2B Linear Algebra

Ex Sheet 5

True/False [1]

a) Every $n \times n$ matrix A is the change-of-basis matrix for some change of basis for $\mathbb{R}^n$.

b) If the change-of-basis matrix from a basis $\mathcal{B}$ to another basis $\mathcal{B}'$ is diagonal, then the coordinate vector of each vector with respect to $\mathcal{B}'$ is a scalar multiple of its coordinate vector with respect to $\mathcal{B}$.

c) If $\mathcal{B}$ is an ordered basis for a vector space V, then the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{B}$ is the identity.

d) Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then T_A is the function that assigns to each vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ the vector $\mathbf{y} = (x_1 + x_2, x_2) \in \mathbb{R}^2$.

e) Let $A \in M_{2\times 3}(\mathbb{R})$. Then T_A is a function from $\mathbb{R}^2$ to $\mathbb{R}^3$.

f) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then $\mathrm{range}(T) = \mathbb{R}^m$.

g) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear then $T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) - T(\boldsymbol{v})$.

h) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear then $T(2\boldsymbol{u}) = T(\boldsymbol{u}) + T(\boldsymbol{u})$.

i) For all linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^n$, we have $S \circ T = T \circ S$.

j) Let A and B be two $n \times n$ matrices over $\mathbb{R}$. If $AB = BA$ then $T_A \circ T_B = T_B \circ T_A$.

k) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Then $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is linear.

l) A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if it is a matrix transformation.

[1] **True/False Questions**

Every Exercise Sheet will have a section containing true/false questions. They are designed to test your understanding from lectures. You should look at your lecture notes and/or the textbook to help you answer these questions, but you should not need to write anything to work out the solution.

Solutions to True/False

a) F b) F c) T d) T e) F f) F g) F h) T i) F j) T k) T l) T

Tutorial Exercises

T1

a) You are given that $\mathcal{B} : e_1, e_1 + e_2, e_1 + e_3, e_1 + e_4$ is an ordered basis for the vector space $V = \mathbb{R}^4$. Find the vector $\boldsymbol{v} \in \mathbb{R}^4$ so that the coordinate vector of $\boldsymbol{v}$ with respect to the basis $\mathcal{B}$ is

$$[\boldsymbol{v}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 4 \end{bmatrix}$$

b) Find the coordinate vector for the vector $\boldsymbol{w} = (-2, 3, -5, 1)$ with respect to the ordered basis $\mathcal{B}$ for $\mathbb{R}^4$ given in (a).

c) Find the change-of-basis matrix $\mathcal{P}_{\mathcal{E}\leftarrow\mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{E}$ where $\mathcal{E}$ is the standard basis for V. Use the matrix $\mathcal{P}_{\mathcal{E}\leftarrow\mathcal{B}}$ to check your answer to (a). Now find the change-of-basis matrix from $\mathcal{E}$ to $\mathcal{B}$, and use this matrix to check your answer to (b).

Solution

a) This coordinate vector means that

$$\begin{aligned}
\boldsymbol{v} &= 2\boldsymbol{e}_1 + 0(\boldsymbol{e}_1 + \boldsymbol{e}_2) + 1(\boldsymbol{e}_1 + \boldsymbol{e}_3) + 4(\boldsymbol{e}_1 + \boldsymbol{e}_4) \\
&= 2(1,0,0,0) + 0(1,1,0,0) + 1(1,0,1,0) + 4(1,0,0,1) \\
&= (2,0,0,0) + (0,0,0,0) + (1,0,1,0) + (4,0,0,4) \\
&= (7,0,1,4).
\end{aligned}$$

b) We need to find the scalars c_1, c_2, c_3, c_4 so that

$$\boldsymbol{w} = c_1\boldsymbol{e}_1 + c_2(\boldsymbol{e}_1 + \boldsymbol{e}_2) + c_3(\boldsymbol{e}_1 + \boldsymbol{e}_3) + c_4(\boldsymbol{e}_1 + \boldsymbol{e}_4)$$

This equation is

$$(-2, 3, -5, 1) = c_1(1,0,0,0) + c_2(1,1,0,0) + c_3(1,0,1,0) + c_4(1,0,0,1)$$

which holds if and only if

$$(-2, 3, -5, 1) = (c_1 + c_2 + c_3 + c_4, c_2, c_3, c_4).$$

By comparing components, we get immediately that $c_2 = 3$, $c_3 = -5$ and $c_4 = 1$. Substituting these values into the equation $c_1 + c_2 + c_3 + c_4 = -2$ from the first component we get $c_1 = -1$. Thus the coordinate vector of $\boldsymbol{w}$ with respect to the basis $\mathcal{B}$ is

$$[\boldsymbol{w}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 1 \end{bmatrix}.$$

c) We want to write the vectors in $\mathcal{B}$ in terms of the standard basis $\mathcal{E}$, and the change of basis matrix from $\mathcal{B}$ to $\mathcal{E}$ that we obtain is

$$\mathcal{P}_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The calculations on which this answer is based is:

