Mathematics 1

Semester 2 Week 3: The algebra of sets (and more counting)

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Login: `https://classresponse.gla.ac.uk`

- 10am Gregory: Session number 612
- 11am Kelvin: Session number 613

Unions

Definition

Fix a set S and let A and B be subsets of S.
The union of A and B is

$$A \cup B = \{s \in S \mid s \in A \text{ or } s \in B\}.$$

The union $A \cup B$ is the smallest subset of S that contains both A and B.

Union is like (inclusive) "or".

More generally

We can generalize this to unions of any family of subsets, e.g. given subsets $S_i \subseteq S$ for $i \in I$ we can define

$$\bigcup_{i \in I} S_i = \{s \in S \mid \exists\, i \in I \text{ with } s \in S_i\}.$$

Intersections

Definition

Fix a set S and let A and B be subsets of S.
The intersection of A and B is

$$A \cap B = \{s \in S \mid s \in A \text{ and } s \in B\}.$$

The intersection $A \cap B$ is the largest subset which is contained in both A and B.

Intersection is like “and”.

More generally

We can generalize this to intersect any family of subsets, e.g. given subsets $S_i \subseteq S$ for $i \in I$ we can define

$$\bigcap_{i \in I} S_i = \{s \in S \mid s \in S_i \ \forall\, i \in I\}.$$

A GRIPE

In Liebeck no ambient set for A and B to be subsets of is specified.

WARNING

Strictly speaking unions and intersections only make sense inside of some fixed set.

There are analogous concepts for any pair of sets, but we don't discuss them here.

BUT...

Despite this, we'll often omit the ambient set when it is clear (e.g. $\mathbb{N}, \mathbb{Z}, \mathbb{R}$).

Multiple response question 1

Question

Choose all correct statements:

(A) $\{2m \mid m \in \mathbb{Z}\} \cap \{3n \mid n \in \mathbb{Z}\} = \{6i \mid i \in \mathbb{Z}\}$

(B) $\{rm \mid m \in \mathbb{Z}\} \cap \{sn \mid n \in \mathbb{Z}\} = \{rsi \mid i \in \mathbb{Z}\}$

(C) $\{m \in \mathbb{Z} \mid m \equiv 0 \mod 2\} \cup \{n \in \mathbb{Z} \mid n \equiv 1 \mod 2\} = \mathbb{Z}$

(D) $\{2m \mid m \in \mathbb{Z}\} \cup \{3n \mid n \in \mathbb{Z}\} = \mathbb{Z}$

Solution

The correct statements are: (A) and (C).

(A) An integer r is divisible by 2 and 3 iff it is divisible by $2 \cdot 3 = 6$ (their lcm).

(B) If $t = \mathrm{lcm}(r, s)$ then the intersection is $\{ti \mid i \in \mathbb{Z}\}$.

(C) $\mathbb{Z}$ is the union of the even and odd integers.

(D) 5 is not in the given union (nor is any integer not divisible by 2 or 3).

Disjointness

Definition

Let $A, B \subseteq S$. We say that A and B are disjoint if

$$A \cap B = \varnothing.$$

Example

The subsets of $\mathbb{R}$

$$\{a \in \mathbb{R} \mid a < 1\} \text{ and } \{b \in \mathbb{R} \mid b > 0\}$$

are not disjoint. The subsets of $\mathbb{Z}$

$$\{a \in \mathbb{Z} \mid a < 1\} \text{ and } \{b \in \mathbb{Z} \mid b > 0\}$$

are disjoint.

Multiple choice question 2

Question

Which of the following correctly defines a partition of S?

(A) Subsets $S_1, \ldots, S_k$ such that $S_1 \cup \cdots \cup S_k = S$ and $S_i \cap S_j = \varnothing$ for $1 \leq i, j \leq k$.

(B) Subsets $S_1, \ldots, S_k$ such that $S_1 \cup \cdots \cup S_k = S$ and $S_1 \cap \cdots \cap S_k = \varnothing$.

