University of Glasgow

Friday, 14 December 2018
4.30 pm – 6.00 pm
(1 hour 30 minutes)

DEGREES OF MSci, MEng, BEng, BSc, MA and MA (Social Sciences)

ALGORITHMIC FOUNDATIONS 2: COMPSCI2003

Answer all questions

This examination paper is worth a total of 60 marks.

The use of calculators is not permitted in this examination.

INSTRUCTIONS TO INVIGILATORS: Please collect all exam question papers and exam answer scripts and retain for school to collect. Candidates must not remove exam question papers.

1. **(a)** Using the laws of logical equivalence show that

$$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$$

is a tautology. [5]

Solution: Using the implication law we have:

$$
\begin{array}{lr}
((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r) \equiv \neg((p \vee q) \wedge (\neg p \vee r)) \vee (q \vee r) & \\
\equiv (\neg(p \vee q) \vee \neg(\neg p \vee r)) \vee (q \vee r) & \text{de Morgan law} \\
\equiv ((\neg p \wedge \neg q) \vee (p \wedge \neg r)) \vee (q \vee r) & \text{de Morgan and double negation laws} \\
\equiv (q \vee (\neg p \wedge \neg q)) \vee (r \vee (p \wedge \neg r)) & \text{associative and commutative laws} \\
\equiv (((q \vee \neg p) \wedge (q \vee \neg q)) \vee ((r \vee p) \wedge (r \vee \neg r)) & \text{distributive law (twice)} \\
\equiv ((q \vee \neg p) \wedge \texttt{true}) \vee ((r \vee p) \wedge \texttt{true}) & \text{tautology law (twice)} \\
\equiv (q \vee \neg p) \vee (r \vee p) & \text{identity law (twice)} \\
\equiv (p \vee \neg p) \vee (q \vee r) & \text{associative and commutative laws} \\
\equiv \texttt{true} \vee (q \vee r) & \text{tautology law} \\
\equiv \texttt{true} & \text{domination law}
\end{array}
$$

(b) Define what it means for a function to be injective, surjective and bijective. [3]

Solution: A function is:

- injective if each element of the domain maps to a unique element of the codomain;
- surjective if every element of codomain has a preimage;
- bijective if it is both injective and surjective.

(c) Is the function $f : \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = x+2$ injective and/or surjective? (Justify your answer.) [2]

Solution: The function is injective as if $x \neq y$, then $x+2 \neq y+2$ for all $x, y \in \mathbb{R}$. The function is surjective, since for any $y \in \mathbb{R}$, we have $y-2 \in \mathbb{R}$ and $f(y-2) = y$.

2. Suppose we have the following predicates over the domain (universe of discourse) U of people:

- $P(x)$: x is perfect;
- $F(x)$: x is a friend of mine.

Express the following English statements in logical formulae using the predicates given above.

(a) No one is perfect. [2]

> **Solution:** $\forall x \in U . \neg P(x)$ or alternatively $\neg \exists x \in U . P(x)$.

(b) At least one of your friends is perfect. [2]

> **Solution:** $\exists x \in U . (F(x) \wedge P(x))$
> Note $\exists x \in U . (F(x) \rightarrow P(x))$ is incorrect as this can hold when you do not have any friends.

Express in concise (good) English without variables each of the following propositions:

(c) $\neg \forall x \in U . P(x)$ [2]

> **Solution:** Not everyone is perfect.
>
> This is only one solution there are a number of different correct solutions.

(d) $\forall x \in U . (P(x) \wedge F(x))$. [2]

> **Solution:** Everyone is your friend and is perfect.
>
> This is only one solution there are a number of different correct solutions.

(e) $(\neg \forall x \in U . F(x)) \vee (\exists x \in U . \neg P(x))$ [2]

> **Solution:** Either not everyone is your friend or someone is not perfect.
>
> This is only one solution there are a number of different correct solutions.

3. **(a)** For any real numbers $a, b \in \mathbb{R}$, prove by induction that we have "$(a-b)$ is a factor of $(a^n - b^n)$ for all $n \in \mathbb{Z}^+$".
Justify each step. [5]

Solution: Consider any real numbers $a, b \in \mathbb{R}$ and let $P(n)$ be the predicate: $a - b$ is a factor of $a^n - b^n$.
Base case: $P(1)$ trivially holds as $(a-b)$ is a factor of its self.
Inductive step: We now assume $P(n)$ is true for some $n \in \mathbb{Z}^+$. Considering $n+1$ we have

$$\begin{aligned} a^{n+1} - b^{n+1} &= a\cdot(a^n - b^n) + a\cdot b^n - b^{n+1} && \text{rearranging} \\ &= a\cdot(a^n - b^n) + b^n\cdot(a-b) && \text{rearranging} \end{aligned}$$

Since by induction the first term is a factor of $(a-b)$ and clearly the second term is a factor of $(a-b)$ it following the sum is also a factor of $(a-b)$.
Therefore by the principle of induction we have proved that $P(n)$ holds for all $n \in \mathbb{Z}^+$.

