2C Intro to real analysis 2020/21

Ex Sheet 5

Solutions and Comments [1] [2]

Q1 *Use algebraic properties of limits to evaluate the limits*

$$\lim_{n\to\infty} \frac{3n^2 - 7n + 8}{2n^2 + 6n - 1}, \quad \lim_{n\to\infty} \frac{n^3 + 2n^2 + 3n + 4}{4n^3 + n^2 + 2n + 5}.$$

Make sure you justify your answers.

Remember the strategy of dividing both the denominator and numerator of fractions like these by the dominating terms. I arrive at the following answer[3]:

> For $n \in \mathbb{N}$, we have
>
> $$\frac{3n^2 - 7n + 9}{2n^2 + 6n - 1} = \frac{3 - 7/n + 9/n^2}{2 + 6/n - 1/n^2} \to \frac{3 - 7 \times 0 + 8 \times 0^2}{2 + 6 \times 0 - 1 \times 0^2} = \frac{3}{2},$$
>
> using algebraic properties of limits and the standard limit $1/n \to 0$.

For variety, I'll write the other solution in a slightly different style.

> We have
>
> $$\begin{aligned}\lim_{n\to\infty} \frac{n^3 + 2n^2 + 3n + 4}{4n^3 + n^2 + 2n + 5} &= \lim_{n\to\infty} \frac{1 + 2/n + 3/n^2 + 4/n^3}{4 + 1/n + 2/n^2 + 5/n^3} \\ &= \frac{1 + 2 \times 0 + 3 \times 0^2 + 4 \times 0^3}{4 + 1 \times 1 + 2 \times 0^2 + 5 \times 0^3} = \frac{1}{4},\end{aligned}$$
>
> using algebraic properties of limits and the standard limit $1/n \to 0$.

Note that in both my answers, I make it clear that I've used algebraic properties of limits, and that I'm using the standard limit $\frac{1}{n} \to 0$. It's always good practice to make it clear what facts are used in an argument. From an exam point of view this shows knowledge (of standard limits, and what the algebraic properties are), and it's always good to show the examiner what you know!

Q2 *Use algebraic properties of limits to evaluate the limits*

$$\lim_{n\to\infty} \frac{4n^2 + 2n - 3}{n^3 + n^2 + n + 1}, \quad \lim_{n\to\infty} \frac{3n + 6 - (1/n)}{4n - 5 + (2/n)}.$$

Make sure you justify your answers.

This works in the same way as the previous exercise. The limits are 0 and $\frac{3}{4}$ respectively, and I leave it to you to provide the required justifications.

[1] If you've not seriously tried these exercises, please don't look at these solutions and comments, until you have. You'll get the most benefit from reading these comments, when you've first thought hard about them yourself, even if you get really stuck — don't just try for a few minutes and then look at the solutions to work out how to proceed, you don't learn anywhere near as much that way.

[2] Note that I deliberately do not include formal answers for all questions.

[3] It's important here to be careful to use $=$ and $\to$ correctly. Two standard errors are to write:

$$\lim_{n\to\infty} \frac{3n^2 - 7n + 9}{2n^2 + 6n - 1} = \frac{3 - 7/n + 9/n^2}{2 + 6/n - 1/n^2}$$

and

$$\frac{3 - 7/n + 9/n^2}{2 + 6/n - 1/n^2} = \frac{3 - 7 \times 0 + 8 \times 0^2}{2 + 6 \times 0 - 1 \times 0^2}.$$

Neither of these statements can be right: in the first statement, the expression on the left is the limit, so is a number, while the right hand expression is a function of n. In particular these are not equal. The second expression is wrong for similar reasons.

Q3 *Let* $(x_n)_{n=1}^{\infty}$ *be a sequence such that* $\frac{x_n - 1}{x_n + 1} \to 0$ *as* $n \to \infty$. *Use properties of limits to show*[4] *that* $x_n \to 1$.

[4] Hint: What would you do if $\frac{x_n-1}{x_n+1} = 0$ for all $n \in \mathbb{N}$? You might want to introduce a new sequence $(y_n)_{n=1}^{\infty}$.

I tried to suggest the key idea in the side note, but often with questions like this, where there isn't a similar example in lectures you need to experiment a bit: hence the idea of seeing how to proceed when $x_n = 1$ for all n. The key trick is to define $y_n = \frac{x_n-1}{x_n+1}$, and then rearrange this to make x_n the subject of the formula.

