Higher-Order Programming I: Map and Filter

Principles of Programming Languages
Lecture 8

`CAS CS 320`

Introduction

Administrivia

Assignment 3 is due on **Friday** by 11:59PM.

Assignment 4 will be out later today.

The midterm for this course is in two weeks (2/27).

Objectives

Introduce the notion of higher-order functions and how they can help us write cleaner, more general code.

Examine two classic higher-order functions, map and filter.

Keywords

higher-order functions

first-class values

functions as function parameters

the abstraction principle

map and filter

tail-recursive map and filter

Practice Problem

*Implement a function **split_sorted** which given*

> *l : a sorted **int list***
> *i : **int***

*returns two sorted lists, one which has the elements of **l** which are at most **i**, and the other which has the elements of **l** which are greater than **i**.*

Higher-Order Functions

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2. They can be returned by another function.
3. They can be passed as arguments to another function.

Note. Types are ***not*** first-class values.

An Aside: Robin Popplestone

"He started a PhD at Manchester University before moving to Leeds University. His project was to develop a program for automated theorem proving, but he got caught up in using the university computer to design a boat. He built the boat and set sail for the University of Edinburgh, where he had been offered a research position. A storm hit while crossing the North Sea, and the boat sank. A widely believed story about Popplestone was that he never completed his PhD in mathematics because he lost his thesis manuscript in the boat, although Popplestone refused to corroborate this."

