DP solutions

4.1

Answer

$q_\pi(11, down) = 1$ and -15.0

4.2

Answer

- $v_\pi(15) = -20$
- $v_\pi(13) = -20$

4.3

Answer

$$q_{k+1}(s,a) = \sum_{s',r} p(s',r|s,a) \left[r + \gamma \sum_{a'} \pi(a'|s) q_k(s',a') \right]$$

4.4

Answer

$$\text{use if } old-action \neq \pi(s) \text{ and the Q-value of } \pi(s) > \text{that of } old-action$$

4.5

Answer

1. Initialization

   initialize $Q(s,a)$ and $\pi(s)$ arbitrarily for every $s \in S$ and $a \in A(s)$.
2. Policy Evaluation

   loop:

   the update equation is:

   $Q(s,a) \leftarrow \sum_{s',r} p(s',r|s,a)[r + \gamma \sum_{a'} \pi(a'|s')Q(s',a')]$
3. Policy Improvement

   the update is:

   for each $s \in S$:

   $\pi(s) \leftarrow arg\max_a Q(s,a)$

4.6

Answer

1. no change
2. $V(s) \leftarrow \sum_{a \in A(s)} \pi(a|s) \sum_{s',r} p(s',r|s,a)[r + \gamma V(s')]$
3. $\pi(a|s) = \max(0, 1(a = arg\max_a \sum_{s',r} p(s',r|s,a)[r + \gamma V(s')]) - \epsilon) + \frac{\epsilon}{|A(S)|}$

4.7

In [1]:

```
%matplotlib notebook
import numpy as np
import matplotlib
import matplotlib.pyplot as plt
from math import exp, factorial
import seaborn as sns
# maximum # of cars in each location
MAX_CARS = 10
# maximum # of cars to move during night
MAX_MOVE_OF_CARS = 5
# expectation for rental requests in first location
RENTAL_REQUEST_FIRST_LOC = 3
# expectation for rental requests in second location
RENTAL_REQUEST_SECOND_LOC = 4
# expectation for # of cars returned in first location
RETURNS_FIRST_LOC = 3
# expectation for # of cars returned in second location
RETURNS_SECOND_LOC = 2
DISCOUNT = 0.9
# credit earned by a car
RENTAL_CREDIT = 10
# cost of moving a car
MOVE_CAR_COST = 2

EXTRA_PLACE_COST = 4

actions = np.arange(-MAX_MOVE_OF_CARS, MAX_MOVE_OF_CARS + 1)
POISSON_UPPER_BOUND = 11
# Probability for poisson distribution
# @lam: lambda should be less than 10 for this function
poisson_cache = dict()
def poisson(n, lam):
    global poisson_cache
    key = n * 10 + lam
    if key not in poisson_cache.keys():
        poisson_cache[key] = exp(-lam) * pow(lam, n) / factorial(n)
    return poisson_cache[key]

# @state: [# of cars in first location, # of cars in second location]
# @action: positive if moving cars from first location to second location,
#          negative if moving cars from second location to first location
# @stateValue: state value matrix
# @constant_returned_cars:  if set True, model is simplified such that
#   the # of cars returned in daytime becomes constant
#   rather than a random value from poisson distribution, which will reduce calculation time
#   and leave the optimal policy/value state matrix almost the same
def expected_return(state, action, state_value, constant_returned_cars):
    # initailize total return
    returns = 0.0

    # cost for moving cars
    returns -= MOVE_CAR_COST * abs(action)

    returns += (action > 0) * MOVE_CAR_COST

    returns -= max(state[1] + action - MAX_CARS, 0) * EXTRA_PLACE_COST

    # go through all possible rental requests
    for rental_request_first_loc in range(0, POISSON_UPPER_BOUND):
```

```
        for rental_request_second_loc in range(0, POISSON_UPPER_BOUND):
            # moving cars
            num_of_cars_first_loc = int(min(state[0] - action, MAX_CARS))
            num_of_cars_second_loc = int(min(state[1] + action, MAX_CARS))

            # valid rental requests should be less than actual # of cars
            real_rental_first_loc = min(num_of_cars_first_loc, rental_request_first_loc)
            real_rental_second_loc = min(num_of_cars_second_loc, rental_request_second_loc)

            # get credits for renting
            reward = (real_rental_first_loc + real_rental_second_loc) * RENTAL_CREDIT

            num_of_cars_first_loc -= real_rental_first_loc
            num_of_cars_second_loc -= real_rental_second_loc

            # probability for current combination of rental requests
            prob = poisson(rental_request_first_loc, RENTAL_REQUEST_FIRST_LOC) * \
                   poisson(rental_request_second_loc, RENTAL_REQUEST_SECOND_LOC)

            if constant_returned_cars:
                # get returned cars, those cars can be used for renting tomorrow
                returned_cars_first_loc = RETURNS_FIRST_LOC
                returned_cars_second_loc = RETURNS_SECOND_LOC
                num_of_cars_first_loc = min(num_of_cars_first_loc + returned_cars_first_loc, MAX
_CARS)
                num_of_cars_second_loc = min(num_of_cars_second_loc + returned_cars_second_loc,
MAX_CARS)
                returns += prob * (reward + DISCOUNT * state_value[num_of_cars_first_loc, num_of
_cars_second_loc])
            else:
                for returned_cars_first_loc in range(0, POISSON_UPPER_BOUND):
                    for returned_cars_second_loc in range(0, POISSON_UPPER_BOUND):
                        num_of_cars_first_loc_ = min(num_of_cars_first_loc + returned_cars_first
_loc, MAX_CARS)
                        num_of_cars_second_loc_ = min(num_of_cars_second_loc + returned_cars_sec
ond_loc, MAX_CARS)
                        prob_ = poisson(returned_cars_first_loc, RETURNS_FIRST_LOC) * \
                                poisson(returned_cars_second_loc, RETURNS_SECOND_LOC) * prob
                        returns += prob_ * (reward + DISCOUNT * state_value[num_of_cars_first_lo
c_, num_of_cars_second_loc_])
    return returns
```

