Bayesian Modelling – Problem Sheet 3

Please hand in your solutions for Problems 1-4 by 6pm on Wednesday 06/03/2019

Problem 1 (the Jeffreys-Lindley's paradox)

Let $x_1, ..., x_n$ be n observations that we model as i.i.d. $\mathcal{N}_1(\theta, \sigma^2)$ random variables, with $\theta \in \Theta := \mathbb{R}$. We are interested below in testing $H_0 : \theta = 0$ against $H_1 : \theta \neq 0$. Let $x^{(n)} = (x_1, \dots, x_n)$ and $\bar{x}_n = \frac{1}{n}\sum_{k=1}^n x_k$ be the sample mean.

1. Considering the frequentist approach first, let $t_n = \sqrt{n}(\bar{x}_n/\sigma)$ and compute $p(t_n)$, the p-value of this test.

2. We now consider the Bayesian approach where $\pi(\theta)$ is the prior distribution defined by

$$\pi(\theta) = \rho_0 \mathbb{1}_{\{0\}}(\theta) + (1-\rho_0) g_1(\theta) \mathbb{1}_{\{\theta \neq 0\}}(\theta), \quad \theta \in \Theta$$

with $g_1(\theta)$ the density of the $\mathcal{N}_1(0, \sigma^2)$ distribution and $\rho_0 \in (0, 1)$.

   a) Show that

$$\pi(\{0\}|x^{(n)}) = \left[1 + \frac{1-\rho_0}{\rho_0}\sqrt{\frac{1}{1+n}} \exp\left(\frac{nt_n^2}{2(n+1)}\right)\right]^{-1}$$

   with t_n as in part 1.

   b) Show that the Bayes factor for this test is given by

$$B_{01}^{\pi}(x^{(n)}) = (1+n)^{1/2} \exp\Big(-\frac{nt_n^2}{2(1+n)}\Big)$$

   with t_n as in part 1.

3. Give the value of $p(t_n)$ and of $\pi(\{0\}|x^{(n)})$ when $t_n = 1.96$, $n = 16\,818$ and $\rho_0 = 1/2$. What happens in this case?

4. Assume $t_n = 1.96$ for all $n \geq 1$ and study the behaviour, as $n \to +\infty$, of $p(t_n)$ and of $B_{01}^{\pi}(x^{(n)})$. Explain your results.

Problem 2

Let $x_1, \dots, x_n$ be n observations that we model as i.i.d. $\mathcal{U}(0, \theta)$ random variables, where $\theta \in \Theta := (0, +\infty)$ is an unknown parameter. Let $\pi(\theta)$ be the density of the Pareto(λ_0, β_0) distribution; that is,

$$\pi(\theta) = \frac{\beta_0 \lambda_0^{\beta_0}}{\theta^{\beta_0+1}} \mathbb{1}_{(\lambda_0, +\infty)}(\theta), \quad \theta \in \Theta$$

where $\lambda_0, \beta_0 \in (0, +\infty)$ are known hyperparameters.

1. Show that the posterior distribution of θ given $x := (x_1, \dots, x_n)$ is the Pareto(λ_n, β_n) distribution where
$$\lambda_n = \max\{\lambda_0, x_1, \dots, x_n\}, \quad \beta_n = \beta_0 + n.$$
2. Compute the mean and the median of the posterior distribution of θ given x.
3. Derive the highest posterior density (HPD) region of θ at level $1-\alpha$, with $\alpha \in (0, 1)$.
4. We wish to test the point null hypothesis $H_0 : \theta = \theta^*$ against $H_1 : \theta \neq \theta^*$ for some $\theta^* > \lambda_n$. To this end, we consider the prior distribution $\tilde{\pi}(\theta)$ defined by
$$\tilde{\pi}(\theta) = \rho_0 \mathbb{1}_{\{\theta^*\}}(\theta) + (1 - \rho_0)\pi(\theta)\mathbb{1}_{\{\theta \neq \theta^*\}}(\theta), \quad \theta \in \Theta$$
with $\rho_0 \in (0, 1)$ and $\pi(\theta)$ as above. Compute the Bayes factor for this test.
5. We now want to test $H_0 : \theta \leq \theta^*$ against $H_1 : \theta > \theta^*$ for some $\theta^* > \lambda_n$. Compute the Bayes factor for this test.