(C) Subsets $S_1, \ldots, S_k$ such that $S_1 \cup \cdots \cup S_k = S$ and $S_i \cap S_j = \varnothing$ for each $i \neq j$ with $1 \leq i, j \leq k$.

(D) Subsets $S_1, \ldots, S_k$ such that the S_i are pairwise disjoint.

Solution

The correct answer is (C): it is the only statement that translates to each element of S occurs in precisely one of the S_i.

Distributivity

Intersections and unions satisfy various compatibility conditions, for instance the distributivity law.

Proposition 17.1

Let A, B, and C be subsets of S. Then

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

The "dual" relation also holds.

Proposition

Let A, B, and C be subsets of S. Then

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

WORKING TOGETHER

QUESTION

Let A and S_i for $i \in I$ be subsets of S. Is it true that

$$A \cap (\bigcup_{i \in I} S_i) = \bigcup_{i \in I} (A \cap S_i)?$$

If yes try to prove it, if no find a counterexample.

CLASSRESPONSE VOTE!

Is it true or false?

Solution

Solution - it's true

Suppose $x \in A \cap (\bigcup_{i \in I} S_i)$. Then $x \in A$ and there is a $j \in I$ with $x \in S_j$.
So $x \in A \cap S_j$ and hence in $\bigcup_{i \in I}(A \cap S_i)$.
Suppose $y \in \bigcup_{i \in I}(A \cap S_i)$. Then $y \in A \cap S_k$ for some $k \in I$. So $y \in \bigcup_{i \in I} S_i$ and $y \in A$.
Hence $y \in A \cap (\bigcup_{i \in I} S_i)$.

Difference

Definition

Let A and B be subsets of S. The difference of A and B is

$$A - B = \{x \in A \mid x \notin B\}.$$

This is also often denoted $A \setminus B$.

An important special case of this is:

Definition

The complement of A in S is $S - A$.

That is, the complement of A is all of the elements of S that are not in A.

Multiple choice question 3

Question

Let S be a set and $A, B, C, D \subseteq S$. Which of the following is false?

(A) $A \cap (S - (S - A)) = A$

(B) $(A - B) - C = A - (B - C)$

(C) $S - (A \cup B) = (S - A) \cap (S - B)$

(D) $A - B = A \cap (S - B)$

Solution

(B) is not correct:

$$\begin{aligned}(A - B) - C &= \{a \in A \mid a \notin B \text{ and } a \notin C\} \\ &\neq \{a \in A \mid a \notin B - C\} \\ &= A - (B - C)\end{aligned}$$

if A, B, C not disjoint cf. (A) once we show it is correct!.

Solution continued

Solution

(A) We note that

$$S - (S - A) = S - \{s \in S \mid s \notin A\} = A$$

so $A \cap (S - (S - A)) = A$.

(C) We check that

$$\begin{aligned} S - (A \cup B) &= \{s \in S \mid s \notin A \text{ and } s \notin B\} \\ &= (S - A) \cap (S - B) \end{aligned}$$

(D) $A - B = \{a \in A \mid a \notin B\} = A \cap (S - B)$

Cartesian products

Definition

Let A and B be sets. The Cartesian product of A and B is the set $A \times B$ defined by

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Given sets $A_1, \ldots, A_n$ we can, proceeding by induction, form $A_1 \times \cdots \times A_n$. It has elements

$$A_1 \times \cdots \times A_n = \{(a_1, \ldots, a_n) \mid a_i \in A_i\}.$$

We often write

$$A^n = \underbrace{A \times \cdots \times A}_{n \text{ times}}$$

Example

For instance $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ consists of all triples of real numbers (a, b, c).

Cartesian products II

Let A and B be sets.

Projections

The product $A \times B$ comes with projections

$$\begin{tikzcd} A & A \times B \arrow[l, "p_A"'] \arrow[r, "p_B"] & B \\ a & (a, b) \arrow[l, mapsto] \arrow[r, mapsto] & b \end{tikzcd}$$

These play a special role: the product $A \times B$ is the set which is 'best' at having a function to A and to B. This can be made precise. We won't do it.

