2C Week 3 2020/21

Least upper bounds and greatest lower bounds

Let $A \subseteq \mathbb{R}$ be a set and let M be an upper bound for A. Then any real number K with $K \geq M$ is also an upper bound for A; in particular, if A has an upper bound then it will have infinitely many upper bounds. So if you are asked to prove a set A is bounded above, exhibiting any upper bound M will do, you do not need to try and find the smallest M which works[1].

Our next focus is on looking for the best, or more accurately *least* upper bound; this concept will enable us to describe the final axiom needed to distinguish between $\mathbb{Q}$ and $\mathbb{R}$.

Definition 2.1. Let $A \subseteq \mathbb{R}$ and $M \in \mathbb{R}$. Define M to be a[2] *least upper bound for A* if and only if the following two conditions are satisfied:

a) M is an upper bound for A; and

b) for all upper bounds M' for A, we have $M \leq M'$.

Note that we do not talk of the "maximum element" of A. For example, when $A = (0, 1)$, it is straightforward to see that 1 is a least upper bound for A, but $1 \notin A$, so it is not a maximal element of A. Note also that when $B = (0, 1]$ we again have that 1 is a least upper bound for B, so it is possible for a least upper bound to be in the set, and also for a least upper bound not to be in the set. When we work with least upper bounds, the following reformulation of the second condition is very helpful[3].

Lemma 2.2. *Let $A \subseteq \mathbb{R}$ and let $M \in \mathbb{R}$ be an upper bound for A. Then M is a least upper bound for A if and only if*

$$\forall \varepsilon > 0, \exists a \in A \text{ s.t. } a > M - \varepsilon. \tag{1}$$

Proof. $\Rightarrow$ Let M be a least upper bound for A, and let $\varepsilon > 0$ be arbitrary. Then $M - \varepsilon < M$, so $M - \varepsilon$ is not an upper bound for A. Therefore[4] there exists $a \in A$ with $a > M - \varepsilon$.

$\Leftarrow$ Let M be an upper bound for A and suppose that condition (1) holds. Let $M' < M$, and write $\varepsilon = M - M'$, so that $\epsilon > 0$. Then $M' = M - \varepsilon$, and so the condition shows that there exists $a \in A$ with $a > M'$. Thus M' is not an upper bound for A, and hence M is a least upper bound for A. $\square$

Throughout this course I will be happy for you to use this lemma to prove that M is a least upper bound for a set A without mentioning it[5]. Let's do a quick example.

Example Show that 3 is a least upper bound of $A = \left\{ \frac{3n}{n+1} \mid n \in \mathbb{N} \right\}$.

[1] For example, consider $A = \{\cos(x) + \sin(x+3) \mid x \in \mathbb{R}\}$. We can see that for any $x \in \mathbb{R}$, $\cos(x) + \sin(x+3) \leq 1 + 1 = 2$, so the set A is certainly bounded above by 2. We could work harder to try and find a smaller upper bound for A, but we don't need to in order to show that A is bounded above.

[2] The use of "a" here is deliberate. We write "a" when it is possible that there could be more than one object satisfying a certain definition. In Theorem 2.4 below, we will show that a set has at most one least upper bound. From that point on we can refer to "the" least upper bound (when it exists).

[3] Putting it in the form of a quantified statement, in a style we've discussed how to prove.

[4] Recall that, by negating the definition of M' is an upper bound for A, M' is not an upper bound for A if and only if there exists $a \in A$ with $a > M'$.

[5] So if you want to prove that M is a least upper bound for A, first show M is an upper bound for A, then show $\forall \varepsilon > 0, \exists a \in A$ s.t. $a > M - \varepsilon$.

Solution. We first show that 3 is an upper bound. Given $n \in \mathbb{N}$, we have
$$\frac{3n}{n+1} \leq \frac{3n}{n+1} + \frac{3}{n+1} = 3,$$
so 3 is an upper bound for A. Now, let $\varepsilon > 0$ be arbitrary. We need to show that there exists $n \in \mathbb{N}$ such that
$$\frac{3n}{n+1} > 3 - \varepsilon,$$
as then $a = \frac{3n}{n+1} \in A$, and $a > 3 - \varepsilon$, showing that 3 is a least upper bound for A by Lemma 2.2. Fortunately we've shown that such an n exists in example 1.7! $\square$

We also introduce the analogous concept of greatest lower bounds.

