# ***Likelihood Ratio Test,***

# ***Most Powerful Test,***

# ***Uniformly Most Powerful Test***

A. We first review the general definition of a hypothesis test.

A hypothesis test is like a lawsuit:

$H_0$: the defendant is innocent **versus**

$H_a$: the defendant is guilty

| | | The truth | |
|---|---|---|---|
| | | $H_0$: the defendant is innocent | $H_a$ the defendant is guilty |
| Jury's Decision | $H_0$ | Right decision | **Type II error** |
| | $H_a$ | **Type I error** | Right decision |

## The <u>significance level</u> and the <u>power of the test</u>.

α = P(Type I error) = P(Reject $H_0 \mid H_0$) ← significance level

β = P(Type II error) = P(Fail to reject $H_0 \mid H_a$)

| |
|---|
| Power = 1- β = P(Reject $H_0 \mid H_a$) |

B. Now we derive the likelihood ratio test for the usual two-sided hypotheses for a population mean. (It has a simple null hypothesis and a composite alternative hypothesis.)

<u>Example 1.</u> Please derive the likelihood ratio test for $H_0: \mu = \mu_0$ versus $H_a: \mu \neq \mu_0$, when the population is normal and population variance $\sigma^2$ is known.

Solution:
For a 2-sided test of $H_0: \mu = \mu_0$ versus $H_a: \mu \neq \mu_0$, when the population is normal and population variance $\sigma^2$ is known, we have:

$\varpi = \{\mu : \mu = \mu_0\}$ and $\Omega = \{\mu : -\infty < \mu < \infty\}$

The likelihoods are:

$$L(\varpi) = L(\mu_0)$$
$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left( -\frac{(x_i - \mu_0)^2}{2\sigma^2} \right) = \left( \frac{1}{2\pi\sigma^2} \right)^{\frac{n}{2}} \exp\left[ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu_0)^2 \right]$$

There is no free parameter in $L(\varpi)$, thus $L(\hat{\varpi}) = L(\varpi)$.

$$L(\Omega) = L(\mu)$$
$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left( -\frac{(x_i - \mu)^2}{2\sigma^2} \right) = \left( \frac{1}{2\pi\sigma^2} \right)^{\frac{n}{2}} \exp\left[ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 \right]$$

There is only one free parameter μ in $L(\Omega)$. Now we shall find the value of μ that maximizes the log likelihood
$\ln L(\Omega) = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2$ .
By solving $\frac{d \ln L(\Omega)}{d\mu} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu) = 0$, we have $\hat{\mu} = \bar{x}$
It is easy to verify that $\hat{\mu} = \bar{x}$ indeed maximizes the loglikelihood, and thus the likelihood function.

Therefore the likelihood ratio is:
$$\lambda = \frac{L(\hat{\varpi})}{L(\hat{\Omega})} = \frac{L(\mu_0)}{\max_{\mu} L(\mu)}$$
$$= \frac{L(\mu_0)}{L(\hat{\mu})} = \frac{\left( \frac{1}{2\pi\sigma^2} \right)^{\frac{n}{2}} \exp\left[ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu_0)^2 \right]}{\left( \frac{1}{2\pi\sigma^2} \right)^{\frac{n}{2}} \exp\left[ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 \right]}$$
$$= \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} \left[ (x_i - \mu_0)^2 - (x_i - \bar{x})^2 \right] \right\}$$
$$= \exp\left[ -\frac{1}{2} \left( \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \right)^2 \right] = \exp\left[ -\frac{1}{2} (z_0)^2 \right]$$

Therefore, the likelihood ratio test that will reject $H_0$ when $\lambda \le \lambda^*$ is equivalent to the z-test that will reject $H_0$ when $|Z_0| \ge c$, where c can be determined by the significance level α as $c = z_{\alpha/2}$.

