2B Linear Algebra

Ex Sheet 8

1 True/False

a) For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with $\mathbf{x} \neq 0$, we have that $\frac{|\mathbf{x} \cdot \mathbf{y}|}{\|\mathbf{x}\|} \leq \|\mathbf{y}\|$.

b) For any vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and any scalar d, we have $\mathbf{x} \cdot (d\mathbf{y}) = (d\mathbf{x}) \cdot \mathbf{y}$.

c) If θ is the angle between vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, and $\theta > \frac{\pi}{2}$, then $\mathbf{x} \cdot \mathbf{y} > 0$.

d) There exists three non-zero mutually orthogonal vectors in $\mathbb{R}^2$.

e) The standard unit vectors in $\mathbb{R}^n$ are mutually orthogonal.

f) If $\mathbf{v}$ is orthogonal to the vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ in $\mathbb{R}^m$, then $\mathbf{v}$ is orthogonal to $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

g) Starting from any five given vectors in $\mathbb{R}^5$, the Gram-Schmidt process can produce five orthonormal vectors.

h) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a set of non-zero mutually orthogonal vectors in $\mathbb{R}^m$, the Gram-Schmidt process would just produce the same set of vectors.

i) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a set of orthonormal vectors in $\mathbb{R}^m$, the Gram-Schmidt process would just produce the same set of vectors.

j) Let Q be an orthogonal $n \times n$ matrix, and $\mathbf{x}$ and $\mathbf{y}$ vectors in $\mathbb{R}^n$. Then $Q\mathbf{x} \cdot Q\mathbf{y} \leq \|\mathbf{x}\| \|\mathbf{y}\|$.

k) Let Q_1, Q_2, Q_3, Q_4 be orthogonal $n \times n$ matrices, all with the same determinant. Then the determinant of $Q_1 Q_2 Q_3 Q_4$ is 1.

l) The sum of two orthogonal $n \times n$ matrices is an orthogonal $n \times n$ matrix.

m) The subset of all orthogonal matrices is a subspace of the space of all matrices.

[1] **True/False Questions**
Every Exercise Sheet will have a section containing true/false questions. They are designed to test your understanding from lectures. You should look at your lecture notes and/or the textbook to help you answer these questions, but you should not need to write anything to work out the solution.

Solutions to True/False

a) T b) T c) F d) F e) T f) T g) F h) F i) T j) T k) T l) F m)F

Tutorial Exercises

T1 Suppose that $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors such that

$$\mathbf{u} \cdot \mathbf{v} = 2, \ \mathbf{v} \cdot \mathbf{w} = -3, \ \mathbf{u} \cdot \mathbf{w} = 5,$$

$$\|\mathbf{u}\| = 1, \ \|\mathbf{v}\| = 2, \ \|\mathbf{w}\| = 7.$$

Evaluate

a) $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{v}+\mathbf{w})$

b) $(2\mathbf{v}-\mathbf{w})\cdot(3\mathbf{u}+2\mathbf{w})$

c) $(\mathbf{u}-\mathbf{v}-2\mathbf{w})\cdot(4\mathbf{u}+\mathbf{v})$

d) $\|\mathbf{u}+\mathbf{v}\|$

e) $\|2\mathbf{w}-\mathbf{v}\|$

f) $\|\mathbf{u}-2\mathbf{v}+4\mathbf{w}\|$

Solution

a) Expanding, we get $\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}+\|\mathbf{v}\|^2+\mathbf{v}\cdot\mathbf{w}=2+5+4-3=8$.

b) Expanding, we get $6\mathbf{v}\cdot\mathbf{u}+4\mathbf{v}\cdot\mathbf{w}-3\mathbf{w}\cdot\mathbf{u}-2\|\mathbf{w}\|^2=12-12-15-98=-113$.

c) Expanding, and gathering like terms, we get $4\|\mathbf{u}\|^2-3\mathbf{u}\cdot\mathbf{v}-\|\mathbf{v}\|^2-8\mathbf{w}\cdot\mathbf{u}-2\mathbf{w}\cdot\mathbf{v}=4-6-4-40+6=-40$.