Definition 2.3. Let $A \subseteq \mathbb{R}$ and $m \in \mathbb{R}$. Define m to be a *greatest lower bound for A* if and only if:

a) m is an lower bound for A; and

b) for all lower bounds m' for A, we have $m' \leq m$.

In the same way as Lemma 2.2, given a lower bound m for A, m is a greatest lower bound for A if and only if
$$\forall \varepsilon > 0, \exists a \in A \text{ s.t. } a < m + \varepsilon.$$

We also note that least upper bounds, and greatest lower bounds are unique (when they exist).

Theorem 2.4. *Let $A \subseteq \mathbb{R}$. Then A has at most one least upper bound and at most one greatest lower bound*[6].

[6] This is a uniqueness theorem, and we prove it using a standard strategy for theorems of this sort: we assume there are two least upper bounds, and show that they are equal. Compare this with the proof that the identity element in a group is unique from 2F.

Proof. Let us consider uniqueness of least upper bounds. Suppose that M and M' are least upper bounds for A. Then, as M is a least upper bound for A, and M' is an upper bound for A, we have $M \leq M'$. Similarly, as M' is a least upper bound for A, and M is an upper bound for A, we have $M' \leq M$. Therefore $M = M'$.

The proof of the uniqueness of greatest lower bounds (when they exist) is similar, it is left as an exercise. $\square$

Now we know that least upper bounds and greatest lower bounds are unique (when they exist), it makes sense to refer to "the" least upper bound and "the" greatest lower bound, respectively. We also introduce some terminology which we'll use henceforth[7]. When M is the least upper bound for A, we call M the *supremum* of A and write $M = \sup(A)$. When m is the greatest lower bound for A we call m the infimum of A and write $M = \inf(A)$.

[7] The terminology "least upper bound" is useful as it describes the definition precisely, but its a bit cumbersome: supremum and infimum are more standard.

Example Show that
$$\inf \left\{ \frac{n+1}{n} \mid n \in \mathbb{N} \right\} = 1.$$

Solution. Let $A = \left\{ \frac{n+1}{n} \mid n \in \mathbb{N} \right\}$. We have to prove two statements, namely that 1 is a lower bound for A, and secondly that it is the greatest lower bound for A[8]. Firstly, for $n \in \mathbb{N}$, we have

$$\frac{n+1}{n} \geq 1,$$

so that 1 is a lower bound[9] for A. Now let $\varepsilon > 0$ be arbitrary[10]. For $n \in \mathbb{N}$, we have

$$\begin{aligned} \frac{n+1}{n} < 1 + \varepsilon &\Leftrightarrow n + 1 < n + n\varepsilon \\ &\Leftrightarrow \frac{1}{\varepsilon} < n. \end{aligned}$$

Now take [11] $n \in \mathbb{N}$ with $n > \frac{1}{\varepsilon}$, so that $\frac{n+1}{n} \in A$ and $\frac{n+1}{n} < 1 + \varepsilon$. Therefore we have $\inf(A) = 1$. □

[8] We expect the first statement to be easier to prove than the second statement.

[9] In this case, this was quite straightforward, but nevertheless we should still write enough so that someone reading our work can see that we have checked that 1 is a lower bound.

[10] We are going to prove the quantified statement $\forall \varepsilon > 0, \exists a \in A$ such that $a < 1 + \varepsilon$, so we start by letting $\varepsilon > 0$ be arbitrary.

[11] The point is that we can see that such an n exists, as the natural numbers are not bounded above (see the discussion of Archimedes axiom later), in contrast it wasn't so immediate that there is an $n \in \mathbb{N}$ such that $\frac{n+1}{n} < 1 + \varepsilon$.

Let's do one more example. This time the set we will establish the supremum of is specified in terms of two other sets.

Example Let A and B be two subsets of $\mathbb{R}$ such that $\sup(A)$ and $\sup(B)$ exists. Define

$$C = \{3a + b \mid a \in A, b \in B\}.$$

Show that $\sup(C)$ exists and $\sup(C) = 3\sup(A) + \sup(B)$.

