27/30 A2

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Assessed Coursework

Course Name	Algorithmic Foundations 2			
Coursework Number	1			
Deadline	Time:	16:30	Date:	26/10/2020
% Contribution to final course mark	10%			
Solo or Group ✓	Solo	✓	Group	
Anticipated Hours	10			
Submission Instructions	Use the latex template and submit the generated pdf through moodle (do not submit the source latex file). Failure to follow the submission instructions will lead to a penalty of 2 bands.			
Please Note: This Coursework cannot be Re-Assessed				

Code of Assessment Rules for Coursework Submission

Deadlines for the submission of coursework which is to be formally assessed will be published in course documentation, and work which is submitted later than the deadline will be subject to penalty as set out below.

The primary grade and secondary band awarded for coursework which is submitted after the published deadline will be calculated as follows:

(i) in respect of work submitted not more than five working days after the deadline
   a. the work will be assessed in the usual way;
   b. the primary grade and secondary band so determined will then be reduced by two secondary bands for each working day (or part of a working day) the work was submitted late.

(ii) work submitted more than five working days after the deadline will be awarded Grade H.

Penalties for late submission of coursework will not be imposed if good cause is established for the late submission. You should submit documents supporting good cause via MyCampus.

Penalty for non-adherence to Submission Instructions is 2 bands

You must complete an "Own Work" form via https://studentltc.dcs.gla.ac.uk/ for all coursework

Algorithmic Foundations 2

Assessed Exercise 1

Notes for guidance

1. There are two assessed exercises. Each is worth 10% of your final grade for this module. Your answers must be the result of your own individual efforts.
2. Please use the latex template and submit your generated pdf via moodle (do not submit the latex source file).
3. Please ensure you have filled out your tutorial group, name and student id.
4. **Failure to follow the submission instructions will lead to a penalty for non-adherence to submission instructions of 2 bands**.
5. As stated on the cover sheet deadline for completing this assessed exercise is **16:30 Monday, November 23, 2020**.
6. The exercise is marked out of 30 using the included marking scheme. Credit will be given for partial answers.

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1. Using the laws of logical equivalence show:

   (a) $\neg(p \vee (q \wedge \neg r)) \wedge q \equiv (\neg p \wedge q) \wedge r$ [4]

Solution:
By De Morgan's law we have:

$$
\begin{aligned}
\neg(p \vee (q \wedge \neg r)) \wedge q \;&\equiv\; (\neg p \wedge \neg(q \wedge \neg r))) \wedge q \\
\equiv\;& (\neg p \wedge (\neg q \vee r)) \wedge q && \text{De Morgan's law} \\
\equiv\;& ((\neg p \wedge \neg q) \vee (\neg p \wedge r)) \wedge q && \text{distributivity law} \\
\equiv\;& (\neg p \wedge \neg q \wedge q) \vee (\neg p \wedge r \wedge q) && \text{distributivity law} \\
\equiv\;& \texttt{false} \vee (\neg p \wedge r \wedge q) && \text{contradiction law} \\
\equiv\;& \neg p \wedge r \wedge q && \text{identity law} \\
\equiv\;& \neg p \wedge q \wedge r && \text{commutativity law} \\
\equiv\;& (\neg p \wedge q) \wedge r && \text{associativity law}
\end{aligned}
$$

4

   (b) $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$ [4]

Solution:
By the implication law we have:

$$
\begin{aligned}
(p \to r) \vee (q \to r) &\equiv (\neg p \vee r) \vee (\neg q \vee r) \\
\equiv\;& \neg p \vee r \vee \neg q \vee r && \text{associativity law} \\
\equiv\;& \neg p \vee \neg q \vee r \vee r && \text{commutativity law} \\
\equiv\;& \neg p \vee \neg q \vee r && \text{idempotency law} \\
\equiv\;& \neg(p \wedge q) \vee r && \text{De Morgan's law} \\
\equiv\;& (p \wedge q) \to r && \text{implication law}
\end{aligned}
$$

4

   (c) $\forall x \in U.\,(P(x) \to \neg Q(x)) \equiv \neg\exists x \in U.\,(P(x) \wedge Q(x))$ [2]

Solution: By the implication law we have:

$$
\begin{aligned}
\forall x \in U.\,(P(x) \to \neg Q(x)) &\equiv \forall x \in U.\,(\neg P(x) \vee \neg Q(x)) \\
\equiv\;& \forall x \in U.\,\neg(P(x) \wedge Q(x)) && \text{De Morgan's law} \\
\equiv\;& \neg\exists x \in U.\,(P(x) \wedge Q(x)) && \text{quantifier negation law}
\end{aligned}
$$

2. Assuming the following predicates:

   - $L(x, y)$: x is strictly less than y ($x<y$);
   - $E(x)$: x is even;

- $P(x)$: x is a prime number;
- $EQ(x, y)$: x equals y (x=y);
- $G(x)$: x is greater than zero (x>0);
- $D(x, y)$: x divides y exactly;

determine which of the following formulae are `true` (in your answer include an expression of the formula in concise (good) English without variables).

(a) $\forall x \in \mathbb{N}. (D(2, x) \rightarrow E(x))$ [2]

> **Solution:**
> If a natural number is divisible by 2, it is even, which is true for all natural numbers. Thus, the formula is true.

2

(b) $\exists y \in \mathbb{N}. \forall x \in \mathbb{N}. L(x, y)$ [2]

> **Solution:**
> There is a natural number that is strictly greater than all natural numbers. This is false because, for all $y \in \mathbb{N}$, x can be taken as $x = y + 1$, which is greater than y, which contradicts the formula.

