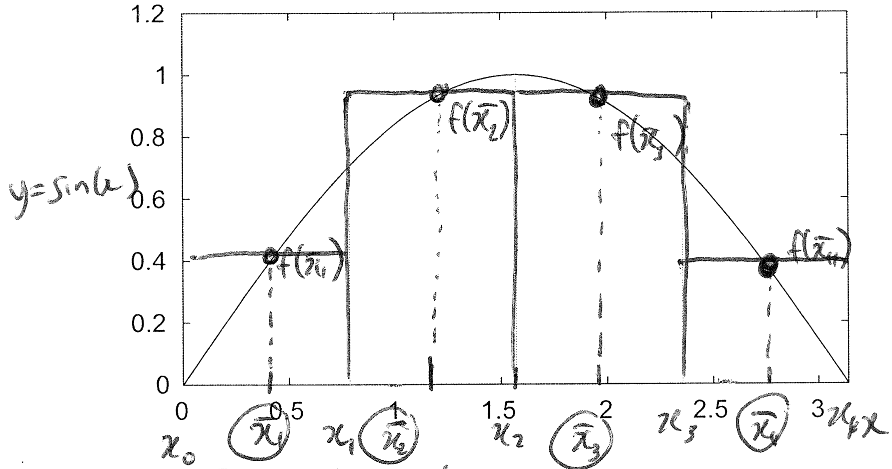
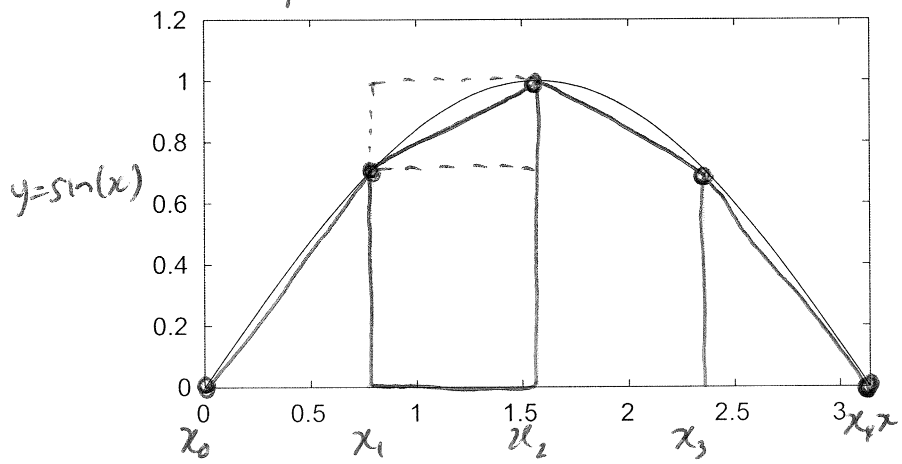
Visualiser 30-01-2020

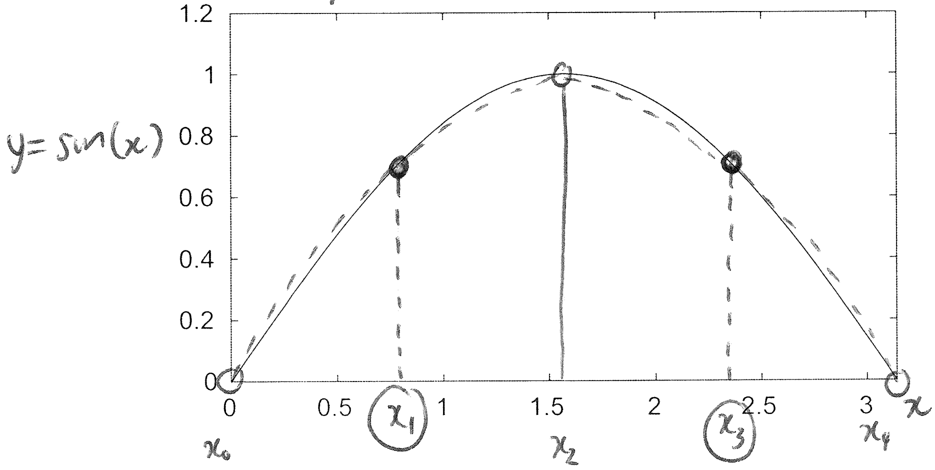
Midpoint rule



Trapezium rule



Simpson's rule



Example 3.3: Compute the integral

$$I = \int_0^{\pi} \sin(x)\, dx$$

using the midpoint rule with four equally-sized intervals.

