2A: Tutorial 3

School of Mathematics and Statistics

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Semester 1 2020–21

Instructions

Access your tutorial lecture

TU01,TU02,TU03,TU04,TU05: October 12th, 09:00-10:00

TU06,TU07,TU08,TU09,TU10: October 12th, 15:00-16:00

using the Zoom plug-in in the Moodle page of the course.

Please mute yourself when you are in the main room of the meeting.

Be ready to work in groups. Having a method of writing on screen, as mouse, tablet or similar is useful.

From the Moodle forum

Let $f = f(x, y)$ and let us compute $\int \frac{\partial^2 f}{\partial x \partial y} dx$.

Let $g = g(x, y)$ be arbitrary. Then

$$\int \frac{\partial g}{\partial x} dx = g(x, y) + c(y), \quad c \text{ arbitrary function}.$$

Indeed $\frac{\partial}{\partial x}(g(x, y) + c(y)) = \frac{\partial g}{\partial x}$.

If $g = \frac{\partial f}{\partial y}$, then $\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial x \partial y}$. Substituting in the previous formula we get

$$\int \frac{\partial^2 f}{\partial x \partial y} dx = \frac{\partial f}{\partial y} + c(y), \quad c \text{ arbitrary function}.$$

Multiple Choice Question 1

Ex Sheet 2, T10

Find n such that the function $f(x, y) = 2xy + x^n y^{2n}$ is a solution of

$$2x^2 \frac{\partial^2 f}{\partial x^2} - y^2 \frac{\partial^2 f}{\partial y^2} + 18f = 36xy.$$

(A) $n = 0$

(B) $n = 0, 2$

(C) $n = \pm 1$

(D) $n = \pm 3$

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$$2x^2 \frac{\partial^2 f}{\partial x^2} - y^2 \frac{\partial^2 f}{\partial y^2} + 18f = 36xy.$$

(A) $n = 0$ **(B)** $n = 0, 2$

(C) $n = \pm 1$ **(D)** $n = \pm 3$

ANSWER: **(D)** The relevant derivatives are

$$f_x = 2y + nx^{n-1}y^{2n}, \qquad f_{xx} = n(n-1)x^{n-2}y^{2n},$$
$$f_y = 2x + 2nx^n y^{2n-1} \qquad f_{yy} = 2n(2n-1)x^n y^{2n-2}.$$

Substitute to give $(18 - 2n^2)x^n y^{2n} + 36xy = 36xy$. So to satisfy this equation we must have $n = \pm 3$.

Multiple Choice Question 2

Unseen question

Find the general solution for $f(x, y, z)$ of

$$\frac{\partial^2 f}{\partial z \partial y} = xyz.$$

(A) $\frac{1}{4}xy^2z^2 + A(x)$

(B) $\frac{1}{4}xy^2z^2 + B(x, y, z)$

(C) $x + C(x, y, z)$

(D) $\frac{1}{4}xy^2z^2 + D(x, y) + E(x, z)$

Multiple Choice Question 2

Unseen question

Find the general solution for $f(x, y, z)$ of

$$\frac{\partial^2 f}{\partial z \partial y} = xyz.$$

(A) $\frac{1}{4}xy^2z^2 + A(x)$ **(B)** $\frac{1}{4}xy^2z^2 + B(x, y, z)$

(C) $x + C(x, y, z)$ **(D)** $\frac{1}{4}xy^2z^2 + D(x, y) + E(x, z)$

ANSWER: **(D)** A partial integration with respect to z gives

$$\frac{\partial f}{\partial y} = \frac{1}{2}xyz^2 + d(x, y).$$

Another partial integration, this time with respect to y gives

$$f = \frac{1}{4}xy^2z^2 + \int d(x, y)\, dy + E(x, z).$$

Multiple Choice Question 2

Unseen question

Find the general solution for $f(x, y, z)$ of

$$\frac{\partial^2 f}{\partial z \partial y} = xyz.$$

(A) $\frac{1}{4}xy^2z^2 + A(x)$

(B) $\frac{1}{4}xy^2z^2 + B(x, y, z)$

(C) $x + C(x, y, z)$

(D) $\frac{1}{4}xy^2z^2 + D(x, y) + E(x, z)$

Denote by $D(x, y) = \int d(x, y)dy$ (this is an arbitrary function of the variables x and y since $d(x, y)$ is also arbitrary). Then

$$f = \frac{1}{4}xy^2z^2 + D(x, y) + E(x, z),$$

where D and E are arbitrary functions of two variables.

