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On Peirce's Notation for the Logic of Relatives

In 1870 C. S. Peirce published his first major contribution to the theory of relations, a paper entitled "Description of a Notation for the Logic of Relatives, resulting from an Amplification of the Conceptions of Boole's Calculus of Logic".[1] This paper, referred to here as NLR, treats of a number of novel topics. Not only is it intended to cover relations of any degree, but also to include numerical algebra, to allow the formulation of concepts and theorems from the differential calculus and eventually to contribute to the foundations of geometry via the notion of a logical quaternion. Nevertheless, NLR has not yet received the attention it deserves; a fact due largely to the obscurity of Peirce's exposition. The aim of the present paper is to clear the way for further study of NLR by clarification and interpretation of his terminology and notation.

For present purposes NLR can be divided into three parts. Firstly, there is a section entitled "General Definitions of the Algebraic Signs"; here the signs denoting (logical) relations between and operations on logical terms are introduced. Despite the title of this section these notions are not defined; rather, purely formal conditions are given which must be satisfied by the operations denoted by the signs. Secondly, there is a section entitled "Application of the Algebraic Signs to Logic"; this section gives the interpretation Peirce assigns to his algebraic signs. Thirdly, there is the rest of the paper in which various results are deduced concerning the interpreted algebraic signs. The present paper is concerned primarily with the first two parts of NLR. The best way of describing Peirce's method of introducing his notation is to say that he offers a formal axiom system (he speaks of "imperative conditions") but does not deduce purely formal results. Instead, an interpretation is immediately assigned to the axioms and the rest of the paper proceeds within this interpretation. Peirce has an unfortunate way of presenting his imperative conditions as though they are definitions; as a consequence there are sometimes discrepancies between the introduction of an algebraic sign and its later interpretation. As regards the algebraic signs:

there are four kinds of multiplication, two kinds of addition and two kinds of involution, as well as operations inverse to these and a variety of special symbols. The algebraic signs will be discussed in the order named, special symbols will be introduced where appropriate.

At the time of writing NLR Peirce was firmly committed to the subject-predicate theory of proposition. This led him to treat relative terms on the same footing as absolute terms: as substantives rather than verbs.[2] Thus he speaks of the logic of *relatives* rather than the logic of *relations*. For the interpretation of NLR it is necessary to make a careful distinction between these two concepts. Verbs, like "loves", are relational terms; nouns, like "lover", are relative terms. If a relation is regarded as a class of ordered pairs, then its domain is a relative and so is its range; the elements of the domain or range may be called *individual* relatives. Thus a lover is an individual relative, the class of lovers is a relative and the class of pairs A:B such that A loves B is a relation. In general Peirce fails to distinguish satisfactorily between individuals (whether individual relatives or not) and classes of these individuals, nor does he draw an explicit distinction between relatives and relations. He divides logical terms into three categories: absolute terms, (simple) relative terms and conjugatives. Absolute terms ("horse", "man") are intended to denote classes of individuals which are not individual relatives. Relative terms ("father", "lover") are, on the whole, intended to denote relatives — that is, classes of dyadic individual relatives. Conjugatives ("giver", "buyer") are similarly intended to denote classes of triadic or higher degree individual relatives. The interpretation of relative terms and conjugatives as denoting relatives rather than relations is an intention more than an accomplished fact. Peirce frequently confuses relatives and relations; whenever his approach (of letting relative terms denote relatives rather than relations) runs into trouble he unhesitatingly provides a relational interpretation instead. The fact that an individual may be an individual *relative* in one respect (e.g. as a father) but not in another respect (e.g. as a man) makes the distinction between classes and relatives a subtle and rather tenuous one. Small wonder therefore that Peirce sometimes takes refuge in a relational interpretation of his relative terms.

In NLR logical terms belonging to different categories are distinguished by using different type-settings; yet another kind of type is used for variables ranging over numbers. No attempt will be made here to imitate these different type-settings; instead, the category to which a logical term belongs will be made clear in context. Peirce uses a number of standard logical terms in his examples, those used here are:

b: black	f: Frenchman	h: horse
m: man	w: woman	
l: lover	*m*: mother	s: servant

g: giver to — of —

Individuals, where distinguished, Peirce denotes by ordinary capital letters. No notion corresponding to that of individual variable appears in his paper, but there are variables x, y, z. . . . which denote classes of individuals, whether individual relatives or not. It is noteworthy that the distinctive typesettings are not used for variables as well. Peirce's algebraic signs are reproduced here; variations in special symbols will be pointed out. Operations will be interpreted set-theoretically; in these interpretations classes are denoted by X, Y, Z. . . . and relations by R, S, T. . . . Relative multiplication is indicated by a semicolon, complementation by an accent, the power set of X by P(X), the empty set by $\curlyvee\!\!\curlywedge$ and X-Y abbreviates $X \cap Y'$.