With four grid intervals $\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}$ and so

$x_0 = 0$, $x_1 = \frac{\pi}{4}$, $x_2 = \frac{\pi}{2}$, $x_3 = \frac{3\pi}{4}$, $x_4 = \pi$

and $\bar{x}_1 = \frac{\pi}{8}$, $\bar{x}_2 = \frac{3\pi}{8}$, $\bar{x}_3 = \frac{5\pi}{8}$, $\bar{x}_4 = \frac{7\pi}{8}$,

so $f(\bar{x}_1) = \sin\left(\frac{\pi}{8}\right)$, $f(\bar{x}_2) = \sin\left(\frac{3\pi}{8}\right)$, $f(\bar{x}_3) = \sin\left(\frac{5\pi}{8}\right)$, $f(\bar{x}_4) = \sin\left(\frac{7\pi}{8}\right)$

Using the midpoint rule

$$M_4 = \Delta x \left[f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + f(\bar{x}_4) \right]$$

$$= \frac{\pi}{4} \left[\sin\left(\frac{\pi}{8}\right) + \sin\left(\frac{3\pi}{8}\right) + \sin\left(\frac{5\pi}{8}\right) + \sin\left(\frac{7\pi}{8}\right) \right]$$

$$= \frac{\pi}{2} \sqrt{1 + \frac{1}{\sqrt{2}}} \approx 2.05234.$$

Example 3.4: Compute the integral

$$I_4 = \int_0^{\pi} \sin(x)\, \mathrm{d}x$$

using the trapezium rule with four equally-sized intervals.

With four grid intervals $\Delta x = \frac{\pi}{4}$ and

$$x_0 = 0, \quad x_1 = \frac{\pi}{4}, \quad x_2 = \frac{\pi}{2}, \quad x_3 = \frac{3\pi}{4}, \quad x_4 = \pi$$

and so $y_0 = f(x_0) = 0$, $y_1 = f(x_1) = \frac{1}{\sqrt{2}}$, $y_2 = f(x_2) = 1$, $y_3 = f(x_3) = \frac{1}{\sqrt{2}}$, $y_4 = f(x_4) = 0$.

Approximating using the trapezium rule

$$\begin{aligned}
T_4 &= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4) \right] \\
&= \frac{\pi}{8}\left[0 + \frac{2}{\sqrt{2}} + 2 + \frac{2}{\sqrt{2}} + 0 \right] \\
&= \frac{\pi}{8}\left[2 + \frac{4}{\sqrt{2}} \right] \\
&= \frac{\pi}{4}\left[1 + \sqrt{2} \right] \\
&\approx 1.89612.
\end{aligned}$$

Example 3.5: Compute the integral

$$I = \int_0^\pi \sin(x)\,\mathrm{d}x$$

using the Simpson's rule with four equally-sized intervals.

With four subintervals $\Delta x = \frac{\pi}{4}$ and

$x_0 = 0$, $x_1 = \frac{\pi}{4}$, $x_2 = \frac{\pi}{2}$, $x_3 = \frac{3\pi}{4}$, $x_4 = \pi$

$y = f(x_0) = 0$, $y_1 = f(x_1) = \frac{1}{\sqrt{2}}$ $y_2 = 1$, $y_3 = \frac{1}{\sqrt{2}}$, $y_4 = 0$.

Applying Simpson's rule with $n=4$

$$S_4 = \frac{\Delta x}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4 \right]$$

$$= \frac{1}{3}\frac{\pi}{4}\left[0 + \frac{4}{\sqrt{2}} + 2 + \frac{4}{\sqrt{2}} + 0 \right]$$

$$= \frac{\pi}{12}\left(4\sqrt{2} + 2\right)$$

$$= \frac{\pi}{6}\left(1 + 2\sqrt{2}\right)$$

$$\approx 2.00456$$