We have

$$\begin{aligned} e_1 &= 1e_1 + 0e_2 + 0e_3 + 0e_4 \\ e_1 + e_2 &= 1e_1 + 1e_2 + 0e_3 + 0e_4 \\ e_1 + e_3 &= 1e_1 + 0e_2 + 1e_3 + 0e_4 \\ e_1 + e_4 &= 1e_1 + 0e_2 + 0e_3 + 1e_4 \end{aligned}$$

so the columns of $\mathcal{P}_{\mathcal{E}\leftarrow\mathcal{B}}$ are given by

$$[e_1]_{\mathcal{E}} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad [e_1+e_2]_{\mathcal{E}} = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \quad [e_1+e_3]_{\mathcal{E}} = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \quad [e_1+e_4]_{\mathcal{E}} = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}$$

To use $\mathcal{P}_{\mathcal{E}\leftarrow\mathcal{B}}$ to check (a), the key calculation is that

$$\begin{pmatrix}1&1&1&1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{bmatrix}2\\0\\1\\4\end{bmatrix} = \begin{bmatrix}7\\0\\1\\4\end{bmatrix}. \tag{1}$$

Use the fact that $\mathcal{P}_{\mathcal{E}\leftarrow\mathcal{B}}[v]_{\mathcal{B}} = [v]_{\mathcal{E}}$. By equation (**??**), we get $[v]_{\mathcal{E}} = \begin{bmatrix}7\\0\\1\\4\end{bmatrix}$. Since $\mathcal{E}$ is the standard basis for $\mathbb{R}^4$ this means $v = (7, 0, 1, 4)$.

The change-of-basis matrix from $\mathcal{E}$ to $\mathcal{B}$ is the inverse of $\mathcal{P}_{\mathcal{E}\leftarrow\mathcal{B}}$, so

$$\mathcal{P}_{\mathcal{B}\leftarrow\mathcal{E}} = \mathcal{P}_{\mathcal{E}\leftarrow\mathcal{B}}^{-1} = \begin{pmatrix}1&1&1&1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}^{-1} = \begin{pmatrix}1&-1&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

To use $\mathcal{P}_{\mathcal{B}\leftarrow\mathcal{E}}$ to check your answer to (b), the key calculation is that

$$\begin{pmatrix}1&-1&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{bmatrix}-2\\3\\-5\\1\end{bmatrix} = \begin{bmatrix}-1\\3\\-5\\1\end{bmatrix}. \tag{2}$$

Use the fact that $\mathcal{P}_{\mathcal{B}\leftarrow\mathcal{E}}[w]_{\mathcal{E}} = [w]_{\mathcal{B}}$. We have $[w]_{\mathcal{E}} = \begin{bmatrix}-2\\3\\-5\\1\end{bmatrix}$ so by equation (**??**), $[w]_{\mathcal{B}} = \begin{bmatrix}-1\\3\\-5\\1\end{bmatrix}$.

T2 Consider ordered bases $\mathcal{B} : (1, 2), (3, -1)$ and $\mathcal{C} : (2, -2), (4, 3)$ for $\mathbb{R}^2$.

a) Find the change of basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{C}$.

b) Find the coordinate vector of $(5, -1)$ with respect to the old basis

$\mathcal{B}$.

c) Find the coordinate vector of $(5,-1)$ with respect to the new basis $\mathcal{C}$, and verify that your answer can be obtained by multiplying together your answers to (a) and (b).

Solution

a) Write the old basis vectors in terms of the new to produce the columns of the change-of-basis matrix. That is, we solve the equations

$$\begin{aligned}(1,2) &= a_1(2,-2)+a_2(4,3)\\ (3,-1) &= b_1(2,-2)+b_2(4,3).\end{aligned}$$

