University of Glasgow

Wednesday, 13 December 2017
1.00 pm – 2.30 pm
(1 hour 30 minutes)

DEGREES OF MSci, MEng, BEng, BSc, MA and MA (Social Sciences)

ALGORITHMIC FOUNDATIONS 2: COMPSCI2003

Answer all questions

This examination paper is worth a total of 60 marks.

The use of calculators is not permitted in this examination.

INSTRUCTIONS TO INVIGILATORS: Please collect all exam question papers and exam answer scripts and retain for school to collect. Candidates must not remove exam question papers.

1. **(a)** The formula

$$\neg(p_1 \to \neg p_2) \to (p_3 \to (p_4 \to \neg p_5))$$

returns `false` for precisely one assignment to the propositions $p_1, \ldots, p_5$. Find this assignment using the laws of logical equivalence and *without* constructing a truth table for the formula. (Justify your answer.) [5]

Solution:

$$\begin{aligned}
&\neg(p_1 \to \neg p_2) \to (p_3 \to (p_4 \to \neg p_5)) && \\
&\quad\equiv \neg(\neg p_1 \vee \neg p_2) \to (p_3 \to (\neg p_4 \vee \neg p_5)) && \text{implication law}\\
&\quad\equiv \neg(\neg p_1 \vee \neg p_2) \to (\neg p_3 \vee (\neg p_4 \vee \neg p_5)) && \text{implication law}\\
&\quad\equiv \neg\neg(\neg p_1 \vee \neg p_2) \vee (\neg p_3 \vee (\neg p_4 \vee \neg p_5)) && \text{implication law}\\
&\quad\equiv (\neg p_1 \vee \neg p_2) \vee (\neg p_3 \vee (\neg p_4 \vee \neg p_5)) && \text{double negation law}\\
&\quad\equiv \neg p_1 \vee \neg p_2 \vee \neg p_3 \vee \neg p_4 \vee \neg p_5 &&
\end{aligned}$$

where the last step from the associative law for $\vee$. Considering this formula we have that the one assignment that returns `false` is when all the propositions $p_1, \ldots, p_5$ are `true`, i.e. all the formulae $\neg p_1, \ldots, \neg p_5$ are `false`.

(b) Define what it means for a function to be injective, surjective and bijective. [3]

Solution: A function is:

- injective if each element of the domain maps to a unique element of the codomain;
- surjective if every element of codomain has a preimage;
- bijective if it is both injective and surjective.

(c) Is the function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x + 2$ injective and/or surjective? (Justify your answer.) [2]

Solution: The function is injective as if $x \neq y$, then $x+2 \neq y+2$ for all $x, y \in \mathbb{R}$. The function is surjective, since for any $y \in \mathbb{R}$, we have $y-2 \in \mathbb{R}$ and $f(y-2) = y$.

2. Let $P(x, y)$ be the predicate *variable x has value y* (i.e. $x = y$), where the universe of discourse is the set of program variables V and set of integers $\mathbb{Z}$. Write each of the following statements using only the above predicate, quantifiers and logical operators:

(a) if program variable x has value 3, then program variable y has value 4; [2]

> **Solution:** $P(x,3) \rightarrow P(y,4)$

(b) every variable has a value between 4 and 6. [2]

> **Solution:** $\forall x \in V. (P(x,4) \vee P(x,5) \vee P(x,6))$

Let $Q(x,y)$ be the predicate $x^2 = y$. Express in English, and determine the truth value of, each of the following propositions:

(c) $Q(4,2)$ [2]

> **Solution:** Four squared equals two.
>
> The proposition is `false`, the square of 4 is 16.

(d) $\exists x \in \mathbb{Z}^+. \forall y \in \mathbb{Z}^+. Q(x,y)$ [2]

> **Solution:** There is a positive integer which when squared equals all other integers (i.e. the square root of all numbers equals the same integer).
>
> The proposition is `false`, for example the square root of 1 and 2 are different.

(e) $\exists x \in \mathbb{Z}^+. \forall y \in \mathbb{Z}^+. Q(y,x)$ [2]

> **Solution:** There is a positive integer which equals the square of all other integers (i.e. the square of all numbers equals the same integer).
>
> The proposition is `false`, for example the square of 1 and 2 are different.

