# Lab 3

# Windowing

### 0. Preface

In this lab, you will explore the tradeoff between time resolution and frequency resolution when performing frequency analysis of signals – aka computing the signal’s spectrum. You will see that we can subtly change the amplitude shape of the time signal signal (which is called windowing) to highlight aspects of the spectrum. We can change how much of the signal is used to calculate the spectrum, and we can add a string of zeros to a relatively short section of signal in order to see some fine detail in the spectrum – this is called ‘zero padding’.

With MATLAB, calculate the magnitude spectrum of 32 samples of a 128 Hz sinusoid at a sampling rate of 2,048 Hz. This results in Figure 1, (Left) with a Kronecker delta function centered right at 128 Hz. This is what the theory predicts.

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| Figure 1: (Left) The magnitude spectrum of a 128 Hz sinusoid sampled at 2048 Hz over . (Right) The magnitude spectrum of a 220 Hz sinusoid  sampled at 2048 Hz over |

Next you need to evaluate the spectrum of 32 samples of a 220 Hz sinusoid sampled at the same rate: 2,048 Hz. You’d expect that to be another Kronecker delta function, but now centered at 220 Hz. However, it is not - see(Figure 1, right): there is energy in every frequency.

Instead of taking only 32 samples of this 220 Hz sinusoid, take 512 samples and evaluate its magnitude spectrum. This time you see (Figure 2) what you expected in the first place: a single Kronecker delta function centered at 220 Hz..

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| Figure 2: The magnitude spectrum of a 220 Hz sinusoid sampled at 2048 Hz over . Now it is properly resolved. |

The DFT of an length-N signal uses only *N* basis functions (phasors) having normalized frequencies, . The DFT of any length-N sampled sinusoid with a frequency that is not one of these *N* frequencies will suffer from *spectral leakage* because such a signal resembles something in every *N*-length DFT basis function.

Another way to think about this is the effect of cropping a periodic sequence to a length that is not an integer number of periods. When you evaluate the discrete Fourier series of this signal (which is nothing more than the DFT), the periodically extended sequence now contains a discontinuity (Figure 3). Spectral leakage is caused by this discontinuity.

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| Figure 3: The periodized waveform (gray) resulting from 32 samples of a 220 Hz sinusoid sampled at 2048 Hz. Arrow points to the discontinuity in the waveform. |

We solved the leakage problem shown in Figure 2, when using 512 samples of the 220 Hz sinusoid. That way, the periodized sequence has no discontinuities, and the expected components are found by the DFT.

However, this synthetic signal is not very realistic because most, if not all, real-world signals vary in time and are certainly not infinite (why?). Thus, one can never find a value *N* for which a real-world sampled signal is periodic —in the strict sense of the word.

Spectral leakage is something that will always exist. This lab will teach you how to “deal with it” so you can “move on”.

### 1. Interpolating the DFT

Although you will never be able to avoid spectral leakage, one way to help is to increase signal length artificially by “zero padding”. Let’s see what happens in that case.

* 1. Using 32 samples of a sinusoid with frequency 220 Hz, sampled at a constant rate of 2,048 Hz, pad zeroes to it to a total length of 1024 samples. Calculate the DFT of the result. Plot this using a line plot, and overlay this with a stem plot of the DFT of the original 32 samples. See Figure 5 for guidance. Include your plot and code in your report.

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| Figure 5: The DFT of the length-32 sinusoid with frequency 220 Hz (stem), and with zeropadding to length-1024 (line). |

Now, we see a peak right at 220 Hz with the correct magnitude. By zero-padding the length-32 sequence we are in effect sampling the discrete-time Fourier transform (DTFT) at a finer resolution than 1/32. This does *not* mean we have cheated the system and solved our frequency resolution problems.

* 1. Using your code from 1.1, reduce the frequency of the sinusoid to 128 Hz and run the code again. Include only the plot and comment on what has happened.

From your results for 1.2, you see that zeropadding is not always a good thing. What was once a single real component at 128 Hz has now become several frequency components, e.g., one at 128 Hz and one at around 32 Hz.

Since multiplication in the time-domain corresponds to a convolution in the frequency domain, the spectrum of an infinite-length 220 Hz sinusoid (which has delta functions at ±220 Hz in its spectrum) is being convolved with the spectrum of the rectangular window (which has the shape of a spectral sinc function). The convolution of the sinc and the impulse at the negative frequency is superposed with that of the positive frequency. This effect is most visible in Figure 5 at the low frequencies where the first “lobe” to the left of the largest one (“main lobe”) is larger than any of the lobes to the right.

Windowing is also occurring in 1.2, resulting in the spectrum you see. This time, the convolution of a 32-sample sinc with the impulse at 128 Hz is such that zeroes are created in every frequency bin except one. After zero-padding this signal however, spurious frequencies are introduced because the original signal is now windowed in such a way that a discontinuity is introduced into the periodized signal.

Spectral leakage may not be a major problem when it is known beforehand that only one real sinusoid is present. As seen in Figure 5, by zero-padding the signal we are able to resolve the exact frequency and amplitude. But what if a signal consists of two or more frequencies that are close together and that have very different amplitudes?

* 1. Using the same code of part 1.1, perform a DFT of a length-32 signal that is the following sum of three sinusoids:



Find the DFT of the signal zero-padded to a length 2048. Overlay the DFT spectrum of length-32 using stem, with the zero-padded DFT spectrum using plot, an example of which is given in Figure 5. Include just your plot.

* 1. Without the knowledge that there are multiple pure tones in this sequence, can you pick out the right frequency components in the magnitude spectrum?