https://en.wikipedia.org/wiki/Robin_Popplestone

Functions as Returned Values

```
# let f x y = x + y;;
val f : int -> int -> int = <fun>
# f 2;;
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Multi-argument functions in OCaml are really single-argument functions which return functions.

Functions as Named Values

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let f = fun x -> fun y -> x + y
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Functions and parameters

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val apply : ('a -> 'b) -> 'a -> 'b = <fun>
# apply add_five 10;;
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This allows us to create new functions which are parametrized by old ones.

Higher-Order Functions Elsewhere

fun $f \to \dfrac{f(x)}{dx}$ **e.g.** $x^2 \mapsto 2x$

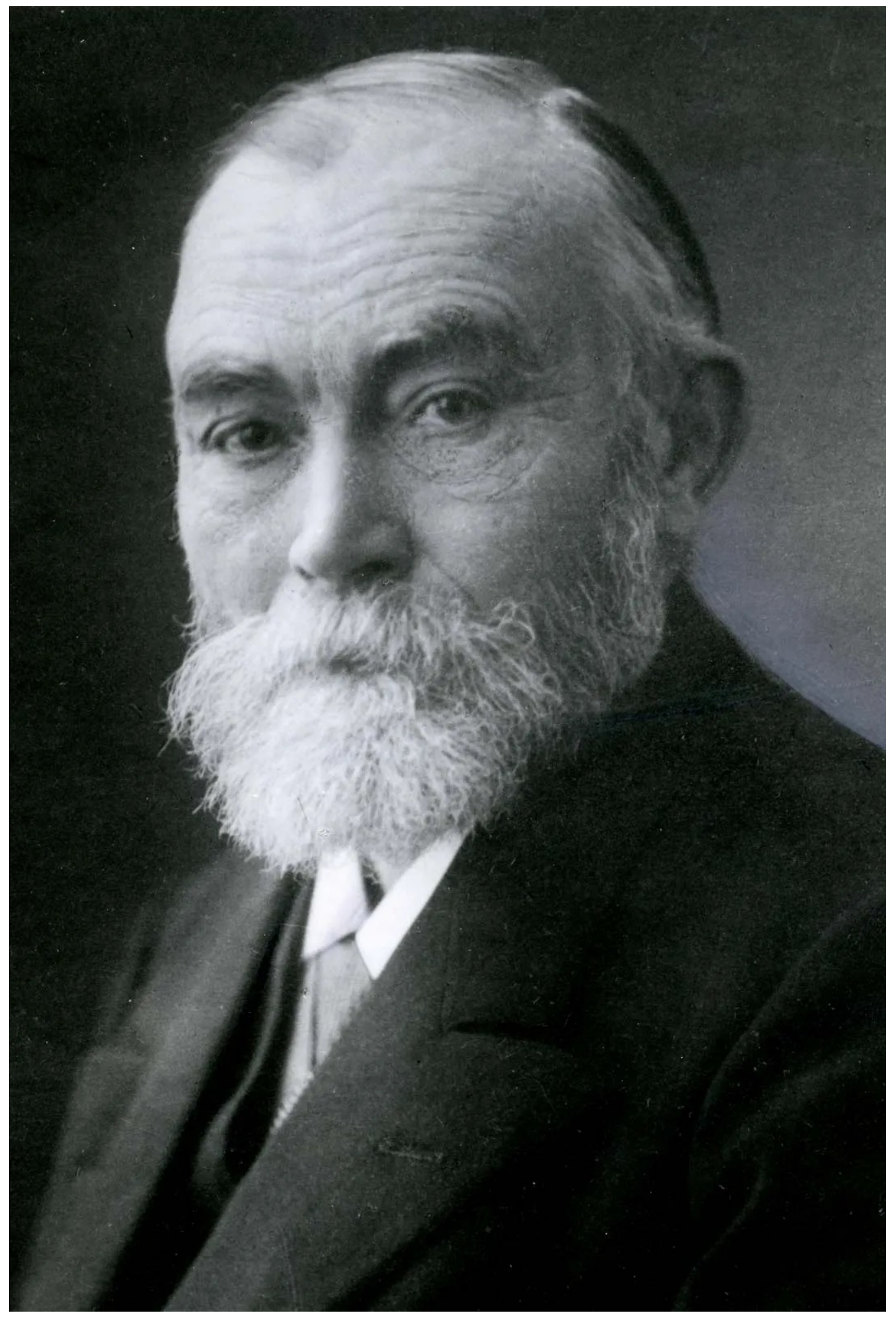
We might think of the type of an derivative as

$$(\mathbb{R} \to \mathbb{R}) \to \mathbb{R} \to \mathbb{R}$$

because it takes one function and produces a new function.

An Aside: What does Higher-Order Mean?

"Like things and functions are different, so are functions whose arguments are functions **radically different** from functions whose arguments must be things. I call the latter functions of first order, the former functions of second order."


Gottlob Frege

https://softwareengineering.stackexchange.com/questions/186035/who-first-coined-the-term-higher-order-function-and-or-first-class-citizen

First-Order Function Types

```
int -> string

t -> t

() -> bool

bool * bool -> bool
```

Second-Order Function Types

```
(int -> string) -> (int -> string)

t -> (s -> t)

(() -> bool) -> bool

bool -> bool -> bool
```

Third-Order Function Types

```
(int -> string) -> (int -> string) -> (int -> string)

(t -> (s -> t)) -> t

((() -> bool) -> bool) -> bool

(bool -> bool -> bool) * bool -> bool
```

And so on...

```
t -> t -> t -> t -> t -> ...
```

The higher-order part comes from the fact that we can do this *ad infinitum*.

(In practice, we rarely use higher than third-order or fourth-order functions.)

The Abstraction Principle

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When we write general programs, we avoid rewriting programs we've (pretty much) written before.

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  match l with
  | [] -> []
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*It doesn't matter if we're reversing a list of **ints** or a list of **strings**, the function is the same.*

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let rec fact n =
  match n with
  | 0 -> 1
  | n -> n * fact (n - 1)

let rec sum n =
  match n with
  | 0 -> 0
  | n -> n + sum (n - 1)
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Some functions cannot be polymorphic.

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  let rec go n =
    match n with
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    | n -> f n (go (n - 1))
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Now we have a single function which we can reuse elsewhere.

demo

(fact_sum.ml)

Another Example: Sorting

```
let rec insert (x : 'a) (l : 'a list) : 'a list =
  match l with
  | [] -> [x]
  | y :: ys -> if x <= y then x :: y :: ys else y :: insert x ys

let rec sort (l : 'a list) : 'a list =
  match l with
  | [] -> []
  | x :: xs -> insert x (sort xs)
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We shouldn't rewrite the whole function.

