Algorithmic Foundations 2 - Tutorial Sheet 9

Graphs and Relations

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1. Consider the following graph:

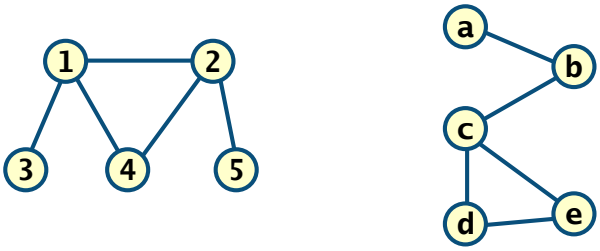
$$G = (\{a, b, c, d, e, f, g\}, \{\{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}, \{d, g\}, \{d, e\}, \{f, g\}, \{e, f\}\})$$

   (a) Draw the graph

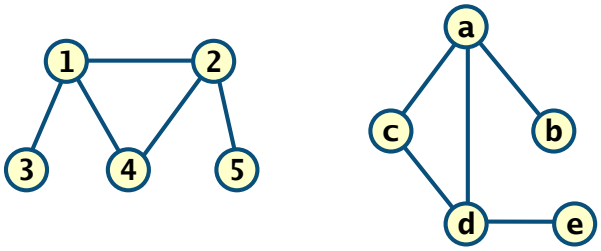
   (b) Is the graph G connected?

2. How many simple undirected graphs are there with 20 vertices and 60 edges?

3. Decide whether or not the two graphs below are isomorphic. Explain your answer.



4. Decide whether or not the two graphs below are isomorphic. Explain your answer.



5. What is an Euler circuit?

6. What is a Hamiltonian circuit?

7. Determine whether each of the following binary relations is

   - reflexive;
   - symmetric;
   - anti-symmetric;
   - transitive.

   (a) The relation R_1 over $\mathbb{N}\times\mathbb{N}$ where $(a, b) \in R_1$ if and only if $a|b$.

   (b) The relation R_2 over $S\times S$ where $S = \{w, x, y, z\}$ and

$$R_2 = \{(w, w), (w, x), (x, w), (x, x), (x, z), (y, y), (z, y), (z, z)\}.$$

(c) The relation R_3 over $\mathbb{Z}\times\mathbb{Z}$ where $(a,b) \in R_3$ if and only if $a \neq b$.

(d) The relation R_4 over $P(X)\times P(X)$ where $X = \{1,2,3,4\}$ and $(S,T) \in R_4$ if and only if $S \subseteq T$.

(e) The relation R_5 over *People*×*People* where *People* is the set of all people and $(a,b) \in R_5$ if and only if a is younger than b.

8. Give an example of a relation on a set that is
   (a) symmetric and anti-symmetric
   (b) neither symmetric nor anti-symmetric

9. Draw the directed graph for the following relations

$$\begin{aligned} R_1 &= \{(1,1),(1,3),(2,1),(2,2),(2,4),(3,1),(3,2),(3,3),(4,1),(4,2),(4,4)\} \\ R_2 &= \{(1,1),(1,2),(1,3),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\} \end{aligned}$$

10. Suppose that the relation R over $A\times A$ is reflexive. Show that R^* is reflexive.
    R^* is the transitive closure of R and is given by $R^* = \cup_{i=1}^{\infty} R^n = R \cup R^2 \cup R^3 \cup R^3 \cup \ldots$.

11. If a relation R over $A\times A$ is irreflexive, then is the relation R^2 necessarily irreflexive?

12. Consider the partially ordered sets:
    - $(P(S), \subseteq)$ where $S = \{a,b,c\}$;
    - $(\{2,3,4,6,8,12,24\}, |)$, i.e. where the relation is the divides relation.

    (a) Draw a Hass diagram for each of the partially ordered sets.
    (b) State both the maximal and minimal elements of each partially ordered set and the greatest and/or least elements when they exist.

Difficult/challenging questions.

13. What is the minimum number of edges required to produce a connected undirected graph?

14. Prove that an undirected graph with more than $(n-1)\cdot(n-2)/2$ edges is connected.

15. Prove that a relation R over $A\times A$ is transitive if and only if R^n is a subset of R for all $n \in \mathbb{Z}^+$.

16. Let R be a relation that is reflexive and transitive. Show that $R^n = R$ for all $n \geq 1$.

17. Let R be a symmetric relation. Show that R^n is symmetric for all $n \in \mathbb{Z}^+$.