Mathematics 2A - December 2020 exam

$\boxed{1}$ $T(x,y,z) = f(v(x,y,z), w(x,y,z))$, where $f = \sqrt{v^2+w^2}$, $v = y + x\cos z$, $w = x + y\sin z$

Computation of ∇T:

$$\frac{\partial T}{\partial x} = \frac{\partial v}{\partial x}\frac{\partial f}{\partial v} + \frac{\partial w}{\partial x}\frac{\partial f}{\partial w} = \frac{\cos z\, v + w}{\sqrt{v^2+w^2}}$$

$$\frac{\partial T}{\partial y} = \frac{\partial v}{\partial y}\frac{\partial f}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial f}{\partial w} = \frac{v + \sin z\, w}{\sqrt{v^2+w^2}}$$

$$\frac{\partial T}{\partial z} = \frac{\partial v}{\partial z}\frac{\partial f}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial f}{\partial w} = \frac{-x\sin z\, v + y\cos z\, w}{\sqrt{v^2+w^2}}$$

Using $v(P) = 3$, $w(P) = 4$ and $f(v(P), w(P)) = 5$ we get

$$\nabla T(P) = \left(\frac{4}{5}, \frac{7}{5}, -\frac{3}{5}\right) \text{ and } |\nabla T(P)| = \frac{1}{5}\sqrt{16+49+9} = \frac{\sqrt{74}}{5}.$$

Fastest rate of change is in the direction of the unit vector

$$\underline{u} = \frac{5}{\sqrt{74}}\nabla T(P) = \frac{1}{\sqrt{74}}(4, 7, -3).$$

$\boxed{2}$ We have $\frac{\partial v}{\partial x} = 2x e^{\tan y}$, $\frac{\partial v}{\partial y} = \frac{x^2}{\cos^2 y} e^{\tan y}$, $\frac{\partial w}{\partial x} = 0$, $\frac{\partial w}{\partial y} = 1$

Let $f(x,y) = F(v,w)$. Then

$$\frac{\partial f}{\partial x} = \frac{\partial v}{\partial x}\frac{\partial F}{\partial v} + \frac{\partial w}{\partial x}\frac{\partial F}{\partial w} = 2x e^{\tan y}\frac{\partial F}{\partial v}$$

$$\frac{\partial f}{\partial y} = \frac{\partial v}{\partial y}\frac{\partial F}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial F}{\partial w} = \frac{x^2}{\cos^2 y} e^{\tan y}\frac{\partial F}{\partial v} + \frac{\partial F}{\partial w}$$

Hence, the PDE becomes

$$\cancel{2x^2 e^{\tan y}\frac{\partial F}{\partial v}} - 2\cos^2 y\left(\cancel{\frac{x^2}{\cos^2 y}e^{\tan y}}\frac{\partial F}{\partial v} + \frac{\partial F}{\partial w}\right) = 1$$

$\Longrightarrow \dfrac{\partial F}{\partial w} = -\dfrac{1}{2\cos^2 w}$ whose solution is $F(v,w) = -\frac{1}{2}\tan w + A(v)$.

Thus $f(x,y) = F\big(v(x,y), w(x,y)\big) = -\frac{1}{2}\tan y + A\left(x^2 e^{\tan y}\right)$.

$\boxed{3}$ i) $\underline{F} \times \underline{G} = \begin{vmatrix} i & j & k \\ F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \end{vmatrix} = \left(F_2G_3 - F_3G_2,\ F_3G_1 - F_1G_3,\ F_1G_2 - F_2G_1\right)$

Hence, $\operatorname{div}(\underline{F}\times\underline{G}) = \dfrac{\partial F_2}{\partial x}G_3 + F_2\dfrac{\partial G_3}{\partial x} - \dfrac{\partial F_3}{\partial x}G_2 - F_3\dfrac{\partial G_2}{\partial x}$

$+ \dfrac{\partial F_3}{\partial y}G_1 + F_3\dfrac{\partial G_1}{\partial y} - \dfrac{\partial F_1}{\partial y}G_3 - F_1\dfrac{\partial G_3}{\partial y} + \dfrac{\partial F_1}{\partial z}G_2 + F_1\dfrac{\partial G_2}{\partial z} - \dfrac{\partial F_2}{\partial z}G_1 - F_2\dfrac{\partial G_1}{\partial z}$

$= F_1\left(\dfrac{\partial G_2}{\partial z} - \dfrac{\partial G_3}{\partial y}\right) + F_2\left(\dfrac{\partial G_3}{\partial x} - \dfrac{\partial G_1}{\partial z}\right) + F_3\left(\dfrac{\partial G_1}{\partial y} - \dfrac{\partial G_2}{\partial x}\right)$

$- \left(\dfrac{\partial F_2}{\partial z} - \dfrac{\partial F_3}{\partial y}\right)G_1 - \left(\dfrac{\partial F_3}{\partial x} - \dfrac{\partial F_1}{\partial z}\right)G_2 - \left(\dfrac{\partial F_1}{\partial y} - \dfrac{\partial F_2}{\partial x}\right)G_3$

$= -\underline{F}\cdot\operatorname{curl}(\underline{G}) + \operatorname{curl}(\underline{F})\cdot\underline{G}$.

ii) if $\underline{F}$, $\underline{G}$ conservative, then $\underline{F} = \nabla f$, $\underline{G} = \nabla g$ for some functions f, g.

