1,1,1), radius 1



Hemisphere, centre (1,1,0), radius 1

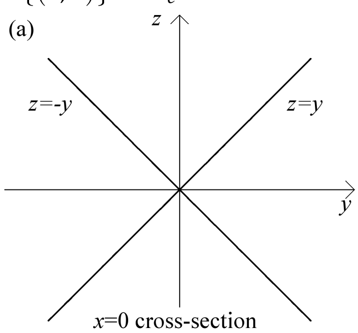
T5 By considering the level curves and cross sections $x = 0$ and $y = 0$, sketch the surfaces

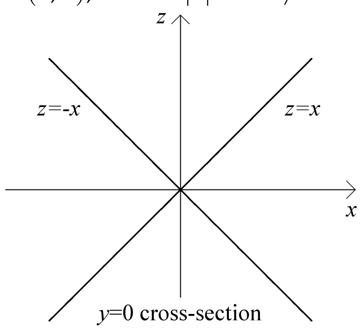
(a) $x^2 + y^2 - z^2 = 0$, (b) $z = x^2 + y^2$, (c) $2x^2 + y^2 + z^2 = 1$.

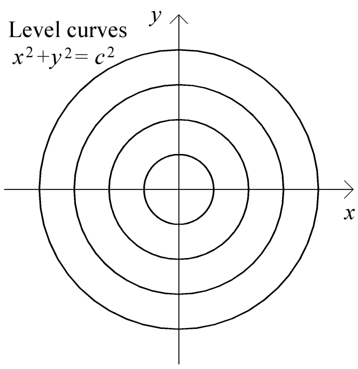
Which surface is the paraboloid and which is the ellipsoid?

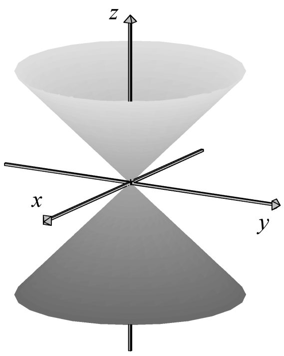
Solution

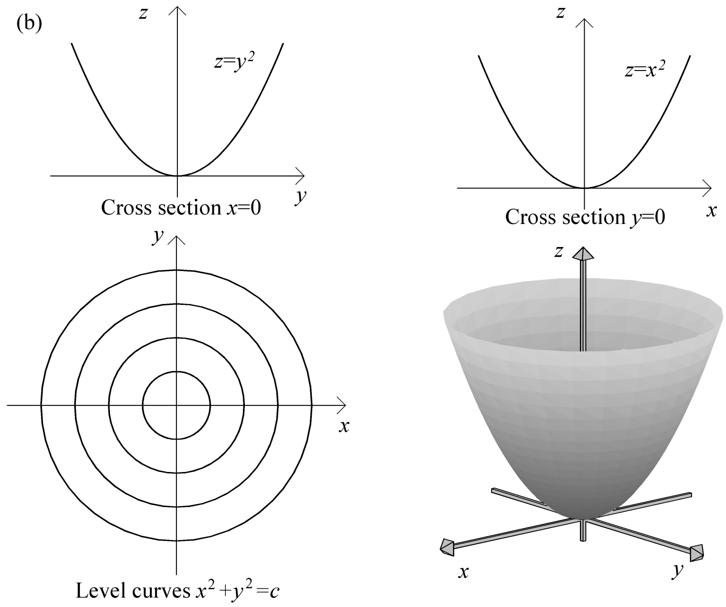
(a) Cross section $x = 0$: $z = \pm y$; Cross section $y = 0$: $z = \pm x$; Level curves: $x^2 + y^2 = c^2$, so that $L_0 = \{(0,0)\}$ or L_c is the circle centre $(0,0)$, radius $|c|$ for $c \neq 0$.