## C. MP Test, UMP Test, and the Neyman-Pearson Lemma

Now considering some one-sided tests where we have :

$H_0: \mu = \mu_0$ **versus** $H_a: \mu > \mu_0$

**or,** $H_0: \mu \le \mu_0$ **versus** $H_a: \mu > \mu_0$

**Given the composite null hypothesis for the second one-sided test, we need to expand our definition of the significance level as follows:**

For $H_0: \theta \in \omega_0$ versus $H_a: \theta \in \omega_1$

Let the random sample be denoted by $\boldsymbol{X} = (X_1, X_2, \cdots, X_n)'$. Suppose the rejection region is $\boldsymbol{C}$ such that: (*note: the rejection region can be defined in terms of the sample points directly instead of the test statistic values)

We reject $H_0$ if $\boldsymbol{X} \in \boldsymbol{C}$
We fail to reject $H_0$ if $\boldsymbol{X} \in \boldsymbol{C}^C$

**Definition:** The **size** or significance level of the test is defined as:

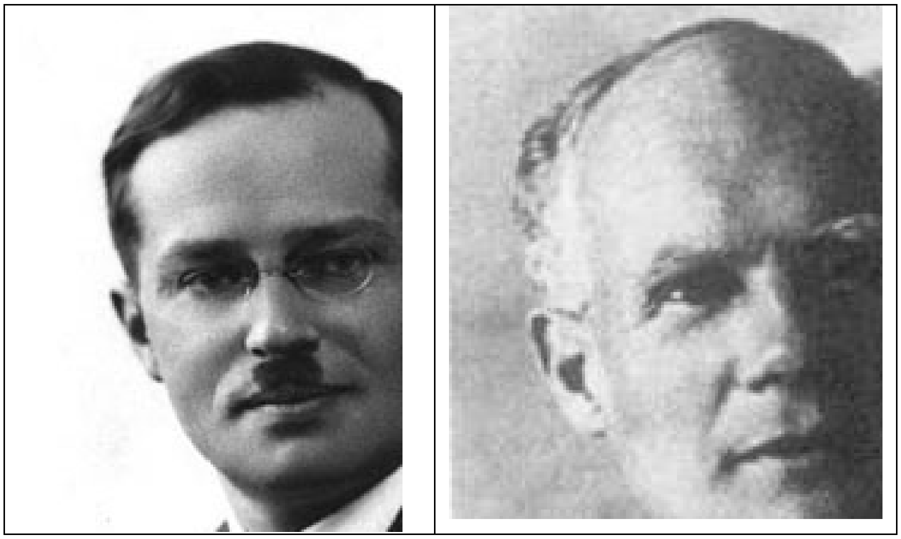
$$\boldsymbol{\alpha} = \max_{\theta \in \omega_0} \boldsymbol{P}(\boldsymbol{X} \in \boldsymbol{C}) = \max_{\theta} \boldsymbol{P_\theta}(\boldsymbol{Reject}\ H_0 | \theta \in \omega_0)$$

**Definition:** The power of a test is given by

$$\mathbf{1} - \boldsymbol{\beta} = \boldsymbol{\gamma}(\theta) = \boldsymbol{P_\theta}(\boldsymbol{Reject}\ H_0 | \theta \in \omega_1)$$

**Definition:** A **most powerful test** corresponds to the best rejection region $\boldsymbol{C}$ such that in testing a simple null hypothesis $H_0: \theta = \theta'$ versus a simple alternative hypothesis $H_a: \theta = \theta''$, this test has the largest possible test power $(\mathbf{1} - \boldsymbol{\beta})$ among all tests of size $\boldsymbol{\alpha}$.

**Theorem (Neyman-Pearson Lemma).** The likelihood ratio test for a simple null hypothesis $H_0: \theta = \theta'$ versus a simple alternative hypothesis $H_a: \theta = \theta''$ is a most powerful test.



Jerzy Neyman (left) and Egon Pearson (right)

**Definition:** A **uniformly most powerful test** in testing a simple null hypothesis $H_0: \theta = \theta'$ versus a composite alternative hypothesis $H_a: \theta \in \omega_1$, or in testing a composite null hypothesis $H_0: \theta \in \omega_0$ versus a composite alternative $H_a: \theta \in \omega_1$ , is a size $\boldsymbol{\alpha}$ test such that this test has the largest possible test power $\boldsymbol{\gamma}(\theta'')$ for every simple alternative hypothesis: $H_a: \theta = \theta'' \in \omega_1$, among all such tests of size $\boldsymbol{\alpha}$.