Peirce employs the usual logical relations: inclusion (denoted by $\prec$), proper inclusion ($<$) and identity ($=$). The imperative conditions on these signs are the transitivity of $\prec$ and the fact that $<$ and $=$ are definable in terms of it (3.47-49). Peirce interprets $\prec$ as follows:

> Thus
>
> $\mathrm{f} \prec \mathrm{m}$
>
> means "every Frenchman is a man", without saying whether there are any other men or not. So,
>
> $m \prec \mathrm{l}$
>
> will mean that every mother of anything is a lover of the same thing . . . . (3.66)

This passage shows how Peirce borrows from the logic of relations. If *m* and l are relatives then

$m \prec \mathrm{l}$

should mean that every mother is a lover, regardless of whom she is mother of or lover of. Instead, the interpretation is relational.

Of the four kinds of multiplication the most fundamental is relative multiplication, indicated by the juxtaposition of letters. It is introduced under the imperative conditions of being associative and doubly distributive with respect to addition; its interpretation is:

> I shall adopt for the conception of multiplication *the application of a relation*, in such a way that, for example, lw shall denote whatever is lover of a woman. (3.68)

The identity relation is a unit with respect to relative multiplication; Peirce denotes it by a heavily printed numeral one (3.68); it will be denoted here by the symbol "*1*". The introduction of this term again shows how Peirce borrows from the logic of relations: as a relative *1* would coincide with the universal class. Note that Peirce feels free to use an absolute term as the multiplicand in a relative product: if R is a relation and X a class then

$$R;X = \{x \mid (\exists y)[\langle x,y\rangle \in R \ \&\ y \in X]\}$$

Coupled with the distinction between absolute and relative terms this leads to the conclusion that Peirce's is a multisorted system: its operations can combine terms belonging to different categories.

The remaining three kinds of multiplication are all introduced as variants of relative multiplication. The first of these is indicated by a comma and is introduced as though it is the special case where relative multiplication is commutative (3.53). A term of the form

x,y

must therefore be interpreted not as two terms combined by an operation which is denoted by a comma, but as the relative product of two terms, the first of which is modified in some way, this modification being indicated by the comma. The intended interpretation of the comma product is set-theoretical intersection (3.74), an operation applicable equally to absolute and relative terms. It was pointed out above that for Peirce the multiplicand in a relative product may be an absolute term; in order to introduce intersection of absolute terms as a variant of relative multiplication he must devise some means whereby the multiplier in a relative product can be an absolute term as well. This is where the modification of the multiplier comes in: if it is an absolute term the comma indicates its conversion to a relative term. This conversion is effected by, almost literally, leaving room for some qualification of the term. Thus "man" becomes "man that is —", which is denoted by "m," and is regarded as a relative term. Filling the gap in such a term ("man that is black") is then, on the face of it, very similar to the application of a relation ("servant of a black"); close enough to satisfy Peirce at any rate. Since a man that is black is also a black that is a man, which is a black man, this multiplication is commutative and coincides with the intersection of classes.

Surprisingly enough Peirce's method is fundamentally sound. Given a class X (the class of men, for example) it can be converted into a relation X_i by considering it as a subrelation of the identity relation:

$$X_i = \{<x,x> \mid x \epsilon X\}$$

This conversion corresponds to Peirce's: in $<x,x>$ regard the first component as the bearer of the predicate defining the class (manhood) and the second as the bearer of an unspecified predicate, then X_i corresponds to "man that is —". And the relative multiplication of such a relation with a class yields the intersection of the two classes:

$$\begin{aligned} X_i;Y &= \{x \mid (\exists y)[<x,y> \epsilon X_i \ \& \ y \epsilon Y]\} \\ &= \{x \mid (\exists y)[x \epsilon X \ \& \ x = y \ \& \ y \epsilon Y]\} \\ &= \{x \mid x \epsilon X \ \& \ x \epsilon Y\} \\ &= X \cap Y \end{aligned}$$

There is another method of converting classes into relations, also introduced by Peirce (3.220 and 3.311). For a class X let

$$X' = \{<x,y> \mid x \epsilon X\} \quad \text{and} \quad 'X = \{<x,y> \mid y \epsilon X\}$$

then $X';'Y = X' \cap 'Y$. But this yields the intersection not of the classes but of their representations.

