

a) Consider the function $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$T(x, y, z) = (2x + z, -y, 3z).$$

i) Show that T is a linear transformation

To be linear, T has to hold under addition of two vectors and scalar multiplication. For addition of two vectors $\langle x, y, z \rangle$ and $\langle x', y', z' \rangle$,

$$\begin{aligned} T(\langle x, y, z \rangle + \langle x', y', z' \rangle) &= T(\langle x + x', y + y', z + z' \rangle) \\ &= \langle 2(x + x') + z + z', -(y + y'), 3(z + z') \rangle \\ &= \langle 2x + z, -y, 3z \rangle + \langle 2x' + z', -y', 3z' \rangle \\ &= T(\langle x, y, z \rangle) + T(\langle x', y', z' \rangle). \end{aligned}$$

Then, for multiplication with a scalar $c \in \mathbb{R}$,

$$\begin{aligned} T(c\langle x, y, z \rangle) &= T(\langle cx, cy, cz \rangle) \\ &= \langle 2cx + cz, -cy, 3cz \rangle \\ &= c\langle 2x + z, -y, 3z \rangle \\ &= cT(\langle x, y, z \rangle). \end{aligned}$$

Thus, T is a linear transformation.

ii) Write down the standard matrix for the transformation T (denoted by $[T]$).

The standard matrix can be found by transforming the unit vectors in \mathbb{R}^3 by T :

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}.$$

Thus, the standard matrix for T is

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

iii) Is $(-2, 3, 6) \in \text{range}(T)$? If it is, find the vector (x, y, z) that is mapped to $(-2, 3, 6)$ by T .

To see if the vector is in the range of T , the following system of linear equations has to be consistent:

$$\begin{aligned} -2 &= 2x + z, \\ 3 &= -y, \\ 6 &= 3z, \end{aligned}$$

the solution to which is the vector $\langle -2, -3, 2 \rangle$. Since the system of linear equations has one solution, $\langle -2, 3, 6 \rangle$ is in the range of T . The vector $\langle -2, -3, 2 \rangle$ is also the vector mapped to $(-2, 3, 6)$ by T , as required.

b) Consider the following sets of vectors,

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\},$$

$$C = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\},$$

both of which form an ordered basis for \mathbb{R}^3 .

i) Find the coordinates of

$$\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$$

with respect to the ordered basis B.

To do this, the linear system of equations based on basis B

$$\begin{aligned} 3 &= x + z, \\ 2 &= y + z, \\ 3 &= x + y, \end{aligned}$$

where scalars $x, y, z \in \mathbb{R}$ are the multiples of the vectors in B. The solution to this system is the vector

$$[\mathbf{v}]_B = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

ii) Find the change of basis matrix $P_{C \leftarrow B}$.

Expressing each vector in B as a linear combination of vectors in C:

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

From the scalars in these equations, the change of basis matrix $P_{C \leftarrow B}$ can be found to be

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$

- iii) Using parts (i) and (ii), find the coordinates of \mathbf{v} with respect to the ordered basis C.

Matrix multiplication can be used to find

$$\begin{aligned} [\mathbf{v}]_C &= P_{C \leftarrow B} [\mathbf{v}]_B \\ &= \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}. \end{aligned}$$

- c) Let C and D be $n \times n$ matrices and $\mathbf{x} \in \mathbb{R}^n$. Suppose that $CD = DC$ and that $\det(D) \neq 0$. Show that, if $D\mathbf{x}$ is an eigenvector of C with corresponding eigenvalue $\lambda \in \mathbb{R}$, then \mathbf{x} is an eigenvector of C, and find the corresponding eigenvalue.

Since $\det(D) \neq 0$, D is an invertible matrix. Furthermore, the assumption can be written as

$$\begin{aligned} C(D\mathbf{x}) &= \lambda(D\mathbf{x}) \\ \Rightarrow CD\mathbf{x} &= \lambda D\mathbf{x}. \end{aligned}$$

Because $CD = DC$,

$$DC\mathbf{x} = \lambda D\mathbf{x}.$$

Since D is invertible, it can be multiplied on the left side to make

$$D^{-1}DC\mathbf{x} = D^{-1}\lambda D\mathbf{x}.$$

Then, since λ is a scalar, it can be rearranged to make

$$D^{-1}DC\mathbf{x} = \lambda D^{-1}D\mathbf{x}.$$

Since $D^{-1}D = I_n$,

$$C\mathbf{x} = \lambda\mathbf{x}.$$

Thus, \mathbf{x} is an eigenvector of C with corresponding eigenvalue λ , as required.