2A degree exam 2016–17, solutions

1. (i) With the property of g given, first perform a partial derivative with respect to λ, using the chain rule we have

$$x\frac{\partial g}{\partial x}(\lambda x, \lambda y) + y\frac{\partial g}{\partial y}(\lambda x, \lambda y) = n\lambda^{n-1}g(x, y)$$

notice that the derivatives of g on the left-hand side are evaluated at $(\lambda x, \lambda y)$ not at (x, y). Now perform the derivative with respect to x to obtain

$$\lambda\frac{\partial g}{\partial x}(\lambda x, \lambda y) = \lambda^n\frac{\partial g}{\partial x}(x, y)$$

and

$$\lambda\frac{\partial g}{\partial y}(\lambda x, \lambda y) = \lambda^n\frac{\partial g}{\partial y}(x, y)$$

then by substitution of the two expressions above into the first (the derivative with respect to λ) we obtain

$$x\lambda^{n-1}\frac{\partial g}{\partial x}(x, y) + y\lambda^{n-1}\frac{\partial g}{\partial y}(x, y) = n\lambda^{n-1}g(x, y)$$

cancellation of λ^{n-1} gives

$$x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = ng$$

(both sides evaluated at (x, y)) as required. An alternative solution is to evaluate the derivative with respect to λ at $\lambda = 1$ which gives the same result.

This question (taking different derivatives of expressions) is related to Ex 1.7 and Ex 1.9 in the lecture notes. Exercise sheet 2, T2, T3, F2 are also related (in that they give practice in application of the chain rule).

(ii) We write that $f(x, y)$ (the solution of the PDE) is given in terms of a new function F composed with the change of variables given, so

$$f(x, y) = F(u(x, y), v(x, y)).$$

First apply the chain rule

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial u}{\partial x}\frac{\partial F}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial F}{\partial v}\\ \frac{\partial f}{\partial y} &= \frac{\partial u}{\partial y}\frac{\partial F}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial F}{\partial v}\end{aligned}$$

in the case of the change of variables given in the question we have

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 1, \quad \frac{\partial v}{\partial x} = \frac{1}{x+y} - \frac{x}{(x+y)^2} = \frac{y}{(x+y)^2}, \quad \frac{\partial v}{\partial y} = -\frac{x}{(x+y)^2}.$$

Combine these derivatives with the chain rule to give

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial F}{\partial u} + \frac{y}{(x+y)^2}\frac{\partial F}{\partial v}\\ \frac{\partial f}{\partial y} &= \frac{\partial F}{\partial u} - \frac{x}{(x+y)^2}\frac{\partial F}{\partial v}.\end{aligned}$$

Now substitute in the PDE given

$$x\left(\frac{\partial F}{\partial u}+\frac{y}{(x+y)^2}\frac{\partial F}{\partial v}\right)+y\left(\frac{\partial F}{\partial u}-\frac{x}{(x+y)^2}\frac{\partial F}{\partial v}\right)=x\log(x+y)$$

there is cancellation in the $F_{,v}$ term and we are left with

$$\begin{aligned}(x+y)\frac{\partial F}{\partial u}&=x\log(x+y)\\ \frac{\partial F}{\partial u}&=\frac{x}{x+y}(x+y)\end{aligned}$$

use the change of variables to write the new PDE as

$$\frac{\partial F}{\partial u}=v\log u$$

which has general solution (after a partial integration with respect to u) of

$$F(u,v)=v\int\log u\,du+A(v)=v\,(u\log u-u)+A(v)$$

where integration by parts has been used and where A is an arbitrary function of one variable. Now the solution of the original PDE is

$$\begin{aligned}f(x,y)&=\frac{x}{x+y}(x+y)\left(\log(x+y)-1\right)+A\left(\frac{x}{x+y}\right)\\ &=x\left(\log(x+y)-1\right)+A\left(\frac{x}{x+y}\right).\end{aligned}$$

This question is similar to Ex 1.13 and 1.14 in the lecture notes. Exercise sheet 3, T1 and F1 are also relevant (further practice of solving PDEs by change of variable).

