Mathematics 1: Week 2
Matrix Algebra

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Session number: 570 (10am) 571 (11am) 572 (11am OF)

11. MULTIPLE CHOICE QUESTION

Reduce the following augmented matrix to reduced row echelon form and compute the value of the constant a

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 3 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 5 & a \\ 0 & 1 & -3 & 5 \end{array}\right].$$

$(A):\ a=7$, $(B):\ a=-7$, $(C):\ a=5$, $(D):$ None of the above.

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$(A):\ a = 7$, $(B):\ a = -7$, $(C):\ a = 5$, $(D):$ None of the above.

Solution: (B)

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 3 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 5 & -7 \\ 0 & 1 & -3 & 5 \end{array}\right].$$

Matrix Operations

In last week's lectures we introduced the concept of a matrix

Definition

A matrix is a rectangular array of numbers called the entries or elements of the matrix.

For a matrix A we write the i, j^{th} entry as a_{ij}. Symbolically this is expressed by $A = (a_{ij})$. So if A is an $m \times n$ matrix then

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

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$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Example

$A = \begin{bmatrix} 2 & 0 & 5 \\ 1 & 4 & -1 \end{bmatrix}$. Here $a_{11} = 2$, $a_{21} = 1$, $a_{12} = 0$ and $a_{2,3} = -1$.

Matrix Operations

If the columns of A are the column vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ in $\mathbb{R}^m$ we write

$$A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n],$$

and if the rows are row vectors $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_m$ in $\mathbb{R}^n$ we write

$$A = \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_m \end{bmatrix}.$$

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$$A = \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_m \end{bmatrix}.$$

A square matrix ($m = n$) is diagonal if and only if its off-diagonal elements are zero. We write $\mathbb{I}_n$ for the $n \times n$ identity matrix: the diagonal matrix with 1's down the diagonals and 0's elsewhere.

Matrix Operations

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Examples

$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 4 \\ 0 & 0 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ all square. B and C diagonal, and $C = \mathbb{I}_3$.

Matrix addition and scalar multiplication

Just like with vectors, matrix addition and scalar multiplication are performed entrywise.

Precisely:

If $A = (a_{ij})$ and $B = (b_{ij})$ are $m \times n$ matrices and c is a scalar, then $A + B$ and cA are $m \times n$ matrices given by

$$\begin{aligned} A + B &= (a_{ij} + b_{ij}), \\ cA &= c(a_{ij}) = (ca_{ij}). \end{aligned}$$

As with vectors, we write $-A$ for the matrix $(-1)A$, and write $A - B$ rather than $A + (-1)B$.

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12. MULTIPLE CHOICE QUESTION

Consider the matrices A, B and S

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & -1 & 2 \\ 5 & 4 & -2 \end{bmatrix}, \quad S = \begin{bmatrix} -2 & 3 & -2 \\ -5 & 0 & 4 \end{bmatrix}$$

where $S = aA + bB$. Compute a, b

(A) : $a = 2,\ b = 1,$ (B) : $a = 2, b = 2,$

(C) : $a = 1, b = 2,$ (D) : None of the above.

Class response

12. Multiple choice question

Consider the matrices A, B and S

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where $S = aA + bB$. Compute a, b

(A) : $a = 2$, $b = 1$, (B) : $a = 2, b = 2$,

(C) : $a = 1, b = 2$, (D) : None of the above.

Solution: (D). We compute $S = 2A - B$.

Matrix multiplication

Matrix multiplication:

If A is an $m \times n$ matrix and B is an $n \times r$ matrix then $C = AB$ is the $m \times r$ matrix with $(i, j)^{th}$ entry given by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Note that c_{ij} is the dot product of the i-th row of A with the j-th column of B.

Remember

We never multiply matrices componentwise.

Matrix-vector multiplication

Recall:

- A row vector in $\mathbb{R}^n$ is a $1 \times n$ matrix.
- A column vector in $\mathbb{R}^n$ is an $n \times 1$ matrix.

