2A Multivariable Calculus 2020

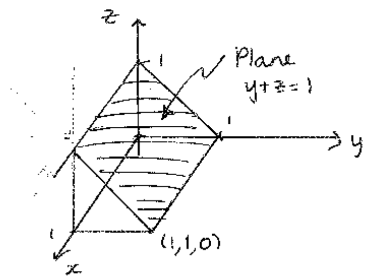
Ex Sheet 6

Tutorial Exercises

T1 Sketch the wedge shaped region W (in the first octant) enclosed by the five planes $x = 0$, $y = 0$, $z = 0$, $x = 1$ and $y + z = 1$. Then evaluate

$$\iiint_W xy \, dxdydz.$$

Solution



Hence the integral is

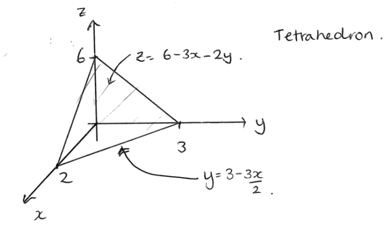
$$\begin{aligned} I &= \int_0^1 dx \int_0^1 dy \int_0^{1-y} xy \, dz = \int_0^1 x \, dx \int_0^1 y[z]_0^{1-y} \, dy \\ &= \int_0^1 x \, dx \int_0^1 y - y^2 \, dy = \int_0^1 x[\frac{y^2}{2} - \frac{y^3}{3}]_0^1 \, dx \\ &= \int_0^1 \frac{1}{6} x \, dx = \frac{1}{6}[\frac{x^2}{2}]_0^1 = \frac{1}{12}. \end{aligned}$$

T2 Sketch the solids whose volume is given by the following integrals

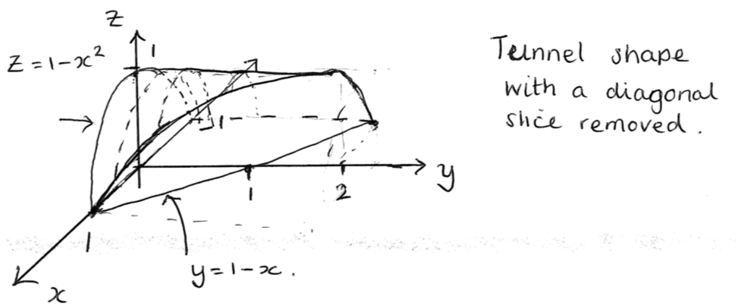
(a) $\displaystyle\int_0^2 dx \int_0^{3-3x/2} dy \int_0^{6-3x-2y} 1 \, dz$ (b) $\displaystyle\int_{-1}^1 dx \int_0^{1-x} dy \int_0^{1-x^2} 1 \, dz$

Solution

(a)



(b)



T3 A solid shell of variable density is in the form of the region lying between the spheres $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 9$. The density ρ of the shell at the point (x,y,z) is given by $\rho(x,y,z) = \sqrt{x^2+y^2+z^2}$. Find the mass of the shell.

Solution

Mass is given by the triple integral of the density ($\sqrt{x^2+y^2+z^2}$) over the volume V. Hence,

$$\text{Mass} = \int\int\int_V \sqrt{x^2+y^2+z^2}\, dxdydz = \int_0^{2\pi} d\theta \int_0^{\pi} d\phi \int_1^3 \rho^3 \sin\phi \, d\rho$$
$$= 2\pi \int_0^{\pi} \sin\phi \, d\phi \int_1^3 \rho^3 \, d\rho = 4\pi [\frac{\rho^4}{4}]_1^3 = 80\pi.$$

T4 Evaluate

$$\iiint z^2 \, dxdydz$$

throughout

a) the part of the sphere $x^2+y^2+z^2 = a^2$ $(a > 0)$ in the first octant,

b) the complete interior of the sphere $x^2+y^2+z^2 = a^2$ $(a > 0)$.

