Bayesian Modelling – Problem Sheet 3 (Solutions)

Problem 1 (the Jeffreys-Lindley's paradox)

1. Under H_0, $T_n \sim \mathcal{N}_1(0,1)$ and thus, with $Z \sim \mathcal{N}_1(0,1)$,

$$p(t_n) = \mathbb{P}(|Z| > |t_n|) = 2\Phi(-|t_n|)$$

where Φ is the c.d.f. of the $\mathcal{N}_1(0,1)$ distribution.

2. a) By definition,

$$\pi(\{0\}|x^{(n)}) = \frac{\rho_0 f(x^{(n)}|0)}{\int_\Theta f(x^{(n)}|\theta)\pi(\theta)\mathrm{d}\theta} = \frac{\rho_0 f(x^{(n)}|0)}{\rho_0 f(x^{(n)}|0) + (1-\rho_0)\int_\Theta f(x^{(n)}|\theta)g_1(\theta)\mathrm{d}\theta}$$

where

$$f(x^{(n)}|0) = \frac{1}{(2\pi\sigma^2)^{n/2}}\exp\Big(-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2\Big)$$

and

$$\begin{aligned}
\int_\Theta f(x^{(n)}|\theta)g_1(\theta)\mathrm{d}\theta &= \frac{1}{(2\pi\sigma^2)^{\frac{n+1}{2}}}\int_\Theta \exp\Big\{-\frac{1}{2\sigma^2}\Big(\sum_{i=1}^n (x_i-\theta)^2 + \theta^2\Big)\Big\}\mathrm{d}\theta \\
&= \frac{1}{(2\pi\sigma^2)^{\frac{n+1}{2}}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2}\int_\Theta \exp\Big\{-\frac{1}{2\sigma^2}\big(\theta^2(n+1) - 2\theta n\bar{x}_n\big)\Big\}\mathrm{d}\theta \\
&= \frac{1}{(2\pi\sigma^2)^{n/2}}\sqrt{\frac{1}{n+1}}\exp\Big\{-\frac{1}{2\sigma^2}\Big(\sum_{i=1}^n x_i^2 - \frac{(n\bar{x}_n)^2}{n+1}\Big)\Big\}.
\end{aligned}$$

Therefore,

$$\pi(\{0\}|x^{(n)}) = \frac{1}{1 + \frac{1-\rho_0}{\rho_0}(n+1)^{-1/2} e^{\frac{n(\sqrt{n}\bar{x}_n/\sigma)^2}{2(n+1)}}} = \Big[1 + \frac{1-\rho_0}{\rho_0}\sqrt{\frac{1}{n+1}}\exp\Big(\frac{nt_n^2}{2(n+1)}\Big)\Big]^{-1}.$$

b) We have

$$\begin{aligned}
B_{01}^\pi(x^{(n)}) = \frac{f(x^{(n)}|0)}{\int_\Theta f(x^{(n)}|\theta)g_1(\theta)\mathrm{d}\theta} &= (n+1)^{1/2}\exp\Big(-\frac{(n\bar{x}_n)^2}{2\sigma^2(n+1)}\Big) \\
&= (n+1)^{1/2}\exp\Big(-\frac{nt_n^2}{2(n+1)}\Big).
\end{aligned}$$

3. When $t_n = 1.96$, $n = 16\,818$ and $\rho_0 = 1/2$, we have

$$p(1.96) = 0.05, \quad \pi(\{0\}|x^{(n)}) = 0.95.$$

Therefore, as a frequentist statistician we are at 95% confident that $\theta \neq 0$ while, as a Bayesian statistician, we are 95% confident that $\theta = 0$!

4. We have

$$\lim_{n\to+\infty} p(t_n) = p(1.96) = 0.05, \quad \lim_{n\to+\infty} B_{01}^{\pi}(x^{(n)}) = +\infty$$

and therefore a Bayesian statistician always accepts H_0 for n sufficiently large while a frequentist statistician remains at 95% confident that H_0 is wrong.

