Algorithmic Foundations 2 - Tutorial Sheet 3

Functions, Sequences, Summation, Integers and Division

Functions

1. Determine which of the following are functions (with the given domain and codomain).
   (a) $f_1 : \mathbb{N} \to \mathbb{N}$ where $f_1(n) = \sqrt{n}$

   > **Solution:** `false` - the codomain does not match the function, e.g. $f_1(2) = \sqrt{2} \notin \mathbb{N}$

   (b) $f_2 : \mathbb{R} \to \mathbb{R}$ where $f_2(x) = \sqrt{x}$

   > **Solution:** `false` - the domain does not match the function, e.g. $f_2(-2) = \sqrt{-2} \notin \mathbb{R}$

   (c) $f_3 : \mathbb{N} \to \mathbb{N}$ where $f_3(n)$ equals any integer greater than n

   > **Solution:** `false` - there is no unique value for any given element of the domain

   (d) $f_4 : \mathbb{Z} \to \mathbb{R}$ where $f_4(x) = 1/(x-5)$

   > **Solution:** `false` - the codomain does not match the function, e.g. $f_4(5) = 1/0 \notin \mathbb{R}$

   (e) $f_5 : \mathbb{R} \to \mathbb{R}$ where $f_5(x) = 1/(x^2-5)$

   > **Solution:** `false` - the codomain does not match the function, e.g. $f_5(\sqrt{5}) = 1/0 \notin \mathbb{R}$

   (f) $f_6 : \mathbb{Z} \to \mathbb{Z}$ where $f_6(x) = 1/(x^2-5)$

   > **Solution:** `false` - the codomain does not match the function, e.g. $f_6(1) = -1/4 \notin \mathbb{Z}$

   (g) $f_7 : \mathbb{R} \to \mathbb{R}$ where $f_7(x) = \begin{cases} x+2 & \text{if } x \geq 0 \\ x-1 & \text{if } x \leq 4 \end{cases}$

   > **Solution:** `false` - there is no unique value when $0 \leq x \leq 4$

   (h) $f_8 : \mathbb{R} \to \mathbb{R}$ where $f_8(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ x-1 & \text{if } x \geq 4 \end{cases}$

   > **Solution:** `false` - the domain does not match the function as there is no value when $2 < x < 4$

2. (a) Give an example function $g_1 : \mathbb{Z} \to \mathbb{Z}$ that is injective and not surjective.

   > **Solution:** $g_1(x) = 2 \cdot x$ – not surjective since there is no x such that $g(x) = 1$

   (b) Give an example function $g_2 : \mathbb{Z} \to \mathbb{Z}$ that is surjective but not injective.

Solution: $g_2(x) = \lfloor x/2 \rfloor$ – not injective since for example $g_2(2) = g_2(3) = 1$.

(c) Give an example function $g_3 : \mathbb{Z} \to \mathbb{N}$ that is bijective.

Solution:
$$g_3(x) = \begin{cases} -2{\cdot}x & \text{if } x{\leq}0 \\ 2{\cdot}x - 1 & \text{otherwise} \end{cases}$$
This is similar to the mapping from natural numbers to even numbers from the lecture. Essentially we map negative integers to the even natural numbers and positive integers to the odd natural numbers (and 0 to 0).

(d) Give an example function $g_4 : \mathbb{N} \to \mathbb{Z}$ that is bijective.

Solution:
$$g_4(x) = \begin{cases} -x/2 & \text{if } x \text{ is even} \\ (x{+}1)/2 & \text{otherwise} \end{cases}$$
This is the reverse of the previous solution (and is in fact the inverse of g_3 which exists as g_3 is a bijection)

(e) Give an example function $g_5 : \mathbb{Z} \to \mathbb{N}$ that is injective but not surjective.

Solution:
$$g_5(x) = \begin{cases} -2{\cdot}x & \text{if } x{\leq}0 \\ 2{\cdot}x{+}1 & \text{otherwise} \end{cases}$$
Not surjective since, there does not exist x such that $g_5(x) = 1$.