Solution

(a) Hence the integral is

$$I = \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi \int_0^a \rho^4 \cos^2\phi \sin\phi \, d\rho$$
$$= \int_0^{\pi/2} d\theta \int_0^{\pi/2} \cos^2\phi \sin\phi \, d\phi \int_0^a \rho^4 \, d\rho = \frac{\pi}{2} . \frac{1.1}{3.1} [\frac{\rho^5}{5}]_0^a = \frac{\pi a^5}{30}.$$

(b) Hence the integral is

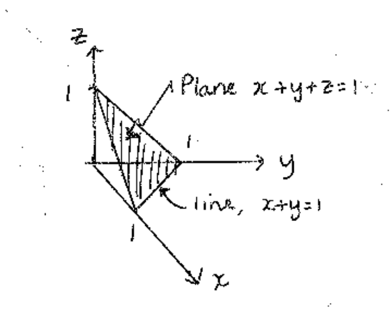
$$I = \int_0^{2\pi} d\theta \int_0^{\pi} d\phi \int_0^a \rho^4 \cos^2\phi \sin\phi \, d\rho$$
$$= \int_0^{2\pi} d\theta \int_0^{\pi} \cos^2\phi \sin\phi \, d\phi \int_0^a \rho^4 \, d\rho$$
$$= 4\pi \int_0^{\pi/2} \cos^2\phi \sin\phi \, d\phi . \frac{a^5}{5} = \frac{4\pi a^5}{5} . \frac{1.1}{3.1} = \frac{4\pi a^5}{15}.$$

T5 Evaluate

$$\iiint_T y \; dxdydz$$

throughout the tetrahedron T given by $x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1$.

Solution



Hence the integral is

$$I = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} y \, dz = \int_0^1 dx \int_0^{1-x} y[z]_0^{1-x-y} \, dy$$
$$= \int_0^1 dx \int_0^{1-x} y((1-x) - y) \, dy = \int_0^1 [\frac{1}{2}y^2(1-x) - \frac{1}{3}y^3]_0^{1-x} \, dx$$
$$= \int_0^1 \frac{1}{6}(1-x)^3 \, dx = -\frac{1}{6}[\frac{(1-x)^4}{4}]_0^1 = \frac{1}{24}.$$

T6 Use triple integration to express the volume of the solid that is bounded by the given surfaces and evaluate the volume:

a) $z = x^2 + y^2 - 3,\ z = -x^2 - y^2 + 5,$

b) $y = x^2,\ z = -y + 4,\ z = 0.$

Solution

(a) The projection of the volume onto the x, y-plane is given by a circle centre 0 radius 2. This is because the two surfaces meet at the widest point of the volume. They meet when $x^2 + y^2 - 3 = -x^2 - y^2 + 5$, rearranging this gives $x^2 + y^2 = 4$. We will call this region D. As we wish to find a volume the integrand is 1.

$$\text{Volume} = \iint_D dx\, dy \int_{x^2+y^2-3}^{-x^2-y^2+5} 1\, dz = \iint_D -2x^2 - 2y^2 + 8\, dxdy$$

Since D is a disc we carryout the remaining double integral using polar coordinates. D can be described by $0 \le r \le 2$ and $0 \le \theta \le 2\pi$ giving:

$$\text{volume} = \int_0^{2\pi} d\theta \int_0^2 (-2r^2 + 8) r\, dr = 2\pi [-\frac{2r^4}{4} + 4r^2]_0^2 = 16\pi.$$

(b) The projection of the volume onto the x, y-plane is given by $x^2 \le y \le 4$ and $-2 \le x \le 2$. This is because the widest point where $z = -y + 4$ meets the plane $y = x^2$ is at $z = 0$. The top and bottom of the volume are given by the surfaces $z = -y + 4$ and $z = 0$ respectively. Taking the integrand to be 1 because we wish to find the volume gives:

$$\begin{aligned}\text{Volume } &= \int_{-2}^{2} dx \int_{x^2}^{4} dy \int_0^{-y+4} 1\ dz = \int_{-2}^{2} dx \int_{x^2}^{4} -y + 4\ dy \\ &= \int_{-2}^{2} \left[\frac{-y^2}{2} + 4y\right]_{x^2}^{4}\ dx = \int_{-2}^{2} 8 + \frac{x^4}{2} - 4x^2\ dx = \frac{256}{15}\end{aligned}$$

T7 Find the mass of the solid of constant density ρ that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$ and $x = 1$.

