Bayesian Modelling – Problem Sheet 4 (Solutions)

Problem 1

1. a) Since π is a p.d.f. on B_δ we have, by Jensen inequality,

$$\log \int_{B_\delta} \prod_{i=1}^{n} \frac{\tilde{f}(x_i|\theta)}{\tilde{f}(x_i|\theta_0)} \pi(\theta) \mathrm{d}\theta \geq \sum_{i=1}^{n} \int_{B_\delta} \log \frac{\tilde{f}(x_i|\theta)}{\tilde{f}(x_i|\theta_0)} \pi(\theta) \mathrm{d}\theta, \quad \forall x^{(n)} \in \mathcal{X}_1^n$$

so that

$$D_{n,\delta,C} \subset \left\{ x \in \mathcal{X}_1^n : \sum_{i=1}^{n} \int_{B_\delta} \log \frac{\tilde{f}(x_i|\theta)}{\tilde{f}(x_i|\theta_0)} \pi(\theta) \mathrm{d}\theta \leq -(1+C) n \delta^2 \right\}.$$

Therefore,

$$\begin{aligned}
\mathbb{P}_{\theta_0}(D_{n,\delta,C}) &\leq \mathbb{P}_{\theta_0}\left(\sum_{i=1}^{n} \int_{B_\delta} \log \frac{\tilde{f}(X_i|\theta)}{\tilde{f}(X_i|\theta_0)} \pi(\theta) \mathrm{d}\theta \leq -(1+C) n \delta^2 \right) \\
&= \mathbb{P}_{\theta_0}\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\int_{B_\delta} \log \frac{\tilde{f}(X_i|\theta)}{\tilde{f}(X_i|\theta_0)} + \int_{B_\delta} KL(\theta_0|\theta) \right] \pi(\theta) \mathrm{d}\theta \right. \\
&\qquad \left. \leq \sqrt{n} \int_{B_\delta} KL(\theta_0|\theta) \pi(\theta) \mathrm{d}\theta - \sqrt{n}(1+C)\delta^2 \right) \\
&\leq \mathbb{P}_{\theta_0}\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\int_{B_\delta} \log \frac{\tilde{f}(X_i|\theta)}{\tilde{f}(X_i|\theta_0)} + \int_{B_\delta} KL(\theta_0|\theta) \right] \pi(\theta) \mathrm{d}\theta \leq -\sqrt{n} C \delta^2 \right)
\end{aligned}$$

where the second inequality holds since $KL(\theta_0|\theta) \leq \delta^2$ for every $\theta \in B_\delta$.

Moreover,

$$\begin{aligned}
&\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\int_{B_\delta} \log \frac{\tilde{f}(X_i|\theta)}{\tilde{f}(X_i|\theta_0)} + \int_{B_\delta} KL(\theta_0|\theta) \right] \pi(\theta) \mathrm{d}\theta \\
&= \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^{n} \int_{B_\delta} \log \frac{\tilde{f}(X_i|\theta)}{\tilde{f}(X_i|\theta_0)} \pi(\theta) \mathrm{d}\theta - \int_{B_\delta} \mathbb{E}_{\theta_0}\left[\log \frac{\tilde{f}(X_1|\theta)}{\tilde{f}(X_1|\theta_0)} \right] \pi(\theta) \mathrm{d}\theta \right)
\end{aligned}$$

showing that

$$
\begin{aligned}
&\mathbb{P}_{\theta_0}(D_{n,\delta,C})\\
&\leq \mathbb{P}_{\theta_0}\Big(\frac{1}{n}\sum_{i=1}^n\int_{B_\delta}\log\frac{\tilde f(X_i|\theta)}{\tilde f(X_i|\theta_0)}\pi(\theta)\mathrm{d}\theta-\int_{B_\delta}\mathbb{E}_{\theta_0}\Big[\log\frac{\tilde f(X_1|\theta)}{\tilde f(X_1|\theta_0)}\Big]\pi(\theta)\mathrm{d}\theta\leq -C\delta^2\Big)\\
&\leq \mathbb{P}_{\theta_0}\Big(\Big|\frac{1}{n}\sum_{i=1}^n\int_{B_\delta}\log\frac{\tilde f(X_i|\theta)}{\tilde f(X_i|\theta_0)}\pi(\theta)\mathrm{d}\theta-\int_{B_\delta}\mathbb{E}_{\theta_0}\Big[\log\frac{\tilde f(X_1|\theta)}{\tilde f(X_1|\theta_0)}\Big]\pi(\theta)\mathrm{d}\theta\Big|\geq C\delta^2\Big)\\
&= \mathbb{P}_{\theta_0}\Big(\Big|\frac{1}{n}\sum_{i=1}^n\int_{B_\delta}\log\frac{\tilde f(X_i|\theta)}{\tilde f(X_i|\theta_0)}\pi(\theta)\mathrm{d}\theta-\mathbb{E}_{\theta_0}\Big[\int_{B_\delta}\log\frac{\tilde f(X_1|\theta)}{\tilde f(X_1|\theta_0)}\pi(\theta)\mathrm{d}\theta\Big]\Big|\geq C\delta^2\Big)
\end{aligned}
$$

where the equality uses Fubini's theorem.

b) Using Fubini's theorem and Jensen's inequality, we have

$$
\begin{aligned}
\mathbb{E}_{\theta_0}\Big[\Big(\int_{B_\delta}\log\frac{\tilde f(X_1|\theta)}{\tilde f(X_1|\theta_0)}\pi(\theta)\mathrm{d}\theta\Big)^2\Big] &\leq \mathbb{E}_{\theta_0}\Big[\int_{B_\delta}\Big(\log\frac{\tilde f(X_1|\theta)}{\tilde f(X_1|\theta_0)}\Big)^2\pi(\theta)\mathrm{d}\theta\Big]\\
&= \int_{B_\delta}\mathbb{E}_{\theta_0}\Big[\Big(\log\frac{\tilde f(X_1|\theta)}{\tilde f(X_1|\theta_0)}\Big)^2\Big]\pi(\theta)\mathrm{d}\theta\\
&\leq \delta^2
\end{aligned}
$$

where the last inequality uses the definition of the set B_δ.

c) Using Markov's inequality with $p=2$ (see Problem 3) and the results in parts 1.a) and 1.b), we have

$$
\begin{aligned}
\mathbb{P}_{\theta_0}(D_{n,\delta,C}) &\leq \frac{\mathrm{Var}_{\theta_0}\Big(\frac{1}{n}\sum_{i=1}^n\int_{B_\delta}\log\frac{\tilde f(X_i|\theta)}{\tilde f(X_i|\theta_0)}\pi(\theta)\mathrm{d}\theta\Big)}{C^2\delta^4}\\
&= \frac{\mathrm{Var}_{\theta_0}\Big(\int_{B_\delta}\log\frac{\tilde f(X_1|\theta)}{\tilde f(X_1|\theta_0)}\pi(\theta)\mathrm{d}\theta\Big)}{C^2 n\delta^4}\\
&\leq \frac{\mathbb{E}_{\theta_0}\Big[\Big(\int_{B_\delta}\log\frac{\tilde f(X_1|\theta)}{\tilde f(X_1|\theta_0)}\pi(\theta)\mathrm{d}\theta\Big)^2\Big]}{C^2 n\delta^4}\\
&\leq \frac{1}{C^2 n\delta^2}
\end{aligned}
$$

where the last inequality uses the definition of the set B_δ.