2. (i) Implicit differentiation with respect to x and y separately gives

$$2r\frac{\partial r}{\partial x}=2x+0,\qquad 2r\frac{\partial r}{\partial y}=0+2y$$

so

$$\frac{\partial r}{\partial x}=\frac{x}{r},\qquad \frac{\partial r}{\partial y}=\frac{y}{r}$$

as required. To calculate the gradient recall that

$$\nabla\phi=\left(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y}\right)$$

and using the chain rule we have

$$\nabla\phi=\left(\frac{\partial r}{\partial x}\Phi'(r),\frac{\partial r}{\partial y}\Phi'(r)\right)$$

applying the result from the implicit differentiation gives

$$\nabla\phi=\Phi'(r)\left(\frac{x}{r},\frac{y}{r}\right)=\frac{1}{r}\Phi'(r)\,(x,y)=\frac{1}{r}\mathbf{x}\Phi'(r)$$

as required.

This question is bookwork from Ex 3.2 in the lecture notes.

(ii) The nabla identity is

$$\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + (\nabla f) \cdot \mathbf{F}.$$

In order to calculate $\nabla^2\phi$ we recall that

$$\nabla^2\phi = \nabla \cdot (\nabla\phi)$$

and that we already have $\nabla\phi$ from part (i). Then we note that setting $f = \Phi'(r)/r$ and $\mathbf{F} = \mathbf{x}$ we can use the nabla identity. We need

$$\nabla \cdot \mathbf{x}, \qquad \nabla\left(\frac{1}{r}\Phi'(r)\right)$$

for the first

$$\nabla \cdot \mathbf{x} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 1 + 1 = 2$$

and for the second we note that $\Phi'(r)/r$ is just another function of r (exactly like $\Phi(r)$ in the first part) so we can apply the result from the first part to obtain

$$\nabla\left(\frac{\Phi'(r)}{r}\right) = \frac{1}{r}\mathbf{x}\left(\frac{\Phi'(r)}{r}\right)'.$$

Applying these results we see

$$\begin{aligned}
\nabla^2\phi &= \nabla \cdot (\nabla\phi) \\
&= \nabla \cdot \left(\frac{1}{r}\Phi'(r)\mathbf{x}\right) \\
&= \frac{2}{r}\Phi'(r) + \mathbf{x} \cdot \left(\frac{1}{r}\mathbf{x}\left(\frac{\Phi'(r)}{r}\right)'\right) \\
&= \frac{2}{r}\Phi'(r) + r\left(\frac{\Phi'(r)}{r}\right)' \\
&= \frac{2}{r}\Phi'(r) - \frac{1}{r}\Phi'(r) + \Phi''(r) \\
&= \frac{1}{r}\left(r\Phi'(r)\right)'
\end{aligned}$$

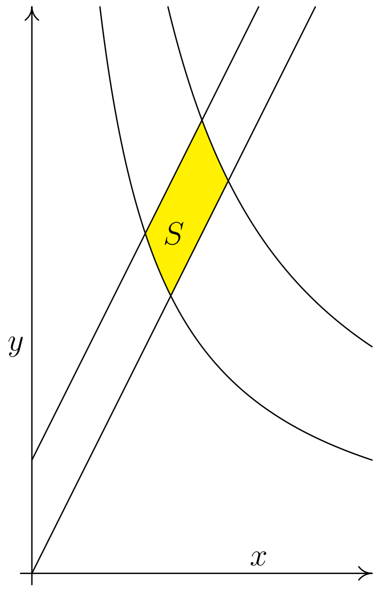
as required.