Solution

The mass is found by integrating the density over the volume.

$$\begin{aligned}\text{mass} = \iiint \rho\ dV &= \int_{-1}^{1} dy \int_{y^2}^{1} dx \int_0^x \rho\ dz \\ &= \rho \int_{-1}^{1} dy \int_{y^2}^{1} x\ dx = \rho \int_{-1}^{1} \left[\frac{x^2}{2}\right]_{y^2}^{1}\ dy \\ &= \frac{\rho}{2} \int_{-1}^{1} 1 - y^4\ dy = \frac{4\rho}{5}.\end{aligned}$$

T8 Evaluate

$$\iiint_R \sqrt{x^2 + y^2 + z^2} e^{-(x^2+y^2+z^2)}\ dxdydz$$

where R is the interior of the sphere $x^2 + y^2 + z^2 = 1$.

Solution

$$
\begin{aligned}
I &= \int_0^{2\pi} d\theta \int_0^{\pi} d\phi \int_0^1 \rho^3 e^{-\rho^2} \sin\phi \, d\rho \\
&= 2\pi \int_0^{\pi} \sin\phi \, d\phi \int_0^1 \rho^3 e^{-\rho^2} \, d\rho \\
&= 4\pi \int_0^1 u e^{-u} \frac{1}{2} \, du, \qquad (\text{using } u = \rho^2 \text{ and } \frac{1}{2} du = \rho \, d\rho) \\
&= 2\pi \big[-ue^{-u} + \int e^{-u}.1 \, du\big]_0^1, \qquad (\text{using integration by parts.}) \\
&= 2\pi \big[-ue^{-u} - e^{-u}\big]_0^1 = 2\pi\big[1 - \frac{2}{e}\big].
\end{aligned}
$$

T9 Let V be the interior of the sphere $x^2 + y^2 + z^2 = 1$. Without doing any integration, explain why

$$\iiint_V x^2 \, dxdydz = \iiint_V y^2 \, dxdydz = \iiint_V z^2 \, dxdydz,$$

and why

$$\iiint_V z \, dxdydz \;=\; 0 \quad \text{and} \quad \iiint_V z^3 \, dxdydz = 0.$$

Solution

$\int\int\int z^2 \, dxdydz$ gives the mass of a sphere $x^2 + y^2 + z^2 \leq 1$, which is symmetrical about $(0, 0, 0)$ and has density z^2 at (x, y, z). Turning the x, y, z axes so that x becomes the y, the y the z, and z the x, then in the new coordinates the density will be x^2 at (x, y, z), but the mass of the sphere will be unchanged, because all we have done is change coordinates. The mass is now expressed as $\int\int\int x^2 \, dxdydz$ in the new coordinates and so the two integrals are equal. Similarly for y^2.

For $\int\int\int z \, dxdydz$ the integrand is as often positive as it is negative, and in a symmetrical way. So the answer is zero. For the same reason, the same is true when the integrand is any odd power of z.

T10 Evaluate
$$\iiint_R \frac{z}{\sqrt{x^2 + y^2 + z^2}} \, dxdydz$$
where R is the interior of the sphere $x^2 + y^2 + z^2 = 2z$.

Solution

Since, $x^2 + y^2 + z^2 = 2z$, in spherical coordinates this is $r = 2\cos\phi$.

$$\begin{aligned}
I &= \int_0^{2\pi} d\theta \int_0^{\pi/2} d\phi \int_0^{2\cos\phi} \rho^2 \cos\phi \sin\phi \, d\rho \\
&= 2\pi \int_0^{\pi/2} \cos\phi \sin\phi \, d\phi \int_0^{2\cos\phi} \rho^2 \, d\rho = 2\pi \int_0^{\pi/2} \cos\phi \sin\phi [\frac{\rho^3}{3}]_0^{2\cos\phi} \, d\phi \\
&= 2\pi \int^{\pi/2} \cos\phi \sin\phi \frac{8\cos^3\phi}{3} \, d\phi = \frac{16\pi}{3} \int_0^{\pi/2} \cos^4\phi \sin\phi \, d\phi = \frac{16\pi}{3} \frac{3.1}{5.3.1} = \frac{16\pi}{15}.
\end{aligned}$$

T11 Evaluate

$$\iiint_R \frac{1}{(x^2 + y^2 + z^2)^2} \, dxdydz$$

where R is the region in the first octant *outside* the sphere $x^2 + y^2 + z^2 = 1$.