Each equation gives a system of two equations in two unknowns which we can solve to give

$$[(1,2)]_{\mathcal{C}} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{14} \\ \frac{3}{7} \end{bmatrix} \quad \text{and} \quad [(3,-1)]_{\mathcal{C}} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \frac{13}{14} \\ \frac{2}{7} \end{bmatrix}, \quad \text{hence} \quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} -\frac{5}{14} & \frac{13}{14} \\ \frac{3}{7} & \frac{2}{7} \end{pmatrix}$$

b) Here we solve the system of two equations in two unknowns coming from the equation

$$(5,-1) = c_1(1,2)+c_2(3,-1).$$

You can show that

$$[(5,-1)]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} \\ \frac{11}{7} \end{bmatrix}.$$

c) Here we solve the system of two equations in two unknowns coming from the equation

$$(5,-1) = c_1'(2,-2)+c_2'(4,3).$$

You can show that

$$[(5,-1)]_{\mathcal{C}} = \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} \frac{19}{14} \\ \frac{4}{7} \end{bmatrix}.$$

It remains to notice that

$$[(5,-1)]_{\mathcal{C}} = \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} \frac{19}{14} \\ \frac{4}{7} \end{bmatrix} = \begin{pmatrix} -\frac{5}{14} & \frac{13}{14} \\ \frac{3}{7} & \frac{2}{7} \end{pmatrix} \begin{bmatrix} \frac{2}{7} \\ \frac{11}{7} \end{bmatrix} = P_{\mathcal{C}\leftarrow\mathcal{B}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P_{\mathcal{C}\leftarrow\mathcal{B}}[(5,-1)]_{\mathcal{B}}$$

as required.

T3 Suppose that $\mathcal{B}$ and $\mathcal{C}$ are ordered bases for a 3-dimensional vector space V and that $[\boldsymbol{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. If the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

find $[\boldsymbol{v}]_{\mathcal{C}}$.

Solution

We just need to compute that

$$[v]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[v]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ 9 \end{bmatrix}.$$

T4 Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

a) Prove that $T(\mathbf{0}) = \mathbf{0}$.

b) Prove that for all $\boldsymbol{v}, \boldsymbol{v'} \in \mathbb{R}^n$, $T(\boldsymbol{v} - \boldsymbol{v'}) = T(\boldsymbol{v}) - T(\boldsymbol{v}')$.

c) Prove by induction on k that for all $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ and all $c_1, \ldots, c_k \in \mathbb{R}$,

$$T(c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_k) = c_1T(\boldsymbol{v}_1) + \cdots + c_kT(\boldsymbol{v}_k).$$

Solution

a) Since T is linear, $T(\lambda\mathbf{0}) = \lambda T(\mathbf{0})$ for all $\lambda \in \mathbb{R}$. But $\lambda\mathbf{0} = \mathbf{0}$ for any $\lambda \in \mathbb{R}$, and $0\boldsymbol{w} = \mathbf{0}$ for any $\boldsymbol{w} \in \mathbb{R}^m$. Thus putting $\lambda = 0$ we obtain

$$T(\mathbf{0}) = T(0\mathbf{0}) = 0T(\mathbf{0}) = \mathbf{0}.$$

Alternatively,

$$T(\mathbf{0}) = T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0}).$$

Now subtract $T(\mathbf{0})$ from both sides to obtain $\mathbf{0} = T(\mathbf{0})$.

b) The key is to note that $\boldsymbol{v} - \boldsymbol{v'} = \boldsymbol{v} + (-1)\boldsymbol{v'}$. Then as T is linear we have

$$T(\boldsymbol{v} - \boldsymbol{v'}) = T(\boldsymbol{v} + (-1)\boldsymbol{v'}) = T(\boldsymbol{v}) + T((-1)\boldsymbol{v'}) = T(\boldsymbol{v}) + (-1)T(\boldsymbol{v'}) = T(\boldsymbol{v}) - T(\boldsymbol{v'}).$$

c) When $k = 1$ the statement is that $T(c_1\boldsymbol{v}_1) = c_1T(\boldsymbol{v}_1)$, which holds since T is linear. Assume the statement is true for $k \geq 1$. Then, for any scalars $c_1, \ldots, c_{k+1}$ and vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1}$ we have

$$\begin{aligned} T(c_1\boldsymbol{v}_1 + \cdots + c_{k+1}\boldsymbol{v}_{k+1}) &= T((c_1\boldsymbol{v}_1 + \cdots c_k\boldsymbol{v}_k) + c_{k+1}\boldsymbol{v}_{k+1}) \\ &= T(c_1\boldsymbol{v}_1 + \cdots c_k\boldsymbol{v}_k) + T(c_{k+1}\boldsymbol{v}_{k+1}) \end{aligned}$$

since T is linear and so $T(\boldsymbol{v} + \boldsymbol{v}') = T(\boldsymbol{v}) + T(\boldsymbol{v}')$ for any vectors $\boldsymbol{v}$ and $\boldsymbol{v}'$, in particular for $\boldsymbol{v} = c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k$ and $\boldsymbol{v}' = c_{k+1}\boldsymbol{v}_{k+1}$. The inductive hypothesis implies that

