University of Glasgow

Wednesday, 7 December 2016
09.30 am – 11.00 am
(1 hour 30 minutes)

DEGREES OF MSci, MEng, BEng, BSc, MA and MA (Social Sciences)

ALGORITHMIC FOUNDATIONS 2: COMPSCI2003

Answer all questions

This examination paper is worth a total of 60 marks.

The use of calculators is not permitted in this examination.

INSTRUCTIONS TO INVIGILATORS: Please collect all exam question papers and exam answer scripts and retain for school to collect. Candidates must not remove exam question papers.

1. **(a)** Prove that $(p \wedge \neg q) \rightarrow q$ and $(p \wedge \neg q) \rightarrow \neg p$ are equivalent using laws of logical equivalence. Justify each step. [4]

(b) Give a simpler form of the two logical statements in **(a)** containing only one logical connective. [1]

(c) Prove for any positive integer $n \in \mathbb{Z}^+$: "n is even if and only if $7 \cdot n + 4$ is even". State what proof methods you use. [5]

2. Suppose the domain of discourse P is all people and we have the following predicates:

- $F(x)$: x is friendly;
- $T(x)$: x is tall;
- $A(x)$: x is angry.

Express the following English statements in logical formulae using the above predicates.

(a) Some people are not angry. [2]

(b) All tall people are friendly. [2]

(c) No friendly people are angry. [2]

Express in concise (good) English without variables each of the following logical formulae.

(d) $\exists x \in P. (T(x) \wedge \neg A(x))$ [2]

(e) $\forall x \in P. (T(x) \rightarrow (F(x) \vee \neg A(x))$ [2]

3. **(a)** Give a recursive definition with initial conditions for the set $\{0.5, 0.05, 0.005, \ldots\}$. [1]

(b) Give a recursive definition for the set of integers not divisible by 4.

[1]

A non-empty proper binary tree over X (where X is a data set) can be inductively defined as follows:

- **base case:** if $x \in X$, then $\texttt{node}(\texttt{nil}, \texttt{nil}, x)$ is a non-empty complete binary tree;
- **inductive step:** if t_1 and t_2 are non-empty complete binary trees and $x \in X$, then $\texttt{node}(t_1, t_2, x)$ is a non-empty complete binary tree.

(c) Give a recursive function `nn` which returns the number of nodes in a non-empty proper binary tree. [2]

(d) Give a recursive function `ne` which returns the number of edges in a non-empty proper binary tree. [2]

(e) Using the above functions, prove using induction that the number of nodes in a non-empty proper binary tree equals 1 plus the number of edges (i.e. $\texttt{nn}(t) = 1 + \texttt{ne}(t)$ for all non-empty proper binary trees t). Justify each step. [4]

4. **(a)** A car number plate has two forms:

(i) three digits followed by three letters;

(ii) four letters followed by two digits.

How many car number plates are there (you can leave your answer in powers of 10 and 26). Explain your answer. [2]

(b) How may ways are there to choose 4 pieces of fruit from a bowl that contains 6 bananas, 5 apples, 7 pears, and 4 oranges? Explain your answer. [3]

(c) How many 8 bit strings are there, that start with a 0 or end with 111 (you can leave you answer in powers of 2)? Explain your answer. [3]

(d) How many students must be in a class to guarantee that at least 4 were born on the same day of the week? Explain your answer. [2]

5. Suppose that you have three urns that you cannot see into. Urn_1 has 9 green balls and 1 red. Urn_2 has 5 green and 5 blue. Urn_3 has 2 green, 4 red, and 4 blue.

(a) If you select a ball from Urn_1, what is the probability you select a green ball? [1]

(b) If you select a ball from each urn what is the probability you select three green balls? [2]

(c) If you select a ball from Urn_1 and a ball from Urn_3, what is the probability you select at least one red ball? [3]

(d) Suppose you randomly select an urn and then randomly select a ball from it. Given the ball you drew was green, what is the probability that it came from Urn_1? [4]

6. (a) Define what we mean when we say that two graphs are isomorphic. [3]

(b) Draw two isomorphic graphs (e.g. with four vertices) one in which the edges cross and one in which the edges do not cross. Explain how the graphs are isomorphic. [3]

Define what it means for a relation R over a set A to be:

(c) symmetric; [1]

(d) anti-symmetric. [1]

Give an example of a relation on a set that is:

(e) both symmetric and anti-symmetric; [1]

(f) neither symmetric nor anti-symmetric. [1]