2C Week 5 2020/21

HANDOUT

Properties of limits

In this section we examine how limits interact with the algebraic operations and the order structure on the real numbers, and obtain rules which demonstrate that our intuition for how limits should behave is valid. These rules provide methods for calculating limits[1] and we use these to establish some standard limits.

Intuitively, we don't expect a sequence to be able to converge to two separate values. The next result justifies our intuition.

[1] which you should feel free to use whenever a question does not require you to work directly from the definition. Still, you should always indicate to the person reading your work which rules you are using.

Theorem 3.8. *Let $(x_n)_{n=1}^{\infty}$ be a convergent sequence. Then its limit is unique.*

Proof. Suppose that[2] $x_n \to L$ and $x_n \to M$ as $n \to \infty$ for some $L, M \in \mathbb{R}$. Suppose that $L \neq M$, and take $\varepsilon = \frac{|L-M|}{2}$. Then the definition of convergence gives $n_1, n_2 \in \mathbb{N}$ with

$$\begin{aligned} n \geq n_1 &\implies |x_n - L| < \varepsilon \\ n \geq n_2 &\implies |x_n - M| < \varepsilon. \end{aligned}$$

[2] This is the usual method for starting a uniqueness proof; we introduce two quantities which both satisfy the relevant condition and show that they are the same. This idea was already used to prove that suprema are unique in chapter 2.

Take[3] $n = \max(n_1, n_2)$. Then

$$|L - M| = |L - x_n + x_n - M| \leq |L - x_n| + |x_n - M| < \varepsilon + \varepsilon = |L - M|,$$

a contradiction. Therefore $L = M$. □

[3] The "max trick" again.

Note the similarity of the above argument with the proof that $(-1)^n$ does not converge: the key idea is to take ε to be half the distance between L and M, just as we took ε to be half the distance between 1 and -1 in Example **??**.

Theorem 3.9 (Convergent sequences are bounded)**.** *Let $(x_n)_{n=1}^{\infty}$ be a convergent sequence. Then $(x_n)_{n=1}^{\infty}$ is bounded.*

Proof. Suppose that $x_n \to L$ as $n \to \infty$. Taking $\varepsilon = 1$ in the definition of convergence[4] there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, we have $|x_n - L| < \varepsilon$. In particular, for $n \geq n_0$ we have[5]

$$|x_n| = |x_n - L + L| \leq |x_n - L| + |L| < 1 + |L|.$$

Define $M = \max(|x_1|, |x_2|, \ldots, |x_{n_0-1}|, |L| + 1) > 0$. Then for all $n \in \mathbb{N}$, $|x_n| \leq M$, so that $(x_n)_{n=1}^{\infty}$ is bounded. □

[4] Note that the value 1 was a choice I made, to see whether the exact value matters you could try changing the value of ε. You should find that the proof works no matter what value of $\varepsilon > 0$ you take here.

[5] Note that the estimate does not yet prove that $(x_n)_{n=1}^{\infty}$ is bounded; in particular we may not have $|x_n| \leq 1 + |L|$ for all $n \in \mathbb{N}$, we only have this for those n with $n \geq n_0$.

The converse to Theorem 3.9 is false: not every bounded sequence converges. For example, taking $z_n = (-1)^n$, gives an example of a bounded sequence which does not converge. The Bolzano–Weierstrass Theorem below encapsulates what can be said about bounded sequences with respect to convergence.

We now come to the key algebraic properties of limits, which show how limits of sequence interact with the algebraic operations of addition, subtraction, multiplication, and division.

Theorem 3.10 (Algebraic properties of limits)**.** *Let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be sequences with limits L and M, respectively.*

a) For $\lambda \in \mathbb{R}$, we have $\lambda x_n \to \lambda L$ as $n \to \infty$;

b) $x_n + y_n \to L + M$ as $n \to \infty$;

c) $x_n y_n \to LM$ as $n \to \infty$;

d) If $M \neq 0$, then $x_n / y_n \to L/M$ as $n \to \infty$.

