2A degree exam 2018–19, solutions

1. The function $F(t) = f(g_1(t), g_2(t))$ and using the chain rule we have

$$F'(t) = g_1'(t)\frac{\partial f}{\partial x}(g_1, g_2) + g_2'(t)\frac{\partial f}{\partial y}(g_1, g_2).$$

In our case $g_1 = \cosh t$ and $g_2 = \sinh t$ and we have $g_1' = \sinh t$ and $g_2' = \cosh t$. We also have $f = x^2 - y^2$ so $f_x = 2x$ and $f_y = -2y$. Therefore

$$F'(t) = (\sinh t)\,(2\cosh t) + (\cosh t)\,(-2\sinh t) = 0.$$

2. We seek to simplify the PDE by writing $f(x, y) = F(v(x, y), w(x, y))$. Then with $v = x^3/y$ and $w = xy$ we have, using the chain rule,

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial v}{\partial x}\frac{\partial F}{\partial v} + \frac{\partial w}{\partial x}\frac{\partial F}{\partial w} = \frac{3x^2}{y}\frac{\partial F}{\partial v} + y\frac{\partial F}{\partial w} \\
\frac{\partial f}{\partial y} &= \frac{\partial v}{\partial y}\frac{\partial F}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial F}{\partial w} = -\frac{x^3}{y^2}\frac{\partial F}{\partial v} + x\frac{\partial F}{\partial w}.
\end{aligned}$$

Substitute into the PDE $xf_x + 3yf_y = x^4$ to give

$$x\left(\cancel{\frac{3x^2}{y}F_v} + yF_w\right) + 3y\left(\cancel{-\frac{x^3}{y^2}F_v} + xF_w\right) = x^4$$

which simplifies to $4xyF_w = x^4$ and using the change of variable to eliminate x and y we obtain the simplified PDE

$$\frac{\partial F}{\partial w} = \frac{1}{4}v$$

a partial integration gives the general solution as

$$F(v, w) = \frac{1}{4}vw + A(v)$$

where A is an aribtrary function of a single variable. The solution to the original PDE is therefore

$$f(x, y) = \frac{1}{4}x^4 + A\left(\frac{x^3}{y}\right).$$

3. The gradient of $\Phi(x, y, z) = \phi(r)$ is

$$\text{grad }\phi(r) = \left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z}\right)$$

using the chain rule we have

$$\text{grad }\phi(r) = \phi'(r)\left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right) = \phi'(r)\,\text{grad}\, r.$$

And using implicit differentiation on $r^2 = x^2 + y^2 + z^2$ we have

$$2r\,\text{grad }r = \mathbf{x}, \qquad \text{grad}\, r = \frac{\mathbf{x}}{r}.$$

Then

$$\text{grad } \phi(r) = \mathbf{x}\frac{1}{r}\phi'(r).$$

Considering the vector field $\mathbf{F} = r\mathbf{x}$, to show that it is conservative we note that $\mathbf{F}$ is defined everywhere in $\mathbb{R}^3$ with continuous derivatives, so it is enough to check the curl of $\mathbf{F}$. If the curl is zero then $\mathbf{F}$ is conservative. Using the nabla identity

$$\nabla \times (a\mathbf{A}) = a\nabla \times \mathbf{A} + \nabla a \times \mathbf{A}$$

we have

$$\nabla \times \mathbf{F} = r\nabla \times \mathbf{x} + \nabla r \times \mathbf{x}.$$

We have already calculated $\nabla r = \mathbf{x}/r$ and we note that $\mathbf{x} \times \mathbf{x} = \mathbf{0}$ (property of the vector product). The curl of $\mathbf{x}$ is

$$\nabla \times \mathbf{x} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial z \\ x & y & z \end{vmatrix} = \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}, \frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}, \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = \mathbf{0}.$$

