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Q1: Let $\varepsilon > 0$. For $x \in \mathbb{R} \setminus \{\frac{3}{2}\}$ we have

$$|f(x) - f(2)| = \left|\frac{x^2+x+2}{2x-3} - 8\right| = \left|\frac{x^2-15x+26}{2x-3}\right| = \frac{|x-13||x-2|}{|2x-3|}.$$

Now $|x-2| < 1$ implies $-1 < x-2 < 1$; hence, $-12 < x-13 < -10$ and $-1 < 2x-3 < 3$. Therefore, we obtain

$$\frac{|x-13||x-2|}{|2x-3|} < 12|x-2|,$$

provided $|x-2| < 1$. Taking $\delta = \min(1, \frac{\varepsilon}{12})$, we get that $|x-2| < \delta$ implies $|f(x) - f(2)| < \varepsilon$. Thus, f is continuous at 2, as required.

Q2: Let $\varepsilon > 0$. For $x \in \mathbb{R} \setminus \{0\}$ we have

$$|f(x) - f(0)| = |x \sin(\tfrac{1}{x}) - 0| = |x||\sin \tfrac{1}{x}|.$$

On the other hand, for $x = 0$, we have $|f(0) - f(0)| = 0$, which definitely is less than $\delta > 0$. Furthermore, since $|\sin \frac{1}{x}| \leq 1$,

$$|x||\sin \tfrac{1}{x}| \leq |x|.$$

Taking $\delta = \varepsilon$, we get that $|x| < \delta$ implies $|f(x) - f(0)| < \varepsilon$; thus, f is continuous at 0.

For $g(x)$, let $\varepsilon = \frac{1}{2}$ and $x = \frac{1}{\frac{\pi}{2} + 2k\pi}$ for $k \in \mathbb{N}$. We have

$$|x - 0| = \left|\frac{1}{\frac{\pi}{2} + 2k\pi}\right| < \delta$$

when $k > \frac{2 - \pi\delta}{4\pi\delta}$. In this case,

$$|g(x) - g(0)| = |\sin(\tfrac{\pi}{2} + 2k\pi)| = 1 > \tfrac{1}{2} = \varepsilon.$$

Thus, g is not continuous at 0, as required.

Q_3: a) Since $0=0+0$,

$$f(0+0)=f(0)+f(0)$$
$$\Rightarrow f(0)=f(0)+f(0)$$
$$\Rightarrow f(0)=0$$

b) Let us first prove that $f(n)=nf(1)$ for $n\in\mathbb{N}$ by induction. In the base case $n=0$, $f(0)=0\cdot f(1)=0$. Then assume that $f(n)=nf(1)$ is true. Then,

$$f(n)=n\,f(1)$$
$$\Leftrightarrow f(1)+f(n)=n\,f(1)+f(1)$$
$$\Leftrightarrow f(n+1)=(n+1)f(1).$$

Thus, $f(n+1)$ holds true. Then, by mathematical induction, $f(n)=nf(1)$.
Furthermore, by expressing a rational number as $\frac{n}{m}$ for $n, m\in\mathbb{N}$,

$$f(\tfrac{n}{m})=f(\sum_{i=0}^{n}\tfrac{1}{m})=n\cdot f(\tfrac{1}{m})$$

by using the given satisfactory condition and the previous result. Then $f(\frac{1}{m})$ can be found by expressing $f(1)$:

$$f(1)=f(\tfrac{m}{m})=m\cdot f(\tfrac{1}{m})$$
$$\Leftrightarrow f(\tfrac{1}{m})=\tfrac{1}{m}f(1).$$

Therefore, $f(\frac{n}{m})=\frac{n}{m}f(1)$. To ~~qu~~ check for negative values, $f((-1)\cdot\frac{n}{m})=\frac{n}{m}f(-1)$, where $f(-1)$ can be found from the condition

$$f(+1-1)=f(0)=f(1)+f(-1)$$
$$\Leftrightarrow f(-1)=f(0)-f(1)=-f(1).$$

Thus, $f(-\frac{n}{m})=-\frac{n}{m}f(1)$. Therefore, $f(q)=qf(1)$ for all $q\in\mathbb{Q}$.

c) We may assume that $\forall x\in\mathbb{R}, \exists (q_n)_{n=1}^{\infty}$ with $q_n\to x$ as $n\to\infty$. From part b), $f(q_n)=q_n f(1)$ for $q\in\mathbb{Q}$. Since $q_n\in dom(f)$ and $x\in dom(f)$ and f is continuous, $q_n\to x\Rightarrow f(q_n)\to f(x)$. Thus, $f(x)=xf(1)$ for all $x\in\mathbb{R}$, as required.
by the sequential characterisation of continuity