**Now we derive the likelihood ratio test for the first one-sided hypotheses.**

__**Example 2.**__ Let $f(x_i|\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}, for\ i = 1, \cdots, n,$ where $\sigma^2$ is known. Derive the likelihood ratio test for the hypothesis $H_0: \mu = \mu_0\ vs. H_a: \mu > \mu_0$, and show whether it is a UMP test or not.

__**Solution:**__

$$\boldsymbol{\omega} = \{\mu: \mu = \mu_0\}$$
$$\boldsymbol{\Omega} = \{\mu: \mu \geq \mu_0\}$$

__**Note: Now that we are not just dealing with the two-sided hypothesis, it is important to know that the more general definition of $\boldsymbol{\omega}$ is the set of all unknown parameter values under $H_0$,**__ __while__ __**$\boldsymbol{\Omega}$ is the set of all unknown parameter values under the union of $H_0$**__ __and__ __**$H_a$.**__

Therefore we have:

$$\begin{cases} L_\omega = f(x_1, x_2, \cdots, x_n|\mu = \mu_0) = \prod_{i=1}^{n} f(x_i|\mu = \mu_0) \\ = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu_0)^2}{2\sigma^2}} = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_0)^2} \\ L_\Omega = f(x_1, x_2, \cdots, x_n|\mu \geq \mu_0) = \prod_{i=1}^{n} f(x_i|\mu \geq \mu_0) \\ = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2} \end{cases}$$

Since there is no free parameter in $L_\omega$, $\sup(L_\omega) = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_0)^2}$.

$$\because \begin{cases} \frac{\partial logL_\Omega}{\partial\mu} = \frac{n}{\sigma^2}(\bar{x} - \mu) \\ \frac{\partial^2 logL_\Omega}{\partial\mu^2} = -\frac{n}{\sigma^2} < 0 \end{cases}, \therefore \sup(L_\Omega)$$

$$= \begin{cases} (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2}, if\ \bar{x} > \mu_0 \\ (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_0)^2}, if\ \bar{x} \leq \mu_0 \end{cases}.$$

$$\therefore LR = \frac{\sup(L_\omega)}{\sup(L_\Omega)} = \begin{cases} e^{-\frac{n}{2\sigma^2}(\bar{x}-\mu_0)^2}, if\ \bar{x} > \mu_0 \\ 1, if\ \bar{x} \leq \mu_0 \end{cases}.$$

$$\therefore P(LR \leq c^*|H_0) = \alpha \Leftrightarrow P\left(Z_0 = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \geq \sqrt{-2logc^*}\middle| H_0\right) = \alpha.$$

$\therefore$ Reject $H_0$ in favor of $H_a$ if $Z_0 = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} \geq Z_\alpha$

Furthermore, it is easy to show that for each $\mu_1 > \mu_0$, the likelihood ratio test of $H_0: \mu = \mu_0\ vs.\ H_a: \mu = \mu_1$ rejects the null hypothesis at the significance level α for $Z_0 = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} \geq Z_\alpha$

By the Neyman-Pearson Lemma, the likelihood ratio test is also the most powerful test.

Now since for each $\mu_1 > \mu_0$, the most powerful size α test of $H_0: \mu = \mu_0\ vs.\ H_a: \mu = \mu_1$ rejects the null hypothesis for $Z_0 = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} \geq Z_\alpha$. Since this same test function is most powerful for each $\mu_1 > \mu_0$, this test is UMP for $H_0: \mu = \mu_0\ vs.\ H_a: \mu > \mu_0$.

### D. Monotone Likelihood Ratio, the Karlin-Rubin Theorem, and the Exponential Family.

A condition under which the UMP tests exists is when the family of distributions being considered possesses a property called *monotone likelihood ratio*.

