2C Intro to real analysis 2020/21

Ex Sheet 8

Solutions and Comments [1] [2]

Q1 *Use Leibniz's test to show that each of the series below converge. Justify your answers.*

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-9}$;

b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+4}$;

c) $\sum_{n=1}^{\infty} (-1)^{n} \frac{n}{n^2+7}$.

We must first introduce our notation, and then check all three conditions in Leibniz's test. As we are told that the test will work in the question, it doesn't matter in which order we check the conditions. I'll first check[3] that the terms of a series converge to zero.

a) Let $a_n = \frac{1}{2n-9}$. We have

$$a_n = \frac{1/n}{2-9/n} \to \frac{0}{2-0} = 0,$$

as $n \to \infty$. For $n \geq 5$, we have $a_n \geq 0$. Finally,

$$a_n - a_{n+1} = \frac{1}{2n-9} - \frac{1}{2(n+1)-9} = \frac{(2n-7)-(2n-9)}{(2n-9)(2n-7)}$$
$$= \frac{2}{(2n-9)(2n-7)} \geq 0,$$

for $n \geq 5$. Thus $a_{n+1} \leq a_n$ for $n \geq 5$. Therefore by Leibniz's test, the series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges.

In the second example we need some strategy for seeing that $\sqrt{n}(n+5) - \sqrt{n+1}(n+4)$ is eventually positive. One way of seeing this is below.

b) Let $a_n = \frac{\sqrt{n}}{n+4}$ and note that $a_n \geq 0$ for all $n \in \mathbb{N}$. We have

$$a_n = \frac{\sqrt{n}}{n+4} = \frac{n^{-1/2}}{1+4/n} \to \frac{0}{1+0} = 0,$$

as $n \to \infty$. Moreover

$$a_n - a_{n+1} = \frac{\sqrt{n}}{n+4} - \frac{\sqrt{n+1}}{n+5} = \frac{\sqrt{n}(n+5) - \sqrt{n+1}(n+4)}{(n+4)(n+5)}.$$

Now one checks that

$$n(n+5)^2 \geq (n+1)(n+4)^2$$

[1] If you've not seriously tried these exercises, please don't look at these solutions and comments, until you have. You'll get the most benefit from reading these comments, when you've first thought hard about them yourself, even if you get really stuck — don't just try for a few minutes and then look at the solutions to work out how to proceed, you don't learn anywhere near as much that way.

[2] Note that I deliberately do not include formal answers for all questions.

[3] unless it's simply immediate that $a_n \geq 0$ when I note that straightaway.

for $n \geq 5$. This implies $\sqrt{n}(n+5) \geq \sqrt{n+1}(n+4)$ and hence also $a_n \geq a_{n+1}$ for $n \geq 5$. Therefore $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges by Leibniz's test.

In the final example a factor of $(-1)^n$ appears instead of $(-1)^{n-1}$. In order to apply the Leibniz test we'll therefore first consider the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+7}$, and then use the fact that multiplying everything by -1 does not affect convergence[4].

[4] More precisely, we'll use the general result that if $\sum_{n=1}^{\infty} x_n$ converges then also $\sum_{n=1}^{\infty}(\lambda x_n)$ converges for any $\lambda \in \mathbb{R}$.

c) Let $a_n = \frac{n}{n^2+7}$ and note $a_n \geq 0$ for all n. We have

$$a_n = \frac{n}{n^2+7} = \frac{1/n}{1+7/n^2} \to \frac{0}{1+0} = 0,$$

as $n \to \infty$. Also

$$\begin{aligned} a_n - a_{n+1} &= \frac{n}{n^2+7} - \frac{n+1}{n^2+2n+8} \\ &= \frac{n(n^2+2n+8)-(n+1)(n^2+7)}{(n^2+7)(n^2+2n+8)} \\ &= \frac{n^2+n-7}{(n^2+7)(n^2+2n+8)} \geq 0, \end{aligned}$$

for $n \geq 3$. Thus $(a_n)_{n=1}^{\infty}$ is eventually decreasing, and hence $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+7}$ converges by Leibniz's test. Therefore, the series $\sum_{n=1}^{\infty} -(-1)^{n-1}\frac{n}{n^2+7} = \sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+7}$ converges as well using algebraic properties of limits.