For absolute terms the conversion to a relative term is necessary in order to use relative multiplication. For relative terms there is no such necessity, yet, in order to be consistent in his claim that intersection is a special case of relative multiplication, Peirce is obliged to apply his method to relative terms as well. Accordingly, "lover" becomes "lover that is —", the idea being that this gives l an extra correlate, so that "l," denotes a conjugative. As with absolute terms, filling the gap yields intersection: a lover that is a servant is also a servant that is a lover and is both lover and servant. On the other hand, since l, is really triadic Peirce can claim, as he must, that this is consistent with his primary notion of relative multiplication as the application of a relation. For,

$$l{,}sw$$

can now be interpreted as the application of the conjugative l, to the two terms s and w (lover of a woman that is a servant of that woman), instead of the application of the relative l,s to the term w (lover-and-servant of a woman). There is an ambiguity here: in isolation l,s must denote a relative term, but upon relative multiplication of this term with another the whole must be interpreted as the result of the application of a conjugative.

It was claimed earlier that there are discrepancies between the introduction of some operations and their subsequent interpretation; the comma product is a case in point. Its introduction as *nothing but* commutative relative multiplication is at variance with its interpretation as intersection. For relations the identification of intersection with commutative relative multiplication is manifestly false. As regards relatives this identification is closer to the mark: for relatives it holds in general that

$$xy \subseteq x,$$

hence, from the premiss $xy = yx$ it follows that

$$xy \subseteq x \cap y.$$

But the converse inclusion does not follow, hence the identification of xy and $x \cap y$ is not valid.

The third kind of multiplication is introduced as "invertible" multiplication; discussion of this operation is postponed until after the discussion of addition. The fourth kind of multiplication is called "functional", it is introduced as follows:

> Functional multiplication is the application of an operation to a function. It may be written like ordinary multiplication; but then there will generally be certain points where the associative principle does not hold. (3.55)

Non-associativity appears to be the only reason for distinguishing functional from ordinary relative multiplication; the only example Peirce offers of functional multiplication in the logic of relatives is multiplication by a conjugative (3.72). The two terms

$$gl(sw) \qquad \text{and} \qquad g(ls)w$$

do not in general denote the same entity: according to the first a servant of a woman was given to a lover, according to the second a woman was given to a lover of a servant. Since the juxtaposition of letters indicates relative multiplication, which is associative, this "non-associativity" must be accounted for in some way and Peirce does so by distinguishing multiplication by a conjugative as a special kind of relative multiplication.

In multiplication by a conjugative the various correlates must be distinguished from each other, and Peirce gives two methods of doing so: indicating the correlates by numerical subscripts (3.69) and using typographical symbols as marks of reference (3.71). The first method is used later to define some special symbols; it is illustrated by the example:

$$g_{12}\, l_1 wh \qquad g_{11}\, l_2 hw \qquad g_{2-1} h l_1 w$$

A letter has as many subscripts as the relative it denotes has correlates; the first subscript to a letter shows how many letters to the right the first correlate stands, the second how many further on the second correlate stands and so on. This can be done in various ways by permuting the letters and changing their subscripts. Thus each of the expressions above denotes the relative consisting of all those who gave a horse to a lover of a woman. By pursuing this method Peirce arrives at two special symbols. Adding "O" as a subscript to a relative term indicates that the term is its own correlate: "l_o" denotes a lover who loves himself, a self-lover. (This according to Peirce. In fact, "l_o" should denote a *class* of self-lovers.) Adding "∞" as a subscript to a relative term indicates that the term has a correlate, but this correlate is "indeterminate": "l_∞" denotes a lover of some indeterminate thing, a lover of something. (Here again Peirce confuses individuals and classes.) Now add "O" as a subscript to the special symbol "*1*", then "$\mathit{1}_o$" must denote a self-identical thing. But anything is identical with itself, and so, Peirce concludes, "$\mathit{1}_o$" denotes "anything" (3.73). Similarly, "$\mathit{1}_\infty$" denotes something that is identical with some undetermined thing, hence it denotes "something" (3.73). Finally, $\mathit{1}_o$ is denoted by the special symbol "1" and $\mathit{1}_\infty$ by the special symbol "I". If these notions seem unduly obscure it should be kept in mind that Peirce was working in the period before quantificational logic. The symbol "I" embodies one of his many attempts to express particular propositions.[3] What he is groping for is the existential quantifier, a notion he only hit upon years later. The present notation is clumsy since it does not allow any means of referring back to that "something" identified by I; Peirce had to use typographical symbols to accomplish this purpose. As regards 1, from the way Peirce arrives at this symbol it is already clear that its intended interpretation is the universal class. This is confirmed by the remark: "Boole's unity is my 1, that is, it represents whatever is " (3.74).

