2C Week 8 2020/21

Handout

Tests for convergence III: the Leibniz test

Let's study another useful criterion for convergence.

Theorem 4.16 (Leibniz's Test). *Let $(x_n)_{n=1}^{\infty}$ be a sequence and suppose that there exists an $n_0 \in \mathbb{N}$ such that*

a) $x_n \geq 0$ *for all* $n \geq n_0$*, that is, the sequence is eventually positive;*

b) $x_{n+1} \leq x_n$ *for all* $n \geq n_0$*, that is, the sequence is eventually decreasing;*

c) $x_n \to 0$ *as* $n \to \infty$*.*

Then the series $\sum_{n=1}^{\infty}(-1)^n x_n$ *converges*[1]*.*

For the proof of this theorem we need the following fact about sequence limits. Suppose $(s_n)_{n=1}^{\infty}$ is a sequence and $L \in \mathbb{R}$ has the property that both $s_{2n} \to L$ and $s_{2n-1} \to L$ as $n \to \infty$. That is, the subsequence of even terms converges to L, and the subsequence of odd terms converges to L, too. Then[2] $s_n \to L$ as $n \to \infty$.

Proof. Without loss of generality, by changing the initial terms of $(x_n)_{n=1}^{\infty}$ if necessary, we may assume that $n_0 = 1$. As usual we set $s_n = \sum_{j=1}^{n}(-1)^j x_j$. Then

$$s_{2(n+1)} = s_{2n+2} = s_{2n} - x_{2n+1} + x_{2n+2} \leq s_{2n}$$

for all $n \in \mathbb{N}$, since $0 \leq x_{2n+2} \leq x_{2n+1}$. Thus the sequence $(s_{2n})_{n=1}^{\infty}$ of even partial sums is decreasing. Similarly, we have

$$s_{2(n+1)-1} = s_{2n+1} = s_{2n-1} + x_{2n} - x_{2n+1} \geq s_{2n-1}$$

for all $n \in \mathbb{N}$, since $0 \leq x_{2n+1} \leq x_{2n}$, so that $(s_{2n-1})_{n=1}^{\infty}$ is increasing. Furthermore, $s_{2n} \geq s_{2n-1}$ for all $n \in \mathbb{N}$ and hence

$$s_2 \geq s_4 \geq s_6 \geq \cdots \geq s_{2n} \geq s_{2n-1} \geq \cdots \geq s_1.$$

It follows that $(s_{2n})_{n=1}^{\infty}$ is bounded below by s_1 and $(s_{2n-1})_{n=1}^{\infty}$ is bounded above by s_2. According to the monotone convergence theorem, we have $\lim_{n\to\infty} s_{2n} = S$ and $\lim_{n\to\infty} s_{2n-1} = T$ for some $S, T \in \mathbb{R}$. Moreover $s_{2n} = s_{2n-1} + x_{2n} \to T$ since $x_n \to 0$ as $n \to \infty$. Thus $S = T$, and hence $(s_n)_{n=1}^{\infty}$ converges. $\square$

Example 4.17. Show that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^3-5}$$

converges.

[1] The same could also be said for the series $\sum_{n=1}^{\infty}(-1)^{n-1}x_n$, by multiplying through by -1 and using algebraic properties of limits. So, as one would expect, it doesn't matter whether it's the odd terms which are eventually negative and evens which are eventually positive, or the other way around, what's important is that the terms are eventually alternating. Alternatively, you can easily check directly from the definition that omitting a finite number of terms from a series does not affect whether or not it converges (although it can affect the sum)

[2] You should try and prove this, directly from the definition of convergence.

Solution. Let $x_n = \frac{n^2}{n^3-5}$. Then

$$x_n = \frac{n^2}{n^3-5} \geq 0$$

for $n \geq 2$. Moreover,

$$\begin{aligned} x_n - x_{n+1} &= \frac{n^2}{n^3-5} - \frac{(n+1)^2}{(n+1)^3-5} \\ &= \frac{n^4+2n^3+n^2+10n+5}{(n^3-5)(n^3+3n^2+3n-4)} > 0 \end{aligned}$$

for $n \geq 2$. Hence the sequence $(x_n)_{n=1}^{\infty}$ is eventually decreasing. Finally, we have

$$x_n = \frac{n^2}{n^3-5} = \frac{1/n}{1-5/n^3} \to 0$$

as $n \to \infty$. Hence, by Leibniz's test, the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^3-5}$ converges. □

Absolute convergence

In this section we study absolute convergence, a property which turns out to be stronger than convergence. Let's start by giving some definitions.

Definition 4.18. A series $\sum_{n=1}^{\infty} x_n$ is said to be absolutely convergent if the series $\sum_{n=1}^{\infty} |x_n|$ is convergent. We say that $\sum_{n=1}^{\infty} x_n$ is conditionally convergent if it is convergent, but not absolutely convergent.

