2A Multivariable Calculus 2020

Ex Sheet 8

Tutorial Exercises

T1 Let $\mathbf{r} = (x, y, z)$, $r = |\mathbf{r}|$ and $\mathbf{a}$ be a constant vector. Prove the following results

(a) $\text{div }\mathbf{r} = 3$, (b) $\text{div }(\mathbf{a} \times \mathbf{r}) = 0$, (c) $\text{div }(r^n\mathbf{a}) = nr^{n-2}(\mathbf{a} \cdot \mathbf{r})$.

Solution

$$\text{(a) div }\mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

(b) $\mathbf{a} \times \mathbf{r} = (c_2z - c_3y, c_3x - c_1z, c_1y - c_2x)$, so $\text{div }(\mathbf{a} \times \mathbf{r}) = \dfrac{\partial c_2z - c_3y}{\partial x} + \dfrac{\partial c_3x - c_1z}{\partial y} + \dfrac{\partial c_1y - c_2x}{\partial z} = 0.$

$$\text{(c) div }(r^n\mathbf{a}) = r^n \text{ div }\mathbf{a} + \text{grad}(r^n) \cdot \mathbf{a} = 0 + nr^{n-2}(\mathbf{a} \cdot \mathbf{r})$$

(using the result from Ex 3.2 and Ex 3.10 from the notes).

T2 Find the divergence and curl of the vector fields

(a) $\mathbf{F} = (3xyz^2, 2xy^3, -x^2yz)$, (b) $\mathbf{G} = (e^{xz}, x^2 + y^2, yz)$,

at an arbitrary point and at $P(1, 1, 1)$.

Solution

(a) $\text{div }\mathbf{F} = \nabla \cdot \mathbf{F} = 3yz^2 + 6xy^2 - x^2y = 8$ at $(1, 1, 1)$ and

$$\text{curl }\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3xyz^2 & 2xy^3 & -x^2yz \end{vmatrix} = (-x^2z - 0)\mathbf{i} + (6xyz + 2xyz)\mathbf{j} + (2y^3 - 3xz^2)\mathbf{k}$$

$$= (-x^2z, 8xyz, 2y^3 - 3xz^2) = (-1, 8, -1) \text{ at } (1, 1, 1).$$

(b) $\text{div }\mathbf{G} = \nabla \cdot \mathbf{G} = ze^{xz} + 2y + y = ze^{xz} + 3y = e + 3$ at $(1, 1, 1)$ and

$$\text{curl }\mathbf{G} = \nabla \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{xz} & x^2 + y^2 & yz \end{vmatrix} = (z - 0)\mathbf{i} + (xe^{xz} - 0)\mathbf{j} + (2x - 0)\mathbf{k}$$

$$= (z, xe^{xz}, 2x) = (1, e, 2) \text{ at } (1, 1, 1).$$

T3 Which of the following vector fields are irrotational?

a) $\mathbf{F} = (yz, xz, xy)$,

b) $\mathbf{G} = \sin xy\,\mathbf{i} + \cos yz\,\mathbf{j} + \sin xz\,\mathbf{k}$,

c) $\mathbf{H} = y^2z\,\mathbf{i} + 2xyz\,\mathbf{j} + xy^2\,\mathbf{k}$.

Solution

(a) $\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = (x-x)\mathbf{i} + (y-y)\mathbf{j} + (z-z)\mathbf{k} = \mathbf{0}$. Therefore $\mathbf{F}$ is irrotational.

(b) $\operatorname{curl}\mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin xy & \cos yz & \sin xz \end{vmatrix} = (0 + y\sin(yz))\mathbf{i} - (z\cos(xz) - 0)\mathbf{j} + (0 - x\cos(xy))\mathbf{k} \neq \mathbf{0}$.
Therefore $\mathbf{G}$ is not irrotational.

(c) $\operatorname{curl}\mathbf{H} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z & 2xyz & xy^2 \end{vmatrix} = (2xy - 2xy)\mathbf{i} + (y^2 - y^2)\mathbf{j} + (2yz - 2yz)\mathbf{k} = \mathbf{0}$. Therefore $\mathbf{H}$ is irrotational.

T4 Calculate the curl of the vector field $\mathbf{F}$ and state whether the vector field is irrotational.

a) $\mathbf{F} = xz\mathbf{i} - y^3\mathbf{j} + xyz\mathbf{k}$,

b) $\mathbf{F} = \cos^2(x)\mathbf{i} - \sin(y)\mathbf{j} + z^4\mathbf{k}$,

c) $\mathbf{F} = \ln(x+z)\mathbf{i} - e^{y^2}\mathbf{j} + xy\mathbf{k})$.