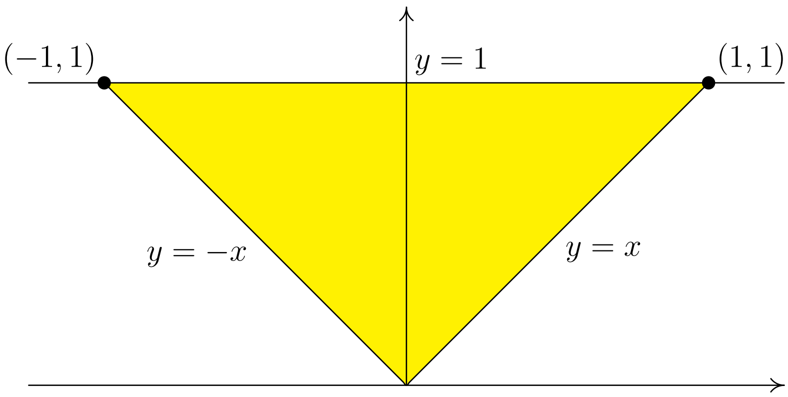
Together these results mean that $\nabla \times \mathbf{F} = \mathbf{0}$ so $\mathbf{F}$ is conservative.

As $\mathbf{F}$ is conservative then there exists a ϕ whose gradient is $r\mathbf{x}$. As we know from the first part that $\phi(r)$ has gradient $\mathbf{x}\phi'(r)/r$ then setting $\phi'(r)/r = r$ will give the required potential. So

$$\phi'(r) = r^2, \qquad \phi(r) = \frac{1}{3}r^3 + c$$

is the potential.

4. The sketch is shown below.



The integral is written as a type-I integral (y integral first) and so we convert to type-II. The type-II description of the region is $-y \le x \le y$ and $0 \le y \le 1$, therefore the integral can be written

$$I = \int_{-1}^{1} \left(\int_{|x|}^{1} y^2 e^{xy}\, dy \right) dx = \int_0^1 \left(\int_{-y}^{y} y^2 e^{xy}\, dx \right) dy.$$

Performing the iterated integral gives

$$\int_0^1 \left(\int_{-y}^{y} y^2 e^{xy}\, dx \right) dy = \int_0^1 \left[y e^{xy} \right]_{x=-y}^{x=y} dy = \int_0^1 y e^{y^2} - y e^{-y^2}\, dy = \left[\frac{1}{2} e^{y^2} + \frac{1}{2} e^{-y^2} \right]_0^1 = \frac{e}{2} + \frac{1}{2e} - 1.$$

5. The boundaries of the region D suggest the change of variable $v = ye^{-x}$ and $w = xy$ as then the region D' in the v-w plane is rectangular $D' = [1, e] \times [1, e]$. In order to transform the integral we need the Jacobian of the change of variable

$$\frac{\partial(x,y)}{\partial(v,w)} = \left(\frac{\partial(v,w)}{\partial(x,y)}\right)^{-1}$$

and
$$\frac{\partial(v,w)}{\partial(x,y)} = \frac{\partial v}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial v}{\partial y}\frac{\partial w}{\partial x} = \left(-ye^{-x}\right)x - \left(e^{-x}\right)y = -(1+x)ye^{-x}$$

therefore in the region D,

$$|J| = \frac{1}{(1+x)ye{-}x}$$

and so the integral becomes (using the change of variable to eliminate x and y

$$\iint_{D'} \frac{1}{v}\, dvdw.$$

As the region D' is rectangular and the integrand is separable we have the integral becomes

$$\left(\int_1^e \frac{1}{v}\, dv\right)\left(\int_1^e 1\, dw\right) = [\log v]_1^e\, [w]_1^e = e - 1.$$

6. The projection of the three-dimensional region onto the x-y plane ($z = 0$) gives a region bounded by $x = 0$, $y = 0$ and $hx + hy = hl$ which give $x + y = l$. This is a triangular region in the x-y plane. Therefore the region can be described by the inequalities

$$0 \le z \le \frac{h}{l}(l - x - y), \quad 0 \le y \le l - x, \quad 0 \le x \le l$$

and so the triple integral can be written as the iterated integral

$$\begin{aligned}
\int_0^l \left(\int_0^{l-x}\left(\int_0^{\frac{h}{l}(l-x-y)} z\, dz\right) dy\right) dx &= \int_0^l \left(\int_0^{l-x} \left[\frac{1}{2}z^2\right]_0^{\frac{h}{l}(l-x-y)} dy\right) dx \\
&= \int_0^l \left(\int_0^{l-x} \frac{h^2}{2l^2}(l-x-y)^2\, dy\right) dx \\
&= \int_0^l \left[-\frac{h^2}{6l^2}(l-x-y)^3\right]_0^{l-x} dx \\
&= \frac{h^2}{6l^2}\int_0^l (l-x)^3\, dx = \frac{h^2}{24l^2}\left[-(l-x)^4\right]_0^l \\
&= \frac{1}{24}h^2l^2.
\end{aligned}$$