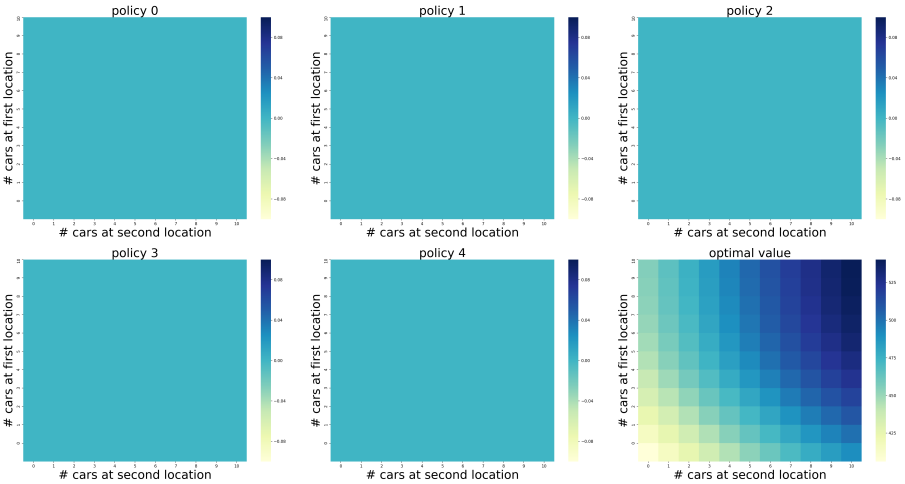
In [12]:

```
def q_4_7():
    value = np.zeros((MAX_CARS+1, MAX_CARS+1))
    policy = np.zeros(value.shape, dtype=np.int)

    iterations = 0

    _, axes = plt.subplots(2,3,figsize=(40,20))
    plt.subplots_adjust(wspace=0.1, hspace=0.2)
    axes = axes.flatten()
    while True:
        print(iterations)
        fig = sns.heatmap(np.flipud(policy), cmap="YlGnBu", ax=axes[iterations])
        fig.set_ylabel('# cars at first location', fontsize=30)
        fig.set_yticks(list(reversed(range(MAX_CARS + 1))))
        fig.set_xlabel('# cars at second location', fontsize=30)
        fig.set_title('policy %d' % (iterations), fontsize=30)
        delta = 0
        while True:
            new_value = value.copy()
            for i in range(MAX_CARS + 1):
                for j in range(MAX_CARS + 1):
                    new_value[i, j] = expected_return([i, j], policy[i, j], new_value,
                                                      True)
            delta = np.abs(new_value-value).sum()
            value = new_value
            print("delta:", delta)
            if delta < 1e-10:
                break
        new_policy = policy.copy()
        policy_changed = False
        for i in range(MAX_CARS + 1):
            for j in range(MAX_CARS + 1):
                actions_rewards = []
                for action in actions:
                    if (action >= 0 and i >= action) or (action < 0 and j >= abs(action)):
                        actions_rewards.append(expected_return([i,j],action,value,True))
                    else:
                        actions_rewards.append(-float('inf'))
                new_policy[i,j] = actions[np.argmax(actions_rewards)]
        policy_changed = (policy != new_policy).sum()
        print("policy_changed:", policy_changed)
        if not policy_changed or iterations >= 3:
            fig = sns.heatmap(np.flipud(value), cmap="YlGnBu", ax=axes[-1])
            fig.set_ylabel('# cars at first location', fontsize=30)
            fig.set_yticks(list(reversed(range(MAX_CARS + 1))))
            fig.set_xlabel('# cars at second location', fontsize=30)
            fig.set_title('optimal value', fontsize=30)

            # policy
            fig = sns.heatmap(np.flipud(policy), cmap="YlGnBu", ax=axes[iterations+1])
            fig.set_ylabel('# cars at first location', fontsize=30)
            fig.set_yticks(list(reversed(range(MAX_CARS + 1))))
            fig.set_xlabel('# cars at second location', fontsize=30)
            fig.set_title('policy %d' % (iterations+1), fontsize=30)


            break
        iterations += 1
    plt.savefig("./img/4_7.png")
```

