2C Intro to real analysis 2020/21 — Ex Sheet 10

Solutions and Comments [1] [2]

Q1 *Let f, g be real functions which are continuous at c and let $\lambda \in \mathbb{R}$.*

a) *Prove that λf (which is defined by $(\lambda f)(x) = \lambda f(x)$ for $x \in \mathrm{dom}(f)$) is continuous at c.*

b) *Prove that fg (which is defined by $(fg)(x) = f(x)g(x)$ for $x \in \mathrm{dom}(fg) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$) is continuous at c.*

In this question it is really useful to work with the sequential characterisation of continuity. Let's remember that f is continuous at $c \in \mathrm{dom}(f)$ if for every sequence $(x_n)_{n=1}^{\infty}$ in $\mathrm{dom}(f)$ with $x_n \to c$ we have $f(x_n) \to f(c)$ as $n \to \infty$.

I'll start with a discussion of b), where we aim to verify this condition for the function fg.

> Consider an arbitrary sequence $(x_n)_{n=1}^{\infty}$ in $\mathrm{dom}(fg) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$ such that $x_n \to c$. Then
>
> $$(fg)(x_n) = f(x_n)g(x_n) \to f(x)g(x) = (fg)(x)$$
>
> by algebraic properties of sequence limits. Therefore, by the sequential characterisation of continuity, the function fg is continuous at c.

Part a) can be proved in a similar way using the sequential characterisation of continuity. For variety, let me prove this part directly from the definition.

> Suppose first that $\lambda = 0$. Let $\varepsilon > 0$ be arbitrary, and take $\delta = 1$. Then
>
> $$|x - c| < \delta \implies |(\lambda f)(x) - (\lambda f)(c)| = |0 - 0| = 0 < \varepsilon.$$
>
> Thus λf is continuous at c.
>
> Now suppose $\lambda \neq 0$. Let $\varepsilon > 0$ be arbitrary, and use continuity of f at c to find $\delta > 0$ such that
>
> $$|x - c| < \delta \implies |f(x) - f(c)| < \frac{\varepsilon}{|\lambda|}.$$
>
> Then
>
> $$|x - c| < \delta \implies |(\lambda f)(x) - (\lambda f)(c)| = |\lambda||f(x) - f(c)| < \varepsilon,$$
>
> so (λf) is continuous at c.

[1] If you've not seriously tried these exercises, please don't look at these solutions and comments, until you have. You'll get the most benefit from reading these comments, when you've first thought hard about them yourself, even if you get really stuck — don't just try for a few minutes and then look at the solutions to work out how to proceed, you don't learn anywhere near as much that way.

[2] Note that I deliberately do not include formal answers for all questions.

Notice that in the case $\lambda = 0$ I could actually have picked any positive value for δ instead of 1.

Q2 *Let $f : [0,1] \to [0,1]$ be continuous. Use the intermediate value theorem to prove that:*

a) *there exists $c \in [0,1]$ with $f(c) = c^3$;*

b) *there exists $c \in [0,1]$ with $f(c) = \sqrt{1-c^2}$.*

In these questions, we want to identify a new function to which we can apply the intermediate value theorem. Often a good strategy is to rearrange the equation: instead of solving $f(c) = c^3$, we solve $f(c) - c^3 = 0$. This leads us to define the function g in the first answer below. Note that I explicitly observe that g is continuous (a hypothesis of the intermediate value theorem) and that $g(0) \geq 0$ and $g(1) \leq 0$ so that the intermediate value theorem applies. This shows the person reading my work that I understand the statement of the intermediate value theorem.

> *a*) Define $g : [0,1] \to [0,1]$ by $g(x) = f(x) - x^3$. Since f is continuous, and the function $x \mapsto x^3$ is continuous, it follows that g is continuous, as it is the difference of two continuous functions. Since $f(0), f(1) \in [0,1]$, we have
>
> $$g(0) = f(0) - 0 \geq 0 - 0 = 0, \text{ and } g(1) = f(1) - 1 \leq 1 - 1 = 0.$$
>
> Therefore, by the intermediate value theorem, there exists $c \in [0,1]$ with $g(c) = 0$. This means $f(c) = c^3$.

Part *b*) is similar. Define $h(x) = f(x) - \sqrt{1-x^2}$. This time, $h(1) = f(1) - 0 \geq 0$ and $h(0) = f(0) - \sqrt{1} \leq 1 - 1 = 0$. I leave it to you to write out the details.

Q3 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(2) + f(5) = 0$ and $f(3) + f(4) = 6$. Show*[3] *that there exists $u, v \in \mathbb{R}$ with $f(u) + f(v) = 1$.*

[3] Hint: Consider the function $g(x) = f(x) + f(7-x)$.