For $n \in \mathbb{N}$, define
$$y_n = \frac{x_n - 1}{x_n + 1}.$$
Then $y_n \to 0$ as $n \to \infty$ by the hypothesis of the exercise. Now $y_n(x_n + 1) = x_n - 1$, and hence $x_n(1 - y_n) = (1 + y_n)$. Note that the map $t \mapsto \frac{t-1}{t+1}$ is defined from $\mathbb{R} \setminus \{-1\} \to \mathbb{R}$, and that 1 does not lie in the range of this map, since $t - 1 \neq t + 1$ for all $t \in \mathbb{R}$. Therefore $y_n \neq 1$ for all n, and so
$$x_n = \frac{1 + y_n}{1 - y_n}.$$
Since $y_n \to 0$ as $n \to \infty$, algebraic properties of limits give
$$x_n \to \frac{1+0}{1-0} = 1,$$
as $n \to \infty$.

Q4 Let $\alpha > 0$. Prove[5] directly from the definition that $n^{-\alpha} \to 0$ as $n \to \infty$.

[5] In this question you may use laws of indices freely; we have not yet defined n^{α} rigorously when α is irrational!

This question is similar to the proof that $1/\sqrt{n} \to 0$ from the previous exercise sheet. We follow the usual strategy[6] of fixing an arbitrary value of $\varepsilon > 0$ and then working out how large n needs to be by examining the inequality $|n^{-\alpha} - 0| < \varepsilon$.

[6] I can't emphasise enough that you must know the definition of convergence by heart. Don't try and memorise patterns for proving that sequences converge directly from the definition. Instead learn the definition, and if you get stuck, start by writing down the definition so you can see exactly what has to be proved.

Let $\varepsilon > 0$ be arbitrary. Then
$$|n^{-\alpha} - 0| = \frac{1}{n^{\alpha}} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon^{1/\alpha}} < n.$$
Thus take $n_0 \in \mathbb{N}$ with $n_0 > 1/\varepsilon^{1/\alpha}$. Then for $n \geq n_0$, we have $|n^{-\alpha}| < \varepsilon$, so $n^{-\alpha} \to 0$ as $n \to \infty$.

Q5 *Use properties of limits to evaluate*
$$\lim_{n\to\infty} \frac{2^n + 5^n}{3^n + 5^n}.$$
Use properties of limits to evaluate
$$\lim_{n\to\infty} \frac{x^n - 1}{x^n + 1}$$

where $x \neq -1$ is a real constant[7].

[7] Your answer may depend on x.

In the first part we follow the strategy of dividing both the numerator and denominator of the fraction by the dominating term, in this case 5^n.

> We have
> $$\frac{2^n + 5^n}{3^n + 5^n} = \frac{(2/5)^n + 1}{(3/5)^n + 1} \to \frac{0+1}{0+1} = 1$$
> as $n \to \infty$, using the standard limit $x^n \to 0$ for $|x| < 1$.

For the second part we must be a little careful about the value of x. For example, if $x = 1$, we see that the expression is always 0.

> When $x = 1$, we have $(x^n - 1)/(x^n + 1) = 0$ for all n, so that $\lim_{n\to\infty} \frac{x^n-1}{x^n+1} = 0$. When $|x| < 1$, we can use the standard limit $x^n \to 0$ to see that
> $$\lim_{n\to\infty} \frac{x^n - 1}{x^n + 1} = \frac{0-1}{0+1} = -1.$$
> When $|x| > 1$, divide numerator and denominator by x^n to get
> $$\lim_{n\to\infty} \frac{x^n - 1}{x^n + 1} = \lim_{n\to\infty} \frac{1 - (1/x)^n}{1 + (1/x)^n} = \frac{1-0}{1+0} = 1,$$
> using the standard limit $(1/x)^n \to 0$, which is valid as $|1/x| < 1$.

Note that in my answer, I'm careful to make sure I don't write something like[8] $\frac{x^n-1}{x^n+1} = \frac{0-1}{0+1}$.

[8] Always make sure you don't write that an expression in n is equal to its limit. Once you've taken a limit, you should get a number, not an expression in n.

Q6 *Use the sandwich principle to prove*[9] *that*

$$\lim_{n\to\infty} \frac{5^n}{n!} = 0; \quad \lim_{n\to\infty} \left(\frac{5}{\sqrt{n}}\right)^n = 0.$$

[9] For the first part, note that $0 \leq 5^n/n!$ for all n, so you want to find a sequence z_n which we know converges to 0 such that $5^n/n! \leq z_n$ for large values of n. Try writing out a typical element of the sequence, i.e. $5^n/n!$ for some large value of n, and compare it to a sequence of the form K/n for a suitable constant K.