Problem 3

Let $x \in \mathbb{R}$ be an observation that we model as a Cauchy$(\theta, 1)$ random variable, so that the likelihood function is given by
$$f(x|\theta) = \frac{1}{\pi(1 + (x - \theta)^2)}, \quad \theta \in \Theta := \mathbb{R},$$
and let $\pi(\theta) \propto 1$ be the Laplace's prior.

Hint: For what follows it is useful to recall that
$$\frac{\mathrm{d}}{\mathrm{d}z} \arctan(z) = \frac{1}{1 + z^2}, \quad z \in \mathbb{R}.$$

1. Compute the posterior distribution of θ given x and its median.
2. Derive the highest posterior density region of θ at level $1 - \alpha$, with $\alpha \in (0, 1)$.
3. We wish to test the hypothesis $H_0 : \theta < \theta^*$ against its alternative $H_1 : \theta \geq \theta^*$ for some $\theta^* \in \mathbb{R}$. Derive the corresponding Bayesian test associated with the $a_0 - a_1$ loss function.
4. We now want to test the point null hypothesis $H_0 : \theta = 0$ against its alternative $H_1 : \theta \neq 0$. For that purpose we now consider the prior density $\pi(\theta)$ defined by
$$\pi(\theta) = \rho_0 \mathbb{1}_{\{0\}}(\theta) + (1 - \rho_0)\, c\, \mathbb{1}_{\{\theta \neq 0\}}, \quad \theta \in \Theta$$
where $c > 0$ and $\rho_0 \in (0, 1)$. Compute the Bayes factor for this test and explain how it is influenced by the parameter c.

Problem 4

Let $X_1, \ldots, X_n$ be i.i.d. $\mathcal{N}_1(\theta, 1)$ random variables, with $\theta \in \Theta := \mathbb{R}$. We are interested below to test $H_0 : \theta = 0$ against $H_1 : \theta \neq 0$. To this end, we consider the prior distribution $\pi(\theta)$ defined by

$$\pi(\theta) = \rho_0 \mathbb{1}_{\{0\}}(\theta) + (1 - \rho_0) g_1(\theta) \mathbb{1}_{\{\theta \neq 0\}}(\theta), \quad \theta \in \Theta$$

where $\rho_0 \in (0, 1)$ and $g_1(\theta)$ is the density of the $\mathcal{N}_1(\mu_0, \sigma_0^2)$ distribution. Let $X^{(n)} = (X_1, \ldots, X_n)$ and $\bar{X}_n = \frac{1}{n} \sum_{k=1}^n X_k$.

1. Show that

$$\log B_{01}^{\pi}(X^{(n)}) = \log \Omega(X^{(n)}) + \frac{n(\bar{X}_n - \mu_0)^2}{2\sigma_0^2(n + 1/\sigma_0^2)} + \frac{1}{2} \log \left(n\sigma_0^2 + 1\right)$$

   where

$$\Omega(X^{(n)}) = \exp\Big(- \frac{\sum_{k=1}^n X_k^2}{2} + \frac{\sum_{k=1}^n (X_k - \bar{X}_n)^2}{2} \Big)$$

   is the likelihood ratio test statistic for the test $H_0 : \theta = 0$ against $H_1 : \theta \neq 0$.

2. Show that under H_0 we have, as $n \to +\infty$

$$\log B_{01}^{\pi}(X^{(n)}) \to +\infty \quad \text{(in } \mathbb{P}_0\text{-probability)}$$

   at speed $\log n$; that is

$$\frac{\log B_{01}^{\pi}(X^{(n)})}{\log n} = c + o_{\mathbb{P}_0}(1)$$

   for a constant $c \in (0, +\infty)$.

3. Show that under H_1 we have

$$\lim_{n \to +\infty} \log B_{01}^{\pi}(X^{(n)}) = -\infty \quad \mathbb{P}_\theta\text{-almost surely}$$

   at speed $-n$; that is,

$$\lim_{n \to +\infty} \frac{\log B_{01}^{\pi}(X^{(n)})}{-n} = c, \quad \mathbb{P}_\theta\text{-almost surely}$$

   for a constant $c \in (0, +\infty)$.