The hint does it all for us. We need to find a solution $x \in \mathbb{R}$ to $g(x) = 1$.

> Define $g(x) = f(x) + f(7-x)$, so that g is the sum of two continuous functions and hence continuous. Since $g(2) = 0$ and $g(3) = 6$, the intermediate value theorem applies and there exists $u \in [0,3]$ with $g(u) = 1$. Define $v = 7 - u$ so that $f(u) + f(v) = 1$.

Q4 *Let $f : (0,1) \to (0,1)$ be a continuous function satisfying $f(x) < x$ for all $x \in (0,1)$. Fix $x_1 \in (0,1)$ and define a sequence $(x_n)_{n=1}^{\infty}$ recursively by $x_{n+1} = f(x_n)$ for $n \in \mathbb{N}$.*

a) *Use the monotone convergence theorem to show that $(x_n)_{n=1}^{\infty}$ converges to some limit L.*

b) What property of limits ensures that $L \in [0, x_1]$?

c) Use the sequential characterisation of continuity to prove[4] *that $L = 0$.*

[4] Hint: Work by contradiction and suppose $L > 0$ and use the recursion rule.

> *a*) We have $x_{n+1} = f(x_n) < x_n$ for all n, so the sequence $(x_n)_{n=1}^{\infty}$ is strictly decreasing. Since each $x_n \in (0, 1)$, the sequence is bounded below by 0 so converges by the monotone convergence theorem. Write $L = \lim_{n\to\infty} x_n$.
> *b*) Properties of limits and order ensure that $0 \leq L \leq x_1$ as each x_n has $0 < x_n \leq x_1$.
> *c*) Suppose $L > 0$. Then $x_n \to L$, so by the sequential characterisation of continuity $x_{n+1} = f(x_n) \to f(L) < L$. But also $x_{n+1} \to L$, so $L < L$, a contradiction. Therefore $L = 0$.

Q5 *Let $f : \mathbb{R} \to \mathbb{R}$ satisfy*

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$. Show that

a) $f(0) = 0$.

b) $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

c) $f(x - y) = f(x) - f(y)$ for all $x, y \in \mathbb{R}$.

d) $f(r) = rf(1)$, for $r \in \mathbb{Q}$.

Now suppose f is continuous at 0. Use the sequential characterisation of continuity to show:

e) f is continuous on $\mathbb{R}$.

f) $f(x) = xf(1)$ for all $x \in \mathbb{R}$. You may assume that for any $x \in \mathbb{R}$, there is a sequence $(r_n)_{n=1}^{\infty}$ of rational numbers with $r_n \to x$ as $n \to \infty$.

The first few steps involve applying the *additivity condition* $f(x + y) = f(x) + f(y)$ repeatedly:

> *a*) Take $x = y = 0$, then we have $f(0 + 0) = f(0) + f(0)$, so $2f(0) = f(0)$, and hence $f(0) = 0$.
> *b*) Take $y = -x$, then we have $0 = f(0) = f(x) + f(-x)$. Rearranging gives $f(-x) = -f(x)$.
> *c*) Using the previous step and the additivity condition, we obtain $f(x - y) = f(x + (-y)) = f(x) + f(-y) = f(x) - f(y)$.

The next step is to handle the rational case.

d) Fix $m \in \mathbb{N}$. We claim that $f(\frac{n}{m}) = nf(\frac{1}{m})$ for all $n \in \mathbb{N}$. Indeed, this holds for $n = 1$, and if $f(\frac{n}{m}) = nf(\frac{1}{m})$ for some $n \in \mathbb{N}$, then $f(\frac{n+1}{m}) = f(\frac{n}{m}) + f(\frac{1}{m}) = (n+1)f(\frac{1}{m})$ by the additivity condition, so the assertion follows by induction.

Now, taking $n = m$ in the previous calculation shows $f(1) = mf(\frac{1}{m})$, so that $f(\frac{1}{m}) = \frac{1}{m}f(1)$. Combining this with the previous paragraph yields $f(\frac{n}{m}) = \frac{n}{m}f(1)$ for $m, n \in \mathbb{N}$.

Finally, $f(r) = rf(1)$ holds for any $r \in \mathbb{Q}$, as we have proved this above when $r > 0$, it is obviously true for $r = 0$, and the case of $r < 0$ follows from the $r > 0$ case and *b*).