$$T(c_1\boldsymbol{v}_1 + \cdots c_k\boldsymbol{v}_k) = c_1T(\boldsymbol{v}_1) + \cdots + c_kT(\boldsymbol{v}_k)$$

and the linearity of T implies that

$$T(c_{k+1}\boldsymbol{v}_{k+1}) = c_{k+1}T(\boldsymbol{v}_{k+1}).$$

Therefore
$$T(c_1v_1 + \cdots + c_{k+1}v_{k+1}) = c_1T(v_1) + \cdots + c_kT(v_k) + c_{k+1}T(v_{k+1})$$
as required.

T5 For each of the the following functions, determine whether it is a linear transformation. If it is a linear transformation you should prove this, and if it is not a linear transformation you should give a counterexample.

a) $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = ax$, where $a \in \mathbb{R}$.

b) $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = ax + b$, where $a, b \in \mathbb{R}$ and $b \neq 0$.

c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (|x|, |y|)$.

d) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x, y, z) = (y, z, x)$.

e) $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by $T(w, x, y, z) = (3w, 2x, y)$.

f) $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by $T(x, y, z) = (z^2, x + y)$.

g) $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by $T(x, y) = (y - 1, x + 2y, 2x + y)$.

h) $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ given by $T(x, y) = (7x, x - y, 2y, 2x - 5y)$.

Solution

a) Let $x, y \in \mathbb{R}$. Then $T(x + y) = a(x + y) = ax + ay = T(x) + T(y)$. Now let $x \in \mathbb{R}$ and let c be a scalar. Then $T(cx) = a(cx) = c(ax) = cT(x)$. Therefore T is linear.

b) We have $T(1) = a + b$ and $T(2) = 2a + b$. But $2T(1) = 2(a + b) = 2a + 2b \neq 2a + b = T(2)$ since $b \neq 0$. Thus T is not linear. Alternatively, use T6(a): since $T(0) = b \neq 0$, the function T is not linear.

c) We have $T(1, 1) = (1, 1)$ and $T(-1, -1) = (1, 1)$, so $T(1, 1) + T(-1, -1) = (2, 2)$. But
$$T((1, 1) + (-1, -1)) = T(0, 0) = (0, 0) \neq (2, 2) = T(1, 1) + T(-1, -1).$$
So T is not linear.

d) Let $(x, y, z), (x', y', z') \in \mathbb{R}^3$. Then
$$\begin{aligned} T((x, y, z) + (x', y', z')) &= T(x + x', y + y', z + z') \\ &= (y + y', z + z', x + x') \\ &= (y, z, x) + (y', z', x') \\ &= T(x, y, z) + (x', y', z'). \end{aligned}$$
Now let $(x, y, z) \in \mathbb{R}^3$ and let $c \in \mathbb{R}$. Then
$$T(c(x, y, z)) = T(cx, cy, cz) = (cy, cz, cx) = c(y, z, x) = cT(x, y, z).$$
Therefore T is a linear transformation.

e) Let $(w,x,y,z),(w',x',y',z') \in \mathbb{R}^4$. Then

$$\begin{aligned} T((w,x,y,z)+(w',x',y',z')) &= T(w+w',x+x',y+y',z+z') \\ &= (3(w+w'),2(x+x'),y+y') \\ &= (3w+3w',2x+2x',y+y') \\ &= (3w,2x,y)+(3w',2x',y') \\ &= T(w,x,y,z)+T(w',x',y',z'). \end{aligned}$$

Now let $(w,x,y,z) \in \mathbb{R}^4$ and let $c \in \mathbb{R}$. Then

$$T(c(w,x,y,z)) = T(cw,cx,cy,cz) = (3(cw),2(cx),cy) = (c(3w),c(2x),cy) = c(3w,2x,y) = cT(w,x,y,z).$$

Therefore T is a linear transformation.

f) We have

$$T\big(2(0,\,0,\,1)\big) = T(0,\,0,\,2) = (4,0),$$

however

$$2T(0,\,0,\,1) = 2(1,\,0) = (2,\,0).$$

So T is not a linear mapping since $T\big(2(0,\,0,\,1)\big) \neq 2T(0,\,0,\,1)$.

g) Note that $T(0,\,0) = (-1,\,0,\,0) \neq (0,0,0)$, so T is not a linear map by T6(a).