Answer

In [13]:

```
q_4_7()
```


policy 0
policy 1
policy 2
policy 3
policy 4
optimal value
cars at first location
cars at second location

0
delta: 9529.606671661502
delta: 8024.010091631562
delta: 6283.614642053212
delta: 5113.226733767827
delta: 4291.926258695471
delta: 3657.229426767363
delta: 3133.5252787803174
delta: 2688.2670121865685
delta: 2305.5766356374443
delta: 1975.7947433312145
delta: 1691.741111170458
delta: 1447.4205042482984
delta: 1237.5939605802484
delta: 1057.6346591974336
delta: 903.4646068175791
delta: 771.5083850630579
delta: 658.6478872774259
delta: 562.1761883853521
delta: 479.751963729502
delta: 409.35602275086154
delta: 349.2510036269969
delta: 297.9447601450647
delta: 254.1575985453207
delta: 216.79328269127245
delta: 184.91358604889865
delta: 157.71609656069597
delta: 134.51495165904652
delta: 114.7241787982594
delta: 97.84333089773406
delta: 83.44512872295047
delta: 71.16484884130733
delta: 60.69122347736737
delta: 51.75864561957712
delta: 44.1404981121907
delta: 37.6434487066706
delta: 32.10257395633204
delta: 27.377193414328246
delta: 23.34731194568792
delta: 19.91058225631747
delta: 16.979712167419052
delta: 14.480251922772709
delta: 12.348706103302334
delta: 10.530922717601982
delta: 8.980718908243034
delta: 7.658708607456106
delta: 6.5313025265390365
delta: 5.569855187565963
delta: 4.7499374042895965
delta: 4.050715781460383
delta: 3.4544235029713946
delta: 2.9459089877058773
delta: 2.5122509615326294
delta: 2.1424301765446785
delta: 1.827049444113186
delta: 1.5580948730172395
delta: 1.3287322497449736
delta: 1.133133389404179
delta: 0.9663280464183686
delta: 0.8240776233890301
delta: 0.7027674697137059