Using previous formula we get

$$\operatorname{div}(\underline{F}\times\underline{G}) = \underline{F}\cdot\underbrace{\cancel{\operatorname{curl}(\nabla g)}}_{=0} - \underbrace{\cancel{\operatorname{curl}(\nabla f)}}_{=0}\cdot\underline{G} = 0$$

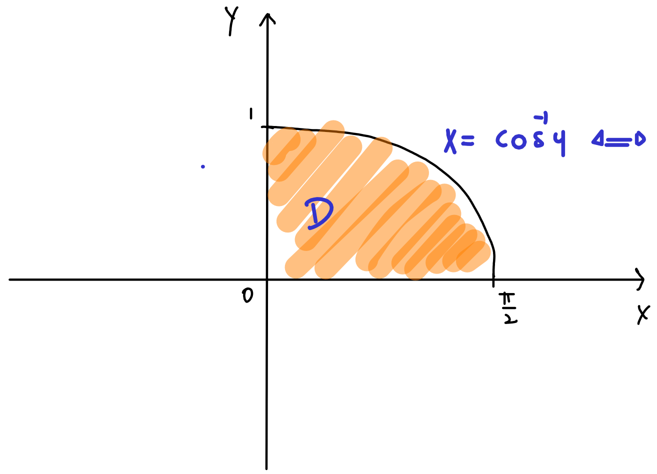
Nabla identities: $\operatorname{curl}(\nabla\phi) = 0$, ϕ function

Since $\operatorname{div}(\underline{F}\times\underline{G}) = 0$, then $\underline{F}\times\underline{G}$ is incompressible.

$\boxed{4}$ The region of integration is $D = \{ 0 \le x \le \cos^{-1} y,\ 0 \le y \le 1 \}$.

(type II description)

A sketch of the region



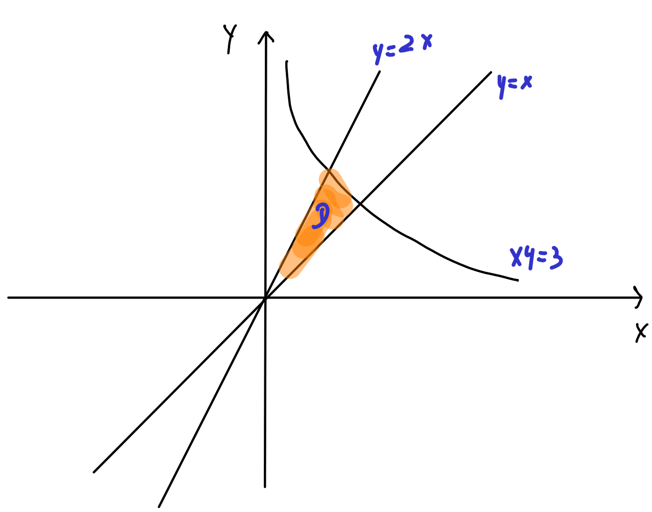
As a type I region $D = \{ 0 \le x \le \frac{\pi}{2},\ 0 \le y \le \cos x \}$.

Inverting the order of integration we get

$$\int_0^1 dy \int_0^{\cos^{-1} y} \frac{1}{\cos x}\, dx = \int_0^{\pi/2} dx \int_0^{\cos x} \frac{1}{\cos x}\, dy$$

$$= \int_0^{\pi/2} \left[\frac{y}{\cos x} \right]_{y=0}^{y=\cos x} dx = \int_0^{\pi/2} dx = \frac{\pi}{2}.$$

$\boxed{5}$ The region of integration is



The sketch suggests to set

$$\begin{cases} v = xy \\ w = \dfrac{y}{x} \end{cases}$$

In these new coordinates, the region of integration is rectangular
$D' = \{0 \le v \le 3,\ 1 \le w \le 2\} = [0,3] \times [1,2]$

Note that

$$\det \begin{pmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{pmatrix} = \det \begin{pmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{pmatrix} = \frac{2y}{x} = 2w$$

Hence, $|J| = \left| \frac{1}{2w} \right| = \frac{1}{2w}$ (since $w > 0$).

