$$Y = \underset{n\times 1}{} \; C \cdot b + E \cdot \alpha + X\beta + W\eta + \varepsilon$$

Y: $n\times 1$, C: $n\times q_1$, b: $q_1\times 1$, E: $n\times q_2$, α: $q_2\times 1$, X: $n\times 1$, β: 1×1, W: $n\times q_2$, η: $q_2\times 1$, ε: $n\times 1$

$\alpha, b, \tilde{v}_i, \beta, \eta, S_1, S_2,$ P1

$\tau, \tilde{\eta}_1^2, \tilde{\eta}_2^2.$

$$y_i = C_i \cdot b + E_i \cdot \alpha + x_i \cdot \beta + w_i \cdot \eta + \xi_2 \tau^{-\frac{1}{2}} \sqrt{\tilde{v}_i}\, z_i, \quad i = 1, \cdots, n$$

$$\tilde{v} \mid \tau \sim \prod_{i=1}^{n} \tau \exp(-\tau \tilde{v}_i) \qquad \tilde{v}_i \sim EXP(\tau)$$

$$z \sim \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} z_i^2\right) \qquad z_i \sim N(0,1)$$

$$\underline{\beta, S_1} \mid \underline{\tilde{\eta}_1^2} \sim \frac{1}{\sqrt{2\pi S_1}} \exp\left(-\frac{\beta^2}{2S_1}\right) \cdot \frac{\tilde{\eta}_1^2}{2} \exp\left(-\frac{\tilde{\eta}_1^2}{2} S_1\right)$$

$$\underline{\eta, S_2} \mid \underline{\tilde{\eta}_2^2} \sim \prod_{k=1}^{q_2} \frac{1}{\sqrt{2\pi S_{2k}}} \exp\left(-\frac{\eta_k^2}{2S_{2k}}\right) \prod_{k=1}^{q_2} \frac{\tilde{\eta}_2^2}{2} \exp\left(-\frac{\tilde{\eta}_2^2}{2} S_{2k}\right)$$

$$\tau \sim \text{gamma}(a, b)$$

$$\tilde{\eta}_1^2 \sim \text{gamma}(c_1, d_1), \qquad \tilde{\eta}_2^2 \sim \text{gamma}(c_2, d_2)$$

$$\alpha \sim \cancel{N_{q_2}(0, \Sigma\alpha_0)} \prod_{k=1}^{q_2} \frac{1}{\sqrt{2\pi \alpha_0}} \exp\left(-\frac{\alpha_k^2}{2\alpha_0}\right)$$

$$b \sim \cancel{N_{q_1}(0, \Sigma b_0)} \prod_{i=1}^{q_1} \frac{1}{\sqrt{2\pi b_0}} \exp\left(-\frac{b_i^2}{2b_0}\right)$$

$$f(y \mid -) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \tau^{-1} \xi_2^2 \tilde{v}_i}} \exp\left\{-\frac{(y_i - C_i b - E_i \alpha - x_i\beta - w_i \eta)^2}{2\tau^{-1}\xi_2^2 \tilde{v}_i}\right\}$$

$$= \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} \frac{(y_i - C_i b - E_i\alpha - x_i\beta - w_i\eta)^2}{\tau^{-1}\xi_2^2\tilde{v}_i}\right\} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\tau^{-1}\xi_2^2\tilde{v}_i}}$$