This is similar to the answer to (c), mapping the negative integers to the even natural numbers and positive integers to odd natural numbers. However, to prevent the function from being surjective, we omit 1 from the range (notice '+1' in the second case whereas in (c) we had '-1').

(f) Give an example function $g_6 : \mathbb{N} \to \mathbb{Z}$ that is surjective and not injective.

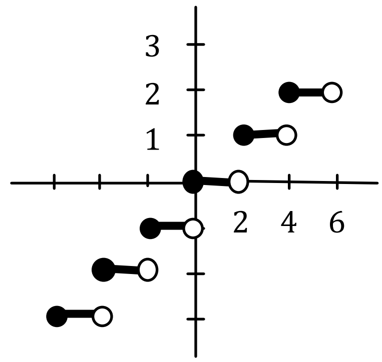
Solution:
$$g_6(x) = \begin{cases} -x/2 & \text{if } x \text{ is even} \\ (x{-}1)/2 & \text{otherwise} \end{cases}$$
Not injective since, for example, $g_6(1){=}g_6(0)$.

This uses the answer to (d), in this case including 0 in the range when restricting to both the even and odd numbers (notice the use of '-1' in the second case while '+1' was used in (d)).

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \lfloor x/2 \rfloor$.

(a) Draw the graph of f.

Solution:



(b) Is f injective?

Solution: No $f(x) = f(y)$ for $x, y \in [n, 2{\cdot}n - 1)$ for any $n \in \mathbb{Z}$

(c) Is f surjective?

Solution: No, for example there does not exist x such that $f(x) = 1/2$.

(d) If $S = \{x \mid x \in \mathbb{R} \wedge 1 \leq x \leq 6\}$ what is $f(S)$?

Solution: $f(S) = \{0, 1, 2, 3\}$

(e) If $T = \{3, 4, 5\}$, what is $f^{-1}(T)$?

Solution: f^{-1} does not exist as the function is not bijective (it is neither injective nor surjective)

4. Suppose $f : A \to B$ and $g : B \to C$ where $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$, $C = \{2, 7, 10\}$ and f and g are defined by:

$$\begin{aligned} f &= \{(1, b), (2, a), (3, a), (4, b)\} \\ g &= \{(a, 10), (b, 7), (c, 2)\}\,. \end{aligned}$$

(a) Find $g \circ f$.

Solution: $\{(1, 7), (2, 10), (3, 10), (4, 7)\}$

(b) Find g^{-1}.

Solution: g is bijective so inverse exists and is given by $\{(2, c), (7, b), (10, a)\}$

5. For the following functions, find the inverse or explain why the function has no inverse.

(a) $h_1 : \mathbb{Z} \to \mathbb{Z}$ where $h_1(x) = x \bmod 10$

Solution: Function is neither injective nor surjective, for example $h_1(10) = h_1(20)$ and there does not exist x such that $h_1(x) = 11$, and hence inverse does not exist.

(b) $h_2 : A \to B$ where $A = \{a, b, c\}$, $B = \{1, 2, 3\}$ and $h_2 = \{(a, 2), (b, 1), (c, 3)\}$

> **Solution:** Function is injective and bijective so inverse exists and is given by $h_2^{-1} = \{(1, b), (2, a), (3, c)\}$

(c) $h_3 : \mathbb{R} \to \mathbb{R}$ where $h_3(x) = 3{\cdot}x{-}5$

> **Solution:** Function is injective and bijective so inverse exists and is given by $h_3^{-1}(x) = (5 + x)/3$

(d) $h_4 : \mathbb{R} \to \mathbb{R}$ where $h_4(x) = \lfloor 2{\cdot}x \rfloor$

> **Solution:** Function is neither injective or surjective, for example, $h_4(1) = h_4(1.1)$ and there does not exist an x such that $h_4(x) = 0.1$, therefore inverse does not exist.