delta: 0.5993150338376267
delta: 0.5110915356207784
delta: 0.43585516865903173
delta: 0.3716941355502854
delta: 0.3169780685166188
delta: 0.2703166011618805
delta: 0.2305240387934191
delta: 0.1965892292804483
delta: 0.16764986926182246
delta: 0.14297059257137334
delta: 0.12192428421292334
delta: 0.10397614495036578
delta: 0.08867010193711167
delta: 0.07561721943017119
delta: 0.06448581591297398
delta: 0.05499303575868453
delta: 0.0468976617225394
delta: 0.03999398549848365
delta: 0.03410658049944004
delta: 0.02908584426950256
delta: 0.024804196846673676
delta: 0.02115283898706366
delta: 0.018038987521606487
delta: 0.01538351761365675
delta: 0.01311895213069647
delta: 0.011187747129667969
delta: 0.009540829521029082
delta: 0.008136350153108651
delta: 0.006938620342964441
delta: 0.005917205061393815
delta: 0.005046149512622833
delta: 0.004303319661005389
delta: 0.003669839770168437
delta: 0.003129612719078523
delta: 0.0026689110031270 5
delta: 0.0022760279227895808
delta: 0.0019409800736411853
delta: 0.001655253732451456
delta: 0.0014115883643626148
delta: 0.0012037923116849925
delta: 0.0010265853364899158
delta: 0.000875464520447 6219
delta: 0.0007465897866723026
delta: 0.0006366863755715713
delta: 0.000542961501366 7899
delta: 0.00046303363023980637
delta: 0.00039487170619167955
delta: 0.00033674371798042557
delta: 0.000287171726178454992
delta: 0.00024489871202604263
delta: 0.000208847852377 4029
delta: 0.00017810394479056413
delta: 0.00015188574764124496
delta: 0.00012952704611279842
delta: 0.00011045972843248819
delta: 9.419925277143193e-05
delta: 8.033242380633965e-05
delta: 6.850692147963855e-05
delta: 5.842220815566179e-05
delta: 4.982203637382554e-05
delta: 4.248787001870369e-05

delta: 3.6233346065728256e-05
delta: 3.0899538160156226e-05
delta: 2.6350899531735195e-05
delta: 2.2471855402272922e-05
delta: 1.9163845024650072e-05
delta: 1.6342769185939687e-05
delta: 1.3936988409568585e-05
delta: 1.1885378114584455e-05
delta: 1.0135762011032057e-05
delta: 8.643711282729782e-06
delta: 7.3712890866772796e-06
delta: 6.286183065640216e-06
delta: 5.360817908695026e-06
delta: 4.571664703689748e-06
delta: 3.898677732649958e-06
delta: 3.324765032175492e-06
delta: 2.835341092577437e-06
delta: 2.4179516913136467e-06
delta: 2.0620130385395896e-06
delta: 1.7584688407623617e-06
delta: 1.499605957633321e-06
delta: 1.278846752938989e-06
delta: 1.0905995964094473e-06
delta: 9.300632086706173e-07
delta: 7.931386107884464e-07
delta: 6.76379272590566e-07
delta: 5.768138748862839e-07
delta: 4.919053822050046e-07
delta: 4.1951295770559227e-07
delta: 3.577410438992956e-07
delta: 3.0508101644954877e-07
delta: 2.601656206024927e-07
delta: 2.2187794002093142e-07
delta: 1.892099703582062e-07
delta: 1.6135322766785976e-07
delta: 1.375994997943053e-07
delta: 1.1735755833797157e-07
delta: 1.000764768832596e-07
delta: 8.533720574632753e-08
delta: 7.276770475073135e-08
delta: 6.207005753822159e-08
delta: 5.292457672112505e-08
delta: 4.513549356488511e-08
delta: 3.848958840535488e-08
delta: 3.283071237092372e-08
delta: 2.800055654006428e-08
delta: 2.386877895332873e-08
delta: 2.0348977614048636e-08
delta: 1.736015065034735e-08
delta: 1.4802083114773268e-08
delta: 1.2628902368305717e-08
delta: 1.0770577318908181e-08
delta: 9.189875527226832e-09
delta: 7.827225090295542e-09
delta: 6.6806364884541836e-09
delta: 5.692129434464732e-09
delta: 4.84988049720414e-09
delta: 4.141668341617333e-09
delta: 3.5395260056247935e-09
delta: 3.0168507691996638e-09
delta: 2.5721647034515627e-09

```
delta: 2.192734882555669e-09
delta: 1.8733885553956497e-09
delta: 1.6054855223046616e-09
delta: 1.3577050594903994e-09
delta: 1.1553993317647837e-09
delta: 9.926566235662904e-10
delta: 8.378719940083101e-10
delta: 7.226503839774523e-10
delta: 6.166942512209062e-10
delta: 5.111360223963857e-10
delta: 4.381490725791082e-10
delta: 3.743707566172816e-10
delta: 3.311697582830675e-10
delta: 2.743831828411203e-10
delta: 2.3788970793248154e-10
delta: 2.020783540501725e-10
delta: 1.801367943699006e-10
delta: 1.4921397450962104e-10
delta: 1.2960299500264227e-10
delta: 1.0902567737502977e-10
delta: 8.930101103032939e-11
policy_changed: 87
1
delta: 8.265033102361485e-11
policy_changed: 87
2
delta: 6.320988177321851e-11
policy_changed: 87
3
delta: 5.3717030823463574e-11
policy_changed: 87
```

4.8

Answer

I think this unexpected way can maximize the probability of winning this game, since the probability of winning when the capital is 40 is 0.4, which is quite high.