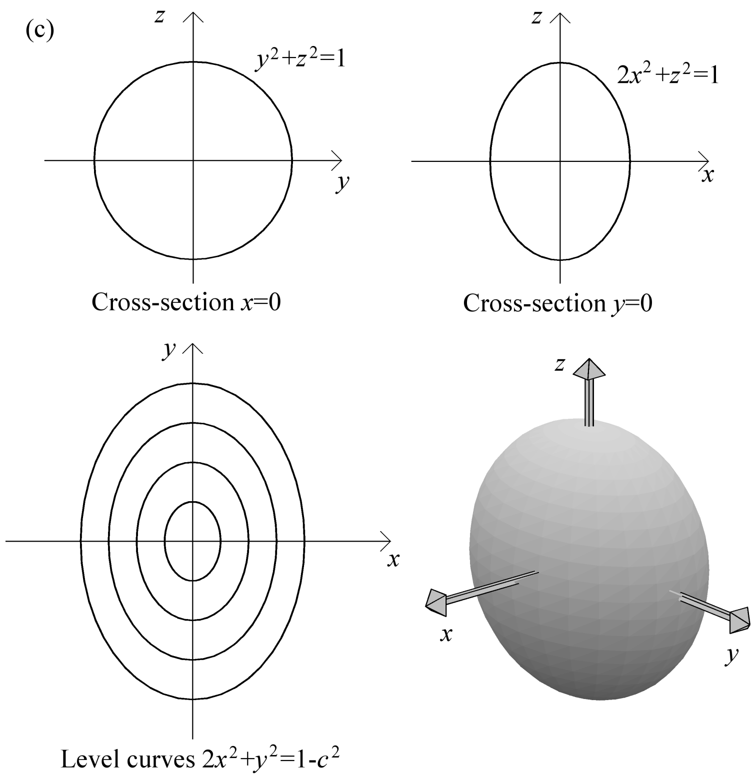








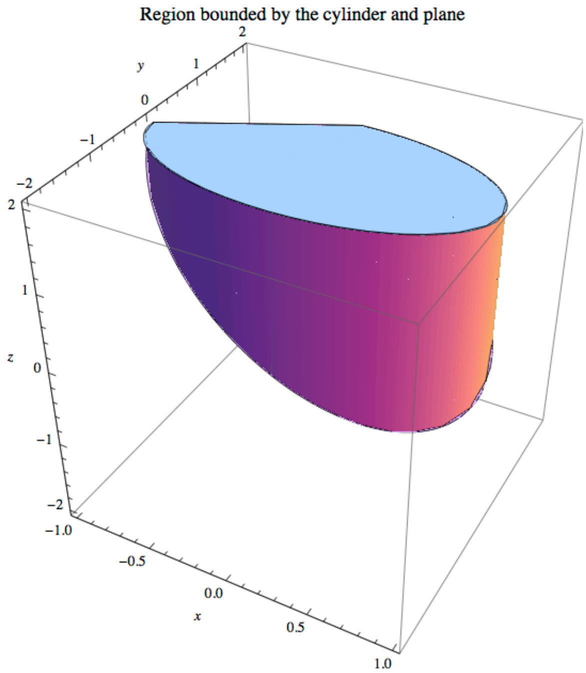




(b) Cross section $x = 0$: $z = y^2$ (parabola); Cross section $y = 0$: $z = x^2$ (parabola); Level curves: $x^2 + y^2 = c$, so that $L_c = \emptyset$ for $c < 0$ or $L_0 = \{(0,0)\}$ or L_c is the circle centre $(0,0)$, radius $\sqrt{c}$ for $c > 0$. This is paraboloid.
(c) Cross section $x = 0$: $y^2 + z^2 = 1$ (circle); Cross section $y = 0$: $2x^2 + z^2 = 1$ (ellipse); Level curves: $2x^2 + y^2 = 1 - c^2$, so that $L_{\pm 1} = \{(0,0)\}$, L_c is an ellipse for $|c| \leq 1$ or $L_c = \emptyset$ for $|c| > 1$. This is an ellipsoid.

T6 Sketch the region bounded by the cylinder $x^2 + y^2 = 1$ and the planes $x - y + z = 1$ and $z = 2$.

Solution



T7 Find all partial derivatives of the functions

(a) $f(x, y) = x\cos(xy + x)$, (b) $g(s, t) = \dfrac{st}{s + t}$, (c) $r(u, v) = (uv + v)^3$,

(d) $h(x, y, z) = \dfrac{yz + zx + xy}{xyz}$, (e) $q(x, y, z) = xe^{-(x^2+y^2)}$.

Can you find a way to rewrite $h(x, y, z)$ in (d) so that calculating its partial derivatives is very easy?

Solution

(a) $f_x = \cos(xy + x) - x(y + 1)\sin(xy + x)$, $f_y = -x^2\sin(xy + x)$,
(b) $g_s = \dfrac{t(s + t) - st.1}{(s + t)^2} = \dfrac{t^2}{(s + t)^2}$, $g_t = \dfrac{s^2}{(s + t)^2}$.
(c) $r_u = 3v(uv + v)^2$, $r_v = 3(u + 1)(uv + v)^2$.
(d) $h = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$. Hence $h_x = -\dfrac{1}{x^2}$, $h_y = -\dfrac{1}{y^2}$ and $h_z = -\dfrac{1}{z^2}$.
(e) $q_x = (1 - 2x^2)e^{-(x^2+y^2)}$, $q_y = -2xye^{-(x^2+y^2)}$ and $q_z = 0$.

T8 Let $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$. Show that

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} = 4(x^2 + y^2).$$

Solution

$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} = 2x \cdot 2x - 2y \cdot (-2y) = 4(x^2 + y^2).$