Note that in the last statement we have to insist that $M \neq 0$, so that we do not try to divide by zero. We don't demand that $y_n \neq 0$ for all n, so the sequence $(\frac{x_n}{y_n})_n$ might not be defined for all $n \in \mathbb{N}$. However, as $y_n \to M \neq 0$, there exists $n_1 \in \mathbb{N}$ such that for $n \geq n_1$, we have $y_n \neq 0$ (take $\varepsilon = |M|$ in the definition of convergence), and so the sequence $(\frac{x_n}{y_n})$ is defined for $n \geq n_1$, and that's all we need to discuss the convergence of $\frac{x_n}{y_n}$ to $\frac{L}{M}$.

I'm going to prove the second and third statement, and encourage you to see if you can prove the other two statements. The strategy we use is similar in spirit to the ideas used to show that if A and B are non-empty subsets of $\mathbb{R}$ which are bounded above, and $C = \{a + b \mid a \in A, b \in B\}$, then $\sup(C) = \sup(A) + \sup(B)$.

Proof of b). Let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be sequences with limits L and M, respectively. Let $\varepsilon > 0$ be arbitrary. Then[6] $\frac{\varepsilon}{2} > 0$, and by the definition of convergence there exists $n_1, n_2 \in \mathbb{N}$ such that

$$\begin{aligned} n \geq n_1 &\implies |x_n - L| < \frac{\varepsilon}{2} \\ n \geq n_2 &\implies |y_n - M| < \frac{\varepsilon}{2}. \end{aligned}$$

Take $n_0 = \max(n_1, n_2)$. Then, for $n \geq n_0$, we have

$$\begin{aligned} |(x_n + y_n) - (L + M)| &= |(x_n - L) + (y_n - M)| \\ &\leq |x_n - L| + |y_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $x_n + y_n \to L + M$ as $n \to \infty$. $\square$

The key thing that made this proof work was the use of the triangle inequality $|(x_n - y_n) + (L - M)| \leq |x_n - L| + |y_n - M|$ at the end, as this enables me to control the distance from $x_n + y_n$ to $L + M$ in terms to the two distances I know something about, namely $|x_n - L|$ and $|y_n - M|$. I had found this inequality before embarking on writing my proof.

For the third part, we will want to make $|x_n y_n - LM|$ smaller than some quantity ε, based on our ability to control $|x_n - L|$ and $|y_n - M|$.

[6] I'm going to use $\frac{\varepsilon}{2}$ in the definition that $x_n \to L$ and $y_n \to M$. Can you see why I want to take $\frac{\varepsilon}{2}$ here?

This leads us to add and subtract the term x_nM leading to the critical inequality. More precisely, we consider

$$\begin{aligned}|x_ny_n - LM| &= |x_ny_n - x_nM + x_nM - LM| \\ &\leq |x_n||y_n - M| + |x_n - L||M|.\end{aligned}$$

With this in mind, we can write down a proof[7].

Proof of c). Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be sequences with limits L and M respectively. Let $\varepsilon > 0$ be arbitrary. By Theorem 3.9, the sequence $(x_n)_{n=1}^{\infty}$ is bounded, so there exists $K > 0$ such that $|x_n| \leq K$ for all $n \in \mathbb{N}$. As $x_n \to L$ and $y_n \to M$, there exists $n_1, n_2 \in \mathbb{N}$ such that[8]

$$\begin{aligned}n \geq n_1 &\implies |x_n - L| < \frac{\varepsilon}{2(|M|+1)} \\ n \geq n_2 &\implies |y_n - M| < \frac{\varepsilon}{2K}.\end{aligned}$$

Take $n_0 = \max(n_1, n_2)$. For $n \in \mathbb{N}$ with $n \geq n_0$, we have

$$\begin{aligned}|x_ny_n - LM| &= |x_ny_n - x_nM + x_nM - LM| \\ &\leq |x_n||y_n - M| + |x_n - L||M| \\ &\leq K|y_n - M| + |x_n - L|(|M|+1) \\ &< K\frac{\varepsilon}{2K} + \frac{\varepsilon}{2(|M|+1)}(|M|+1) = \varepsilon,\end{aligned}$$

so that $x_ny_n \to LM$ as $n \to \infty$. □

For part d) you'll need to find a key inequality before you start, relating

$$\left|\frac{x_n}{y_n} - \frac{L}{M}\right|$$

to $|x_n - L|$ and $|y_n - M|$. Try writing the expression above as a single fraction, and then look to add and subtract a term, as in part c). You'll need to take care to insist that y_n is bounded away from 0 for large enough[9] n.

