# Testing with general prior distributions
# Some clarifications

## 1 Preliminary remarks

### A first preliminary remark

In Chapter 4 we first assume that we work with well-defined prior probability distributions. The theory is derived under this assumption and, in a second step, is extended to improper prior distributions. Similarly, in Chapter 1 we introduced the posterior distribution $\pi(\theta|x) \propto f(x|\theta)\pi(\theta)$ using Bayes rule and considering a proper prior density $\pi(\theta)$. Then, in Chapter 3, we extended the definition of the posterior distribution to improper prior densities $\pi(\theta)$ for which

$$\int_{\Theta} f(x|\theta)\pi(\theta)\mathrm{d}\theta < +\infty. \tag{1}$$

**Remark:** Condition (1) is always verified if $\pi(\theta)$ is a proper prior density.

### A second preliminary remark

In Chapter 4, assuming first that $\pi(\theta)$ is a proper prior density is particularly important to define the Bayes factor

$$B_{01}^{\pi}(x) = \frac{\pi(\Theta_0|x)}{\pi(\Theta_1|x)}\frac{\pi(\Theta_1)}{\pi(\Theta_0)}. \tag{2}$$

Indeed, if $\pi(\theta)$ is improper then $\pi(\Theta_i) = +\infty$ for at least one $i \in \{0, 1\}$ and the above definition of the Bayes factor is meaningless.

### A third preliminary remark

In most of Chapter 4 we consider a proper prior density $\pi(\theta)$ such that

$$\pi(\theta) = \rho_0\pi_0(\theta) + (1 - \rho_0)\pi_1(\theta), \quad \theta \in \Theta \tag{3}$$

where $\rho_0 \in (0,1)$ and where, for $i \in \{0,1\}$, $\pi_i(\theta)$ is p.d.f. on $\Theta_i$ (i.e. $\pi_i(\Theta_i) = 1$ and $\pi_i(\theta) = 0$ for all $\theta \notin \Theta_i$). However, there is no loss of generality in doing this. Indeed, every proper prior density $\pi(\theta)$ can be written as in (3) by taking $\rho_0 = \pi(\Theta_0)$ and

$$\pi_0(\theta) = \frac{\pi(\theta)\mathbb{1}_{\Theta_0}(\theta)}{\pi(\Theta_0)}, \quad \pi_1(\theta) = \frac{\pi(\theta)\mathbb{1}_{\Theta_1}(\theta)}{\pi(\Theta_1)}, \quad \theta \in \Theta.$$

In the context of hypothesis testing, the advantage of specifying $\pi(\theta)$ using (3) is that we can then separately choose the elements of $\pi(\theta)$ the decision rule $\delta^\pi(x)$ depends on, namely

- $\rho_0$, the prior probability of $H_0$ that enters in the acceptance level of Proposition 4.2;
- $\pi_0(\theta)$ and $\pi_1(\theta)$, the two elements of $\pi(\theta)$ that enter in the definition (2) of the Bayes factor.

# 2 Hypothesis testing with general prior densities

We now consider a (potentially improper) prior density $\pi(\theta)$ that can be written as in (3) for some $\rho_0 \in (0,1)$ and some (potentially improper) prior density $\pi_i(\theta)$ on $\Theta_i$, $i \in \{0,1\}$.

## 2.1 Preliminary remark

We saw above that every proper prior density $\pi(\theta)$ can be written as in (3). This is however not true for improper prior densities.

To see this let $\pi(\theta) = c$ for all $\theta \in \Theta$. Then, for $\rho_0 \in (0,1)$ we can write $\pi(\theta)$ as in (3) if and only if

$$\pi_0(\theta) = \frac{c}{\rho_0}\mathbb{1}_{\Theta_0}(\theta), \quad \pi_1(\theta) = \frac{c}{1-\rho_0}\mathbb{1}_{\Theta_1}(\theta), \quad \theta \in \Theta.$$

Assume now that $\Theta_0$ is bounded and let $c_0 = \int_{\Theta_0} \mathrm{d}\theta < +\infty$. In this case,

$$\pi_0(\Theta_0) = \frac{c\,c_0}{\rho_0}$$

and hence, unless $\rho_0 = c\,c_0$, the density $\pi_0(\theta)$ is neither proper nor improper. Notice that if $c > 1/c_0$ then there exists no $\rho \in (0,1)$ such that $\pi_0(\Theta_0) = 1$. Consequently, when $\Theta_0$ is bounded and $\pi(\theta) = c$ for all $\theta \in \Theta$ and some $c > \big(\int_{\Theta_0} \mathrm{d}\theta\big)^{-1}$, there exist no proper prior density $\pi_0(\theta)$ and $\rho_0 \in (0,1)$ such that $\pi(\theta)$ can be written as in (3).

**Remark:** In the above example (3) can hold for $\pi_0(\theta)$ such that $\pi_0(\Theta_0) \neq 1$ but in this case $\pi_0(\theta)$ is neither proper nor improper, and is thus not considered as a valid prior density on $\Theta_0$.

## 2.2 Bayes factor for general prior densities

If $\pi_i(\Theta_i) = +\infty$ for at least one $i \in \{0, 1\}$ then the definition (2) of the Bayes factor is meaningless, as mentioned above. The extension of the definition (2) of the Bayes factor to improper prior densities is based on the equality

$$B_{01}^{\pi}(x) = \frac{m_0(x)}{m_1(x)} = \frac{\int_{\Theta} f(x|\theta)\pi_0(\theta)\mathrm{d}\theta}{\int_{\Theta} f(x|\theta)\pi_1(\theta)\mathrm{d}\theta} \tag{4}$$

established for proper prior densities. Equality (4) allows to define the Bayes factor for improper prior distributions, provided that (1) holds.

## 2.3 Proposition 4.2 with improper prior densities

The result of Proposition 4.2 does not directly hold for an improper prior density $\pi(\theta)$. This observation can be deduced from the fact that, as $\pi(\Theta_i) = +\infty$ for at least one $i \in \{0, 1\}$, the acceptance level $\frac{a_1\pi(\Theta_1)}{a_0\pi(\Theta_0)}$ that appears in Proposition 4.2 is either 0 or infinite.

The following result (whose proof is left as an exercise) extends Proposition 4.2 to the case where $\pi_0(\theta)$ and $\pi_1(\theta)$ are potentially improper prior densities.

**Proposition 1.** *Let $\pi(\theta)$ be as in (3) for some $\rho_0 \in (0, 1)$ and some prior density $\pi_i(\theta)$ on $\Theta_i$, $i = 0, 1$. Then, the decision rule $\delta^{\pi} : \mathcal{X} \to \{0, 1\}$ associated with the $a_0$–$a_1$ loss function can be defined, for $x \in \mathcal{X}$, by*

$$\delta^{\pi}(x) = \begin{cases} 1 & \text{if } B_{01}^{\pi}(x) \geq \frac{a_1(1-\rho_0)}{a_0\rho_0}, \\ 0 & \text{otherwise.} \end{cases}$$