2. If $\pi(B_\delta)=0$ then we trivially have $\mathbb{P}_{\theta_0}(D_{n,\delta,C})=0$ and thus$\mathbb{P}_{\theta_0}(D_{n,\delta,C})\leq\frac{1}{C^2n\delta^2}$ for all $n\geq 1$, as required.

   Assume now that $\pi(B_\delta)>0$ and let $\tilde\pi:\Theta\to\mathbb{R}$ be defined by

$$
\tilde\pi(\theta)=\frac{\pi(\theta)}{\pi(B_\delta)}\mathbb{1}_{B_\delta}(\theta),\quad \theta\in\Theta.
$$

Then,

$$\begin{aligned}
\mathbb{P}_{\theta_0}(D_{n,\delta,C}) &= \mathbb{P}_{\theta_0}\bigg(\int_\Theta \prod_{i=1}^n \frac{\tilde{f}(X_i|\theta)}{\tilde{f}(X_i|\theta_0)} \pi(\theta)\mathrm{d}\theta \leq \pi\big(B_\delta\big) e^{-(1+C)n\delta^2} \bigg\}\bigg) \\
&\leq \mathbb{P}_{\theta_0}\bigg(\int_{B_\delta} \prod_{i=1}^n \frac{\tilde{f}(X_i|\theta)}{\tilde{f}(X_i|\theta_0)} \pi(\theta)\mathrm{d}\theta \leq \pi\big(B_\delta\big) e^{-(1+C)n\delta^2} \bigg) \\
&= \mathbb{P}_{\theta_0}\bigg(\int_{B_\delta} \prod_{i=1}^n \frac{\tilde{f}(X_i|\theta)}{\tilde{f}(X_i|\theta_0)} \tilde{\pi}(\theta)\mathrm{d}\theta \leq e^{-(1+C)n\delta^2} \bigg) \\
&\leq \frac{1}{C^2 n\delta^2}
\end{aligned}$$

where the last inequality uses the result of part 1.

Problem 2

1. a) To simplify the notation let $V_n = \{\theta : \epsilon_n \leq \|\theta - \theta_0\| < \epsilon\}$. Then,

$$\begin{aligned}
\pi\big(V_n|X^{(n)}\big) &= \phi_n(X^{(n)})\pi(V_n|X^{(n)}) + \mathbb{1}_{D^c_{n,\delta_n,1}}(X^{(n)})\big(1 - \phi_n(X^{(n)})\big)\pi(V_n|X^{(n)}) \\
&\quad + \mathbb{1}_{D_{n,\delta_n,1}}(X^{(n)})\big(1 - \phi_n(X^{(n)})\big)\pi(V_n|X^{(n)}) \\
&\leq \phi_n(X^{(n)}) + \mathbb{1}_{D^c_{n,\delta_n,1}}(X^{(n)})\big(1 - \phi_n(X^{(n)})\big)\pi(V_n|X^{(n)}) + \mathbb{1}_{D_{n,\delta_n,1}}(X^{(n)})
\end{aligned}$$

where, under (C2), $\limsup_{n\to+\infty} \mathbb{E}_{\theta_0}[\phi_n(X^{(n)})] = 0$. Using the result of Problem 1,

$$\mathbb{E}_{\theta_0}[\mathbb{1}_{D_{n,\delta_n,1}}(X^{(n)})] = \mathbb{P}_{\theta_0}(D_{n,\delta_n,1}) \leq \frac{1}{M^2 M_n^2}, \quad \forall n \in \mathbb{N}$$

and therefore, since $\lim_{n\to+\infty} M_n = +\infty$,

$$\limsup_{n\to+\infty} \mathbb{E}_{\theta_0}[\mathbb{1}_{D_{n,\delta_n,1}}(X^{(n)})] = 0.$$

Consequently,

$$\limsup_{n\to+\infty} \mathbb{E}_{\theta_0}\big[\pi(V_n|X^{(n)})\big] \leq \limsup_{n\to+\infty} \mathbb{E}_{\theta_0}\big[\mathbb{1}_{D^c_{n,\delta_n,1}}(X^{(n)})(1-\phi_n(X^{(n)}))\pi(V_n|X^{(n)})\big].$$

b) Let $j \in \{1, \dots, J\}$. Then,

$$\begin{aligned}
&\mathbb{E}_{\theta_0}\big[\mathbb{1}_{D^c_{n,\delta_n,1}}(X^{(n)})(1 - \phi_n(X^{(n)}))\pi(\Theta_{n,j}|X^{(n)})\big] \\
&= \int_{\mathcal{X}_1^n} \mathbb{1}_{D^c_{n,\delta_n,1}}(x^{(n)})(1 - \phi_n(x^{(n)})) \frac{\int_{\Theta_{n,j}} \pi(\theta) \prod_{i=1}^n \frac{\tilde{f}(x_i|\theta)}{\tilde{f}(x_i|\theta_0)}\mathrm{d}\theta}{\int_\Theta \pi(\theta) \prod_{i=1}^n \frac{\tilde{f}(x_i|\theta)}{\tilde{f}(x_i|\theta_0)}\mathrm{d}\theta} \prod_{k=1}^n \tilde{f}(x_k|\theta_0)\mathrm{d}x_k \\
&= \int_{\mathcal{X}_1^n} \mathbb{1}_{D^c_{n,\delta_n,1}}(x^{(n)})(1 - \phi_n(x^{(n)})) \frac{\int_{\Theta_{n,j}} \pi(\theta) \prod_{i=1}^n \tilde{f}(x_i|\theta)\mathrm{d}\theta}{\int_\Theta \pi(\theta) \prod_{i=1}^n \frac{\tilde{f}(x_i|\theta)}{\tilde{f}(x_i|\theta_0)}\mathrm{d}\theta} \prod_{k=1}^n \mathrm{d}x_k \\
&\leq \frac{e^{2M^2M_n^2}}{\pi(B_{\delta_n})} \int_{\mathcal{X}_1^n} (1 - \phi_n(x^{(n)})) \int_{\Theta_{n,j}} \pi(\theta) \prod_{i=1}^n \tilde{f}(x_i|\theta)\mathrm{d}\theta \prod_{k=1}^n \mathrm{d}x_k
\end{aligned}$$

where the inequality uses the definition of the set $D_{n,\delta_n,1}$.