Multiple choice question 4

Question

Which of the following statements is incorrect?

(A) The empty product, i.e. the product of no sets, is $\{*\}$, a set with a single element.

(B) Any product $A \times \varnothing$ is empty.

(C) If $A \subseteq S$ and $B \subseteq T$ then $A \times B \subseteq S \times T$.

(D) $A \times B = B \times A$.

Solution

The incorrect statement is (D): unless $A = B$ these sets do not have the same elements. (a, b) and (b, a) are not the same element if $b \neq a$!

$A \times B$ and $B \times A$ are the same in a weaker sense; we will return to this.

Solution continued

Solution

(A) This is like the statement $a^0 = 1$ or $0! = 1$. One can prove it, but that will have to wait!

(B) Any product $A \times \varnothing$ is empty.
The empty set contains no elements, so there are no possible pairs (a, b) with $b \in \varnothing$.

(C) If $A \subseteq S$ and $B \subseteq T$ then $A \times B \subseteq S \times T$.
$A \times B$ is the subset of $S \times T$ consisting of all pairs (s, t) with $s \in A$ and $t \in B$.

Recall

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We discussed several important definitions last time:

- Unions and intersections
- Disjointness
- Distributivity $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- Differences of sets
- Cartesian products

Today we'll do some counting.

Finite and infinite sets

Definition

We call a set S finite if it has a finite number of elements. If S has n elements we write

$$|S| = n.$$

We call this the cardinality of S.

Definition

We call a set T infinite if T is not finite. Infinite sets also have cardinalities; we'll come to this later.

Example

The sets $\varnothing$, $\{1, 2, \ldots, n\}$, and $\{x \in \mathbb{R} \mid x^2 = 1\}$ are finite.
The sets $\mathbb{R}$ and $\{n \in \mathbb{Z} \mid n \text{ is divisible by } 2\}$ are infinite.

Inclusion-Exclusion

Proposition 17.2

Let A and B be finite subsets of S. Then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Proof

Let $A \cap B = \{c_1, \dots, c_k\}$, so that $|A \cap B| = k$.
We can thus write
$A = \{a_1, \dots, a_m, c_1, \dots, c_k\}$, $B = \{b_1, \dots, b_n, c_1, \dots, c_k\}$ and so
$A \cup B = \{a_1, \dots, a_m, b_1, \dots, b_n, c_1, \dots, c_k\}$.
Counting the elements in each set we see:
$|A \cup B| = m + n + k$
and
$|A| + |B| - |A \cap B| = (m + k) + (n + k) - k = m + n + k$.

Multiple choice question 1

Question

There are 28 bands playing at a black and viking metal festival. 16 of the bands play black metal and 9 of the bands play both black and viking metal. How many of the bands play viking metal?

(A) 21

(B) 28

(C) 12

(D) 9

Solution

Let B be the set of black metal bands and V the set of viking metal bands. Then

$$|V| = |B \cup V| - |B| + |B \cap V| = 28 - 16 + 9 = 21.$$

General inclusion-exclusion

We've just seen that:

Proposition 17.2

Let A and B be finite subsets of S. Then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

What if there are more subsets?

An example

Example

Suppose we have $A, B, C \subseteq S$. Then $A \cup B \cup C = A \cup (B \cup C)$.

$$|A \cup (B \cup C)| = |A| + |B \cup C| - |A \cap (B \cup C)|.$$

By distributivity $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
Using inclusion-exclusion again

$$|(A \cap B) \cup (A \cap C)| = |A \cap B| + |A \cap C| - |A \cap B \cap C|$$

and

$$|B \cup C| = |B| + |C| - |B \cap C|.$$

We can combine these expressions to find that

$$|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$

The general theorem

Theorem 17.1

Let n be a positive integer, and let $A_1, \ldots, A_n$ be finite subsets of S. Then

$$|A_1 \cup \cdots \cup A_n| = c_1 - c_2 + c_3 - \cdots + (-1)^{n-1} c_n,$$

where for $1 \leq i \leq n$, the number c_i is the sum of the sizes of the i-fold intersections

$$c_i = \sum_{\{k_1,\ldots,k_i\} \subseteq \{1,\ldots,n\}} |A_{k_1} \cap \cdots \cap A_{k_i}|.$$