The use of 1 shows up very clearly Peirce's confusion of relatives and relations. From the following three results of his:

$x,y = y,x$	((9) of 3.81)
$x,1 = x$	((47) of 3.86)
If $b,x = a$ then $a \prec b$	((88) of 3.91)

it follows easily that:

$$x \prec 1$$

Since x can be a relative term, this says that whatever is denoted by a relative term must be contained in the universal class. The most natural interpretation of "universal class" in Peirce's case is the class of all individuals. If this is accepted then it follows that relative terms must denote relatives, not relations, since a relation cannot be contained in a class of individuals. On the other hand, from the relational interpretation of $\prec$ it may be argued that "universal class" must be understood in the modern sense of containing the universal relation as a subset.[4] In this case a relative term might well denote a relation. However, the latter argument posits an unlikely degree of sophistication on Peirce's part.

The two kinds of addition Peirce employs are denoted, respectively, by the signs "+," and "+". The former is introduced under the imperative conditions of associativity and commutativity (3.51); it is clear that its intended interpretation is set-theoretical union (3.67). The sign itself, together with this interpretation, is carried over from two earlier papers of Peirce's ([4] and [5]). So is the special symbol "O", which denotes a class that "does not go beyond any class, that is *nothing* or nonentity" (3.5). In NLR O is introduced by the condition:

$$x +, O = x \tag{3.67}$$

It is also defined in terms of + (3.82). This second kind of addition is introduced as "invertible" addition (3.52). Its imperative conditions are associativity, commutativity and the fact that the inverse operation is "determinative". That is:

$$\text{If } x + z_1 = y \quad \text{and} \quad x + z_2 = y \text{ then } z_1 = z_2$$

The interpretation Peirce assigns to + is highly interesting; its explication requires a short digression concerning the algebra of Boole.

Boolean algebra in its present form is not just a precise version of Boole's original system; it is rather the outcome of simplifications effected by Boole's successors. This is pointed out in [2], where the question of what Boole's original system really is about is treated at length. The laws regarding those operations Boole actually uses are collected together and viewed as an axiom system, the aim being to find out what kind of logical structure this axiom system formalizes. It turns out that sense can be made of Boole's system by taking the basic notion to be that of a *heap* rather than a class. A heap is a collection of objects in which more than one example of an object may occur, such as the set of roots of a polynomial equation in which multiplicities are counted. Hence the notion of a heap is an extension of the notion of a class: elements may be counted more than once. A heap can be represented notationally by attaching a numerical coefficient to each of its elements, this coefficient giving the multiplicity of that element in the heap. Operations (addition and multiplication) on heaps are then definable from operations on these coefficients; by allowing the coefficients to take on negative as well as positive values one obtains the notion of a *signed* heap. The axiom system for Boole's original system is an axiom system for signed heaps, and the structure it formalizes is that of a commutative ring with unit and no nonzero nilpotents. For Boole, a sum $x + y$ is "interpretable" only in case the collections represented by x and y have no members in common. Nevertheless, he does not shrink from using "uninterpretable" sums like $x + x$ in the intermediary steps of a deduction, as long as both the original data and the end result are "interpretable". Such a method is justifiable by the embeddability of one domain of elements in another — for example, the natural numbers in the reals — and this is the role played by the notion of a heap. In the class calculus Boole's terminology of "interpretable" and "uninterpretable" is accurate; upon embedding classes into heaps even the "uninterpretable" expressions become interpretable. Thus, if x is a class, $x + x$ is a heap consisting of all the elements of x, each of these elements being counted twice. Note that if x and y are disjoint classes their heap addition coincides with their union.