In addition, using Tonelli's theorem,

$$\begin{aligned}
&\int_{\mathcal{X}_1^n} (1-\phi_n(x^{(n)})) \int_{\Theta_{n,j}} \pi(\theta) \prod_{i=1}^n \tilde{f}(x_i|\theta)\mathrm{d}\theta \prod_{k=1}^n \mathrm{d}x_k \\
&\quad = \int_{\Theta_{n,j}} \Big(\int_{\mathcal{X}_1^n} (1-\phi_n(x^{(n)})) \prod_{i=1}^n \tilde{f}(x_i|\theta)\mathrm{d}x_i \Big) \pi(\theta)\mathrm{d}\theta \\
&\quad = \int_{\Theta_{n,j}} \mathbb{E}_\theta\big[1-\phi_n(X^{(n)})\big]\pi(\theta)\mathrm{d}\theta
\end{aligned}$$

where, under (C2), we have for n large enough

$$\mathbb{E}_\theta\big[1-\phi_n(X^{(n)})\big] \leq e^{-j^2\, M_n^2 D}, \quad \forall \theta \in \Theta_{n,j}, \quad \forall j \in \{1,\dots,J\}.$$

Consequently, for n large enough,

$$\mathbb{E}_{\theta_0}\big[\mathbb{1}_{D^c_{n,\delta_n,1}}(X^{(n)})(1-\phi_n(X^{(n)}))\pi(V_n|X^{(n)})\big] \leq \frac{\pi(\Theta_{n,j})}{\pi(B_{\delta_n})}\, e^{2M^2M_n^2 - Dj^2M_n^2}.$$

c) Under (C3), for every $\delta > 0$ small enough we have

$$B_{\delta/c_\star}(\theta_0) := \{\theta :\, \|\theta-\theta_0\| < \delta/c_\star\} \subset B_\delta$$

and thus, for n large enough,

$$\pi(B_{\delta_n}) \geq \pi\big(B_{\delta_n/c_\star}(\theta_0)\big) \geq \mathrm{Vol}\big(B_{\delta_n/c_\star}(\theta_0)\big) \inf_{\theta\in B_{\delta_n/c_\star}(\theta_0)} \pi(\theta)$$

where $\mathrm{Vol}\big(B_{\delta_n/c_\star}(\theta_0)\big)$ denotes the volume of the set $B_{\delta_n/c_\star}(\theta_0)$. Under (C1), $\pi(\theta)$ is continuous and strictly positive $(\theta_0-\delta_\pi, \theta_0+\delta_\pi)$, for some $\delta_\pi > 0$, and thus for n large enough (i.e. for n such that $\delta_n/c_\star < \delta_\pi$),

$$\inf_{\theta\in B_{\delta_n/c_\star}(\theta_0)} \pi(\theta) \geq c_\pi$$

for some constant $c_\pi > 0$. Using the fact that $\mathrm{Vol}\big(B_{\delta_n/c_\star}(\theta_0)\big) \geq \underline{c}(\delta_n/c_\star)^d$ for some constant $\underline{c} > 0$, we conclude that there exists a constant $\underline{L} > 0$ such that $\pi(B_{\delta_n}) \geq \underline{L}\delta_n^d$ for all n large enough.

d) As per in part 1.c),

$$\pi(\Theta_{n,j}) \leq \mathrm{Vol}\big(\Theta_{n,j}\big) \sup_{\theta\in\Theta_{n,j}} \pi(\theta), \quad \forall j \in \{1,\dots,J\}.$$

Without loss of generality, we can assume that $\epsilon < \delta_\pi/4$ (with δ_π as in (C1)) so that, for n large enough,

$$\Theta_{n,j} \subset B_{\delta_\pi/2}(\theta_0) := \{\theta :\, \|\theta-\theta_0\| < \delta_\pi/2\}, \quad \forall j = 1,\dots,J.$$

Hence, for n large enough and using the continuity of $\pi(\theta)$ on $B_{\delta_\pi/2}(\theta_0)$, we have

$$\sup_{\theta\in\Theta_{n,j}} \pi(\theta) \le \sup_{\theta\in B_{\delta_\pi/2}(\theta_0)} \pi(\theta) < +\infty, \qquad \forall j = 1,\dots,J.$$

Together with the fact that $\mathrm{Vol}\big(\Theta_{n,j}\big) \le \bar{c}\epsilon_n^d$ for some finite constant $\bar{c} > 0$, this concludes to show that there exists a finite constant $\bar{L} > 0$ such that $\pi(\Theta_{n,j}) \le \bar{L}\epsilon_n^d$ for all $j = 1,\dots,J$.

e) The results in parts 2.d) and 2.e) imply that, for n large enough,

$$\frac{\pi(\Theta_{n,j})}{\pi(B_{\delta_n})} \le \frac{\bar{L}}{\underline{L}}\frac{\epsilon_n^d}{\delta_n^d} = \frac{\bar{L}}{\underline{L}\,M^d}, \quad \forall j\in\{1,\dots,J\}$$

which, together with the result of part 1.b), shows that for n large enough,

$$\begin{aligned}
&\mathbb{E}_{\theta_0}\big[\mathbb{1}_{D^c_{n,\delta_n,1}}(X^{(n)})(1-\phi_n(X^{(n)}))\pi\big(\{\theta:\, \epsilon_n \le \|\theta-\theta_0\| < \epsilon\}|X^{(n)}\big)\big]\\
&\le \sum_{j=1}^{J}\mathbb{E}_{\theta_0}\big[\mathbb{1}_{D^c_{n,\delta_n,1}}(X^{(n)})(1-\phi_n(X^{(n)}))\pi\big(\Theta_{n,j}|X^{(n)}\big)\big]\\
&\le \frac{\bar{L}}{\underline{L}\,M^d}\sum_{j=1}^{J} e^{2M^2M_n^2 - Dj^2M_n^2}\\
&= \frac{\bar{L}}{\underline{L}\,M^d} e^{2M^2M_n^2 - DM_n^2}\sum_{j=1}^{J} e^{-D(j^2-1)M_n^2}\\
&\le \frac{\bar{L}}{\underline{L}\,M^d} e^{M_n^2(2M^2-D)}\sum_{j=1}^{\infty} e^{-D(j^2-1)M_n^2}.
\end{aligned}$$

Since $M = \sqrt{D/4}$ we have

$$2M^2 - D = -\frac{D}{2}$$

and thus, for n large enough,

$$\mathbb{E}_{\theta_0}\big[\mathbb{1}_{D^c_{n,\delta_n,1}}(X^{(n)})(1-\phi_n(X^{(n)}))\pi\big(\{\theta:\, \epsilon_n \le \|\theta-\theta_0\| < \epsilon\}|X^{(n)}\big)\big] \le C\,\frac{\bar{L}}{\underline{L}\,M^d}e^{-\frac{D}{2}M_n^2}$$

with $C = \sum_{j=1}^{\infty} e^{-2D(j^2-1)} < +\infty$.
Therefore

$$\limsup_{n\to+\infty}\mathbb{E}_{\theta_0}\big[\mathbb{1}_{D^c_{n,\delta_n,1}}(X^{(n)})(1-\phi_n(X^{(n)}))\pi\big(\{\theta:\, \epsilon_n \le \|\theta-\theta_0\| < \epsilon\}|X^{(n)}\big)\big] = 0$$

which, together with the result in part 1.a), implies that

$$\limsup_{n\to+\infty}\mathbb{E}_{\theta_0}\big[\pi\big(\{\theta:\, \epsilon_n \le \|\theta-\theta_0\| < \epsilon\}|X^{(n)}\big)\big] = 0.$$

2. a) Proceeding as in part 1.b) we get, for n large enough.

$$\mathbb{E}_{\theta_0}\big[\mathbb{1}_{D^c_{n,\gamma_n,1}}(X^{(n)})(1-\phi_n(X^{(n)}))\pi\big(\{\theta:\ \|\theta-\theta_0\|\geq\epsilon|X^{(n)}\big)\big]\leq\frac{e^{2\tilde{M}^2\log(n)-Dn\epsilon^2}}{\pi(B_{\gamma_n})}.$$

For n sufficiently large,

$$\frac{e^{2\tilde{M}^2\log(n)-Dn\epsilon^2}}{\pi(B_{\gamma_n})}\leq\frac{e^{-\log(n)2\tilde{M}^2}}{\pi(B_{\gamma_n})}$$

where, using the result of part 1.c), $\pi(B_{\gamma_n})\geq\underline{L}\gamma_n^d$ for some $\underline{L}>0$ and for n large enough.

