2C Intro to real analysis 2020/21

Ex Sheet 4

Solutions and Comments [1] [2]

Q1 *Give an example of a real sequence $(x_n)_{n=1}^{\infty}$ which satisfies $x_1 = 3, x_2 = \pi, x_3 = e, x_4 = -4$.*

This is an almost silly question. It's point is to remind you that we don't need to define a sequence by means of a formula and instead all you need to do is specify every term in the sequence. I would give the answer

> Define
> $$x_n = \begin{cases} 3, & n = 1 \\ \pi, & n = 2 \\ e, & n = 3 \\ -4, & n = 4 \\ 0, & n \geq 5. \end{cases}$$
> Then $(x_n)_{n=1}^{\infty}$ satisfies the requirements.

Note that to define a sequence you must define x_n for every $n \in \mathbb{N}$, it is not enough just to define x_1, x_2, x_3, x_4.

Q2 *Show[3] directly from the definition that*

$$\lim_{n\to\infty} \frac{n-2}{n+3} = 1, \quad \lim_{n\to\infty} \frac{2n^3 - 3n + 5}{n^3 - 2n^2 + 2} = 2.$$

First let me emphasise the importance of the words "**directly from the definition**". To compute the first limit it's far easier to use properties of limits[4] as follows. For $n \in \mathbb{N}$, we have

$$\frac{n-2}{n+3} = \frac{1 - \frac{2}{n}}{1 + \frac{3}{n}} \to \frac{1 - 2 \times 0}{1 + 3 \times 0} = 1,$$

as $n \to \infty$. However, in order for you to really understand the definition of convergent sequences, and to appreciate the usefulness of general results on properties of limits, it's important to do some exercises which use the definition directly[5].

To answer the question, we must recall the definition of $x_n \to L$ as $n \to \infty$. Namely:

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall (n \in \mathbb{N} \text{ with } n \geq n_0), |x_n - L| < \varepsilon.$$

If you didn't know this definition, do make sure you learn it.

The first part of the definition tells us how to start our answer. We must prove that a certain statement holds for all $\varepsilon > 0$, so we start by fixing an arbitrary value of ε.

[1] If you've not seriously tried these exercises, please don't look at these solutions and comments, until you have. You'll get the most benefit from reading these comments, when you've first thought hard about them yourself, even if you get really stuck — don't just try for a few minutes and then look at the solutions to work out how to proceed, you don't learn anywhere near as much that way.

[2] Note that I deliberately do not include formal answers for all questions.

[3] You may use the polynomial estimation lemma if it helps, particularly for the second part.

[4] We will see examples of this type during lectures.

[5] If you gave the answer above in the exam when the question said "directly from the definition" it would be worth no marks!

Let $\varepsilon > 0$ be arbitrary.

Now we need to find an $n_0 \in \mathbb{N}$ such that for values of n larger than n_0, the sequence lies within the ε-error band around L. We do this by simplifying the difference between the sequence and L, and estimating the resulting quantity to make it easier to work with.

For $n \in \mathbb{N}$, we have

$$\left|\frac{n-2}{n+3} - 1\right| = \left|\frac{-5}{n+3}\right| = \frac{5}{n+3} < \frac{5}{n} < \varepsilon,$$

provided $n > 5/\varepsilon$. Take $n_0 \in \mathbb{N}$ with $n_0 > 5/\varepsilon$, so that if $n \geq n_0$ we have

$$\left|\frac{n-2}{n+3} - 1\right| < \varepsilon.$$

Therefore $\lim_{n\to\infty} \frac{n-2}{n+3} = 1$.

Note that the step $\frac{5}{n+3} < \frac{5}{n}$ is not necessary. You could say that we have

$$\left|\frac{n-2}{n+3} - 1\right| = \left|\frac{-5}{n+3}\right| = \frac{5}{n+3},$$

and so we take $n_0 \in \mathbb{N}$ with $n_0 > \frac{5}{\varepsilon} - 3$. In this way we conclude that $n \geq n_0$ implies $\frac{5}{n+3} < \varepsilon$. Remember that there will be many correct expressions for n_0 in terms of ε — what matters is you find an expression for n_0 which works, not that you get the same one as me.

Now for the second limit. The additional ingredient you might want to use here is the polynomial estimation lemma 1.9, to systematically estimate the resulting fraction.

