Deep Optimal Stopping

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Introduction

This report reviews the application of deep optimal stopping for a Bermudan max-call option, following the paper of Becker, Cheridito, and Jentzen outlined in section 1. The primary objective of the paper is to obtain the lower and the upper bounds of optimal stopping values for various aspects. The principal problem solved by this work is to predict the stopping rule of a Bermudan option by employing deep learning techniques.

The Bermudan option is one of the flexible financial structures, and its holders can exercise it at several discrete time instants; thus, optimal stopping is essential for achieving maximum payoff. The idea behind the deep learning technique is to provide a set of rules that helps to find samples of financial market decision-making based on approximating the efficient stopping rule.

Methodology

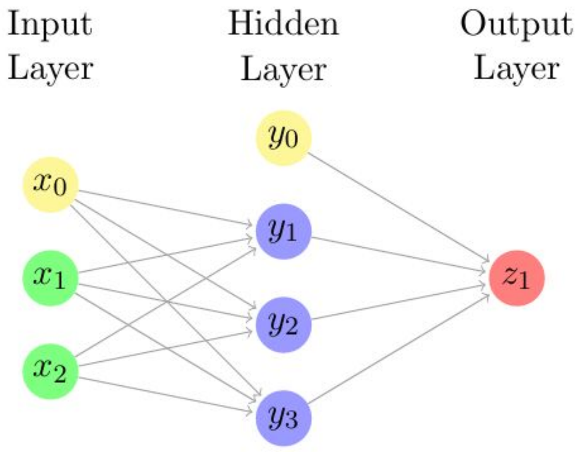
The fundamental approach involves using a neural network that helps state whether to continue or pause at different times. The optimal stopping problem is to determine the best time (τ) to stop a process to maximize a reward or minimize a loss: $sup_{\tau} E[g(\tau , X_{\tau})]$

The asset paths, indeed, are modeled through the Black-Scholes model. $X_t^i = (r - \delta_i)dt + \sigma_i dW_t^i$

This model also uses geometric Brownian motion for the underlying assets. Multiple paths are created for each dimensional value (equal to the number of assets) with a constant number of exercise opportunity points "N." The Black-Scholes model is Tractable and encompasses some significant characteristics of stock prices.

Neural Network Architecture

The proposed method used a feedforward neural network to approximate the optimal stopping rule. The neural network architecture consists of:



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$$\mathcal{L}(\theta) = -\mathbb{E}[R(\tau) \cdot p(\tau;\theta)] + \lambda \cdot \mathbb{E}[p(\tau;\theta)]$$

Training and Evaluation

It had been trained with the asset paths with an **8192** batch size, and then the more extensive set of paths comprising **4096000** was used to get a converging result. The training was carried out for **3000** epochs to improve the visualization and make the model learn properly. As for stopping and continuation values, standard evaluation procedures were used to approximate the option value lower **(L)** and upper **(U)** bounds.

$$\hat{L} = \frac{1}{K_L} \sum_{k=1}^{K_L} g(l^k, y^k_{l^k})$$

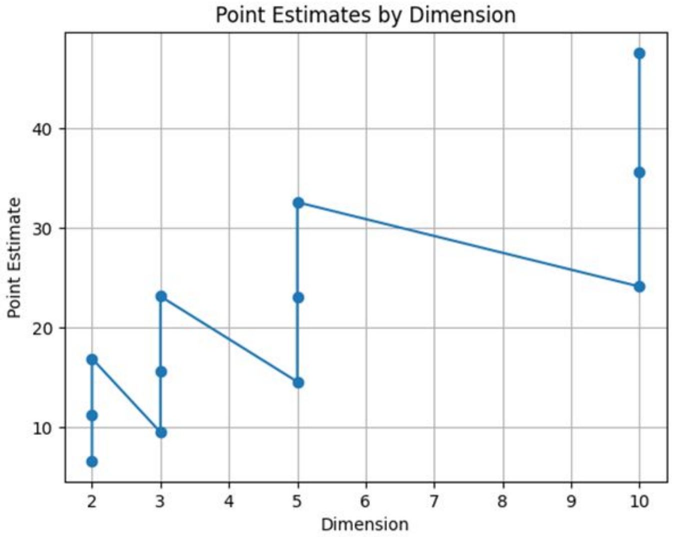
$$L = E[g(\tau_\Theta, X_{\tau_\Theta})]$$

$$\hat{U} = \frac{1}{K_U} \sum_{k=1}^{K_U} \max_{0 \le n \le N} \left(g\left(n, z_n^k\right) - M_n^k \right)$$

$$U = E\left[\max_{0 \le n \le N} \left(g(n, X_n) - M_n - \epsilon_n \right) \right]$$

Numerical Result

It depicts Point Estimates of the Bermudan max-call option value by increasing Dimensions. From the above trend, an increase in the dimensionality of both matrices is accompanied by significant fluctuations in the point estimates used in estimating the optimal stopping value. In particular, higher dimensions imply more significant point estimates, which suggests that the stopping strategy is more challenging to assess.