Rule for matrix multiplication applies to matrix-vector multiplication:

- If $\mathbf{a}$ is a row vector in $\mathbb{R}^n$ and B is an $n \times r$ matrix then we can compute $\mathbf{a}B$
- Similarly, if A is an $m \times n$ matrix and $\mathbf{b}$ is a column vector in $\mathbb{R}^n$ then we can compute $A\mathbf{b}$

Matrix-vector multiplication

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- If **a** is a row vector in $\mathbb{R}^n$ and B is an $n \times r$ matrix then we can compute $\mathbf{a}B$
- Similarly, if A is an $m \times n$ matrix and **b** is a column vector in $\mathbb{R}^n$ then we can compute $A\mathbf{b}$

For the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

we form the a_{ij} into a matrix A, the x_j into a column vector $\mathbf{x}$ and the b_j into a column vector $\mathbf{b}$; the system becomes $A\mathbf{x} = \mathbf{b}$.

Recall

Our definition of matrix-vector multiplication explains why we must use column vectors for **x** and **b**

Class response

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13. Multi-select question

Consider the matrices A, B and C and vectors $\mathbf{a}$ and $\mathbf{b}$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 2 & 1 \\ -1 & 3 & 9 \end{bmatrix},$$

$$\mathbf{a} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

Which of the following operations are admissible?

(A) : $A\mathbf{b}$, (B) : AB, (C) : BC, (D) : $\mathbf{a}C$, (E) : AC, (F) : $C\mathbf{a}$.

13. MULTI-SELECT QUESTION

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$$\mathbf{a} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

Which of the following operations are admissible?

$(A): A\mathbf{b}, \ (B): AB, \ (C): BC, \ (D): \mathbf{a}C, \ (E): AC, \ (F): C\mathbf{a}.$

Solution: (A) and (C) are admissible.

[EXAMPLE 2.1]

Matrix Powers

If A is an $n \times n$ matrix and k is a positive integer then

$$A^0 = \mathbb{I}_n, \qquad A^2 = AA, \qquad A^k = \underbrace{AA \cdots A}_{k \text{ times}}.$$

Example

For $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, we have,

$$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \; A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \text{ and } A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

The transpose of a matrix

Definition

The transpose of an $m \times n$ matrix A is the $n \times m$ matrix A^T obtained by interchanging the rows and columns of A. Entrywise: $(A^\mathsf{T})_{ij} = A_{ji}$.

The transpose of a matrix

Definition

The transpose of an $m \times n$ matrix A is the $n \times m$ matrix A^{T} obtained by interchanging the rows and columns of A.
Entrywise: $(A^{\mathsf{T}})_{ij} = A_{ji}$.

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14. Free response question

Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$. What is A^{T}_{12}?

The transpose of a matrix

Definition

The transpose of an $m \times n$ matrix A is the $n \times m$ matrix A^T obtained by interchanging the rows and columns of A.
Entrywise: $(A^\mathsf{T})_{ij} = A_{ji}$.

14. Free response question

Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$. What is A^T_{12}?

Solution: $A^\mathsf{T}_{12} = A_{21} = 1$.

Symmetric matrices

Definition

A square matrix is symmetric if $A^{\mathsf{T}} = A$. That is, A is symmetric if and only if it is equal to its own transpose. Entrywise $A = (a_{ij})$ is symmetric if and only if $a_{ij} = a_{ji}$ for all i and j.

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15. MULTIPLE CHOICE QUESTION

Which of $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 7 \\ 1 & 7 & 5 \end{bmatrix}$ are symmetric?

(A) Neither; (B) Matrix A only; (C) Matrix B only; (D) Both

Symmetric matrices

Definition

A square matrix is symmetric if $A^{\mathsf{T}} = A$. That is, A is symmetric if and only if it is equal to its own transpose. Entrywise $A = (a_{ij})$ is symmetric if and only if $a_{ij} = a_{ji}$ for all i and j.

15. Multiple choice question

Which of $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 7 \\ 1 & 7 & 5 \end{bmatrix}$ are symmetric?

(A) Neither; (B) Matrix A only; (C) Matrix B only; (D) Both

Solution: (C).

The addition and scalar multiplication rules for $m \times n$ matrices obey the following algebraic properties.

THEOREM 3.2

Let A, B and C be $m \times n$ matrices and c and d be scalars. Then

(A) $A + B = B + A$,

(B) $(A + B) + C = A + (B + C)$,

(C) $A + 0 = A$,

(D) $A + (-A) = 0$,

(E) $c(A + B) = cA + cB$,

(F) $(c + d)A = cA + dA$,

(G) $c(dA) = (cd)A$,

(H) $1A = A$.