To get back to the interpretation of Peirce's sign $+$: the operation corresponding to this sign is precisely the operation of heap addition. This can be seen as follows. An examination of Peirce's use of this sign shows that the operation corresponding to it must conform to three requirements. Firstly, it must allow the use of the form

$$1 + x.$$

Such terms occur most prominently in Peirce's treatment of "infinitesimal" relatives (3.100 ff.); it is clear that they are intended to be interpretable and should not reduce either to 1 or to x. Secondly, the operation represented by "$+$" must satisfy the formula

$$x +, y = x + y - x,y \qquad ((24) \text{ of } 3.81)$$

where the minus sign represents the operation inverse to $+$ (3.57). Thirdly, a sum of the form

$$x + x$$

should not be a logical term (3.75). By a "logical term" Peirce means anything which falls under one of his three categories; in particular then such a sum should not denote a class, even when x is a class. These three requirements rule out any standard interpretation of $+$. Heap addition, however, satisfies all three. The sum $1 + x$ consists of all elements of 1 with the elements of x counted twice. In $x + y$ all elements belonging to x,y are counted twice, subtracting these elements once yields $x +, y$. (Subtraction will be discussed further along with the other inverse operations.) Thirdly, the sum $x + x$ consists of all elements of x, each counted twice. This is a heap but not a class, hence not a logical term in Peirce's sense. Besides the fact that heap addition fits the requirements of Peirce's sign + there is also some direct evidence that this is the interpretation he had in mind: he speaks of the elements of such a sum being "counted over twice" (3.104) and in an earlier paper he spoke of elements being "taken into account twice over" (3.3). Heap addition is indeed not newly introduced in NLR. Already in [4] Peirce gives the formula characterizing heap addition:

$$a + b = (a +, b) + (a,b) \qquad ((28) \text{ of } 3.13)$$

But the notions of a heap and of heap addition are not well developed. For Peirce a heap is something arising from the "arithmetical" (i.e. heap) addition of two classes. The addition of heaps which are not classes is not explicitly considered. The simplicity of Peirce's notion of heap addition lends support to the contention that he thinks of himself as dealing with relatives rather than relations. In $x + y$, x and y may be relative terms. If these denote relatives then heap addition is immediately applicable. If they were to denote relations, however, this would require of Peirce that he was not only aware that a relation may be regarded as a class of ordered pairs, but also that he was prepared to allow multiplicities of ordered pairs. It is, finally, an interesting question whether Peirce, on reading Boole, interpreted Boole's addition as heap addition, or whether he thought of these two operations as being distinct.

It remains to discuss that multiplication introduced as "invertible" and indicated by a dot (3.54). Its only imperative condition is that the corresponding inverse operation must be "determinative". The interpretation of this multiplication comes in a passage which should be quoted in full.

> The sum $x + x$ generally denotes no logical term. But $x,\infty + x,\infty$ may be considered as denoting some two x's. It is natural to write

$$x + x = 2.x$$

and $$x, \infty + x, \infty = 2.x, \infty$$

where the dot shows that this multiplication is invertible. We may also use the antique figures so that

$$2.x, \infty = 2x$$

just as $$1 \infty = \mathrm{I}$$

Then 2 alone will denote some two things. (3.75)

The term 2.x is thus seen to be interpreted as "twice-x": all elements of x counted or considered twice. The interpretation of "invertible" multiplication, therefore, is that it abbreviates the repeated application of heap addition. "Invertible" multiplication will be referred to here simply as the *dot-product*; it should be noted that the dot product differs from heap multiplication. Since Peirce considers heap operations only as applied to classes it is not surprising that he did not come up with heap multiplication: the heap multiplication of two classes yields their intersection. The dot product is also used in connection with individuals. From the earlier discussion of the comma and the sign ∞ it is clear that "x, ∞" means "an x that is some undetermined thing", which is shortened to "some x". Adding (by $+$) two such x's then yields "some two x's" and by analogy this is abbreviated to "$2.x, \infty$". Again, this usage is extended to introduce further special symbols: an antique numeral denotes that number of unspecified things.

The interpretations of "invertible" addition and multiplication are further examples of the discrepancy sometimes occurring between the way in which Peirce introduces his signs and their interpretation. As introduced, "invertible" addition would seem to be defined as a special case of logical addition ($+,$) and "invertible" multiplication as a special case of relative multiplication. Yet their interpretations (as heap addition and the dot product) are different operations altogether. An editorial footnote to the introduction of the dot product as "invertible" multiplication identifies this operation with a kind of multiplication which Peirce had introduced in [5] as "arithmetical" multiplication (3.21). This identification should be ignored: the operation in question is concerned with events and their probabilities and, unlike the dot product, has nothing to do with heap addition.