d) $\|\mathbf{u}+\mathbf{v}\|^2=(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})=\|\mathbf{u}\|^2+2\mathbf{u}\cdot\mathbf{v}+\|v\|^2=1+4+4=9$. Hence $\|\mathbf{u}+\mathbf{v}\|=3$.

e) $\|2\mathbf{w}-\mathbf{v}\|^2=(2\mathbf{w}-\mathbf{v})\cdot(2\mathbf{w}-\mathbf{v})=4\|\mathbf{w}\|^2-4\mathbf{w}\cdot\mathbf{v}+\|\mathbf{v}\|^2=196+12+4=212$. Hence $\|2\mathbf{w}-\mathbf{v}\|=\sqrt{212}$.

f) $\|\mathbf{u}-2\mathbf{v}+4\mathbf{w}\|^2=\|\mathbf{u}\|^2-4\mathbf{u}\cdot\mathbf{v}+8\mathbf{u}\cdot\mathbf{w}+4\|\mathbf{v}\|^2-16\mathbf{v}\cdot\mathbf{w}+16\|\mathbf{w}\|^2=1-8+40+16+48+784=881$. Hence $\|u-2v+4w\|=\sqrt{881}$.

T2 Which of the following sets of vectors are orthogonal? Which are orthonormal?

a) $\left\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}2\\-1\end{pmatrix}\right\}$

b) $\left\{\begin{pmatrix}1\\3\end{pmatrix},\begin{pmatrix}4\\5\end{pmatrix}\right\}$

c) $\left\{\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix},\begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix}\right\}$

Solution

a) Orthogonal

b) None

c) Orthonormal

T3 Let $\mathbf{x}$ and $\mathbf{y}$ be non-zero vectors in $\mathbb{R}^n$.

a) Prove that $\mathbf{y}=c\mathbf{x}+\mathbf{w}$ for some scalar c and some vector $\mathbf{w}$ orthogonal to $\mathbf{x}$.

b) Show that the scalar c and vector $\mathbf{w}$ in part (a) are unique. I.e. show that if $\mathbf{y} = d\mathbf{x} + \mathbf{v}$ where d is a scalar and $\mathbf{v}$ a vector orthogonal to $\mathbf{x}$ then $d = c$ and $\mathbf{v} = \mathbf{w}$. (Hint: compute $\mathbf{x} \cdot \mathbf{y}$).

Solution

a) We know that once c is chosen $\mathbf{w}$ is forced to be $\mathbf{y} - c\mathbf{x}$. Hence we would also want $\mathbf{w} \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{y} - c\|\mathbf{x}\|^2 = 0$. Set $c = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|^2}$, and $\mathbf{w} = \mathbf{y} - c\mathbf{x}$.

b) Assuming we can also write $\mathbf{y} = d\mathbf{x} + \mathbf{v}$, we have that $\mathbf{x} \cdot \mathbf{y} = c\|\mathbf{x}\|^2$ but also $\mathbf{x} \cdot y = d\|\mathbf{x}\|^2$. Hence $c = d$. This also means $\mathbf{v} = \mathbf{y} - d\mathbf{x} = \mathbf{y} - c\mathbf{x} = \mathbf{w}$.

T4 Let U be the subspace of $\mathbb{R}^4$ with the basis

$$(1, 0, 1, 0), \quad (0, 1, -1, 0), \quad (0, 0, 1, 1).$$

Use the Gram-Schmidt process to find an orthonormal basis for U.

Solution

Let

$$\mathbf{y}_1 = (1, 0, 1, 0), \quad \mathbf{y}_2 = (0, 1, -1, 0), \quad \mathbf{y}_3 = (0, 0, 1, 1).$$

Put $\mathbf{x}_1 = \mathbf{y}_1 = (1, 0, 1, 0)$ to get started. Next put

$$\mathbf{x}_2 = \mathbf{y}_2 - \frac{\mathbf{x}_1 \cdot \mathbf{y}_2}{\mathbf{x}_1 \cdot \mathbf{x}_1}\mathbf{x}_1.$$

Then

$$\mathbf{x}_2 = (0, 1, -1, 0) - \frac{-1}{2}(1, 0, 1, 0) = \frac{1}{2}(1, 2, -1, 0).$$