Explanation of the paradox: Remark first that if t_n remains unchanged while n increases then $|\bar{x}_n|$ decreases with n. Now, this paradox arises because while the frequentist approach tests H_0 without any references to H_1 (i.e. the p-value does not depend on H_1) the Bayesian approach compares the inference under H_0 and under H_1. Since Θ_1 contains all possible values of θ that are not zero (and in particular values of θ such that $|\theta| \gg 0$), the Bayesian approach concludes that H_0 is more and more in agreement with the observations as $|\bar{x}_n|$ decreases.

It is however worth mentioning that while, in this example, the confidence of a frequentist statistician in H_0 remains unchanged as n increases he however considers that it is more and more likely that $|\theta|$ is small when the sample size increases. In other words, if for some arbitrary $\epsilon > 0$ we consider the test $H_0' : \theta \in (-\epsilon, \epsilon)$ against the alternative $H_1' : \theta \notin (-\epsilon, \epsilon)$, then both the Bayesian statistician and the frequentist statistician will be more and more confident that the null hypothesis H_0' is true as n increases. Therefore, the paradox described in this example will only arise for some specific testing problems where, informally speaking, Θ_1 is much bigger than Θ_0.

Problem 2

1. The likelihood function for x_1 is

$$\tilde{f}(x_1|\theta) = \frac{1}{\theta}\mathbb{1}_{(0,\theta)}(x_1) = \frac{1}{\theta}\mathbb{1}_{(x_1,+\infty)}(\theta), \quad \theta \in \Theta$$

and thus

$$f(x|\theta) = \prod_{k=1}^{n} \tilde{f}(x_k|\theta) = \frac{1}{\theta^n}\mathbb{1}_{(\max\{x_1,\dots,x_n\},+\infty)}(\theta).$$

Therefore

$$\begin{aligned}\pi(\theta|x) &\propto f(x|\theta)\pi(\theta) \propto \frac{1}{\theta^{\beta_0+1}}\mathbb{1}_{(\lambda_0,+\infty)}(\theta)\frac{1}{\theta^n}\mathbb{1}_{(\max\{x_1,\dots,x_n\},+\infty)}(\theta)\\ &\propto \frac{1}{\theta^{\beta_0+n+1}}\mathbb{1}_{(\max\{\lambda_0,x_1,\dots,x_n\},+\infty)}(\theta)\end{aligned}$$

and we deduce that $\theta|x \sim \text{Pareto}(\lambda_n, \beta_n)$ where

$$\lambda_n = \max\{\lambda_0, x_1, \ldots, x_n\} \quad \beta_n = \beta_0 + n.$$

2. For the posterior mean, we have

$$\mathbb{E}_\pi[\theta|x] = \int_0^\infty \theta\,\pi(\theta|x)\mathrm{d}\theta = \beta_n\lambda_n^{\beta_n}\int_{\lambda_n}^\infty \theta^{-\beta_n}\mathrm{d}\theta = \lambda_n\frac{\beta_n}{\beta_n - 1}.$$

The posterior median $\delta^\pi_{\text{med}}(x)$ is such that

$$\int_{\delta^\pi_{\text{med}}(x)}^\infty \pi(\theta|x)\mathrm{d}\theta = \frac{1}{2}$$

and thus, since

$$\int_{\delta^\pi_{\text{med}}(x)}^\infty \pi(\theta|x)\mathrm{d}\theta = \beta_n\lambda_n^{\beta_n}\int_{\delta^\pi_{\text{med}}(x)}^\infty \theta^{-(\beta_n+1)}\mathrm{d}\theta = \left(\frac{\lambda_n}{\delta^\pi_{\text{med}}(x)}\right)^{\beta_n}$$

we conclude that $\delta^\pi_{\text{med}}(x) = 2^{1/\beta_n}\lambda_n$.

