Friday, 14th December, 2018 1.00 pm *to* 2.30 pm

EXAMINATION FOR THE DEGREES OF
M.A. AND B.Sc.

Mathematics 2A - Multivariable Calculus

An electronic calculator may be used provided that it does not have a facility for either textual storage or display, or for graphical display.

Candidates must attempt all questions.

1. The function $g : \mathbb{R} \to \mathbb{R}^2$ can be written $g(t) = (g_1(t), g_2(t))$. The function $F : \mathbb{R} \to \mathbb{R}$ is the composition of $f : \mathbb{R}^2 \to \mathbb{R}$ and g. Write down the derivative of F in terms of derivatives of f and g.

   Use the chain rule to calculate dF/dt in the particular case $g(t) = (\cosh t, \sinh t)$ and $f(x, y) = x^2 - y^2$. *You must calculate all relevant derivatives of g and f for full credit.* **5**

2. If $f(x, y) = F(v(x, y), w(x, y))$, write down the chain rule for the calculation of $\partial f/\partial x$ and $\partial f/\partial y$.

   Find the general solution $f(x, y)$ of the partial differential equation
   $$x\frac{\partial f}{\partial x} + 3y\frac{\partial f}{\partial y} = x^4$$
   by using the change of variables $v = x^3/y$ and $w = xy$. **8**

3. Let ϕ be a differentiable function of one variable, let $\mathbf{x} = (x, y, z)$ and $r = |\mathbf{x}|$. Calculate
   $$\text{grad } \phi(r).$$
   Confirm that the vector field $\mathbf{F} = r\mathbf{x}$ is conservative and use the result above to find a potential for $\mathbf{F}$. **8**

   HINT: You may use any nabla identities without proof, but they must be stated clearly.

4. Sketch the region of integration for
   $$I = \int_{-1}^{1} \left(\int_{|x|}^{1} y^2 e^{xy}\, dy \right) dx.$$
   Rewrite the integral by changing the order of integration, hence evaluate the integral. **7**

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5. By making a change of variable, calculate

$$\iint_D (1+x)\,dxdy$$

where D is the region that lies between the curves $y = e^x$, $y = e^{x+1}$, $xy = 1$ and $xy = e$. **8**

6. Calculate

$$\iiint_V z\,dxdydz$$

where V is the tetrahedron formed by the planes $x = 0$, $y = 0$, $z = 0$ and $hx + hy + lz = hl$ where $h > 0$ and $l > 0$ are constants. **6**

7. Consider the part of the surface $z = 1 + x^2 - y^2$ that lies inside $x^2 + y^2 = 1$. Calculate the surface area. *HINT: After formulating the integral you might find it helpful to switch to polar coordinates.* **6**

8. State and use Green's theorem to calculate

$$\oint y^2\,dx + x^2\,dy$$

where the line integral is taken along path consisting of the line joining $(0, 0)$ to $(1, 0)$, followed by the line joining $(1, 0)$ to $(0, 2)$, followed by the line joining $(0, 2)$ to $(0, 0)$. **6**

9. State and use the divergence theorem to calculate

$$\iint_S \mathbf{f}\cdot\mathbf{n}\,dS$$

where $\mathbf{f} = (yx, y^3 - y, z(1 - y))$ and S is the surface of the sphere centred at the origin, radius 1. *HINT: After applying the divergence theorem, you might find it helpful to use spherical polar coordinates.* **6**

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