Next put

$$\mathbf{x}_3 = \mathbf{y}_3 - \frac{\mathbf{x}_1 \cdot \mathbf{y}_3}{\mathbf{x}_1 \cdot \mathbf{x}_1}\mathbf{x}_1 - \frac{\mathbf{x}_2 \cdot \mathbf{y}_3}{\mathbf{x}_2 \cdot \mathbf{x}_2}\mathbf{x}_2.$$

Then

$$\begin{aligned}
\mathbf{x}_3 &= (0, 0, 1, 1) - \frac{1}{2}(1, 0, 1, 0) - \frac{-\frac{1}{2}}{\frac{6}{4}}\frac{1}{2}(1, 2, -1, 0) \\
&= (0, 0, 1, 1) - \frac{1}{2}(1, 0, 1, 0) + \frac{1}{6}(1, 2, -1, 0) \\
&= \frac{1}{6}(-2, 2, 2, 6) \\
&= \frac{1}{3}(-1, 1, 1, 3).
\end{aligned}$$

The vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ have lengths

$$\sqrt{(1^2 + 0^2 + 1^2 + 0^2)} = \sqrt{2}, \qquad \frac{1}{2}\sqrt{(1^2 + 2^2 + (-1)^2 + 0^2)} = \frac{1}{2}\sqrt{6}$$

$$\text{and} \qquad \frac{1}{3}\sqrt{((-1)^2 + 1^2 + 1^2 + 3^2)} = \frac{1}{3}\sqrt{12} = \frac{2}{3}\sqrt{3}.$$

So
$$\frac{1}{\sqrt{2}}(1, 0, 1, 0), \quad \frac{1}{\sqrt{6}}(1, 2, -1, 0), \quad \frac{1}{2\sqrt{3}}(-1, 1, 1, 3)$$
is an orthonormal basis for U.

T5 Let U be the subspace of $\mathbb{R}^4$ spanned by
$$(1, -1, 0, 0), \quad (1, 1, 1, -3), \quad (0, 1, -1, 0), \quad (0, 0, 1, -1).$$
Find a basis for U and then use the Gram-Schmidt process to find an orthonormal basis for U.

Solution

The Mathematics-2B course included two methods for finding a basis for U. The first method involves writing the given vectors as the columns of a matrix; the second method involves writing the given vectors as the rows of a matrix.

Method 1

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -3 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -3 & 0 & -1 \end{bmatrix} \qquad [R_2 \to R_2 + R_1]$$
$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & -3 & 0 & -1 \end{bmatrix} \qquad [R_2 \leftrightarrow R_3]$$
$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -3 & 2 \end{bmatrix} \qquad \begin{bmatrix} R_3 \to R_3 - 2R_2 \text{ then} \\ R_4 \to R_4 + 3R_2 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad [R_4 \to R_4 + R_3]$$

The last matrix is an echelon matrix with leading entries in columns 1, 2 and 3. So the given vectors corresponding to columns 1, 2 and 3 of the first matrix,
$$\text{i.e.} \quad (1, -1, 0, 0), \quad (1, 1, 1, -3), \quad (0, 1, -1, 0)$$
form a basis of U.

Let
$$\mathbf{y}_1 = (1, -1, 0, 0), \quad \mathbf{y}_2 = (1, 1, 1, -3), \quad \mathbf{y}_3 = (0, 1, -1, 0).$$

Put $\mathbf{x}_1 = \mathbf{y}_1 = (1, -1, 0, 0)$. Next put

$$\mathbf{x}_2 = \mathbf{y}_2 - \frac{\mathbf{x}_1 \cdot \mathbf{y}_2}{\mathbf{x}_1 \cdot \mathbf{x}_1}\,\mathbf{x}_1.$$

Then

$$\mathbf{x}_2 = (1, 1, 1, -3) - \frac{0}{2}(1, -1, 0, 0) = (1, 1, 1, -3).$$

Next put

$$\mathbf{x}_3 = \mathbf{y}_3 - \frac{\mathbf{x}_1 \cdot \mathbf{y}_3}{\mathbf{x}_1 \cdot \mathbf{x}_1}\,\mathbf{x}_1 - \frac{\mathbf{x}_2 \cdot \mathbf{y}_3}{\mathbf{x}_2 \cdot \mathbf{x}_2}\,\mathbf{x}_2.$$

