Mathematics 2A - December Exam 2019

i) $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$, $x \neq 0$

$$\frac{\partial f}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\,\frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{-y\cdot 2x}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-x\cdot 2y}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

Hence,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{2xy - 2xy}{(x^2+y^2)^2} = 0$$

2) Chain rule needed is

$$\frac{\partial f}{\partial x} = \frac{\partial v}{\partial x}\frac{\partial F}{\partial v} + \frac{\partial w}{\partial x}\frac{\partial F}{\partial w}$$

$$\frac{\partial f}{\partial y} = \frac{\partial v}{\partial y}\frac{\partial F}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial F}{\partial w}$$

since $v = xy$ and $w = \frac{x}{y}$ we got

$$\frac{\partial v}{\partial x} = y, \quad \frac{\partial v}{\partial y} = x, \quad \frac{\partial w}{\partial x} = \frac{1}{y}, \quad \frac{\partial w}{\partial y} = -\frac{x}{y^2}$$

Then

$$\frac{\partial f}{\partial x} = y\frac{\partial F}{\partial v} + \frac{1}{y}\frac{\partial F}{\partial w}, \quad \frac{\partial f}{\partial y} = x\frac{\partial F}{\partial v} - \frac{x}{y^2}\frac{\partial F}{\partial w}$$

Substituting into the PDE we get

$$x\left(\cancel{y\frac{\partial F}{\partial v}} + \frac{1}{y}\frac{\partial F}{\partial w}\right) - y\left(\cancel{x\frac{\partial F}{\partial v}} - \frac{x}{y^2}\frac{\partial F}{\partial w}\right) = 2x^2$$

$$\Rightarrow \cancel{2}\,\frac{\cancel{x}}{y}\frac{\partial F}{\partial w} = \cancel{2}x^{\cancel{2}} \quad \Rightarrow \frac{\partial F}{\partial w} = xy = v$$

Hence $F(v,w) = vw + A(v)$, A arbitrary function of 1 variable

$$\Rightarrow f(x,y) = F\left(xy, \frac{x}{y}\right) = x^2 + A(xy)$$

3) write $\underline{c} = (c_1, c_2, c_3)$

$$\underline{c}\cdot\underline{r} = c_1 x + c_2 y + c_3 z \Rightarrow \nabla(\underline{c}\cdot\underline{r}) = (c_1, c_2, c_3) = \underline{c}$$

$$\underline{c}\times\underline{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ c_1 & c_2 & c_3 \\ x & y & z \end{vmatrix} = \left(c_2 z - c_3 y,\ c_3 x - c_1 z,\ c_1 y - c_2 x\right)$$

$$\operatorname{curl}(\underline{c}\times\underline{r}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ c_2 z - c_3 y & c_3 x - c_1 z & c_1 y - c_2 x \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(c_1 y - c_2 x) - \frac{\partial}{\partial z}(c_3 x - c_1 z),\ \frac{\partial}{\partial z}(c_2 z - c_3 y) - \frac{\partial}{\partial x}(c_1 y - c_2 x),\right.$$
$$\left. \frac{\partial}{\partial x}(c_3 x - c_1 z) - \frac{\partial}{\partial y}(c_1 z - c_3 y) \right)$$

$$= \left(c_1 - (-c_1),\ c_2 - (-c_2),\ c_3 - (-c_3) \right) = (2c_1, 2c_2, 2c_3) = 2\underline{c}$$

We have

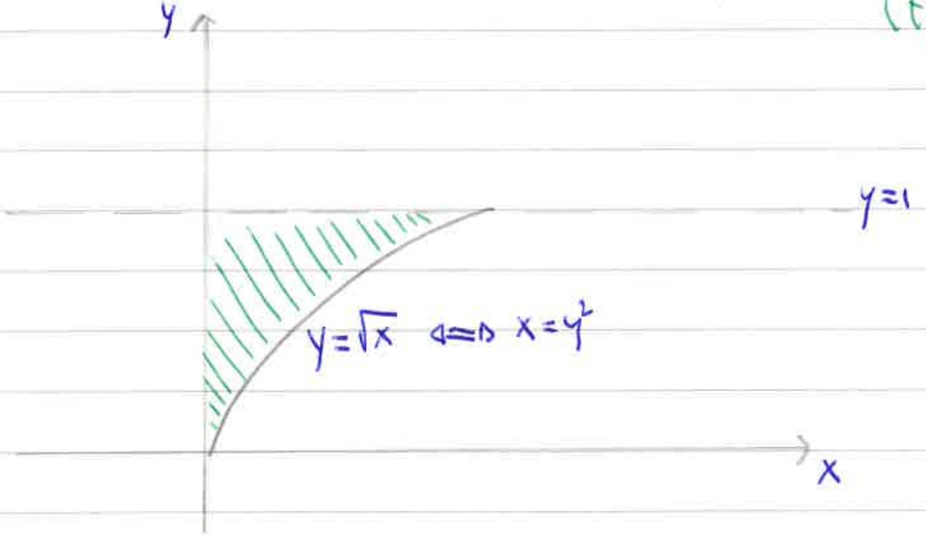
using the identity

$$\operatorname{div}\left((\underline{c}\times\underline{r}) \times \nabla(\underline{c}\cdot\underline{r})\right) = \operatorname{curl}(\underline{c}\times\underline{r})\cdot\nabla(\underline{c}\cdot\underline{r}) - (\underline{c}\times\underline{r})\cdot\operatorname{curl}(\nabla(\underline{c}\cdot\underline{r}))$$

