2C Intro to real analysis 2020/21

Ex Sheet 1

Solutions and Comments [1] [2]

Q1 *Let P, Q and R be statements. Construct a truth table for*

$$(\text{not}(P \textit{ or } Q) \textit{ and } R).$$

First note that our truth table should have 8 rows. We can see this as there are 2 possibilities for the truth of P (true or false), and 2 for the truth of Q, leading to 4 possibilities for P and Q combined. For each of these 4 possibilities there are 2 possible truth values for R, so we see that there are 8 cases to consider in total.

The truth table is:

P	Q	R	$(\text{not}(P \text{ or } Q) \text{ and } R)$
T	T	T	F
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

If you found this exercise tricky, you should consider working in steps. First write down the truth table for $(P \text{ or } Q)$. Then write down the negation of this line to get the truth table for $\text{not}(P \text{ or } Q)$, and only then complete the table for $(\text{not}(P \text{ or } Q) \text{ and } R)$. That is, fill in

P	Q	R	$(P \text{ or } Q)$	$\text{not}(P \text{ or } Q)$	$(\text{not}(P \text{ or } Q) \text{ and } R)$
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

Q2 *Let P and Q be statements. Use truth tables to show*[3] *that* $\text{not}(P \textit{ and } Q)$ *is equivalent to* $(\text{not}(P) \textit{ or } \text{not}(Q))$.

Unlike the previous question, here you need to persuade the person reading your work that the two truth tables for $\text{not}(P \text{ and } Q)$ and

[1] If you've not seriously tried these exercises, please don't look at these solutions and comments, until you have. You'll get the most benefit from reading these comments, when you've first thought hard about them yourself, even if you get really stuck — don't just try for a few minutes and then look at the solutions to work out how to proceed, you don't learn anywhere near as much that way.

[2] Note that I deliberately do not include formal answers for all questions.

[3] You want to show something in this question, not just write down two truth tables. Therefore you should write some sentences, and provide enough detail in your truth tables so that someone reading it can see that your argument is correct.

(not(P) or not(Q)) are identical, and simply writing these down with no calculation or explanation doesn't really do that. I would suggest putting in the intermediate lines so that the logic is clear.

The truth table for not(P and Q) is

P	Q	P and Q	not(P and Q)
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

while the truth table for (not(P) or not(Q)) is

P	Q	not(P)	not(Q)	(not(P) or not(Q))
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Therefore the two statements are equivalent.

Q3 *Negate the statement "If $x^2 + 2x + 1 = 0$, then $x = -1$".*

In previous years, this question has generally been done rather poorly, with a standard incorrect answer being "If $x^2 + 2x + 1 = 0$, then $x \neq -1$". As the statement is an "if ... then ..." construction it is an implication. Let us write P for the statement "$x^2 + 2x + 1 = 0$" and Q for the statement $x = -1$, so that the question asks for the negation of $P \implies Q$. This negation is P and not(Q)[4]. Thus the answer here is[5]:

The negation of the statements is "$x^2 + 2x + 1 = 0$ and $x \neq -1$".

[4] if you have a tendency to get this wrong, then you really need to think about negating implications carefully - remember the only way an implication can be false is when the hypothesis P is true, and the conclusion Q is false.

[5] Whenever we negate an implication we expect an answer of the form "... and not ...".

Q4 *Consider the implication*

$$P: \quad x > y \implies x^2 > y^2,$$

for integers x and y.

a) *Write down the contrapositive of P, the negation of P and the converse Q of P.*

b) *Give integers x, y for which P is false.*

c) *Give integers x, y for which Q is false.*

The contrapositive of P is

$$x^2 \leq y^2 \implies x \leq y$$

and the converse of P, denoted Q, is

$$Q : x^2 > y^2 \implies x > y.$$

Remembering what it means for an implication to be false[6], for P to be false we need $x > y$ and $x^2 \leq y^2$. This will only be possible when $y < 0$, so that leads me to choose a negative value of y, say $y = -2$. I now look for a value of x which works, and here $x = 1$ does the trick. This gives me an answer[7].

[6] see the previous question

[7] it is not the unique answer, there are other values of x and y which also satisfy the requirements.

Take $x = 1$ and $y = -2$. Then $x > y$ and $x^2 = 1 \leq 4 = y^2$, so P is false for these values of x and y.

Note how my answer clearly specifies the values of x and y and shows that they work. You always want to make life as easy as possible for the person reading your work.

A pair of x, y such that Q is false is given by $x = -2$ and $y = 1$. Then $x^2 = 4 > 1 = y^2$ but $x \leq y$.

Don't be afraid to experiment when answering this sort of question: try some values of x and y and see what happens.

Q5 *Consider the two statements*[8]

$$P: \quad \forall x > 0, \exists y > 0 \text{ s.t. } y < x.$$
$$Q: \quad \exists y > 0 \text{ s.t. } \forall x > 0, y < x.$$

Which, if any, of these statements are true? Justify your answer by either proving P or its negation, and similarly for Q.

[8] Remember that throughout this course quantifications like "$\forall x > 0$" mean "for all $x \in \mathbb{R}$ with $x > 0$".

P is true and Q is false.

In particular, note that one cannot change the order of the universal and existential quantifiers in general: the order of the "$\forall a > 0$" and "$\exists y > 0$" matters. This is an important thing to remember.