Let us revisit the examples we have seen above using properties of limits. A key strategy when taking limits of fractions, as in the question below, is to identify the dominating terms as n gets large, and divide the numerator and denominator by this quantity.

Example 3.11. Use properties of limits to find

$$\lim_{n\to\infty} \frac{3n^3 + 4n + 5}{5n^3 - 3n^2 + 4}.$$

Solution. The dominating terms in the numerator and denominator are $3n^3$ and $5n^3$, respectively. For $n \in \mathbb{N}$, we have

$$\begin{aligned}\frac{3n^3 + 4n + 5}{5n^3 - 3n^2 + 4} &= \frac{3 + \frac{4}{n^2} + \frac{5}{n^3}}{5 - \frac{3}{n} + \frac{4}{n^3}} \\ &= \frac{3 + 4\left(\frac{1}{n}\right)^2 + 5\left(\frac{1}{n}\right)^3}{5 - 3\left(\frac{1}{n}\right) + 4\left(\frac{1}{n}\right)^3} \\ &\to \frac{3 + 4 \times 0^2 + 5 \times 0^3}{5 - 3 \times 0 + 4 \times 0^3} = \frac{3}{5},\end{aligned}$$

[7] We do have an additional difficulty to overcome. We must not try and ask for n large enough so that $|y_n - M| < \frac{\varepsilon}{2|x_n|}$, as the quantity we put into the definition that $y_n \to M$ can not depend on n. For this reason we use Theorem 3.9 to find $K > 0$ such that $|x_n| \leq K$ for all n.

[8] I use $2(|M|+1)$ rather than $2|M|$ below as we don't know that $M \neq 0$; this way we don't have to handle this issue separately.

[9] It'll probably help to show first that there exists $n_1 \in \mathbb{N}$, such that for $n \geq n_1$, $|y_n| \geq \frac{|M|}{2}$. If you get stuck, there is a proof in [ERA].

as $n \to \infty$, using the standard limit $\frac{1}{n} \to 0$ as $n \to \infty$, and algebraic properties of limits[10]. $\square$

In the solution above it's important to use the symbols, $\to$ and $=$ correctly. Remember that when we take limits, we use the symbol $\to$ to indicate this: in particular notice that

$$\frac{3+4\left(\frac{1}{n}\right)^2+5\left(\frac{1}{n}\right)^3}{5-3\left(\frac{1}{n}\right)+4\left(\frac{1}{n}\right)^3} \neq \frac{3+4\times 0^2+5\times 0^3}{5-3\times 0+4\times 0^3},$$

as the former is an expression in n, the latter a number. Make sure you don't write $=$ unless two quantities are equal[11].

[10] It's always a good idea to indicate what you are using.

[11] It's also important to make sure you do connect all these mathematical expressions, you certainly can't write them in a list down the side of the page without any indication of how they're related.

The Sandwich Principle and standard limits

Before coming back to more examples of this type, it's useful to have more standard limits at our disposal. To do this, we investigate how limits interact with the order properties of $\mathbb{R}$.

Theorem 3.12 (Limits and order). *Let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be sequences, and suppose $x_n \to L$ and $y_n \to M$ as $n \to \infty$. If there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $x_n \leq y_n$, then $L \leq M$.*

Proof. Let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be sequences, let $N \in \mathbb{N}$ be such that $x_n \leq y_n$ for $n \geq N$, and suppose $x_n \to L$ and $y_n \to M$ as $n \to \infty$. Suppose that $L > M$, and take $\varepsilon = \frac{L-M}{2}$. Then there exists $n_1, n_2 \in \mathbb{N}$ such that

$$\begin{aligned} n \geq n_1 &\implies |x_n - L| < \varepsilon; \\ n \geq n_2 &\implies |y_n - M| < \varepsilon. \end{aligned}$$

