2A Multivariable Calculus 2020

Ex Sheet 3

Tutorial Exercises

T1 Find the general solution of the PDE

$$\frac{\partial \phi}{\partial x} = y\cos(2x+y),$$

where ϕ is a function of two independent variables x and y.

T2 Find the general solution of the PDE

$$\frac{\partial f}{\partial y} = xy\exp(y^2) + 4\log x,$$

where f is a function of two independent variables x and y.

Solution

$$f(x,y) = \frac{x}{2}\exp(y^2) + 4y\log x + A(x), \text{ where } A \text{ is an arbitrary function.}$$

Solution

$$\phi(x,y) = \frac{y}{2}\sin(2x+y) + A(y), \text{ where } A \text{ is an arbitrary function.}$$

T3 By making the change of variables indicated, find the general solution of each of the following partial differential equations.

a) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 6xy$. Change to $u = \frac{y}{x}$ and $v = x$

b) $2x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} = 2xy$. Change to $u = xy^2$, and $v = y$

Solution

(a) The chain rule gives

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = \frac{-y}{x^2}\frac{\partial f}{\partial u} + 1\cdot\frac{\partial f}{\partial v}\,.$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial f}{\partial u}\frac{1}{x} + 0\,.$$

Therefore the PDE is

$$x\left(\frac{-y}{x^2}\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right) + y\left(\frac{1}{x}\frac{\partial f}{\partial u}\right) = 6xy$$

i.e. $f_v = 6y = 6x\frac{y}{x} = 6uv$. Integrating with respect to v gives $f = 3uv^2 + \phi(u)$. Hence the general solution is $f = 3xy + \phi(\frac{y}{x})$, where ϕ is an arbitrary function of one variable.

(b)

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = y^2\frac{\partial f}{\partial u} + 0\,.$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 2xy\frac{\partial f}{\partial u} + 1\cdot\frac{\partial f}{\partial v}\,.$$

Therefore the PDE is

$$2xy^2\frac{\partial f}{\partial u} - 2xy^2\frac{\partial f}{\partial u} - y\frac{\partial f}{\partial v} = 2xy$$

i.e. $z_v = -2x$. Since $x = u/y^2 = u/v^2$, we have $f_v = -2u/v^2$ and $f = \frac{2u}{v} + \phi(u)$. Hence the general solution is $f = 2xy + \phi(xy^2)$, where ϕ is an arbitrary function.

T4 By making the change of variables indicated, find the general solution of each of the following partial differential equations.

a) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3y(y^2 - x^2)$. Change to $u = x$, $v = \frac{y}{x}$.

b) $2x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{6x^4}{y^2}$. Change to $u = \frac{x}{y^2}$, and $v = x$

c) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{4x^3}{y}$. Change to $u = \frac{x}{y}$, $v = x$.

Solution

(a)

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\left(\frac{-y}{x^2}\right)\,.$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0 + \frac{\partial f}{\partial v}\frac{1}{x}\,.$$

Therefore the PDE is

$$x\frac{\partial f}{\partial u} - \frac{y}{x}\frac{\partial f}{\partial v} + \frac{y}{x}\frac{\partial f}{\partial v} = 3y(y^2 - x^2)$$

i.e. $f_u = 3\frac{y}{x}(y^2 - x^2) = 3v((uv)^2 - u^2) = 3u^2v^3 - 3u^2v$ and $f = u^3v^3 - u^3v + \phi(v)$. Hence the general solution is $f = y^3 - x^2y + \phi(\frac{y}{x})$, where ϕ is an arbitrary function.

(b)

$$\frac{\partial z}{\partial x} = \frac{1}{y^2}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \quad \frac{\partial z}{\partial y} = \frac{-2x}{y^3}\frac{\partial z}{\partial u}.$$

Therefore the PDE is $\frac{2x}{y^2}z_u + 2xz_v - \frac{2x}{y^2}z_u = \frac{6x^4}{y^2}$, i.e. $z_v = \frac{3x^3}{y^2} = 3uv^2$. Hence, $z = uv^3 + \phi(u)$, i.e. $z = \frac{x^4}{y^2} + \phi\left(\frac{x}{y^2}\right)$.

(c)

$$\frac{\partial z}{\partial x} = \frac{1}{y}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \quad \frac{\partial z}{\partial y} = \frac{-x}{y^2}\frac{\partial z}{\partial u}.$$

Therefore the PDE is $\frac{x}{y}z_u + xz_v - \frac{x}{y}z_u = \frac{4x^3}{y}$, i.e. $z_v = \frac{4x^2}{y} = 4uv$. Hence, $z = 2uv^2 + \phi(u)$, i.e. $z = 2\frac{x^3}{y} + \phi\left(\frac{x}{y}\right)$.