Q2 *Determine whether $\sum_{n=1}^{\infty} x_n$ is absolutely convergent, conditionally convergent or divergent when x_n is:*

a) $\frac{\cos(n\pi)}{n}$;

b) $\frac{\cos(n^2)}{n^2+5}$;

c) $\frac{(-3)^n+2^n}{3^n+2^n}$;

d) $\frac{(-1)^{n-1}}{n-1-\sqrt{n}}$.

In this question we need to follow our methods for deciding which test for convergence to follow[5].

[5] The theme of this set of exercises.

For *a*), note that $\cos(n\pi)$ is just a more complicated way of writing $(-1)^n$. Once we spot this, Leibniz's test comes into play. For *b*), we use the bound $|\cos(n^2)| \leq 1$, and then a comparison test with $\sum_{n=1}^{\infty}\frac{1}{n^2}$ to obtain absolute convergence. *c*) is a good example of the importance of checking whether the terms go to zero before doing anything more complicated.

a) Note that $|\cos(n\pi)| = 1$ for all n so that $\sum_{n=1}^{\infty} \left|\frac{\cos(n\pi)}{n}\right|$ is the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges. Thus the series is not absolutely convergent. Now write $a_n = \frac{1}{n}$, so that $a_n \geq 0$ for all n; moreover $(a_n)_{n=1}^{\infty}$ is decreasing and $a_n \to 0$. By Leibniz's test $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges, and then so too does $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} = -\sum_{n=1}^{\infty}(-1)^{n-1}a_n$. Thus this series is conditionally convergent.

b) We have

$$0 \leq \left|\frac{\cos(n^2)}{n^2+5}\right| \leq \frac{1}{n^2+5} \leq \frac{1}{n^2}.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the series $\sum_{n=1}^{\infty} \frac{\cos(n^2)}{n^2+5}$ is absolutely convergent by the comparison test.

c) Let $a_n = \frac{(-3)^n+2^n}{3^n+2^n}$. Then

$$|a_n| = \frac{3^n+(-2)^n}{3^n+2^n} = \frac{1+(-2/3)^n}{1+(2/3)^n} \to \frac{1+0}{1+0} = 1,$$

as $n \to \infty$. Hence $a_n \not\to 0$, so the series $\sum_{n=1}^{\infty} \frac{(-3)^n+2^n}{3^n+2^n}$ diverges.

d) We have

$$\left|\frac{(-1)^{n-1}}{n-1-\sqrt{n}}\right| = \frac{1}{n-1-\sqrt{n}} \geq \frac{1}{n}$$

for $n \geq 3$ (when $n \geq 1+\sqrt{n}$). Thus by comparison with the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n-1-\sqrt{n}}$ is not absolutely convergent.

Now set $a_n = \frac{1}{n-1-\sqrt{n}}$ so that $a_n \geq 0$ for $n \geq 3$. We have $a_n \to 0$ as $n \to \infty$ and

$$\begin{aligned} a_n - a_{n+1} &= \frac{1}{n-1-\sqrt{n}} - \frac{1}{n-\sqrt{n+1}} \\ &= \frac{n-\sqrt{n+1}-(n-1-\sqrt{n})}{(n-1-\sqrt{n})(n-\sqrt{n+1})} \\ &= \frac{1-(\sqrt{n+1}-\sqrt{n})}{(n-1-\sqrt{n})(n-\sqrt{n+1})}. \end{aligned}$$

Now $\sqrt{n+1}-\sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \leq 1$ for all $n \in \mathbb{N}$, so that $1-(\sqrt{n+1}-\sqrt{n}) \geq 0$ for all n. Also $n-1 \geq \sqrt{n}$ for all $n \geq 3$, as can be seen by squaring both sides. Similarly, note that $n \geq \sqrt{n+1}$ for $n \geq 2$.

In conclusion, for $n \geq 3$, we have $a_n \geq a_{n+1}$. Thus $(a_n)_{n=1}^{\infty}$ is eventually decreasing, and hence $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is conditionally convergent, by Leibniz's test.