To prove continuity on $\mathbb{R}$, we use the sequential characterisation and the additivity condition.

e) Suppose that $(x_n)_{n=1}^{\infty}$ is a real sequence with $x_n \to x$ as $n \to \infty$. Then $x_n - x \to 0$, so by continuity of f at 0, we have $f(x_n - x) \to f(0) = 0$. Using the additivity condition and algebraic properties of limits, this gives $f(x_n) - f(x) \to 0$, so that $f(x_n) \to f(x)$ as $n \to \infty$. Therefore f is continuous on $\mathbb{R}$.

f) Let $x \in \mathbb{R}$, and let $(r_n)_{n=1}^{\infty}$ be a sequence of rational numbers with $r_n \to x$ as $n \to \infty$. By the sequential characterisation of continuity, $f(r_n) \to f(x)$. Since $f(r_n) = r_n f(1)$ by *d*), and $r_n \to x$, we have $xf(1) = f(x)$ by uniqueness of limits.

Q6 *We have shown in lectures that if $[a, b]$ is a closed bounded interval and $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded. In this question I ask you to check that all these conditions are needed. Provide examples of:*

a) a continuous function $f : (0, 1) \to \mathbb{R}$ which is not bounded.

b) a continuous function $f : [1, \infty) \to \mathbb{R}$ which is not bounded.

c) a function $f : [0, 1] \to \mathbb{R}$ which is not bounded.

Here we need to think of some unbounded functions. There's many choices you can make, but you need to think of some functions which blow up on the specified domains, which leads to my choices in *a*) and *b*).

a) Define $f : (0, 1) \to \mathbb{R}$ by $f(x) = \frac{1}{x}$. Then f is continuous since it is a rational function, and it is unbounded because $f(\frac{1}{n}) = n$ for each $n \in \mathbb{N}$, and the natural numbers are unbounded.

b) Define $f : [1, \infty) \to \mathbb{R}$ by $f(x) = x$. Then f is continuous and unbounded.

For *c*), we can't write down a continuous function, as a continuous function on $[0, 1]$ is necessarily bounded. Here's one way to proceed.

Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} n, & x = \frac{1}{n} \text{ for } n \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

Q7 *For the continuous functions below, determine whether or not they are uniformly continuous*[5].

a) $f : (0,1) \to \mathbb{R}$ *given by* $f(x) = x^2$.

b) $g : (0,1) \to \mathbb{R}$ *given by* $g(x) = \frac{1}{x}$.

c) $h : [1,\infty) \to \mathbb{R}$ *given by* $h(x) = \frac{1}{x}$.

The first example is similar to an example in the lecture notes. However, since the domain is restricted to the interval $(0,1)$, the function turns out to be uniformly continuous[6].

The function f is uniformly continuous. Let $\varepsilon > 0$ be arbitrary and take $\delta = \frac{\varepsilon}{2}$. Then for $x, c \in (0,1)$ with $|x - c| < \delta$ we have

$$|x^2 - c^2| = |x - c||x + c| < 2|x - c| < \varepsilon,$$

as $|x + c| \leq |x| + |c| < 2$. Therefore f is uniformly continuous.

For the functions g and h, a picture suggests that on $(1,\infty)$ the function $\frac{1}{x}$ is uniformly continuous, while on $(0,1)$ it isn't. Let's prove these claims formally.

The function g is not uniformly continuous. Let $\varepsilon = 1$, and let $\delta > 0$ be arbitrary. For $n \in \mathbb{N}$, consider $c_n = \frac{1}{n}$ and $x_n = \frac{2}{n}$. Then

$$|g(c_n) - g(x_n)| = |n - \frac{n}{2}| = \frac{n}{2} \geq \varepsilon$$

for $n \geq 2$. Since $|x_n - c_n| = \frac{1}{n}$, taking n with $n > \max(2, \frac{1}{\delta})$, we have $|x_n - c_n| < \delta$ and $|g(x_n) - g(c_n)| \geq \varepsilon$.

The function h is uniformly continuous. Let $\varepsilon > 0$ be arbitrary. For $x, c > 1$, we have

$$\left|\frac{1}{x} - \frac{1}{c}\right| = \frac{|x - c|}{|x||c|} < |x - c|,$$

as $|x| > 1$ and $|c| > 1$. We take $\delta = \varepsilon$, so that $|x - c| < \delta \implies |h(x) - h(c)| < \varepsilon$. It follows that h is uniformly continuous.

[5] I would encourage you to draw a picture and try and decide whether the δ in the definition of continuity needs to depend on c or not, before trying to justify your answers.

[6] You could use that f extends to a continuous function on the closed and bounded interval $[0,1]$, so is uniformly continuous, by the theorem that continuous functions on bounded closed intervals are uniformly continuous.