3. $C^\pi_\alpha(x)$, the highest posterior density (HPD) region of θ at level $1-\alpha$, is given by

$$C^\pi_\alpha(x) = \{\theta : \pi(\theta|x) \geq \gamma_\alpha\}$$

where $\gamma_\alpha > 0$ is such that (because $\pi(\theta|x)$ is a continuous density)

$$\pi(C^\pi_\alpha(x)|x) = 1 - \alpha. \tag{1}$$

We first note that

$$\pi(\theta|x) \geq \gamma_\alpha \Longleftrightarrow \frac{\beta_n\lambda_n^{\beta_n}\mathbb{1}_{(\lambda_n,+\infty)}(\theta)}{\theta^{\beta_n+1}} \geq \gamma_\alpha \Longleftrightarrow \lambda_n \leq \theta \leq b_\alpha$$

where

$$b_\alpha = \left(\frac{\beta_n\lambda_n^{\beta_n}}{\gamma_\alpha}\right)^{1/(\beta_n+1)}.$$

Therefore, $C^\pi_\alpha(x) = [\lambda_n, b_\alpha]$ and it remains to find b_α such that (1) holds.
Since

$$\pi(C_\alpha(x)|x) = \int_{\lambda_n}^{b_\alpha} \pi(\theta|x)\mathrm{d}\theta = \beta_n\lambda_n^{\beta_n}\int_{\lambda_n}^{b_\alpha}\theta^{-(\beta_n+1)}\mathrm{d}\theta = 1 - \left(\frac{\lambda_n}{b_\alpha}\right)^{\beta_n}$$

we have $\pi(C^\pi_\alpha(x)|x) = 1-\alpha$ if and only if

$$\left(\frac{\lambda_n}{b_\alpha}\right)^{\beta_n} = \alpha \Longleftrightarrow b_\alpha = \frac{\lambda_n}{\alpha^{1/\beta_n}}$$

so that $C^\pi_\alpha(x) = [\lambda_n, \frac{\lambda_n}{\alpha^{1/\beta_n}}]$.

4. We have
$$B_{01}^{\pi}(x) = \frac{f(x|\theta^*)}{\int_0^{\infty} f(x|\theta)\pi(\theta)\mathrm{d}\theta} = \frac{\pi(\theta^*|x)}{\pi(\theta^*)} = \frac{\beta_n \lambda_n^{\beta_n}}{\beta_0 \lambda_0^{\beta_0}}(\theta^*)^{\beta_0-\beta_n}.$$

5. Let $\Theta_0 = (0, \theta^*)$. Then, by definition,
$$B_{01}^{\pi}(x) = \frac{\pi(\Theta_0|x)}{1-\pi(\Theta_0|x)} \frac{1-\pi(\Theta_0)}{\pi(\Theta_0)}$$
where
$$\pi(\Theta_0|x) = \int_{\lambda_n}^{\theta^*} \pi(\theta|x)\mathrm{d}\theta = \beta_n \lambda_n^{\beta_n} \int_{\lambda_n}^{\theta^*} \theta^{-\beta_n-1}\mathrm{d}\theta = 1 - \left(\frac{\lambda_n}{\theta^*}\right)^{\beta_n}$$
and, similarly,
$$\pi(\Theta_0) = \int_{\lambda_0}^{\theta^*} \pi(\theta)\mathrm{d}\theta = 1 - \left(\frac{\lambda_0}{\theta^*}\right)^{\beta_0}$$
Consequently,
$$B_{01}^{\pi}(x) = \frac{\left(\frac{\theta^*}{\lambda_n}\right)^{\beta_n} - 1}{\left(\frac{\theta^*}{\lambda_0}\right)^{\beta_0} - 1}.$$

Problem 3

1. The posterior distribution is given by
$$\pi(\theta|x) \propto f(x|\theta)\pi(\theta) \propto \frac{1}{\pi(1+(\theta-x)^2)}$$
and thus $\theta|x \sim \text{Cauchy}(x, 1)$.