Q3 *Determine whether each of the series below converge or diverge. Justify your answer, clearly indicating the relevant tests or results that you use.*

In all of these questions, we need to think carefully to choose the right test, to avoid wasting time on something which won't work. Don't be afraid to work intuitively, until you can work out how to proceed, and follow the guide in the lecture notes describing which tests work best in different situations. When you are writing your answers try to make sure you demonstrate your knowledge to the reader, by writing in such a way that it shows you know what the statements of the convergence tests are.

a) $\sum_{n=1}^{\infty} \binom{3n}{n} \frac{1}{7^n}$.

Here it's not so clear whether or not the terms converge to zero, so we should press on to the next step in the process. Both of the expressions 7^n and $\binom{3n}{n}$ suggest[6] using the ratio test, as there will be significant cancellation in $\frac{a_{n+1}}{a_n}$.

[6] Recall that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ for $n, r \in \mathbb{N}$ with $r \leq n$.

> Let $a_n = \binom{3n}{n} \frac{1}{7^n}$ so that $a_n \geq 0$ for all $n \in \mathbb{N}$. We have
>
> $$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{(3n+3)!}{(n+1)!(2n+2)!7^{n+1}} \frac{n!(2n!)7^n}{(3n)!} \\ &= \frac{(3n+3)(3n+2)(3n+1)}{7(2n+2)(2n+1)(n+1)} \\ &= \frac{(3+\frac{3}{n})(3+\frac{2}{n})(3+\frac{1}{n})}{7(2+\frac{2}{n})(2+\frac{1}{n})(1+\frac{1}{n})} \to \frac{27}{28} < 1, \end{aligned}$$
>
> as $n \to \infty$. Therefore by the ratio test, $\sum_{n=1}^{\infty} \binom{3n}{n} \frac{1}{7^n}$ converges.

It's very easy to make a mistake when calculating this ratio[7].

[7] A standard error is to obtain $\frac{(3n+1)!}{(2n+1)!(n+1)!7^{n+1}}$ for a_{n+1} or to cancel the fraction $\frac{(3n+3)!}{(3n)!}$ as $3n+3$ rather than $(3n+3)(3n+2)(3n+1)$. If you did something like this, then make a note to take care on similar questions.

b) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+1}}$.

The terms here go to zero, and roughly speaking the size of the n-th term is $n^{-1/2}$, so we expect divergence by comparing with $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$.

> Set $a_n = \frac{n}{\sqrt{n^3+1}}$ and $b_n = \frac{1}{n^{1/2}}$, and note that $a_n \geq 0$ and $b_n \geq 0$ for all n. Then
>
> $$\frac{a_n}{b_n} = \frac{n^{3/2}}{\sqrt{n^3+1}} = \frac{1}{\sqrt{1+1/n^3}} \to \frac{1}{\sqrt{1+0}} = 1 > 0,$$
>
> as $n \to \infty$. Since $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges, so too does $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+1}}$ by the limiting version of the comparison test.

Note how we clearly indicate the name of the test we're using, and give information showing that we understand the limiting version of the test (as we clearly indicate that we've checked that $a_n \geq 0$ and $b_n \geq 0$, and note that the limit is strictly positive).

c) $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{1/n}$.

Here the first step in the guide for testing convergence comes to our rescue. A quick check reveals that the terms of the series do not converge to zero.

We have
$$\left(\frac{1}{2}\right)^{1/n} = \frac{1}{2^{1/n}} \to \frac{1}{1} \neq 0,$$
as $n \to \infty$. Therefore the series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{1/n}$ diverges.

d) $\sum_{n=1}^{\infty} \frac{1+\cos(n)}{1+n\sqrt{n}}$.

In this case, once we see that the terms of the series go to zero[8] we start to think about what to do next. Noting that $-1 \leq \cos(n) \leq 1$ for all n, we see that $0 \leq 1+\cos(n) \leq 2$. This leads us to the comparison test, and this is a classic case where the limiting version of the comparison test doesn't help as
$$\frac{1+\cos(n)}{1+n\sqrt{n}} \frac{n^{3/2}}{1}$$
does not converge. We must use the standard version of comparison.

