2A: Tutorial 5

School of Mathematics and Statistics

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Semester 1 2020–21

Instructions

Access your tutorial lecture

TU01,TU02,TU03,TU04,TU05: October 26th, 09:00-10:00

TU06,TU07,TU08,TU09,TU10: October 26th, 15:00-16:00

using the Zoom plug-in in the Moodle page of the course.

Please mute yourself when you are in the main room of the meeting.

Be ready to work in groups. Having a method of writing on screen, as mouse, tablet or similar is useful. Screen sharing is allowed!

Multiple Choice Question 1

Ex Sheet 5, T3(a)

For the following integral, describe the region of integration in polar coordinates.

$$\int_0^1 dy \int_y^{\sqrt{2-y^2}} 3\,(x+y)\;dx.$$

(A) $0 \le r \le \sqrt{2},\ 0 \le \theta \le \frac{\pi}{2}$

(B) $0 \le r \le \sqrt{2},\ 0 \le \theta \le \frac{\pi}{4}$

(C) $0 \le r \le 2,\ 0 \le \theta \le \frac{\pi}{2}$

(D) $0 \le r \le 2,\ 0 \le \theta \le \frac{\pi}{4}$

Multiple Choice Question 1

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For the following integral, describe the region of integration in polar coordinates.

$$\int_0^1 dy \int_y^{\sqrt{2-y^2}} 3\,(x+y)\,dx.$$

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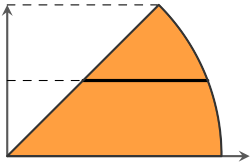
ANSWER: **(B)** The figure shows the domain. The left boundary is $x = y$, the right boundary is $x = \sqrt{2-y^2}$.



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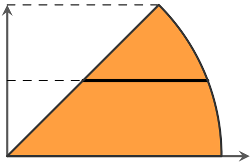
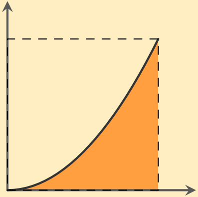
ANSWER: **(B)** $x = y$ is the line $\theta = \pi/4$ and $x = \sqrt{2 - y^2}$ is the curve $r = \sqrt{2}$. The x-axis is $\theta = 0$.



Multiple Choice Question 2

Unseen example

The red region lies between the lines $y = 0$ and $x = 1$ and the curve $y = \sqrt{x}$. Describe the region as a type-II region.



(A) $0 \leq x \leq y^2, 0 \leq y \leq 1$

(B) $y^2 \leq x \leq 1, 0 \leq y \leq 1$

(C) $0 \leq x \leq 1, 0 \leq y \leq x^2$

(D) $0 \leq x \leq 1, 0 \leq y \leq \sqrt{x}$

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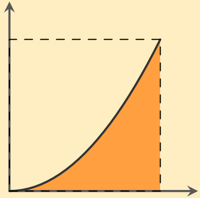

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(D) $0 \leq x \leq 1, 0 \leq y \leq \sqrt{x}$

ANSWER: **(B)**

Ex Sheet 4, T13(a)

By changing the order of integration, evaluate

$$I = \int_0^1 dy \int_y^1 \sinh(x^2)\, dx.$$

Ex Sheet 5, T8

Evaluate

$$I = \iint_D x\sqrt{x^2 + y^2}\, dx\, dy$$

over the finite region D in the first quadrant enclosed by the x-axis, the line $y = \sqrt{3}x$ and the circle $x^2 + y^2 = a^2$, where $a > 0$.

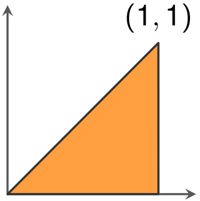
Tutorial questions

Ex Sheet 4, T13(a)

By changing the order of integration, evaluate

$$I = \int_0^1 dy \int_y^1 \sinh(x^2)\, dx.$$

The first step is to express the region of integration as a type-I or type-II region. In this case the integral is type-II and so the type-II description is $y \leq x \leq 1$ and $0 \leq y \leq 1$.