   By definition, the posterior median $\delta_{\text{med}}^{\pi}(x)$ satisfies (since $\pi(\theta|x)$ is a continuous distribution)
$$\int_{\delta_{\text{med}}^{\pi}(x)}^{\infty} \pi(\theta|x)\mathrm{d}\theta = \frac{1}{2}.$$
   Since
$$\begin{aligned}\int_{\delta_{\text{med}}^{\pi}(x)}^{\infty} \pi(\theta|x)\mathrm{d}\theta = \frac{1}{\pi}\int_{\delta_{\text{med}}^{\pi}(x)}^{\infty} \frac{1}{1+(\theta-x)^2}\mathrm{d}\theta &= \frac{1}{\pi}\int_{\delta_{\text{med}}^{\pi}(x)-x}^{\infty} \frac{1}{1+z^2}\mathrm{d}z \\ &= \frac{1}{2} - \frac{1}{\pi}\arctan(\delta_{\pi}(x) - x)\end{aligned}$$
   we conclude that $\arctan(\delta_{\text{med}}^{\pi}(x) - x) = 0$ and thus that $\delta_{\text{med}}^{\pi}(x) = x$.

   Remark: It would be enough to say that $\delta_{\text{med}}^{\pi}(x) = x$ because, for any $\mu \in \mathbb{R}$ and $\sigma^2 \in \mathbb{R}_+$, the Cauchy$(\mu, \sigma^2)$ distribution is symmetric around μ.

2. $C_\alpha^\pi(x)$, the highest posterior density (HPD) region of θ at level $1-\alpha$ is defined by

$$C_\alpha^\pi(x) = \{\theta : \pi(\theta|x) \geq \gamma_\alpha\}$$

where γ_α is a positive quantity such that (because $\pi(\theta|x)$ is a continuous density)

$$\pi\big(C_\alpha(x)|x\big) = 1-\alpha. \tag{2}$$

We have

$$\begin{aligned}\pi(\theta|x) \geq \gamma_\alpha \Longleftrightarrow \frac{1}{\pi(1+(\theta-x)^2)} \geq \gamma_\alpha &\Longleftrightarrow (\theta-x)^2 \leq \frac{1}{\pi\gamma_\alpha} - 1 \\ &\Longleftrightarrow x - b_\alpha \leq \theta \leq x + b_\alpha\end{aligned}$$

where (for $\gamma_\alpha \leq 1/\pi$)

$$b_\alpha = \left(\frac{1}{\pi\gamma_\alpha} - 1\right)^{1/2}.$$

Therefore,

$$\begin{aligned}\pi\big(C_\alpha^\pi(x)|x\big) = \pi\big([x-b_\alpha, x+b_\alpha]|x\big) =& \int_{x-b_\alpha}^{x+b_\alpha} \pi(\theta|x)\mathrm{d}\theta \\ =& \frac{1}{\pi}\int_{x-b_\alpha}^{x+b_\alpha} \frac{1}{1+(\theta-x)^2}\mathrm{d}\theta \\ =& \frac{1}{\pi}\int_{-b_\alpha}^{b_\alpha} \frac{z}{1+z^2}\mathrm{d}z \\ =& \frac{2}{\pi}\arctan(b_\alpha)\end{aligned}$$

and (2) implies that

$$\arctan(b_\alpha) = \frac{\pi(1-\alpha)}{2} \Leftrightarrow b_\alpha = \tan\left(\frac{\pi(1-\alpha)}{2}\right).$$

Hence,

$$C_\alpha^\pi(x) = \left[x - \tan\left(\frac{\pi(1-\alpha)}{2}\right), x + \tan\left(\frac{\pi(1-\alpha)}{2}\right)\right]$$

3. Let $\Theta_0 = (-\infty, \theta^*)$. Then,

$$\begin{aligned}\pi(\Theta_0|x) = \int_{-\infty}^{\theta^*} \pi(\theta|x)\mathrm{d}\theta = \frac{1}{\pi}&\int_{-\infty}^{\theta^*} \frac{1}{1+(\theta-x)^2}\mathrm{d}\theta \\ =& \frac{1}{\pi}\int_{-\infty}^{\theta^*-x} \frac{1}{1+z^2}\mathrm{d}z \\ =& \frac{1}{\pi}\arctan(\theta^*-x) + \frac{1}{2}.\end{aligned}$$

Hence, the Bayes test can be formulated as follows: H_0 is accepted if

$$\frac{1}{\pi}\arctan(\theta^* - x) + \frac{1}{2} \geq \frac{a_1}{a_0 + a_1}$$

and rejected otherwise, where a_i $(i = 0, 1)$ is the cost of rejecting H_i when H_i is true.