The two remaining operations are (ordinary) involution and backward involution. Ordinary involution is indicated by x^y, its imperative conditions are (3.56):

$$(x^y)^z = x^{(yz)} \qquad\qquad x^{y+,z} = x^y,x^z$$

$$(x +, y)^z = x^z +, \Sigma_p x^{z-p},y^p +, y^z$$

The last formula is called the *binomial theorem;* the index p ranges over all non-empty proper subsets of z, the capital sigma indicates repeated logical addition. Peirce interprets involution as follows:

> I shall take involution in such a sense that x^y will denote everything which is an x for every individual of y. (3.77)

Thus, for example, l^s is interpreted as "lover of every servant". Besides giving the interpretation of involution the quotation confirms that Peirce is dealing with relatives: in x^y x must be and y may be a relative term; x^y then denotes a class of *individuals*, each of which is an x of every *individual* that is a y. Further confirmation comes from the fact that Peirce sometimes employs terms of the form

$$I^y$$

(3.100, for example) which make sense only if the "something" identified by I is an individual relative.

For relations R and S the operation corresponding to Peirce's involution is defined by:

$$R^S = \{ \langle x,y\rangle \mid (\forall z)[\langle z,y\rangle \epsilon S \Rightarrow \langle x,z\rangle \epsilon R]\} = (R';S)'$$

Under this definition most of Peirce's results concerning involution are provable for relations. In particular the relational analogue of Peirce's binomial theorem is provable. This is:

$$(R \cup S)^T = R^T \cup \bigcup_{\Lambda C X C T}[R^{T-X} \cap S^X] \cup S^T$$

Or, in simplified form:

$$(R \cup S)^T = \bigcup_{X \epsilon P(T)}[R^{T-X} \cap S^X]$$

The proof requires the following

Lemma: For any relations R, S and T it holds that

$$(R \cap S);T = \bigcap_{X \epsilon P(T)} [R;(T-X) \cup S;X]$$

Proof: $R;(T-X) \cup S;X = [R \cap (S \cup S')];(T-X) \cup [(R \cup R') \cap S];X$
$= (R \cap S);T \cup [(R \cap S');(T-X) \cup (R' \cap S);X]$

On both sides of this equation take the intersection over $X \epsilon P(T)$. Since $(R \cap S);T$ is independent of X this yields

$$\bigcap_{X \epsilon P(T)} [R;(T-X) \cup S;X] = (R \cap S);T \cup \bigcap_{X \epsilon P(T)} [(R \cap S');(T-X) \cup (R' \cap S);X]$$

In order to prove the Lemma it is necessary and sufficient to show that the last term on the right of this equation is empty. So suppose that

$$<a,b> \epsilon \bigcap_{X \epsilon P(T)} [(R \cap S');(T-X) \cup (R' \cap S);X]$$

Then, for every X in P(T), there is some element c such that:

$[<a,c> \epsilon (R \cap S') \;\&\; <c,b> \epsilon X'] \vee [<a,c> \epsilon (R' \cap S) \;\&\; <c,b> \epsilon X]$
and $<c,b> \epsilon T$

Now let $Y = \{<u,v> \mid v = b \;\;\&\;\; <a,u> \epsilon (R \cap S')\}$
and let $Z = Y \cap T$

Then Z is in P(T) and hence there is some element c such that:

$$[<a,c> \epsilon (R \cap S') \;\&\; <c,b> \epsilon Z'] \vee [<a,c> \epsilon (R' \cap S) \;\&\; <c,b> \epsilon Z] \quad (1)$$
and $<c,b> \epsilon T$ (2)

Suppose now that $<c,b> \epsilon Z$. Then the first disjunct of (1) is false, and hence

$$<a,c> \epsilon (R' \cap S) \;\;\&\;\; <c,b> \epsilon Z \quad (3)$$

From the second conjunct of (3) it then follows that $<a,c> \epsilon (R \cap S')$, which contradicts the first conjunct of (3).

So it must hold that $<c,b> \epsilon Z'$. The second disjunct of (1) is then false, and so

$$<a,c> \epsilon (R \cap S') \quad \& \quad <c,b> \epsilon Z' \tag{4}$$

From the second conjunct of (4) it follows that

$$<c,b> \epsilon Y' \quad \text{or} \quad <c,b> \epsilon T'$$

Of these alternatives the second is ruled out by (2), hence it holds that $<c,b> \epsilon Y'$. That is

$$b \neq b \quad \text{or} \quad <a,c> \notin (R \cap S')$$

Since the first alternative is false it must therefore hold that $<a,c> \notin (R \cap S')$. But this contradicts the first conjunct of (4). Q.E.D.