Let $\varepsilon > 0$ be arbitrary. For $n \in \mathbb{N}$, we have

$$\left|\frac{2n^3 - 3n + 5}{n^3 - 2n^2 + 2} - 2\right| = \left|\frac{4n^2 - 3n + 1}{n^3 - 2n^2 + 2}\right| = \frac{4n^2 - 3n + 1}{n^3 - 2n^2 + 2}.$$

Here we are using in the last equality that both the numerator and denominator in the expression are positive for all $n \in \mathbb{N}$. By Lemma 1.9, there exists $n_1, n_2 \in \mathbb{N}$ with

$$\begin{aligned} n \geq n_1 &\implies 2n^2 \leq 4n^2 - 3n + 1 \leq 6n^2; \\ n \geq n_2 &\implies \frac{1}{2}n^3 \leq n^3 - 2n^2 + 2 \leq \frac{3}{2}n^2. \end{aligned}$$

Thus, for $n \geq \max(n_1, n_2)$ we have

$$\left|\frac{2n^3 - 3n + 5}{n^3 - 2n^2 + 2} - 2\right| = \frac{4n^2 - 3n + 1}{n^3 - 2n^2 + 2} \leq \frac{6n^2}{n^3/2} = \frac{12}{n}.$$

Choose $n_0 \in \mathbb{N}$ such that $n_0 > \max(n_1, n_2, 12/\varepsilon)$ so that for $n \geq n_0$,

$$\left|\frac{2n^3 - 3n + 5}{n^3 - 2n^2 + 2} - 2\right| < \varepsilon.$$

Hence
$$\lim_{n\to\infty} \frac{2n^3 - 3n + 5}{n^3 - 2n^2 + 2} = 2.$$

The use of Lemma 1.9 in the answer above isn't necessary. We can alternatively note that
$$\frac{4n^2 - 3n + 1}{n^3 - 2n^2 + 2} \leq \frac{4n^2}{n^3 - n^3/2} = \frac{8}{n}.$$
for $n \geq 4$. To see this, use that $3n \geq 1$ for all $n \in \mathbb{N}$ so that $4n^2 - 3n + 1 \leq 4n^2$. Also $2n^2 \leq n^3/2$ for $n \geq 4$, so that $n^3 - 2n^2 + 2 \geq n^3 - n^3/2$ for these n. We can then take $n_0 \in \mathbb{N}$ with $n_0 > \max(4, \varepsilon/8)$ so that for $n \geq n_0$, we have
$$\left| \frac{2n^3 - 3n + 5}{n^3 - 2n^2 + 2} - 2 \right| < \varepsilon.$$

Q3 *Show*[6] *directly from the definition that*
$$\lim_{n\to\infty} \frac{6n^5 + n^3 - 4n}{2n^5 - 2n^4 + 1} = 3, \quad \lim_{n\to\infty} \frac{3n + (-1)^n}{n} = 3.$$

[6] You may use the polynomial estimation lemma if it helps, particularly for the first part.

The first part of this question appears on the second feedback exercise; so I'll not include a solution here... For the second part:

Let $\varepsilon > 0$ be arbitrary. For $n \in \mathbb{N}$, we have
$$\left| \frac{3n + (-1)^n}{n} - 3 \right| = \left| \frac{(-1)^n}{n} \right| = \frac{1}{n}.$$

Did you notice that $|(-1)^n| = 1$ for all n? A standard mistake here is to forget about the modulus and end up with an expression involving $(-1)^n$ in your choice of n_0. This can't be right: n_0 is allowed to depend on ε but not on n. Now we continue:

So take $n_0 \in \mathbb{N}$ with $n_0 > 1/\varepsilon$. For $n \geq n_0$, we have
$$\left| \frac{3n + (-1)^n}{n} - 3 \right| < \varepsilon,$$
so
$$\lim_{n\to\infty} \frac{3n + (-1)^n}{n} = 3.$$

Q4 *Show that the sequences* $(x_n)_{n=1}^{\infty}$ *and* $(y_n)_{n=1}^{\infty}$ *given by*
$$x_n = \frac{1 + (-1)^n}{2}, \quad y_n = \sin\left(\frac{n\pi}{2}\right)$$
do not converge to any limit.

Originally, I had intended this question to also include the "directly from the definition" to see if you can modify the ideas in the proof that $(-1)^n$ does not converge. However, since I finally decided not to include that in the question there are various possible approaches. For example, you could observe that $(-1)^n = 2x_n - 1$. Therefore, if there exists $L \in \mathbb{R}$ with $x_n \to L$, we would have $(-1)^n \to 2L - 1$ by properties of limits, contradicting the result from lectures that $(-1)^n$ does not converge. Hence $(x_n)_{n=1}^{\infty}$ does not converge.