Changing variables in the integral we get

$$\iint_D \underbrace{(x^2+y^2)}_{= \frac{v}{w} + vw} \, dx\, dy = \iint_{[0,3]\times[1,2]} \left(\frac{v}{w} + vw\right) \frac{1}{2w} \, dv\, dw$$

$$= \frac{1}{2}\left(\int_0^3 v\, dv\right)\left(\int_1^2 \left(\frac{1}{w^2} + 1\right) dw\right) = \frac{1}{2}\left[\frac{v^2}{2}\right]_0^3 \left[-\frac{1}{w} + w\right]_1^2 = \frac{1}{2} \cdot \frac{9}{2} \cdot \frac{3}{2} = \frac{27}{8}.$$

$\boxed{6}$ write $\underline{F} = (P, Q) = (\pi y \sin(\pi x y), \pi x \sin(\pi x y))$

We have that $\frac{\partial P}{\partial y} = \pi \sin(\pi x y) + \pi^2 x y \cos(\pi x y) = \frac{\partial Q}{\partial x}$

thus $\underline{F}$ is conservative.

We find $f = f(x,y)$ such that $\underline{F} = \nabla f$ by solving

$$\begin{cases} \frac{\partial f}{\partial x} = \pi y \sin(\pi x y) & (a) \\ \frac{\partial f}{\partial y} = \pi x \sin(\pi x y) & (b) \end{cases}$$

Solving (a) we get $f(x,y) = -\cos(\pi x y) + A(y)$.

Plugging this partial solution in (b) we get the following equation for $A(y)$:

$$\pi x \cancel{\sin(\pi x y)} + A'(y) = \frac{\partial f}{\partial y} = \pi x \cancel{\sin(\pi x y)} \Rightarrow A'(y) = 0$$

$$\Rightarrow A(y) = c, \quad c \text{ constant}$$

choose any value you like

Hence $\underline{F} = \nabla f$, where $f = -\cos(\pi x y) + \alpha$

The work is given by

$$\int_C \underline{F} \cdot d\underline{r} = \int_C \nabla f \cdot d\underline{r} = f(B) - f(A) =$$

use Theorem from Lectures

$$= -\cos(-5\pi) + \cos(10\pi) = -(-1) + 1 = 2.$$

$\boxed{7}$ Since $0 \leq z \leq 2$, then S is the graph of the function

$z = \sqrt{x^2+y^2}$ for $(x,y) \in D = \{x^2+y^2 \leq 4, \ y \geq 0\}$ (projection of S on the xy plane).

A parameterisation is $\underline{r}(x,y) = (x, y, \sqrt{x^2+y^2})$, $(x,y) \in D$, then

$$\left| \frac{\partial \underline{r}}{\partial x} \times \frac{\partial \underline{r}}{\partial y} \right| = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}} = \sqrt{2}.$$

Hence,

$$\iint_S e^{x^2+y^2+z^2} \, dS = \sqrt{2} \iint_D e^{2(x^2+y^2)} \, dx \, dy = \sqrt{2} \int_0^{\pi} d\theta \int_0^2 r e^{2r^2} \, dr$$

in polar coordinates

D is $[0,2] \times [0,\pi]$

$$= \sqrt{2} \, \pi \cdot \left[\frac{e^{2r^2}}{4} \right]_0^2 = \frac{\sqrt{2}}{4} \pi \left(e^8 - 1\right).$$

$\boxed{8}$ Write $(2020 + xy^2\cos x^2 - y)\,dx + (y\sin x^2 + e^{y^2 - \tan y})\,dy$

$$= P\,dx + Q\,dy.$$

Then, $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 2xy\cos x^2 - 2xy\cos x^2 + 1 = 1$

Using Green's Theorem we get:

$$\oint_C P\,dx + Q\,dy = \iint_{\text{Isle of Arran}} 1\,dx\,dy = \text{Area}(\text{Isle of Arran}) = 432.$$

$\boxed{9}$ Let $\underline{n}$ be the outward normal vector field to S.

By the divergence theorem

$$\iint_S \underline{F}\cdot\underline{n}\,dS = \iiint_V \operatorname{div}(\underline{F})\,dx\,dy\,dz \underset{\operatorname{div}(\underline{F}) = y}{=} \iiint_V y\,dx\,dy\,dz$$

In the new variables we have $V' = \{0 \le \theta \le 2\pi,\ 1 \le u \le e,\ 0 \le r \le \frac{1}{\sqrt{u}}\}$

The Jacobian of the transformation is the same as for polar coordinates: $|J| = r$.

Hence, $$\iiint_V y\,dx\,dy\,dz = \int_0^{2\pi} d\theta \int_1^e du \int_0^{\frac{1}{\sqrt{u}}} u\,r\,dr$$

$$= \cancel{2}\pi \int_1^e \left[u\,\frac{r^2}{\cancel{2}}\right]_{r=0}^{r=\frac{1}{\sqrt{u}}} du = \pi \int_1^e du = \pi(e-1)$$