How did I see that P is true? Firstly it's very reasonable to draw some pictures and try some examples to see what is going on. Statement P says for every strictly positive real number x (remember the implicit assumption that x is always real when we say $x > 0$) there is some real number y between 0 and x. You should be able to persuade yourself that this is true, perhaps by drawing a number line. You can also try some examples: what y works when $x = 1$? What about when $x = 1/10$? You should be on the lookout for a general strategy for finding the y in terms of x: for example taking y to be a half of x works. This leads us to the proof:

Proof of P. Let $x > 0$ be arbitrary[9]. Define[10] a number y by $y = x/2$. We then have[11] $y > 0$ and $y < x$. Thus P is true[12]. □

To prove that Q is false you should first do some experiments to see that this is the case. This might lead you directly to a proof. You could also try a proof by contradiction, by starting with "Suppose there exists $y > 0$ such that for all $x > 0$ we have $y < x \ldots$". Another way to proceed which may be useful is to start by writing down the formal negation of Q, which is

$$\forall y > 0, \exists x > 0 \text{ s.t. } x \leq y.$$

You can then follow the strategy for proving P to prove the negation of Q. I'll leave the details to you.

Q6 *Consider the two statements*

$$R : \forall x \in \mathbb{R}, \exists y \in \mathbb{R} \textit{ such that } \forall z \in \mathbb{R}, z > y \implies z > x;$$
$$S : \exists y \in \mathbb{R} \textit{ such that } \forall x \in \mathbb{R}, \forall z \in \mathbb{R}, z > y \implies z > x.$$

Determine whether R is true or false. Prove your assertion. Determine whether S is true or false. Prove your assertion.

> R is true and S is false.

You can try and see this by experimenting on a number line and aim for formal proofs in the same way as the previous question. The other thing you might do to get a feel for what is going on is to start by understanding the last chunk of the statements. That is, for real numbers x, y consider the statement $R(x, y)$ given by "$\forall z \in \mathbb{R}, z > y \implies z > x$". Try to understand when this is true[13]. You could start with some examples, say $x = 1$ and $y = 2$ to get a feel for this. Some experimenting should lead you to guess that $R(x, y)$ is equivalent to $y \geq x$, which is the case[14]. You could either try and prove this, and then use the strategies from question 6 to complete your proof. Alternatively, I can use this guessed equivalence to reach the conclusion that R is probably true (given $x \in \mathbb{R}$, we certainly can find $y \in \mathbb{R}$ with $y \geq x$), and in fact this gives me a proof.

Proof of R. Let $x \in \mathbb{R}$ be arbitrary and take $y = x$. Given $z \in \mathbb{R}$ with $z > y$, we have $z > x$, so that $\forall z \in \mathbb{R}, z > y \implies z > x$. Thus R is true. □

Note that I proved the actual statement R and I didn't use the guessed equivalence to do any more than help me understand what's going on — the guessed equivalence doesn't form part of my proof. If this was a question to be marked, it would be just the proof above I'd hand in.

I'll leave it to you to write down a formal proof that S is false.[15]

[9] Note how the proof begins. We want to prove a statement of the form "$\forall x > 0, R(x)$", so we start by fixing an arbitrary x which has $x > 0$, but make no further assumptions about x. If we can prove the statement $R(x)$ for this x, which in our case is "$\exists y > 0$ s.t. $y < x$", we will have proved it for all $x > 0$ as we made no additional assumptions about our x. You should always make sure you introduce your symbols in this way, it's not logically correct to start talking about x without introducing it.

[10] The value $x > 0$ is now fixed, but arbitrary. We have to show that "$\exists y > 0$ s.t. $y < x$". To do this, I should be aiming to tell you what such a y is: the value of y can of course depend on x. Here it depends on x by means of a formula, but in a different question, you might split into cases depending on the value of x and define y in different ways in each case.

[11] It's always good to be really explicit and show the reader that our value of y satisfies the two conditions we need — essentially you want the person reading your proof not to have to think about it, just to be able to see that it works.

[12] The proof is complete as we've shown that for any $x > 0$, we can find the required $y > 0$, but it's good to make this clear to the reader by ending with a conclusion.

[13] which will depend on the values of the dummy variables x and y

[14] Can you prove this?

[15] Again you might want to start by writing down an alternative form for the negation of S.

Q7 *Let $A \subset \mathbb{R}$ and $M \in \mathbb{R}$. By definition, the statement that M is a supremum of A is*

$$(\forall a \in A, a \leq M) \text{ and } (\forall \varepsilon \in \mathbb{R} \text{ with } \varepsilon > 0, \exists a \in A \text{ such that } a > M - \varepsilon).$$

Write down the negation of this statement. Try some examples of M for the interval $A = [1, 3]$ and check when the statement is true and false[16].

[16] We'll be learning more about suprema later in the course.

The negation of the statement is

$$(\exists a \in A \text{ s.t. } a > M) \text{ or } (\exists \varepsilon > 0 \text{ s.t. } \forall a \in A, a \leq M - \varepsilon).$$

This is another question that if you found tricky you should proceed in stages. Try writing P for "$(\forall a \in A, a \leq M)$" and Q for "$\forall \varepsilon > 0, \exists a \in A,$ such that $a > M - \varepsilon$". The first step is to write down the negation of the statement "P and Q" in terms of the negation of P and the negation of Q. Then compute these negations, possibly doing Q in a couple of stages. This should lead you to the answer I gave.

The very last part of the exercise was extension material at this point, and we will cover it carefully in chapter 2 of the lecture notes. If you tried this you should have found that the statement was only true when $M = 3$.