Assignment due date: Feb 10, 2017, 11:59pm

Hand-in to be submitted electronically in PDF format with code to the CDF server by the above due date

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I hereby affirm that all the solutions I provide, both in writing and in code, for this assignment are my own. I have properly cited and noted any reference material I used to arrive at my solution and have not share my work with anyone else. I am also aware that should my code be copied from somewhere else, whether found online, from a previous or current student and submitted as my own, it will be reported to the department.

Fan Xie.
Signature

(Note: -3 marks penalty for not completing properly the above section)

Part 1 total marks: 50
Part 2 total marks: 50
Total: 100

Modified by E. Garfinkiel, D. Levin Winter 2017
(adapted from F. Estrada)

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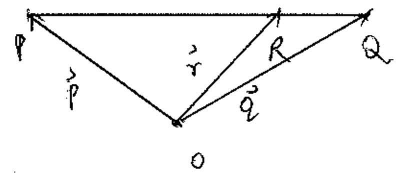
Part 1. A).

1.1)



$\because \vec{PR} = 4\vec{RQ}$

$\vec{PR} = \vec{r} - \vec{p}$

$\vec{RQ} = \vec{q} - \vec{r}$

$\Rightarrow (\vec{r}-\vec{p}) = 4\cdot(\vec{q}-\vec{r})$

$\Rightarrow \underline{\underline{\vec{r} = \frac{1}{5}\vec{p} + \frac{4}{5}\vec{q}}}$

1.2) $\vec{p} + t\cdot(\vec{q}-\vec{p})$, where $\vec{p} = \vec{OP}$, $\vec{q} = \vec{OQ}$, $0 \le t \le 1$.

1.3). To test if an arbitrary point $R(x, y)$ is on the line segment or not:

① $\min(x_0, x_1) \le x \le \max(x_0, x_1)$
and $\min(y_0, y_1) \le y \le \max(y_0, y_1)$.

② $f(x,y) = 0$, where
$f(x,y) = (y-y_0)(x_1-x_0) - (y_1-y_0)(x-x_0)$
i.e. the implicit curve function $f(x,y)=0$.

A point has to satisfy the above constraints to be on the line segment. Otherwise it is not on the line segment.

2.1). From the above, $f(x,y) = (y-y_0)(x_1-x_0) - (y_1-y_0)(x-x_0)$

let $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \vec{n}_0$, the gradient vector.

$\Rightarrow \vec{n}_0 = (-(y_1-y_0), (x_1-x_0))$

$\therefore \vec{n} = \frac{1}{|\vec{n}_0|}\vec{n}_0$, the unit normal vector.

$$\underline{\underline{\hat{n} = \frac{1}{\sqrt{(y_1-y_0)^2+(x_1-x_0)^2}} \cdot (-(y_1-y_0), x_1-x_0)}}$$

2.2). the distance between an arbitrary point $R(x, y)$ and PQ is equal to the projection of PR on the unit normal vector $\vec{n}$.

$$\Rightarrow d_{R\text{-}PQ} = \frac{\vec{PR} \cdot \vec{n}}{|\vec{n}|} = \frac{(y_0 - y_1,\ x_1 - x_0) \cdot ((x - x_0),\ (y_0 - y_0))}{\sqrt{(x_1 - x_0)^2 + (y - y_0)^2}}$$

$$d_{R\text{-}PQ} = \frac{1}{\sqrt{(x_1 - x_0)^2 + (y - y_0)^2}} \left[(y_0 - y_1)(x - x_0) + (y - y_0)(x_1 - x_0) \right]$$

B). 1). Rotations and Translations :

Not commutative.

Proof by contradiction:

$$T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$TR = \begin{bmatrix} \cos\theta & -\sin\theta & 1 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$RT = \begin{bmatrix} \cos\theta & -\sin\theta & \cos\theta \\ \sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$

$TR \neq RT$. Therefore ~~Ro~~ rotation and translation are not commutative.

