2C Intro to real analysis 2020/21

Ex Sheet 4

Feedback and solutions

Q1 *Let $f : \mathbb{R} \setminus \{\frac{3}{2}\} \to \mathbb{R}$ be given by $f(x) = \frac{x^2+x+2}{2x-3}$. Show, directly from the definition, that f is continuous at 2.*

This question is similar to a load of examples we've done in lectures, but it was one of those questions where you have to be careful with what value of r you use with $|x-2| < r$.

Let $\varepsilon > 0$ be arbitrary. We have

$$|f(x) - f(2)| = \left| \frac{x^2+x+2}{2x-3} - 8 \right| = \left| \frac{x^2 - 15x + 26}{2x-3} \right| = \frac{|x-2||x-13|}{|2x-3|}.$$

Now, we have

$$\begin{aligned} |x-2| < \frac{1}{4} &\implies -\frac{1}{4} < x - 2 < \frac{1}{4} \\ &\implies \frac{1}{2} < 2x - 3 < \frac{3}{2} \\ &\implies \frac{1}{2} < |2x-3| < \frac{3}{2} \\ &\implies \frac{1}{|2x-3|} < 2. \end{aligned}$$

Also,

$$\begin{aligned} |x-2| < \frac{1}{4} &\implies -\frac{1}{4} < x - 2 < \frac{1}{4} \\ &\implies -\frac{45}{4} < x - 13 < -\frac{43}{4} \\ &\implies |x-13| < \frac{45}{4}. \end{aligned}$$

Take $\delta = \min(\frac{1}{4}, \frac{2\varepsilon}{45})$. Then for $|x-2| < \delta$, we have

$$|f(x) - f(2)| = \frac{|x-13|}{|2x-3|}|x-2| \leq \frac{45}{2}|x-2| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, f is continuous at 2.

Why did I choose $|x-2| < \frac{1}{4}$? I could have started with

$$|x-2| < 1 \implies -1 < x-2 < 1 \implies -1 < 2x-3 < 3. \qquad (1)$$

The problem is that if $2x-3$ lies in the interval $[-1, 3]$, then we could have $2x - 3 = 0$, so that all we can deduce from (1) is that [1]

$$0 \leq |2x-3| < 3,$$

and this does not give an upper bound for $\frac{1}{|2x-3|}$. Therefore we need to consider $|x-2| < r$ with r small enough that the resulting interval

[1] If you find it hard to see this, I would encourage you to draw a picture of the modulus function $y = |x|$ and mark the region from $x = -1$ to $x = 3$ on your graph and identify the minimum and maximum values of the modulus on this interval from your picture.

$-2r+1 < 2x-3 < 2r+1$ does not contain 0, or have 0 as an end point. That is, we need $1-2r > 0$, so taking any r with $r < \frac{1}{2}$ will work.[2]

Q2 *Let $f : \mathbb{R} \to \mathbb{R}$ be given by*

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Show directly from the definition that f is continuous at 0.

Let $g : \mathbb{R} \to \mathbb{R}$ be given by

$$g(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Is g continuous at 0? Justify your answer with a proof.

In the first part of this question, the $\sin\left(\frac{1}{x}\right)$ is a complete red herring, as $|\sin(y)| \leq 1$ for all $y \in \mathbb{R}$. This leads to the following answer.

> Let $\varepsilon > 0$ and take $\delta = \varepsilon$. Given $x \in \mathbb{R}$ with $|x-0| < \delta$, either $x = 0$, when $|f(x) - f(0)| = 0 < \varepsilon$, or $x \neq 0$, when
>
> $$|f(x) - f(0)| = |x \sin\left(\frac{1}{x}\right)| \leq |x| < \varepsilon.$$
>
> Since $\varepsilon > 0$ was arbitrary, f is continuous at 0.

Note in the answer above, it was important to split into two cases to avoid a potential division by 0.[3]

In the second part of the question, the $\sin\left(\frac{1}{x}\right)$ is certainly not a red herring. The point is that as x gets close to 0, $\frac{1}{x}$ goes off to infinity[4], and so $\sin\left(\frac{1}{x}\right)$ will move between $+1$ and -1 infinitely many times near 0.[5] In the answer below, I use this fact to identify a sequence $(x_n)_{n=1}^{\infty}$ converging to 0, so that $g(x_n) = 1$ for all n.

> For $n \in \mathbb{N}$, let $x_n = \frac{1}{(2n+\frac{1}{2})\pi)}$, so that $x_n \to 0$ as $n \to \infty$. We have $g(x_n) = 1$ for all $n \in \mathbb{N}$, so that $g(x_n) \to 1 \neq 0 = g(0)$. Therefore g is not continuous at 0.

Q3 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function which satisfies $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.*

a) *By considering $x = y = 0$, find $f(0)$.*

b) *Show that $f(q) = qf(1)$ for $q \in \mathbb{Q}$.*[6]

c) *Suppose additionally that f is continuous. Show that $f(x) = xf(1)$ for all $x \in \mathbb{R}$.*[7]

[2] You can see that the two key points on the real line are $x = 2$ (the root of $x-2=0$) and $x = \frac{3}{2}$ (the root of $2x-3=0$). These points are distance $1/2$ apart; it is no coincidence that we need to take r smaller than the distance between these points.

[3] Did you take care of this point in your solution?

[4] we won't make this precise

[5] Try drawing a graph, or getting a computer to sketch one for you.

[6] I'd suggest first establishing why $f(n) = nf(1)$ for $n \in \mathbb{N}$, then seeing why $f(\frac{n}{m}) = \frac{n}{m}f(1)$ for $n, m \in \mathbb{N}$, and finally dealing with the negative rational numbers.

[7] In this question you may assume that for any real number x, there is a sequence $(q_n)_{n=1}^{\infty}$ of rational numbers with $q_n \to x$ without proof. You can find the ideas used to prove this claim in exercise sheet 9.

(a) We have $0 = 0 + 0$, so that $f(0) = f(0) + f(0)$ which implies that $f(0) = 0$.

(b) We claim that $f(n) = nf(1)$ for all $n \in \mathbb{N}$. Certainly this holds when $n = 1$, so suppose inductively that $f(n) = nf(1)$ for some $n \in \mathbb{N}$. Then $f(n+1) = f(n) + f(1) = nf(1) + f(1) = (n+1)f(1)$. This proofs the claim by induction.

Now let $m, n \in \mathbb{N}$. As $n = m \times \frac{n}{m}$, we have $f(n) = m \times f(\frac{n}{m})$ (arguing as in the previous paragraph). Hence $f(\frac{n}{m}) = \frac{1}{m}f(n) = \frac{n}{m}f(1)$.

For $x \in \mathbb{R}$, note that $x + (-x) = 0$. Therefore $f(x) + f(-x) = f(0) = 0$, and hence $f(-x) = -f(x)$. Therefore $f(q) = qf(1)$ for all rational numbers q.

(c) Given $x \in \mathbb{R}$, let $(q_n)_{n=1}^{\infty}$ be a sequence of rational numbers with $q_n \to x$ as $n \to \infty$. By the sequential characterisation of continuity, $f(q_n) \to f(x)$. By part (b), $f(q_n) = q_n f(1)$, and hence $f(q_n) \to xf(1)$. By uniqueness of limits, $f(x) = xf(1)$.