[8] due to the $\frac{1}{1+n\sqrt{n}}$ part

We have
$$0 \leq \frac{1+\cos(n)}{1+n\sqrt{n}} \leq \frac{2}{n^{3/2}},$$
for all $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, so too does $\sum_{n=1}^{\infty} \frac{1+\cos(n)}{1+n\sqrt{n}}$ by the comparison test.

e) $\sum_{n=1}^{\infty} \frac{n+(-1)^n}{n^3+2}$.

The $(-1)^n$ here is a red herring - the numerator alternates between $n+1$ and $n-1$ and so (for large n) is roughly just n. This suggests comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Set $a_n = \frac{n+(-1)^n}{n^3+2}, b_n = \frac{1}{n^2}$, so that $a_n \geq 0$ and $b_n \geq 0$ for all $n \in \mathbb{N}$. Then
$$\frac{a_n}{b_n} = \frac{n^3+n^2(-1)^n}{n^3+2} = \frac{1+(-1)^n/n}{1+2/n^3} \to \frac{1+0}{1+0} = 1,$$
as $n \to \infty$ (we can use $0 \leq |(-1)^n/n^2| \leq 1/n^2 \to 0$ and the sandwich principle to see that $(-1)^n/n^2 \to 0$ as $n \to \infty$). As $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so too does $\sum_{n=1}^{\infty} \frac{n+(-1)^n}{n^3+2}$ by the limit version of the comparison test.

f) $\sum_{n=1}^{\infty} \frac{(n!)^3 27^n}{(3n)!}$.

This question brings the ratio test to mind: there will be extensive cancellation in $\frac{a_{n+1}}{a_n}$. Note though that if we take the limit $\frac{a_{n+1}}{a_n}$ we get the value 1, so we need to look at the ratio more carefully and see that $\frac{a_{n+1}}{a_n} \geq 1$ for all $n \in \mathbb{N}$.

Let $a_n = \frac{(n!)^3 27^n}{(3n)!}$, so that $a_n \geq 0$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{27^{n+1}((n+1)!)^3}{(3n+3)!} \frac{(3n)!}{27^n(n!)^3} \\ &= \frac{27(n+1)^3}{(3n+3)(3n+2)(3n+1)} \\ &= \frac{3n+3}{3n+3}\frac{3n+3}{3n+2}\frac{3n+3}{3n+1} \geq 1,\end{aligned}$$

for all $n \in \mathbb{N}$. By the ratio test it follows that $\sum_{n=1}^\infty \frac{(n!)^3 27^n}{(3n)!}$ diverges.

Note how I try to write the inequalities above so that we can see that the final ratio is at least 1 without having to think too hard about why this is the case.

g) $\sum_{n=1}^\infty \frac{\cos(n\pi)}{\sqrt{n}}$.

A standard thing to remember here is that $\cos(n\pi) = (-1)^n$, so that the terms in this series strictly alternate positive and negative. Thus the Leibniz test comes to mind. If instead you'd tried to show that the series is absolutely convergent using $|\cos(n\pi)| \leq 1$, you'd be out of luck: this series turns out to be conditionally convergent.

Set $a_n = \frac{1}{\sqrt{n}}$, so that $a_n \geq 0$, $(a_n)_{n=1}^\infty$ is decreasing and $a_n \to 0$ as $n \to \infty$. Therefore $\sum_{n=1}^\infty (-1)^{n-1} a_n$ converges by Leibniz's test, and hence $\sum_{n=1}^\infty \frac{\cos(n\pi)}{\sqrt{n}} = -\sum_{n=1}^\infty (-1)^{n-1} a_n$ converges.

h) $\sum_{n=1}^\infty (-1)^{n-1} \frac{n+2}{n^2+16}$.

This is another Leibniz question. We can see the terms go to zero, so it's just a matter of checking that the positive part $\frac{n+2}{n^2+16}$ gives a decreasing sequence.