Solution

(a) $\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & -y^3 & xyz \end{vmatrix} = (xz - 0)\mathbf{i} + (x - yz)\mathbf{j} + (0-0)\mathbf{k} = (xz, -yz, 0)$. Therefore $\mathbf{F}$ is NOT irrotational.

(b) $\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos^2 x & -\sin y & z^4 \end{vmatrix} = (0-0)\mathbf{i} + (0-0)\mathbf{j} + (0-0)$=$\mathbf{0}$. Therefore $\mathbf{F}$ is irrotational.

(c) $\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \ln(x+z) & -e^{yz} & xy \end{vmatrix} = (x + ye^{yz})\mathbf{i} + (\frac{1}{x+z} - y)\mathbf{j} + (0-0)\mathbf{k} = (x+ye^{yz}, \frac{1-xy-zy}{x+z}, 0).$
Therefore $\mathbf{F}$ is NOT irrotational.

T5 Let $\mathbf{F} = (x^2y, yz, x+z)$. Find

(i) $\operatorname{curl}\operatorname{curl} \mathbf{F}$, (ii) $\operatorname{grad}\operatorname{div} \mathbf{F}$.

Solution

We have

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & yz & x+z \end{vmatrix} = (0-y, 0-1, 0-x^2) = -(y, 1, x^2),$$

and

$$\operatorname{div} \mathbf{F} = \frac{\partial(x^2y)}{\partial x} + \frac{\partial(yz)}{\partial y} + \frac{\partial(x+z)}{\partial z} = 2xy + z + 1.$$

Hence (i),

$$\operatorname{curl}\operatorname{curl} \mathbf{F} = -\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 1 & x^2 \end{vmatrix} = -(0-0, 0-2x, 0-1) = (0, 2x, 1),$$

and (ii),

$$\operatorname{grad}\operatorname{div} \mathbf{F} = \left(\frac{\partial(2xy+z+1)}{\partial x}, \frac{\partial(2xy+z+1)}{\partial y}, \frac{\partial(2xy+z+1)}{\partial z}\right) = (2y, 2x, 1).$$

T6 Let $\mathbf{r} = (x, y, z)$ and $r = \sqrt{x^2+y^2+z^2}$.

a) Find $\operatorname{div}(r^2\mathbf{r})$.

b) Show that for any smooth function f,

$$\operatorname{grad} f(r) = \frac{f'(r)}{r}\mathbf{r}.$$

c) Show that $\operatorname{curl} \mathbf{r} = \mathbf{0}$. Deduce that $\operatorname{curl} f(r)\mathbf{r} = \mathbf{0}$ for any smooth function f.

d) Determine $\operatorname{grad}(\log r)$ and deduce that $\nabla^2(\log r) = 1/r^2$.

Solution

a) Using the identity

$$\operatorname{div}(f\mathbf{F}) = f \operatorname{div}\mathbf{F} + \operatorname{grad} f \cdot \mathbf{F},$$

we get

$$\operatorname{div}(r^2\mathbf{r}) = r^2 \operatorname{div}\mathbf{r} + \operatorname{grad}(r^2) \cdot \mathbf{r}.$$

Since $\mathbf{r} = (x, y, z)$ and $r = \sqrt{x^2 + y^2 + z^2}$, we also have $\operatorname{div}\mathbf{r} = 3$ and $\operatorname{grad}(r^2) = 2\mathbf{r}$ (see example in the lecture notes: $\operatorname{grad}(r^n) = nr^{n-2}\mathbf{r}$). Therefore,

$$\operatorname{div}(r^2\mathbf{r}) = 3r^2 + 2\mathbf{r} \cdot \mathbf{r} = 5r^2.$$

b) $\operatorname{grad} f(r) = f'(r)\left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right) = f'(r)\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = \frac{f'(r)}{r}\mathbf{r}$ as required.

c) We have

$$\operatorname{curl}\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = (0-0)\mathbf{i} + (0-0)\mathbf{j} + (0-0)\mathbf{k} = \mathbf{0}.$$

Using the identity

$$\operatorname{curl}(f\mathbf{F}) = f \operatorname{curl}\mathbf{F} + \operatorname{grad} f \times \mathbf{F},$$

we get

$$\operatorname{curl}(f(r)\mathbf{r}) = f(r) \operatorname{curl}\mathbf{r} + \operatorname{grad} f(r) \times \mathbf{r} = \frac{f'(r)}{r}\mathbf{r} \times \mathbf{r} = \mathbf{0}.$$

since $\mathbf{r} \times \mathbf{r} = \mathbf{0}$.