The strategy will be to show that the number $M = 3\sup(A) + \sup(B)$ satisfies the definition to be a least upper bound[12], i.e. that M is an upper bound and that it satisfies the statement $\forall \varepsilon > 0, \exists c \in C$ with $c > M - \varepsilon$. To do this we will need to use the defining properties of $\sup(A)$ and $\sup(B)$.

[12] As you'll see, to prove that M is an upper bound for C, we will use that $\sup(A)$ is an upper bound for A, and that $\sup(B)$ is an upper bound for B. Then to prove the second statement, we use the corresponding statements for $\sup(A)$ and $\sup(B)$: i.e. that $\forall \varepsilon > 0, \exists a \in A$ s.t. $a > \sup(A) - \varepsilon$ and $\forall \varepsilon > 0, \exists b \in B$ s.t. $b > \sup(B) - \varepsilon$.

Solution. Let $M = 3\sup(A) + \sup(B)$. For $a \in A$ and $b \in B$, we have $a \leq \sup(A)$ and $b \leq \sup(B)$ (as $\sup(A)$ and $\sup(B)$ are upper bounds of A and B, respectively). Therefore $3a + b \leq 3\sup(A) + \sup(B) = M$, i.e. M is an upper bound for C.

Now let $\varepsilon > 0$ be arbitrary. Then[13] $\frac{\varepsilon}{4} > 0$. As $\sup(A)$ is a least upper bound for A, there exists $a \in A$ with $a > \sup(A) - \frac{\varepsilon}{4}$, and as $\sup(B)$ is a least upper bound for B, there exists $b \in B$ with $b > \sup(B) - \frac{\varepsilon}{4}$. Therefore[14]

$$\begin{aligned} 3a + b &> 3\left(\sup(A) - \frac{\varepsilon}{4}\right) + \left(\sup(B) - \frac{\varepsilon}{4}\right) \\ &= 3\sup(A) + \sup(B) - \varepsilon = M - \varepsilon. \end{aligned}$$

Since $3a + b \in C$, this shows that M is the least upper bound for C, so that $\sup(C)$ exists and is equal to $M = 3\sup(A) + \sup(B)$. □

[13] We will use $\frac{\varepsilon}{4}$ in the definition of least upper bound of A and B respectively, so here I am emphasising that we can do this as $\frac{\varepsilon}{4} > 0$.

[14] In this calculation, we use $\frac{3\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon$, and it is because of this I made the choice to use $\frac{\varepsilon}{4}$ in the first place. I'd seen this coming, when I made the first choice.

The completeness axiom

We now turn to the key additional axiom for the real numbers which distinguishes $\mathbb{R}$ from $\mathbb{Q}$, namely, we insist that those subsets of $\mathbb{R}$ which could possibly have suprema do have suprema.

Axiom 2.5 (The completeness axiom)**.** Every non-empty subset of $\mathbb{R}$ which is bounded above has a supremum.

Note that there are two conditions needed before we learn that $A \subseteq \mathbb{R}$ has a supremum. Firstly, the set A needs to be non-empty[15]. Secondly, if A is not bounded above, it certainly can not have a least upper bound [16].

We will see why the completeness axiom is required at the end of the chapter, when we will use it to show that there exists $x \in \mathbb{R}$ with $x^2 = 2$. Since we know that such an x is necessarily irrational, we can not use the earlier axioms to prove the completeness axiom (as the rational numbers do not satisfy the completeness axiom).

We start with an immediate consequence of the axiom, namely the corresponding statement for greatest lower bounds[17].

Theorem 2.6. *Every non-empty subset of $\mathbb{R}$ which is bounded below has an infimum.*

The proof I give here is rather similar to the example of $C = \{3a + b \mid a \in A, b \in B\}$ above. For a different proof see Theorem 1.4.7 in ERA.

Proof. Let A be a non-empty subset of $\mathbb{R}$ which is bounded below. Define $B = \{-a \mid a \in A\}$. Since A is non-empty, there exists $a \in A$, so that $-a \in B$ and hence B is non-empty. Since A is bounded below, take $m \in \mathbb{R}$ such that for all $a \in A, m \leq a$. Then $-a \leq -m$ for all $a \in A$, so that B is bounded above. Therefore $\sup(B)$ exists by the completeness axiom[18].