Let me give an answer directly from the definition[7]. Note that the sequence $(x_n)_{n=1}^{\infty}$ alternates between the values 0 and 1 (according to whether n is odd or even). We choose a value of ε so that the error bands of width ε around 0 and 1 do not overlap. Taking $\varepsilon = 1/2$ works, as does any smaller value of $\varepsilon > 0$.

[7] Later in the chapter we will see a method using subsequences which gives a more efficient way of answering this type of question.

Note that in the example in lectures, namely that $((-1)^n)_{n=1}^{\infty}$ does not converge, we took $\varepsilon = 1$. Here the two values the sequence takes are $+1$ and -1, and $\varepsilon = 1$ is chosen to be exactly one half of the distance between these values.

Suppose there exists $L \in \mathbb{R}$ such that $x_n \to L$ as $n \to \infty$. Take $\varepsilon = 1/2 > 0$ in the definition of convergence to find $n_0 \in \mathbb{N}$ such that for $n \in \mathbb{N}$,

$$n \geq n_0 \implies |x_n - L| < 1/2.$$

Take $n \in \mathbb{N}$ with $n \geq n_0$ and n odd to see that $|0 - L| < 1/2$. Similarly, take $n \in \mathbb{N}$ with $n \geq n_0$ and n even to see that $|1 - L| < 1/2$. Then, using the triangle inequality, we have

$$\begin{aligned} 1 = |0-1| = |0 - L + L - 1| &\leq |0 - L| + |L - 1| \\ &= |0 - L| + |1 - L| < \frac{1}{2} + \frac{1}{2} = 1, \end{aligned}$$

a contradiction. Hence $(x_n)_{n=1}^{\infty}$ does not converge.

For the second sequence (y_n), note that y_n behaves as follows,

$$1, 0, -1, 0, 1, 0, -1, 0, 1, \ldots.$$

You should now be able to write down a proof that (y_n) does not converge in a similar way to the previous example. For instance, you could take $\varepsilon = 1/2$ (halfway between 1 and 0) and use values of n of the form $4k + 1$ (where $y_n = 1$) and even values of n (where $y_n = 0$). There are other ways to proceed. For example, using $\varepsilon = 1$ and values of n of the form $4k + 1$ and $4k + 3$ for $k \in \mathbb{N}$ works as well.

Q5 *Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be real sequences and $L, M \in \mathbb{R}$ such that*

$$\lim_{n\to\infty} x_n = L, \quad \text{and} \quad \lim_{n\to\infty} y_n = M.$$

Prove[8] directly from the definition that

$$\lim_{n\to\infty} (2x_n - 3y_n) = 2L - 3M.$$

[8] As a hint, you may want to write

$$|(2x_n - 3y_n) - (2L - 3M)| = |2(x_n - L) - 3(y_n - M)|$$

and apply the triangle inequality.

We must show

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, (n \geq n_0 \implies |2x_n - 3y_n - (2L - 3M)| < \varepsilon),$$

so the first line of our proof should be automatic.

Let $\varepsilon > 0$ be arbitrary.

We now want to find n_0 such that

$$n \geq n_0 \implies |2x_n - 3y_n - (2L - 3M)| < \varepsilon.$$

To do this, we start by looking at the quantity $|2x_n - 3y_n - (2L - 3M)|$, remembering that our hypothesis $x_n \to L$ and $y_n \to M$ allow us to control the quantities $|x_n - L|$ and $|y_n - M|$. This leads us to the key application of the triangle inequality. Namely, for $n \in \mathbb{N}$, we have

$$\begin{aligned} |2x_n - 3y_n - (2L - 3M)| &= |2(x_n - L) - 3(y_n - M)| \\ &\leq 2|x_n - L| + 3|y_n - M|. \end{aligned}$$

We can ensure that the right hand side of this inequality is at most ε by making n large enough. For instance, if we choose n_0 such that $|x_n - L| < \frac{\varepsilon}{5}$ and $|y_n - M| < \frac{\varepsilon}{5}$ for $n \geq n_0$ this will be the case. Let us now formally write down the rest of the answer:

Then $\frac{\varepsilon}{5} > 0$. Since $x_n \to L$ and $y_n \to M$ as $n \to \infty$, there exists $n_1, n_2 \in \mathbb{N}$, such that

$$\begin{aligned} \forall n \in \mathbb{N}, (n \geq n_1 &\implies |x_n - L| < \frac{\varepsilon}{5} \\ \forall n \in \mathbb{N}, (n \geq n_2 &\implies |y_n - M| < \frac{\varepsilon}{5}. \end{aligned}$$

