2A: Tutorial 9

School of Mathematics and Statistics

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Semester 1 2020–21

INSTRUCTIONS

Access your tutorial lecture

TU01,TU02,TU03,TU04,TU05: November 23rd, 09:00-10:00

TU06,TU07,TU08,TU09,TU10: November 23rd, 15:00-16:00

using the Zoom plug-in in the Moodle page of the course.

Please mute yourself when you are in the main room of the meeting.

Be ready to work in groups. Having a method of writing on screen, as mouse, tablet or similar is useful. Screen sharing is allowed!

Multiple Choice Question 1

Unseen question

Which of the following is **not** a parametric description of the curve consisting of the semi-circle in $y \geq 0$ from $(1, 0)$ to $(-1, 0)$ that is part of the circle centred at the origin, radius 1.

(A) $(\cos 2t, \sin 2t)$, $t \in [0, \pi/2]$

(B) $(-t, \sqrt{1 - t^2})$, $t \in [-1, 1]$

(C) $(\cos \pi t^2, \sin \pi t^2)$, $t \in [0, 1]$

(D) $(1 + t, 1 - t)/\sqrt{2(1 + t^2)}$, $t \in (-\infty, 1]$

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ANSWER: **(D)** In all cases the points lie on the circle centre $(0, 0)$, radius 1. We can check this by confirming that $x^2 + y^2 = 1$ is satisfied.

$$\cos^2 2t + \sin^2 2t = 1, \quad (-t)^2 + \left(\sqrt{1-t^2}\right) = t^2 + 1 - t^2 = 1$$

and

$$\cos^2 \pi t^2 + \sin^2 \pi t^2 = 1, \quad \frac{(1+t)^2}{2(1+t^2)} + \frac{(1-t)^2}{2(1+t^2)} = 1.$$

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ANSWER: **(D)**

The only question is whether the range of t given gives the curve we're interested in and that the direction is correct. In **(A)** the start and end are $(1,0)$ and $(-1,0)$, and the same is true for **(B)** and **(C)**. For **(D)** we have for $t \to -\infty$ the point tends to $(-1,1)/\sqrt{2}$ and at $t = 1$ then point is $(1,0)$ so the part of the circle is clockwise between these points — not a semi-circle and the wrong direction.

Multiple Choice Question 2

Ex Sheet 8, T3 (related)

Which one of the following vector fields is irrotational?

(A) $(z, y, -x)$

(B) $(y^2 z, 2xyz^2, xy^2)$

(C) (yz, xz, xy)

(D) $(\sin xy, \cos yz, \sin xz)$

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ANSWER: **(C)** A vector field is irrotational if its curl is zero. We can compute the curl in each case:

(A)

$$(0, 2, 0)\,;$$

(B)

$$\left(2xy - 4xyz, 0, 2yz^2 - 2yz\right)\,;$$

(C)

$$(0, 0, 0)\,;$$

(D)

$$\left(y \sin yz, -z \cos xz, -x \cos xy\right)\,.$$

Tutorial questions

Ex Sheet 9, T5

Evaluate

$$\int_C xy^2\,dx + x^4y\,dy$$

where C is part of the parabola $y = 2x^2$ from $(0,0)$ to $(1,2)$.

Ex Sheet 9, T11

Use Green's Theorem to calculate

$$\int_C 2xy^3\,dx + 3x^2\,dy$$

where C is the square with vertices at $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ traversed anticlockwise.

Ex Sheet 9, T5

Evaluate

$$\int_C xy^2\,dx + x^4y\,dy$$

where C is part of the parabola $y = 2x^2$ from $(0,0)$ to $(1,2)$.

A parametric description of C is $\mathbf{r}(t) = (t, 2t^2)$ for $t \in [0,1]$, so

$$\begin{aligned}\int_C xy^2\,dx + x^4y\,dy &= \int_0^1 \left[t\left(2t^2\right)^2 + t^4\left(2t^2\right)\cdot 4t\right]\,dt \\ &= \int_0^1 4t^5 + 8t^7\,dt = \left[\frac{2}{3}t^6 + t^8\right]_0^1 = \frac{5}{3}\,.\end{aligned}$$

Tutorial questions

Ex Sheet 9, T11

Use Green's Theorem to calculate

$$\int_C 2xy^3\,dx + 3x^2\,dy$$

where C is the square with vertices at $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ traversed anticlockwise.

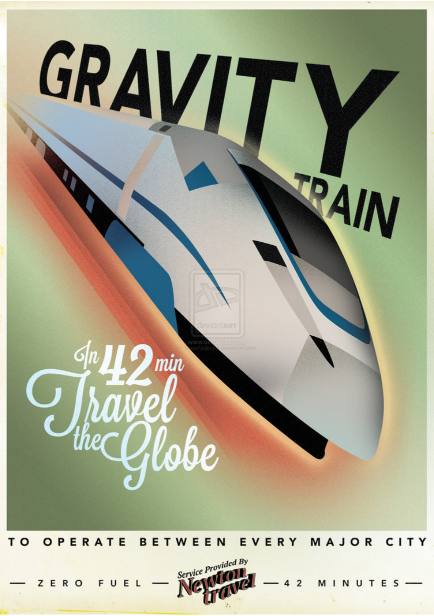
Green's Theorem gives

$$\int_C 2xy^3\,dx + 3x^2\,dy = \iint_D 6x(1-y^2)\,dxdy$$

As the region is rectangular and the integrand is separable the integral is the product of two one-dimensional integrals

$$\left(\int_0^1 6x\,dx\right)\left(\int_0^1 1-y^2\,dy\right) = \left[3x^2\right]_0^1\left[y-\frac{1}{3}y^3\right]_0^1 = 2.$$

Bonus picture



Bonus question

Gravity train

The travel time for travel through a tunnel C through the Earth starting from rest, assuming spherical symmetry, uniform density, Newtonian gravity and conservation of energy (no energy loss to friction in the tunnel) is

$$T[C] = \int_C \frac{1}{\sqrt{1-r^2}}\, ds$$

where units are chosen so that the radius of the Earth is 1 and a unit time is $\sqrt{R/g_0} \approx 13\text{min}\,28\text{s}$ (g_0 is the surface acceleration due to gravity). Show that any straight line connecting two points on the surface has $T[C] = \pi$.

Bonus question

Without loss of generality place one end of the tunnel at $(-\sin\theta, \cos\theta)$ and the other end at $(\sin\theta, \cos\theta)$ ($0 \le \theta \le \pi/2$, so that the angular separation is 2θ) and a parametric description of the line is

$$\mathbf{r} = (t\sin\theta, \cos\theta), \qquad \dot{\mathbf{r}} = (\sin\theta, 0), \qquad |\dot{\mathbf{r}}| = \sin\theta$$

with $t \in [-1, 1]$. Also

$$1 - r^2 = 1 - |\mathbf{r}|^2 = \sin^2\theta\left(1 - t^2\right)$$

so

$$T[\text{line}] = \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\,dt = \left[\sin^{-1} t\right]_{-1}^{1} = \pi$$

independent of angular separation. This means that the travel time for straight line tunnels on a spherical planet of uniform density ρ is

$$T = \sqrt{\frac{3\pi}{4G\rho}}.$$

(For the Earth, $T \approx 42,2$ minutes. Depending on where you go, it may be much faster than traveling with a plane!)