Do not be mistaken —as many are— that adding more zeroes to your input will give you finer frequency resolution. In effect, it makes your spectrum converge to the Fourier transform, or DTFT, of the sampled and thus periodically extended signal. Zero-padding does not give you more resolution in the sense that you would be able to differentiate between closely spaced components. Fortunately, there are ways to remedy the effects of spectral leakage. This will be studied next.

### 2. Frequency Content of Windows

Since multiplication in the time-domain is convolution in the frequency domain, one obviously wants a window that has an impulse-like spectrum. Alas, the only window that has this property is an infinitely long rectangular window, which is not realistic, and in fact leads to complete uncertainty in how a signal varies with time. With careful window design, however, one can fashion a window that will relieve the problems above.

There are literally hundreds of windows one can use to help analyze or process data.. We will focus on four more popular ones.

A length-*N* rectangular window is defined as:

. (1)

A length-*N* triangular window, where *N* is odd, is defined as:

. (2)

A length-*N* sine window is defined as:

. (3)

A length-*N* Hann window is defined as:

. (4)

You will now analyze the frequency content of these four common windows. After that, you will see how these windows perform in resolving the frequency content of the signal you synthesised in 1.3.

2.1 Program the four windows above. Using *N*=31 for the triangular window, and *N*=32 for the other windows, plot each window on the same graph (using plot instead of stem). Label each plot using gtext. Include this plot in your report.

2.2 Now evaluate the DFT of each window at 1,024 points (fft(w,1024)). In four different diagrams, plot the normalized magnitude spectrum (not dB) as a function of normalized frequency for each window, but before doing so you will want to use fftshift to move the zero frequency to the center. The normalized frequency interval is . Your figures should look like those in Figure 7. Include your figure in your report.

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| Figure 7: The frequency content of the four window functions given by (1-4) |

One immediately apparent fact is that the windows each have different shapes in the frequency domain – they are different functions of frequency. The main-lobe is thinnest for the rectangular window, and widest for the triangular and Hann windows. The sidelobes are at different levels for each one, too.

The subtle differences between the windows can be made clearer by using a logarithmic magnitude scale. In this case, the magnitude spectrum can be transformed into a decibel spectrum using the following normalization:

. (5)

A function that is normalized in this way has a maximum value of zero, and everything else is negative—so don’t be alarmed.

2.3 For the windows in part 2.2, plot the normalized dB spectral function of the rectangular and triangular windows, eqs. (1)-(2), on the *same* graph. Limit the magnitude axis to [-80:5] dB, and the frequency axis to [-0.2:0.2]. Include your plot with a legend, as well as your code. Your figure should look similar to that in Figure 8.

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| Figure 8: The frequency content of the rectangular and triangular windows |
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2.4 Find the width of the main lobe for these two windows. What is the width of the main lobe for the triangular window in terms of the width for the rectangular window? Do you have any intuition why this is so?

2.5 At what magnitude (in dB) is the height of the first side-lobe for the rectangular and triangular windows?

2.6 Repeat 2.3 for the sine and Hann window functions, eqs. (3)-(4). Include your plot with a legend. No need to include your code.

2.7 What is the width of the main lobe for the sine and Hann windows?

2.8 At what magnitude (in dB) is the height of the first side-lobe for the sine and Hann window?

From the above, we can clearly see that, out of these four windows, the rectangular window has the thinnest main lobe but the highest sidelobes. It also has the “slowest” decay of the sidelobes – this is known as ‘side-lobe roll-off (rate)’. The triangular window has a steeper decay of the sidelobes, but the main lobe is twice as wide as that of the rectangular window. The Hann window has a similar main lobe width, but has the lowest magnitude sidelobes of all these four windows.

### 3. Application of windowing to signal analysis

Now let’s apply these windows to the problems observed above in 1.4.

3.1 Apply windowing to the 32-sample sequence you created in 1.3 using the rectangular, triangular, sine, and Hann windows. Plot each spectrum (four plots) using a stem plot (not in dB). Plot the normalized magnitude for each case (abs(X)./max(abs(X))). However, this time do *not* use fftshift. Use the proper frequency indices, i.e., f = n\*Fs/N. Plot only the frequencies between 0 and the Nyquist frequency. Include your code and plot. Your plot should look like Figure 9: The normalized magnitude DFT of the length-32 windowed sum of three sinusoids with frequencies 128, 220, and 525 Hz.

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| Figure 9: The normalized magnitude DFT of the length-32 windowed sum of three sinusoids with frequencies 128, 220, and 525 Hz. |

Obviously, we still cannot resolve the component at frequency 220 Hz because that frequency has fallen in a crack. The Hann and sine windowings indicate that energy is present at 525 Hz, which does not appear to be caused by an artefact of the windowing. Having just 32 samples to begin with, this is the best we can do with one window. Let’s go one step further and zero-pad each sequence to a length of 2,048.

3.2 For each of the windowed signals in 3.1, zero-pad each sequence to a length of 2,048 and evaluate the DFT (you can do both at the same time by fft(x,2048)). Plot each magnitude spectrum (four plots) using plot on a normalized dB magnitude scale. Plot only the frequencies between 0 and the Nyquist frequency. Include your code and plot.

3.3 Answer the following questions:

3.3.1 Knowing how the main lobes depend on the type of window, and how the sidelobes decay, for which one(s) of these windows can you say with certainty that this signal has three components?

3.3.2 For which window(s) can you accurately determine the frequency of the middle amplitude sinusoid?

3.3.3 Why are the sine and Hann windows good at revealing the lowest amplitude component, but not the middle amplitude component?

3.3.4 Why should the lowest amplitude component have a normalized dB level of –40 dB?