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MAT 169
Fall Semester 2009-10
Lecture 31 Notes

These notes correspond to Section 9.2 in the text.

Arc Length of Parametrically Defined Curves

In the last lecture we learned how to compute the arc length of a curve described by an equation of the form $y = f(x)$, where $a \leq x \leq b$. The arc length L of such a curve is given by the definite integral

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Now, suppose that this curve can also be defined by parametric equations

$$x = g(t), \quad y = h(t), \tag{1}$$

where $c \leq t \leq d$. It follows that

$$y = h(t) = f(g(t)),$$

and therefore, by the Chain Rule,

$$\frac{dy}{dt} = h'(t) = f'(g(t))g'(t) = f'(g(t))\frac{dx}{dt} = f'(x)\frac{dx}{dt}.$$

In the integral defining the arc length of the curve, we make the substitution $x = g(t)$ and obtain

$$\begin{aligned} L &= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \\ &= \int_c^d \sqrt{1 + [f'(g(t))]^2}\, g'(t)\, dt \\ &= \int_c^d \sqrt{[g'(t)]^2 + [f'(g(t))g'(t)]^2}\, dt \\ &= \int_c^d \sqrt{[g'(t)]^2 + [h'(t)]^2}\, dt \\ &= \int_c^d \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \end{aligned}$$

It turns out that this formula for the arc length applies to *any* curve that is defined by parametric equations of the form (1), as long as x and y are differentiable functions of the parameter t. To derive the formula in the general case, one can proceed as in the case of a curve defined by an equation of the form $y = f(x)$, and define the arc length as the limit as $n \to \infty$ of the sum of the lengths of n line segments whose endpoints lie on the curve.

Example Compute the length of the curve

$$x = 2\cos^2\theta, \quad y = 2\cos\theta\sin\theta,$$

where $0 \leq \theta \leq \pi$.

Solution This curve is plotted in Figure 1; it is a circle of radius 1 centered at the point $(1, 0)$. It follows that its length, which we will denote by L, is the circumference of the circle, which is 2π. Using the arc length formula, we can obtain the same result as follows:

$$\begin{aligned}
L &= \int_0^\pi \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\
&= \int_0^\pi \sqrt{(-4\cos\theta\sin\theta)^2 + \left(2\cos^2\theta - 2\sin^2\theta\right)^2}\, d\theta \\
&= 2\int_0^\pi \sqrt{(2\cos\theta\sin\theta)^2 + \left(\cos^2\theta - \sin^2\theta\right)^2}\, d\theta \\
&= 2\int_0^\pi \sqrt{4\cos^2\theta\sin^2\theta + \cos^4\theta - 2\cos^2\theta\sin^2\theta + \sin^4\theta}\, d\theta \\
&= 2\int_0^\pi \sqrt{\cos^4\theta + 2\cos^2\theta\sin^2\theta + \sin^4\theta}\, d\theta \\
&= 2\int_0^\pi \sqrt{(\cos^2\theta + \sin^2\theta)^2}\, d\theta \\
&= 2\int_0^\pi \sqrt{1^2}\, d\theta \\
&= 2\int_0^\pi d\theta \\
&= 2\pi.
\end{aligned}$$

Note: Double-angle and half-angle formulas could have been used in this example, but little would have been gained except during the differentiation stage, so I chose not to use them. □

Example Compute the length of the curve

$$x = t\sin t, \quad y = t\cos t$$

from $t = 0$ to $t = 2\pi$.