3. **(a)** Use mathematical induction to show that the following holds for all $n \in \mathbb{Z}^+$ (positive integers):

$$\sum_{k=1}^{n} k^2 = \frac{n\cdot(n+1)\cdot(2\cdot n+1)}{6}$$

Justify each step. [5]

Solution: Let $P(n)$ be the predicate:

$$\sum_{k=1}^{n} k^2 = \frac{n\cdot(n+1)\cdot(2\cdot n+1)}{6}.$$

Base case: $P(1)$ holds since $1 = 6/6 = 1\cdot 2\cdot 3/6 = 1\cdot(1+1)\cdot(2+1)/6$.

Inductive step: We now assume $P(n)$ is true for some $n \in \mathbb{Z}^+$. Considering $n+1$ we have:

$$\begin{aligned}
\sum_{k=1}^{n+1} k^2 &= \left(\sum_{k=1}^{n} k^2\right) + (n+1)^2 \\
&= \frac{n\cdot(n+1)\cdot(2\cdot n+1)}{6} + (n+1)^2 && \text{by the inductive hypothesis} \\
&= \frac{n\cdot(n+1)\cdot(2\cdot n+1) + 6\cdot(n+1)^2}{6} && \text{rearranging} \\
&= \frac{(n+1)\cdot\big(n\cdot(2\cdot n+1) + 6\cdot(n+1)\big)}{6} && \text{rearranging} \\
&= \frac{(n+1)\cdot\big(2\cdot n^2 + n + 6\cdot n + 6\big)}{6} && \text{rearranging} \\
&= \frac{(n+1)\cdot\big(2\cdot n^2 + 7\cdot n + 6\big)}{6} && \text{rearranging} \\
&= \frac{(n+1)\cdot\big((n+2)\cdot(2\cdot n+3)\big)}{6} && \text{rearranging} \\
&= \frac{(n+1)\cdot((n+1)+1)\cdot(2\cdot(n+1)+1)}{6} && \text{rearranging}
\end{aligned}$$

and hence $P(n+1)$ holds.
Therefore by the principle of induction we have proved that $P(n)$ holds for all $n \in \mathbb{Z}^+$.

(b) Prove the triangular inequality, which states that if x and y are real numbers then $|x+y| \leq |x| + |y|$ (where $|z|$ is the absolute value of z). State what method of proof you used. [5]

Solution: We use a proof by cases. There are 4 cases to consider:

1. if x and y are greater than or equal to zero, then

   $$|x+y| = x+y = |x| + |y|$$

   and the result holds in this case.

2. if x is greater than zero and y is negative, then

$$|x+y| \leq \max\{|x|,|y|\} \leq |x|+|y|$$

and the result holds in this case.

3. if x is negative and y is greater than zero, then the result follows similarly to the case above.

4. if x and y are negative, then

$$|x+y| = (-x)+(-y) = |x|+|y|$$

and the result holds in this case.

Since these are all the cases to consider the result holds.

4. **(a)** A car number plate has two forms:

(i) three digits followed by three letters;

(ii) four letters followed by two digits.

How many car number plates are there (you can leave your answer in powers of 10 and 26). Explain your answer. [2]

Solution: Using the product rule there are $10{\cdot}10{\cdot}10{\cdot}26{\cdot}26{\cdot}26$ combinations of three digits followed by three letters and $26{\cdot}26{\cdot}26{\cdot}26{\cdot}10{\cdot}10$ combinations of four letters followed by two digits, since these are disjoint using the sum rule the total number of number plates equals:

$$10^3{\cdot}26^3 + 26^4{\cdot}10^2 = 10^2{\cdot}26^3{\cdot}(10+26) = 10^2{\cdot}26^3{\cdot}36\,.$$

(b) How may ways are there to choose 4 pieces of fruit from a bowl that contains 6 bananas, 5 apples, 7 pears, and 4 oranges? Explain your answer. [3]

Solution: We can solve this using the "stars and bars" approach or noticing the fact this is a case of combinations with repetitions.

Using the stars and bars approach, we have 4 regions separated by bars $-|-|-|-$ where each region corresponds to a type of fruit. The stars correspond to the quantity of the item chosen, i.e. $-|*|**|*$ would correspond to one item of class two, two items of class three, and one item of class four. There are 7! ways to permute the 4 stars and 3 bars. But the stars are indistinguishable as are the bars, therefore we over-count the stars by 4! and the bars by 3!. Therefore there are $7!/(3!{\cdot}4!)$ ways, i.e. 35 ways.