h) For $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} T\big(\lambda(x_1,\,y_1)+(x_2,\,y_2)\big) &= T(\lambda x_1+x_2,\,\lambda y_1+y_2) \\ &= \big(7(\lambda x_1+x_2),\,(\lambda x_1+x_2)-(\lambda y_1+y_2),\,2(\lambda y_1+y_2),\,2(\lambda x_1+x_2)-5(\lambda y_1+y_2)\big) \\ &= (7\lambda x_1+7x_2,\,\lambda x_1+x_2-\lambda y_1-y_2,\,2\lambda y_1+2y_2,\,2\lambda x_1+2x_2-5\lambda y_1-5y_2) \\ &= (7\lambda x_1,\,\lambda x_1-\lambda y_1,\,2\lambda y_1,\,2\lambda x_1-5\lambda y_1)+(7x_2,\,x_2-y_2,\,2y_2,\,2x_2-5y_2) \\ &= \lambda(7x_1,\,x_1-y_1,\,2y_1,\,2x_1-5y_1)+(7x_2,\,x_2-y_2,\,2y_2,\,2x_2-5y_2) \\ &= \lambda T(x_1,\,y_1)+T(x_2,\,y_2). \end{aligned}$$

This is enough to show that T is a linear map, because special cases include the two defining properties of a linear map, namely

$$\begin{aligned} T\big((x_1,y_1)+(x_2,y_2)\big) &= T(x_1,y_1)+T(x_2,y_2) \\ T\big(\lambda(x_1,y_1)\big) &= \lambda T(x_1,y_1). \end{aligned}$$

T6 Find the standard matrix $[T]$ for each function T in T5 which is a linear transformation.

Solution

The linear maps are from parts a), d), e) and h).

For part a), we have $T : \mathbb{R} \to \mathbb{R}$ so $[T]$ will be the 1×1 matrix $[a]$.

For part d), we have $T(e_1) = T(1,0,0) = (0,0,1)$, $T(e_2) = T(0,1,0) = (1,0,0)$ and $T(e_3) =$

$T(0,0,1) = (0,1,0)$. The standard matrix for T is the matrix $[T]$ with ith column given by $T(\boldsymbol{e}_i)$:

$$[T] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

For part e), we have $T(\boldsymbol{e}_1) = T(1,0,0,0) = (3,0,0)$, $T(\boldsymbol{e}_2) = T(0,1,0,0) = (0,2,0)$, $T(\boldsymbol{e}_3) = T(0,0,1,0) = (0,0,1)$ and $T(\boldsymbol{e}_4) = T(0,0,0,1) = (0,0,0)$. The standard matrix for T is the matrix $[T]$ with ith column given by $T(\boldsymbol{e}_i)$:

$$[T] = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

For part h), we have $T(\boldsymbol{e}_1) = T(1,0) = (7,1,0,2)$ and $T(\boldsymbol{e}_2) = T(0,1) = (0,-1,2,-5)$. The standard matrix for T is the matrix $[T]$ with ith column given by $T(\boldsymbol{e}_i)$:

$$[T] = \begin{bmatrix} 7 & 0 \\ 1 & -1 \\ 0 & 2 \\ 2 & -5 \end{bmatrix}.$$

T7 Let

$$A = \begin{pmatrix} 4 & 3 \\ 2 & -1 \\ 0 & 9 \end{pmatrix}$$

and let T_A be the corresponding matrix transformation.

a) Determine the m and n so that $T_A : \mathbb{R}^n \to \mathbb{R}^m$.

b) For $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, find a formula for $T_A(\boldsymbol{x}) \in \mathbb{R}^m$.

Now repeat this question for the following matrices:

$$B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & -5 & 8 \\ 1 & -2 & 17 & 6 \\ 8 & 2 & 3 & 4 \end{pmatrix}.$$

Solution

For the matrix A:

a) $m = 3$ and $n = 2$.

b) For $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$ we have

$$T_A(\boldsymbol{x}) = A\boldsymbol{x} = \begin{pmatrix} 4 & 3 \\ 2 & -1 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (4x_1 + 3x_2, 2x_1 - x_2, 9x_2) \in \mathbb{R}^3.$$

For the matrix B:

a) $m = 2$ and $n = 2$.

b) For $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$ we have

$$T_B(\boldsymbol{x}) = B\boldsymbol{x} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (2x_1 - x_2, -x_1 + 2x_2) \in \mathbb{R}^2.$$

For the matrix C:

a) $m = 3$ and $n = 4$.

b) For $\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ we have

$$T_C(\boldsymbol{x}) = C\boldsymbol{x} = \begin{pmatrix} 1 & 2 & -5 & 8 \\ 1 & -2 & 17 & 6 \\ 8 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$= (x_1 + 2x_2 - 5x_3 + 8x_4, x_1 - 2x_2 + 17x_3 + 6x_4, 8x_1 + 2x_2 + 3x_3 + 4x_4) \in \mathbb{R}^3.$$