(e) $h_5 : \mathbb{Z} \to \mathbb{Z}$ where $h_5(x) = \begin{cases} x + 2 & \text{if } x \geq 5 \\ x{-}1 & \text{if } x \leq 4. \end{cases}$

> **Solution:** Function is injective but not surjective, for example, there does not exist an x such that $h_5(x) = 4$, and hence inverse does not exist.

Difficult/challenging questions (Functions and Sequences).

6. If f and $f \circ g$ are injective, does it follow that g is injective?

> **Solution:** In this case the answer is yes. Let $g : A \to B$ and $f : B \to C$ and suppose for a contradiction g is not injective. Therefore there exists $a_1 \neq a_2 \in A$ such that $g(a_1) = g(a_2)$. Since $g(a_1) = g(a_2)$ we have $f(g(a_1)) = f(g(a_2))$. It therefore follows by defintion of function composition that $(f \circ g)(a_1) = (f \circ g)(a_2)$, which contradicts the fact that $f \circ g$ is injective. (Other proof methods are also possible.)

7. If g and $f \circ g$ are injective, does it follow that f is injective?

> **Solution:** The answer is no and the reason is that g may map only onto the part of f's domain for which f is injective over this subdomain.
>
> As an example consider $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^2$ and where $g(x) : \mathbb{Z}^+ \to \mathbb{Z}$ where $g(x) = x$. Clearly we have both g and $f \circ g$ are injective, while f is not. (Many other counter examples are possible.)

8. If f and $f \circ g$ are surjective, does it follow that g is surjective?

> **Solution:** In this case the answer is no as we may only need a subset of g's range (f's domain) to map onto the entire range of f.
>
> For example consider $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = |x|$ and $g : \mathbb{Z} \to \mathbb{N}$ where $g(x) = |x|$. Clearly we have both f and $f \circ g$ are surjective, while g is not. (Many other counter examples are possible.)

9. If g and $f \circ g$ are surjective, does it follow that f is surjective?

> **Solution:** Suppose $g : A \to B$ and $f : B \to C$. Now since for any $c \in C$, there exists $a \in A$ such that $(f \circ g)(a) = c$ since $f \circ g$ is surjective. Hence by definition of $f \circ g$ we have that $f(g(a)) = c$, i.e. there exists $b = f(a)$ such that $f(b) = c$. Since c was arbitrary we have f is surjective. (Other proof methods are also possible.)

Sequences, Summation, Integers and Division

10. What are the terms a_0, a_1, a_2 and a_3 of the sequence $\langle a_n \rangle_{n \in \mathbb{N}}$, when a_n equals:

    (a) $(-2)^n$?

    > **Solution:** 1, -2, 4, -8

    (b) 3?

    > **Solution:** 3, 3, 3, 3

    (c) $7 + 4^n$?

    > **Solution:** 8, 11, 23, 71

    (d) $2^n + (-2)^n$?

    > **Solution:** 2, 0, 8, 0

11. For each of the following lists of integers, provide a simple formula that generates the terms of an integer sequence that begins with the given list.

    (a) 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, ...

    > **Solution:** $a_0 = 7$ and $a_n = a_{n-1} + 4$ for $n \geq 1$.
    >
    > This is an arithmetic progression in which the nth term is given by $a_n = 7 + 4 \cdot n$ for $n \geq 0$.

    (b) 2, 6, 18, 54, 162, 486, 1458, 4374, 13122, 39366, ...

> **Solution:** $a_0 = 2$ and $a_n = 3{\cdot}a_{n-1}$ for $n \geq 1$.
>
> This is a geometric progression in which the nth term is given by $a_n = 2{\cdot}3^n$ for $n \geq 0$.

(c) 5, 12, 19, 26, 33, 40, 47, 54, 61, 68, ...

> **Solution:** $a_0 = 5$ and $a_n = a_{n-1} + 7$ for $n \geq 1$.
>
> This is an arithmetic progression in which the nth term is given by $a_n = 5 + 7{\cdot}n$ for $n \geq 0$.

(d) 6, 24, 96, 384, 1536, 6144, 24576, 98304, 393216, 1572864, ...

