Excellent work.

MATH 15/15

SOLVE

TOTAL 20/20

a) Consider the function $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x, y, z) = (2x + z, -y, 3z).$$

I didn't take any marks off for this, but be careful with this notation.

i) Show that T is a linear transformation

To be linear, T has to ~~hold under~~ be closed under addition ~~of two vectors~~ and scalar multiplication. For addition of two vectors $\langle x, y, z\rangle$ and $\langle x', y', z'\rangle$, $\in \mathbb{R}^3$

Don't use these brackets for vectors. This notation is reserved for inner products.

$$\begin{aligned} T(\langle x, y, z\rangle + \langle x', y', z'\rangle) &= T(\langle x + x', y + y', z + z'\rangle) \\ &= \langle 2(x + x') + z + z', -(y + y'), 3(z + z')\rangle \\ &= \langle 2x + z, -y, 3z\rangle + \langle 2x' + z', -y', 3z'\rangle \\ &= T(\langle x, y, z\rangle) + T(\langle x', y', z'\rangle). \end{aligned}$$

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Then, for multiplication with a scalar $c \in \mathbb{R}$,

$$\begin{aligned} T(c\langle x, y, z\rangle) &= T(\langle cx, cy, cz\rangle) \\ &= \langle 2cx + cz, -cy, 3cz\rangle \\ &= c\langle 2x + z, -y, 3z\rangle \\ &= cT(\langle x, y, z\rangle). \end{aligned}$$

Thus, T is a linear transformation.

ii) Write down the standard matrix for the transformation T (denoted by $[T]$).

The standard matrix can be found by transforming the unit vectors in $\mathbb{R}^3$ by T:

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\3\end{bmatrix}.$$

Thus, the standard matrix for T is

$$\begin{bmatrix}2 & 0 & 1\\0 & -1 & 0\\0 & 0 & 3\end{bmatrix}.$$

iii) Is $(-2,3,6) \in \text{range}(T)$? If it is, find the vector (x, y, z) that is mapped to $(-2,3,6)$ by T.

To see if the vector is in the range of T, the following system of linear equations has to be consistent:

$$\begin{aligned} -2 &= 2x + z, \\ 3 &= -y, \\ 6 &= 3z, \end{aligned}$$

the solution to which is the vector $\langle -2, -3, 2\rangle$. Since the system of linear equations has one solution, $\langle -2,3,6\rangle$ is in the range of T. The vector $\langle -2, -3, 2\rangle$ is also the vector mapped to $(-2,3,6)$ by T, as required.

b) Consider the following sets of vectors,

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\},$$

$$C = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\},$$

both of which form an ordered basis for $\mathbb{R}^3$.

i) Find the coordinates of

$$\boldsymbol{v} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$$

with respect to the ordered basis B.

To do this, the linear system of equations based on basis B

$$3 = x + z,$$
$$2 = y + z,$$
$$3 = x + y,$$

where scalars $x, y, z \in \mathbb{R}$ are the multiples of the vectors in B. The solution to this system is the vector

$$[\boldsymbol{v}]_B = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}. \checkmark$$

ii) Find the change of basis matrix $P_{C \leftarrow B}$.

Expressing each vector in B as a linear combination of vectors in C:

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$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

From the scalars in these equations, the change of basis matrix $P_{C \leftarrow B}$ can be found to be

$$\checkmark \quad \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$

iii) Using parts (i) and (ii), find the coordinates of ***v*** with respect to the ordered basis C.

Matrix multiplication can be used to find

$$[\boldsymbol{v}]_C = P_{C\leftarrow B}[\boldsymbol{v}]_B$$
$$= \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}.$$

c) Let C and D be $n \times n$ matrices and $\boldsymbol{x} \in \mathbb{R}^n$. Suppose that $CD = DC$ and that $\det(D) \neq 0$. Show that, if $D\boldsymbol{x}$ is an eigenvector of C with corresponding eigenvalue $\lambda \in R$, then ***x*** is an eigenvector of C, and find the corresponding eigenvalue.

Since $\det(D) \neq 0$, D is an invertible matrix. Furthermore, the assumption can be written as

$$C(D\boldsymbol{x}) = \lambda(D\boldsymbol{x})$$
$$\Rightarrow CD\boldsymbol{x} = \lambda D\boldsymbol{x}.$$

Because $CD = DC$,

$$DC\boldsymbol{x} = \lambda D\boldsymbol{x}.$$

Since D is invertible, it can be multiplied on the left side to make

$$D^{-1}DC\boldsymbol{x} = D^{-1}\lambda D\boldsymbol{x}.$$

Then, since λ is a scalar, it can be rearranged to make

$$D^{-1}DC\boldsymbol{x} = \lambda D^{-1}D\boldsymbol{x}.$$

Since $D^{-1}D = I_n$,

$$C\boldsymbol{x} = \lambda\boldsymbol{x}.$$

Thus, ***x*** is an eigenvector of C with corresponding eigenvalue λ, as required.

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Great.