2C Intro to real analysis 2020/21

Ex Sheet 2

Solutions and Comments [1] [2]

Q1 *Find $K \in \mathbb{R}$ so that the implication*

$$|x+2| \le 1 \implies \left|\frac{x+5}{x-1}+1\right| \le K|x+2|$$

is true for all $x \in \mathbb{R}$. As always, make sure you fully justify your answer.

This exercise is very similar to example 1.8 from lectures. The first step is to simplify the expression $|\frac{x+5}{x-1}+1|$ which we will need to bound, which we do by

$$\left|\frac{x+5}{x-1}+1\right| = \left|\frac{x+5+x-1}{x-1}\right| = \frac{2|x+2|}{|x-1|}.$$

Now look at what we have and what the question asks us to do. We must find a constant $K \in \mathbb{R}$ such that $\frac{2|x+2|}{|x-1|} \le K|x+2|$ whenever $|x+2| \le 1$. We can see that there's already a factor of $|x+2|$ on both sides of the inequality we're trying to establish, so the question now amounts to finding an upper bound for $\frac{2}{|x-1|}$ when $|x+2| \le 1$. To see what happens you could draw a graph of $\frac{2}{|x-1|}$; this should show you what sort of upper bound to expect.

Now we compute what happens to $\frac{2}{|x-1|}$ when $|x+2| \le 1$. First,

$$\begin{aligned} |x+2| \le 1 &\implies -1 \le x+2 \le 1 \\ &\implies -4 \le x-1 \le -2. \end{aligned}$$

Since we need information about $|x-1|$ we need to convert the information we've gained about $x-1$ to information about $|x-1|$. In this case, as $-4 \le x-1 \le -2$, the number $x-1$ is negative, so $|x-1| = -(x-1)$ and $2 \le |x-1| \le 4$. You should be careful to think about this sort of step, for example if we'd reached $-4 \le x-1 \le 2$, say, then we get $|x-1| \le 4$, but we don't get a helpful lower bound on $|x-1|$ — all we can say is $0 \le |x-1|$. The difference here is that 0 lies between -4 and 2, whereas in the question we'd got bounds which are both negative.

Finally, we need to get an upper bound on $\frac{2}{|x-1|}$, and this will come from a lower bound on $|x-1|$. Precisely,

$$2 \le |x-1| \le 4 \implies \frac{1}{4} \le \frac{1}{|x-1|} \le \frac{1}{2} \implies \frac{1}{2} \le \frac{2}{|x-1|} \le 1.$$

This is another place to be careful[3].

Now we've arrived at an upper bound $\frac{2}{|x-1|} \le 1$ under the assumption that $|x+2| \le 1$, it's time to write this up properly. The solution I'd hand in is:

[1] If you've not seriously tried the exercises, please don't look at these solutions and comments, until you have. You'll get the most benefit from reading these comments, when you've first thought hard about them yourself, even if you get really stuck — don't just try for a few minutes and then look at the solutions to work out how to proceed, you don't learn anywhere near as much that way.

[2] Note that I deliberately do not include formal answers for all questions.

[3] A standard mistake is to rearrange the inequalities $2 \le |x-1| \le 4$ incorrectly as $\frac{1}{2} \le \frac{1}{|x-1|} \le \frac{1}{4}$.

We have
$$\left|\frac{x+5}{x-1}+1\right| = \left|\frac{x+5+x-1}{x-1}\right| = \frac{2|x+2|}{|x-1|}.$$
Now, we have
$$\begin{aligned}|x+2| \leq 1 &\implies -1 \leq x+2 \leq 1\\ &\implies -4 \leq x-1 \leq -2\\ &\implies 2 \leq |x-1| \leq 4.\end{aligned}$$
So, assuming that $x \in \mathbb{R}$ satisfies $|x+2| \leq 1$, we have
$$\frac{2|x+2|}{|x-1|} \leq \frac{2|x+2|}{2} = |x+2|.$$
Therefore, if we take $K = 1$ then we have
$$|x+2| \leq 1 \implies \left|\frac{x+5}{x-1}+1\right| \leq K|x+2|.$$

Note that any value of K with $K \geq 1$ is also correct, provided it's suitably justified.