Therefore, for n large enough

$$\begin{aligned}\mathbb{E}_{\theta_0}\big[\mathbb{1}_{D^c_{n,\gamma_n,1}}&(X^{(n)})(1-\phi_n(X^{(n)}))\pi\big(\{\theta:\ \|\theta-\theta_0\|\geq\epsilon|X^{(n)}\big)\big]\\&\leq\frac{e^{-\log(n)2\tilde{M}^2}}{\underline{L}\tilde{M}^d}(\log(n))^{-d/2}n^{d/2}\\&=\frac{e^{-\log(n)2\tilde{M}^2+\frac{d}{2}\log(n)}}{\underline{L}\tilde{M}^d}(\log(n))^{-d/2}.\end{aligned}$$

b) Since $\tilde{M}=\sqrt{d/2}$, $-2\tilde{M}^2+\frac{d}{2}=-\frac{d}{2}$ and thus, for n large enough and using the result of part 2.a),

$$\mathbb{E}_{\theta_0}\big[\mathbb{1}_{D^c_{n,\gamma_n,1}}(X^{(n)})(1-\phi_n(X^{(n)}))\pi\big(\{\theta:\ \|\theta-\theta_0\|\geq\epsilon|X^{(n)}\big)\big]\leq\frac{e^{-\frac{d}{2}\log(n)}}{\underline{L}\tilde{M}^d}(\log(n))^{-d/2}$$

so that

$$\limsup_{n\to+\infty}\mathbb{E}_{\theta_0}\big[\mathbb{1}_{D^c_{n,\gamma_n,1}}(X^{(n)})(1-\phi_n(X^{(n)}))\pi\big(\{\theta:\ \|\theta-\theta_0\|\geq\epsilon|X^{(n)}\big)\big]=0.$$

Therefore,

$$\begin{aligned}\limsup_{n\to+\infty}&\,\mathbb{E}_{\theta_0}\big[\pi\big(\{\theta:\ \|\theta-\theta_0\|\geq\epsilon|X^{(n)}\big)\big]\\&\leq\limsup_{n\to+\infty}\mathbb{E}_{\theta_0}\big[\mathbb{1}_{D^c_{n,\gamma_n,1}}(X^{(n)})(1-\phi_n(X^{(n)}))\pi\big(\{\theta:\ \|\theta-\theta_0\|\geq\epsilon|X^{(n)}\big)\big]\\&=0\end{aligned}$$

where the inequality uses similar computations as in part 1.a).

3. Using the results of parts 1 and 2 we have

$$\begin{aligned}\limsup_{n\to+\infty}&\,\mathbb{E}_{\theta_0}\big[\pi\big(\{\theta:\|\theta-\theta_0\|\geq M_n\,n^{-1/2}\}\,\big|X^{(n)}\big)\big]\\&\leq\limsup_{n\to+\infty}\mathbb{E}_{\theta_0}\big[\pi\big(\{\theta:M_n\,n^{-1/2}\leq\|\theta-\theta_0\|<\epsilon\}\,\big|X^{(n)}\big)\big]\\&+\limsup_{n\to+\infty}\mathbb{E}_{\theta_0}\big[\pi\big(\{\theta:\|\theta-\theta_0\|\geq\epsilon\,\big|X^{(n)}\big)\big]\\&=0.\end{aligned}$$

Therefore, for every $\delta > 0$ we have,

$$\begin{aligned}
\limsup_{n\to+\infty} \mathbb{P}_{\theta_0}\Big(\pi\big(\{\theta :\|\theta-\theta_0\| \geq M_n\, n^{-1/2}\}\,\big|X^{(n)}\big) \geq \delta\Big) \\
&\leq \frac{\limsup_{n\to+\infty} \mathbb{E}_{\theta_0}\big[\pi\big(\{\theta : \|\theta-\theta_0\| \geq M_n\, n^{-1/2}\}\,\big|X^{(n)}\big)\big]}{\delta} \\
&= 0,
\end{aligned}$$

where the inequality uses Markov's inequality (see Problem 3), showing that

$$\pi\big(\{\theta : \|\theta-\theta_0\| \geq M_n\, n^{-1/2}\}\,\big|X^{(n)}\big) \to 0, \quad \text{in } \mathbb{P}_{\theta_0}\text{-probability.}$$

Problem 3

- (**Markov's inequality**) We have

  $$\mathbb{E}[|X|^p] = \mathbb{E}[|X|^p \mathbb{1}(|X| < \epsilon)] + \mathbb{E}[|X|^p \mathbb{1}(|X| \geq \epsilon)] \geq \mathbb{E}[|X|^p \mathbb{1}(X \geq \epsilon)] \geq \epsilon^p \mathbb{E}[\mathbb{1}(|X| \geq \epsilon)]$$

  where the second inequality holds because $p \geq 1$. Consequently

  $$\mathbb{E}[\mathbb{1}(|X| \geq \epsilon)] = \mathbb{P}(|X| \geq \epsilon) \leq \frac{\mathbb{E}[|X|^p]}{\epsilon^p}$$

  showing the result.

- (**Law of large numbers**)
  1. a) We have

     $$\begin{aligned}
     \mathbb{E}\big[\big(\sum_{k=1}^n Z_k\big)^4\big] &= \mathbb{E}\Big[\sum_{k_1=1}^n \sum_{k_2=1}^n \sum_{k_3=1}^n \sum_{k_4=1}^n Z_{k_1} Z_{k_2} Z_{k_3} Z_{k_4}\Big] \\
     &= \sum_{k_1=1}^n \sum_{k_2=1}^n \sum_{k_3=1}^n \sum_{k_4=1}^n \mathbb{E}\big[Z_{k_1} Z_{k_2} Z_{k_3} Z_{k_4}\big]
     \end{aligned}$$

     where, since $\mathbb{E}[Z_1] = 0$, $\mathbb{E}\big[Z_{k_1} Z_{k_2} Z_{k_3} Z_{k_4}\big] = 0$ if and only if there exists a $j \in \{1, \ldots, 4\}$ such that $k_j \neq k_i$ for all $i \in \{1, \ldots, 4\} \setminus \{j\}$. In other words, $\mathbb{E}\big[Z_{k_1} Z_{k_2} Z_{k_3} Z_{k_4}\big] \neq 0$ if and only if we are in one of the following 4 situations:

     $$\begin{aligned}
     k_1 = k_2 = k_3 = k_4, &\quad k_1 = k_2 \neq k_3 = k_4 \\
     k_1 = k_3 \neq k_2 = k_4, &\quad k_1 = k_4 \neq k_2 = k_3.
     \end{aligned}$$

