What is the probability of these events when we randomly select a permutation of $\{1, 2, 3, 4\}$?
Hint: Use symmetry.

(a) `1` comes before `4` in the permutation.

Here `1` coming before `4` and `4` comes before `1` are equally likely and one must occur, and therefore the probability is $\frac{1}{2}$.

(b) `4` comes before `1` in the permutation.

For the same reason as for (a) the probability is $\frac{1}{2}$.

(c) `4` comes before `1` and `4` comes before `2` in the permutation.

Here there are three possible equally likely outcomes either `1`, `2` or `4` comes first and hence the probability is $\frac{1}{3}$.

(d) `4` comes before `1`, `4` comes before `2`, and `4` comes before `3` in the permutation.

Here there are four possible equally likely outcomes either `1`, `2`, `3` or `4` comes first and hence the probability is $\frac{1}{4}$.

(e) `4` comes before `3` and `2` comes before `1` in the permutation.

The probabilities `4` comes before `3` and `2` comes before `1` are each $\frac{1}{2}$ as for (a) and (b) and sence the events are indepedent the probability of both occuring is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

What is the probability of these events when we randomly select a permutation of the 26 lowercase letters of the English alphabet?

First let us think about all the possible permutations, here we r-permutations from a set of size n where $n = 26$ and $r = 26$ and therefore the total number is 26!.

- The permutation consists of the letters in reverse alphabetic order.

  Only one permutation meets this requirement, and hence the probability is $\frac{1}{26!}$.

- `z` is the first letter of the permutation.

  If we fix `z` as the first element then we have a permutation of size 25 with 25 characters so get the probability $\frac{25!}{26!}$.

- `z` comes before `a` in the permutation.

  Here we can just use the fact `z` coming before `a` is as likely as `a` comes before `z` and one of these must occur. Therefore the probability is $\frac{1}{2}$.

- `a` immediately comes before `z` in the permutation.

  If we think of `az` as a single character then we have a permutation of size 25 with 25 characters so get the probability $\frac{25!}{26!}$.

- `a` immediately comes before `m`, which immediately comes before `z` in the permutation.

  If we think of `amz` as a single character then we have a permutation of size 25 with 25 characters so get the probability $\frac{24!}{26!}$.

- `m`, `n`, and `o` are in their original places in the permutation.

  This means we have 23 spaces to fill using 23 chararacters, and hence the probability is (23!)/(26!).

Monty Hall puzzle. A prize behind one of the three doors (d_1, d_2 and d_3) each with probability $\frac{1}{3}$.

1. You select a door.
2. Monty Hall opens one of the two doors you did not select that he knows is a door without the prize behind, selecting at random if neither has a prize behind.
3. Monty then asks you whether you would like to switch doors.

Suppose that:

- W is the random variable whose value is the winning door;
- M denote the random variable corresponding to the door that Monty opens
- you choose door d_i.

(a) What is the probability that you will win the prize if you never switch doors?

This has probability $\frac{1}{3}$ as the prize behind one of the three doors each with probability $\frac{1}{3}$.

(b) What is the probability that you will win the prize if you always switch doors?

This has probability 2/3 as if you choose the right door initially after swapping you will have chosen the wrong door, while if you selected one of the doors without the prize, Monty will then open the other door without the prize and by switching you will be switching to the door with the prize.

(c) Find $\mathbf{P}[M = d_k \mid W = d_j]$ for $j = 1, 2, 3$ and $k = 1, 2, 3$ when $d_i \neq d_j$ i.e. when you did not initially choose the winning door.

Here since you have chosen one door and the prize is behind one other, there is only one door Monty can choose i.e. we have:

$$\mathbf{P}[M = d_k \mid W = d_j] = \begin{cases} 1 & \text{if } k \neq i \text{ and } k \neq j \\ 0 & \text{otherwise} \end{cases}$$

(d) Find $\mathbf{P}[M = d_k \mid W = d_j]$ for $j = 1, 2, 3$ and $k = 1, 2, 3$ when $d_i = d_j$ i.e. when you did choose the winning door.

Here since you have chosen the same dorr the prize is behind, there are two doors Monty can choose and he chooses them at random when there is choice we have:

$$\mathbf{P}[M = d_k \mid W = d_j] = \begin{cases} \frac{1}{2} & \text{if } k \neq i = j \\ 0 & \text{otherwise} \end{cases}$$

(e) Use Bayes' theorem to find $\mathbf{P}[W = d_j \mid M = d_k]$ where j and k are distinct and $d_i \neq d_j$ i.e. when you did not choose the winning door. **Note.** Monty will never choose the winning door to reveal so the probability j and k are equal is aways 0.