Let $a_n = \frac{n+2}{n^2+16}$ so that $a_n \geq 0$ for all $n \in \mathbb{N}$ and

$$a_n = \frac{1/n + 2/n^2}{1 + 16/n^2} \to \frac{0+0}{1+0} = 0,$$

as $n \to \infty$. Now

$$\begin{aligned}a_n - a_{n+1} &= \frac{(n+2)(n^2+2n+17) - (n+3)(n^2+16)}{(n^2+16)(n^2+2n+17)} \\ &= \frac{n^2+5n-14}{(n^2+16)(n^2+2n+17)} \geq 0\end{aligned}$$

for $n \geq 2$. Therefore $(a_n)_{n=1}^\infty$ is eventually decreasing, and so $\sum_{n=1}^\infty (-1)^{n-1} a_n$ converges by the Leibniz test.

i) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$. [9]

Following the hint leads us to the comparison test.

For $n \in \mathbb{N}$, we have

$$0 \leq \frac{\sqrt{n+1}-\sqrt{n-1}}{n} = \frac{2}{n(\sqrt{n+1}+\sqrt{n-1})} \leq \frac{2}{n^{3/2}}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, so too does the series $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$.

[9] Hint: look at the method used in the section on the sandwich principle for sequences, when we computed $\lim_{n\to\infty}(\sqrt{n+4}-\sqrt{n})$.

j) $\sum_{n=1}^{\infty} \frac{\cos(n^2)}{n^2+5}$.

Here we use the fact that $|\cos(n^2)| \leq 1$ and look for absolute convergence — we can see that this will work, as the n^2+5 term in the denominator suggests comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Be careful not to compare $\sum_{n=1}^{\infty} \frac{\cos(n^2)}{n^2+5}$ directly — the comparison test only works for series with eventually positive terms.

For $n \in \mathbb{N}$, we have

$$0 \leq \left|\frac{\cos(n^2)}{n^2+5}\right| \leq \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the comparison test shows that $\sum_{n=1}^{\infty} \frac{\cos(n^2)}{n^2+5}$ is absolutely convergent and hence converges.

k) $\sum_{n=1}^{\infty} \frac{n^{1/n}}{n+1}$.

It's not clear whether or not $a_n \to 0$, but we can compare with the harmonic series.

Let $a_n = \frac{n^{1/n}}{n+1}$. Since

$$a_n \geq \frac{1}{n+1}$$

for all $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges, by the comparison test so too does $\sum_{n=1}^{\infty} \frac{n^{1/n}}{n+1}$.

l) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n+1}{3n+1}$.

This is a classic case where we can save time by checking whether the terms of an alternating series converge to zero before trying the other parts of the Leibniz test.

We have

$$\left|(-1)^n \frac{2n+1}{3n+1}\right| = \frac{2+1/n}{3+1/n} \to \frac{2}{3} \neq 0,$$

as $n \to \infty$. Since the terms of the series do not converge to zero, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n+1}{3n+1}$ diverges.

m) $\sum_{n=1}^{\infty}\frac{n^n}{(n+1)!}$.

This is one which suggests the ratio test, but it's a little tricky to compute the limit of the ratios.

Let $a_n = \frac{n^n}{(n+1)!}$, so that $a_n \geq 0$ for all $n \in \mathbb{N}$. We have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+2)!}\frac{(n+1)!}{n^n} = \left(\frac{n+1}{n}\right)^n \frac{n+1}{n+2} \geq \frac{(n+1)^2}{n(n+2)} \geq 1,$$

for all $n \in \mathbb{N}$. Therefore by the ratio test, $\sum_{n=1}^{\infty}\frac{n^n}{(n+1)!}$ diverges.

As a side remark, be careful in computing

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n.$$

You can't evaluate this limit by first taking $\lim_{n\to\infty}\left(\frac{n+1}{n}\right) = 1$ and then taking a second limit $\lim_{n\to\infty} 1^n = 1$: both limiting processes happen at the same time, you can't split them up in this way[10].

[10] In fact, $\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e$, which we could use in the ratio test, but since we haven't proved this yet (it'll appear in the 3rd year Analysis course) we simply showed that the ratio is always at least one.

n) $\sum_{n=1}^{\infty}\frac{(-3)^n n^2}{2^{2n}}$.

Set $a_n = \left|\frac{(-3)^n n^2}{2^{2n}}\right|$. Then

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}(n+1)^2}{2^{2n+2}}\frac{2^{2n}}{3^n n^2} = \frac{3(n+1)^2}{4n^2} \to \frac{3}{4},$$

as $n \to \infty$. As $3/4 < 1$, the series $\sum_{n=1}^{\infty}\frac{(-3)^n n^2}{2^{2n}}$ is absolutely convergent, and hence convergent.

o) $\sum_{n=1}^{\infty}\frac{4^n}{n!+3^n}$.