     Therefore,

     $$\begin{aligned}
     &\sum_{k_1=1}^n \sum_{k_2=1}^n \sum_{k_3=1}^n \sum_{k_4=1}^n \mathbb{E}\big[Z_{k_1} Z_{k_2} Z_{k_3} Z_{k_4}\big] \\
     &\qquad = \sum_{k_1=1}^n \mathbb{E}\big[Z_{k_1}^4\big] + 3\Big(2\sum_{k_1=1}^{n-1} \sum_{k_2=k_1+1}^n \mathbb{E}\big[Z_{k_1}^2 Z_{k_2}^2\big]\Big) \\
     &\qquad = n\mathbb{E}[Z_1^4] + 3(n-1)n\mathbb{E}[Z_1^2]^2.
     \end{aligned}$$

b) For every $\epsilon > 0$ we have, using Markov's inequality with $p = 4$,

$$\begin{aligned}\mathbb{P}\Big(\Big|\frac{1}{n}\sum_{k=1}^{n} Z_k\Big| \geq \epsilon\Big) &\leq \frac{\mathbb{E}\big[\big(\sum_{k=1}^{n} Z_k\big)^4\big]}{n^4\epsilon^4} \\ &= \frac{n\mathbb{E}[Z_1^4] + 3n(n-1)\mathbb{E}[Z_1^2]^2}{n^4\epsilon^4} \\ &\leq \frac{c}{n^2\epsilon^4}\end{aligned}$$

for some finite constant $c > 0$. Therefore, since $\sum_{n=1}^{\infty} n^{-2} < +\infty$, it follows that

$$\sum_{n=1}^{\infty}\mathbb{P}\Big(\Big|\frac{1}{n}\sum_{k=1}^{n} Z_k\Big| \geq \epsilon\Big) \leq \sum_{n=1}^{\infty}\frac{c}{n^2\epsilon^4} < +\infty, \quad \forall \epsilon > 0$$

and thus, by Lemma 6.3 of the lecture notes, $\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n} Z_k = 0$, $\mathbb{P}$-almost surely.

2. Let $\tilde{Z}_k = Z_k - \mathbb{E}[Z_1]$ be such that $\mathbb{E}[\tilde{Z}_k] = 0$. Then, using the result of part 1.b) we have, $\mathbb{P}$-almost surely,

$$0 = \lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\tilde{Z}_k = \lim_{n\to+\infty}\Big(\frac{1}{n}\sum_{k=1}^{n} Z_k - \mathbb{E}[Z_1]\Big) = \lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n} Z_k - \mathbb{E}[Z_1]$$

showing that $\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n} Z_k = \mathbb{E}[Z_1]$, $\mathbb{P}$-almost surely.

Problem 4

1. To simplify the notation let $V_\epsilon = \{\theta : \|\theta - \theta_0\| \geq \epsilon\}$. Then,

$$\begin{aligned}\pi\big(V_\epsilon|X^{(n)}\big) &= \phi_n(X^{(n)})\pi(V_\epsilon|X^{(n)}) + \mathbb{1}_{D_{n,\delta,1}^c}(X^{(n)})\big(1 - \phi_n(X^{(n)})\big)\pi(V_\epsilon|X^{(n)}) \\ &\quad + \mathbb{1}_{D_{n,\delta,1}}(X^{(n)})\big(1 - \phi_n(X^{(n)})\big)\pi(V_\epsilon|X^{(n)}) \\ &\leq \phi_n(X^{(n)}) + \mathbb{1}_{D_{n,\delta,1}^c}(X^{(n)})\big(1 - \phi_n(X^{(n)})\big)\pi(V_\epsilon|X^{(n)}) + \mathbb{1}_{D_{n,\delta,1}}(X^{(n)}).\end{aligned}$$

Under Condition (A2) of Theorem 6.1, $\limsup_{n\to+\infty}\mathbb{E}_{\theta_0}[\phi_n(X^{(n)})] = 0$ while, using the result of Problem 1,

$$\limsup_{n\to+\infty}\mathbb{E}_{\theta_0}\big[\mathbb{1}_{D_{n,\delta,1}}(X^{(n)})\big] = \limsup_{n\to+\infty}\mathbb{P}_{\theta_0}\big(D_{n,\delta,1}\big) \leq \limsup_{n\to+\infty}\frac{1}{n\delta^2} = 0.$$

Therefore,

$$\limsup_{n\to+\infty}\mathbb{E}_{\theta_0}\big[\pi\big(V_\epsilon|X^{(n)}\big)\big] \leq \limsup_{n\to+\infty}\mathbb{E}_{\theta_0}\Big[\mathbb{1}_{D_{n,\delta,1}^c}(X^{(n)})\big(1 - \phi_n(X^{(n)})\big)\pi(V_\epsilon|X^{(n)})\Big].$$

2. Let $\mathcal{X}_1 = \mathbb{R}$, then

$$
\begin{aligned}
&\mathbb{E}_{\theta_0}\big[\mathbb{1}_{D^c_{n,\delta_n,1}}(X^{(n)})(1-\phi_n(X^{(n)}))\pi(V_\epsilon|X^{(n)})\big] \\
&= \int_{\mathcal{X}_1^n} \mathbb{1}_{D^c_{n,\delta_n,1}}(x^{(n)})(1-\phi_n(x^{(n)}))\frac{\int_{V_\epsilon}\pi(\theta)\prod_{i=1}^n \frac{\tilde f(x_i|\theta)}{\tilde f(x_i|\theta_0)}\mathrm{d}\theta}{\int_\Theta \pi(\theta)\prod_{i=1}^n \frac{\tilde f(x_i|\theta)}{\tilde f(x_i|\theta_0)}\mathrm{d}\theta}\prod_{k=1}^n \tilde f(x_k|\theta_0)\mathrm{d}x_k \\
&= \int_{\mathcal{X}_1^n} \mathbb{1}_{D^c_{n,\delta_n,1}}(x^{(n)})(1-\phi_n(x^{(n)}))\frac{\int_{V_\epsilon}\pi(\theta)\prod_{i=1}^n \tilde f(x_i|\theta)\mathrm{d}\theta}{\int_\Theta \pi(\theta)\prod_{i=1}^n \frac{\tilde f(x_i|\theta)}{\tilde f(x_i|\theta_0)}\mathrm{d}\theta}\prod_{k=1}^n \mathrm{d}x_k \\
&\leq \frac{e^{2n\delta^2}}{\pi(B_\delta)}\int_{\mathcal{X}_1^n}(1-\phi_n(x^{(n)}))\int_{V_\epsilon}\pi(\theta)\prod_{i=1}^n \tilde f(x_i|\theta)\mathrm{d}\theta\prod_{k=1}^n \mathrm{d}x_k
\end{aligned}
$$

where the inequality uses the definition of the set $D_{n,\delta_n,1}$.