Solution

$$\begin{aligned}
I &= \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi \int_1^{\infty} \frac{1}{\rho^2} \sin\phi \, d\rho \\
&= \frac{\pi}{2} \int_0^{\pi/2} \sin\phi \, d\phi \int_1^{\infty} \frac{1}{\rho^2} \, d\rho = \frac{\pi}{2} . \frac{1}{1} [\frac{-1}{\rho}]_1^{\infty} = \frac{\pi}{2}.
\end{aligned}$$

T12 Find the volume of the region lying inside the cylinder $x^2 + 4y^2 = 4$ above the xy-plane, and below the plane $2 + x$.

Solution

$$\begin{aligned}
\text{Volume} &= \int\int\int_V 1 \, dxdydz = \int_{-2}^{2} dx \int_{-\sqrt{1-x^2/4}}^{\sqrt{1-x^2/4}} dy \int_0^{2+x} 1dz \\
&= \int_{-2}^{2} [2y + xy]_{y=-\sqrt{1-x^2/4}}^{y=\sqrt{1-x^2/4}} \, dx = 4\pi.
\end{aligned}$$

The final integral can be done using the substitution $x = 2\cos u$.

T13 Find $\iiint_R z \, dV$, over the region R satisfying $x^2 + y^2 \leq z \leq \sqrt{2 - x^2 - y^2}$.

Solution

The domain of integration consists of a paraboloid with a spherical top. So we use spherical polar coordinates to solve the problem. To do the integration we split up the domain into two sections, A and B. Firstly the sphere meets the paraboloid when $x^2 + y^2 = z = \sqrt{2 - x^2 - y^2} = \sqrt{2 - z}$, so $z^2 + z - 2 = 0$, since $z \geq 0$ we have $z = 1$ is the only solution. Considering the $x = 0$ cross section of the paraboloid we determine $y = 1$ when $z = 1$, hence the sphere and paraboloid meet at the angle $\phi = \pi/4$. So region A is given by $0 \leq \rho \leq \sqrt{2}$, $0 \leq \phi \leq \pi/4$ and $0 \leq \theta \leq 2\pi$. For $\phi > \pi/4$ the radius is determined by the paraboloid, $z = x^2 + y^2$, in spherical polar coordinates this gives $\rho = \frac{\cos\phi}{\sin^2\rho}$. So the second section of the domain, B, is given by $0 \leq \rho \leq \frac{\cos\phi}{\sin^2\rho}$, $\pi/4 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq 2\pi$.

Hence, $\iiint_V z\, dxdydz = \iiint_A z\, dxdydz + \iiint_B z\, dxdydz$

$$\begin{aligned}\iiint_A z\, dxdydz &= \int_0^{2\pi} d\theta \int_0^{\pi/4} d\phi \int_0^{\sqrt{2}} \rho^3 \cos\phi \sin\phi\, d\rho \\ &= 2\pi \int_0^{\pi/4} \cos\phi \sin\phi\, d\phi [\frac{\rho^4}{4}]_0^{\sqrt{2}} = \pi/2.\end{aligned}$$

$$\begin{aligned}\iiint_B z\, dxdydz &= \int_0^{2\pi} d\theta \int_{\pi/4}^{\pi/2} d\phi \int_0^{\frac{\cos\phi}{\sin^2\phi}} \rho^3 \cos\phi \sin\phi\, d\rho \\ &= 2\pi \int_{\pi/4}^{\pi/2} \cot^5\phi\, \mathrm{cosec}^2\phi\, d\phi = \pi/(12).\end{aligned}$$

(The last integral is calculated using the substitution $u = \cot\phi$.) Hence the integral is $\pi/2 + \pi/12 = 7\pi/12$.