(b) Prove that the following statement is correct: "for any integer n, if $3\cdot n+2$ is odd, then $9\cdot n+5$ is even".
Justify each step. [4]

Solution: There are a number of different approaches to proving `odd`$(3\cdot n+2)$ implies `even`$(9\cdot n+5)$. We list some below.

Direct Proof 1. For $3\cdot n+2$ to be odd, $3\cdot n$ must be odd, because an odd number plus an even number is odd. For $3\cdot n$ to be odd n must be odd, because if n was even then $3\cdot n$ would also be even. Consequently we can express n as $2\cdot k+1$ for some integer k. Therefore we have:

$$9\cdot n+5 = 18\cdot k+9+5 = 2\cdot(9\cdot k+7)$$

this is even.

Direct Proof 2. Assume $3\cdot n+2 = 2\cdot k+1$, rearranging we have $3\cdot n = 2\cdot k-1$, and hence:

$$9\cdot n+5 = 3\cdot(2\cdot k-1)+5 = 6\cdot k+2 = 2\cdot(3\cdot k+1)$$

which is even.

Proof by contradiction. Let us assume for a contradiction that `odd`$(3\cdot n+2)$ and `odd`$(9\cdot n+5)$. Similarly to the derivation in the second direct proof, since $3\cdot n+2 = 2\cdot k+1$ we have $9\cdot n+5 = 2\cdot(3\cdot k+1)$ which contradicts our assumption.

Indirect Proof. To prove by an indirect proof we will show `odd`$(9\cdot n+5)$ implies `even`$(3\cdot n+2)$. Therefore suppose $9\cdot n+5 = 2k+1$ for some integer k. Now rearranging we have $9n = 2\cdot k-2 = 2\cdot(k+1)$. Therefore $9\cdot n$ is even, and since 9 is odd, it follows that n must be even. Consequently $n = 2{\cdot}l$ for some integer l, and hence $3{\cdot}n+2 = 6{\cdot}l+2 = 2{\cdot}(3{\cdot}l+1)$ which is even as required.

(c) What proof technique did you use in part (b) above? [1]

Solution: We have given above 4 possible answers for this question, 2 use direct proof one indirect, and one by contradiction.

4. (a) How many ways are there for a robot to travel in *xyz* space from the origin $(0,0,0)$ to the point $(3,2,4)$ by taking 9 steps, where each step corresponds to either:

- moving one unit in the positive x direction;
- moving one unit in the positive y direction;
- moving one unit in the positive z direction.

(Moving in the negative x, y or z direction is prohibited, so that no backtracking is allowed).

Explain your answer. [4]

Solution: The robot can make a sequence of moves such as *xxxyyzzzz* or *xyxzyzxzz*, i.e. a move of length $3+2+4=9$. There are 9! possible permutations of such a sequence. However, we are over-counting as the three x's are indistinguishable as are the two y's and the four z's. Therefore we have:

$$\begin{aligned}\frac{9!}{3!2!4!} &= \frac{9{\cdot}8{\cdot}7{\cdot}6{\cdot}5}{3!2!} \\ &= \frac{9{\cdot}8{\cdot}7{\cdot}6{\cdot}5}{6{\cdot}2} \\ &= \frac{9{\cdot}8{\cdot}7{\cdot}5}{2} \\ &= 9{\cdot}4{\cdot}7{\cdot}5 \\ &= 1260\end{aligned}$$

possible ways to travel.

No marks will be lost for returning the answer $9{\cdot}4{\cdot}7{\cdot}5$.

(b) Suppose that every telephone number in the world is assigned a number that contains:

- a country code which is either a single digit (x_1), two digits (x_1x_2) or three digits ($x_1x_2x_3$);
- followed by a 10-digit telephone number of the form $n_1x_4x_5 - n_2n_3n_4 - x_6x_7x_8x_9$

where $x_i \in \{0,1,2,3,\dots,9\}$ and $n_i \in \{2,3,4\}$.

How many different telephone numbers would be available worldwide under this numbering scheme?

Explain your answer. [4]

Solution: Using the sum and product rule there are:

$$10 + 10 \cdot 10 + 10 \cdot 10 \cdot 10 = 1110$$

country codes. Within a country, using the product rule there are

$$3 \cdot 10 \cdot 10 \cdot 3 \cdot 3 \cdot 3 \cdot 10 \cdot 10 \cdot 10 = 3^4 \cdot 10^6$$

unique numbers. Therefore combining these results with the product rule we have:

$$1110 \cdot 3^4 \cdot 10^6$$

unique numbers.

(c) How many students must be in a class to guarantee that at least 5 were born on the same day of the week?

Explain your answer. [2]

Solution: Using the pigeonhole problem, if n objects are placed into k containers, then there is at least one box containing $\lceil n/k \rceil$ objects. In this case there are $k = 7$ containers (days of the week) and we require $5 = \lceil n/7 \rceil$. Through simple calculations we find n must be at least 29.