Q2 *Use the polynomial estimation lemma from the lecture notes to show that there exist $K, R > 0$ such that*
$$x \geq R \implies \frac{4x^2-3x+2}{5x+1} \geq Kx$$
is true for all $x \in \mathbb{R}$. As always, make sure you fully justify your answer.

This example is similar to example 1.10 from lectures. Our first step is to use Lemma 1.9 twice, to get bounds on the polynomials appearing in the fraction that we must bound below. Thus

By Lemma 1.9, there exists $R_1 > 0$ such that
$$x \geq R_1 \implies \frac{1}{2}4x^2 \leq 4x^2 - 3x + 2 \leq \frac{3}{2}4x^2,$$
and there exists $R_2 > 0$ such that
$$x \geq R_2 \implies \frac{1}{2}5x \leq 5x+1 \leq \frac{3}{2}5x.$$

We want to use both of the inequalities above, and so we need both the hypotheses $x \geq R_1$ and $x \geq R_2$ to hold. It is for this reason we take $R = \max(R_1, R_2)$, so that if $x \geq R$, then $x \geq R_1$ and $x \geq R_2$. Then we need to use both inequalities to get a lower bound on $\frac{4x^2-3x+2}{5x+1}$. When you do this, remember that we need to estimate $\frac{1}{5x+1}$ from below, and that $\frac{1}{2}5x \leq 5x+1 \leq \frac{3}{2}5x$ implies $\frac{2}{15x} \leq \frac{1}{5x+1} \leq \frac{2}{5x}$ for $x > 0$. A possible mistake here is to get the inequalities the wrong way round and claim that $\frac{2}{5x} \leq \frac{1}{5x+1} \leq \frac{2}{15x}$, this would lead to the incorrect value $K = \frac{4}{5}$.

Now we finish the answer as follows:

Take $R = \max(R_1, R_2)$. Then

$$\begin{aligned} x \geq R &\implies \frac{1}{2}4x^2 \leq 4x^2 - 3x + 2 \leq \frac{3}{2}4x^2 \text{ and } \frac{1}{2}5x \leq 5x + 1 \leq \frac{3}{2}5x \\ &\implies \frac{4x^2 - 3x + 2}{5x + 1} \geq \frac{4x^2/2}{15x/2} = \frac{4x}{15}. \end{aligned}$$

Thus we can take $K = \frac{4}{15}$.

Q3 *Show that there exist $K, R > 0$ such that*

$$x \geq R \implies \left| \frac{3x^2 - 5x + 1}{2x^2 + x + 1} - \frac{3}{2} \right| \leq \frac{K}{x}.$$

As always, make sure you fully justify your answer.

This question is very similar to the previous one - the only extra step is that we need to simplify the expression $\left| \frac{3x^2-5x+1}{2x^2+x+1} - \frac{3}{2} \right|$ before proceeding as above, and we'll need to take care of the modulus. Let me go straight to the answer this time.

We have

$$\begin{aligned} \left| \frac{3x^2 - 5x + 1}{2x^2 + x + 1} - \frac{3}{2} \right| &= \left| \frac{(6x^2 - 10x + 2) - (6x^2 + 3x + 3)}{4x^2 + 2x + 2} \right| \\ &= \left| \frac{-13x - 1}{4x^2 + 2x + 2} \right| = \frac{|13x + 1|}{|4x^2 + 2x + 2|}. \end{aligned}$$