Then

$$\begin{aligned}\mathbf{x}_3 &= (0, 1, -1, 0) - \frac{-1}{2}(1, -1, 0, 0) - \frac{0}{12}(1, 1, 1, -3)\\ &= \frac{1}{2}(1, 1, -2, 0).\end{aligned}$$

The vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ have lengths

$$\sqrt{(1^2 + (-1)^2 + 0^2 + 0^2)} = \sqrt{2}, \qquad \sqrt{(1^2 + 1^2 + 1^2 + (-3)^2)} = \sqrt{12} = 2\sqrt{3}$$

$$\text{and} \qquad \frac{1}{2}\sqrt{(1^2 + 1^2 + (-2)^2 + 0^2)} = \frac{1}{2}\sqrt{6}.$$

So

$$\frac{1}{\sqrt{2}}(1, -1, 0, 0), \quad \frac{1}{2\sqrt{3}}(1, 1, 1, -3), \quad \frac{1}{\sqrt{6}}(1, 1, -2, 0)$$

is an orthonormal basis for U.

<u>Method 2</u>

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & -3 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & -3 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad [R_2 \to R_2 - R_1]$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad [R_2 \leftrightarrow R_3]$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad [R_3 \to R_3 - 2R_2]$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad [R_3 \to \tfrac{1}{3}R_3]$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad [R_4 \to R_4 - R_3]$$

The last matrix is an echelon matrix. So

$$(1, -1, 0, 0), \quad (0, 1, -1, 0), \quad (0, 0, 1, -1)$$

is a basis of U.

Let

$$\mathbf{y}_1 = (1, -1, 0, 0), \quad \mathbf{y}_2 = (0, 1, -1, 0), \quad \mathbf{y}_3 = (0, 0, 1, -1).$$

Put $\mathbf{x}_1 = \mathbf{y}_1 = (1, -1, 0, 0)$. Next put

$$\mathbf{x}_2 = \mathbf{y}_2 - \frac{\mathbf{x}_1 \cdot \mathbf{y}_2}{\mathbf{x}_1 \cdot \mathbf{x}_1}\,\mathbf{x}_1.$$

Then

$$\mathbf{x}_2 = (0, 1, -1, 0) - \frac{-1}{2}(1, -1, 0, 0) = \frac{1}{2}(1, 1, -2, 0).$$

Next put

$$\mathbf{x}_3 = \mathbf{y}_3 - \frac{\mathbf{x}_1 \cdot \mathbf{y}_3}{\mathbf{x}_1 \cdot \mathbf{x}_1}\,\mathbf{x}_1 - \frac{\mathbf{x}_2 \cdot \mathbf{y}_3}{\mathbf{x}_2 \cdot \mathbf{x}_2}\,\mathbf{x}_2.$$

Then

$$\begin{aligned} \mathbf{x}_3 &= (0, 0, 1, -1) - \frac{0}{2}(1, -1, 0, 0) - \frac{-1}{\frac{6}{4}}\,\frac{1}{2}(1, 1, -2, 0) \\ &= (0, 0, 1, -1) + \frac{1}{3}(1, 1, -2, 0) \\ &= \frac{1}{3}(1, 1, 1, -3). \end{aligned}$$

The vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ have lengths

$$\sqrt{(1^2 + (-1)^2 + 0^2 + 0^2)} = \sqrt{2}, \qquad \frac{1}{2}\sqrt{(1^2 + 1^2 + (-2)^2 + 0^2)} = \frac{1}{2}\sqrt{6}$$

$$\text{and} \qquad \frac{1}{3}\sqrt{(1^2 + 1^2 + 1^2 + (-3)^2)} = \frac{1}{3}\sqrt{12} = \frac{2}{3}\sqrt{3}.$$

So

$$\frac{1}{\sqrt{2}}(1, -1, 0, 0), \quad \frac{1}{\sqrt{6}}(1, 1, -2, 0), \quad \frac{1}{2\sqrt{3}}(1, 1, 1, -3)$$

is an orthonormal basis for U. (It is just a rearrangement of the first orthonormal basis we found.)