**Definition:** Let $f(\boldsymbol{x}|\theta)$ be the joint pdf of the sample $\boldsymbol{X} = (X_1, X_2, \cdots, X_n)'$. Then $f(\boldsymbol{x}|\theta)$ **is** said to have a monotone likelihood ratio in the statistic $T(\boldsymbol{X})$ if for any choice $\theta_1 < \theta_2$ of parameter values, the likelihood ratio $f(\boldsymbol{x}|\theta_2)/f(\boldsymbol{x}|\theta_1)$ depends on values of the data $\boldsymbol{X}$ only through the value of statistic $T(\boldsymbol{X})$ and, in addition, this ratio is a monotone function of $T(\boldsymbol{X})$.

**Theorem (Karlin-Rubin).** Suppose that $f(\boldsymbol{x}|\theta)$ has an increasing monotone likelihood ratio for the statistic $T(\boldsymbol{X})$. Let $\alpha$ and $k_\alpha$ be chosen so that $\boldsymbol{\alpha} = \boldsymbol{P_\theta}(T(\boldsymbol{X}) \geq \boldsymbol{k_\alpha})$**. Then**

$$\boldsymbol{C} = \{\boldsymbol{x}: T(\boldsymbol{x}) \geq \boldsymbol{k_\alpha}\}$$

is the rejection region (*also called 'critical region') for a UMP test for the one-sided tests of: $H_0: \theta \leq \theta_0$ versus $H_1: \theta > \theta_0$.

**Theorem.** For a Regular Exponential Family with only one unknown parameter θ, and a population p.d.f.:

$$f(x;\theta) = c(\theta) \cdot g(x) \cdot \exp[t(x)w(\theta)]$$

If $w(\theta)$ is a is an increasing function of $\theta$, then we have a monotone likelihood ratio in the statistic $T(\underline{X}) = \sum_{i=1}^{n} t(X_i)$.
{*Recall $T(\underline{X}) = \sum_{i=1}^{n} t(X_i)$} is also complete sufficient for $\theta$}

**Examples of a family with monotone likelihood ratio:**
For $X_1, \ldots, X_n$ iid Exponential ($\lambda$),

$$\frac{p(\boldsymbol{x}|\lambda_1)}{p(\boldsymbol{x}|\lambda_0)} = \frac{\prod_{i=1}^{n} \lambda_1 e^{-\lambda_1 X_i}}{\prod_{i=1}^{n} \lambda_0 e^{-\lambda_0 X_i}} = \left(\frac{\lambda_1}{\lambda_0}\right)^n \exp\left(-(\lambda_1 - \lambda_0)\sum_{i=1}^{n} X_i\right)$$

For $\lambda_1 > \lambda_0$, $\dfrac{p(\boldsymbol{x} \mid \lambda_1)}{p(\boldsymbol{x} \mid \lambda_0)}$ is an increasing function of $-\sum_{i=1}^{n} X_i$ so the family has monotone likelihood ratio in $T(\boldsymbol{x}) = -\sum_{i=1}^{n} X_i$.

**<u>Example 3.</u>** Let $X_1, X_2, \dots X_n \sim N(\mu, \sigma^2)$, where $\sigma^2$ is known

1. Find the LRT for $H_0: \mu \le \mu_0 \quad vs \quad H_a: \mu > \mu_0$
2. Show the test in (1) is a UMP test.

**<u>Solution:</u>**

$$\boldsymbol{\omega} = \{\mu: \mu \le \mu_0\}$$
$$\boldsymbol{\Omega} = \{\mu: -\infty < \mu < \infty\}$$

**<u>Note: Now that we are not just dealing with the two-sided hypothesis, it is important to know that the more general definition of $\boldsymbol{\omega}$ is the set of all unknown parameter values under $H_0$</u>**<u>, while</u> **<u>$\boldsymbol{\Omega}$ is the set of all unknown parameter values under the union of $H_0$ and $H_a$</u>**<u>.</u>

(1) The ratio of the likelihood is shown as the following,

$$\lambda(x) = \frac{L(\widehat{\omega})}{L(\widehat{\Omega})}$$

$L(\widehat{\omega})$ is the maximum likelihood under $H_0: \mu \le \mu_0$

$$L(\widehat{\omega}) = \begin{cases} \left(\dfrac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left(-\dfrac{\sum(x_i - \bar{x})^2}{2\sigma^2}\right), & \bar{x} < \mu_0 \\ \left(\dfrac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left(-\dfrac{\sum(x_i - \mu_0)^2}{2\sigma^2}\right), & \bar{x} \ge \mu_0 \end{cases}$$