4. By definition,

$$B_{01}^{\pi}(x) = \frac{m_0(x)}{m_1(x)}$$

where

$$m_0(x) = f(x|0) = \frac{1}{\pi(1 + x^2)}$$

and

$$m_1(x) = c \int_{\mathbb{R}} \frac{1}{\pi(1 + (x - \theta)^2)} \mathrm{d}\theta = c.$$

Therefore,

$$B_{01}^{\pi}(x) = \frac{1}{c\,\pi\,(1 + x^2)}$$

so that $B_{01}^{\pi}(x)$ increases (resp. decreases) as c decreases (resp. increases). The reason for this phenomenon is that the prior density

$$\pi(\theta) = \rho_0 \mathbb{1}_{\{0\}}(\theta) + (1 - \rho_0)\, c\, \mathbb{1}_{\{\theta \neq 0\}}, \quad \theta \in \Theta$$

gives relatively more (resp. less) 'weight' to H_0 when c decreases (resp. increases). (Remark: We write 'weight' between comas because $\pi(\Theta_1) = +\infty$, $\Theta_1 = \mathbb{R} \setminus \{0\}$.)

Problem 4

1. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Then, by definition,

$$B_{01}^{\pi}(x) = \frac{m_0(x)}{m_1(x)}$$

where

$$m_0(x) = f(x|0) = \frac{1}{(2\pi)^{n/2}} \exp\Big(-\frac{1}{2}\sum_{i=1}^{n} x_i^2\Big).$$

and

$$
\begin{aligned}
m_1(x) &= \int_\Theta f(x|\theta) g_1(\theta) \mathrm{d}\theta \\
&= \frac{1}{(2\pi)^{n/2}} \frac{1}{\sqrt{2\pi\sigma_0^2}} \int_\Theta \exp\Big(-\frac{1}{2}\sum_{i=1}^n (x_i-\theta)^2 - \frac{1}{2\sigma_0^2}(\theta-\mu_0)^2 \Big) \mathrm{d}\theta \\
&= \frac{1}{(2\pi)^{n/2}} \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{1}{2}\left(\sum_{i=1}^n x_i^2 + \frac{\mu_0^2}{\sigma_0^2}\right)} \int_\Theta \exp\Big\{ -\frac{1}{2}\Big(\theta^2 (n+1/\sigma_0^2) - 2\theta(n\bar{x}_n + \mu_0/\sigma_0^2) \Big) \Big\} \mathrm{d}\theta \\
&= \sqrt{\frac{\sigma_n^2}{(2\pi)^n \sigma_0^2}} \exp\Big\{ -\frac{1}{2}\Big(\sum_{i=1}^n x_i^2 + \frac{\mu_0^2}{\sigma_0^2} - \frac{\mu_n^2}{\sigma_n^2} \Big) \Big\}
\end{aligned}
$$

where

$$\sigma_n^2 = \frac{1}{n+1/\sigma_0^2}, \quad \mu_n = \sigma_n^2 \big(n\bar{x}_n + \mu_0/\sigma_0^2 \big).$$