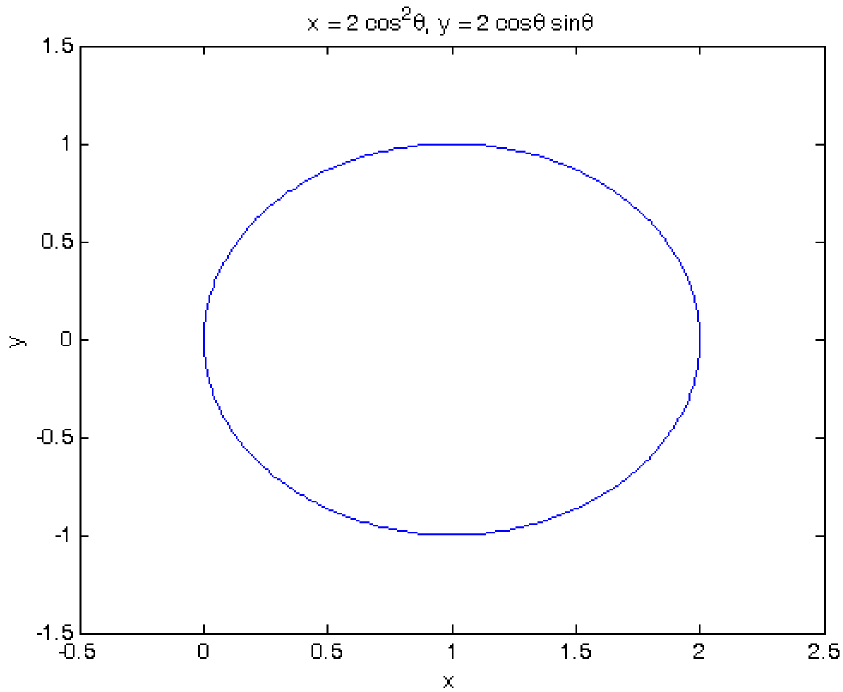

Figure 1: Curve defined by $x = \cos^2\theta$, $y = 2\cos\theta\sin\theta$

Solution The length L is given by the integral

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{(\sin t + t\cos t)^2 + (\cos t - t\sin t)^2}\,dt \\
&= \int_0^{2\pi} \sqrt{\sin^2 t + 2\sin t\cos t + t^2\cos^2 t + \cos^2 t - 2\sin t\cos t + t^2\sin^2 t}\,dt \\
&= \int_0^{2\pi} \sqrt{(\sin^2 t + \cos^2 t) + t^2(\cos^2 t + \sin^2 t)}\,dt \\
&= \int_0^{2\pi} \sqrt{1 + t^2}\,dt \\
&= \int_0^{\tan^{-1}(2\pi)} \sqrt{1 + \tan^2\theta}\sec^2\theta\,d\theta \\
&= \int_0^{\tan^{-1}(2\pi)} \sec^3\theta\,d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right]\Big|_0^{\tan^{-1}(2\pi)} \\
&= \frac{1}{2}\left[\sqrt{1+\tan^2\theta}\tan\theta + \ln|\sqrt{1+\tan^2\theta}+\tan\theta|\right]\Big|_0^{\tan^{-1}(2\pi)} \\
&= \frac{1}{2}\left[\sqrt{1+(2\pi)^2}(2\pi) + \ln|\sqrt{1+(2\pi)^2}+2\pi|\right] \\
&\approx 21.2563.
\end{aligned}
$$

The integral of $\sec^3\theta$ is obtained using integration by parts, with $u = \sec\theta$ and $dv = \sec^2\theta\,d\theta$. The curve is displayed in Figure 2. □

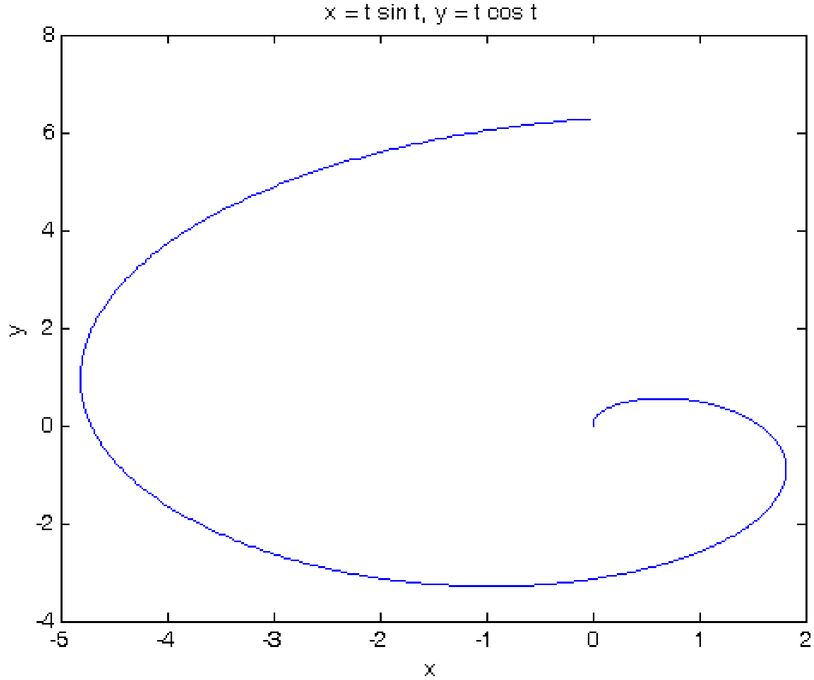

Figure 2: Curve defined by $x = t\sin t$, $y = t\cos t$

Summary

- If a curve is defined by parametric equations $x = g(t)$, $y = h(t)$ for $c \leq t \leq d$, the arc length of the curve is the integral of $\sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{[g'(t)]^2 + [h'(t)]^2}$ from c to d.