$L(\widehat{\Omega})$ is the maximum likelihood under $H_0 \; union \; H_a$

$$L(\widehat{\Omega}) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left(-\frac{\sum(x_i - \bar{x})^2}{2\sigma^2}\right)$$

The ratio of the maximized likelihood is,

$$\lambda(x) = \begin{cases} 1, \quad \bar{x} < \mu_0 \\ exp\left(-\dfrac{\left(\sum(x_i - \mu_0)^2 - \sum(x_i - \bar{x})^2\right)}{2\sigma^2}\right), & \bar{x} \ge \mu_0 \end{cases}$$

$\Rightarrow$

$$\lambda(x) = \begin{cases} 1, & \bar{x} < \mu_0 \\ e^{-\frac{n(\bar{x}-\mu_0)^2}{2\sigma^2}}, & \bar{x} \ge \mu_0 \end{cases}$$

By LRT, we reject null hypothesis if $\lambda(x) < c$. Generally, c is chosen to be less than 1. The rejection region is,

$$R = \left\{X : e^{-\frac{n(\bar{x}-\mu_0)^2}{2\sigma^2}} < c\right\} = \left\{X : \bar{X} > \mu_0 + \sqrt{-\frac{2\sigma^2}{n}\ln c}\right\}$$

(2) We can use the Karlin-Rubin theorem to show that the LRT is UMP.

***First***, we need to show the size for the LRT test.

$$\alpha = \sup_{\mu \le \mu_0} P(X \in R)$$

In which,

$$P(X \in R) = P\left(\bar{X} > \mu_0 + \sqrt{-\frac{2\sigma^2}{n}\ln c}\right)$$

$$= P\left(\frac{\bar{X}-\mu}{\sqrt{\sigma^2/n}} > \frac{\mu_0-\mu}{\sqrt{\sigma^2/n}} + \sqrt{-2\ln c}\right)$$

$$= P\left(Z > \frac{\mu_0-\mu}{\sqrt{\sigma^2/n}} + \sqrt{-2\ln c}\right)$$

where z follows standard normal distribution,

$P\left(Z > \frac{\mu_0-\mu}{\sqrt{\sigma^2/n}} + \sqrt{-2\ln c}\right)$ is an increasing function of $\mu$. So

$$\alpha = \sup_{\mu \le \mu_0} P(X \in R) = P\left(Z > \frac{\mu_0-\mu}{\sqrt{\sigma^2/n}} + \sqrt{-2\ln c} \quad | \mu = \mu_0\right)$$

$$= P\left(Z > \sqrt{-2\ln c} \quad | \mu = \mu_0\right)$$

So the LRT test a size $\alpha$ test where $\alpha = P\left(Z > \sqrt{-2\ln c}\right)$

***Next***, we prove the family has monotone likelihood ratio (MLR) in $\bar{X}$. It is straightforward to show that $\bar{X}$ is a sufficient statistics for $\mu$ because we have an exponential family. For any pair $\mu_2 > \mu_1$, the ratio of the likelihood is:

$$\frac{f(\boldsymbol{x}|\mu_2)}{f(\boldsymbol{x}|\mu_1)} = \frac{\left(\frac{n}{2\pi\sigma^2}\right)^{1/2} \exp\left(-\frac{(\bar{x}-\mu_2)^2}{2\sigma^2/n}\right)}{\left(\frac{n}{2\pi\sigma^2}\right)^{1/2} \exp\left(-\frac{(\bar{x}-\mu_1)^2}{2\sigma^2/n}\right)}$$
$$= exp\left(\frac{2\sigma^2}{n}((\bar{x}-\mu_1)^2 - (\bar{x}-\mu_2)^2)\right)$$
$$= \exp(\frac{2\sigma^2}{n}(\mu_2-\mu_1)(2\bar{x}-\mu_1-\mu_2))$$

It is increasing in $\bar{X}$**.** So the family has monotone likelihood ratio (MLR) in $\bar{X}$.

By the Karlin-Rubin theorem, we know that the LRT test is also an UMP test.