$$= 2\underline{c}\cdot\underline{c} - (\underline{c}\times\underline{r})\cdot\underbrace{\cancel{\operatorname{curl}(\underline{c})}}_{=0} = 2\underline{c}\cdot\underline{c} = 2(c_1^2 + c_2^2 + c_3^2) \geq 0$$

4) the region of integration is $D = \{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 1,\ \sqrt{x}\le y\le 1\}$

(type I. domain)



As a type II-domain we have

$D = \{(x,y)\in\mathbb{R}^2 \mid 0\le x\le y^2,\ 0\le y\le 1\}$

Then, the integral becomes

$$\int_0^1 \left(\int_0^{y^2} e^{\frac{x}{y}}\, dx\right) dy = \int_0^1 \left[\, y\, e^{\frac{x}{y}} \right]_{x=0}^{x=y^2} dy$$

$$= \int_0^1 y\,(e^y-1)\, dy = \Big[\, y\,(e^y-y)\Big]_0^1 - \int_0^1 (e^y-y)\, dy$$

integration by parts

$$= e-1-\left[e^y-\frac{y^2}{2}\right]_0^1 = \cancel{e}-\cancel{1}-\cancel{e}+\frac{1}{2}\cancel{+1} = \frac{1}{2}$$

5) The text suggests to set

$$v = xy \qquad w = xy^2$$

Then the region D is described as

$$\{(v,w) \in \mathbb{R}^2 \mid 1 \le v \le e,\ 1 \le w \le 2\} = [1,e] \times [1,2]$$

We have $\dfrac{\partial(v,w)}{\partial(x,y)} = \det\begin{pmatrix} y & x \\ y^2 & 2xy \end{pmatrix} = 2xy^2 - xy^2 = xy^2$

$$\Rightarrow |J| = \frac{1}{|xy^2|} = \frac{1}{xy^2} \qquad (\text{since in the domain } x > 0)$$

Using the change of variable, formula for double integrals one gets

$$\iint_D xy^3\, dx\, dy = \iint_{\substack{[1,e] \times [1,2] \\ v \quad\ \ w}} xy^3 \cdot \frac{1}{xy^2}\, dv\, dw = \iint_{[1,e]\times[1,2]} y\, dv\, dw$$

$$= \iint_{[1,e]\times[1,2]} \frac{w}{v}\, dv\, dw = \left(\int_1^e \frac{dv}{v}\right)\left(\int_1^2 w\, dw\right)$$

$$= \Big[\log v\Big]_1^e \left[\frac{w^2}{2}\right]_1^2 = (1-0)\left(2 - \frac{1}{2}\right) = \frac{3}{2}$$

6) Parametrise the sphere as

$$r(\theta, \phi) = 2\left(\cos\theta \sin\phi,\ \sin\theta \sin\phi,\ \cos\phi\right) \qquad (*)$$

where $0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi$

The equation $z=0$ using $(*)$ becomes $\cos\phi = 0 \Rightarrow \phi = \frac{\pi}{2}$

(since $0 \leq \phi \leq \pi$)

so we get $0 \leq \phi \leq \frac{\pi}{2}$

The equation $x^2+y^2=1$ using $(*)$ becomes

$$\left(2\cos\theta \sin\phi\right)^2 + \left(2\sin\theta \sin\phi\right)^2 = 1$$

$$\parallel$$

$$4\left(\cos^2\theta + \sin^2\theta\right)\sin^2\phi = 4\sin^2\phi$$

$\Rightarrow \sin^2\phi = \frac{1}{4} \Rightarrow \sin\phi = \frac{1}{2}$ (since $0 \leq \phi \leq \frac{\pi}{2}$)

which has solution $\phi = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

So we get the further constraint $0 \leq \phi \leq \frac{\pi}{6}$.

Hence S is parametrized by $r(\theta, \phi)$ as in $(*)$

with $0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \frac{\pi}{6}$

Hence $$\iint_S y^2\, dS = \iint_{[0,2\pi]\times[0,\frac{\pi}{6}]} \left(2\sin\theta \sin\phi\right)^2 \underbrace{\left| r_\theta \times r_\phi \right|}_{\parallel \atop 4\sin\phi} d\theta\, d\phi$$

$$= 16 \iint_{[0,2\pi]\times[0,\frac{\pi}{6}]} \sin^2\theta \sin^3\phi \, d\theta \, d\phi$$

$$= 16 \left(\int_0^{2\pi} \sin^2\theta \, d\theta \right) \left(\int_0^{\frac{\pi}{6}} \sin^3\phi \, d\phi \right)$$

$$= 16 \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_0^{2\pi} \left[\frac{\cos^3\phi}{3} - \cos\phi \right]_0^{\pi/6}$$

$$= 16 \pi \left(\frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{2} - \frac{1}{3} + 1 \right)$$