The challenge in this question is to identify the sequence $(z_n)_{n=1}^{\infty}$ which dominates $5^n/n!$ and clearly goes to zero. My first step is to look at $5^n/n!$ more carefully, writing it as

$$\frac{5^n}{n!} = \frac{5}{1}\frac{5}{2}\frac{5}{3}\frac{5}{4}\frac{5}{5}\frac{5}{6}\cdots\frac{5}{n-1}\frac{5}{n}.$$

I've deliberately written it in this way as there are n numbers being multiplied in both the denominator and numerator. The last term $5/n$ here already goes to zero, so we just need to see what happens to the other terms. Note that the terms $5/6, 5/7, \ldots, 5/(n-1)$ are all less than 1, so

$$\frac{5}{6}\cdots\frac{5}{n-1} \leq 1$$

and so, for $n \geq 5$,

$$\frac{5^n}{n!} = \frac{5}{1}\frac{5}{2}\frac{5}{3}\frac{5}{4}\frac{5}{5}\frac{5}{6}\cdots\frac{5}{n-1}\frac{5}{n} \leq \frac{5}{1}\frac{5}{2}\frac{5}{3}\frac{5}{4}\frac{5}{5}\frac{5}{n} = \frac{5^5}{4!}\frac{1}{n}.$$

This gives me the inequality needed to use the sandwich principle. There are many strategies you could have used here, as there are many expressions which dominate $5^n/n!$ and go to zero. So the thing to do is to experiment[10] a bit and see what happens.

[10] If you're stuck, do write out the first few terms. Look for the appearance of something that's a standard limit, in this case the factor $\frac{5}{n}$. Remember we're looking to find an expression which is bigger than $\frac{5^n}{n!}$, and still goes to zero, but is also much simpler, hence the idea of removing all the terms $\frac{5}{5}, \frac{5}{6}, \cdots, \frac{5}{n-1}$.

For $n \geq 5$, we have

$$0 \leq \frac{5^n}{n!} = \frac{5}{1}\frac{5}{2}\frac{5}{3}\frac{5}{4}\frac{5}{5}\frac{5}{6}\cdots\frac{5}{n-1}\frac{5}{n} \leq \frac{5}{1}\frac{5}{2}\frac{5}{3}\frac{5}{4}\frac{5}{5}\frac{5}{n} = \frac{5^5}{4!}\frac{1}{n} \to 0$$

as n tends to ∞, using the standard limit $1/n \to 0$. Therefore by the sandwich principle $5^n/n! \to 0$.

In the second expression, does the n-th power really matter? We know that $5/\sqrt{n} \to 0$ as $n \to \infty$, by using the standard limit $1/\sqrt{n} \to 0$. But once $5/\sqrt{n} < 1$, which happens for $n > 25$, we have

$$\left(\frac{5}{\sqrt{n}}\right)^n < \frac{5}{\sqrt{n}}.$$

This leads us to the following answer, which is not the only possible way to proceed[11].

[11] For example, you could also note that for $n > 26$, we have

$$0 \leq \left(\frac{5}{\sqrt{n}}\right)^n \leq \left(\frac{5}{\sqrt{26}}\right)^n \to 0,$$

using the standard limit $x^n \to 0$ for $|x| < 1$. The sandwich principle again gives $\left(\frac{5}{\sqrt{n}}\right)^n \to 0$.

For $n \geq 25$, we have

$$0 \leq \left(\frac{5}{\sqrt{n}}\right)^n \leq \frac{5}{\sqrt{n}} \to 0,$$

as $n \to \infty$, using the standard limit $1/\sqrt{n} \to 0$. Therefore by the sandwich principle $\left(\frac{5}{\sqrt{n}}\right)^n \to 0$ as $n \to \infty$.

Q7 *Use the sandwich principle to find*

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}.$$

Justify your answer.

Let $0 < a < b$. Use the sandwich principle to show that

$$\lim_{n\to\infty} (a^n + b^n)^{1/n} = b.$$

These are similar to examples from lectures.

We have

$$0 \leq \sqrt{n+1} - \sqrt{n} = \left(\sqrt{n+1} - \sqrt{n}\right)\left(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)$$
$$= \frac{1}{\sqrt{n+1}+\sqrt{n}} \leq \frac{1}{\sqrt{n}} \to 0,$$

using the standard limit $n^{-1/2} \to 0$. Therefore by the sandwich principle, $\sqrt{n+1} - \sqrt{n} \to 0$ as $n \to \infty$.

> As $0 < a < b$, we have
> $$b \leq (a^n + b^n)^{1/n} \leq (b^n + b^n)^{1/n} = 2^{1/n} b \to b$$
> as $n \to \infty$, using the standard limit $2^{1/n} \to 1$ as $n \to \infty$. Therefore by the sandwich principle $(a^n + b^n)^{1/n} \to b$ as $n \to \infty$.