In addition, using Tonelli's theorem,

$$
\begin{aligned}
&\int_{\mathcal{X}_1^n}(1-\phi_n(x^{(n)}))\int_{V_\epsilon}\pi(\theta)\prod_{i=1}^n \tilde f(x_i|\theta)\mathrm{d}\theta\prod_{k=1}^n \mathrm{d}x_k \\
&\quad= \int_{V_\epsilon}\Big(\int_{\mathcal{X}_1^n}(1-\phi_n(x^{(n)}))\prod_{i=1}^n \tilde f(x_i|\theta)\mathrm{d}x_i\Big)\pi(\theta)\mathrm{d}\theta \\
&\quad= \int_{V_\epsilon}\mathbb{E}_\theta\big[1-\phi_n(X^{(n)})\big]\pi(\theta)\mathrm{d}\theta \\
&\quad\leq \sup_{\theta\in V_\epsilon}\mathbb{E}_\theta\big[1-\phi_n(X^{(n)})\big]
\end{aligned}
$$

where, under Condition (A2) of Theorem 6.1,

$$\mathbb{E}_\theta\big[1-\phi_n(X^{(n)})\big] \leq e^{-nD_2}.$$

Consequently,

$$\mathbb{E}_{\theta_0}\big[\mathbb{1}_{D^c_{n,\delta_n,1}}(X^{(n)})(1-\phi_n(X^{(n)}))\pi(V_\epsilon|X^{(n)})\big] \leq \frac{e^{2n\delta^2}}{\pi(B_\delta)}e^{-nD_2}.$$

3. We can for instance assume that $\pi(\theta) > 0$ for all $\theta \in \Theta = \mathbb{R}^d$. Indeed, in this case we have, assuming first that $\delta \in (0,\gamma]$,

$$\pi(B_\delta) \geq \pi\big(\{\theta:\, \|\theta-\theta_0\| < \delta/c_\star\}\big) > 0$$

where the second inequality holds since the set $\{\theta:\, \|\theta-\theta_0\| < \delta/c_\star\}$ has a strictly positive volume while $\pi(\theta) > 0$ for all $\theta \in \Theta$.

We now consider the case $\delta > \gamma$. Since $B_\gamma \subset B_\delta$ we have

$$\pi(B_\delta) \geq \pi(B_\gamma) \geq \pi\big(\{\theta:\, \|\theta-\theta_0\| < \gamma/c_\star\}\big) > 0$$

as required.

4. Let $\epsilon > 0$ and $\delta = \sqrt{D_2/4}$ (remark that $D_2 > 0$ depends on $\epsilon > 0$). Then, $2\delta^2 - D_2 = -D_2/2$ and thus, using the result of part 2 and the assumption on $\pi(\theta)$,

$$\mathbb{E}_{\theta_0}\big[\mathbb{1}_{D^c_{n,\delta_n,1}}(X^{(n)})(1-\phi_n(X^{(n)}))\pi(V_\epsilon|X^{(n)})\big] \le \frac{e^{-n\frac{D}{2}}}{\pi(B_\delta)} \le \frac{e^{-n\frac{D}{2}}}{c_\delta}, \quad \forall n \ge 1$$

for some $c_\delta > 0$. Therefore,

$$\limsup_{n\to+\infty} \mathbb{E}_{\theta_0}\big[\mathbb{1}_{D^c_{n,\delta_n,1}}(X^{(n)})(1-\phi_n(X^{(n)}))\pi(V_\epsilon|X^{(n)})\big] = 0$$

which, together with the result of part 1, shows that

$$\limsup_{n\to+\infty} \mathbb{E}_{\theta_0}\big[\pi(V_\epsilon|X^{(n)})\big] = 0.$$

5. Let $\epsilon > 0$. Then, for every $\delta > 0$ we have,

$$\limsup_{n\to+\infty} \mathbb{P}_{\theta_0}\big(\pi(V_\epsilon|X^{(n)}) \ge \delta\big) \le \frac{\limsup_{n\to+\infty} \mathbb{E}_{\theta_0}\big[\pi(V_\epsilon|X^{(n)}))\big]}{\delta} = 0$$

where the inequality uses Markov's inequality (see Problem 3) and the equality uses the result of part 4. This shows that for every $\epsilon > 0$ we have

$$\pi\big(\{\theta : \|\theta-\theta_0\| \ge \epsilon\}\,\big|X^{(n)}\big) \to 0, \quad \text{in } \mathbb{P}_{\theta_0}\text{-probability},$$

as required.

Problem 5

1. a) Let $x > 0$. Then,

$$\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}\mathrm{d}t \le \frac{1}{\sqrt{2\pi}}\int_x^\infty \frac{t}{x}e^{-t^2/2}\mathrm{d}t = -\frac{1}{x\sqrt{2\pi}}e^{-t^2/2}\Big|_x^\infty = \frac{e^{-x^2/2}}{x\sqrt{2\pi}}.$$

   b) Let $\epsilon > 0$. Then,

$$\mathbb{E}_{\theta_0}[\psi_n(X^{(n)})] = \mathbb{P}_{\theta_0}\big(|\bar{X}_n - \theta_0| \ge \epsilon/2\big) = \mathbb{P}_{\theta_0}\big(|\bar{X}_n - \mathbb{E}_{\theta_0}[X_1]| \ge \epsilon/2\big) = \mathbb{P}\big(|Z| \ge \sqrt{n}\epsilon/2\big)$$

   where $Z \sim \mathcal{N}_1(0,1)$. Hence, using the result in part 1.a),

$$\mathbb{E}_{\theta_0}[\psi_n(X^{(n)})] = \mathbb{P}\big(|Z| \ge \sqrt{n}\epsilon/2\big) = 2\mathbb{P}\big(Z \ge \sqrt{n}\epsilon/2\big) \le 4\frac{e^{-n\epsilon^2/8}}{\sqrt{n}\epsilon\sqrt{2\pi}}.$$

Next, let $\theta \in \mathbb{R}$ be such that $|\theta - \theta_0| \geq \epsilon$. Then, using the *hint*, we have

$$\begin{aligned}
\mathbb{E}_\theta[1 - \psi_n(X^{(n)})] &= \mathbb{P}_\theta\big(|\bar{X}_n - \theta_0| < \epsilon/2\big) \\
&\leq \mathbb{P}_\theta\big(|\theta - \theta_0| - |\bar{X}_n - \theta| < \epsilon/2\big) \\
&= \mathbb{P}_\theta\big(|\bar{X}_n - \theta| > |\theta - \theta_0| - \epsilon/2\big) \\
&\leq \mathbb{P}_\theta\big(|\bar{X}_n - \theta| > |\theta - \theta_0| - |\theta - \theta_0|/2\big) \\
&= \mathbb{P}_\theta\big(|\bar{X}_n - \mathbb{E}_\theta[X_1]| > |\theta - \theta_0|/2\big) \\
&= \mathbb{P}_\theta\big(|\bar{X}_n - \mathbb{E}_\theta[X_1]| \geq |\theta - \theta_0|/2\big) \\
&= \mathbb{P}\big(|Z| \geq \sqrt{n}|\theta - \theta_0|/2\big)
\end{aligned}$$

where $Z \sim \mathcal{N}_1(0, 1)$. Hence, using the result in part 1.a),

$$\mathbb{E}_\theta[1 - \psi_n(X^{(n)})] \leq 4\frac{e^{-n|\theta-\theta_0|^2/8}}{\sqrt{n}|\theta - \theta_0|\sqrt{2\pi}}.$$

c) It is easily checked that, for $D_1 > 0$ and $D_2 > 0$, small enough we have

$$4\frac{e^{-n\epsilon^2/8}}{\sqrt{n}\epsilon\sqrt{2\pi}} \leq e^{-nD_1}, \quad \forall n \geq 1$$

and,

$$\sup_{\theta:|\theta-\theta_0|\geq\epsilon} 4\frac{e^{-n|\theta-\theta_0|^2/8}}{\sqrt{n}|\theta - \theta_0|\sqrt{2\pi}} \leq 4\frac{e^{-n\epsilon^2/8}}{\sqrt{n}\epsilon\sqrt{2\pi}} \leq e^{-nD_2}, \quad \forall n \geq 1.$$

Therefore, Condition (A2) of Theorem 6.1 holds for $\phi_n = \psi_n$, with ψ_n as in part 1.b).