<u>Additional Remark</u>

What would happen if we applied the Gram-Schmidt process without first finding a basis for U? In order to investigate this, let

$$\mathbf{y}_1 = (1, -1, 0, 0), \quad \mathbf{y}_2 = (1, 1, 1, -3), \quad \mathbf{y}_3 = (0, 1, -1, 0), \quad \mathbf{y}_4 = (0, 0, 1, -1).$$

As before, put $\mathbf{x}_1 = \mathbf{y}_1 = (1, -1, 0, 0)$. Next put

$$\mathbf{x}_2 = \mathbf{y}_2 - \frac{\mathbf{x}_1 \cdot \mathbf{y}_2}{\mathbf{x}_1 \cdot \mathbf{x}_1}\mathbf{x}_1.$$

Then

$$\mathbf{x}_2 = (1, 1, 1, -3) - \frac{0}{2}(1, -1, 0, 0) = (1, 1, 1, -3).$$

Next put

$$\mathbf{x}_3 = \mathbf{y}_3 - \frac{\mathbf{x}_1 \cdot \mathbf{y}_3}{\mathbf{x}_1 \cdot \mathbf{x}_1}\mathbf{x}_1 - \frac{\mathbf{x}_2 \cdot \mathbf{y}_3}{\mathbf{x}_2 \cdot \mathbf{x}_2}\mathbf{x}_2.$$

Then

$$\begin{aligned}\mathbf{x}_3 &= (0, 1, -1, 0) - \frac{-1}{2}(1, -1, 0, 0) - \frac{0}{12}(1, 1, 1, -3)\\ &= \frac{1}{2}(1, 1, -2, 0).\end{aligned}$$

Next put

$$\mathbf{x}_4 = \mathbf{y}_4 - \frac{\mathbf{x}_1 \cdot \mathbf{y}_4}{\mathbf{x}_1 \cdot \mathbf{x}_1}\mathbf{x}_1 - \frac{\mathbf{x}_2 \cdot \mathbf{y}_4}{\mathbf{x}_2 \cdot \mathbf{x}_2}\mathbf{x}_2 - \frac{\mathbf{x}_3 \cdot \mathbf{y}_4}{\mathbf{x}_3 \cdot \mathbf{x}_3}\mathbf{x}_3.$$

Then

$$\begin{aligned}\mathbf{x}_4 &= (0, 0, 1, -1) - \frac{0}{2}(1, -1, 0, 0) - \frac{4}{12}(1, 1, 1, -3) - \frac{-1}{\frac{6}{4}}\frac{1}{2}(1, 1, -2, 0)\\ &= (0, 0, 1, -1) - \frac{1}{3}(1, 1, 1, -3) + \frac{1}{3}(1, 1, -2, 0)\\ &= (0, 0, 0, 0).\end{aligned}$$

So the Gram-Schmidt process fails to extend the orthogonal list of non-zero vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ we have already produced. The process failed because $\mathbf{y}_4$ is in the subspace of U spanned by $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, contrary to the requirements for the choice of the vector $\mathbf{y}_4$ to continue the process. (It is impossible to extend the list $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ since it is already a basis of U.) We found that

$$\mathbf{y}_4 = \frac{\mathbf{x}_1 \cdot \mathbf{y}_4}{\mathbf{x}_1 \cdot \mathbf{x}_1}\mathbf{x}_1 + \frac{\mathbf{x}_2 \cdot \mathbf{y}_4}{\mathbf{x}_2 \cdot \mathbf{x}_2}\mathbf{x}_2 + \frac{\mathbf{x}_3 \cdot \mathbf{y}_4}{\mathbf{x}_3 \cdot \mathbf{x}_3}\mathbf{x}_3.$$

T6 Which of the following matrices are orthogonal. For those that are, find their inverses.

a) $\begin{pmatrix} 0 & 0 & -1 \\ 0 & \sqrt{5} & 0 \\ \pi & 0 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

c) Let $a = \sqrt{4 + \sin^2(\theta)}$, and $b = \sqrt{\cos^2(\theta)\sin^2(\theta) + 1}$ and consider
$\begin{pmatrix} \frac{2}{a} & \frac{\cos(\theta)\sin(\theta)}{b} \\ \frac{\sin(2\theta)}{a} & \frac{-1}{b} \end{pmatrix}$

Solution

a) Not orthogonal

b) Orthogonal, and the transpose is the inverse

c) Orthogonal, and the transpose is the inverse

T7 Let Q be an orthogonal $n \times n$ matrix. What is $\text{col}(A)$?