Here are a few examples of absolutely convergent series.

Example. *The series $\sum_{n=1}^{\infty} \frac{1}{n^2}\cos(n)$ is absolutely convergent.*

Solution. If we write $x_n = \frac{1}{n^2}\cos(n)$, then clearly $|x_n| \leq \frac{1}{n^2}$ for all $n \in \mathbb{N}$. Hence $\sum_{n=1}^{\infty} |\frac{1}{n^2}\cos(n)|$ converges by the comparison test. By definition, this means $\sum_{n=1}^{\infty} \frac{1}{n^2}\cos(n)$ is absolutely convergent. □

Example. *The series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ is absolutely convergent for all $x \in \mathbb{R}$.*

Solution. Let us write $y_n = \frac{x^n}{n!}$. Then we have $|y_n| = \frac{|x|^n}{n!}$. Hence $\sum_{n=1}^{\infty} |y_n|$ is convergent according to the second example in 4.15. That is, $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ is absolutely convergent. □

We will see next that absolute convergence of a series implies its convergence. That is, absolute convergence is a stronger property than convergence.

Theorem 4.19. *Let $\sum_{n=1}^{\infty} x_n$ be an absolutely convergent series. Then $\sum_{n=1}^{\infty} x_n$ converges.*

Proof. For any $n \in \mathbb{N}$ we have $-|x_n| \leq x_n \leq |x_n|$, so that $0 \leq x_n + |x_n| \leq 2|x_n|$. Since $\sum_{n=1}^{\infty} |x_n|$ converges by assumption, also $2\sum_{n=1}^{\infty} |x_n|$ converges. Hence $\sum_{n=1}^{\infty} x_n + |x_n|$ converges by the comparison test. Finally, since

$$x_n = (x_n + |x_n|) - |x_n|,$$

also $\sum_{n=1}^{\infty} x_n$ converges using algebraic properties. □

The converse of Theorem 4.19 does not hold. That is, there are convergent series which are not absolutely convergent.

Example. *The series* $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ *is convergent, but not absolutely convergent.*

Solution. The series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ is convergent by Leibniz's test, but it is not absolutely convergent because the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent. □

Let's have a look at some further examples.

Example. *Determine whether the series*

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^3-5}$$

is absolutely convergent, conditionally convergent or divergent.

Solution. According to Leibniz's test, the series converges. Since

$$\frac{n^2}{n^3-5} \geq \frac{n^2}{n^3} = \frac{1}{n}$$

for $n \geq 3$, the series $\sum_{n=1}^{\infty}\frac{n^2}{n^3-5}$ diverges by the comparison test. Hence $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^3-5}$ is conditionally convergent. □

Example. *Determine whether the series*

$$\sum_{n=1}^{\infty}\frac{(-2)^n}{n3^n}$$

is absolutely convergent, conditionally convergent or divergent.

Solution. Writing $x_n = \frac{(-2)^n}{n3^n}$ we have $|x_n| = \frac{2^n}{n3^n}$ and

$$\frac{|x_{n+1}|}{|x_n|} = \frac{2^{n+1}}{(n+1)3^{n+1}}\frac{n3^n}{2^n} = \frac{2}{3}\frac{n}{n+1} \to \frac{2}{3}$$

as $n \to \infty$. Hence $\sum_{n=1}^{\infty}\frac{(-2)^n}{n3^n}$ is absolutely convergent according to the limit version of the ratio test. □

Guide for testing convergence

We have seen a number of criteria for testing convergence. So, given a series $\sum_{n=1}^{\infty} a_n$, how should we decide which of these criteria to apply?

Here is a guide for how to systematically make use of the methods we have discussed.

1. Does $a_n \to 0$?

No The series diverges.

Yes Proceed to 2.

2. Are the a_n eventually positive?

Yes

- Is the form of the a_n's amenable to the ratio test (do we find expressions like $x^n, n!, \binom{2n}{n}$ in the definition of a_n, so that there will be cancellations in a_{n+1}/a_n)? If so consider the *ratio test*.
- Otherwise look at the *comparison test*. You may compare the series to one of the standard series we have seen above. Remember in particular that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$, and that $\sum_{n=0}^{\infty} x^n$ converges if and only if $|x| < 1$.

No

- Do the terms alternate in sign $+, -, +, -, +, \dots$? If so consider Leibniz's test.
- Otherwise try to see if the series is absolutely convergent. Use the above methods for $\sum_{n=1}^{\infty} |a_n|$.

It worthwhile to point out that this is only a rough guide, in the sense that it is not guaranteed that we'll find out about convergence/divergence using the above methods — still it helps in many examples, in particular in the examples you'll find on the exercise sheets.