T8 Let A and B be as in T7.

a) Find the matrix AB.

b) Determine the k and l so that $T_{AB} : \mathbb{R}^l \to \mathbb{R}^k$. For $\boldsymbol{x} = (x_1, \ldots, x_l) \in \mathbb{R}^l$, find a formula for $T_{AB}(\boldsymbol{x}) \in \mathbb{R}^k$.

c) Determine the p and q so that $T_A \circ T_B : \mathbb{R}^q \to \mathbb{R}^p$. For $\boldsymbol{x} = (x_1, \ldots, x_q) \in \mathbb{R}^q$, find a formula for $(T_A \circ T_B)(\boldsymbol{x}) \in \mathbb{R}^p$ using the formulas for T_A and T_B in exercise T8. Is your final answer the same as part b)?

Solution

a)

$$AB = \begin{pmatrix} 4 & 3 \\ 2 & -1 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 5 & -4 \\ -9 & 18 \end{pmatrix}.$$

b) $k = 3$ and $l = 2$, and for $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$ we have

$$T_{AB}(\boldsymbol{x}) = (AB)\boldsymbol{x} = \begin{pmatrix} 5 & 2 \\ 5 & -4 \\ -9 & 18 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (5x_1 + 2x_2, 5x_1 - 4x_2, -9x_1 + 18x_2) \in \mathbb{R}^3.$$

c) $p = 3$ and $q = 2$, and for $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$ we have

$$\begin{aligned} (T_A \circ T_B)(\boldsymbol{x}) &= T_A(T_B(\boldsymbol{x})) \\ &= T_A(2x_1 - x_2, -x_1 + 2x_2) \\ &= (4(2x_1 - x_2) + 3(-x_1 + 2x_2), 2(2x_1 - x_2) - (-x_1 + 2x_2), 9(-x_1 + 2x_2)) \\ &= (5x_1 + 2x_2, 5x_1 - 4x_2, -9x_1 + 18x_2) \in \mathbb{R}^3. \end{aligned}$$

Yes, this final answer is the same as in part b).

T9 Let B be as in T12.

a) Find the matrix B^{-1} and hence find a formula for $T_{B^{-1}}(\boldsymbol{x}) \in \mathbb{R}^2$, where $\boldsymbol{x} \in \mathbb{R}^2$.

b) Use the formulas for T_B and $T_{B^{-1}}$ to show that $(T_B \circ T_{B^{-1}})(\boldsymbol{x}) = \boldsymbol{x}$ and $(T_{B^{-1}} \circ T_B)(\boldsymbol{x}) = \boldsymbol{x}$, for all $\boldsymbol{x} \in \mathbb{R}^2$. (This shows that T_B is invertible with inverse $(T_B)^{-1} = T_{B^{-1}}$.)

Solution

a) We have

$$B^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

and so for $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$

$$T_{B^{-1}}(\boldsymbol{x}) = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(\frac{2}{3}x_1 + \frac{1}{3}x_2, \frac{1}{3}x_1 + \frac{2}{3}x_2\right) \in \mathbb{R}^2.$$

b) For the first composition we have

$$\begin{aligned}
(T_B \circ T_{B^{-1}})(\boldsymbol{x}) &= T_B(T_{B^{-1}}(\boldsymbol{x})) \\
&= T_B\left(\frac{2}{3}x_1 + \frac{1}{3}x_2, \frac{1}{3}x_1 + \frac{2}{3}x_2\right) \\
&= \left(2\left(\frac{2}{3}x_1 + \frac{1}{3}x_2\right) - \left(\frac{1}{3}x_1 + \frac{2}{3}x_2\right), -\left(\frac{2}{3}x_1 + \frac{1}{3}x_2\right) + 2\left(\frac{1}{3}x_1 + \frac{2}{3}x_2\right)\right) \\
&= (x_1, x_2) \\
&= \boldsymbol{x}.
\end{aligned}$$

For the second composition we have

$$\begin{aligned}
(T_{B^{-1}} \circ T_B)(\boldsymbol{x}) &= T_{B^{-1}}(T_B(\boldsymbol{x})) \\
&= T_{B^{-1}}(2x_1 - x_2, -x_1 + 2x_2) \\
&= \left(\frac{2}{3}(2x_1 - x_2) + \frac{1}{3}(-x_1 + 2x_2), \frac{1}{3}(2x_1 - x_2) + \frac{2}{3}(-x_1 + 2x_2)\right) \\
&= (x_1, x_2) \\
&= \boldsymbol{x}.
\end{aligned}$$

T10 Consider the real vector space $\mathbb{R}^3$ and the ordered basis

$$\mathcal{B} \colon (1, -1, 1), (1, 1, 0), (2, 1, 0).$$

Find a formula for the coordinates of a vector $\boldsymbol{x} = (x, y, z)$ with respect to $\mathcal{B}$.