Note that for $x \geq 0$, we have $13x + 1 \geq 0$ and $4x^2 + 2x + 2 \geq 0$ so that

$$\frac{|13x + 1|}{|4x^2 + 2x + 2|} = \frac{13x + 1}{4x^2 + 2x + 2}.$$

Now, by Lemma 1.9, there exists $R_1 > 0$ and $R_2 > 0$ such that

$$x \geq R_1 \implies \frac{1}{2}13x \leq 13x + 1 \leq \frac{3}{2}13x$$

and

$$x \geq R_2 \implies \frac{1}{2}4x^2 \leq 4x^2 + 2x + 2 \leq \frac{3}{2}4x^2.$$

Take $R = \max(R_1, R_2)$, so that

$$x \geq R \implies \frac{13x + 1}{4x^2 + 2x + 2} \leq \frac{39x/2}{4x^2/2} = \frac{39}{4x}.$$

Thus we can take $K = 39/4$, and then

$$x \geq R \implies \left| \frac{3x^2 - 5x + 1}{2x^2 + x + 1} - \frac{3}{2} \right| \leq \frac{39}{4x}.$$

Again notice the importance of using the correct inequalities when estimating the numerator.

There are other estimates one could have done here. For example, for $x \geq 1$, we have

$$\frac{13x+1}{4x^2+2x+2} \leq \frac{13x+x}{4x^2} = \frac{7}{2x},$$

so taking $R = 1$ and $K = 7/2$ also works. In particular, it's worth pointing out that there are many valid pairs of K and R for which the implication is valid.

Q4 *Show that $A = \{3x - 2y + (1/z) \mid x, y, z \in (2,4)\}$ is bounded.*

With this question (and the following one), the first thing to do is to make sure you know what you are being asked to do. We should look up the definition of bounded sets, and related results. When we do this, we discover that a set $A \subseteq \mathbb{R}$ is *bounded* if and only if it is both bounded above and bounded below. We then look up these definitions: A is bounded above if and only if there exists $M \in \mathbb{R}$ such that for all $a \in A, a \leq M$; similarly, A is bounded below if and only if there exists $m \in \mathbb{R}$ such that for all $a \in A, m \leq a$. So we now try to find values of M and m which work.

Let $x, y, z \in (2,4)$ be arbitrary. Then

$$6 < 3x < 12, \quad -8 < -2y < -4, \text{ and } \frac{1}{4} < \frac{1}{z} < \frac{1}{2}.$$

Adding these inequalities, we have

$$-\frac{7}{4} < 3x - 2y + \frac{1}{z} < \frac{17}{2},$$

so A is bounded above by $17/2$ and below by $-7/4$. Hence A is bounded.

Alternatively, recall that Lemma 2.5 gave the alternative characterisation of bounded sets: A is bounded if and only if there exists $K > 0$ such that for all $a \in A, |a| \leq K$. This gives the following answer.

Let $x, y, z \in (2,4)$ be arbitrary. Then, using the triangle inequality,

$$\begin{aligned}|3x - 2y + 1/z| &\leq |3x| + |-2y| + |1/z| = 3|x| + 2|y| + |1/z| \\ &< 3 \cdot 4 + 2 \cdot 4 + 1 \cdot \frac{1}{2} = \frac{41}{2},\end{aligned}$$

so A is bounded.

Note that the upper bound $41/2$ we get in the second answer isn't the same as in our first answer. Indeed, the first answer gives us the least upper bound of A, whereas in the second answer we just get some upper bound. This doesn't make the first answer better — we only had to show that A is bounded, not find the least upper bound and greatest lower bounds. Both answers are equally good.

Q5 *Show that $B = \{\frac{1}{x+1} \mid x < -1\}$ is bounded above but not bounded below.*

Firstly, the bounded above bit:

For $x < -1$, we have $\frac{1}{x+1} < 0$, so 0 is an upper bound for B.

Now for the lower bound. If you find this question confusing, it's good to be clear about what we are trying to do. The statement that B is bounded below is

$$\exists m \in \mathbb{R} \text{ s.t. } \forall b \in B, m \leq b.$$

Thus the statement that B is not bounded below is

$$\forall m \in \mathbb{R}, \exists b \in B \text{ s.t. } b < m.$$

Given the general form of an element of B, this becomes:

$$\forall m \in \mathbb{R}, \exists x < -1 \text{ s.t. } \frac{1}{x+1} < m$$

and so this is the statement that we will prove. Thus the first line of our argument is to introduce an arbitrary value of $m \in \mathbb{R}$, and we will then find a value of $x < -1$ (which will depend on m) so that $\frac{1}{x+1} < m$.