Rearrangements

Given a series $\sum_{n=1}^{\infty} x_n$, we can form a new series by *rearranging* the terms x_n. For instance, we could consider

$$x_1 + x_2 + x_4 + x_3 + x_6 + x_8 + x_5 + x_{10} + x_{12} + \cdots ,$$

A rearrangement corresponds to a relabeling of the indices of the terms of the series[3]. To make that precise we give the following definition.

[3] In particular, all terms x_n appear exactly once in a rearrangement, we are neither allowed to write e.g. $x_1 + x_2 + x_2 + x_4 + x_3 + \cdots$, nor to leave out any x_n.

Definition 4.20. Let $\sum_{n=1}^{\infty} x_n$ be a series. We call a a series $\sum_{n=1}^{\infty} y_n$ a rearrangement of $\sum_{n=1}^{\infty} x_n$ if there exists a bijection $\theta : \mathbb{N} \to \mathbb{N}$ such that $y_n = x_{\theta(n)}$.

A natural question one may ask is how a rearrangement affects convergence. We will present two results in this direction.

Theorem 4.21. *Let $\sum_{n=1}^{\infty} x_n$ be an absolutely convergent series. Then any rearrangement $\sum_{n=1}^{\infty} x_{\theta(n)}$ of $\sum_{n=1}^{\infty} x_n$ is also absolutely convergent and converges to the same sum.*

Proof. Let us write $y_n = x_{\theta(n)}$, so that $\sum_{n=1}^{\infty} x_{\theta(n)} = \sum_{n=1}^{\infty} y_n$.

Assume first that $x_n \geq 0$ for all $n \in \mathbb{N}$. Given any $k \in \mathbb{N}$, let m_k be the maximum of the numbers $\theta(1), \dots, \theta(k)$. Then

$$\sum_{j=1}^{k} y_j = \sum_{j=1}^{k} x_{\theta(j)} \leq \sum_{j=1}^{m_k} x_j \leq \sum_{j=1}^{\infty} x_j,$$

that is, the sequence of partial sums $s_k = \sum_{j=1}^{k} y_j$ is bounded. Moreover it is increasing because all terms x_n are positive. Hence the series $\sum_{n=1}^{\infty} y_n$ is convergent by the monotone convergence theorem. By ordering properties of limits, we also see that $\sum_{n=1}^{\infty} y_n \leq \sum_{n=1}^{\infty} x_n$. Since $\sum_{n=1}^{\infty} x_n$ may also be considered as a rearrangement of $\sum_{n=1}^{\infty} y_n$, a symmetric argument to the above shows that $\sum_{n=1}^{\infty} x_n \leq \sum_{n=1}^{\infty} y_n$, so these sums are equal.

Now consider the general case. For $x \in \mathbb{R}$ we set

$$x^+ = \max\{0, x\}, \quad x^- = \max\{-x, 0\}.$$

Then[4] $x^+ - x^- = x$ and $x^+ + x^- = |x|$. Since

$$0 \leq x_n^+ \leq |x_n|, \qquad 0 \leq x_n^- \leq |x_n|,$$

the series $\sum_{n=1}^{\infty} x_n^+$ and $\sum_{n=1}^{\infty} x_n^-$ converge according to the comparison test. Note that y_n^+ is a rearrangement of x_n^+, and similarly y_n^- is a rearrangement of x_n^-. By the argument above, it follows that $\sum_{n=1}^{\infty} y_n^+$ and $\sum_{n=1}^{\infty} y_n^-$ converge as well and sum to the same corresponding values. This implies that $\sum_{n=1}^{\infty} |y_n| = \sum_{n=1}^{\infty} y_n^+ + y_n^-$ converges too, so $\sum_{n=1}^{\infty} y_n$ converges absolutely. Moreover, we have that

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} x_n^+ - \sum_{n=1}^{\infty} x_n^- = \sum_{n=1}^{\infty} y_n^+ - \sum_{n=1}^{\infty} y_n^- = \sum_{n=1}^{\infty} y_n,$$

where the first and last equalities above follow from Theorem **??**. □

[4] Check this claim by distinguishing the cases $x \geq 0$ and $x < 0$.

We conclude this section with two remarks on rearrangements of conditionally convergent sequences. It turns out that if the requirement of absolute convergence of $\sum_{n=1}^{\infty} x_n$ is weakened to conditional convergence, then things go badly wrong. In fact, the following result holds[5].

[5] A rather strange and surprising result at first sight!

Theorem 4.22. *Let $\sum_{n=1}^{\infty} x_n$ be conditionally convergent and let $T \in \mathbb{R}$ be arbitrary. Then there exists a rearrangement of $\sum_{n=1}^{\infty} x_n$ which converges to T.*

We will not prove this theorem here[6]. For conditionally convergent series, there are also rearrangements which are divergent.

[6] For a proof and more information see chapter 3 in Rudin's book.