Take $n \geq \max(N, n_1, n_2)$. Then $|x_n - L| < \varepsilon$, so $x_n > L - \varepsilon = M + \varepsilon$. Also $|y_n - M| < \varepsilon$, so $y_n < M + \varepsilon$. Combining these inequalities gives $x_n > y_n$, which contradicts the fact that $x_n \leq y_n$ for $n \geq N$. Therefore $L \leq M$. $\square$

Most often we use this when one sequence is constant, for example, taking $x_n = 0$ for all n, the proposition says that if $(y_n)_{n=1}^\infty$ is a sequence of eventually positive terms, with $y_n \to M$ as $n \to \infty$, then $M \geq 0$. Note that we must use inequalities of the form $\leq$ and $\geq$ in these results: the information $y_n > 0$ for all n, and $y_n \to M$ as $n \to \infty$, does not[12] imply $M > 0$.

[12] Consider $y_n = \frac{1}{n}$.

Theorem 3.13 (The sandwich principle). *Let $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty$ and $(z_n)_{n=1}^\infty$ be sequences and suppose that $x_n \to L$ and $z_n \to L$ as $n \to \infty$. If there exists $N \in \mathbb{N}$ such that*

$$n \geq N \implies x_n \leq y_n \leq z_n,$$

then $\lim_{n\to\infty} y_n = L$.

Proof. Let $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty, (z_n)_{n=1}^\infty, L$ and N be as in the statement of the theorem, and let $\varepsilon > 0$ be arbitrary. Then there exists $n_1, n_2 \in \mathbb{N}$

such that

$$n \geq n_1 \implies |x_n - L| < \varepsilon \implies x_n > L - \varepsilon.$$
$$n \geq n_2 \implies |z_n - L| < \varepsilon \implies z_n < L + \varepsilon.$$

Then take $n_0 = \max(n_1, n_2, N)$. For $n \geq n_0$, we have

$$L - \varepsilon < x_n \leq y_n \leq z_n < L + \varepsilon \implies |y_n - L| < \varepsilon.$$

Therefore $y_n \to L$ as $n \to \infty$. □

We shall often apply the sandwich principle when one of the sequences $(x_n)_{n=1}^{\infty}$ or $(z_n)_{n=1}^{\infty}$ is constant; the challenge is then to find the other sequence which should be easier to work with[13] and relatively straightforward to provide the required inequalities. Don't be afraid to put some large values of n into your calculator to make sure you're aiming for the right limit.

[13] For instance, formed from standard limits.

Example 3.14. Evaluate

$$\lim_{n\to\infty}(\sqrt{n+4} - \sqrt{n}), \quad \lim_{n\to\infty}(3^n + 5^n)^{1/n}.$$

Solution. For the first example, we use a standard trick for working with surds, namely multiplying by $\frac{\sqrt{n+4}+\sqrt{n}}{\sqrt{n+4}+\sqrt{n}}$ in order to use the difference between two squares formula[14]. For $n \in \mathbb{N}$, we have

$$0 \leq (\sqrt{n+4} - \sqrt{n}) = \frac{4}{\sqrt{n+4}+\sqrt{n}} \leq \frac{4}{\sqrt{n}} \to 0,$$

as $n \to \infty$.[15] By the sandwich principle,

$$\lim_{n\to\infty}(\sqrt{n+4} - \sqrt{n}) = 0.$$

[14] I'm expecting to try and prove that the limit is 0, using a calculator if necessary to see this. So I would like to write the original expression using terms like $\frac{K}{\sqrt{n}}$, as we know that these converge to 0.

[15] Can you prove that $\frac{4}{\sqrt{n}} \to 0$ as $n \to \infty$ directly from the definition?

For the second, again I use a calculator if necessary to decide that the limit of this sequence will be 5, which we get from $(5^n)^{1/n}$. Thus what is happening is that the 5^n term dominates, and the 3^n term becomes insignificant for large n. This way of thinking leads to the following calculation:

$$5 = (0^n + 5^n)^{1/n} \leq (3^n + 5^n)^{1/n} \leq (5^n + 5^n)^{1/n} = 5 \times 2^{1/n} \to 5,$$

as $n \to \infty$, using the standard limit[16] $2^{1/n} \to 1$. Therefore, by the sandwich principle

$$\lim_{n\to\infty}(3^n + 5^n)^{1/n} = 5$$

as claimed. □

[16] See the exercises, and the comments on standard limits below.