> **Solution:** $a_0 = 6$ and $a_n = 4{\cdot}a_{n-1}$ for $n \geq 1$.
>
> This is a geometric progression in which the nth term is given by $a_n = 6{\cdot}4^n$ for $n \geq 0$.

12. Find the value of each of the following sums. In your calculations, using the theorems:

- $\sum_{i=1}^{n} i = n{\cdot}(n+1)/2$
- if r=1, then $\sum_{i=0}^{n} a{\cdot}r^i = (n+1){\cdot}a$
- if r≠1, then $\sum_{i=0}^{n} a{\cdot}r^i = (a{\cdot}r^{n+1} - a)/(r-1)$.

(a) $\sum_{j=0}^{8}(1 + (-1)^j)$

> **Solution:**
>
> $$\begin{aligned} \textstyle\sum_{j=0}^{8}(1 + (-1)^j) &= \textstyle\sum_{j=0}^{8}1 + \sum_{j=0}^{8}(-1)^j \\ &= 9 + (1{\cdot}(-1)^9 - 1)/(-1-1) \\ &= 9 + (-2)/(-2) \\ &= 9 + 1 \\ &= 10 \end{aligned}$$

(b) $\sum_{j=1}^{4}(2^j + 3{\cdot}j)$

> **Solution:**
>
> $$\begin{aligned} \textstyle\sum_{j=1}^{4}(2^j + 3{\cdot}j) &= \textstyle\sum_{j=1}^{4}2^j + 3{\cdot}\left(\sum_{j=1}^{4}j\right) \\ &= \textstyle\sum_{j=0}^{3}2{\cdot}2^j + 3{\cdot}\left(\sum_{j=1}^{4}j\right) \\ &= (2{\cdot}2^4 - 1)/(2-1) + 3{\cdot}(4{\cdot}5/2) \\ &= (32-2)/1 + 3{\cdot}10 \\ &= 30 + 30 \\ &= 60 \end{aligned}$$

(c) $\sum_{j=0}^{5} 5{\cdot}2^j$

Solution:

$$\sum_{j=0}^{5} 5{\cdot}2^j = (5{\cdot}2^6 - 5)/(2-1) = 315$$

(d) $\sum_{i=0}^{2} \sum_{j=0}^{3} (2{\cdot}i + 3{\cdot}j)$

Solution:

$$\begin{aligned}
\sum_{i=0}^{2}\sum_{j=0}^{3}(2{\cdot}i+3{\cdot}j) &= \left(\sum_{i=0}^{2}\sum_{j=0}^{3}2{\cdot}i\right) + \left(\sum_{i=0}^{2}\sum_{j=0}^{3}3{\cdot}j\right) \\
&= 2{\cdot}\left(\sum_{i=0}^{2}i{\cdot}\left(\sum_{j=0}^{3}1\right)\right) + 3{\cdot}\left(\sum_{i=0}^{2}\left(\sum_{j=0}^{3}j\right)\right) \\
&= 2{\cdot}\left(\sum_{i=0}^{2}i{\cdot}4\right) + 3{\cdot}\left(\sum_{i=0}^{2}(4{\cdot}3)/2\right) \\
&= 8{\cdot}\left(\sum_{i=0}^{2}i\right) + 18{\cdot}\left(\sum_{i=0}^{2}1\right) \\
&= 8{\cdot}(3{\cdot}2)/2 + 18{\cdot}3 \\
&= 78
\end{aligned}$$

(e) $\sum_{i=0}^{3} \sum_{j=0}^{2} (3{\cdot}i + 2{\cdot}j)$

Solution: 78 - this is the same as (d) after swapping i and j, the order of the summations and the commutativity of addition, i.e. $a{+}b = b{+}a$.

(f) $\sum_{j=0}^{19} 7{\cdot}8^j$

Solution:

$$\sum_{j=0}^{19} 7{\cdot}8^j \;=\; (7{\cdot}8^{20} - 7)/7 \;=\; 8^{20} - 1$$

13. Prove or disprove each of the following statements.

(a) If $a|b$ and $c|d$, then $(a{+}c)|(b{+}d)$.