Solution

The coordinate vector of $\boldsymbol{x}$ with respect to the basis $\mathcal{B}$ is

$$[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

where the coordinates λ_i are the unique scalars which satisfy

$$(x, y, z) = \lambda_1(1, -1, 1) + \lambda_2(1, 1, 0) + \lambda_3(2, 1, 0),$$

in other words the λ_i are the solutions of the system

$$\begin{aligned} \lambda_1 + \lambda_2 + 2\lambda_3 &= x \\ -\lambda_1 + \lambda_2 \ \ + \lambda_3 &= y \\ \lambda_1 \qquad\qquad\quad &= z. \end{aligned}$$

Elementary row operations show that

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & x \\ -1 & 1 & 1 & y \\ 1 & 0 & 0 & z \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 0 & 0 & z \\ 0 & 1 & 0 & -x + 2y + 3z \\ 0 & 0 & 1 & x - y - 2z \end{array}\right],$$

so the coordinate vector that we're looking for is

$$[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} z \\ -x + 2y + 3z \\ x - y - 2z \end{bmatrix}.$$

T11 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix $A = [T]$. Prove that $\mathrm{range}(T) = \mathrm{col}(A)$.

Solution

We first show that $\mathrm{range}(T) \subseteq \mathrm{col}(A)$. For this, let $\boldsymbol{w} \in \mathbb{R}^m$ be in $\mathrm{range}(T)$. Then by definition of the range, $\boldsymbol{w} = T(\boldsymbol{v})$ for some $\boldsymbol{v} \in \mathbb{R}^n$. By definition of A, we have $T(\boldsymbol{v}) = A\boldsymbol{v}$ and so $A\boldsymbol{v} = \boldsymbol{w}$. Let $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$. Then $\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + \cdots + v_n\boldsymbol{e}_n$. So

$$\boldsymbol{w} = A\boldsymbol{v} = A(v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + \cdots + v_n\boldsymbol{e}_n) = v_1A\boldsymbol{e}_1 + v_2A\boldsymbol{e}_2 + \cdots + v_nA\boldsymbol{e}_n.$$

Now $A\boldsymbol{e}_i$ is the ith column of the matrix A, so we have expressed $\boldsymbol{w}$ as a linear combination of the columns of A. Therefore $\boldsymbol{w}$ is in $\mathrm{col}(A)$ as required.

We now show that $\mathrm{col}(A) \subseteq \mathrm{range}(T)$. For this, let $\boldsymbol{w} \in \mathbb{R}^m$ be in $\mathrm{col}(A)$ and let the columns of A be $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$. Then by definition of the column space, there are scalars $c_1, c_2, \ldots, c_n$ so that

$$\boldsymbol{w} = c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2 + \cdots + c_n\boldsymbol{a}_n.$$

Now the ith column of A is $A\boldsymbol{e}_i$, hence we have $\boldsymbol{a}_i = A\boldsymbol{e}_i = T(\boldsymbol{e}_i)$. Thus

$$\boldsymbol{w} = c_1 T(\boldsymbol{e}_1) + c_2 T(\boldsymbol{e}_2) + \cdots + c_n T(\boldsymbol{e}_n).$$

As T is a linear map, the right-hand side is equal to $T(c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + \cdots + c_n\boldsymbol{e}_n)$. Let $\boldsymbol{v} = c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + \cdots + c_n\boldsymbol{e}_n$, then we have $\boldsymbol{w} = T(\boldsymbol{v})$. Thus $\boldsymbol{w}$ is in range(T) as required.

We conclude that range$(T) = \text{col}(A)$.

