An Introduction to the Relational Axiom System (RAS)

A Preliminary Exploration of Pure-Relation Foundations

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Abstract

This document introduces the *Relational Axiom System* (RAS), a foundation that discards the traditional "object-first" perspective and treats **relations** as the sole fundamental building blocks. We explicitly state the core axioms in a first-order logical framework, illustrate how mathematical constructs can be expressed purely in relational terms, and provide a more detailed formal proof of the Cantor–Bernstein Theorem in RAS. Future research directions and open challenges in mathematics, physics, and computational complexity are also discussed.

1 Introduction

Set theory has long been the foundation of modern mathematics, where objects are considered primary and relations/functions are derived from them. The **Relational Axiom System (RAS)** proposes a different approach: *relations* are fundamental, and any notion of "object" is merely a construct emerging from relations.

The motivations for RAS include:

- **Philosophical:** Aligning with process philosophies and network-based views by eliminating the necessity of standalone entities.
- **Mathematical:** Unifying diverse structures via relational operations such as composition and expansion.
- **Physical/Computational:** Providing natural descriptions for quantum entanglement, distributed systems, and neural networks.

This document presents:

1. The two core axioms of RAS and derived relational constructs.
2. A formal illustration of Cantor–Bernstein in RAS with a detailed proof.
3. Potential applications and open problems.

2 Core Axioms of RAS

In RAS, everything is a *relation*. Formally, we assume a single-sorted first-order language where:

- All variables range over relations (not objects).

- A ternary predicate $\mathrm{App}(R, x, y)$ expresses that "(x, y) belongs to relation R".
- Logical equality $=$ is available, or an identity relation can be introduced.

Axiom 1: Existence (All Existence is Relational)

Axiom 1 (Existence). *In any valid model of RAS, every entity is a relation. No irreducible 'objects' exist.*

$$\forall R \quad (\mathrm{Rel}(R)).$$

Alternatively, if we assume a single-sorted domain, $\mathrm{Rel}(R)$ is trivially true for all R.

Axiom 2: Relational Expansion

Axiom 2 (Relational Expansion). *Relations can always combine, evolve, or generate higher-order relations.* A key example is **composition**, denoted by:

$$\mathrm{Comp}(R_1, R_2).$$

We postulate:

$$\forall R_1, R_2 \quad \exists R_3 \quad (R_3 = \mathrm{Comp}(R_1, R_2)).$$

And the application rule:

$$\forall x, z \quad \mathrm{App}(\mathrm{Comp}(R_1, R_2), x, z) \iff \exists y \quad (\mathrm{App}(R_1, x, y) \wedge \mathrm{App}(R_2, y, z)).$$

3 Cantor–Bernstein Theorem in RAS

We now reconstruct the classical Cantor–Bernstein Theorem using RAS.

Theorem 1 (Cantor–Bernstein in RAS)**.** *If R_{AB} is a single-valued injective relation linking A to B, and R_{BA} is an injective relation linking B to A, then there exists a bijective relation $R_\star : A \leftrightarrow B$.*

Proof.

1. Define an iterative sequence of relations:

$$R_0 = R_{AB}, \quad R_{n+1} = R_{AB} \circ R_{BA} \circ R_n.$$

This sequence represents the alternating applications of R_{AB} and R_{BA}, gradually extending a matching between elements of A and B.

2. Show that the sequence stabilizes. Define the limiting relation as:

$$R_\infty = \bigcup_n R_n.$$

This ensures that all elements reachable through R_{AB} and R_{BA} will eventually be included in the relation.

3. **Injectivity:** Since each R_n is injective by construction, their union R_∞ remains injective. If $x_1 R_\infty y$ and $x_2 R_\infty y$, then for some finite n, x_1 and x_2 would both relate to y under R_n, contradicting injectivity of R_n.

4. **Surjectivity:** Any $b \in B$ must eventually be reached by some $a \in A$ through repeated applications of R_{AB} and R_{BA}. If there existed an unmatched element in B, then by construction of R_∞, it would eventually be matched, ensuring completeness.

5. **Final Matching and Notation Consistency:** To align with the meta-logical framework used in this paper, we redefine the relation application explicitly:

$$\forall x, y, \quad \mathrm{App}(R_\star, x, y) \iff \mathrm{App}(R_\infty, x, y).$$

   This ensures uniform notation across RAS statements.

6. Define $R_\star = R_\infty$ as the final relation. Since it is both injective and surjective, it is bijective.

Thus, there exists a bijective relation $R_\star : A \leftrightarrow B$, completing the proof. $\square$

4 Future Research Directions

4.1 Computational Complexity in RAS

One open question is whether RAS allows for non-recursive computations, possibly defining new computational complexity classes. Specifically:

- Can RAS formally express hypercomputational models?
- Is there an equivalent to polynomial-time complexity in purely relational terms?

4.2 RAS in Quantum Mechanics and Bell's Inequality

A natural test case for RAS physics is Bell's inequality. If quantum entanglement can be described purely in relational terms, we should be able to reconstruct:

- Quantum nonlocality as a relational phenomenon.
- Bell's inequality violations without assuming wavefunction realism.

A small-scale relational reconstruction of Bell experiments can help clarify these ideas.

5 Future Directions and Open Problems

5.1 Quantum Non-Locality

RAS could provide a foundation for relational descriptions of quantum entanglement and non-local correlations.

5.2 Computational Complexity

Can RAS yield an alternative perspective on P vs NP or new approaches to computational problems?

5.3 Relational Neural Networks

Viewing neural networks as pure relational structures could lead to novel learning paradigms.

5.4 Formalization in Proof Assistants

Formalizing RAS in Coq or Lean could help rigorously establish its properties.

6 Conclusion

The **Relational Axiom System (RAS)** challenges the conventional object-based framework by proposing a purely relational foundation. Although still in its early stages, RAS presents intriguing possibilities for mathematics, physics, and computation.