Let $n_0 = \max(n_1, n_2)$. For $n \in \mathbb{N}$, with $n \geq n_0$, we have $n \geq n_1$ and $n \geq n_2$. Therefore

$$\begin{aligned} |2x_n - 3y_n - (2L - 3M)| &= |2(x_n - L) - 3(y_n - M)| \\ &\leq 2|x_n - L| + 3|y_n - M| < 2\frac{\varepsilon}{5} + 3\frac{\varepsilon}{5} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this means $2x_n - 3y_n \to 2L - 3M$ as $n \to \infty$.

Q6 *Show directly from the definition that* $\frac{1}{\sqrt{n}} \to 0$ *as* $n \to \infty$.

Let $\varepsilon > 0$ be arbitrary. For $n \in \mathbb{N}$, we have

$$\left| \frac{1}{\sqrt{n}} - 0 \right| = \frac{1}{\sqrt{n}} < \varepsilon \iff n > \frac{1}{\varepsilon^2}.$$

Therefore we choose $n_0 \in \mathbb{N}$ with $n_0 > \frac{1}{\varepsilon^2}$. For $n \geq n_0$, we have

$$\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon,$$

and so $1/\sqrt{n} \to 0$ as $n \to \infty$.

This question is very similar to question 1, and indeed all questions where we establish convergence directly from the definition. We fix an arbitrary value of $\varepsilon > 0$ and examine the difference $|1/\sqrt{n} - 0|$ of a typical element of the sequence from the proposed limit. We aim to show that this quantity is at most ε, provided we make n sufficiently large. In this case, this amounts to rearranging $\frac{1}{\sqrt{n}} < \varepsilon$ to make n the subject of the inequality.

Q7 *Let $(x_n)_{n=1}^{\infty}$ be a real sequence and let $L \in \mathbb{R}$. For each of the three properties below, decide whether the property implies that $x_n \to L$ (either prove that it does, or give an example of a sequence with the property which does not converge to L), and decide whether $x_n \to L$ implies that the property holds (again, either prove that the implication is true, or give an example of a sequence $(x_n)_{n=1}^{\infty}$ and L with $x_n \to L$ but for which the property fails to hold)*[9].

$$\forall \varepsilon > 0, \forall n_0 \in \mathbb{N}, \exists n \in \mathbb{N} \text{ s.t. } (n \geq n_0 \text{ and } |x_n - L| < \varepsilon). \tag{1}$$

$$\exists \varepsilon > 0 \text{ s.t. } \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, (n \geq n_0 \implies |x_n - L| < \varepsilon). \tag{2}$$

$$\exists n_0 \in \mathbb{N} \text{ s.t. } \forall \varepsilon > 0, \forall n \in \mathbb{N}, (n \geq n_0 \implies |x_n - L| < \varepsilon). \tag{3}$$

[9] The point of this question is that the definition of convergence is designed to capture our intuitive definition of which sequences should converge and which should not. I often see this definition misstated in exams (sometimes in one of these three forms), so it's useful to think about what happens when we change the definition a little bit.

Remember that the definition of $x_n \to L$ as $n \to \infty$ is

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, (n \geq n_0 \implies |x_n - L| < \varepsilon).$$

Let's try and understand what each of the quantified statements means.

The difference between condition (1) and the definition of convergence is that the types of quantifiers on the n_0 and n have changed. Suppose condition (1) holds for $(x_n)_{n=1}^{\infty}$ and L. Then for any possible ε, and any $n_0 \in \mathbb{N}$, we can find *at least one* $n \in \mathbb{N}$ with $n \geq n_0$, such that $|x_n - L| < \varepsilon$. Remember that this last condition means that the distance between x_n and L is at most ε. By repeatedly changing n_0 and ε, we see that there will be infinitely many values of n for which x_n gets close to L as n gets large. However, since for each ε and n_0, the condition only gives that there exists some $n \geq n_0$ with $|x_n - L| < \varepsilon$ (as opposed to all n having this condition), it need *not* be the case that all of the values of x_n get close to L as n gets large: therefore we shouldn't expect condition (1) to imply convergence.

I now turn these thoughts into an example, by thinking of a sequence where infinitely many of the values get close to a number L as n gets large, but the sequence doesn't converge: for example I could arrange for $|x_n - L|$ to be small when n is even, but not when n is odd.