Hence,

$$m_1(x) = \frac{1}{(2\pi)^{n/2}} \Big(\frac{1}{n\sigma_0^2+1} \Big)^{1/2} \exp\Big\{ -\frac{1}{2}\Big(\sum_{i=1}^n x_i^2 + \frac{\mu_0^2}{\sigma_0^2} \Big) + \frac{(n\bar{x}_n + \mu_0/\sigma_0^2)^2}{2(n+1/\sigma_0^2)} \Big\}$$

so that

$$B_{01}^\pi(x) = (n\sigma_0^2+1)^{1/2} \exp\Big(\frac{\mu_0^2}{2\sigma_0^2} - \frac{(n\bar{x}_n + \mu_0/\sigma_0^2)^2}{2(n+1/\sigma_0^2)} \Big).$$

Note that

$$\frac{1}{2\sigma_0^2(n+1/\sigma_0^2)} + \frac{n}{2(n+1/\sigma_0^2)} = \frac{1}{2}$$

so that

$$
\begin{aligned}
&\frac{(n\bar{x}_n + \mu_0/\sigma_0^2)^2}{2(n+1/\sigma_0^2)} - \frac{\mu_0^2}{2\sigma_0^2} \\
&\qquad = \frac{(n\bar{x}_n)^2 + (\mu_0/\sigma_0^2)^2 + 2n\bar{x}_n\mu_0/\sigma_0^2}{2(n+1/\sigma_0^2)} - \frac{\mu_0^2}{2\sigma_0^2} \\
&\qquad = \frac{n}{2(n+1/\sigma_0^2)} n\bar{x}_n^2 + \frac{(\mu_0/\sigma_0^2)^2 + 2n\bar{x}_n\mu_0/\sigma_0^2}{2(n+1/\sigma_0^2)} - \frac{\mu_0^2}{2\sigma_0^2} \\
&\qquad = \frac{n\bar{x}_n^2}{2} - \frac{n}{2\sigma_0^2(n+1/\sigma_0^2)}\bar{x}_n^2 - \frac{n}{2\sigma_0^2(n+1/\sigma_0^2)}\mu_0^2 + \frac{n}{2\sigma_0^2(n+1/\sigma_0^2)} 2n\bar{x}_n\mu_0 \\
&\qquad = \frac{n\bar{x}_n^2}{2} - \frac{n(\bar{x}_n - \mu_0)^2}{2\sigma_0^2(n+1/\sigma_0^2)}.
\end{aligned}
$$

Finally, using the identity

$$n\bar{x}_n^2 = \sum_{i=1}^n x_i^2 - \sum_{i=1}^n (x_i - \bar{x}_n)^2$$

we obtain

$$B_{01}^{\pi}(x) = (n\sigma_0^2 + 1)^{1/2} \exp\left(\frac{\sum_{i=1}^n (x_i - \bar{x}_n)^2}{2} - \frac{\sum_{i=1}^n x_i^2}{2} + \frac{n(\bar{x}_n - \mu_0)^2}{2\sigma_0^2(n + 1/\sigma_0^2)} \right)$$

and thus

$$\log B_{01}^{\pi}(x) = -\frac{\sum_{i=1}^n x_i^2}{2} + \frac{\sum_{i=1}^n (x_i - \bar{x}_n)^2}{2} + \frac{n(\bar{x}_n - \mu_0)^2}{2\sigma_0^2(n + 1/\sigma_0^2)} + \frac{1}{2}\log\left(n\sigma_0^2 + 1\right).$$

2. We first remark that

$$-\frac{\sum_{i=1}^n X_i^2}{2} + \frac{\sum_{i=1}^n (X_i - \bar{X}_n)^2}{2} = -\frac{n\bar{X}_n^2}{2} \tag{3}$$

so that, for any $n \geq 2$, we have

$$\frac{\log B_{01}^{\pi}(X^{(n)})}{\log(n)} = -\frac{n\bar{X}_n^2}{2\log(n)} + \frac{n(\bar{X}_n - \mu_0)^2}{\log(n) 2\sigma_0^2(n + 1/\sigma_0^2)} + \frac{1}{2}\frac{\log\left(n\sigma_0^2 + 1\right)}{\log(n)}. \tag{4}$$

