MATHS 2B - FEEDBACK EXERCISE 1 (2020-21) SOLUTIONS

Q1) **METHOD 1**

VECTORS $\underline{u} = (1,-1,5)$, $\underline{v} = (2,3,-1)$ AND $\underline{w} = (-3,-7,7)$ ARE LINEARLY INDEPENDENT IF THE ONLY SOLUTION TO THE EQUATION

$$c_1\underline{u} + c_2\underline{v} + c_3\underline{w} = \underline{0}$$

FOR SCALARS $c_1, c_2, c_3 \in \mathbb{R}$ IS THE TRIVIAL SOLUTION $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.

WE HAVE:

$$c_1(1,-1,5) + c_2(2,3,-1) + c_3(-3,-7,7) = (0,0,0).$$

WE OBTAIN THE FOLLOWING SYSTEM OF EQUATIONS FOR c_1, c_2, c_3:

$$\begin{aligned} c_1 + 2c_2 - 3c_3 &= 0, \\ -c_1 + 3c_2 - 7c_3 &= 0, \\ 5c_1 - c_2 + 7c_3 &= 0. \end{aligned}$$

THESE CAN BE WRITTEN IN THE FORM OF AN AUGMENTED MATRIX AS:

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ -1 & 3 & -7 & 0 \\ 5 & -1 & 7 & 0 \end{array}\right)$$

ROW OPERATIONS GIVE:

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ -1 & 3 & -7 & 0 \\ 5 & -1 & 7 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

WE HENCE DEDUCE THAT $c_1 = -c_3$, $c_2 = 2c_3$. SETTING $c_3 = t \in \mathbb{R}$ THEN $c_1 = -t$, $c_2 = 2t$.

IT FOLLOWS THAT $\forall t \in \mathbb{R}$,

$$-t\underline{u} + 2t\underline{v} + t\underline{w} = \underline{0}.$$

IT FOLLOWS THAT AS THERE EXISTS A NON-TRIVIAL SOLUTION (IN FACT THERE ARE INFINITELY MANY) TO THE PROBLEM $c_1\underline{u} + c_2\underline{v} + c_3\underline{w} = \underline{0}$ THEN THE VECTORS $\underline{u}, \underline{v}$ AND $\underline{w}$ ARE <u>NOT</u> LINEARLY INDEPENDENT - THEY ARE LINEARLY DEPENDENT.

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METHOD 2

6 0

ALTERNATIVELY, NOTICE THAT $\underline{w} = \underline{u} - 2\underline{v}$ (I.E. $\underline{w}$ CAN BE WRITTEN AS A LINEAR COMBINATION OF $\underline{u}$ AND $\underline{v}$) AND INVOKE THEOREM 2.5.

Q2) IF $B \in \mathrm{Span}(A_1, A_2)$ THEN THERE EXISTS SCALARS $c_1, c_2 \in \mathbb{R}$ SUCH THAT

$$c_1A_1 + c_2A_2 = B.$$

WE WISH TO FIND c_1, c_2 SUCH THAT

$$c_1\begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix} + c_2\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -3 & 8 \\ 4 & -7 \end{pmatrix}.$$

IT FOLLOWS THAT WE MUST SOLVE THE EQUATIONS

$$\begin{aligned} c_1 + 3c_2 &= -3, \\ 4c_1 + 2c_2 &= 8, \\ 2c_1 + c_2 &= 4, \\ -c_1 + 2c_2 &= -7. \end{aligned}$$

THIS CAN BE WRITTEN IN THE FORM OF AN AUGMENTED MATRIX AS

$$\left(\begin{array}{cc|c} 1 & 3 & -3 \\ 4 & 2 & 8 \\ 2 & 1 & 4 \\ -1 & 2 & -7 \end{array}\right).$$

WHICH REDUCES UNDER ROW OPERATIONS TO

$$\left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

GIVING $c_1 = 3$, $c_2 = -2$. IT FOLLOWS THAT $B = 3A_1 - 2A_2$ AND SO $B \in \mathrm{Span}\{A_1, A_2\}$.

OR RECOGNISE IMMEDIATELY THAT $B = 3A_1 - 2A_2$ (WITHOUT SOLVING EQUATIONS FOR c_1, c_2) TO JUSTIFY.

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Q3) WE HAVE

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B=\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

WE THEN HAVE

$$A^{-1}=\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad B^{-1}=\frac{1}{eh-fg}\begin{pmatrix} h & -f \\ -g & e \end{pmatrix}.$$

IT FOLLOWS THAT

$$(A^{-1}+\lambda B^{-1})^T=\left[\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}+\frac{\lambda}{eh-fg}\begin{pmatrix} h & -f \\ -g & e \end{pmatrix}\right]^T$$
$$=\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}^T+\frac{\lambda}{eh-fg}\begin{pmatrix} h & -f \\ -g & e \end{pmatrix}^T$$
$$=\frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}+\frac{\lambda}{eh-fg}\begin{pmatrix} h & -g \\ -f & e \end{pmatrix}.$$

WE NOTE THAT

$$A^T=\begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad B^T=\begin{pmatrix} e & g \\ f & h \end{pmatrix}$$

AND

$$(A^T)^{-1}=\frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}, \quad (B^T)^{-1}=\frac{1}{eh-fg}\begin{pmatrix} h & -g \\ -f & e \end{pmatrix}$$

SO

$$(A^T)^{-1}+\lambda(B^T)^{-1}=\frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}+\frac{\lambda}{eh-fg}\begin{pmatrix} h & -g \\ -f & e \end{pmatrix}$$
$$=(A^{-1}+\lambda B^{-1})^T.$$

WE CONCLUDE THAT $(A^{-1}+\lambda B^{-1})^T=(A^T)^{-1}+\lambda(B^T)^{-1}$. IF A AND B ARE SYMMETRIC THEN $A=A^T$, $B=B^T$ SO

$$(A^{-1}+\lambda B^{-1})^T=(A^T)^{-1}+\lambda(B^T)^{-1}$$
$$=A^{-1}+\lambda B^{-1}$$

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ALTERNATIVE SOL$^{\underline{N}}$ TO (C)

$$(A^{-1}+\lambda B^{-1})^T=(A^{-1})^T+(\lambda B^{-1})^T$$ by Thm 3.4 (b) which states $(A+B)^T=A^T+B^T$.

$$=(A^{-1})^T+\lambda(B^{-1})^T$$ by Thm 3.4 (c) which states $(kA)^T=kA^T$, $k\in\mathbb{R}$.

$$=(A^T)^{-1}+\lambda(B^T)^{-1}$$ By Thm 3.9 which states $(A^T)^{-1}=(A^{-1})^T$.

$\square$.