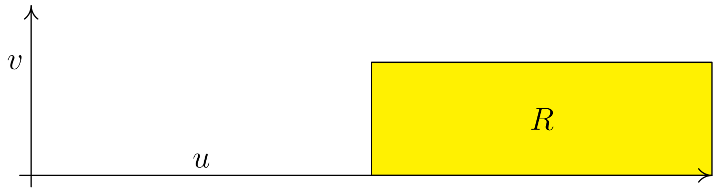
This involves recall of definitions and identities (nabla identity and definition of Laplacian) and is similar to Exercise sheet 8, T1, T2 and Exercise sheet 2, F2.

3. (i) In this part recall that we are trying to find a change of variables that makes the region rectangular. So we are looking for two functions $u(x,y)$ and $v(x,y)$ such that the boundaries can be described as $u(x,y) = a$, $u(x,y) = b$, $v(x,y) = c$ and $v(x,y) = d$. A simple rearrangement of the curves defining the four parts of the boundary of the region gives the following candidate change of variables

$$u(x,y) = xy, \qquad v(x,y) = y - 2x$$

then the boundaries of the region R are $u = 3$, $u = 6$, $v = 0$ and $v = 1$. The regions are sketched below.





This question is about identifying changes of variables for double integrals. It is similar to parts of to Ex 2.9 and 2.10 in the lecture notes. Exercise sheet 5 T2, T3, T4, F1, F2, F3 and F4 are all relevant.

(ii) In order to compute the double integral using the change of variables we must calculate the *Jacobian* of the transformation. As we have found $u(x, y)$ and $v(x, y)$ in part (i) we use that

$$J = \left(\frac{\partial\,(u, v)}{\partial\,(x, y)}\right)^{-1}$$

and we calculate the relevant partial derivatives

$$\frac{\partial u}{\partial x} = y, \quad \frac{\partial u}{\partial y} = x, \quad \frac{\partial v}{\partial x} = -2, \quad \frac{\partial v}{\partial y} = 1$$

and so

$$J = (y \cdot 1 - (-2) \cdot x)^{-1} = \frac{1}{y + 2x}$$

this is always positive in the region S so $|J| = J$ for our calculation. Then

$$\iint_S (y + 2x)^3 \; dxdy = \iint_R (y + 2x)^3 \, \frac{1}{(y + 2x)} \, dudv$$

where the Jacobian has been used. We must now express the integrand in terms of u and v. Note that

$$(y + 2x)^3 \, \frac{1}{(y + 2x)} = (y + 2x)^2 = (y - 2x)^2 + 8xy$$

using the hint with $a = y$ and $b = 2x$. Now we see

$$(y + 2x)^2 = (y - 2x)^2 + 8xy = v^2 + 8u$$

and our integral becomes

$$\begin{aligned}\iint_R v^2 + 8u \, dudv &= \int_0^1 \int_3^6 v^2 + 8u \, dudv \\ &= \int_0^1 \left[v^2 u + 4u^2\right]_3^6 \, dv \\ &= \int_0^1 3v^2 + 108 \, dv = \left[v^3 + 108v\right]_0^1 = 109.\end{aligned}$$

This question is about performing double integrals using a change of variable. It is similar to Ex 2.9 and 2.10 in the lecture notes. Exercise sheet 5 T2, T3, T4, F1, F2, F3 and F4 are all relevant. The level of difficulty is slightly lower than the equivalent question in the 2015–16 paper.

(iii) There are different ways to proceed. One way is as follows, first notice that the order of integration is y, then z, then x. Notice that the limits in z are constants (don't depend on x) and so we can reorder the integration to first perform the z integral then y and x (preserving the relative order of y and x. As the integrand is a function of z times a function of x and y we see that

$$\begin{aligned}\int_0^4 dx \int_1^2 dz \int_{\sqrt{x}}^2 \frac{xz^2}{1+y^5}\,dy &= \left(\int_1^2 z^2\,dz\right)\int_0^4 dx \int_{\sqrt{x}}^2 \frac{x}{1+y^5}\,dy \\ &= \left[\frac{1}{3}z^3\right]_1^2 \int_0^4 dx \int_{\sqrt{x}}^2 \frac{x}{1+y^5}\,dy \\ &= \frac{7}{3}\int_0^4 dx \int_{\sqrt{x}}^2 \frac{x}{1+y^5}\,dy\end{aligned}$$