[PROOF of (a)]

Properties of matrix multiplication

Matrix multiplication behaves differently from multiplication of numbers. In general multiplication is not commutative.
Also, we could have $A^2 = 0$ even if $A \neq 0$.

Theorem 3.3

Let A, B and C be matrices and k be a scalar. The following identities hold whenever the operations involved can be performed.

(A) $A(BC) = (AB)C$,

(B) $A(B + C) = AB + AC$,

(C) $(A + B)C = AC + BC$,

(D) $k(AB) = (kA)B = A(kB)$,

(E) $\mathbb{I}_m A = A = A\mathbb{I}_n$ if A is $m \times n$.

ALGEBRAIC RULES FOR TRANSPOSE

THEOREM 3.4

Let A and B be matrices. The following identities hold whenever the operations involved can be performed.

(A) $(A^{\mathsf{T}})^{\mathsf{T}} = A$,

(B) $(A+B)^{\mathsf{T}} = A^{\mathsf{T}} + B^{\mathsf{T}}$,

(C) $(kA)^{\mathsf{T}} = k(A^{\mathsf{T}})$,

(D) $(AB)^{\mathsf{T}} = B^{\mathsf{T}}A^{\mathsf{T}}$,

(E) $(A^m)^{\mathsf{T}} = (A^{\mathsf{T}})^m$ for all integers $m \geq 0$.

[PROOF of (d)]

THEOREM 3.5

(A) If A is a square matrix then $A + A^{\mathsf{T}}$ is a symmetric matrix,

(B) For any matrix A, AA^{T} and $A^{\mathsf{T}}A$ are symmetric matrices.

[PROOF]

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16. MULTIPLE CHOICE QUESTION

Consider the square matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ h & i & j \end{bmatrix}$. Which of the following is $\mathrm{Tr}(A)$?

(A) : aej; (B) : $a+e+j$; (C) : $\begin{bmatrix} a & d & h \\ b & e & i \\ c & f & j \end{bmatrix}$; (D) : None of the above

16. MULTIPLE CHOICE QUESTION

Consider the square matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ h & i & j \end{bmatrix}$. Which of the following is $\mathrm{Tr}(A)$?

(A) : aej; (B) : $a+e+j$; (C) : $\begin{bmatrix} a & d & h \\ b & e & i \\ c & f & j \end{bmatrix}$; (D) : None of the above

Solution: (B): $\mathrm{Tr}(A) = a + e + j$.

TRACE OF A MATRIX

The trace of a square matrix (denoted $\mathrm{Tr}(A)$ or $\mathrm{tr}(A)$) is the sum of the elements on the leading diagonal.

Definition

If A is an $n \times n$ matrix, an inverse of A is an $n \times n$ matrix A' such that

$$AA' = I_n, \qquad \text{and} \qquad A'A = I_n.$$

If A' exists we say A is invertible. If no inverse exists, then we say that A is not invertible.

Inverses are unique

Theorem 3.6

If an $n \times n$ matrix A is invertible then its inverse is unique.

[PROOF]

Notation

If A is invertible we write A^{-1} for its inverse.

Important Warning

Never write $\frac{1}{A}$ for the inverse of a matrix A. Matrices are not numbers, and we have not defined an operation of division, only addition, subtraction and multiplication.

THEOREM 3.7

If A is an invertible $n \times n$ matrix then the system of linear equations given by $A\mathbf{x} = \mathbf{b}$ has the unique solution given by $\mathbf{x} = A^{-1}\mathbf{b}$.

[PROOF]

Inverses and determinants

Theorem 3.8

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then A is invertible if $ad - bc \neq 0$, in which case

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$ then A is not invertible.

[PROOF]

Determinant

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we call $ad - bc$ the determinant of A, so that

$$\det(A) = ad - bc.$$

Theorem 3.8 says that A is invertible iff $\det(A) \neq 0$.

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17. MULTIPLE CHOICE QUESTION

Consider the matrices $A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix}$. Decide which of the following statements is true.