On the other hand, noticing that this is an r-combination from a set of n elements where n=r=4, we have the total number of ways equals:

$$C(n{+}r{-}1,r) = C(4{+}4{-}1,4) = C(7,4) = \frac{7!}{3!{\cdot}4!} = \frac{7{\cdot}6{\cdot}5}{3{\cdot}2{\cdot}1} = 7{\cdot}5 = 35$$

(c) How many sequences of 8 bits are there, that start with a 0 or end with 111 (you can leave you answer in powers of 2)? Explain your answer. [3]

Solution: There are 2^7 bit strings of the form $0*******$ and 2^5 bit strings of the form $*****111$. But both of these include the strings $0****111$, i.e. 2^4 strings. Therefore there are $2^7+2^5-2^4$ strings that begin with one 0's or end with three 1's.

For completeness we can compute the actual number as

$$2^7+2^5-2^4 = 2^4\cdot(2^3+2-1) = 16\cdot(8+1) = 16\cdot 9 = 144$$

(d) How many students must be in a class to guarantee that at least 5 were born on the same day of the week? Explain your answer. [2]

Solution: We have 7 containers, i.e. the days of the week, and we need to know the minimum number objects n, i.e. the students, such that there are 2 objects in one container. Using the pigeon hole principle we are looking for the smallest n such that $\lceil n/7 \rceil \geq 5$. Since $28/7 = 4$ it follows that we need at least 29 students.

5. Suppose you have one deck of playing cards consisting of 8 playing cards with 4 aces, 2 kings and 2 queens and a second deck of 8 playing cards which has 1 ace, 4 kings and 3 queens.

(a) What is the probability you randomly select an ace from the first deck. [1]

Solution: Let A_i be the event you select an ace from the ith deck. Clearly we have $\mathbf{P}[A_1] = 4/8 = 1/2$.

(b) Suppose you randomly select a card from both decks, what is the probability you select two queens. [2]

Solution: Let Q_i be the event you select an queen from the ith deck. Since the events Q_1 and Q_2 are independent we have:

$$\mathbf{P}[Q_1 \cap Q_2] = \mathbf{P}[Q_1]\cdot\mathbf{P}[Q_2] = \frac{2}{8}\cdot\frac{3}{8} = \frac{3}{32}$$

(c) Suppose you randomly select two cards from the first deck without replacement, what is the probability the second card you select is a king. [3]

Solution: Let K_i^j be the event you select an king from the ith deck with your jth selection.

$$\mathbf{P}[K_1^2] = \mathbf{P}[K_1^2 \cap K_1^1] + \mathbf{P}[K_1^2 \cap \neg K_1^1] = \frac{1}{7}\cdot\frac{2}{8} + \frac{2}{7}\cdot\frac{6}{8} = \frac{14}{56} = \frac{7}{28}$$

(d) Suppose you randomly choose a card from the first deck, add it to the second deck and then random choose a card from the second deck. If the card drawn from the second deck was an ace, what is the probability that the card selected from the first deck was also an ace? [4]

> **Solution:** Using Bayes' law we have:
>
> $$\begin{aligned}\mathbf{P}[A_1 \mid A_2] &= \frac{\mathbf{P}[A_2 \mid A_1]\mathbf{P}[A_1]}{\mathbf{P}[A_2 \mid A_1]\mathbf{P}[A_1] + \mathbf{P}[A_2 \mid \neg A_1]\mathbf{P}[\neg A_1]} \\ &= \frac{2/9{\cdot}4/8}{2/9{\cdot}4/8 + 1/9{\cdot}4/8} \\ &= \frac{2/9}{2/9+1/9} \\ &= \frac{2}{3}\end{aligned}$$

6. Define what it means for a graph $G = (V,E)$ to be:

(a) connected; [2]

> **Solution:** A graph is connected if every pair of vertices is joined by a path.

(b) a clique (complete). [2]

> **Solution:** A graph G is complete (a clique) if every pair of vertices is joined by an edge.

For each relation below, determine if the relation is reflexive, symmetric and/or antisymmetric, explain your answer.

(c) $R_1 = \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid x+y = 0\}$ [3]

> **Solution:** For the relation R_1 we have:
>
> - it is not reflexive since for example $(1,1) \notin R_1$;
> - it is symmetric since $x+y = y+x$ for all $x,y \in \mathbb{R}$;
> - it is not antisymmetric since $(1,-1),(-1,1) \in R_1$.

(d) $R_2 = \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid x{=}1\}$ [3]

Solution: For the relation R_2 we have:

- it is not reflexive since for example $(2,2) \notin R_2$;
- it is not symmetric since for example $(1,2) \in R_2$ and $(2,1) \notin R_2$;
- it is antisymmetric since for $(x,y),(y,x) \in R_2$, then $x{=}y{=}1$.