Given this Lemma the proof of the binomial theorem follows easily from the characterization of involution in terms of relative product. Here the Lemma proves the binomial theorem, but Peirce did it the other way round: the Lemma is one of the results he deduced from his binomial theorem ((132) of 3.116). The proof given here is a modification of a proof given by Schröder ([8], §29).

Backward involution is indicated by ${}^{x}y$, the discussion of this operation Peirce added in the process of printing, hence it is not introduced along with the other operations in the first part of NLR. For relatives ${}^{x}y$ is interpreted as denoting an x only of y's (3.112); for relations it can be defined by:

$$R_S = \{<x,y> \mid (\forall z)[<x,z> \epsilon R \Rightarrow <z,y> \epsilon S]\} = (R;S')'$$

Peirce does not define any of his involutions in terms of relative multiplication; that he is nevertheless well aware of the connections between these operations is shown by some of his results:

$l^s = 1-(1-l)s$ ((124) of 3.112)

${}^{l}s = 1-l(1-s)$ ((125) of 3.112)

$m^n = {}^{(1-m)}(1-n)$ (3.112)

The last result appears in a table exhibiting the relationships between complements, converses and involutions. The notion of converse receives scant treatment in the whole of NLR, probably because Peirce regarded it not as an operation on relatives but as being itself a special kind of relative (3.147). It is only used in the account of backward involution added in print.

For most of his operations Peirce also introduces an inverse operation; it remains to discuss these notions. Given a binary operation *, a (right) inverse operation is, strictly speaking, only definable if the equation

$$x^*z = y$$

always has a unique solution for z. That is, both the following conditions must hold:

$$(\exists z)[x^*z = y],\ \forall x,y \tag{1}$$
$$\text{If } x^*z_1 = y \quad \text{and} \quad x^*z_1 = y \quad \text{then} \quad z_1 = z_2,\ \forall x,y,z_1,z_2. \tag{2}$$

In this case the inverse operation $\div$ is defined by:

$$y \div x \overset{\text{def.}}{=} (7z)[x^*z = y] \tag{3}$$

If * is considered as a group operation then its inverse operation can be more simply defined in terms of inverse elements:

$$y \div x = x^{-1*}y$$

which then yields the same operation as (3). In his early logical work Peirce deviates from this standard view of inverses in two respects. Firstly, he does not regard (2) as a necessary condition for the definition of an inverse operation; instead, the general definition of an inverse operation is:

$$\text{If} \quad x^*z = y \quad \text{then} \quad y \div x = z \tag{4}$$

((10) of 3.5 and (13) of 3.6, for example). In case (2) holds the operation * is said to be "invertible", the inverse operation $\div$ is then "determinative" and has as value that unique z such that:

$$x^*z = y \tag{5}$$

If (2) does not hold the inverse operation is still defined (by (4)) but it is "indeterminative"' $y \div x$ can take on any one of the values z satisfying (5). Note that, if an "indeterminative" inverse operation is not to be entirely meaningless, the values z satisfying (5) should form some non-empty proper subset of the universal class. Clearly it would be advantageous to specify this set by delimiting the range of values that $y \div x$ can assume. In NLR Peirce does not do this, but since he takes the notion of invertibility for granted it may be assumed that delimitations of inverse operations obtained in earlier papers are still applicable.

The second point at which Peirce, in his early work, deviates from the standard view of inverses lies with the emphasis on inverse *operations* rather than inverse *elements*. The rationale behind this is the intended analogy of the logic of relatives with arithmetic: Peirce wants his operations on relatives to correspond as closely as possible to arithmetical operations (3.61). In his later work, where Peirce turned his attention to developing a general "algebra of logic" without any specific connections with arithmetic, the inverse operations do not appear. In fact, by 1883 Peirce was thoroughly dissatisfied with inverse operations and held the opinion that such operations have value only when introduced in terms of inverse elements — a view opposed by Schröder.[5]

The inverse operations will be discussed in the same order as the operations: multiplication, addition, involution. Since relative multiplication is not commutative it has both a left and a right inverse operation, which Peirce denotes, respectively, by:

$$x{:}y \quad \text{and} \quad \frac{y}{x} \tag{3.58}$$

Peirce fails to point out that these inverse operations are indeterminative (both for relatives and for relations), he therefore does not consider the question of delimiting the values over which they range. This question was taken up by Schröder, who showed that $R;X = S$ $\left(\text{that is, } X = \frac{S}{R}\right)$ iff:

(i) $S \subseteq R;(R\ \ ;S')'$, and
(ii) For some relation Y,
$X = (R^u;S')' \cap [Y \cup [R^u;[R;(R^u;S' \cup Y')']' \cap S]]$