Under H_0, the observations $X_1, \dots, X_n$ are i.i.d. $\mathcal{N}_1(0,1)$ random variables and therefore $\sqrt{n}\bar{X}_n \sim \mathcal{N}_1(0,1)$. Consequently, for every $\epsilon > 0$ we have

$$\mathbb{P}_0\Big(\Big|\frac{n\bar{X}_n^2}{2\log(n)}\Big| \geq \epsilon\Big) = \mathbb{P}_0\Big(|\sqrt{n}\bar{X}_n| \geq \sqrt{2\log(n)\epsilon}\Big) = 2\Phi\Big(-\sqrt{2\log(n)\epsilon}\Big) \tag{5}$$

where $\Psi(\cdot)$ is the c.d.f. of the $\mathcal{N}_1(0,1)$ distribution. Since $\lim_{x\to+\infty}\Phi(-x) = 0$, we have, using (5),

$$\lim_{n\to+\infty} \mathbb{P}_0\Big(\Big|\frac{n\bar{X}_n^2}{2\log(n)}\Big| \geq \epsilon\Big) = 0, \quad \forall \epsilon > 0. \tag{6}$$

Hence, $\frac{n\bar{X}_n^2}{2\log(n)} \to 0$ in $\mathbb{P}_0$-probability.

Next, we remark that, for any $\theta \in \Theta$ we have, by the law of large numbers,

$$\lim_{n\to+\infty} \bar{X}_n = \mathbb{E}_\theta[X_1] = \theta, \quad \mathbb{P}_\theta\text{-almost surely}$$

so that

$$\lim_{n\to+\infty} \frac{n(\bar{X}_n - \mu_0)^2}{2\sigma_0^2(n + 1/\sigma_0^2)} = \frac{(\theta - \mu_0)^2}{2\sigma_0^2}, \quad \mathbb{P}_\theta\text{-almost surely}, \quad \forall \theta \in \Theta. \tag{7}$$

Using this result with $\theta = 0$ yields

$$\lim_{n\to+\infty} \frac{n(\bar{X}_n - \mu_0)^2}{\log(n) 2\sigma_0^2(n + 1/\sigma_0^2)} = 0, \quad \mathbb{P}_0\text{-almost surely}. \tag{8}$$

Lastly, using l'Hospital's rule, it is easily checked that

$$\lim_{n\to+\infty} \frac{\log\left(n\sigma_0^2+1\right)}{2\log(n)} = \frac{1}{2\sigma_0^2} \tag{9}$$

and therefore, using (4), (6), (8) and (9), we conclude that

$$\frac{\log B_{01}^{\pi}(X^{(n)})}{\log(n)} \to \frac{1}{2\sigma_0^2} > 0, \quad \text{in } \mathbb{P}_0\text{-probability.}$$

3. Using (3) we have

$$\frac{\log B_{01}^{\pi}(X^{(n)})}{-n} = \frac{\bar{X}_n^2}{2} - \frac{(\bar{X}_n-\mu_0)^2}{2\sigma_0^2(n+1/\sigma_0^2)} - \frac{1}{2}\frac{\log\left(n\sigma_0^2+1\right)}{n}$$

where

$$\lim_{n\to+\infty} \frac{1}{2}\frac{\log\left(n\sigma_0^2+1\right)}{n} = 0.$$

Under H_1, the observations $X_1,\ldots,X_n$ are i.i.d. $\mathcal{N}_1(\theta,1)$ random variables for some $\theta \neq 0$, and thus, by the law of large numbers, $\bar{X}_n^2 \to \theta^2 > 0$, $\mathbb{P}_\theta$-almost surely. Using (7),

$$\lim_{n\to+\infty} \frac{(\bar{X}_n-\mu_0)^2}{2\sigma_0^2(n+1/\sigma_0^2)} = 0, \quad \mathbb{P}_\theta\text{-almost surely}$$

and therefore

$$\lim_{n\to+\infty} \frac{\log B_{01}^{\pi}(X^{(n)})}{-n} = \frac{\theta^2}{2} > 0, \quad \mathbb{P}_\theta\text{-almost surely.}$$