5. Suppose there are two boxes of balls, the first box contains three white balls and three blue balls, while the second contains four white and two blue ball.

(a) What is the probability you randonly select a blue ball from the second box. [1]

> **Solution:** Let B_i be the event you select a blue ball from box i. Clearly we have $\mathbf{P}[B_1] = 2/6 = 1/3$.

(b) If you select a ball from both boxes at random, what is the probability you select two white balls. [2]

> **Solution:** Let W_i be the event you select a while ball from box i. Since the events W_1 and W_2 are independent we have:
>
> $$\mathbf{P}[W_1 \cap W_2] = \frac{1}{2} \cdot \frac{4}{6} = \frac{1}{3}$$

(c) If you select a ball from both boxes at random, what is the probability you select two of different colours. [3]

> **Solution:** Let D be the event the balls selected are different colours. We have:
>
> $$\mathbf{P}[D] = \mathbf{P}[W_1 \cap B_2] + \mathbf{P}[B_1 \cap W_2] = \frac{1}{2} \cdot \frac{2}{6} + \frac{1}{2} \cdot \frac{4}{6} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

(d) Suppose you first choose a box at random and then select a ball from that box at random. What is the probability that a ball from the first box was chosen, given you selected a blue ball. [4]

> **Solution:** Let A_i be the event choose the ith box and B a blue ball is chosen. We want to find $\mathbf{P}[A_1 \mid B]$. Clearly $\mathbf{P}[A_i] = 1/2$ for $1 \leq i \leq 2$, $\mathbf{P}[B \mid A_1] = 3/6$ and $\mathbf{P}[B \mid A_2] = 2/6$. Now since $\mathbf{P}[A_1] + \mathbf{P}[A_2] = 1$, using Bayes' law we have:
>
> $$\mathbf{P}[A_1 \mid B] = \frac{\mathbf{P}[B \mid A_1]\mathbf{P}[A_1]}{\mathbf{P}[B \mid A_1]\mathbf{P}[A_1] + \mathbf{P}[B \mid A_2]\mathbf{P}[A_2]} = \frac{1/2 \cdot 1/2}{1/2 \cdot 1/2 + 1/3 \cdot 1/2} = \frac{1/2}{5/6} = \frac{3}{5}.$$

6. Assume that we have a list l, and are given the functions:

- $\texttt{head}(l)$ which returns the first element of a non-empty list;
- $\texttt{tail}(l)$ which returns the tail of a non-empty list;
- $\texttt{isEmpty}(l)$ returns `true` if the list is empty and `false` otherwise.

For example if l equals $\langle 5,3,4,2,7,8,3,4\rangle$, then $\texttt{head}(l)$ would return 5, $\texttt{tail}(l)$ would return $\langle 3,4,2,7,8,3,4\rangle$, and $\texttt{isEmpty}(l)$ would return `false`.

Using the above functions, in a pseudo code of your choice:

(a) write a recursive function $\texttt{sum}(l)$, that returns the summation of the elements in a list. For example, $\texttt{sum}(\langle 1,5,2,3\rangle)$ returns $1+5+2+3=11$. [2]

Solution:

$$\texttt{sum}(l) = \textbf{if } \texttt{isEmpty}(l) \textbf{ then } 0 \textbf{ else } \texttt{head}(l) + \texttt{sum}(\texttt{tail}(l))$$

(b) write a recursive function $\texttt{present}(e,l)$, that returns `true` if e appears in the list l and `false` otherwise.
For example, $\texttt{present}(6, \langle 1,5,2,3\rangle)$ returns `false` and $\texttt{present}(4, \langle 1,2,3,1,2,4,2\rangle)$ returns `true`. [2]

Solution:

$$\begin{aligned}\texttt{present}(e,l) = \ & \textbf{if } \texttt{isEmpty}(l) \textbf{ then } \texttt{false} \textbf{ else} \\ & \qquad Equals(e, \texttt{head}(l)) \vee \texttt{present}(e, \texttt{tail}(l))\end{aligned}$$

where $Equals(x,y)$ is the predicate that returns `true` if and only if $x{=}y$.

For each relation below over all people, determine if the relation is symmetric, antisymmetric, and/or transitive.
Justify your answer.

(c) $(a,b) \in R_1$ if and only if a was born on the same day as b. [3]

Solution: For the relation R_1 we have:

- it is symmetric (if a was born on the same day as b, then b was certainly born on the same day as a;
- it is not anti-symmetric (there exists distinct people a and b that were born on the same day so we have $(a,b) \in R_1$ and $(b,a) \in R_1$);

- it is transitive (if a was born on the same day as b and b was born on the same day as c, then clearly a was born on the same day as c.

(d) $(a,b) \in R_2$ if a is (strictly) taller than b. [3]

Solution: For the relation R_2 we have:

- it is not symmetric (if a is taller than b, then b is not taller than a);
- it is anti-symmetric (if a is taller than b, then b is not taller than a);
- it is transitive (if a is taller than b and b is taller than c, then a is taller than c).

END OF QUESTION PAPER