The next step is to express the region as the other type. In our case we must express as a type-I region: $0 \le y \le x$ and $0 \le x \le 1$. Then reformulate the integral

$$
\begin{aligned}
I &= \int_0^1 \left(\int_0^x \sinh(x^2)\, dy \right)\, dx \\
&= \int_0^1 \left[y \sinh(x^2) \right]_{y=0}^{y=x}\, dx \\
&= \int_0^1 x \sinh(x^2)\, dx = \left[\frac{1}{2} \cosh(x^2) \right]_0^1 = \frac{1}{2} (\cosh 1 - 1)\,.
\end{aligned}
$$

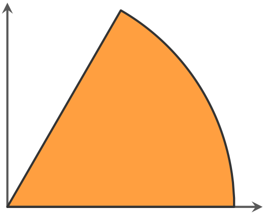
EX SHEET 5, T8

Evaluate

$$I = \iint_D x\sqrt{x^2 + y^2}\, dx\, dy$$

over the finite region D in the first quadrant enclosed by the x-axis, the line $y = \sqrt{3}x$ and the circle $x^2 + y^2 = a^2$, where $a > 0$.

The domain is



The straight line $y = \sqrt{3}x$ becomes $\theta = \tan^{-1}\sqrt{3} = \pi/3$ and the circle radius is $r = a$. Hence, in polar coordinates the region is described as $D' = \{0 \le r \le a\, 0 \le \theta \le \pi/3\}$.

Therefore the integral becomes

$$\begin{aligned} I &= \iint_{D'} r\cos\theta\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\, r\, drd\theta \\ &= \iint_{D'} r^3\cos\theta\, drd\theta\,. \end{aligned}$$

The integrand is separable and the region of integration rectangular so

$$I = \left(\int_0^a r^3\,dr\right)\left(\int_0^{\pi/3}\cos\theta\,d\theta\right) = \left[\frac{1}{4}r^4\right]_0^a [\sin\theta]_0^{\pi/3} = \frac{a^4\sqrt{3}}{8}.$$

Ex Sheet 5, T13

By making a suitable change of variables, evaluate

$$I = \iint_D xy \, dx \, dy$$

where D is the region enclosed by the two hyperbolas $xy = 1$ and $xy = 7$ and the two parabolas $y = 2x^2$ and $y = 4x^2$.

EX SHEET 5, T13

By making a suitable change of variables, evaluate

$$I = \iint_D xy \, dx \, dy$$

where D is the region enclosed by the two hyperbolas $xy = 1$ and $xy = 7$ and the two parabolas $y = 2x^2$ and $y = 4x^2$.

The region of integration suggests the change of variable $u = xy$ and $v = y/x^2$. The relevant partial derivatives are

$$\frac{\partial u}{\partial x} = y \qquad \frac{\partial v}{\partial x} = -\frac{2y}{x^3}$$

$$\frac{\partial u}{\partial y} = x \qquad \frac{\partial v}{\partial y} = \frac{1}{x^2}$$

So that $\partial(u, v)/\partial(x, y) = u_x v_y - u_y v_x = 3y/x^2$.

Then

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{x^2}{3y} = \frac{1}{3v}$$

and the integrand becomes u.

With this change of variable, the region D is changed into the region $D' = \{1 \le u \le 7\,, 2 \le v \le 4\}$ (a rectangular region).

Note that you could also have inverted the change of variables, in this case $y = (u^2 v)^{1/3}$ and $x = (u/v)^{1/3}$, to get $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{3v}$.

The integrand is separable and the region D' is rectangular so

$$I = \iint_{D'} \frac{u}{3v}\, dudv = \left(\int_1^7 u\, du\right)\left(\int_2^4 \frac{1}{3v}\, dv\right)$$

$$= \left[\frac{1}{2}u^2\right]_1^7 \left[\frac{1}{3}\log v\right]_2^4 = 8\log 2\,.$$