Solution: `false` - for example, consider $a = b = c = 1$ and $d = 2$, then 1 divides both 1 and 2, but 2 does not divide 3.

(b) If $a|b$ and $b|c$, then $a|c$.

Solution: `true` - If $b = a{\cdot}k$ and $c = b{\cdot}l$, then $c = a{\cdot}(k{\cdot}l)$, so $a|c$

(c) If $a|c$ and $b|c$, then $(a{+}b)|c$.

Solution: `false` - for example, consider $a = b = c = 1$, then 1 divides 1, but 2 does not divide 1

(d) If $a|b$ and $c|d$, then $(a{\cdot}c)|(b{+}d)$.

Solution: `false` - for example, consider $a = b = 2$ and $c = d = 1$, then 2 divides 2 and 1 divides 1, but 2 does not divide 3

(e) If $a|b$ and $b|a$, then $a{=}b$.

> **Solution:** `false` - for example, consider $a = 1$ and $b = -1$, then 1 divides -1 and -1 divides 1, but 1 and -1 are not equal

(f) If $a|(b+c)$, then $a|b$ and $a|c$.

> **Solution:** `false` - for example, consider $a = 2, b = c = 3$, then 2 divides 6, but 2 does not divide 3

(g) If $a|b{\cdot}c$, then $a|b$ or $a|c$.

> **Solution:** `false` - for example, consider $a = 4$, $b = 2$, $c = 6$, then 4 divides 12, but 4 divides neither 2 nor 6

(h) If $a|c$ and $b|c$, then $(a{\cdot}b)|c^2$.

> **Solution:** `true` - if $c = a{\cdot}k$ and $c = b{\cdot}l$, then $c^2 = (a{\cdot}b){\cdot}(k{\cdot}l)$, so $(a{\cdot}b)|c^2$

14. Find the unique prime factorisation of 2940.

> **Solution:** Following the approach in the lecture:
>
> - $N = 2940$, $p = 2$ and $rootN = 54$
> - `print`(2)
> - $N = 1470$, $p = 2$ and $rootN = 38$
> - `print`(2)
> - $N = 735$, $p = 2$ and $rootN = 27$
> - $p =$ `nextPrime`(2)
> - `print`(3)
> - $N = 245$, $p = 3$ and $rootN = 15$
> - $p =$ `nextPrime`(3)
> - `print`(5)
> - $N = 49$, $p = 5$ and $rootN = 7$
> - $p =$ `nextPrime`(5)
> - `print`(7)
> - $N = 7$, $p = 5$ and $rootN = 2$
> - `print`(7)
>
> This yields the prime factorisation $2940 = 2^2{\cdot}3{\cdot}5{\cdot}7^2$

Difficult/challenging questions (Sequences, Summation, Integers and Division).

15. Provide a simple formula that generates the terms of the integer sequence 3, 6, 11, 18, 27, 38, 51, 66, 83, 102, ...

> **Solution:** $a_1 = 3$ and $a_n = a_{n-1} + (2{\cdot}n{-}1)$ for $n \geq 2$ (i.e. the next value is the previous value plus the next odd number starting from 3, so 3+3, 6+5, 11+7, ...)
>
> For information, the nth term is given by $a_n = n^2 + 2$ for $n \geq 1$.

16. Let $\langle a_j \rangle_{j \in \mathbb{N}}$ be an arithmetic progression, given by $a_0 = c$ and $a_j = a_{j-1} + d$ for $j{\geq}1$ (where c and d are constants).