2

(c) $\exists x \in \mathbb{Z}^+. \forall y \in \mathbb{Z}^+. (G(x) \rightarrow (P(y) \wedge L(x, y)))$ [2]

> **Solution:**
> There is a positive integer that is less than all positive integer primes. The statement is true for $x = 1$ because the smallest positive prime is 2.

4.1

Next, using the above predicates and quantifiers where necessary, express the following English statements in logic.

(d) "Any non-zero integer divides itself" [2]

> **Solution:**
> $\forall x \in (\mathbb{Z} - \{0\}). D(x, x)$

2

(e) "A prime number's only positive factors are 1 and itself." [2]

> **Solution:**
> $\forall x \in \mathbb{Z}. \forall y \in \mathbb{Z}^+. (P(x) \rightarrow (D(y, x) \wedge (EQ(y, 1) \vee EQ(y, x))))$

4.2

3. Prove that $(B \backslash A) \cup (C \backslash A) = (B \cup C) \backslash A$ using:

(a) a containment proof. [3]

Solution:
First we show $(B \backslash A) \cup (C \backslash A) \subseteq (B \cup C) \backslash A$. Considering any x, by definition of union we have:

$$
\begin{aligned}
& x \in (B\backslash A) \cup (C\backslash A) \;\Rightarrow\; \text{either } x \in B\backslash A \text{ or } x \in C\backslash A \\
\Rightarrow\;& \text{either } x \in B \cap \neg A \text{ or } x \in C \cap \neg A && \text{by definition of set difference} \\
\Rightarrow\;& \text{either } x \in B \text{ and } x \notin A \text{ or } x \in C \text{ and } x \notin A && \text{by definition of intersection} \\
\Rightarrow\;& (\text{either } x \in B \text{ or } x \in C) \text{ and } x \notin A && \text{distributivity law} \\
\Rightarrow\;& x \in B \cup C \text{ and } x \notin A && \text{by definition of union} \\
\Rightarrow\;& x \in (B \cup C) \cap \neg A && \text{by definition of intersection} \\
\Rightarrow\;& x \in (B \cup C) \backslash A && \text{by definition of set difference}
\end{aligned}
$$

3

thus, $(B\backslash A) \cup (C\backslash A) \subseteq (B \cup C)\backslash A$, as required.
To complete the proof, we show $(B \cup C)\backslash A \subseteq (B\backslash A) \cup (C\backslash A)$. Considering any x, by definition of set difference we have:

$$
\begin{aligned}
& x \in (B \cup C)\backslash A \;\Rightarrow\; x \in (B \cup C) \cap \neg A \\
\Rightarrow\;& x \in (B \cup C) \text{ and } x \notin A && \text{by definition of intersection} \\
\Rightarrow\;& (\text{either } x \in B \text{ or } x \in C) \text{ and } x \notin A && \text{by definition of union} \\
\Rightarrow\;& \text{either } x \in B \text{ and } x \notin A \text{ or } x \in C \text{ and } x \notin A && \text{distributivity law} \\
\Rightarrow\;& \text{either } x \in B \cap \neg A \text{ or } x \in C \cap \neg A && \text{by definition of intersection} \\
\Rightarrow\;& \text{either } x \in B\backslash A \text{ or } x \in C\backslash A && \text{by definition of set difference} \\
\Rightarrow\;& (B\backslash A) \cup (C\backslash A) && \text{by definition of union}
\end{aligned}
$$

thus, $(B \cup C)\backslash A \subseteq (B\backslash A) \cup (C\backslash A)$, completing the proof.

(b) using set builder notation and logical equivalences. [3]

Solution:

$$
\begin{aligned}
(B \cup C)\backslash A &= \{x \,|\, x \in (B \cup C)\backslash A\} \\
&= \{x \,|\, x \in (B \cup C) \wedge \neg A\} && \text{by definition of set difference} \\
&= \{x \,|\, x \in (B \vee C) \wedge \neg A\} && \text{by definition of union} \\
&= \{x \,|\, x \in (B \wedge \neg A) \vee (C \wedge \neg A)\} && \text{distributivity law} \\
&= \{x \,|\, x \in (B\backslash A) \vee (C\backslash A)\} && \text{by definition of set difference} \\
&= \{x \,|\, x \in (B\backslash A) \cup (C\backslash A)\} && \text{by definition of union}
\end{aligned}
$$

3

5.1

4. For each of the following functions find the inverse or explain why no inverse exists.

   (a) $f : \mathbb{N} \to \mathbb{N}$ where $f(x) = 4 \cdot x^2 + 1$ [2]

   Solution:
   In order for the function to have an inverse, it must be bijective, i.e. it must be both injective and surjective. The function is not surjective because there is no $x \in \mathbb{N}$ in the domain for $f(x) = 1$ in the codomain $\mathbb{N}$; thus, there is no inverse.

   2

   (b) $g : \mathbb{Z} \to \mathbb{Z}$ where $g(x) = x + 7$ [2]

   Solution:
   To find the inverse, swap x and $g(x)$:

   $$x = g^{-1}(x) + 7$$
   $$\Leftrightarrow g^{-1}(x) = x - 7$$

   2

Index of comments

3.1 careful with brackets applying multiple laws and reference all that you use

3.2 this is more than just the quantifier law see the handouts and model solutions

4.1 not quite - see the model solutions and if you want to discuss please let me know

4.2 not quite this state if a number is prime than all positive integers divide this prime and equals 1 or the prime

5.1 be careful mixing logical operators and set operators I have not penalised you as the arguments are clear but bare this in mind in the future