EXAMINATION FOR THE DEGREES OF
M.A. AND B.Sc.

Mathematics 2B - Linear Algebra

An electronic calculator may be used provided that it does not have a facility for either textual storage or display, or for graphical display.

Candidates must attempt all questions.

Specimen exam paper - SOLUTIONS

1. We consider a subset of $\mathbb{R}^3$ in the form

$$\mathcal{C} = \{[1,1,0],[0,1,1],[0,0,1]\}$$

(i) Show that $\mathcal{C}$ forms a basis for $\mathbb{R}^3$.

(ii) Construct the change-of-basis matrix $P_{\mathcal{C}\leftarrow\mathcal{E}}$, where $\mathcal{E}$ is the standard basis for $\mathbb{R}^3$

(iii) Consider the vector $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \in \mathbb{R}^3$. Using your solution to part(ii), or otherwise, compute the coordinates of $\mathbf{w}$ with respect to the basis $\mathcal{C}$.

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(i) We first show that the vectors in $\mathcal{C}$ are linearly independent by considering solutions to the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0},$$

where c_1, c_2 and c_3 are scalars. We form the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Hence, the only solution to the homogeneous system is $c_1 = c_2 = c_3 = 0$, so the vectors are linearly independent. Since $\dim(\mathbb{R}^3) = 3$ and we have three

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linearly independent vectors, these vectors must form a basis for $\mathbb{R}^3$ using a theorem from lectures.

(ii) We let C be the matrix with columns formed from the coordinate vectors of the vectors in $\mathcal{C}$ with respect to the standard basis $\mathcal{E}$. Further, we let E be the matrix with columns as the vectors in the standard basis $\mathcal{E}$. We compute the change-of-basis matrix using the Gauss-Jordan method

$$[C|E] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] = [\mathbb{I}|P_{\mathcal{C}\leftarrow\mathcal{E}}]$$

so we have

$$P_{\mathcal{C}\leftarrow\mathcal{E}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

(iii) We have $[\mathbf{w}]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$. Hence, using the change-of-basis matrix

$$[\mathbf{w}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{E}}[\mathbf{w}]_{\mathcal{E}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

2. Consider two $n \times n$ matrices C and D, where D is invertible. Prove that the matrix C is non-invertible if and only if the matrix DC is non-invertible.

. Suppose C is non-invertible. Then C must have a zero eigenvalue by Thm 4.16. We let $\mathbf{x}$ be the eigenvector corresponding to the zero eigenvalue and so $C\mathbf{x} = 0\mathbf{x} = \mathbf{0}$. Multiplying both sides by the matrix D, we obtain $DC\mathbf{x} = D\mathbf{0} = \mathbf{0} = 0\mathbf{x}$ and so DC has a zero eigenvalue. Hence, DC must be non-invertible by Thm 4.16.
Conversely suppose DC is non-invertible. Then by Thm 4.16 DC has a zero eigenvalue with corresponding eigenvector $\mathbf{y}$ and so $DC\mathbf{y} = \mathbf{0} = 0\mathbf{y}$. Since D is invertible, then $C\mathbf{y} = D^{-1}\mathbf{0} = 0\mathbf{y}$ and so C is non-invertible (by Thm 4.16) as it has a zero eigenvalue.
Hence, C is non-invertible if and only if the matrix DC is non-invertible.

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3. Consider an $n \times n$ matrix A. Suppose that A has an eigenvalue λ. Show that the eigenspace E_λ is a subspace of $\mathbb{R}^n$.

. By definition the eigenspace E_λ contains the eigenvectors of A with eigenvalue λ as well as the zero vector. Consider two vectors $\mathbf{x}$ and $\mathbf{x}$ which belong to E_λ. Hence $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \lambda\mathbf{y}$. We consider

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \lambda\mathbf{x} + \lambda\mathbf{y} = \lambda(\mathbf{x} + \mathbf{y}),$$

so $\mathbf{x} + \mathbf{y} \in E_\lambda$ and the subset is closed under addition. Consider a scalar c, then

$$A(c\mathbf{x}) = cA\mathbf{x} = c\lambda\mathbf{x} = \lambda(c\mathbf{x}),$$

so $c\mathbf{x} \in E_\lambda$ and the subset is closed under scalar multiplication. Hence, E_λ forms a subspace of $\mathbb{R}^n$.

