Homework #5, due May 11, 2020

1. Consider the following 100×100 images f_{ij}. For the corrupted region D, include all (i, j) satisfying $1 \le i \le 10$ or $i \ge 90, i$ divisible by 2. Also, for pixels (i, j) satisfying $20 \le i \le 80$, include in D those that additionally satisfy $s + \sum_{r=0}^{s-1} r \le j < s + \sum_{r=0}^{s} r$, for some $s = 1, \ldots, 12$.

   Approximately solve TV diminishing flow to steady state in the corrupted region. Use initial intensities of 0.5 in the corrupted region, von Neumann boundary conditions at the boundary of the image, time stepsize $\Delta t = \frac{\epsilon}{4}$, with $\epsilon = 0.001$, and stop at iteration k satisfying
$$||f^{(k)} - f^{(k-1)}||_\infty = \max_{i,j} |f_{ij}^{(k)} - f_{ij}^{(k-1)}| \le 10^{-6}.$$

   Plot the final image. Also write down the calculated intensity at pixel $(50, 70)$ and the time step k satisfying the stopping condition.

   (a)
$$f_{ij} = \begin{cases} 0.75 & \text{, if } 0 \le i \le 50 \\ 0.25 & \text{, if } 50 < i < 100. \end{cases}$$

   (b) $f_{ij} = \frac{i+j}{198}$

2. Use the same method, with the same initial condition, boundary condition, time stepsize, ϵ, and stop condition, on the following images. For each image, add the following somewhere in your corrupted region:

   - three horizontal lines of three pixel height
   - five vertical lines of two pixel width
   - two rectangles
   - one circle

   Plot the initial image, with intensities 0.5 in the corrupted region, and the final image. Also write down the time step satisfying the stopping condition.

   (a) "rose.bmp"
   (b) "squid.bmp"
   (c) "oak.bmp"
   (d) "buildings.bmp"

3. For the image, corrupted region, and algorithm of Problem 1a:

   (a) Investigate whether smaller stopping condition will greatly affect the solution by trying instead stopping condition $\le 10^{-8}$. Comment on whether the final image looks different (if it does, plot it), and write down the time step satisfying the stopping condition.

(b) Investigate whether larger ϵ will greatly affect the solution by trying instead $\epsilon = 0.01$ with stopping condition $\leq 10^{-11}$. Comment on whether the final image looks different (if it does, plot it), and write down the time step satisfying the stopping condition.

(c) Investigate whether larger Δt will greatly affect the solution by trying instead $\epsilon = 0.001$ and $\Delta t = \epsilon/2$, with stopping condition either $\leq 10^{-6}$ or running a maximum of $50,000$ time steps. Comment on whether the final image looks different (if it does, plot it), and write down the time step satisfying the stopping condition and the final difference $||f^{(k)} - f^{(k-1)}||_\infty$.

4. Let $\delta > 0$. Show

$$f_t = \nabla \cdot \frac{\nabla f}{\sqrt{|\nabla|^2 + \delta^2}}$$

decreases the energy

$$E(f) = \int_0^H \int_0^W \sqrt{|\nabla f|^2 + \delta^2}\, dx\ dy$$

in the sense $(E(f))_t \leq 0$, for von Neumann boundary conditions at the boundary of the rectangular domain $[0, H] \times [0, W]$.

5. Let $\delta > 0$. Show

$$\nabla \cdot \frac{\nabla f}{\sqrt{|\nabla f|^2 + \delta^2}} = \frac{1}{\sqrt{|\nabla f|^2 + \delta^2}} \left[\Delta f - \frac{\nabla f^T \nabla^2 f \nabla f}{|\nabla f|^2 + \delta^2} \right]$$

where

$$\nabla^2 f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix},$$

by simplifying the expression.

6. Use Taylor series to show the central differencing formula

$$(f_{xy})_{ij} = \frac{f_{i+1,j+1} - f_{i+1,j-1} - f_{i-1,j+1} + f_{i-1,j-1}}{4h^2} + \mathcal{O}(h^2)$$

on a uniform grid with stepsize h.

7. (Math 279) Handle

$$f_t = \frac{1}{\sqrt{|\nabla f|^2 + \delta^2}} \left[\Delta f - \frac{\nabla f^T \nabla^2 f \nabla f}{|\nabla f|^2 + \delta^2} \right]$$

through the alternate discretization of using 2nd order central differencing for all the derivative terms $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ (use Problem 6 for f_{xy}). Use this numerical scheme to perform your inpainting. Test it on the image and corrupted region of Problem 1a, perhaps with $h = \delta = 0.1$ (or smaller), and see if $\Delta t = \delta h^2/4$ works (or whether Δt needs to be smaller). Also test it on an image and corrupted region of your choice. For both, compare your results to the original discretization and form your conclusions.