4.9

In [52]:

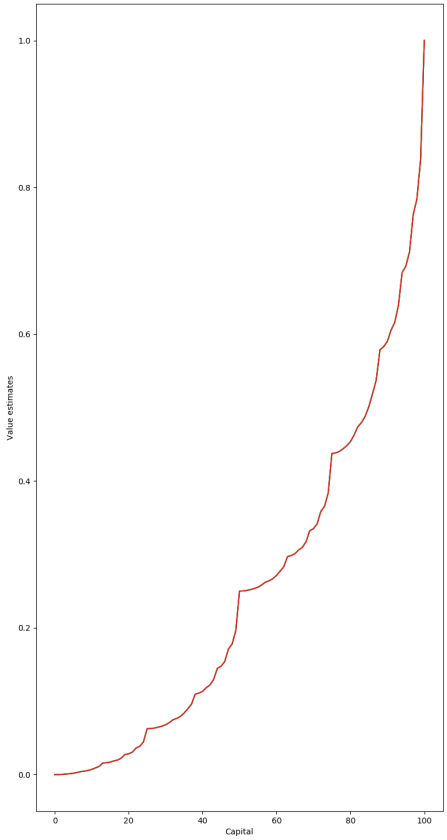
```
# %matplotlib notebook
import numpy as np
import matplotlib
import matplotlib.pyplot as plt
import warnings
warnings.filterwarnings("ignore")

# goal
GOAL = 100

# all states, including state 0 and state 100
STATES = np.arange(GOAL + 1)

# probability of head
HEAD_PROBS = [0.25,0.55]
THETAS = np.arange(-12,-8)

def q_4_9():
    # state value

    state_value = np.zeros(GOAL + 1)
    state_value[0] = 0
    state_value[GOAL] = 1.0
    plt.figure(figsize=(20, 40))
    for i,HEAD_PROB in enumerate(HEAD_PROBS):
        plt.subplot(1,2,i+1)
        plt.title("HEAD_PROB is %f" %HEAD_PROB)
        for THETA in THETAS:
            # value iteration
            while True:
                delta = 0.0
                for state in STATES[1:GOAL]:
                    # get possilbe actions for current state
                    actions = np.arange(min(state, GOAL - state) + 1)
                    action_returns = []
                    for action in actions:
                        action_returns.append(
                            HEAD_PROB * state_value[state + action] + (1 - HEAD_PROB) * state_va
lue[state - action])
                    new_value = np.max(action_returns)
                    delta += np.abs(state_value[state] - new_value)
                    # update state value
                    state_value[state] = new_value
                if delta < float("1e%d"%THETA):
                    break

            # compute the optimal policy
            policy = np.zeros(GOAL + 1)
            for state in STATES[1:GOAL]:
                actions = np.arange(min(state, GOAL - state) + 1)
                action_returns = []
                for action in actions:
                    action_returns.append(
                        HEAD_PROB * state_value[state + action] + (1 - HEAD_PROB) * state_value[
state - action])
                action_returns[0] = 0
                policy[state] = actions[np.argmax(np.round(action_returns, 5))]
```

