Maths: 14/15

S|O|L|V|E
1|1|1|1|1

Tot: 19/20

Well done!

a) Gram-Schmidt process.

Let the vectors in the basis for U be $\boldsymbol{w_1} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $\boldsymbol{w_2} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$, and $\boldsymbol{w_3} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}$. Then find vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3} \in U$ by the Gram-Schmidt process:

$$\boldsymbol{v_1} = \boldsymbol{w_1} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix},$$

$$\boldsymbol{v_2} = \boldsymbol{w_2} - \left(\frac{\boldsymbol{w_2} \cdot \boldsymbol{v_1}}{\boldsymbol{v_1} \cdot \boldsymbol{v_1}}\right) \boldsymbol{v_1} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix},$$

$$\boldsymbol{v_3} = \boldsymbol{w_3} - \left(\frac{\boldsymbol{w_3} \cdot \boldsymbol{v_1}}{\boldsymbol{v_1} \cdot \boldsymbol{v_1}}\right) \boldsymbol{v_1} - \left(\frac{\boldsymbol{w_3} \cdot \boldsymbol{v_2}}{\boldsymbol{v_2} \cdot \boldsymbol{v_2}}\right) \boldsymbol{v_2} = \begin{pmatrix} 1 \\ -1/2 \\ 1/2 \\ 1 \end{pmatrix}.$$

Thus, an orthogonal basis for U consists of the vectors $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1/2 \\ 1/2 \\ 1 \end{pmatrix}$. To check that they are orthogonal, calculate dot products between all vectors:

$$\boldsymbol{v_1} \cdot \boldsymbol{v_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} = 1 - 1 = 0,$$

$$\boldsymbol{v_1} \cdot \boldsymbol{v_3} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1/2 \\ 1/2 \\ 1 \end{pmatrix} = -\frac{1}{2} + \frac{1}{2} = 0,$$

$$\boldsymbol{v_2} \cdot \boldsymbol{v_3} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1/2 \\ 1/2 \\ 1 \end{pmatrix} = -\frac{1}{2} - \frac{1}{2} + 1 = 0.$$

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Therefore, the vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3} \in U$ are orthogonal.

b) 2×2 matrix A.

i) Diagonalisation.

Since the characteristic polynomial of A is $p_A(\lambda) = (\lambda - 2)(\lambda - 1)$, the eigenspaces E_2, E_1 can be found from eigenvalues $\lambda_2 = 2, \lambda_1 = 1$, respectively. When $\lambda = 2$,

$$A - 2I_2 = \begin{bmatrix} -1 & 0 \\ 4 & 0 \end{bmatrix}.$$

Then

$$E_2 = \mathrm{null}(A - 2I_2) = \left\{\begin{pmatrix}0\\t\end{pmatrix} : t \in \mathbb{R}\right\} = \mathrm{span}\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}.$$

When $\lambda = 1$,

$$A - 1I_2 = \begin{bmatrix}0 & 0\\4 & 1\end{bmatrix}.$$

Then

$$E_1 = \mathrm{null}(A - 1I_2) = \left\{\begin{pmatrix}t\\-4t\end{pmatrix} : t \in \mathbb{R}\right\} = \mathrm{span}\left\{\begin{pmatrix}1\\-4\end{pmatrix}\right\}.$$

Let $\boldsymbol{v_2} = \begin{pmatrix}0\\1\end{pmatrix}$ and $\boldsymbol{v_1} = \begin{pmatrix}1\\-4\end{pmatrix}$. Since A is a 2×2 matrix with 2 linearly independent eigenvectors $\boldsymbol{v_1}, \boldsymbol{v_2}$, A is diagonalisable. Furthermore, let P be the matrix $P = [\boldsymbol{v_2}, \boldsymbol{v_1}]$ and let D be the matrix $D = \mathrm{diag}(\lambda_2, \lambda_1)$. Then the equation $P^{-1}AP = D$ is satisfied by the diagonalisation of A and the fact that $\det(P) = -1 \neq 0$, which means that it is invertible. Show it.

ii) A^4.

First, find P^{-1} to be $\begin{bmatrix}4 & 1\\1 & 0\end{bmatrix}$. A^4 can then be expressed as

Prove the formula

$$A^4 = PD^4P^{-1} = \begin{bmatrix}0 & 1\\1 & -4\end{bmatrix}\begin{bmatrix}2^4 & 0\\0 & 1^4\end{bmatrix}\begin{bmatrix}4 & 1\\1 & 0\end{bmatrix}$$
$$= \begin{bmatrix}1 & 0\\60 & 16\end{bmatrix}.$$

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c) Proof about $A^2 = I_n$.

To prove that A is symmetric if and only if A is orthogonal, one needs to prove it as an implication that holds both ways. By a property of invertibility, given that $A^2 = I_n$, $A = A^{-1}$. Firstly, assuming A is symmetric, $A = A^T$. Thus, $A^T = A^{-1}$, which is true if and only if A is orthogonal. Secondly, assuming A is orthogonal, $A^{-1} = A^T$. Thus, from the given identity, $A = A^T$, which is the definition of symmetry. Therefore, A is symmetric if and only if A is orthogonal, as required.

If you want to type your answers try using LaTex!

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