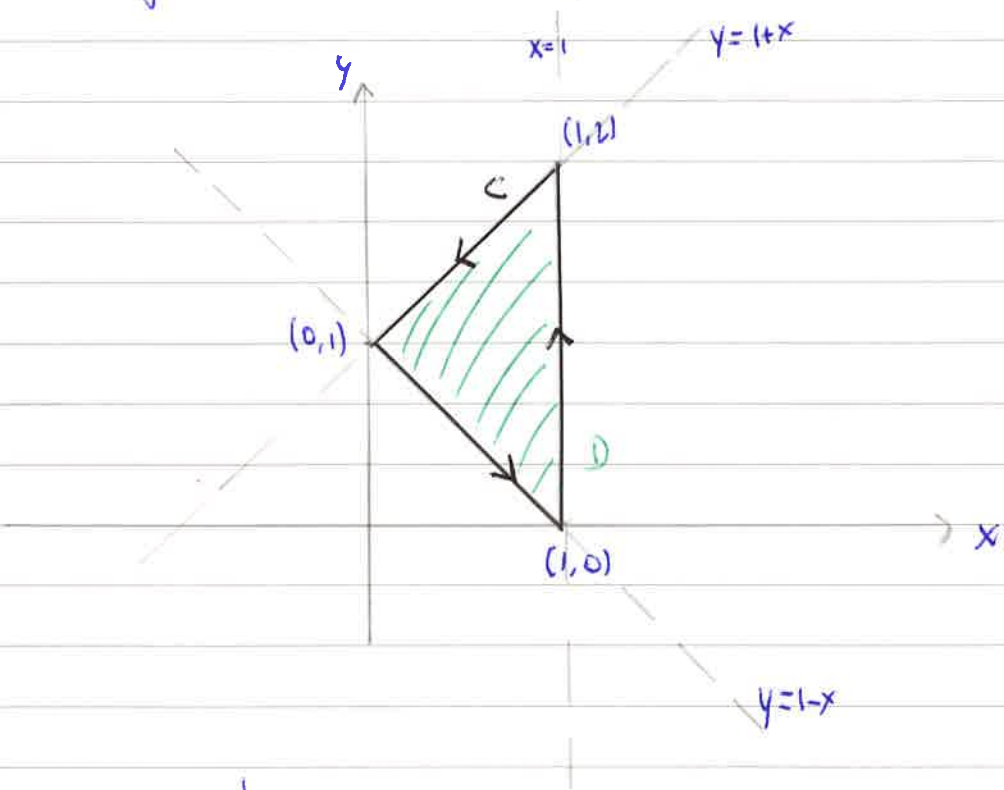
$$= \pi \left(\frac{32}{3} - 6\sqrt{3} \right)$$

7) Green Theorem : let C be a closed, simple positive oriented curve. Then

$$\oint_C P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where D is the region enclosed by C.

In this case



As a type I region we have

$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ 1-x \le y \le 1+x\}$$

Applying Green Theorem we get

$$\oint_C (2013 + 2x^2 y \sinh x^2)\,dx + (x\cosh(x^2) + \tan^{-1}(y))\,dy =$$

$$= \iint_D \left(\cosh(x^2) + \cancel{2x^2 \sinh(x^2)} - \cancel{2x^2 \sinh(x^2)} \right) dx\, dy$$

$$= \iint_D \cosh(x^2)\, dx\, dy = \int_0^1 \left(\int_{1-x}^{1+x} \cosh(x^2)\, dy \right) dx$$

$$= \int_0^1 \Big[\cosh(x^2)\, y \Big]_{y=1-x}^{y=1+x} dx = \int_0^1 2x \cosh(x^2)\, dx$$

$$= \Big[\sinh(x^2) \Big]_0^1 = \frac{e - e^{-1}}{2}$$

8) Divergence Theorem: let S be a closed orientable surface with outward pointing normal $\underline{n}$. Then

$$\iint_S \underline{F} \cdot \underline{n} \, dS = \iiint_V \operatorname{div} \underline{F} \, dx\,dy\,dz$$

where V is the region enclosed by S.

In our case $\operatorname{div} \underline{F} = \cancel{z \sin(yz)} + z - \cancel{z \sin(yz)} = z$

and V may be described as

$V = \{ (x,y,z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1, \ 0 \leq y \leq 1-x, \ 0 \leq z \leq 1-x-y \}$

By the divergence theorem we get

$$\iint_S \underline{F} \cdot \underline{n} \, dS = \iiint_V z \, dx\,dy\,dz = \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} z \, dz \right) dy \right) dx$$

$$= \frac{1}{2} \int_0^1 \left(\int_0^{1-x} (1-x-y)^2 \, dy \right) dx = \frac{1}{6} \int_0^1 (1-x)^3 \, dx = \frac{1}{24}$$