d) Using (b), $\operatorname{grad}(\log r) = \mathbf{r}/r^2$. Therefore

$$\begin{aligned} \nabla^2(\log r) &= \operatorname{div}\operatorname{grad}(\log r) = \nabla \cdot (r^{-2}\mathbf{r}) \\ &= r^{-2}\nabla \cdot \mathbf{r} + \operatorname{grad}(r^{-2}) \cdot \mathbf{r} = 3r^{-2} - 2r^{-4}\mathbf{r} \cdot \mathbf{r} \qquad = 1/r^2, \end{aligned}$$

as required.

T7 Let $\mathbf{r} = (x, y, z)$, $r = |\mathbf{r}|$, and $\mathbf{a}$ be a constant vector, show that

$$\operatorname{grad}\left(\frac{\mathbf{a} \cdot \mathbf{r}}{r^3}\right) = \frac{1}{r^3}\mathbf{a} - \frac{3(\mathbf{a} \cdot \mathbf{r})}{r^5}\mathbf{r}.$$

Using this result, show

$$\operatorname{grad}\left(\frac{\mathbf{a} \cdot \mathbf{r}}{r^3}\right) + \operatorname{curl}\left(\frac{\mathbf{a} \times \mathbf{r}}{r^3}\right) = \mathbf{0}.$$

Solution

Using the identities

$$\operatorname{grad}(fg) = f \operatorname{grad}(g) + \operatorname{grad}(f)g, \quad \operatorname{curl}(f\mathbf{F}) = f \operatorname{curl}\mathbf{F} + \operatorname{grad} f \times \mathbf{F},$$

we have

$$\begin{aligned}\operatorname{grad}\big(r^{-3}(\mathbf{a}\cdot\mathbf{r})\big) &= r^{-3}\operatorname{grad}(\mathbf{a}\cdot\mathbf{r}) + \operatorname{grad}(r^{-3})(\mathbf{a}\cdot\mathbf{r})\\ &= r^{-3}\mathbf{a} - 3r^{-5}(\mathbf{a}\cdot\mathbf{r})\mathbf{r},\end{aligned}$$

and

$$\begin{aligned}\operatorname{curl}\big(r^{-3}(\mathbf{a}\times\mathbf{r})\big) &= r^{-3}\operatorname{curl}(\mathbf{a}\times\mathbf{r}) + \operatorname{grad}(r^{-3})\times(\mathbf{a}\times\mathbf{r})\\ &= 2r^{-3}\mathbf{a} - 3r^{-5}\mathbf{r}\times(\mathbf{a}\times\mathbf{r})\\ &= 2r^{-3}\mathbf{a} - 3r^{-5}\big((\mathbf{r}\cdot\mathbf{r})\mathbf{a} - (\mathbf{r}\cdot\mathbf{a})\mathbf{r}\big)\\ &= -r^{-3}\mathbf{a} + 3r^{-5}(\mathbf{a}\cdot\mathbf{r})\mathbf{r}.\end{aligned}$$

Hence the sum is the zero vector as required.

T8 Let f be a smooth scalar field and $\mathbf{F}$ a smooth vector field. Prove the identity

$$\operatorname{div}(f\mathbf{F}) = f\operatorname{div}\mathbf{F} + \operatorname{grad} f\cdot\mathbf{F}.$$

Solution

Let $\mathbf{F} = (F_1, F_2, F_3)$. Then

$$\begin{aligned}\operatorname{div}(f\mathbf{F}) = \nabla\cdot(f\mathbf{F}) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(fF_1, fF_2, fF_3)\\ &= \frac{\partial}{\partial x}(fF_1) + \frac{\partial}{\partial y}(fF_2) + \frac{\partial}{\partial z}(fF_3)\\ &= f\frac{\partial F_1}{\partial x} + f\frac{\partial F_2}{\partial y} + f\frac{\partial F_3}{\partial z} + \frac{\partial f}{\partial x}F_1 + \frac{\partial f}{\partial y}F_2 + \frac{\partial f}{\partial z}F_3\\ &= f(\nabla\cdot\mathbf{F}) + (\nabla f)\cdot\mathbf{F} = f\operatorname{div}\mathbf{F} + \operatorname{grad} f\cdot\mathbf{F},\end{aligned}$$

as required.