We now claim that $-\sup(B)$ is the greatest lower bound of A, and so $\inf(A)$ exists. For $a \in A$, we have $-a \in B$, so that $-a \leq \sup(B)$, and hence $-\sup(B) \leq a$. Therefore $-\sup(B)$ is a lower bound for A. Now let $\varepsilon > 0$ be arbitrary. There exists $a \in A$ such that $-a > \sup(B) - \varepsilon$, so that $a < -\sup(B) + \varepsilon$, and so $-\sup(B)$ is the greatest lower bound for A, as required. $\square$

Two consequences of completeness

We end this chapter with two consequences of completeness. The first is that the natural numbers are not bounded above. Archimedes observed that an additional ingredient beyond the usual properties of addition, multiplication (which today we encode in the field axioms) is needed to describe the real numbers, and so introduced this fact as an axiom[19].

Let us see how Archimedes axiom follows from our completeness axiom.

Theorem 2.7 (Archimedes axiom)**.** *The natural numbers $\mathbb{N} \subseteq \mathbb{R}$ are not bounded above. In particular, given any $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$.*

Proof. We argue by contradiction, so suppose that $\mathbb{N}$ is bounded above in $\mathbb{R}$. Then as $\mathbb{N}$ is certainly non-empty, $M = \sup(\mathbb{N})$ exists

[15] The empty set is bounded above, as any $M \in \mathbb{R}$ is an upper bound for the empty set — this also shows that the empty set does not have a least upper bound.

[16] Don't forget these two conditions, as I expect you to be able to state the completeness axiom precisely.

[17] It's reasonable to ask why Theorem 2.6 is a theorem and not an axiom. The reason is that mathematicians want to use the smallest possible set of axioms to describe $\mathbb{R}$, and prove other properties from these axioms. Since we can prove Theorem 2.6 from the completeness axiom (together with the other axioms for $\mathbb{R}$), we should do so.

An equally valid approach is to make Theorem 2.6 the axiom, and prove from this axiom that every non-empty subset of $\mathbb{R}$ which is bounded above has a supremum.

[18] Note that I check that B satisfies the hypotheses of the completeness axiom before I assert that $\sup(B)$ exists.

[19] We can prove directly that $\mathbb{N}$ is not bounded above in $\mathbb{Q}$ as follows: let $q \in \mathbb{Q}$ be arbitrary. If $q \leq 0$, then $1 \in \mathbb{N}$ has $1 > q$, so q is not an upper bound for $\mathbb{N}$. If $q > 0$, write $q = \frac{m}{n}$ for $m, n \in \mathbb{N}$. Then $q = \frac{m}{n} < m + 1 \in \mathbb{N}$, so q is not an upper bound for $\mathbb{N}$. The reason this argument works here is that we have an explicit form for every element of $\mathbb{Q}$; this is no longer the case for $\mathbb{R}$.

We do need an extra axiom beyond the ordered field axioms to see that the natural numbers are not bounded above in $\mathbb{R}$. In his book 'A companion to analysis: A Second First and First Second Course in Analysis', Tom Körner constructs an example of an ordered field in which the natural numbers are bounded above. In particular, this field has an infinitesimal element $\alpha > 0$ with the property that by the completeness axiom. As M is an upper bound for $\mathbb{N}$, we have $n \le M$ for all $n \in \mathbb{N}$. Given $n \in \mathbb{N}$, we have $n+1 \in \mathbb{N}$. Therefore $n+1 \le M$, and hence $n \le M-1$. This proves that $M-1$ is also an upper bound for $\mathbb{N}$, but since $M-1 < M$, this contradicts the fact that M is the least upper bound for $\mathbb{N}$. Therefore $\mathbb{N}$ is not bounded above. ☐

We have already used Archimedes axiom repeatedly in examples. In example 1.7, we proved that

$$\forall \varepsilon > 0, \exists n \in \mathbb{N} \text{ s.t. } \frac{3n}{n+1} < 3 - \varepsilon.$$

To do this we showed that for a fixed value of $\varepsilon > 0$, we had

$$\frac{3n}{n+1} < 3 - \varepsilon \Leftrightarrow n > \frac{3}{\varepsilon} - 1.$$

Now we can see that it is possible to take $n \in \mathbb{N}$ with $n > \frac{3}{\varepsilon} - 1$, as $\frac{3}{\varepsilon} - 1$ is not an upper bound for $\mathbb{N}$.