2. Let $\phi_n(X^{(n)}) = \mathbb{1}(|\bar{X}_n - \theta_0| \geq M_n/(2\sqrt{n}))$. Then, using the result in part 1.b)

$$\limsup_{n\to+\infty} \mathbb{E}_{\theta_0}[\phi_n(X^{(n)})] \leq \limsup_{n\to+\infty} \frac{4\,e^{-M_n^2/8}}{M_n\sqrt{2\pi}} = 0$$

while, for all $n \in \mathbb{N}$,

$$\mathbb{E}_\theta[1 - \phi_n(X^{(n)})] \leq 4\frac{e^{-n|\theta-\theta_0|^2/8}}{\sqrt{n}|\theta - \theta_0|\sqrt{2\pi}} \leq 4\frac{e^{-n|\theta-\theta_0|^2/8}}{M_n\sqrt{2\pi}}$$

for all θ such that $|\theta - \theta_0| \geq M_n/\sqrt{n}$. For n large enough $4/(M_n\sqrt{2\pi}) \leq 1$ since $\lim_{n\to+\infty} M_n = +\infty$ and thus, for $\epsilon > 0$ arbitrary, we have for n large enough,

$$\mathbb{E}_\theta[1 - \phi_n(X^{(n)})] \leq e^{-n|\theta-\theta_0|^2/8} \leq e^{-n\frac{1}{8}(|\theta-\theta_0|^2\wedge\epsilon^2)}$$

for all θ such that $|\theta - \theta_0| \geq M_n/\sqrt{n}$.

3. a) Let $(\tilde{\theta}, \theta) \in \mathbb{R}^2$ and remark first that

$$\begin{aligned}\mathbb{E}_{\tilde{\theta}}[\log \tilde{f}(X_1|\tilde{\theta})] = \mathbb{E}_{\tilde{\theta}}\Big[-\frac{1}{2}\log(2\pi) - \frac{(X_1-\tilde{\theta})^2}{2}\Big] &= -\frac{1}{2}\log(2\pi) - \frac{1}{2}\mathrm{Var}_{\tilde{\theta}}(X_1) \\ &= -\frac{1}{2}\log(2\pi) - \frac{1}{2}.\end{aligned}$$

In addition

$$\begin{aligned}\mathbb{E}_{\tilde{\theta}}[\log \tilde{f}(X_1|\theta)] &= \mathbb{E}_{\tilde{\theta}}\Big[-\frac{1}{2}\log(2\pi) - \frac{(X_1-\theta)^2}{2}\Big] \\ &= -\frac{1}{2}\log(2\pi) - \frac{1}{2}\mathbb{E}_{\tilde{\theta}}\big[\big((X_1-\tilde{\theta}) + (\tilde{\theta}-\theta)\big)^2\big] \\ &= -\frac{1}{2}\log(2\pi) - \frac{1}{2}\mathrm{Var}_{\tilde{\theta}}(X_1) - (\tilde{\theta}-\theta)\mathbb{E}_{\tilde{\theta}}[X_1-\tilde{\theta}] - \frac{1}{2}(\tilde{\theta}-\theta)^2 \\ &= -\frac{1}{2}\log(2\pi) - \frac{1}{2} - \frac{1}{2}(\tilde{\theta}-\theta)^2\end{aligned}$$

so that

$$KL(\tilde{\theta}|\theta) = \mathbb{E}_{\tilde{\theta}}\left[\log\frac{\tilde{f}(X_1|\tilde{\theta})}{\tilde{f}(X_1|\theta)}\right] = \mathbb{E}_{\tilde{\theta}}[\log \tilde{f}(X_1|\tilde{\theta})] - \mathbb{E}_{\tilde{\theta}}[\log \tilde{f}(X_1|\theta)] = \frac{1}{2}(\tilde{\theta}-\theta)^2.$$

b) Let $x_1 \in \mathbb{R}$ and note that

$$\begin{aligned}\Big(\log \tilde{f}(x_1|\tilde{\theta}) - \log \tilde{f}(x_1|\theta)\Big)^2 &= \frac{1}{4}\Big((x_1-\theta)^2 - (x_1-\tilde{\theta})^2\Big)^2 \\ &= \frac{1}{4}\Big((\theta^2-\tilde{\theta}^2) - 2x_1(\theta-\tilde{\theta})\Big)^2 \\ &= \frac{1}{4}\Big((\theta^2-\tilde{\theta}^2)^2 + 4x_1^2(\theta-\tilde{\theta})^2 - 4x_1(\theta-\tilde{\theta})(\theta^2-\tilde{\theta}^2)\Big).\end{aligned}$$

Therefore, using the fact that $\mathbb{E}_{\tilde{\theta}}[X_1^2] = 1 + \tilde{\theta}^2$, we have

$$\begin{aligned}&\mathbb{E}_{\tilde{\theta}}\left[\left(\log\frac{\tilde{f}(X_1|\tilde{\theta})}{\tilde{f}(X_1|\theta)}\right)^2\right] \\ &= \frac{1}{4}\Big((\theta^2-\tilde{\theta}^2)^2 + 4(1+\tilde{\theta}^2)(\theta-\tilde{\theta})^2 - 4\tilde{\theta}(\theta-\tilde{\theta})(\theta^2-\tilde{\theta}^2)\Big) \\ &= \frac{1}{4}\Big(\big((\theta-\tilde{\theta})(\theta+\tilde{\theta})\big)^2 + 4(1+\tilde{\theta}^2)(\theta-\tilde{\theta})^2 - 4\tilde{\theta}(\theta-\tilde{\theta})^2(\theta+\tilde{\theta})\Big) \\ &= \frac{(\theta-\tilde{\theta})^2}{4}\Big((\theta+\tilde{\theta})^2 + 4(1+\tilde{\theta}^2) - 4\tilde{\theta}(\theta+\tilde{\theta})\Big).\end{aligned}$$

c) Let $v = \theta - \theta_0$. Then, using the result in part 3.b), we have

$$\mathbb{E}_{\theta_0}\left[\left(\log\frac{\tilde{f}(X_1|\theta_0)}{\tilde{f}(X_1|\theta)}\right)^2\right] = \frac{v^2}{4}\Big((v+2\theta_0)^2 + 4(1+\theta_0^2) - 4\theta_0(v+2\theta_0)\Big).$$

Let $\bar{v} > 0$ be arbitrary. Then,

$$\mathbb{E}_{\theta_0}\left[\left(\log\frac{\tilde{f}(X_1|\theta_0)}{\tilde{f}(X_1|\theta)}\right)^2\right] \leq (\theta-\theta_0)^2\,\frac{C_{\bar{v}}}{4},\quad C_{\bar{v}} := (\bar{v}+2\theta_0)^2+4(1+\theta_0^2)+4\big|\theta_0(\bar{v}+2\theta_0)\big|$$

for all $\theta \in \Theta$ verifying $|\theta - \theta_0| \leq \bar{v}$.