T12 Let T be a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. Prove that T is linear if and only if for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ and all scalars $\lambda \in \mathbb{R}$,

$$T(\lambda\boldsymbol{u} + \boldsymbol{v}) = \lambda T(\boldsymbol{u}) + T(\boldsymbol{v}).$$

Solution

Assume that T is linear. Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then by definition of linearity,

$$T(\lambda\boldsymbol{u} + \boldsymbol{v}) = T(\lambda\boldsymbol{u}) + T(\boldsymbol{v}) = \lambda T(\boldsymbol{u}) + T(\boldsymbol{v}).$$

Now assume that for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ and all scalars $\lambda \in \mathbb{R}$,

$$T(\lambda\boldsymbol{u} + \boldsymbol{v}) = \lambda T(\boldsymbol{u}) + T(\boldsymbol{v}).$$

Then in the special case that $\lambda = 1$, we have

$$T(\boldsymbol{u} + \boldsymbol{v}) = T(1\boldsymbol{u} + \boldsymbol{v}) = 1T(\boldsymbol{u}) + T(\boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v}).$$

Thus for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, we have $T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$. Now in the special case that $\boldsymbol{v} = \boldsymbol{0}$, we have

$$T(\lambda\boldsymbol{u}) = T(\lambda\boldsymbol{u} + \boldsymbol{0}) = \lambda T(\boldsymbol{u}) + T(\boldsymbol{0}).$$

We would like to deduce that $T(\lambda\boldsymbol{u}) = \lambda T(\boldsymbol{u})$ since $T(\boldsymbol{0}) = \boldsymbol{0}$, but we cannot use T6(a) since we have not yet proved that T is linear. However observe that in the special case $\lambda = 1$ and $\boldsymbol{u} = \boldsymbol{v} = \boldsymbol{0}$, we get

$$T(\boldsymbol{0}) = T(1\boldsymbol{0} + \boldsymbol{0}) = 1T(\boldsymbol{0}) + T(\boldsymbol{0}) = 2T(\boldsymbol{0}).$$

Subtract $T(\boldsymbol{0})$ from both sides of this to get $T(\boldsymbol{0}) = \boldsymbol{0}$ as desired. Therefore for all $\boldsymbol{u} \in \mathbb{R}^n$ and all scalars $\lambda \in \mathbb{R}$, we have $T(\lambda\boldsymbol{u}) = \lambda T(\boldsymbol{u})$. We conclude that T is linear.

T13 Answer the following questions using the criterion for linearity in T12, rather than any results about matrix transformations.

a) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^p$ be linear transformations. Prove that $S \circ T$ is linear.

b) Let $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. Define a map $S + T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ by

$$(S + T)(\boldsymbol{x}) = S(\boldsymbol{x}) + T(\boldsymbol{x}).$$

Prove that $S + T$ is linear.

c) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $c \in \mathbb{R}$ be a scalar. Define a map cT from $\mathbb{R}^n$ to $\mathbb{R}^m$ by

$$(cT)(\boldsymbol{x}) = c(T(\boldsymbol{x})).$$

Prove that cT is linear.

Solution

a) Let $\boldsymbol{u}, \boldsymbol{u}' \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then since both T and S are linear

$$(S \circ T)(\lambda \boldsymbol{u} + \boldsymbol{u}') = S(T(\lambda \boldsymbol{u} + \boldsymbol{u}')) = S(\lambda T(\boldsymbol{u}) + T(\boldsymbol{u}')) = \lambda S(T(\boldsymbol{u})) + S(T(\boldsymbol{u}')) = \lambda((S \circ T)(\boldsymbol{u})) + (S \circ T)(\boldsymbol{u}').$$

Hence $S \circ T$ is linear.

b) Let $\boldsymbol{u}, \boldsymbol{u}' \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then since both T and S are linear

$$\begin{aligned}(S+T)(\lambda \boldsymbol{u} + \boldsymbol{u}') &= S(\lambda \boldsymbol{u} + \boldsymbol{u}') + T(\lambda \boldsymbol{u} + \boldsymbol{u}') \\ &= \lambda S(\boldsymbol{u}) + S(\boldsymbol{u}') + \lambda T(\boldsymbol{u}) + T(\boldsymbol{u}') \\ &= \lambda(S(\boldsymbol{u}) + T(\boldsymbol{u})) + (S(\boldsymbol{u}') + T(\boldsymbol{u}')) \\ &= \lambda((S+T)(\boldsymbol{u})) + (S+T)(\boldsymbol{u}').\end{aligned}$$

Hence $S + T$ is linear.

c) Let $\boldsymbol{u}, \boldsymbol{u}' \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then since T is linear

$$\begin{aligned}(cT)(\lambda \boldsymbol{u} + \boldsymbol{u}') &= c(T(\lambda \boldsymbol{u} + \boldsymbol{u}')) \\ &= c(\lambda T(\boldsymbol{u}) + T(\boldsymbol{u}')) \\ &= c(\lambda T(\boldsymbol{u})) + cT(\boldsymbol{u}') \\ &= \lambda(cT(\boldsymbol{u})) + cT(\boldsymbol{u}') \\ &= \lambda((cT)(\boldsymbol{u})) + (cT)(\boldsymbol{u}').\end{aligned}$$

Hence cT is linear.