- $$b_j \mid - \propto \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i - C_i b - E_i\alpha - x_i\beta - w_i\eta)^2}{\tau^{-1}\xi_2^2\tilde{v}_i}\right\} \exp\left(-\frac{b_j^2}{2b_0}\right), \quad j = 1, \cdots q_1$$

$$\propto \exp\left\{-\frac{1}{2}\left[\left(\sum_{i=1}^{n}\frac{\tau C_{ij}^2}{\xi_2^2\tilde{v}_i} + \frac{1}{b_0}\right) b_j^2 - 2\sum_{i=1}^{n}\frac{\tau \tilde{y}_{ij} C_{ij}}{\xi_2^2\tilde{v}_i} b_j\right]\right\}$$

$$\tilde{y}_{ij} = y_i - E_i\alpha - x_i\beta - w_i\eta - \sum_{j'=1, j'\neq j}^{q_1} x_{ij'}\beta_{j'} = \mu + C_{ij} b_j$$

$$\sigma_j^2 = \left(\sum_{i=1}^{n}\frac{\tau C_{ij}^2}{\xi_2^2\tilde{v}_i} + \frac{1}{b_0}\right)^{-1}, \qquad \mu_j = \left(\sum_{i=1}^{n}\frac{\tau\tilde{y}_{ij}C_{ij}}{\xi_2^2\tilde{v}_i}\right)\cdot\sigma_j^2$$

$$\sim N(\mu_j, \sigma_j^2)$$

- $\alpha_k | - \propto \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} \frac{(y_i - C_i b - E_i\alpha - X_i\beta - W_i\eta)^2}{\tau^{-1}\xi_2^2 \tilde{v}_i}\right\} \exp\left(-\frac{\alpha_k^2}{\alpha_0}\right),\quad k=1,\cdots,q_2$

$$\propto \exp\left\{-\frac{1}{2}\left[\left(\sum_{i=1}^{n} \frac{\tau E_{ik}^2}{\xi_2^2 \tilde{v}_i} + \frac{1}{\alpha_0}\right)\alpha_k^2 - 2\sum_{i=1}^{n} \frac{\tau \tilde{y}_{ik} E_{ik}}{\xi_2^2 \tilde{v}_i}\alpha_k\right]\right\}$$

$$\tilde{y}_{ik} = y_i - C_i b - X_i\beta - W_i\eta - \sum_{j=1, j\neq k}^{q_2} E_{ik}\alpha_k = \mu(y) + E_{ik}\alpha_k$$

$$\sigma_k^2 = \left(\sum_{i=1}^{n} \frac{\tau E_{ik}^2}{\xi_2^2 \tilde{v}_i} + \frac{1}{\alpha_0}\right)^{-1}, \quad \mu_k = \left(\sum_{i=1}^{n} \frac{\tau \tilde{y}_{ik} E_{ik}}{\xi_2^2 \tilde{v}_i}\right)\cdot \sigma_k^2$$

$$\sim N(\mu_k, \sigma_k^2)$$

- $f(\tilde{v}_i | -) \propto f(y|-)\, \pi(\tilde{v}_i | \tau)$

$$\propto \frac{1}{\sqrt{\tilde{v}_i}} \exp\left\{-\frac{(y_i - C_i b - E_i\alpha - X_i\beta - W_i\eta)^2}{2\tau^{-1}\xi_2^2 \tilde{v}_i}\right\} \exp(-\tau \tilde{v}_i)$$

$$\propto \frac{1}{\sqrt{\tilde{v}_i}} \exp\left\{-\frac{1}{2}\left[\big(\underbrace{2\tau}_{a}\big)\tilde{v}_i + \underbrace{\frac{\tau(y_i - C_i b - E_i\alpha - X_i\beta - W_i\eta)^2 \cdot v_i^{-1}}{\xi_2^2}}_{b}\right]\right\}$$

$$\sim \text{inv. Gaussian}\left(\mu = \sqrt{\frac{a}{b}},\ \lambda = a\right)$$

$$\sim \text{inv. Gaussian}\left(\mu = \sqrt{\frac{2\cancel{\tau}\cdot \xi_2^2}{\cancel{\tau}(y_i - C_i b - E_i\alpha - X_i\beta - W_i\eta)^2}},\ \lambda = 2\tau\right)$$