We now have a double integral for which the integral is difficult to perform. So we consider changing the order of integration. The double integral is expressed as a Type-I integral — we will convert it to Type-II. A sketch might help, but in this case it is straightforward to see that as a Type-II region we have $0 \le x \le y^2$ and $0 \le y \le 2$, so

$$\begin{aligned}\int_0^4 dx \int_{\sqrt{x}}^2 \frac{x}{1+y^5}\,dy &= \int_0^2 dy \int_0^{y^2} \frac{x}{1+y^5}\,dx \\ &= \int_0^2 \left[\frac{1}{2(1+y^5)}x^2\right]_0^{y^2} dy \\ &= \frac{1}{2}\int_0^2 \frac{y^4}{1+y^5}\,dy \\ &= \frac{1}{2}\left[\frac{1}{5}\log\left(1+y^5\right)\right]_0^2 \\ &= \frac{1}{10}\log 33\end{aligned}$$

So altogether the triple integral has value

$$\frac{7}{30}\log 33.$$

Another approach is not to use the separability of the integrand but simply reorder the integration to be x then y then z. This can be seen by sketching the three-dimensional region. *For sketching the region see Exercise sheet 6 T2. For changing the order of integrand see Ex 2.5 in the lecture notes and Exercise sheet 4 T1, F1.*

4. (i) Green's theorem states that for a simple, closed and positively oriented plane curve C that encloses a region A we have

$$\int_C P dx + Q dy = \iint_A \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dxdy.$$

The curve in this question satisfies the conditions for Green's theorem and we set $P = -y^2$ and $Q = x^2$ to see that

$$\int_C x^2 dy - y^2 dx = \iint_A 2x - (-2y)\,dxdy = 2\iint_A x + y\,dxdy$$

The region A has boundaries $\theta = 0$ and $0 \le r \le \sqrt{2}$ (the part of the straight line from $(0,0)$ to $(\sqrt{2},0)$) then $r = \sqrt{2}$ for $\theta = 0$ to $\theta = \pi/4$, then $\theta = \pi/4$ and $0 \le r \le \sqrt{2}$ for the straight line from $(1,1)$ to $(0,0)$ (the slope of this line is 1 and therefore $\tan\theta = 1$ and so $\theta = \pi/4$ on this line). In polar coordinates the region A is $0 \le r \le \sqrt{2}$ and $0 \le \theta \le \pi/4$. We now use polar coordinates to transform the integral

$$2\iint_A x + y\, dxdy = 2\int_0^{\pi/4}\int_0^{\sqrt{2}} (r\cos\theta + r\sin\theta)\, r\, drd\theta$$

where we have used that the Jacobian is r and that $x = r\cos\theta$ and $y = r\sin\theta$. Perform the r integral first (the region is rectangular in the r-θ plane and so it doesn't matter the order of integration)

$$\begin{aligned}
2\int_0^{\pi/4}\int_0^{\sqrt{2}} (r\cos\theta + r\sin\theta)\, r\, drd\theta &= 2\int_0^{\pi/4} (\cos\theta + \sin\theta)\left[\frac{1}{3}r^3\right]_0^{\sqrt{2}} d\theta \\
&= \frac{4\sqrt{2}}{3}\int_0^{\pi/4} \cos\theta + \sin\theta\, d\theta \\
&= \frac{4\sqrt{2}}{3}\left[\sin\theta - \cos\theta\right]_0^{\pi/4} \\
&= \frac{4\sqrt{2}}{3}\left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - 0 - (-1)\right] = \frac{4\sqrt{2}}{3}
\end{aligned}$$

This question is similar to Ex 4.4 and 4.5 in the lecture notes (for Green's Thm). Exercise sheet 9 T2, F3, F4 (for Green's Thm). For the polar coordinates part the calculation is very similar to Ex. 2.7 in the lecture notes, Exercise sheet 4 T2, T3, T4(b) (in particular) and F6 are relevant for the polar coordinate calculation.