Our second example shows that the axiom we have introduced is enough to show that the square root of 2 exists.

Theorem 2.8 (Square root 2 exists)**.** *There exists $x \in \mathbb{R}$ with $x^2 = 2$.*

How do we proceed to use the completeness axiom to prove this theorem? To use the completeness axiom to produce certain real numbers, we write down non-empty subsets A which are bounded above, and then get a real number $\sup(A)$. So we should look for a set $A \subset \mathbb{R}$ which we think will have supremum $\sqrt{2}$. Of course, the set $[0, \sqrt{2}]$ will have this property, for instance, but since the point of this theorem is to use the axioms for $\mathbb{R}$ to prove that $\sqrt{2}$ exists, we need to define a set A entirely using the basic algebraic operations and the order relation given to us by the algebraic and order axioms. We should then carefully check that our set satisfies the hypotheses of the completeness axiom, namely that A is non-empty and bounded above. Then $\sup(A)$ will exist, and the final and most difficult step of the proof is to show that $\sup(A)^2 = 2$.

Proof. Define

$$A = \{y \in \mathbb{R} \mid y^2 \le 2\}.$$

As $0^2 \le 2$, we have $0 \in A$, so A is non-empty. Now suppose $y \in \mathbb{R}$ has $y \ge 2$, then $y^2 \ge 4$, so $y \notin A$. Thus 2 is an upper bound for A and hence A is bounded above. Therefore, by the completeness axiom, $M = \sup(A)$ exists. Note that $M \ge 1$ as $1 \in A$ and $M \le 2$ as 2 is an upper bound for A.

Suppose[20] $M^2 < 2$. Choose $\delta > 0$ with $\delta < \min(\frac{2-M^2}{5}, 1)$, and then define[21] $y = M + \delta$. As $\delta < 1$, we have $\delta^2 < \delta$ and so[22]

$$y^2 = (M+\delta)^2 = M^2 + 2M\delta + \delta^2 \le M^2 + 4\delta + \delta = M^2 + 5\delta < 2.$$

Thus $y \in A$, but this is a contradiction as $y > M$. Therefore $M^2 \ge 2$.

Suppose[23] $M^2 > 2$. Choose $\delta > 0$ with $\delta < \min(M, \frac{M^2-2}{2M})$, so that

[20] We aim to reach a contradiction, and we will do this by showing that for a small enough value of δ we have $M + \delta \in A$, as then M will not be an upper bound for A.

[21] This choice of δ was made by doing the calculation first to see what would work. Note also, that unlike some calculations in lectures it's not an equivalent form of $(M+\delta)^2 < 2$, but instead an inequality which implies that $(M+\delta)^2 < 2$.

[22] In this estimate I use that $M \le 2$.

[23] This time the contradiction will come from showing that $M - \delta$ is also an upper bound for A, provided δ is small enough.

$M^2 - 2M\delta > 2$ and $M - \delta > 0$. Then,

$$(M-\delta)^2 = M^2 - 2M\delta + \delta^2 \geq M^2 - 2M\delta > 2.$$

Now if $y \geq (M-\delta)$, then $y^2 \geq (M-\delta)^2 > 2$ so $y \notin A$. Thus $M - \delta$ is an upper bound for A, contradicting the fact that M is the supremum of A.

Since $M^2 < 2$ and $M^2 > 2$ both lead to contradictions, we conclude that $M^2 = 2$, as required. $\square$

We could use the same process to show that for every $x \geq 0$, there exists $y \geq 0$ with $y^2 = x$, and use this to define the square root function[24]. In a similar fashion you should be able to show that for each $x \geq 0$ and each $n \in \mathbb{N}$, there exists $y \geq 0$ with $y^n = x$, so we can define powers of the form $x^{\frac{m}{n}}$ when $m, n \in \mathbb{N}$. Defining what we mean by irrational powers will take far more work, and will be done in the third year course.

[24] Note that the uniqueness of y doesn't require completeness: it can be deduced from the algebraic axioms, as if $y^2 = z^2$, then $(y-z)(y+z) = 0$, so $y = z$ or $y = -z$. Hence for each $x \geq 0$, there is a unique $y \geq 0$ with $y^2 = x$, and so we really do define a function when we do this.