While the factors $4^n, n!$ and 3^{n+1} all suggest the ratio test, it's not so clear how to simplify

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}(n!+3^n)}{4^n((n+1)!+3^{n+1}}.$$

Instead, if we first note that for large n, the term $n!$ dominates 3^n, this suggests an initial use of the comparison test.

We have

$$0 \leq \frac{4^n}{n!+3^n} \leq \frac{4^n}{n!}.$$

Now set $a_n = \frac{4^n}{n!}$, so that $a_n \geq 0$ and

$$\frac{a_{n+1}}{a_n} = \frac{4}{n+1} \to 0,$$

as $n \to \infty$. The ratio test shows that $\sum_{n=1}^{\infty} a_n$ converges, and then the comparison test gives the convergence of $\sum_{n=1}^{\infty} \frac{4^n}{n!+3^n}$.

Q4

a) Let $(a_n)_{n=1}^{\infty}$ be a sequence with $a_n \geq 0$ for all n. Show that if $\sum_{n=1}^{\infty} a_n$ converges, then so too does $\sum_{n=1}^{\infty} a_n^2$.

b) Give an example of a series $\sum_{n=1}^{\infty} a_n$ such that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_n^3$ converge while $\sum_{n=1}^{\infty} a_n^2$ diverges.

The first part is an example of the comparison test in action, and the entire question illustrates why we need to work with eventually positive terms in the comparison test. Once we know that $\sum_{n=1}^{\infty} a_n$ converges, we will have $a_n \to 0$ as $n \to \infty$. In particular, for large n, we have $0 \leq a_n < 1$, which ensures that $a_n^2 \leq a_n$. For the second part we take advantage of Leibniz's test.

a) Suppose that $a_n \geq 0$ for all n and that $\sum_{n=1}^{\infty} a_n$ converges. Then $a_n \to 0$, so taking $\varepsilon = 1$ in the definition of convergence gives $n_0 \in \mathbb{N}$ such that $a_n < 1$ for $n \geq n_0$. In particular, we have $a_n^2 \leq a_n$ for $n \geq n_0$. Therefore $\sum_{n=0}^{\infty} a_n^2$ converges by the comparison test.

b) Take $a_n = (-1)^{n-1}\frac{1}{\sqrt{n}}$. Then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_n^3$ converge by Leibniz's test, while $\sum_{n=1}^{\infty} a_n^2$ is the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges.

Q5 *This exercise is a brain teaser, it is harder than the other exercises and will not feature on the exam.*
Consider the series

$$1 + \frac{1}{2} + \cdots + \frac{1}{8} + \frac{1}{10} + \cdots + \frac{1}{18} + \frac{1}{20} + \cdots + \frac{1}{88} + \frac{1}{100} + \cdots \qquad (1)$$

obtained by summing $\frac{1}{n}$ over all those n for which 9 does not appear in the decimal expansion of n. Prove that this series converges.

I'll provide an outline solution. The trick is to group the terms in the series by the number of digits in the denominator. For each k, there are precisely $8 \times 9^{k-1}$ natural numbers which have exactly[11] k digits none of which is a 9. For such a natural number n with k digits we have $n \geq 10^{k-1}$, so that $\frac{1}{n} \leq 10^{1-k}$. Write T_k for the sum of $\frac{1}{n}$ over all $n \in \mathbb{N}$, with at most k digits none of which are 9, then

$$T_k \leq \sum_{r=1}^{k} 8 \times 9^{r-1} 10^{1-r}$$

[11] This is a counting argument which you should be familiar from first year courses. There are eight ways to choose the first digit from $\{1, 2, \ldots, 8\}$, and nine ways to choose the subsequent $k-1$ digits from $\{0, 1, 2, \ldots, 8\}$.

Since the geometric series $\sum_{r=1}^{\infty} 8 \times 9^{r-1}10^{1-r}$ converges, the sequence $(T_k)_{k=1}^{\infty}$ is bounded above. From this we can see that the series (1) converges.