In order to record another useful limit, we present the following lemma.

Lemma 3.15. *Let $(x_n)_{n=1}^{\infty}$ be a sequence and $L \in \mathbb{R}$. Then*

$$x_n \to L \text{ as } n \to \infty \quad \Leftrightarrow \quad |x_n - L| \to 0 \text{ as } n \to \infty.$$

The lemma is proved by noting that $||x_n - L| - 0| = |x_n - L|$. I leave it to you to write up the details.

Theorem 3.16. *Let $x \in \mathbb{R}$. Then*

$$x^n \to 0 \iff |x| < 1.$$

Proof. Suppose that $|x| \geq 1$. Then $|x|^n \geq 1$ for all n, and hence[17] $|x|^n \nrightarrow 0$. Therefore $x^n \nrightarrow 0$.

Conversely, when $x = 0$, we have $x^n = 0$ for all n, so that $x^n \to 0$, so suppose that $0 < |x| < 1$. Write $\frac{1}{|x|} = 1 + K$ for some $K > 0$. Then, using the binomial expansion, we have

$$\frac{1}{|x|^n} = (1+K)^n = 1 + nK + \cdots + K^n \geq nK.$$

Therefore

$$0 \leq |x|^n \leq \frac{1}{nK} \to 0,$$

as $n \to \infty$. By the sandwich principle, $|x|^n \to 0$ as $n \to \infty$. This implies $x^n \to 0$ due to Lemma 3.15. □

[17] Can you prove this directly from the definition? What value of ε should you take?

Example 3.17. Evaluate

$$\lim_{n\to\infty} \frac{5^n + 3^n}{5^n - 3^n}.$$

This works in a very similar way to previous "algebraic properties" type questions. Identify the dominating term[18] and divide denominator and numerator by this term[19].

[18] In this question that's 5^n.

[19] As with the other algebraic properties questions be careful how you write your solution. In particular, make sure you never claim that some expression in n is equal to it's limit, and use the symbols $\lim_{n\to\infty}$, $=$ and $\to$ correctly.

Solution. Using Theorem 3.16 and algebraic properties of limits, we obtain

$$\frac{5^n + 3^n}{5^n - 3^n} = \frac{1 + \left(\frac{3}{5}\right)^n}{1 - \left(\frac{3}{5}\right)^n} \to \frac{1+0}{1-0} = 1,$$

as $n \to \infty$, since $|\frac{3}{5}| < 1$. □

In exercises using the sandwich principle and properties of limits, you should feel free to use standard limits we've either proved above, or on exercises freely in order to compute other limits[20]. The key standard limits are:

[20] though you should always say that you're doing so.

Theorem 3.18 (Standard limits)**.** *The following results hold:*

a) $\frac{1}{n^\alpha} \to 0$ *as* $n \to \infty$ *for any* $\alpha > 0$.

b) $x^n \to 0$ *as* $n \to \infty$ *if and only if* $|x| < 1$.

c) $x^{1/n} \to 1$ *as* $n \to \infty$ *for all* $x > 0$.

Proof. *a*) This is an exercise on sheet 5.
b) This is theorem 3.16.
c) We will only prove this for $x \geq 1$ here[21]. Let us write $x = 1 + c$ for $c \geq 0$. Moreover let $\varepsilon > 0$ be arbitrary. Then $|x^{1/n} - 1| < \varepsilon$ if and only if $x < (1+\varepsilon)^n$. Moreover $1 + n\varepsilon \leq (1+\varepsilon)^n$ by the binomial formula. Hence if we chose $n_0 \in \mathbb{N}$ such that $n_0 > \frac{c}{\varepsilon}$, we obtain

$$x = 1 + c \leq 1 + n\frac{c}{n_0} < 1 + n\varepsilon \leq (1+\varepsilon)^n$$

for $n \geq n_0$. Therefore $x^{1/n} \to 1$ as $n \to \infty$ in this case. □

[21] Can you prove the claim for $0 < x < 1$?