T9 Let $\mathbf{r} = (x, y, z)$, $r = |\mathbf{r}|$, and $\mathbf{a}$ be a constant vector. Prove that

(a) $\operatorname{div}\,((\mathbf{a}\cdot\mathbf{r})\mathbf{a}) = \mathbf{a}^2$, (b) $\operatorname{curl}((\mathbf{a}\cdot\mathbf{r})\mathbf{a}) = \mathbf{0}$,

(c) $\operatorname{div}\,((\mathbf{a}\cdot\mathbf{r})\mathbf{r}) = 4(\mathbf{a}\cdot\mathbf{r})$, (d) $\operatorname{curl}\,((\mathbf{a}\cdot\mathbf{r})\mathbf{r}) = \mathbf{a}\times\mathbf{r}$,

(e) $\operatorname{div}\,((\mathbf{a}\cdot\mathbf{r})(\mathbf{a}\times\mathbf{r})) = 0$, (f) $\operatorname{curl}\,((\mathbf{a}\cdot\mathbf{r})(\mathbf{a}\times\mathbf{r})) = 3(\mathbf{a}\cdot\mathbf{r})\mathbf{a} - \mathbf{a}^2\mathbf{r}$,

(g) $\operatorname{curl}\left(\dfrac{\mathbf{a}\times\mathbf{r}}{r^2}\right) = \dfrac{2(\mathbf{r}\cdot\mathbf{a})}{r^4}\mathbf{r}$, (h) $\operatorname{curl}\,(\mathbf{r}\times(\mathbf{a}\times\mathbf{r})) = 3(\mathbf{r}\times\mathbf{a})$.

Solution

(a) Using the identity

$$\operatorname{div}(f\mathbf{F}) = f\operatorname{div}\mathbf{F} + \operatorname{grad} f\cdot\mathbf{F},$$

we get

$$\operatorname{div}((\mathbf{a}\cdot\mathbf{r})\mathbf{a}) = (\mathbf{a}\cdot\mathbf{r})\operatorname{div}\mathbf{a} + \operatorname{grad}((\mathbf{a}\cdot\mathbf{r}))\cdot\mathbf{a} = (\mathbf{a}\cdot\mathbf{r})0 + \mathbf{a}\cdot\mathbf{a} = \mathbf{a}^2.$$

(b) Using the identity

$$\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}\mathbf{F} - \mathbf{F}\times\operatorname{grad} f,$$

we get

$$\operatorname{curl}((\mathbf{a}\cdot\mathbf{r})\mathbf{a}) = (\mathbf{a}\cdot\mathbf{r})\operatorname{curl}\mathbf{a} - \mathbf{a}\times\operatorname{grad}((\mathbf{a}\cdot\mathbf{r})) = \mathbf{0} - \mathbf{a}\times\mathbf{a} = \mathbf{0}.$$

(c)

$$\operatorname{div}((\mathbf{a}\cdot\mathbf{r})\mathbf{r}) = (\mathbf{a}\cdot\mathbf{r})\operatorname{div}\mathbf{r} + \operatorname{grad}((\mathbf{a}\cdot\mathbf{r}))\cdot\mathbf{r} = (\mathbf{a}\cdot\mathbf{r})3 + \mathbf{a}\cdot\mathbf{r} = 4(\mathbf{a}\cdot\mathbf{r}).$$

(d)

$$\operatorname{curl}((\mathbf{a}\cdot\mathbf{r})\mathbf{r}) = (\mathbf{a}\cdot\mathbf{r})\operatorname{curl}\mathbf{r} - \mathbf{r}\times\operatorname{grad}((\mathbf{a}\cdot\mathbf{r})) = \mathbf{0} - \mathbf{r}\times\mathbf{a} = \mathbf{a}\times\mathbf{r}.$$

(e)

$$\operatorname{div}((\mathbf{a}\cdot\mathbf{r})(\mathbf{a}\times\mathbf{r})) = (\mathbf{a}\cdot\mathbf{r})\operatorname{div}(\mathbf{a}\times\mathbf{r}) + (\mathbf{a}\times\mathbf{r})\cdot\operatorname{grad}((\mathbf{a}\cdot\mathbf{r})) = (\mathbf{a}\cdot\mathbf{r})0 - (\mathbf{a}\times\mathbf{r})\mathbf{a} = 0 - [\mathbf{a},\mathbf{r},\mathbf{a}] = 0.$$