 (a) Write an expression for a_j in terms of c, d and j ($j \geq 0$)

 > **Solution:** $a_j = c + j{\cdot}d$ for $j \geq 0$

 (b) Write an expression for the sum in terms of c, d and n

 > **Solution:** From the solution to the previous part of the question we have:
 >
 > $$\begin{aligned}
 \textstyle\sum_{j=0}^{n} a_j &= \textstyle\sum_{j=0}^{n}(c + j{\cdot}d) & \\
 &= \left(\textstyle\sum_{j=0}^{n} c\right) + \left(\textstyle\sum_{j=0}^{n} j{\cdot}d\right) & \text{rearranging} \\
 &= (n{+}1){\cdot}c + \left(\textstyle\sum_{j=0}^{n} j{\cdot}d\right) & \text{since adding } c \text{ together } n{+}1 \text{ times equals } (n{+}1){\cdot}c \\
 &= (n{+}1){\cdot}c + \left(\textstyle\sum_{j=0}^{n} j\right){\cdot}d & \text{using the fact } x{\cdot}z + y{\cdot}z = (x+y){\cdot}z \\
 &= (n{+}1){\cdot}c + \left(0 + \textstyle\sum_{j=1}^{n} j\right){\cdot}d & \text{rearranging} \\
 &= (n{+}1){\cdot}c + \left(\frac{n{\cdot}(n{+}1)}{2}\right){\cdot}d & \text{using the theorem from the lectures} \\
 &= (n{+}1)\cdot\left(c + \frac{n{\cdot}d}{2}\right) & \text{rearranging.}
 \end{aligned}$$

17. Determine whether each of the following sets are countable. For those that are countably infinite, construct a bjectition between the set of natural numbers and that set.

 (a) the integers 1, 2, 4, 8, 16, 32, ...;

 > **Solution:** The set is countably infinite and one possible bijection is given by $f(i) = 2^i$ for all $i \in \mathbb{N}$.

 (b) integers divisible by 3;

Solution: The set is countably infinite and one possible bijection is given by:

$$f(i) = \begin{cases} 3\cdot(i/2) & \text{if } i \text{ is even} \\ -3\cdot((i+1)/2) & \text{if } i \text{ is odd} \end{cases}$$

for all $i \in \mathbb{N}$.

(c) real numbers whose decimal expansions comprise 3's and 7's;

Solution: This set is uncountable - the proof follows the argument used to show the real numbers between 0 and 1 are uncountable (this approach is called Cantor's diagonal argument). In particular, we assume this set is countable which means that we can list the set:

$$\begin{array}{rcl} r_0 & = & 0.x_0^0 x_1^0 x_2^0 \dots x_i^0 \dots \\ r_1 & = & 0.x_0^1 x_1^1 x_2^1 \dots x_i^1 \dots \\ \vdots & & \vdots \\ r_n & = & 0.x_0^n x_1^n x_2^n \dots x_i^n \dots \\ \vdots & & \vdots \end{array}$$

where $x_i^n \in \{0,3,7\}$ for all $n, j \in \mathbb{N}$. Note x_j^n can equal 0 corresponding to the case when the decimal expansion is finite and we then add zero, e.g. 0.73 is expressed as $0.73000\dots$.

Now, using the list we can create a number $r = 0.x_0 x_1 x_2 \dots x_i \dots$ in the set but not in the list by choosing for each i the value x_i such that:

$$x_i = \begin{cases} 7 & \text{if } x_i^i \in \{3,0\} \\ 3 & \text{otherwise (i.e. if } x_i^i = 7). \end{cases}$$

By construction, r in the set but not equal to r_i for any i, which contradicts the fact that the list contained all the elements of the set. Therefore, we cannot construct such a list and the set must be uncountable.

(d) A subset A of a countable set B.

Solution: As B is countable, we can list the elements of B as $b_0, b_1, b_2, b_3, \dots$. Now since A is a subset of B, there are integers $0 \leq i_0 < i_1 < i_2 < i_3 < \dots$ such that $b_{i_j} \in A$, and if $b_k \in A$ for some k, then $k = i_j$ for some $j \geq 0$.

If A is not finite (i.e. the sequence $i_1, i_2, \dots$ is infinite), then it is countably infinite and one possible bijection is given by $f(j) = b_{i_j}$ for $j \in \mathbb{N}$.