Solution

Since Q is orthogonal, the columns are orthonormal and hence form a basis for $\mathbb{R}^n$. Thus $\text{col}(A) = \mathbb{R}^n$.

T8 Let Q be an orthogonal $n \times n$ matrix. Show that for any n by n matrix A, there exists an n by n matrix B such that $A = BQ$.

Solution

We have $Q^TQ = I$. Let $B = AQ^T$. Then $BQ = AQ^TQ = AI = A$.

T9 Let U be a subspace of $\mathbb{R}^n$. The orthogonal complement of U is the set $U^\perp$ of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in U, i.e.

$$U^\perp = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \cdot \mathbf{u} = 0 \text{ for all } \mathbf{u} \in U\}.$$

Prove that $U^\perp$ is a subspace of $\mathbb{R}^n$ and $U^\perp \cap U = \{0\}$.

Solution

First $\mathbf{0} \in U^\perp$ since, for all $\mathbf{u} \in U$,

$$\mathbf{0} \cdot \mathbf{u} = 0.$$

Now suppose that $\mathbf{v}, \mathbf{w} \in U^\perp$ and α is a scalar. Then, for all $\mathbf{u} \in U$,

$$\mathbf{v} \cdot \mathbf{u} = \mathbf{w} \cdot \mathbf{u} = 0.$$

Therefore

$$(\alpha\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \alpha\mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u} = (\alpha \times 0) + 0 = 0.$$

So $\alpha\mathbf{v} + \mathbf{w} \in U^\perp$. Hence $U^\perp$ is a subspace of V.

Now suppose that $\mathbf{u} \in U \cap U^\perp$. Then

$$\mathbf{u} \cdot \mathbf{u} = 0.$$

Therefore

$$\mathbf{u} = \mathbf{0}.$$

Hence

$$U \cap U^\perp = \{\mathbf{0}\}.$$

T10 Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be an orthonormal basis for $\mathbb{R}^n$. Show that, for all vectors $\mathbf{v} \in \mathbb{R}^n$,

$$\mathbf{v} = \sum_{j=1}^{n} (\mathbf{v} \cdot \mathbf{e}_j)\mathbf{e}_j.$$

Solution

Let $\mathbf{v} \in V$. Since $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ is a basis for V, we can find scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\mathbf{v} = \sum_{j=1}^{n} \alpha_j \mathbf{e}_j.$$

Then, for $k = 1, 2, \ldots, n$,

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{e}_k &= (\sum_{j=1}^{n} \alpha_j \mathbf{e}_j) \cdot \mathbf{e}_k \\
&= \sum_{j=1}^{n} \alpha_j \mathbf{e}_j \cdot \mathbf{e}_k \\
&= \sum_{j=1}^{n} \alpha_j (\mathbf{e}_j \cdot \mathbf{e}_k) \\
&= \alpha_k. && [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \text{ is an orthonormal list}]
\end{aligned}$$

Hence

$$\mathbf{v} = \sum_{j=1}^{n} (\mathbf{v} \cdot \mathbf{e}_j)\mathbf{e}_j.$$

Suppose that we were given an orthogonal basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ instead of an orthonormal one. Then, we would have obtained

$$\mathbf{v} = \sum_{j=1}^{n} \frac{\mathbf{v} \cdot \mathbf{e}_j}{\mathbf{e}_j \cdot \mathbf{e}_j} \mathbf{e}_j.$$

T11 Let A and B be orthogonal $n \times n$ matrices such that $A^T B + B^T A = -I$. Show that $A + B$ is orthogonal.

Solution

$(A + B)^T (A + B) = A^T A + B^T B + A^T B + B^T A = I + I - I = I.$

T12 Let U be a subspace of $\mathbb{R}^n$, and $U^\perp$ the orthogonal complement (as defined in T9). Let $\mathbf{x}$ be some vector in $\mathbb{R}^n$.

a) Use Gram-Schmidt to show that there is a basis $\{\mathbf{f}_1, \mathbf{f}_2, \ldots \mathbf{f}_n\}$ for $\mathbb{R}^n$ such that $\{\mathbf{f}_1, \ldots, \mathbf{f}_m\}$ belong to U and $\{\mathbf{f}_{m+1}, \ldots, \mathbf{f}_n\}$ belong to $U^\perp$.

b) Prove that there is a vector $\mathbf{y} \in U$ and $\mathbf{z} \in U^{\perp}$ such that $\mathbf{x} = \mathbf{y} + \mathbf{z}$.

c) Prove that this choice of $\mathbf{y}$ and $\mathbf{z}$ is unique.