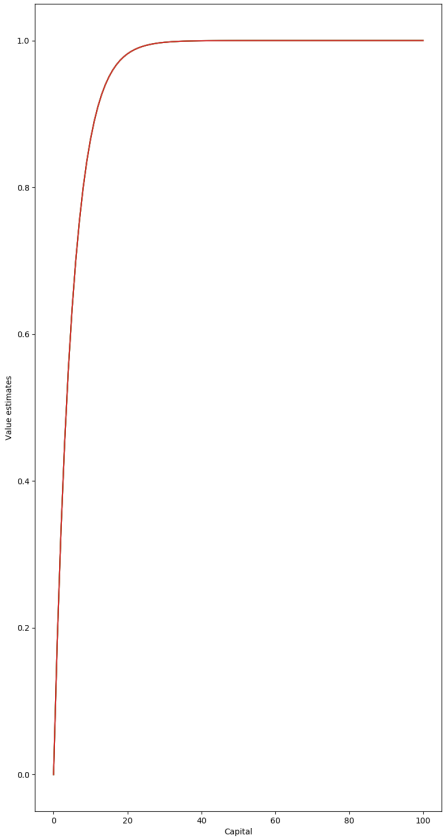
```
            plt.subplot(2, 2, (i+1))
            plt.plot(state_value,label=str(THETA))
            plt.xlabel('Capital')
            plt.ylabel('Value estimates')

            plt.subplot(2, 2, (i+1)+2)
            plt.scatter(STATES, policy,label=str(THETA))
            plt.xlabel('Capital')
            plt.ylabel('Final policy (stake)')
        plt.legend()
    plt.savefig('./img/4_9.png')
```

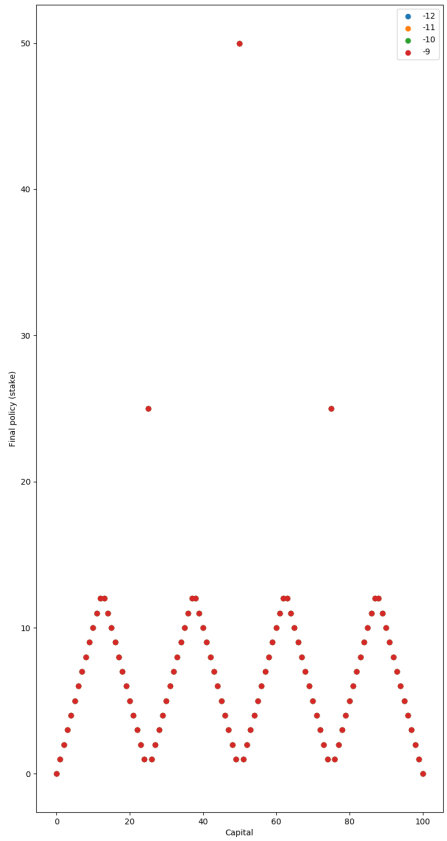
Answer

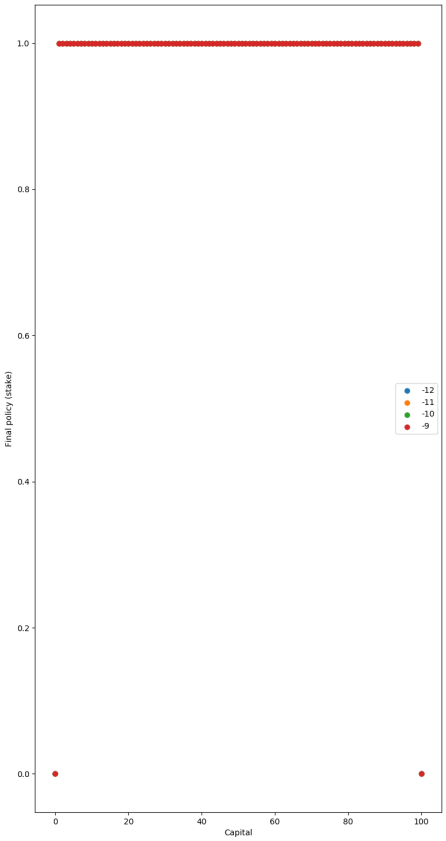
In [53]:

```
q_4_9()
```


Value estimates
Capital
0.0
0.2
0.4
0.6
0.8
1.0
0
20
40
60
80
100


Value estimates
Capital
0.0
0.2
0.4
0.6
0.8
1.0
0
20
40
60
80
100


Final policy (stake)
Capital
-12
-11
-10
-9
0
10
20
30
40
50
0
20
40
60
80
100


Final policy (stake)
Capital
-12
-11
-10
-9
0.0
0.2
0.4
0.6
0.8
1.0
0
20
40
60
80
100

4.10

Answer

$$q_{k+1} = \max_{a'} \sum_{s',r} p(s', r|s, a)[r + \gamma q_k(s', a')]$$