First we choose specific values for i, j and k. Since in this casde i, j and k are distinct let us choose without loss of generality $i = 1$, $j = 2$ and $k = 3$. Now, using Bayes' rule we have:

$$\mathbf{P}[W = d_2 \mid M = d_3] = \frac{\mathbf{P}[M = d_3 \mid W = d_2] \cdot \mathbf{P}[W = d_2]}{\mathbf{P}[M = d_3]}$$

Now we know $\mathbf{P}[M = d_3 \mid W = d_2] = 1$ from part (c) and $\mathbf{P}[W = d_2] = \frac{1}{3}$. It remains to compute $\mathbf{P}[M = d_3]$ which we can do using Partition theorem (the law of total probability) considering the events of the prize being behind the three different doors (the prize must be behind one of the doors and the prize cannot be behind two doors). Therefore we have $\mathbf{P}[M = d_3]$ equals

$$\mathbf{P}[M = d_3|W = d_1]{\cdot}\mathbf{P}[W = d_1] + \mathbf{P}[M = d_3|W = d_2]{\cdot}\mathbf{P}[W = d_2] + \mathbf{P}[M = d_3|W = d_3]{\cdot}\mathbf{P}[W = d_3]$$
$$= \frac{1}{2}{\cdot}\frac{1}{3} + 1{\cdot}\frac{1}{3}0{\cdot}\frac{1}{3} = \frac{1}{2}$$

The fact $\mathbf{P}[M = d_3|W = d_1] = \frac{1}{2}$ follows from (d), $\mathbf{P}[M = d_3|W = d_2] = \frac{1}{2}$ follows from (c), $\mathbf{P}[M = d_3|W = d_3] = 0$ as Monty never chooses the door that the prize is behind. The facts that $\mathbf{P}[W = d_j] = \frac{1}{3}$ for $j = 1, 2, 3$ follow from the fact the prize is placed behind the door at random.

We therefore have that:

$$\mathbf{P}[W = d_2 \mid M = d_3] = \frac{\mathbf{P}[M = d_3 \mid W = d_2]{\cdot}\mathbf{P}[W = d_2]}{\mathbf{P}[M = d_3]} = \frac{1{\cdot}\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}.$$

(f) Use Bayes' theorem to find $\mathbf{P}[W = d_j \mid M = d_k]$ where j and k are distinct and $d_i = W$ i.e. when you did choose the winning door.

First we choose specific values for i, j and k. Since in this casde $i = j$ and k is distinct let us choose without loss of generality $i = j = 1$ and $k = 2$. Now, using Bayes' rule we have:

$$\mathbf{P}[W = d_1 \mid M = d_2] = \frac{\mathbf{P}[M = d_2 \mid W = d_1]{\cdot}\mathbf{P}[W = d_1]}{\mathbf{P}[M = d_2]}$$

Now we know $\mathbf{P}[M = d_2 \mid W = d_1] = \frac{1}{2}$ from part (d) and $\mathbf{P}[W = d_1] = \frac{1}{3}$. It remains to compute $\mathbf{P}[M = d_2]$ which we can do using Partition theorem (the law of total probability) considering the events of the prize being behind the three different doors (the prize must be behind one of the doors and the prize cannot be behind two doors). Therefore we have that $\mathbf{P}[M = d_2]$ equals

$$\mathbf{P}[M = d_2|W = d_1]{\cdot}\mathbf{P}[W = d_1] + \mathbf{P}[M = d_2|W = d_2]{\cdot}\mathbf{P}[W = d_2] + \mathbf{P}[M = d_2|W = d_3]{\cdot}\mathbf{P}[W = d_3]$$
$$= \frac{1}{2}{\cdot}\frac{1}{3} + 0{\cdot}\frac{1}{3}1{\cdot}\frac{1}{3} = \frac{1}{2}$$

The fact $\mathbf{P}[M = d_2|W = d_1] = \frac{1}{2}$ follows from (d), $\mathbf{P}[M = d_2|W = d_2] = 0$ as Monty never chooses the door that the prize is behind and $\mathbf{P}[M = d_3|W = d_2] = 1$ follows from (c). The facts that $\mathbf{P}[W = d_j] = \frac{1}{3}$ for $j = 1, 2, 3$ follow from the fact the prize is placed behind the door at random.

We therefore have that:

$$\mathbf{P}[W = d_1 \mid M = d_2] = \frac{\mathbf{P}[M = d_3 \mid W = d_2]{\cdot}\mathbf{P}[W = d_2]}{\mathbf{P}[M = d_3]} = \frac{\frac{1}{2}{\cdot}\frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}.$$

Suppose that we roll a fair die until a 6 comes up.

(a) What is the probability that we roll the die n times?

Here we need to first roll $n - 1$ without getting a 6 and then roll a siz so the probability is $(\frac{5}{6})^{n-1}{\cdot}\frac{1}{6}$.

(b) What is the probability the game ends?

If $\mathbf{P}[end]$ is the probability of ending we have:

$$\mathbf{P}[end] = \frac{1}{6} + \frac{5}{6}{\cdot}\mathbf{P}[end]$$

since with probability $\frac{1}{6}$ we finish after one throw or we continue. Solving this equation we get $\mathbf{P}[end] = 1$.

(c) What is the expected number of times we roll the die?

If $\mathbf{E}[end]$ is the expected number of rolls we have:

$$\mathbf{E}[end] = 1 + \frac{5}{6} \cdot \mathbf{E}[end]$$

since with probability $\frac{5}{6}$ we continue and roll again. Solving this equation we get $\mathbf{E}[end] = 6$.