2C Intro to real analysis 2020/21

Ex Sheet 4

Deadline 15:00 on Friday 20 November 2020

Your solutions should be handed in electronically through the course Moodle page. Late submissions will not be accepted. Your work must be legible and be in a single PDF file **(only PDF files will be graded)**. A PDF scan of your work can be created by scanning your work on any photocopier in the university, for upload to Moodle. **We will not accept or grade photographed work.**

Q1 Let $f : \mathbb{R} \setminus \{\frac{3}{2}\} \to \mathbb{R}$ be given by $f(x) = \frac{x^2+x+2}{2x-3}$. Show, directly from the definition, that f is continuous at 2.

Q2 Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Show directly from the definition that f is continuous at 0.

Let $g : \mathbb{R} \to \mathbb{R}$ be given by

$$g(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Is g continuous at 0? Justify your answer with a proof.

Q3 Let $f : \mathbb{R} \to \mathbb{R}$ be a function which satisfies $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

a) By considering $x = y = 0$, find $f(0)$.

b) Show that for $f(q) = qf(1)$ for $q \in \mathbb{Q}$.[1]

c) Suppose additionally that f is continuous. Show that $f(x) = xf(1)$ for all $x \in \mathbb{R}$.[2]

[1] I'd suggest first establishing why $f(n) = nf(1)$ for $n \in \mathbb{N}$, then seeing why $f(\frac{n}{m}) = \frac{n}{m}f(1)$ for $n, m \in \mathbb{N}$ and finally dealing with the negative rational numbers.

[2] In this question you may assume that for any real number x, there is a sequence $(q_n)_{n=1}^{\infty}$ of rational numbers with $q_n \to x$ without proof. You can find the ideas used to prove this claim in exercise sheet 9.

Marking

Each question will be marked out of 3, for a total of 9 marks. In each case remember that you are being asked to 'show' or 'prove' something, so you must write your answer in a way that convinces the reader that the statement is true; not just some notes that you made in convincing yourself that it was true. In particular your answer should be in sentences, with all symbols used clearly introduced.

There will be a further 4 marks available across the entire assignment for the mathematical presentation of your work as a logical argument, using correct mathematical notation. For example, small failures to write in sentences, introduce variables and small notational errors will not be penalised in the marks for each question, provided it is clear what is intended, but will certainly contribute to a lower score in the mathematical presentation category. Substantial logical failings which leave your answer unclear, or flawed are likely to lead to lower marks for both the question and the overall mathematical presentation.