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{\v C}ech complexes of hypercube graphs

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ČECH COMPLEXES OF HYPERCUBE GRAPHS

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ABSTRACT. A Čech complex of a finite simple graph G is a nerve complex of balls in the graph, with one ball centered at each vertex. More precisely, let the Čech complex $\mathcal{N}(G,r)$ be the nerve of all closed balls of radius $\frac{r}{2}$ centered at vertices of G, where these balls are drawn in the geometric realization of the graph G (equipped with the shortest path metric). The simplicial complex $\mathcal{N}(G,r)$ is equal to the graph G when $r=1$, and homotopy equivalent to the graph G when r is smaller than half the length of the shortest cycle in G. For higher values of r, the topology of $\mathcal{N}(G,r)$ is not well-understood. We consider the n-dimensional hypercube graphs $\mathbb{I}_n$ with 2^n vertices. Our main results are as follows. First, when $r=2$, we show that the Čech complex $\mathcal{N}(\mathbb{I}_n,2)$ is homotopy equivalent to a wedge of 2-spheres for all $n\geq 1$, and we count the number of 2-spheres appearing in this wedge sum. Second, when $r=3$, we show that $\mathcal{N}(\mathbb{I}_n,3)$ is homotopy equivalent to a simplicial complex of dimension at most 4, and that for $n\geq 4$ the reduced homology of $\mathcal{N}(\mathbb{I}_n,3)$ is nonzero in dimensions 3 and 4, and zero in all other dimensions. Finally, we show that for all $n\geq 1$ and $r\geq 0$, the inclusion $\mathcal{N}(\mathbb{I}_n,r)\hookrightarrow\mathcal{N}(\mathbb{I}_n,r+2)$ is null-homotopic, providing a bound on the length of bars in the persistent homology of Čech complexes of hypercube graphs.

Keywords : Čech complex, persistent homology, collapsibility, hypercube
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1. INTRODUCTION

Let $G=(V(G),E(G))$ be a finite simple connected graph. We can equip the vertex set $V(G)$ with the shortest path metric, which extends to give a shortest path metric on the geometric realization of G, in which the realization of each edge is isometric to the unit interval $[0,1]$. The Čech simplicial complex $\mathcal{N}(G,r)$ is the nerve of all closed balls of radius $\frac{r}{2}$ centered at vertices of G, where these balls are drawn in the geometric realization of the graph G equipped with the shortest path metric; see Figure 1. In other words, the vertex set of $\mathcal{N}(G,r)$ is $V(G)$, and a set of vertices forms a simplex if their corresponding balls have a point of intersection. For a precise definition of $\mathcal{N}(G,r)$, see Definition 2.2.

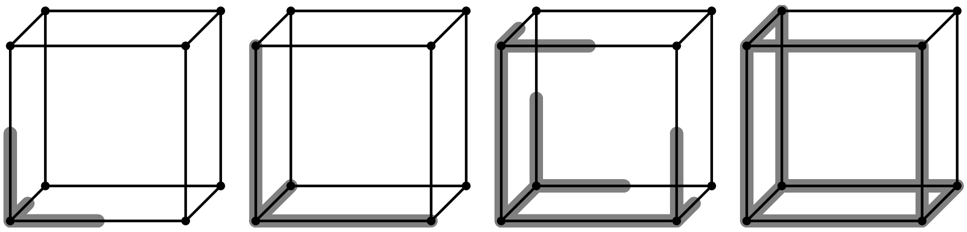

FIGURE 1. Balls of radius $\frac{r}{2}$ about a vertex v in the geometric realization of the 3-dimensional hypercube graph, for $r=1,2,3,4$.

When $r=0$, the Čech simplicial complex $\mathcal{N}(G,0)$ is $V(G)$, a disjoint union of vertices. When $r=1$, the complex $\mathcal{N}(G,1)$ is equal to the graph G. If ℓ is the length of the shortest cycle in the graph G, and if $1\leq r<\frac{\ell}{2}$, then any intersection of balls of radius $\frac{r}{2}$ is either empty or contractible, and hence the nerve lemma applies to guarantee that $\mathcal{N}(G,r)$ is homotopy equivalent to the geometric

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realization of G (Lemma 2.3). But for larger values of r, the hypotheses of the nerve lemma are no longer satisfied, and the topology of $\mathcal{N}(G, r)$ is not well-understood.