7. The projection of the surface onto the x-y plane is the unit disc D, $x^2 + y^2 \le 1$. The surface is the graph $z = 1 + x^2 - y^2$, to convert the surface integral into a double integral we calculate

$$\sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + (2x)^2 + (-2y)^2} = \sqrt{1 + 4\left(x^2 + y^2\right)}.$$

The surface area is

$$\iint_S 1\, dS = \iint_D \sqrt{1 + z_x^2 + z_y^2}\, dxdy = \iint_D \sqrt{1 + 4\left(x^2 + y^2\right)}\, dxdy.$$

The region D and the fact that the integrand is a function of x^2+y^2 suggest using polar coordinates. In polar coordinates the region is $0 \le r \le 1$ and $0 \le \theta \le 2\pi$ and so

$$\iint_D \sqrt{1+4\left(x^2+y^2\right)}\,dxdy = \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}r\,drd\theta$$

and as the region is rectangular (in polar coordinates) and the integrand is separable we have that the surface area of the surface is

$$2\pi\int_0^1 r\left(1+4r^2\right)^{1/2}\,dr = 2\pi\left[\frac{1}{12}\left(1+4r^2\right)^{3/2}\right]_0^1 = \frac{\pi}{6}\left(5\sqrt{5}-1\right).$$

8. Green's theorem states that, for a simple closed curve C that is positively oriented, enclosing a region A we have

$$\oint_C P\,dx + Q\,dy = \iint_A \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dxdy.$$

In the case here, the region A is enclosed by $x=0$, $y=0$ and the line $y = 2-2x$. So applying Green's theorem by identifying $P = y^2$ and $Q = x^2$, we are left to calculate

$$\iint_A 2x-2y\,dxdy = \int_0^1\left(\int_0^{2-2x} 2x-2y\,dy\right)\,dx.$$

Computing the iterated integral gives

$$\int_0^1 \left[2xy-y^2\right]_0^{2(1-x)}\,dx = \int_0^1 4x(1-x)-4(1-x)^2\,dx = \int_0^1 -8x^2+12x-4\,dx = -\frac{8}{3}+6-4 = -\frac{2}{3}.$$

9. The divergence thoerem says that, given $\mathbf{F}$, a vector field in $\mathbb{R}^3$ we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V \nabla\cdot\mathbf{F}\,dV$$

where S is a closed orientable surface enclosing the region V, with outward pointing unit normal $\mathbf{n}$.

In our case

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(yx)\,\frac{\partial}{\partial y}\left(y^3-y\right) + \frac{\partial}{\partial z}\left(z(1-y)\right) = y + \left(3y^2-1\right) + (1-y) = 3y^2.$$

The volume V enclosed by the surface S is the unit ball, which in spherical polar coordinates has $0 \le r \le 1$, $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$. Recall that $y = r\sin\theta\sin\phi$ and the Jacobian is $r^2\sin\phi$, so that the integral becomes

$$\int_0^{2\pi}\left(\int_0^{\pi}\left(\int_0^1 3\left(r\sin\phi\sin\theta\right)^2\cdot r^2\sin\phi\,dr\right)\,d\phi\right)\,d\theta$$

and since the integrand is separable and the limits are all constants we can write this as

$$\left(\int_0^{2\pi}\sin^2\theta\,d\theta\right)\left(\int_0^{\pi}\sin^3\phi\,d\phi\right)\left(\int_0^1 3r^4\,dr\right) = \left[\frac{\theta}{2}-\frac{1}{4}\sin 2\theta\right]_0^{2\pi}\left[-\cos\phi+\frac{1}{3}\cos^3\phi\right]_0^{\pi}\left[\frac{3}{5}r^5\right]_0^1$$
$$= \pi\cdot\frac{4}{3}\cdot\frac{3}{5} = \frac{4\pi}{5}.$$