> Condition (1) does not imply that $x_n \to L$. For example, define $x_n = (-1)^n$ and $L = 1$. We know that $(x_n)_{n=1}^{\infty}$ does not converge from lectures, but condition (1) holds. Indeed, given $\varepsilon > 0$ and $n_0 \in \mathbb{N}$, choose $n \in \mathbb{N}$ with $n \geq n_0$ and n even, so that $|x_n - L| < \varepsilon$, so condition (1) holds.

In the example above the sequence $(x_n)_{n=1}^{\infty}$ was bounded. Can you see how to modify this example so that (1) still holds, but $(x_n)_{n=1}^{\infty}$ is unbounded?

The intuitive discussion above also helps to see that if $x_n \to L$ as $n \to \infty$, then condition (1) holds. Having reached this conclusion, I then take a convergent sequence and give a formal proof that condition (1) holds, following our methods for proving quantified statements[10].

[10] When we define subsequences later in the chapter, see if you can persuade yourself that (1) is equivalent to "$(x_n)_{n=1}^{\infty}$ has a subsequence which converges to L."

> $x_n \to L$ implies condition (1). To see this, suppose $x_n \to L$ and let $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ be arbitrary. Then, by the definition of convergence, there is some $n_1 \in \mathbb{N}$ (note that I use a different variable name as n_0 has been fixed) so that
>
> $$\forall n \in \mathbb{N}, (n \geq n_1 \implies |x_n - L| < \varepsilon).$$
>
> Take $n = \max(n_1, n_0)$. Then $|x_n - L| < \varepsilon$ (as $n \geq n_1$) and $n \geq n_0$, so this shows that condition (1) holds.

In condition (2), the "$\forall \varepsilon > 0$" in the definition of convergence has been replaced by "$\exists \varepsilon > 0$". Therefore, if $x_n \to L$, then condition (2) holds[11]. However the converse implication is not true: indeed condition (2) says that x_n eventually lies within distance ε of L, for some fixed quantity ε. So to find an example of a sequence satisfying condition (2), which doesn't converge, we should take a non-convergent sequence all of whose elements lie within a fixed distance of some number L: again $x_n = (-1)^n$ and $L = 1$ will do the job[12]. In fact, condition (2) holding (for some L) is equivalent to $(x_n)_{n=1}^{\infty}$ being bounded[13].

[11] The definition of convergence says that "$\exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, (n \geq n_0 \implies |x_n - L| < \varepsilon)$" is true for every value of $\varepsilon > 0$, so certainly this is true for *some* value of $\varepsilon > 0$, which is condition (2).

[12] This time I leave it to you to formally verify that this sequence satisfies condition (2).

[13] Can you prove this?

Finally for condition (3), which has been obtained by interchanging the first two quantifiers in the definition of convergence. Remember in the definition of convergence, the value of n_0 which is required is likely to depend on ε: when we verify the definition, we expect to provide an expression for n_0 in terms of ε. In condition (3), a single value of n_0 must work for all values of ε. For this reason if (3) holds, then certainly[14] $x_n \to L$ as $n \to \infty$. To find an example of a convergent sequence $(x_n)_{n=1}^{\infty}$ which doesn't satisfy (3), we should look for an example where the n_0 needs to depend on ε. We don't have to look very far: $x_n = \frac{1}{n}$ will work[15].

[14] Give a formal proof!

[15] In fact, condition (3) is equivalent to "x_n is eventually equal to L", that is, to the statement "$\exists n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_0 \implies x_n = L$." Can you prove this?

Let $x_n = \frac{1}{n}$ and $L = 0$, so that $x_n \to L$ as $n \to \infty$. For this example condition (3) does not hold. To see this, let $n_0 \in \mathbb{N}$ be arbitrary. Take $\varepsilon > 0$ with $\varepsilon < \frac{1}{n_0}$ (for example $\varepsilon = \frac{1}{2n_0}$). Then taking $n = n_0$, we have $n \geq n_0$ and $|x_n - L| = \frac{1}{n_0} > \varepsilon$. This shows that condition (3) does not hold.

Notice that in the example above, I want to prove that condition (3) fails to hold, so I ended up proving the formal negation of (3), namely

$$\forall n_0 \in \mathbb{N}, \exists \varepsilon > 0 \text{ s.t. } \exists n \in \mathbb{N} \text{ s.t. } (n \geq n_0 \text{ and } |x_n - L| \geq \varepsilon).$$