4. Let U be the subspace of $\mathbb{R}^4$ spanned by

$$(1, 1, 0, 0), \quad (2, 0, 1, 0), \quad (1, 0, 2, 1)$$

Use the Gram-schmidt Process to find an orthogonal basis for U.

. We first show that the vectors in $\mathcal{C}$ are linearly independent by considering solutions to the equation

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0},$$

where c_1, c_2 and c_3 are scalars. We form the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Gaussian-Jordan elimination gives the final augmented form as

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so if $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$, then $c_1 = c_2 = c_3 = 0$. So $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is a basis for U.

We now apply the Gram–Schmidt process to construct an orthogonal basis:

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\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \\
\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\
&= \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \\
\mathbf{v}_3 &= \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&= \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \\ 1 \end{bmatrix}.
\end{aligned}
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So $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is an orthonormal basis for U.

5. Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$(x_1, x_2) \to (x_1 + 2x_2, -x_1).$$

Show that T is a linear transformation.

. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ and $c \in \mathbb{R}$, where $\mathbf{x} = (x_1, x_2)$, and $\mathbf{y} = (y_1, y_2)$.. We first show T preserves addition, so we consider

$$T(\mathbf{x} + \mathbf{y}) = T(x_1 + y_1, x_2 + y_2) = ((x_1 + y_1) + 2(x_2 + y_2), -(x_1 + y_1))$$

$$= (x_1+2x_2+y_1+2y_2, -x_1-y_1) = (x_1+2x_2, -x_1)+(y_1+2y_2, -y_1) = T(\mathbf{x})+T\mathbf{y})$$

and so T preserves addition. We then show T preserves scalar multiplication, so we consider

$$T(c\mathbf{x}) = T(cx_1, cx_2) = (cx_1 + 2cx_2, -cx_1) = c(x_1 + 2x_2, -x_1) = cT(\mathbf{x})$$

so T also preserves scalar multiplication and hence T is a linear transformation.

6. Let q be the quadratic form defined by

$$q(x_1, x_2) = -3x_1^2 + 5x_2^2 - 6x_1x_2.$$

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(i) Find a non-singular change of variables $\mathbf{x} = Q\mathbf{y}$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

such that $q(y_1, y_2) = \lambda_1 y_1^2 + \lambda_2 y_2^2$, for some $\lambda_1, \lambda_2 \in \mathbb{R}$.

(ii) Find a diagonal matrix D such $Q^T AQ = D$, where A is the matrix associated to q and determine the rank and the signature of q.

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(i) The quadratic can be written as $\mathbf{x}^T A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^2$ and

$$A = \begin{bmatrix} -3 & -3 \\ -3 & 5 \end{bmatrix}.$$

The eigenvalues of A satisfy $\lambda^2 - 2\lambda - 24 = (\lambda - 6)(\lambda + 4)$, hence $\lambda_1 = 6$, $\lambda_2 = -4$. The corresponding eigenspaces E_6 and E_{-4} are spanned by $\mathbf{v}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\mathbf{v}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ respectively. These vectors are orthonormal and we construct an orthogonal matrix Q whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$Q = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$$

Applying orthogonal diagonalisation, $\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T QQ^T AQQ^T\mathbf{x} = (Q^T\mathbf{x})^T DQ^T\mathbf{x} = \mathbf{y}^T D\mathbf{y}$, where $D = \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix}$ and so the quadratic in the new variables is

$$6(y_1)^2 - 4(y_2)^2,$$

where $Q\mathbf{y} = \mathbf{x}$, since $Q^T = Q^{-1}$.

(ii) From part (i) $D = \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix}$ and the rank of q is 2 and the signature is 0.

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