Using the result in part 3.a) it then follows that Condition (C3) of Problem 2 holds for $c_\star = \max(1/2, C_{\bar{v}}/4)$ and $\delta = \bar{v}$.

Problem 6

1. Let $q_{1-\alpha/2}$ be the $(1-\alpha/2)$-quantile of the $\mathcal{N}_1(0,1)$ distribution. Then, as the posterior distribution is Gaussian, the HPD region at level $1-\alpha$ is the interval

$$\big[\mu_n - q_{1-\alpha/2}\sqrt{\sigma_n^2}\,,\ \mu_n + q_{1-\alpha/2}\sqrt{\sigma_n^2}\big].$$

2. a) Let $\lambda_n = (1+n\sigma_0^2)^{-1}$ and $\bar{X}_n = n^{-1}\sum_{k=1}^n X_k$. Then, using the change of variable formula (see Problem Sheet 2, Problem 1)

$$\begin{aligned}
&\pi^*(s|X^{(n)})\\
&= \frac{1}{\sqrt{n}}\pi\Big(\hat{\theta}_n + \frac{s}{\sqrt{n}}\,|X^{(n)}\Big)\\
&= \sqrt{\frac{1+n\sigma_0^2}{2\pi n\sigma_0^2}}\exp\left(-(1+n\sigma_0^2)\frac{\Big(\lambda_n(\mu_0-\bar{X}_n)+\frac{s}{\sqrt{n}}\Big)^2}{2\sigma_0^2}\right)\\
&= \sqrt{\frac{1+n\sigma_0^2}{2\pi n\sigma_0^2}}\exp\left(-\frac{1+n\sigma_0^2}{2\,\sigma_0^2}\Big(\lambda_n^2(\mu_0-\bar{X}_n)^2+\frac{s^2}{n}+\frac{2s\lambda_n(\mu_0-\bar{X}_n)}{\sqrt{n}}\Big)^2\right)\\
&= \sqrt{\frac{1+n\sigma_0^2}{2\pi n\sigma_0^2}}\exp\left(-\frac{s^2}{2}-\frac{s^2}{2n\sigma_0^2}-\frac{1+n\sigma_0^2}{2\,\sigma_0^2}\Big(\lambda_n^2(\mu_0-\bar{X}_n)^2+\frac{2s\lambda_n(\mu_0-\bar{X}_n)}{\sqrt{n}}\Big)^2\right)
\end{aligned}$$

where, using the law of large numbers, $\lim_{n\to+\infty}\bar{X}_n \to \mathbb{E}[X_1] = \theta_0 \in (-\infty,+\infty)$, $\mathbb{P}_{\theta_0}$-almost surely, while

$$\lim_{n\to+\infty}\sqrt{\frac{1+n\sigma_0^2}{2\pi n\sigma_0^2}} = \frac{1}{\sqrt{2\pi}},\quad \lim_{n\to+\infty}\lambda_n^2 = 0.$$

Hence for all $s \in \mathbb{R}$,

$$\lim_{n\to+\infty}\pi^*(s|X^{(n)}) = \frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{2}},\quad \mathbb{P}_{\theta_0}\text{-almost surely}$$

as required.

b) Since for all $n \geq 1$ we have $\hat{\theta}_n \sim \mathcal{N}_1(\theta_0, n^{-1})$ it follows that

$$\big[\hat{\theta}_n - q_{1-\alpha/2}\, n^{-1/2}, \hat{\theta}_n + q_{1-\alpha/2}\, n^{-1/2}\big] = \big[\bar{X}_n - q_{1-\alpha/2}\, n^{-1/2}, \bar{X}_n + q_{1-\alpha/2}\, n^{-1/2}\big]$$

is a confidence interval at level $1-\alpha$.

c) As n increases the posterior distribution $\pi^*(s|X^{(n)})$ becomes more and more similar to the distribution of $\sqrt{n}(\hat{\theta}_n - \theta_0)$ which is used to compute the confidence interval. Hence, as $n \to +\infty$, the difference between the credible interval and the confidence vanishes.

More formally, remark that

$$|\mu_n - \bar{X}_n| = \frac{|\mu_0 - \bar{X}_n|}{1 + n\sigma_0^2}, \quad |\sigma_n^2 - n^{-1}| = \frac{1}{n(1+n\sigma_0^2)} = \mathcal{O}(n^{-2}) \qquad (1)$$

and let $L_n = \mu_n - q_{1-\alpha/2}\sqrt{\sigma_n^2}$ and $\tilde{L}_n = \bar{X}_n - q_{1-\alpha/2}\, n^{-1/2}$ be the left boundary of the credible interval and of the confidence interval, respectively. Then, by (1) and using the law of large numbers,

$$\big|L_n - \tilde{L}_n\big| = \mathcal{O}(n^{-1}), \quad \mathbb{P}_{\theta_0} - \text{almost surely}$$

meaning that, as n increases, the difference $\big|L_n - \tilde{L}_n\big|$ converges at rate $1/n$, which is much faster than the length of the confidence interval and of the credible interval, which are both order $\mathcal{O}(n^{-1/2})$.

The same result holds for the right boundaries.

3. a) Given the expression of the posterior distribution given in the statement of the question it should be clear that the new assumption on the data generating process does not modify the limiting distribution of $\sqrt{n}(\theta - \hat{\theta}_n)|X^{(n)}$. Thus, the result follows from part 2.a).

b) Using the central limit theorem,

$$\sqrt{n}\big(\hat{\theta}_n - \mathbb{E}_0[X_1]\big) \Rightarrow \mathcal{N}_1(0, \mathrm{Var}_0(X_1)) \qquad (2)$$

with $\mathrm{Var}_0(X_1) = 1.5$, and therefore

$$\big[\bar{X}_n - q_{1-\alpha/2}\sqrt{1.5}\, n^{-1/2}, \bar{X}_n + q_{1-\alpha/2}\sqrt{1.5}\, n^{-1/2}\big]$$

is a confidence interval at level $1-\alpha$.

c) As n increases the posterior distribution $\pi^*(s|X^{(n)})$ does not converge to the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n - \theta_0)$ which is used to compute the confidence interval. Hence, as $n \to +\infty$, the difference between the credible interval and the confidence do not vanish.

More formally, if as per above we let L_n and $\tilde{L}_n$ be the left boundary of the credible interval and of the confidence interval, respectively, it is easily checked that now we have

$$\big|L_n - \tilde{L}_n\big| = \mathcal{O}(n^{-1/2}), \quad \text{almost surely}$$

Hence, the difference $\left|L_n - \tilde{L}_n\right|$ is of same order than the length of the confidence interval and of the credible interval. The same result holds for the right boundaries.

4. The conclusion of this exercise is the following: when the model is well-specified a credible set at level $1-\alpha$ is a valid confidence interval at level $1-\alpha$. When the model is misspecified this is (in general) not the case.