(A): A and B are both invertible;
(B): A is invertible but B is not;
(C): B is invertible but A is not;
(D): Neither A nor B are invertible.

17. Multiple choice question

Consider the matrices $A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix}$. Decide which of the following statements is true.
(A): A and B are both invertible;
(B): A is invertible but B is not;
(C): B is invertible but A is not;
(D): Neither A nor B are invertible.

Solution: (B). We have $\det(A) = -6$ and $\det(B) = 0$, so A is invertible but B is not.

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18. Multiple choice question

Reconsider the matrix $A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$. Select the inverse of A

$(A): \ A^{-1} = \dfrac{1}{-6}\begin{bmatrix} 1 & -5 \\ -2 & 4 \end{bmatrix};$ $(B): \ A^{-1} = \dfrac{1}{-6}\begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix};$

$(C): \ A^{-1} = \dfrac{1}{-6}\begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix};$ $(D):$ None of the above

18. Multiple choice question

Reconsider the matrix $A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$. Select the inverse of A

$(A):\ A^{-1} = \dfrac{1}{-6}\begin{bmatrix} 1 & -5 \\ -2 & 4 \end{bmatrix};\quad (B):\ A^{-1} = \dfrac{1}{-6}\begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix};$

$(C):\ A^{-1} = \dfrac{1}{-6}\begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix};\quad (D):$ None of the above

Solution (B). We have

$$A^{-1} = \frac{1}{-6}\begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix}.$$

EXAMPLE

Solve the system

$$x + 5y = 3, \qquad 2x + 4y = 1,$$

using the inverse of the coefficient matrix.

[EXAMPLE 2.2]

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19. MULTIPLE RESPONSE QUESTION

Consider the linear system 2x+y=1, 2x+y=0. Which of the following statements are true:

(A): The system has a unique solution;
(B): The system has infinitely many solutions;
(C): The system has no solutions;
(D): The system is consistent;
(E): The system is inconsistent;
(F): The coefficient matrix has zero determinant;
(G): The coefficient matrix has non-zero determinant;
(H): The coefficent matrix is invertible.

19. MULTIPLE RESPONSE QUESTION

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(E): The system is inconsistent;
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(G): The coefficient matrix has non-zero determinant;
(H): The coefficent matrix is invertible.

Solutions: (C), (E), (F). We saw in Lecture 1 that this system has no solutions (two parallel lines) and the coefficient matrix is <u>not</u> invertible.

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20. Multiple response question

Consider the linear system 2x+y=1, 4x+2y=2. Which of the following statements are true:

(A): The system has a unique solution;
(B): The system has infinitely many solutions;
(C): The system has no solutions;
(D): The system is consistent;
(E): The system is inconsistent;
(F): The coefficient matrix has zero determinant;
(G): The coefficient matrix has non-zero determinant;
(H): The coefficent matrix is invertible.

20. Multiple response question

Consider the linear system 2x+y=1, 4x+2y=2. Which of the following statements are true:
(A): The system has a unique solution;
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(C): The system has no solutions;
(D): The system is consistent;
(E): The system is inconsistent;
(F): The coefficient matrix has zero determinant;
(G): The coefficient matrix has non-zero determinant;
(H): The coefficent matrix is invertible.

Solutions: (B), (D), (F). We saw in Lecture 1 that this system has infinitely many solutions (the same line) but the coefficient matrix is <u>not</u> invertible.

Properties of invertibility

Theorem 3.9

(a) If A is an invertible matrix, then A^{-1} is invertible and $(A^{-1})^{-1} = A$.

(b) If A is an invertible matrix and $c \neq 0$ is a scalar then cA is invertible and $(cA)^{-1} = \frac{1}{c}A^{-1}$.

(c) If A and B are invertible matrices of the same size, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

(d) If A is an invertible matrix, then A^{T} is invertible and $(A^{\mathsf{T}})^{-1} = (A^{-1})^{\mathsf{T}}$.

(e) If A is invertible matrix then A^n is invertible for all integers $n \geq 0$ and $(A^n)^{-1} = (A^{-1})^n$.

Definition

If A is invertible and $n \geq 0$ an integer we define

$$A^{-n} = (A^{-1})^n = (A^n)^{-1}.$$