Another Example: Sorting

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let insert (le : 'a -> 'a -> bool) =
  let rec go x l =
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An Aside. Also note the arguments. Why doesn't sort take a list?

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It's not a concrete principle. Roughly it says:

- Abstract out core functionality.
- Use higher-order functions to parametrize by functionality specific to the problem.
- (Try to understand the algebra of programming.)

Understanding Check

```
let rec next_word (cs : char list) =
  match cs with
  | [] -> []
  | x :: xs ->
    if (not (x = ' ')) then x :: next_word xs else []

let rec pos_prefix (l : int list) : int list =
  match l with
  | [] -> []
  | x :: xs ->
    if x > 0 then x :: pos_prefix xs else []
```

Write a function which factors out the core functionality of the above two functions.

Map

Simple Example: Update All Users

```
type user = {
  name : string ;
  id : int ;
}

let capitalize = ...

let fix_usernames (us : user list)  =
  List.map (fun u -> { u with name = capitalize u.name }) us
```

`map` is used to apply a function to every element in a list (or other structure).

Definition of Map

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let rec map f l =
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If the list is nonempty, we apply f to its first element, and recurse.

Tail-Recursive Map

```
let rec map_t f l =
  let rec go l acc =
    match l with
    | [] -> List.rev acc
    | x :: xs -> go xs (f x :: acc)
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For a tail-recursive version we can build the list in reverse in **acc** and then reverse it at the end.

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This may seem inefficient, but its just a constant factor slower.

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An Aside. The standard library map is *not* tail-recursive.

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1. There is a function **rev_map**, which is tail-recursive and does give the output in reverse order.

2. **map** is defined somewhat differently to account for side-effects.

We won't dwell on these for now, but it may be worth reading about.

demo

(normalize.ml)

Understanding Check

Using a single call to map, implement function ***pointwise_max*** *which, given*

```
  f : 'a -> int
  g : 'a -> int
  l : 'a list
```

returns ***l*** *with* ***f*** *or* ***g*** *applied to each element, whichever gives a larger value.*

Filter

Simple Example:

```
type user = {
  name : string ;
  id : int ;
  num_likes : int ;
}

let popular (us : user list) (cap : int) =
  List.filter (fun u -> u.num_likes > cap) us
```

filter is used to do grab all elements in a list which satisfy a given property.

Predicates

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A predicate is a function which represents a property.

Example. `let even n = n mod 2 = 0`

Definition of Filter

```
let rec filter p l =
  match l with
  | [] -> []
  | x :: xs ->
    (if p x then [x] else []) @ filter p xs
```

If the list is empty there is nothing to do.

If the first element satisfies our predicate we keep it and recurse.

Otherwise, we drop it and recurse.

Tail Recursive Filter

```
let filter_tail p =
  let rec go acc l =
    match l with
    | [] -> List.rev acc
    | x :: xs -> go ((if p x then [x] else []) @ acc) xs
  in go []
```

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As with map, we have to reverse the output before returning it.

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As with map, we have to reverse the output before returning it.

The standard library implementation of **filter** *is* tail-recursive.

demo

(primes.ml)

Understanding Check

What do the following functions do?

```
let f = List.filter (fun i -> i mod 2 <> 0)

let g l = l |> List.map (fun x -> x * x) |> f

let h p q = List.filter (fun i -> p i && q i)
```

Summary

Higher-order function allow for better abstraction because we can parameterize functions by other functions.

map and filter are very common patterns which can be used to write clean and simple code.