(ii) We recognise that the surface is the graph of a function and the projection is the region D that has $x^2 + y^2 \le 1$. In order to calculate the surface integral we need

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

in order to change the surfae integral into a double integral. We calculate the partial derivatives

$$\frac{\partial z}{\partial x} = 3x^2 - 3y^2, \qquad \frac{\partial z}{\partial y} = -6xy$$

so

$$\begin{aligned}
\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} &= \sqrt{1 + (3x^2 - 3y^2)^2 + (-6xy)^2} \\
&= \sqrt{1 + 9x^4 + 9y^4 - 18x^2y^2 + 36x^2y^2} \\
&= \sqrt{1 + 9\left(x^2 + y^2\right)^2}
\end{aligned}$$

Then the integral becomes

$$\iint_S x^2 + y^2\, dS = \iint_D \left(x^2 + y^2\right)\sqrt{1 + 9\left(x^2 + y^2\right)^2}\, dxdy$$

the projection and the integrand suggest using polar coordinates as D is $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$ in polar coordinates and $x^2 + y^2 = r^2$, so we obtain

$$\begin{aligned}
\int_0^{2\pi} \int_0^1 r^2 \sqrt{1 + 9r^4} r \, dr d\theta &= \int_0^{2\pi} \int_0^1 r^3 \sqrt{1 + 9r^4} \, dr d\theta \\
&= 2\pi \left[\frac{1}{54} \left(1 + 9r^4\right)^{3/2} \right]_0^1 \\
&= \frac{\pi}{27} \left(10^{3/2} - 1\right)
\end{aligned}$$

This question uses the results on surface integrals and then polar coordinates. For surface integration see Ex 4.7 and 4.8 in the lecture notes and Exercise sheet 10 T1, T2, T3, F1, F2, F3, F4. For the polar coordinates see Exercise sheet 4 T2, T3, T4.

(iii) The divergence theorem says that for a bounded volume V with surface S we have

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_V \nabla \cdot \mathbf{F} \, dV.$$

In our case we have

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(3z) = -1 - 1 + 3 = 1.$$

We now consider the best way to describe the volume. The intersection of a cone and a sphere is best described via spherical polar coordinates in this case. We have that $0 \leq r \leq 1$ as we are inside the sphere radius 1 and $0 \leq \theta \leq 2\pi$ as the cone is symmetric about its axis. The boundary of the cone is given by

$$r \cos\phi = \sqrt{3\left(r^2 \sin^2\phi \sin^2\theta + r^2 \sin^2\phi \cos^2\theta\right)} = \sqrt{3} r \sin\phi$$

so $\tan\phi = 1/\sqrt{3}$ so $\phi = \pi/6$. Therefore (as the divergence is 1) the integral is

$$\int_0^{2\pi} \int_0^{\pi/6} \int_0^1 r^2 \sin\phi \, dr d\phi d\theta = \frac{2\pi}{3} \left[-\cos\phi\right]_0^{\pi/6} = \frac{\pi}{3}\left(2 - \sqrt{3}\right).$$

An alternative is to describe the region as

$$\sqrt{3(x^2 + y^2)} \leq z \leq \sqrt{1 - x^2 - y^2}$$

for $(x, y) \in D$ where D is given by $4(x^2 + y^2) \leq 1$ (circle centre origin radius 1/2), followed by a use of polar coordinates for the x-y integral.

For examples involving the divergence theorem see Ex 4.9 and 4.10 in the lecture notes. See also Exercise sheet 10 T4, T5, T6 and F8. For spherical polar examples see Ex 2.14 (paricularly for the cone) and Exercise sheet 6 T4 and F6.