Tutorial questions

Ex Sheet 2, T3

Let $f(x,y) = xy^2 \sin(\frac{x}{y})$. Show that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f$.

Ex Sheet 3, T3(a)

Find the general solution of the following partial differential equation:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 6xy \qquad \text{(change to } u = y/x \text{ and } v = x\text{).}$$

Ex Sheet 2, T3

Let $f(x, y) = xy^2 \sin(\frac{x}{y})$. Show that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f$.

The two derivatives are (using the chain and product rules)

$$f_x = y^2 \sin\left(\frac{x}{y}\right) + xy \cos\left(\frac{x}{y}\right), \qquad f_y = 2xy \sin\left(\frac{x}{y}\right) - x^2 \cos\left(\frac{x}{y}\right).$$

Now

$$\begin{aligned} x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= xy^2 \sin\left(\frac{x}{y}\right) \cancel{+x^2 y \cos\left(\frac{x}{y}\right)} \\ &\quad + 2xy^2 \sin\left(\frac{x}{y}\right) \cancel{-x^2 y \cos\left(\frac{x}{y}\right)} = 3xy^2 \sin\left(\frac{x}{y}\right) = 3f. \end{aligned}$$

Ex Sheet 3, T3(a)

Find the general solution of the following partial differential equation:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 6xy \qquad \text{(change to } u = y/x \text{ and } v = x\text{).}$$

Write the function $f(x, y) = F(u(x, y), v(x, y))$. Then the chain rule gives

$$\frac{\partial f}{\partial x} = -\frac{y}{x^2}\frac{\partial F}{\partial u} + \frac{\partial F}{\partial v}, \qquad \frac{\partial f}{\partial y} = \frac{1}{x}\frac{\partial F}{\partial u}.$$

Substitution into the PDE gives

$$x\left(\cancel{-\frac{y}{x^2}\frac{\partial F}{\partial u}} + \frac{\partial F}{\partial v}\right) + y\left(\cancel{\frac{1}{x}\frac{\partial F}{\partial u}}\right) = 6xy.$$

Now simplify and write y in terms of u and v to get

$$\frac{\partial F}{\partial v} = 6y = 6uv\,.$$

Partial integration with respect to v gives

$$F = 3uv^2 + A(u)\,,$$

where A is an arbitrary function of one variable. Finally the general solution is

$$f(x,y) = F(u(x,y), v(x,y)) = 3\left(\frac{y}{x}\right)x^2 + A\left(\frac{y}{x}\right) = 3xy + A\left(\frac{y}{x}\right).$$

Bonus tutorial question

Wave equation

The wave equation in one dimension, for $\phi(x, t)$, is

$$c^2 \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial t^2} \,,$$

where $c > 0$ is a constant (the wave speed). By making the change of variable $l = x + ct$ and $r = x - ct$ find the general solution.

Bonus tutorial question

Write $\phi(x,t) = F(l(x,t), r(x,t))$, then apply the chain rule to calculate

$$\phi_x = l_x F_l + r_x F_r = F_l + F_r\,, \quad \phi_t = l_t F_l + r_t F_r = cF_l - cF_r\,.$$

A further application of the chain rules gives the second order partial derivatives

$$\phi_{xx} = F_{ll} + F_{rr} + 2F_{lr}, \qquad \phi_{tt} = c^2\left(F_{ll} + F_{rr} - 2F_{lr}\right).$$

Bonus tutorial question

Substitute into the PDE

$$c^2\left(\cancel{F_{ll}} + \cancel{F_{rr}} + 2F_{lr}\right) = c^2\left(\cancel{F_{ll}} + \cancel{F_{rr}} - 2F_{lr}\right)$$

and simplify to obtain

$$F_{lr} = 0\,.$$

Two partial integrations give the solution $F(l, r) = f(l) + g(r)$, where f and g are arbitrary functions of one variable. Finally

$\phi(x, t) = f(x+ct)+g(x-ct)$ (superposition of left and right waves).