Let $m \in \mathbb{R}$ be arbitrary. If $m \geq 0$, then take $x = -2$ so that $\frac{1}{x+1} = -1 < m$. Suppose then that $m < 0$. Then

$$\frac{1}{x+1} < m \Leftrightarrow \frac{1}{m} < x+1 \Leftrightarrow -1 + \frac{1}{m} < x.$$

Therefore we can take $x = -1 + \frac{1}{2m}$, so that $-1 + \frac{1}{m} < x < -1$ and $\frac{1}{x+1} < m$, as required. Thus B is not bounded below.

Q6 *Let $x, y \in \mathbb{R}$. Use the triangle inequality to prove*[4] *the reverse triangle inequality*

$$\big||x| - |y|\big| \leq |x - y|.$$

[4] You may wish to do this in two cases, by proving both $|x| - |y| \leq |x - y|$ and $|y| - |x| \leq |x - y|$. Writing $x = x - y + y$ might help.

The hint provides the motivation explaining where the answer below comes from.

Let $x, y \in \mathbb{R}$. Then, using the triangle inequality, we have

$$|x| = |x - y + y| \leq |x - y| + |y|,$$

and
$$|y| = |y - x + x| \leq |y - x| + |x| = |x - y| + |x|.$$
Therefore both $|x| - |y| \leq |x - y|$ and $|y| - |x| \leq |x - y|$. Hence $\big||x| - |y|\big| \leq |x - y|$, as required.

Q7 *Prove that for $x, a \in \mathbb{R}$,*
$$|x - a| < \frac{1}{2}|a| \implies \frac{1}{2}|a| < |x| < \frac{3}{2}|a|.$$
Is the converse implication always true? Provide a proof or a counterexample.

Here's an approach using the reverse triangle inequality.

Using the reverse triangle inequality for $x, a \in \mathbb{R}$, we have
$$\Big||x| - |a|\Big| < |x - a|.$$
Therefore, using the properties of the modulus function,
$$\begin{aligned} |x - a| < \frac{1}{2}|a| &\implies \Big||x| - |a|\Big| < \frac{1}{2}|a| \\ &\implies -\frac{1}{2}|a| < |x| - |a| < \frac{1}{2}|a| \\ &\implies \frac{1}{2}|a| < |x| < \frac{3}{2}|a|. \end{aligned}$$

Is the converse true? Well, note that the hypotheses of the implication ($|x - a| < \frac{1}{2}|a|$) can only hold when x and a have the same sign, while this is not the case for the conclusion. I'd be looking for a counterexample to the converse when x and a have different signs.

Q8 *(This is an extra exercise, which is not really part of the course). Show*[5] *that there is no order $<$ on $\mathbb{C}$ satisfying the order axioms:*

[5] You may wish to consider what happens if $0 < i$ and if $i < 0$.

a) *for all $a \in \mathbb{C}$, exactly one of the statements $a = 0$, $a > 0$ and $0 > a$ is true.*

b) *for all $a, b \in \mathbb{C}$, with $0 < a$ and $0 < b$, we have $0 < a + b$;*

c) *for all $a, b \in \mathbb{C}$, with $0 < a$ and $0 < b$, we have $0 < ab$;*

Following the hint, let's see what happens if $0 < i$.

Suppose $0 < i$. Then by (c), we have $0 < -1$.

This is not a contradiction — we are not assuming that the order $<$ extends the usual order on $\mathbb{R}$, so we can not assume that $0 < 1$. We must carry on to reach the desired contradiction.

Using (c) again (applied to $0 < -1$ and $0 < i$) we have $0 < -i$. Now using (b), we have $0 < i + (-i) = 0$, and this contradicts (a).

Now assume $0 < -i$, and obtain a contradiction in a similar way.