How many ways are there to choose 6 items from 10 distinct items when ...

(a) ... the items in the choices are ordered and repetition is not allowed?

There are the possible permutations of size r=6 from a set of size n=10. Therefore there are:

$$P(10, 6) = \frac{10!}{(10-6)!} = \frac{10!}{4!} = 151,200$$

ways.

(b) ... the items in the choices are ordered and repetition is allowed?

Here using the product rule there are 10 choices for each position and therefore 10^6 ways.

(c) ... the items in the choices are unordered and repetition is not allowed?

Here we are choosing r=6 items out of a set of size n=10. Therefore there are:

$$C(10, 6) = \frac{10!}{6! \cdot (10-6)!} = \frac{10!}{6! \cdot 4!} = 210$$

ways.

(d) ... the items in the choices are unordered and repetition is allowed?

Here we can use the stars and bars approach as repetition is allowed. There are 6 stars and 9 bars (since 10 distinct items) and therefore there are: $15!/(6! \cdot 9!)$ ways.

How many strings of length 10 over the alphabet $\{a, b, c\}$ have either exactly three a's or exactly four b's?

First we consider the number of strings of length 10 over the alphabet $\{a, b, c\}$ have exactly three a's. First if we consider the options for the a's we have three a's that can be placed in 10 different positions so we have $C(10, 3)$ options. For the remaining 7 positions we can either choose a b or a c (there are precisely 3 a's that we have already specified), so 2 options for 7 positions, and hence 2^7 options. Since we need to do both there the product rule there are $C(10, 3) \cdot 2^7$ ways.

Next we consider the number of strings of length 10 over the alphabet $\{a, b, c\}$ have exactly four b's. First if we consider the options for the b's we have four b's that can be placed in 10 different positions so we have $C(10, 4)$ options. For the remaining 6 positions we can either choose an a or a c (there are precisely 4 b's that we have already specified), so 2 options for 6 positions, and hence 2^6 options. Since we need to do both there the product rule there are $C(10, 4) \cdot 2^6$ ways.

To prevent over counting we also need to calculate the number of strings of length 10 over the alphabet $\{a, b, c\}$ have exactly three a's and four b's. First if we consider the options for the a's we have three a's that can be placed in 10 different positions so we have $C(10, 3)$ options. For the b's we have four b's that be in any of the remaining 7 positions, so we have $C(7, 4)$. For the remaining 6 positions we only choose a c (the a's and b's have already specified), so 1 option for 3 positions, and hence 1 options. Since we need to do both there the product rule there are $C(10, 3) \cdot C(7, 4) \cdot 1$ ways.

Finally using the inclusion exclusion principle the number strings of length 10 over the alphabet $\{a, b, c\}$ have either exactly three a's or exactly four b's equals"

$$C(10, 3) \cdot 2^7 + C(10, 4) \cdot 2^6 - C(10, 3) \cdot C(7, 4) \cdot 1$$

A drawer contains a dozen black, a dozen red socks and a dozen white socks, all unmatched. If you take socks out at random in the dark ...

(a) ... how many socks must you take out to be sure that you have at least two socks of the same color?

This can be solved using the pigeon hole principle. There are three containers representing the colours black, red and white. We want to calculate the fewest number of objects (socks) needed to ensure that at least one of the containers contains 2 objects. By the Generalised Pigeonhole Principle, we need the smallest n such that $\texttt{ceil}(n/3) = 2$, and hence 4 socks are required.

(b) ... how many socks must you take out to be sure that you have at least two black socks?

Here you can pick all the red and white socks before getting a pair of black socks, and therefore $12 + 12 + 2 = 26$ socks are required

Show that there are at least seven people in Glasgow (population: 1.7 million) with the same two initials who were born on the same day of the year (but not necessarily in the same year). Assume that everyone has two initials.

This can be solved using the pigeon hole principle. The contains are for people with the same two initials who were born on the same day of the year. There are 26^2 different initials using the product rule and 366 days (remember the extra day in a leap year). By the Generalised Pigeonhole Principle we have that one container has at least $\texttt{ceil}(1,700,000/(26^2 \cdot 366)) = 7$ objects are required.

How many different strings can be made from the letters in *MISSISSIPPI*, using all the letters?

There are 11 letters, but the 4 I's are indistinguishable, as are the 4 S's and 2 P's. Therefore there are $11!/(5! \cdot 5! \cdot 2!)$ different strings.

How many ways are there to choose a dozen donuts from 20 varieties...

(a) ... if there are no two donuts of the same variety?

These is a permutation of size r=12 from 20 objects, and hence $P(20, 12) = 20!/(20 - 12)! = 20!/8!$ ways.

(b) ... if all donuts are of the same variety?

Here there are 20 ways, as there are 20 varieties and all must be of the same variety.

(c) ... if there are no restrictions?

This is a combination with repetition and therefore we can use the stars and bars approach. There are 12 stars and 19 bars (since 20 distinct varieties) and therefore there are: $31!/(12! \cdot 19!)$ ways.

(d) ... if there are at least two varieties among the dozen donuts chosen?

Here we can just take the answer for (c) and subtract those that do not meet the requirement, i.e. when all the donuts are of the same variety which is the answer to part (a), i.e. there are $1!/(12! \cdot 19!) - 20$ ways.

(e) ... if there must be at least six blueberry-filled donuts?

Here we have fixed 6 donuts, and therefore there are $12 - 6 = 6$ left to choose and these can be of any variety. This is therefore a combination with repetition, using the stars and bars approach, there are 6 stars and 19 bars (since 20 distinct varieties) and therefore there are: $25!/(6! \cdot 19!)$ ways.