Example

Example

For $n = 4$:

c_1 $|A_1| + |A_2| + |A_3| + |A_4|$

c_2 $|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|$

c_3 $|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|$

c_4 $|A_1 \cap A_2 \cap A_3 \cap A_4|$

Multiple Choice Question 2

Question

Let n be a positive integer, and let $A_1, \ldots, A_n$ be finite subsets of S. Which of the following statements is incorrect?

(A) One could prove the Inclusion-Exclusion Theorem inductively

(B) The terms $c_2, \ldots, c_n$ all vanish precisely if the A_i are pairwise disjoint

(C) The formula we obtained is related to the binomial coefficients

(D) The formula fails if for some i and j we have $A_i = A_j$

Solution

(D) is incorrect: as long as the A_i are finite (so the formula makes sense) it works.

The proof

We now discuss the proof of the theorem (following Liebeck).

Proof of Theorem 17.1

Let $x \in A_1 \cup \cdots \cup A_n$. We will show x contributes exactly 1 to the right-hand side of

$$|A_1 \cup \cdots \cup A_n| = c_1 - c_2 + c_3 - \cdots + (-1)^{n-1} c_n.$$

Suppose x belongs to exactly k of the subsets $A_1, \ldots, A_n$. Then x contributes

- k to the sum $c_1 = |A_1| + \cdots |A_n|$.
- In $c_2 = \sum_{i<j} |A_i \cap A_j|$ the element x appears precisely when x is in both A_i and A_j. There are $\binom{k}{2}$ such terms, so x contributes $\binom{k}{2}$.
- In general x contributes $\binom{k}{i}$ to c_i.

PROOF CONTINUED

So the total contribution of x to

$$c_1 - c_2 + c_3 - \cdots + (-1)^{n-1} c_n$$

is $k - \binom{k}{2} + \binom{k}{3} - \cdots + (-1)^{k-1}\binom{k}{k}$. We proved last week (Proposition 16.3) that

$$k - \binom{k}{2} + \binom{k}{3} - \cdots + (-1)^{k-1}\binom{k}{k} = 1.$$

Hence each element of $|A_1 \cup \cdots \cup A_n|$ contributes 1 to the right-hand side. This shows that both sides are equal.

Euler's totient function

Definition

For a positive integer n, define

$$\phi(n) = |\{1 \leq m \leq n \mid \mathrm{hcf}(m, n) = 1\}|.$$

We call ϕ the Euler totient function or Euler ϕ-function.

Example

$\{1 \leq m \leq 10 \mid \mathrm{hcf}(m, 10) = 1\} = \{1, 3, 7, 9\}$. So $\phi(10) = 4$.

A formula for ϕ

Proposition 17.3

Let $n \geq 2$ be an integer with prime factorization

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$

(where the primes p_i are distinct and all $a_i \geq 1$). Then

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right).$$

Multiple response question 3

$$\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_k}\right).$$

Question

Which of the following statements are correct?

(A) For a prime p, $\phi(p) = p - 1$.

(B) For primes p, q we have $\phi(pq) = \phi(p)\phi(q)$.

(C) $\phi(n) + \phi(m) = \phi(n+m)$.

(D) $\phi(15) = 8$.

Question

Solution

(A) For a prime p, $\phi(p) = p - 1$.
For every $m < p$ we have $\operatorname{hcf}(m, p) = 1$.

(B) For primes p, q we have $\phi(pq) = \phi(p)\phi(q)$.
$\phi(pq) = pq\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right) = pq - q - p + 1$ and
$\phi(p)\phi(q) = (p-1)(q-1) = pq - q - p + 1$.