- $f(s_1 | -) \propto \pi(\beta | s_1)\, \pi(s_1 | \tilde{\eta}_1^2)$

$$\propto \frac{1}{\sqrt{2\pi s_1}} \cdot \exp\left(-\frac{\beta^2}{2 s_1}\right) \exp\left(-\frac{\tilde{\eta}_1^2}{2} s_1\right)$$

$$\propto \frac{1}{\sqrt{s_1}} \exp\left\{-\frac{1}{2}\left[\tilde{\eta}_1^2 s_1 + \beta^2 \cdot s_1^{-1}\right]\right\}$$

$$\sim \text{inv. Gaussian}\left(\mu = \sqrt{\frac{\tilde{\eta}_1^2}{\beta^2}},\ \lambda = \tilde{\eta}_1^2\right)$$

$f(s_2 | -)$ 同上

- $f(\beta \mid -) \propto f(y \mid -)\, \pi(\beta \mid S_1)$

$$\propto \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i - x_i\beta - C_ib - E_i\alpha - W_i\eta)^2}{\tau^{-1}\xi_2^2\tilde{U}_i}\right\}\exp\left(-\frac{\beta^2}{2S_1}\right)$$

$$\propto \exp\left\{-\frac{1}{2}\left(\left[\frac{\sum_{i=1}^{n}(y_i - C_ib - E_i\alpha - W_i\eta)^2 - 2x_i(y_i - C_ib - E_i\alpha - W_i\eta)\beta + x_i^2\beta^2}{\tau^{-1}\xi_2^2\tilde{U}_i}\right] - \frac{\beta^2}{S_1}\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\left(\sum_{i=1}^{n}\frac{x_i^2}{\tau^{-1}\xi_2^2\tilde{U}_i} + \frac{1}{S_1}\right)\beta^2 - 2\sum_{i=1}^{n}\frac{\tau(y_i - C_ib - E_i\alpha - W_i\eta)x_i}{\xi_2^2\tilde{U}_i}\beta\right]\right\}$$

$$\sigma^2 = \left(\sum_{i=1}^{n}\frac{\tau x_i^2}{\xi_2^2\tilde{U}_i} + \frac{1}{S_1}\right)^{-1}, \quad \mu = \frac{\sum_{i=1}^{n}\frac{\tau(y_i - C_ib - E_i\alpha - W_i\eta)x_i}{\xi_2^2\tilde{U}_i}}{\sum_{i=1}^{n}\frac{\tau x_i^2}{\xi_2^2\tilde{U}_i} + \frac{1}{S_1}}$$

$$\mu = \left(\sum_{i=1}^{n}\frac{\tau(y_i - C_ib - E_i\alpha - W_i\eta)x_i}{\xi_2^2\cdot\tilde{U}_i}\right)\cdot\sigma^2$$

$k = 1, \cdots, q_2$

- $f(\eta_k \mid -) \propto f(y \mid -)\, \pi(\eta_k \mid S_2)$

$$\propto \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i - x_i\beta - C_ib - E_i\alpha - W_i\eta)^2}{\tau^{-1}\xi_2^2\tilde{U}_i}\right\}\exp\left(-\frac{\eta_k^2}{2S_{2k}}\right)$$

$$\propto \exp\left\{-\frac{1}{2}\left[\left(\sum_{i=1}^{n}\frac{\tau W_{ik}^2}{\xi_2^2\tilde{U}_i} + \frac{1}{S_{2k}}\right)\eta_k^2 - 2\sum_{i=1}^{n}\frac{\tau\tilde{y}_{ik}W_{ik}}{\xi_2^2\tilde{U}_i}\eta_k\right]\right\}$$

$$\tilde{y}_{ik} = y_i - x_i\beta - C_ib - E_i\alpha - \sum_{j=1, j\neq k}^{q_2} W_{ij}\eta_j$$

$$\sigma_k^2 = \left(\sum_{i=1}^{n}\frac{\tau W_{ik}^2}{\xi_2^2\tilde{U}_i} + \frac{1}{S_{2k}}\right)^{-1}, \quad \mu_k = \left(\sum_{i=1}^{n}\frac{\tau\tilde{y}_{ik}W_{ik}}{\xi_2^2\tilde{U}_i}\right)\cdot\sigma_k^2.$$