This delimits the values of the right inverse operation; a similar solution holds for the left inverse operation.[6] No inverse operations are introduced for the other three kinds of multiplication. Since the comma product is equated with Boole's multiplication Peirce probably considered its inverse operation as being generally known. This operation, indicated by (say) "$\div$" is delimited by the condition (3.6):

$$X \subseteq X \div Y \subseteq X \cup Y'$$

The operation inverse to logical addition is indicated by "$\overline{}\!\!,$", this operation is indeterminative and delimited by the condition (3.5):

$$X \cap Y' \subseteq X \overline{,} Y \subseteq X$$

The operation inverse to heap addition is indicated by "—", this operation should be determinative. Since the inverse of heap addition is heap subtraction this must therefore be the interpretation of the minus sign. Heap subtraction is determinative as long as heaps are considered to be signed heaps; that is, coefficients are allowed to have negative as well as positive values. Heap subtraction shows Peirce's method of defining inverse operations to be unsatisfactory. He considers heap addition as an operation on classes; it then satisfies the condition (2):

$$\text{If } x + z_1 = y \quad \text{and} \quad x + z_2 = y \quad \text{then} \quad z_1 = z_2$$

and hence its inverse operation, heap subtraction of classes, should be determinative. That this is not the case is easily seen upon reflecting that heap addition and subtraction are extensions of numerical addition and subtraction: just as n-m is not defined for positive integers when $n \leqslant m$, so x-y is not defined for classes when $x \subseteq y$. That is, condition (1) fails for heap addition when this operation is applied only to classes: the uniqueness of an inverse element is guaranteed, but not its existence. Peirce's condition (4)

$$\text{If} \quad x + z = y \quad \text{then} \quad y - x = z$$

defines y-x in such a way that each element of the class x occurs at least once in the heap y. But if heap subtraction is to be determinative y-x should be defined even when both x and y are classes and $x \not\subset y$. The only way of ensuring this is to allow "negative occurrences" of elements, thus introducing the notion of a signed heap. It should be said that in some places Peirce does seem to be using such a notion (3.77, for example). But this is never explicitly acknowledged.

The use of the minus sign in NLR should not be confused with the operation of complementation. The complement of a set can be regarded as an inverse element (though not a group-theoretic one); since Peirce disregards inverse elements in favour of inverse operations he prefers to introduce complementation as a special application of the minus sign. Namely, in (heap) subtracting a class x from 1 the coefficients of x vanish, so that 1-x is the complement of x (3.106). Peirce also defines complementation in another way, in terms of involution and the special relative "not", denoted by "n" (3.64). The use of n does not amount to the direct introduction of complementation, for n denotes the diversity relation:

$$\langle a,b \rangle \in n$$

should be interpreted as "a is not the same as b" and not as "a is the same as not-b". Nor is n introduced as the complement of *1*, for to do so would defeat the purpose of defining complementation by using n. Given a class or relative x, an element a is not in x if and only if, for every element b in x, a is not the same as b. Hence the involution n^x yields the complement of x (3.137). For this characterization of complementation to be regarded as a definition the operation of involution must be a primitive notion; perhaps this explains why Peirce did not define involution in terms of relative product. Peirce's method can be used to effect a reduction in the number of primitive notions of the calculus of relations. Take the diversity relation N and involution as primitive notions, then:

$$R' = N^R \quad \text{and} \quad I = N' \quad \text{and} \quad R;S = ((R')^S)'$$

It is only rarely that Peirce treats of complements as such (that is, as inverse elements) in NLR. When he does he refers to "non-x" (3.84) or uses the notation "$\bar{x}$" (3.90).

Only the operations inverse to the two kinds of involution remain. For ordinary involution Peirce introduces a left and a right inverse operation (3.60), denoted respectively by:

$$\sqrt[x]{y} \quad \text{and} \quad \log_x y$$

Again, Peirce does not point out the indeterminateness of these operations; they can be delimited by using Schröder's delimitation result for relative multiplication. Peirce does not introduce any operations inverse to backward involution, the same considerations would apply here.

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NOTES

1. [6] in the list of references given below. Reference to Peirce's work is made by the numbered paragraphs in [7].
2. This is pointed out in [1].
3. A subject discussed in [1].
4. A remarks in [3] gives the impression that Peirce introduced the universal relation in NLR. This is not the case. At most his universal element 1 can be said to contain the universal relation as a subset.
5. [8], S.240, SS.260-61, S.175.
6. Ibid. §19 S.265.

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