(C) $\phi(n) + \phi(m) = \phi(n + m)$.
This is false: e.g. $\phi(3) + \phi(5) = 2 + 4 = 6 \neq \phi(8) = 4$.

(D) $\phi(15) = 8$.

$$\phi(15) = \phi(5 \cdot 3) = 15\left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{3}\right) = 8.$$

The proof

We won't go through the proof. But you should read it and Example 17.4 (you should have done this already!). It's a good example of the use of the inclusion-exclusion theorem.

Powerset

Definition

Let S be a set. The powerset of S is

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

i.e. it is the set of subsets of S. It is naturally ordered by inclusion.

Example

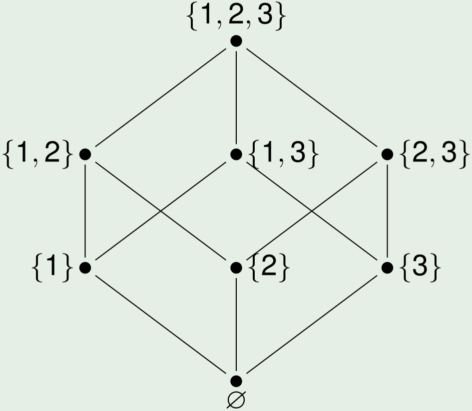
$\mathcal{P}(\{1,2,3\})$ consists of the subsets

$$\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$$

A PICTURE

EXAMPLE

$\mathcal{P}(\{1,2,3\})$



The number of subsets

Proposition 17.4

Let S be a finite set. Then

$$|\mathcal{P}(S)| = 2^{|S|}.$$

So if S has n elements then $\mathcal{P}(S)$ has 2^n elements.

The proof

Proof

Let $S = \{1, 2, \ldots, n\}$. A subset $\{i_1, \ldots, i_k\}$ can be represented uniquely as a binary string of length n, with a 1 in positions $i_1, \ldots, i_k$ and 0 elsewhere.

Example

If $S = \{1, 2, 3\}$ then $\varnothing \longleftrightarrow 000$, $\{1, 3\} \longleftrightarrow 101$ etc...

So there are n choices to make, with 2 possibilities each time. Thus there are 2^n choices, and hence 2^n subsets.

Question

Multiple response question 4

Let S be a finite set, with powerset $\mathcal{P}(S)$. Which of the following statements are correct?

(A) The powerset is closed under taking unions

(B) The powerset is closed under taking intersections

(C) The powerset has a smallest element

(D) The powerset has a greatest element

Solution

All of these are correct!

Bonus meme: union-closed subsets

Let S be a non-empty finite set. We just observed that $\mathcal{P}(S)$ is closed under unions: if $A, B \in \mathcal{P}(S)$ then $A \cup B \in \mathcal{P}(S)$.

Observation

Every element $s \in S$ is in precisely half of the elements of $\mathcal{P}(S)$. Indeed, for $A \in \mathcal{P}(S)$, i.e. $A \subseteq S$, either $s \in A$ or $s \notin A$.
If $s \notin A$ then $s \in S - A$.
So s is in exactly half the subsets of S.

Bonus meme: the union-closed problem

Suppose $\mathcal{U} \subsetneq \mathcal{P}(S)$ is a non-empty collection of subsets closed under unions.

Heuristic

- $\mathcal{P}(S)$ is the largest collection of union-closed subsets of S.
- in $\mathcal{P}(S)$ any element of S is contained in exactly half of the subsets.
- Since $\mathcal{U}$ has fewer subsets, it's easier for an element to be in at least half of the subsets. $\mathcal{P}(S)$ should be as bad as it gets.

Question

Is there an element of S which is in at least half of the subsets in $\mathcal{U}$?

BONUS MEME: THE UNION-CLOSED PROBLEM

This is an open question!

CONJECTURE, FRANKL 1979

For every finite, non-empty, union-closed family of finite sets there exists an element that belongs to at least half of the sets.