University of Glasgow

Friday, 14 December 2018
4.30 pm – 6.00 pm
(1 hour 30 minutes)

DEGREES OF MSci, MEng, BEng, BSc, MA and MA (Social Sciences)

ALGORITHMIC FOUNDATIONS 2: COMPSCI2003

Answer all questions

This examination paper is worth a total of 60 marks.

The use of calculators is not permitted in this examination.

INSTRUCTIONS TO INVIGILATORS: Please collect all exam question papers and exam answer scripts and retain for school to collect. Candidates must not remove exam question papers.

1. **(a)** Using the laws of logical equivalence show that

$$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$$

is a tautology. [5]

(b) Define what it means for a function to be injective, surjective and bijective. [3]

(c) Is the function $f : \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = x + 2$ injective and/or surjective? (Justify your answer.) [2]

2. Suppose we have the following predicates over the domain (universe of discourse) U of people:

- $P(x)$: x is perfect;
- $F(x)$: x is a friend of mine.

Express the following English statements in logical formulae using the predicates given above.

(a) No one is perfect. [2]

(b) At least one of your friends is perfect. [2]

Express in concise (good) English without variables each of the following propositions:

(c) $\neg \forall x \in U . P(x)$ [2]

(d) $\forall x \in U . (P(x) \wedge F(x))$. [2]

(e) $(\neg \forall x \in U . F(x)) \vee (\exists x \in U . \neg P(x))$ [2]

3. **(a)** For any real numbers $a, b \in \mathbb{R}$, prove by induction that we have "$(a - b)$ is a factor of $(a^n - b^n)$ for all $n \in \mathbb{Z}^+$".
Justify each step. [5]

(b) Prove that the following statement is correct: "for any integer n, if $3 \cdot n + 2$ is odd, then $9 \cdot n + 5$ is even".
Justify each step. [4]

(c) What proof technique did you use in part (b) above? [1]

4. **(a)** How many ways are there for a robot to travel in xyz space from the origin $(0,0,0)$ to the point $(3,2,4)$ by taking 9 steps, where each step corresponds to either:

- moving one unit in the positive x direction;
- moving one unit in the positive y direction;
- moving one unit in the positive z direction.

(Moving in the negative x, y or z direction is prohibited, so that no backtracking is allowed).

Explain your answer. [4]

(b) Suppose that every telephone number in the world is assigned a number that contains:

- a country code which is either a single digit (x_1), two digits (x_1x_2) or three digits ($x_1x_2x_3$);
- followed by a 10-digit telephone number of the form $n_1x_4x_5 - n_2n_3n_4 - x_6x_7x_8x_9$

where $x_i \in \{0,1,2,3,\ldots,9\}$ and $n_i \in \{2,3,4\}$.

How many different telephone numbers would be available worldwide under this numbering scheme?

Explain your answer. [4]

(c) How many students must be in a class to guarantee that at least 5 were born on the same day of the week?

Explain your answer. [2]

5. Suppose there are two boxes of balls, the first box contains three white balls and three blue balls, while the second contains four white and two blue ball.

(a) What is the probability you randonly select a blue ball from the second box. [1]

(b) If you select a ball from both boxes at random, what is the probability you select two white balls. [2]

(c) If you select a ball from both boxes at random, what is the probability you select two of different colours. [3]

(d) Suppose you first choose a box at random and then select a ball from that box at random. What is the probability that a ball from the first box was chosen, given you selected a blue ball. [4]

6. Assume that we have a list l, and are given the functions:

- `head`(l) which returns the first element of a non-empty list;
- `tail`(l) which returns the tail of a non-empty list;
- `isEmpty`(l) returns `true` if the list is empty and `false` otherwise.

For example if l equals $\langle 5,3,4,2,7,8,3,4\rangle$, then `head`$(l)$ would return 5, `tail`(l) would return $\langle 3,4,2,7,8,3,4\rangle$, and `isEmpty`$(l)$ would return `false`.

Using the above functions, in a pseudo code of your choice:

(a) write a recursive function `sum`(l), that returns the summation of the elements in a list.
For example, `sum`$(\langle 1,5,2,3\rangle)$ returns $1+5+2+3=11$. [2]

(b) write a recursive function `present`(e,l), that returns `true` if e appears in the list l and `false` otherwise.
For example, `present`$(6,\langle 1,5,2,3\rangle)$ returns `false` and `present`$(4,\langle 1,2,3,1,2,4,2\rangle)$ returns `true`. [2]

For each relation below over all people, determine if the relation is symmetric, antisymmetric, and/or transitive.
Justify your answer.

(c) $(a,b)\in R_1$ if and only if a was born on the same day as b. [3]

(d) $(a,b)\in R_2$ if a is (strictly) taller than b. [3]