Q.E.D

B). 2). Two rotations:

Yes, they are commutative.

Proof: Let $R_1 = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $R_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

be 2 general rotation matrixes.

$$R_1 \cdot R_2 = \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1 \cdot \sin\theta_2 & -\cos\theta_1 \sin\theta_2 - \sin\theta_1 \cos\theta_2 & 0 \\ \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2 & -\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ①$$

$$R_2 \cdot R_1 = \begin{bmatrix} \cos(\theta_2+\theta_1) & -\sin(\theta_2+\theta_1) & 0 \\ \sin(\theta_2+\theta_1) & \cos(\theta_2+\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ②$$

Apparently ① = ② $\Rightarrow$ $R_1 R_2 = R_2 R_1$

Thus, 2 rotations are commutative.

Q.E.D.

5.

(3). Translation, Reflection:

Proof by contradiction:

$$R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$, the reflection matrix about $y = x$.

$$T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$RT = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$TR = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow RT \neq TR$. Q.E.D (Not commutative).

(4). Shearing in x-direction, an uniform scaling

Proof:

Shear: $$\cancel{S}T_1 = \begin{bmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (general case)

Uniform Scale: $(a = b)$ $$T_2 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (general case)

$$T_1 \cdot T_2 = \begin{bmatrix} a & bs & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T_2 T_1 = \begin{bmatrix} a & as & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $a = b$, $T_1 \cdot T_2 = T_2 \cdot T_1$

Therefore, they are commutative.

(5). Rotation, non-uniform scaling.

Proof: let Rotation matrix, $R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

non-uniform scaling, $S = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$RS = \begin{bmatrix} a\cos\theta & -b\sin\theta & 0 \\ a\sin\theta & b\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$SR = \begin{bmatrix} a\cos\theta & -a\sin\theta & 0 \\ b\sin\theta & b\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since in general, $a \neq b$.

$\Rightarrow RS \neq SR$.

$\therefore$ They are not commutative.

C. (1).

$$M = T \cdot R \cdot Sh \cdot S \cdot Re$$

T: translation
R: rotation
Sh: shear
S: scale
Re: reflection.

C. (2). Proof:

~~Assume~~ let $\vec{H}$, $\vec{P}$ be an invertible affine transformation matrix and a point on the triangle before transformation.

i.e. $H = \left[\begin{array}{c|c} \breve{A} & \vec{t} \\ \hline 0 \cdots 0 & 1 \end{array}\right]$, $\vec{P}$ satisfy $T(\alpha,\beta) = \vec{P_1} + \alpha\vec{d_1} + \beta\vec{d_2}$

let $\vec{P'} = H \cdot \vec{P}$, i.e. the point after transformation.

then $\vec{P'}$ is on $T'(\alpha,\beta) = H \cdot T(\alpha,\beta)$

$$= H\vec{P_1} + \alpha H\vec{d_1} + \beta \cdot H\vec{d_2}$$
$$= \vec{P_1'} + \alpha \cdot \vec{d_1'} + \beta \vec{d_2'}$$

where $\vec{P_1'}$ is $\vec{P_1}$ after transformation and $\vec{d_1'}$, $\vec{d_2'}$ are vectors $\vec{d_1}$, $\vec{d_2}$ after transform.

Since α, β are scalar and remain unchanged, $\begin{cases} \alpha+\beta \leq 1 \\ \alpha \geq 0 \\ \beta \geq 0 \end{cases}$

$\Rightarrow$ $T'(\alpha,\beta) = \vec{P_1'} + \alpha\vec{d_1'} + \beta\vec{d_2'}$ still represents a triangle.

Q.E.D.

8.

Part c.

(3) Affine transformations which involve translations can't be represented by cartesian coordinations. In any other case, it can.

Main advantage:

1) the last column in the general form represents the coordinates of the origin after transformation while cartesian form doesn't.

2). Additional projective transformation can be composed on this general form of affine transformation whereas Cartesian form can't.