Solution

a) If U is $\{\mathbf{0}\}$ or $\mathbb{R}^n$ (then $U^{\perp} = \mathbb{R}^n$ or $\{\mathbf{0}\}$ respectively), the problem is easy. Therefore we can assume that we are not is such a case and so we may assume U has a basis $\{\mathbf{f}_1, \mathbf{f}_2, \ldots \mathbf{f}_m\}$, where $1 < m < n$. Using Gram-Schmidt we may produce orthogonal vectors $\mathbf{f}_{m+1}, \mathbf{f}_{m+2} \ldots, \mathbf{f}_n$ such that these are orthogonal to all the f_i's, $1 \leq i \leq m$, and hence belong to $U^{\perp}$, and such that $\{\mathbf{f}_1, \mathbf{f}_2, \ldots \mathbf{f}_m, \mathbf{f}_{m+1}, f_{m+2} \ldots, \mathbf{f}_n\}$ forms a basis for $\mathbb{R}^n$.

b) Given our basis, we have $\mathbf{x} = \alpha_1 \mathbf{f}_1 + \ldots \alpha_n \mathbf{f}_n$ for some scalars $\alpha_1, \ldots, \alpha_n$. Let $\mathbf{y} = \alpha_1 \mathbf{f}_1 + \ldots + \alpha_m \mathbf{f}_m$, and $\mathbf{z} = \alpha_{m+1} \mathbf{f}_{m+1} + \ldots \alpha_n \mathbf{f}_n$, then $\mathbf{x} = \mathbf{y} + \mathbf{z}$ where $\mathbf{y}$ and $\mathbf{z}$ have the required properties.

c) Assume there was another way of writing $\mathbf{x} = \overline{\mathbf{y}} + \overline{\mathbf{z}}$ where $\overline{\mathbf{y}} \in U$ and $\overline{\mathbf{z}} \in U^{\perp}$. Then $\mathbf{0} = \mathbf{x} - \mathbf{x} = (\mathbf{y} + \mathbf{z}) - (\overline{\mathbf{y}} + \overline{\mathbf{z}}) = (\mathbf{y} - \overline{\mathbf{y}}) + (\mathbf{z} - \overline{\mathbf{z}})$. Let $\mathbf{Y} = \mathbf{y} - \overline{\mathbf{y}} \in U$ and $\mathbf{Z} = \mathbf{z} - \overline{\mathbf{z}} \in U^{\perp}$. Then $\mathbf{0} = \mathbf{Y} + \mathbf{Z}$ and so $\mathbf{0} \cdot \mathbf{Y} = 0 = \mathbf{Y} \cdot \mathbf{Y} + \mathbf{Z} \cdot \mathbf{Y} = \|\mathbf{Y}\|^2$. Hence $\mathbf{Y} = \mathbf{0}$, and thus $\mathbf{Z} = \mathbf{0}$. This implies $\mathbf{y} = \overline{\mathbf{y}}$ and $\mathbf{z} = \overline{\mathbf{z}}$.

T13 A permutation f on the set $\{1, 2, \ldots, n\}$ is a bijection

$$f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}.$$

Let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$. Assume there is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ which satisfies $T(\mathbf{e}_i) = \mathbf{e}_{f(i)}, 1 \leq i \leq n$, and let A be the corresponding matrix associated with the transformation.

a) Describe, in general, what the matrix A looks like. (Hint: come up with simple permutations to get a feel for what the matrix is).

b) Hence show that A is orthogonal.

Solution

a) By considering how T acts on the standard unit vectors, we see that it shuffles them around (and the way the shuffle is performed depends on what the permutation is). Hence A looks like the identity matrix but with the columns shuffled around.

b) The columns of A are the standard unit vectors, hence orthogonal.