- $f(s_{2k}|-) \propto \pi(\eta_k | s_{2k})\, \pi(s_{2k} | \tilde{\eta}_2^2)$ $\quad k=1,\cdots,q_2$

$$\propto \frac{1}{\sqrt{2\pi s_{2k}}} \exp\left(-\frac{\eta_k^2}{2 s_{2k}}\right) \exp\left(-\frac{\tilde{\eta}_2^2}{2} s_{2k}\right)$$

$$\propto \frac{1}{\sqrt{s_{2k}}} \exp\left\{-\frac{1}{2}\left[\tilde{\eta}_2^2 \cdot s_{2k} + \eta_k^2 \cdot s_{2k}^{-1}\right]\right\}$$

$$\sim \text{inv. Gaussian}\left(\mu = \sqrt{\frac{\tilde{\eta}_2^2}{\eta_k^2}},\ \lambda = \tilde{\eta}_2^2\right)$$

- $f(\tau|-) \propto f(y|-)\, \pi(\tilde{v}|\tau)\, \pi(\tau)$

$$\propto \tau^{\frac{n}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} \frac{(y_i - C_i b - E_i\alpha - X_i\beta - W_i\eta)^2}{\tau^{-1}\xi_2^2 \tilde{v}_i}\right\} \cdot \tau^n \exp\left(-\tau\sum_{i=1}^{n}\tilde{v}_i\right)$$

$$\cdot\ \tau^{a-1}\exp(-b\tau)$$

$$\propto \tau^{a+\frac{3\cdot n}{2}-1} \exp\left\{-\tau\left[\underline{\sum_{i=1}^{n}\left(\frac{(y_i - C_i b - E_i\alpha - X_i\beta - W_i\eta)^2}{2\xi_2^2\tilde{v}_i} + \tilde{v}_i\right) + b}\right]\right\}$$

$$\sim \text{Gamma}\left(a+\frac{3\cdot n}{2},\ \quad\right)$$

- $f(\tilde{\eta}_1^2|-) \propto \pi(s_1 | \tilde{\eta}_1^2)\, \pi(\tilde{\eta}_1^2)$

$$\propto \frac{\tilde{\eta}_1^2}{2} \exp\left(-\frac{\tilde{\eta}_1^2}{2} s_1\right) (\tilde{\eta}_1^2)^{c_1-1} \exp(-d_1\tilde{\eta}_1^2)$$

$$\propto (\tilde{\eta}_1^2)^{1+c_1-1} \exp\left(-\tilde{\eta}_1^2\left(\frac{s_1}{2} + d_1\right)\right) \sim \text{gamma}\left(1+c_1,\ \frac{s_1}{2}+d_1\right)$$

- $f(\tilde{\eta}_2^2|-) \propto \pi(s_2 | \tilde{\eta}_2^2)\, \pi(\tilde{\eta}_2^2)$

$$\propto \prod_{k=1}^{q_2} \frac{\tilde{\eta}_2^2}{2} \exp\left(-\frac{\tilde{\eta}_2^2}{2} s_{2k}\right) (\tilde{\eta}_2^2)^{c_2-1} \exp(-d_2\tilde{\eta}_2^2)$$

$$\propto (\tilde{\eta}_2^2)^{(q_2+c_2-1)} \exp\left\{-\tilde{\eta}_2^2\left(\sum_{k=1}^{q_2}\frac{s_{2k}}{2} + d_2\right)\right)$$

$$\sim \text{gamma}\left(q_2+c_2,\ \sum_{k=1}^{q_2}\frac{s_{2k}}{2} + d_2\right)$$