	Dimension	s0	L Estimate	U Estimate	Point Estimate	Time (seconds)	95% CI
0	2	90	6.614669	6.650757	6.632713	270.896322	[6.618654478372294, 6.646771455263947]
1	2	100	11.195805	11.212377	11.204091	220.707574	[11.185565226070159, 11.222616681008153]
2	2	110	0.000000	16.923519	8.461760	267.904354	[8.438905138334848, 8.484614010623561]
3	3	90	9.537443	9.538170	9.537806	224.768852	[9.521505073358684, 9.554107545017152]
4	3	100	15.667356	15.667356	15.667356	225.504699	[15.646483068064892, 15.688228757924346]
5	3	110	22.892173	23.150432	23.021302	271.003745	[22.99618536704136, 23.0464188243426]
6	5	90	14.808115	14.591049	14.699582	230.068375	[14.680598591840152, 14.718565513730766]
7	5	100	23.060268	23.060268	23.060268	274.698615	[23.036994694654076, 23.083541904717336]
8	5	110	32.689643	32.575764	32.632704	275.010822	[32.60579965836142, 32.65960737073617]
9	10	90	24.048309	24.135179	24.091744	288.646266	[24.070143557615125, 24.113344622280394]
10	10	100	35.575934	35.575934	35.575934	285.831714	[35.55100248762382, 35.600864659501326]
11	10	110	47.563978	47.563978	47.563978	243.863457	[47.53626567901536, 47.591690946919414]

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Processing Asymmetric Case - Dimension 2, s0 90
Asymmetric - Dimension 2, s0 90 - L: 12.4107, U: 12.3956, Point Estimate: 12.4031, Time: 2.90s
Processing Asymmetric Case - Dimension 2, s0 100
Asymmetric - Dimension 2, s0 100 - L: 16.7908, U: 16.7945, Point Estimate: 16.7927, Time: 2.95s
Processing Asymmetric Case - Dimension 2, s0 110
Asymmetric - Dimension 2, s0 110 - L: 22.3149, U: 22.3783, Point Estimate: 22.3466, Time: 2.89s
Processing Asymmetric Case - Dimension 3, s0 90
Asymmetric - Dimension 3, s0 90 - L: 16.7419, U: 16.7264, Point Estimate: 16.7342, Time: 4.05s
Processing Asymmetric Case - Dimension 3, s0 100
Asymmetric - Dimension 3, s0 100 - L: 23.1318, U: 23.1049, Point Estimate: 23.1184, Time: 4.06s
Processing Asymmetric Case - Dimension 3, s0 110
Asymmetric - Dimension 3, s0 110 - L: 30.9645, U: 30.9399, Point Estimate: 30.9522, Time: 4.16s
Processing Asymmetric Case - Dimension 5, s0 90
Asymmetric - Dimension 5, s0 90 - L: 24.8499, U: 24.8794, Point Estimate: 24.8647, Time: 6.20s
Processing Asymmetric Case - Dimension 5, s0 100
Asymmetric - Dimension 5, s0 100 - L: 34.2091, U: 34.1534, Point Estimate: 34.1812, Time: 6.16s
Processing Asymmetric Case - Dimension 5, s0 110
Asymmetric - Dimension 5, s0 110 - L: 44.7107, U: 44.6745, Point Estimate: 44.6926, Time: 6.19s
Processing Asymmetric Case - Dimension 10, s0 90
Asymmetric - Dimension 10, s0 90 - L: 81.3918, U: 81.6032, Point Estimate: 81.4975, Time: 11.03s
Processing Asymmetric Case - Dimension 10, s0 100
Asymmetric - Dimension 10, s0 100 - L: 99.5927, U: 99.5714, Point Estimate: 99.5820, Time: 11.01s
Processing Asymmetric Case - Dimension 10, s0 110
Asymmetric - Dimension 10, s0 110 - L: 117.9561, U: 117.9628, Point Estimate: 117.9594, Time: 11.47s
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This table offers an overview of the model's performance in determining the optimal stopping rules for a Bermudan max-call option specified by Dimension and the initial asset values s0. It presents the Point Estimate for the option values and the Lower Bound (L Estimate) and Upper Bound (U Estimate). Also, the table contains the Time (seconds) of training and evaluation so the readers can judge the computational complexity for each case. The confidence level together is presented as a 95% Confidence Interval (95%CI), which shows how close each point estimate is to the actual value. Depending on chosen dimensions and asset values, the results show that the model can estimate option values in increasingly complex options while using more advanced features and uncover potential difficulties where the model would find it difficult to decide between the stop and continue. The L and U estimates are also usually close to each other, especially when p is large, so it is possible to suspect the limitations of the model learning that can be further enhanced.

Symmetric case:

S0=90

D	Actual	Estimate
2	8.074	6.633
3	11.287	9.538
5	16.644	14.670
10	26.240	24.092

Asymmetric case :

S0=90

D	Actual	Estimate
2	14.339	12.403
3	19.091	16.734
5	27.662	24.865
10	85.987	81.498

Reference:

Becker, Sebastian, et al. “Deep Optimal Stopping.” *Journal of Machine Learning Research*, vol. 20, no. 74, Apr. 2019, pp. 1–25. https://doi.org/10.3929/ethz-b-000344707.

Pizante, V. (2019). Discussion on Deep Optimal Stopping. *Github*. https://github.com/vanessa-pizante/Deep-Optimal-Stopping/blob/master/DOS%20-%20Report.pdf