Suppose we have the following predicates:

- $P(x)$: x is prime
- $E(x)$: x is even
- $G(x, y)$: $x > y$
- $Eq(x, y)$: $x = y$
- $S(x, y, z) : x + y = z$

Express the following formula in good English (do not use variables and avoid the use of "there exists" and "for all").

- $\forall x \in \mathbb{Z}^+ . \exists x \in \mathbb{Z}^+ . G(y, x)$

  There is an integer greater than all integers.

- $\forall x \in \mathbb{Z}^+ . \exists y \in \mathbb{Z}^+ . \exists z \in \mathbb{Z}^+ . ((E(x) \wedge G(x, 3)) \rightarrow (P(y) \wedge P(z) \wedge S(y, z, x))$

  Any even positive integer greater than 3 can be expressed as the sum of two primes (this is the Goldbach conjecture).

- $\forall x \in \mathbb{Z}^+ . \forall y \in \mathbb{Z}^+ . ((\neg G(x, y) \wedge \neg G(y, x)) \rightarrow Eq(x, y))$

  Given any two positive integers, if neither is greater than the other then they are equal.

Suppose we have the following predicates:

- $P(x)$: x is prime
- $E(x)$: x is even
- $G(x, y)$: $x>y$
- $S(x, y, z) : x + y = z$
- $T(x, y, z)$: $x{\cdot}y = z$

Express the following English statements in logical formulae using the predicates given above over the domain of discourse $\mathbb{Z}^+$.

- Given any two positive integers there exists another positive integer which equals their product.

  $\forall x \in \mathbb{Z}^+ . \forall y \in \mathbb{Z}^+ . \exists z \in \mathbb{Z}^+ . T(x, y, z)$

- There is no largest prime.

  Two possible solutions are: $\forall x \in \mathbb{Z}^+ . \exists y \in \mathbb{Z}^+ . G(y, x)$ and $\neg\exists x \in \mathbb{Z}^+ . \forall y \in \mathbb{Z}^+ . (\neg Eq(x, y) \rightarrow G(x, y))$, other solutions are possible.

- If a positive integer is not prime, it is composite.

  One possible solution is:

  $$\forall x \in \mathbb{Z}^+ . (\neg P(x) \rightarrow \exists y \in \mathbb{Z}^+ . \exists z \in \mathbb{Z}^+ . (G(y, 1) \wedge G(z, 1) \wedge T(y, z, x))$$

  other solutions are possible.

  Note. A positive number is composite if it equals the product of two positive integers both greater than 1.

Prove that $A \cap (B \cup A) = A$ using a containment proof. Explain your steps.

First we show $A \cap (B \cup A) \subseteq A$, therefore consider any $x \in A \cap (A \cup B)$, by definition of intersection we have:

$$\begin{aligned} x \in A \cap (A \cup B) \Rightarrow\ & x \in A \text{ and } x \in A \cup B \\ \Rightarrow\ & x \in A \end{aligned}$$

and hence, since $x \in A \cap (A \cup B)$ was arbitrary, we have $A \cap (B \cup A) \subseteq A$.

For the reverse direction, i.e. showing $A \subseteq A \cap (B \cup A)$, consider any $x \in A$, we have:

$$\begin{aligned} x \in A \Rightarrow\ & x \in A \text{ and } x \in A \\ \Rightarrow\ & x \in A \text{ and } x \in A \cup B && \text{by definition of union} \\ \Rightarrow\ & x \in A \cap (A \cup B) && \text{by definition of intersection} \end{aligned}$$

since $x \in A$ was arbitrary, we have $A \subseteq A \cap (B \cup A)$ completing the proof.

Prove $(A \backslash C) \cup (B \backslash C) = (A \cup B) \backslash C$ using set builder notation and logical equivalences. Explain your steps.

$$
\begin{aligned}
&(A - C) \cup (B - C) \;=\; \{x \mid x \in (A - C) \cup (B - C)\} && \\
&= \{x \mid (x \in A - C) \vee (x \in B - C)\} && \text{definition of union} \\
&= \{x \mid ((x \in A) \wedge (x \notin C)) \vee ((x \in B) \wedge (x \notin C))\} && \text{defn. of set difference} \\
&= \{x \mid ((x \notin C) \wedge (x \in A)) \vee ((x \notin C) \wedge (x \in B))\} && \text{commutative law} \\
&= \{x \mid (x \notin C) \wedge ((x \in A)) \vee (x \in B))\} && \text{distributive law} \\
&= \{x \mid ((x \in A)) \vee (x \in B)) \wedge (x \notin C)\} && \text{commutative law} \\
&= \{x \mid ((x \in A \cup B)) \wedge (x \notin C)\} && \text{defn. of union} \\
&= \{x \mid x \in (A \cup B) - C\} && \text{defn. of set difference} \\
&= (A \cup B) - C &&
\end{aligned}
$$