Q8 *Let A be a non-empty subset of $\mathbb{R}$ which is bounded above by M. Prove*[12] *that M is the least upper bound for A if and only if there exists a sequence $(a_n)_{n=1}^{\infty}$ with $a_n \in A$ for all n and $a_n \to M$.*

[12] There's two directions to prove. Make sure you write down exactly what you know and what you are trying to prove. If you get really stuck then read the proof of theorem 2.3.12 from ERA which gives a similar result for greatest lower bounds.

We have two directions to prove. Let's first prove the implication from right to left (which is the easy direction), so we suppose that there is a sequence $(a_n)_{n=1}^{\infty}$ with $a_n \in A$ for all n and $a_n \to M$. We need to prove that for all $\varepsilon > 0$, there exists $a \in A$ with $a > M - \varepsilon$ and this can be done by taking an element sufficiently far down the sequence $(a_n)_{n=1}^{\infty}$.

> Let M be an upper bound for A. Suppose that there exists a sequence $(a_n)_{n=1}^{\infty}$ with $a_n \in A$ for all n and $a_n \to M$. Then for $\varepsilon > 0$, find $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, we have $|a_n - M| < \varepsilon$. Then $a_{n_0} > M - \varepsilon$, and hence M is the least upper bound for A.

The converse is more interesting. We suppose that M is the supremum of A and have to find the required sequence. Since M is the supremum we know that

$$\forall \varepsilon > 0, \exists n \in \mathbb{N} \text{ s.t. } a > M - \varepsilon.$$

We now use this property, by taking $\varepsilon = 1/n$ for each $n \in \mathbb{N}$, writing a_n for the resulting element of A given by the statement. Note that to see we can do this it really helps to write out the statement we get from M being the supremum.

> Now suppose that M is the supremum of A. Then for each $n \in \mathbb{N}$, take $\varepsilon = 1/n$ in the least upper bound condition for the supremum to find $a_n \in A$ with $a_n > M - 1/n$. Then
> $$M - \frac{1}{n} < a_n \leq M$$
> for all n, so that by the sandwich principle $a_n \to M$.

Q9 *Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be real sequences. From the algebraic properties of limits we know that if both $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ converge, then so too does $(x_n + y_n)_{n=1}^{\infty}$. Investigate what can be said when:*

a) *exactly one of the sequences converges, and the other does not.*

b) *both sequences do not converge.*

Determine whether $(x_n + y_n)_{n=1}^{\infty}$ necessarily *converges or diverges, justifying your answers with proofs or examples as appropriate*[13].

For the first part, we use properties of limits. Note that in my answer, I aim to be clear about what my conclusion is, and how I prove it.

Suppose $(x_n)_{n=1}^{\infty}$ converges to $L \in \mathbb{R}$, and $(y_n)_{n=1}^{\infty}$ does not converge. Define $(z_n)_{n=1}^{\infty}$ by $z_n = x_n + y_n$. Then $(z_n)_{n=1}^{\infty}$ does not converge.

Proof. Note that $y_n = z_n - x_n$. If $(z_n)_{n=1}^{\infty}$ converges to $M \in \mathbb{R}$, then by standard properties of limits we have $y_n = z_n - x_n \to L - M$, a contradiction. Therefore $(z_n)_{n=1}^{\infty}$ does not converge. $\square$

[13] In the first case, let $z_n = x_n + y_n$. If $(x_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ both converge, write y_n in terms of z_n and x_n. Does this argument work in the second case? If not, perhaps you want to look for an example: can you write a convergent sequence as a sum (or perhaps difference) of two sequences which do not converge?

We can see that this strategy doesn't work in the second case, when both $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ fail to converge, as even if $z_n = x_n + y_n$ converges, we would not be able to write x_n or y_n as a difference of two converging sequences. This leads us to look[14] for a counterexample: can we think of a way of adding together two sequences which fail to converge, so that their sum does converge?

[14] You'll need to have standard examples of non-convergent sequence to hand to do this, like $(-1)^n$. This sequence alternates, so a strategy is to arrange for the alternation in the sequence $(x_n)_{n=1}^{\infty}$ to "cancel" with that in the sequence $(y_n)_{n=1}^{\infty}$.

When $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ do not converge, it is possible for the sequence $(z_n)_{n=1}^{\infty}$ given by $z_n = x_n + y_n$ to converge or fail to converge. For example, take $x_n = y_n = (-1)^n$, so that both $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ do not converge and z_n also fails to converge. As a second example take $x_n = (-1)^n$ and $y_n = -(-1)^n$. Again both $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ diverge, but now $z_n = 0$ for all n, and the constant sequence $(z_n)_{n=1}^{\infty}$ converges to 0.