(f)

$$\begin{aligned}\operatorname{curl}((\mathbf{a}\cdot\mathbf{r})(\mathbf{a}\times\mathbf{r})) &= (\mathbf{a}\cdot\mathbf{r})\operatorname{curl}(\mathbf{a}\times\mathbf{r}) - (\mathbf{a}\times\mathbf{r})\times\operatorname{grad}((\mathbf{a}\cdot\mathbf{r})) = (\mathbf{a}\cdot\mathbf{r})2\mathbf{a} - (\mathbf{a}\times\mathbf{r})\times\mathbf{a}\\ &= 2(\mathbf{a}\cdot\mathbf{r})\mathbf{a} + \mathbf{a}\times(\mathbf{a}\times\mathbf{r}) = 2(\mathbf{a}\cdot\mathbf{r})\mathbf{a} + (\mathbf{a}\cdot\mathbf{r})\mathbf{a} - (\mathbf{a}\cdot\mathbf{a})\mathbf{r} = 3(\mathbf{a}\cdot\mathbf{r})\mathbf{a} - \mathbf{a}^2\mathbf{r}.\end{aligned}$$

(g)

$$\begin{aligned}\operatorname{curl}(\frac{1}{r^2}(\mathbf{a}\times\mathbf{r})) &= \frac{1}{r^2}\operatorname{curl}(\mathbf{a}\times\mathbf{r}) - (\mathbf{a}\times\mathbf{r})\times\operatorname{grad}(\frac{1}{r^2}) = \frac{1}{r^2}2\mathbf{a} - (\mathbf{a}\times\mathbf{r})\times(\frac{-2}{r^4}\mathbf{r})\\ &= \frac{2}{r^2}\mathbf{a} - \frac{2}{r^4}\mathbf{r}\times(\mathbf{a}\times\mathbf{r}) = \frac{2}{r^2}\mathbf{a} - \frac{2}{r^4}(r^2\mathbf{a} - (\mathbf{r}\cdot\mathbf{a})\mathbf{r}) = \frac{2}{r^2}\mathbf{a} - \frac{2}{r^2}\mathbf{a} + \frac{2(\mathbf{r}\cdot\mathbf{a})}{r^4}\mathbf{r} = \frac{2(\mathbf{r}\cdot\mathbf{a})}{r^4}\mathbf{r}.\end{aligned}$$

(h)

$$\begin{aligned}\operatorname{curl}(\mathbf{r}\times(\mathbf{a}\times\mathbf{r})) &= \operatorname{curl}(r^2\mathbf{a} - (\mathbf{r}\cdot\mathbf{a})r) = \operatorname{curl}(r^2\mathbf{a}) - \operatorname{curl}(\mathbf{r}\cdot\mathbf{a})\mathbf{r}\\ &= (r^2\operatorname{curl}\mathbf{a} - \mathbf{a}\times\operatorname{grad}(r^2)) - (\mathbf{a}\times\mathbf{r}) = \mathbf{0} - (\mathbf{a}\times 2\mathbf{r}) - (\mathbf{a}\times\mathbf{r}) = -3(\mathbf{a}\times\mathbf{r}) = 3(\mathbf{r}\times\mathbf{a}).\end{aligned}$$

T10 Let $\mathbf{r} = (x,y,z)$, $r = |\mathbf{r}|$, $\lambda\in\mathbb{R}$ and $\mathbf{F} = \lambda r^{-3}\mathbf{r}$. Verify that $\nabla\times\mathbf{F} = \mathbf{0}$ at all points in $\mathbb{R}^3$ except for the origin.

Solution

Using the identity

$$\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}\mathbf{F} + \operatorname{grad} f\times\mathbf{F},$$

we take $f = \lambda r^{-3}$ and $\mathbf{F} = \mathbf{r}$ to give

$$\nabla \times (\lambda r^{-3}\mathbf{r} = \lambda r^{-3}\,\mathrm{curl}\,\mathbf{r} + \mathrm{grad}\left(\lambda r^{-3}\right) = \qquad \mathbf{0} + \lambda(-3)r^{-5}\mathbf{r} = \mathbf{0}.$$

Using $\mathrm{curl}\,\mathbf{r} = \mathbf{0}$ from T2(c) and $\mathrm{grad}\,(r^n) = nr^{n-2}\mathbf{r}$ from Ex 3.2 of the notes. Lastly since $\mathbf{r}$ is parallel to itself we have $\mathbf{r} \times \mathbf{r} = \mathbf{0}$.