In applied and computational topology, Čech complexes are frequently-used tools to approximate the "shape" of a dataset [13]. In the context of this paper, the dataset is the vertex set of a graph, which is a common source of data when modeling social networks, transportation schedules, the routing of messages in a communication network, or the structure of molecules. More generally, one is given a finite dataset X sampled, perhaps with noise, from some underlying space M, which may be a manifold, or a stratified space, or a graph. One would like to approximate topological information about M using only the sampling X. The idea of persistent homology [21, 41, 20] is to "thicken" X depending on some scale parameter $r \geq 0$, and to measure how the topology of these thickenings changes as the scale r increases. Example choices of thickenings are the Vietoris–Rips complexes [38], witness complexes [19], alpha complexes [22, 7], or Čech complexes [20] of X. In the case of Čech complexes, there are multiple options: one defines the Čech complex as the nerve of all balls with centers in X, where those balls are drawn in either X, or in M, or in $\mathbb{R}^m$ (if M happens to be contained in some Euclidean space $\mathbb{R}^m$). There are a wide variety of reconstruction results for several different types of thickenings, which give theoretical or probabilistic guarantees for being able to recover the homology groups or the homotopy type of the underlying space M from these thickenings of X [28, 30, 31, 35]. The most standard such reconstruction result is the nerve lemma for Čech complexes [11]. These reconstruction results, however, typically require the scale parameter r to be sufficiently small compared to the (unknown) curvature of the space M. In accordance with the idea of persistent homology, data science practitioners increase the scale $r \geq 0$ to values much larger than these regimes, in order to see which topological features persist. There is a hence a need to understand these thickenings when the aforementioned reconstruction results no longer apply, including the case of Čech complexes when the balls are large enough so that the nerve lemma no longer applies.

Čech complexes are closely related to Vietoris–Rips complexes, and so we provide a brief introduction to the similarities and differences between these constructions. Because they are constructed as nerve complexes, Čech complexes sometimes satisfy the hypotheses of the nerve theorem [11], in which case the Čech complex is homotopy equivalent to a union of the balls. Though Vietoris–Rips complexes are less likely to satisfy a nerve theorem, they are still equipped with reconstruction guarantees when the scale parameter is not too large [30, 31]. Furthermore, in part because Vietoris–Rips complexes are *clique* or *flag* complexes, their persistent homology can be efficiently computed [8]. In many ways Čech and Vietoris–Rips complexes have similar behavior. Indeed, their filtrations are multiplicatively interleaved [13], and they both satisfy stability results when the underlying dataset is perturbed by a controllable amount with respect to the Gromov–Hausdorff distance [17, 16]. This means that one could study the Čech or Vietoris–Rips persistent homology of manifolds by studying the Čech or Vietoris–Rips complexes of graphs that approximate that manifold. Indeed, this was the approach taken in [2] when studying the Čech and Vietoris–Rips persistent homology of the circle. For G a finite connected graph, it is in general not easy to produce a list of the maximal simplices in a Vietoris–Rips complex of G. However, this is often easier to do for Čech complexes (see for example Lemma 3.1 in the case of hypercube graphs), and this is one reason why we consider Čech complexes here.

The papers [14, 3, 36] study the Vietoris–Rips complexes of hypercube graphs, uncovering some structure when the scale parameter is small or when the dimension of homology is small, though many questions remain unanswered. Questions about the shape of Vietoris–Rips complexes of hypercube graphs originally arose from work by mathematical biologists who were applying topology in order to study genetic trees, medial recombination, and reticulate evolution [23, 24, 12, 15]. In this paper, we consider analogous questions about Čech complexes of hypercube graphs. More is known about the Vietoris–Rips and Čech complexes of cycle graphs, which are always homotopy equivalent to either a single odd-dimensional sphere or to a wedge sum of even-dimensional spheres of the same dimension [1, 4]. The persistent homology of Vietoris–Rips complexes of finite connected graphs are studied in [5] (see also [33]), but less is known about the persistent homology of Čech complexes of graphs. The papers [25, 39] study the 1-dimensional persistence of Čech and Vietoris–Rips complexes of a different flavor, which have an uncountable number of vertices (one for each point in a metric graph or geodesic space), instead of the simplicial complexes with a finite number of vertices that we study here.

In this paper we consider the Čech complexes of n-dimensional hypercube graphs $\mathbb{I}_n$ with 2^n vertices. Our main results are as follows. When $r = 2$, we show in Theorem 4.1 that for all $n \geq 1$, the Čech complex $\mathcal{N}(\mathbb{I}_n, 2)$ is homotopy equivalent to a $2^{n-2}(n^2 - 3n + 4)$-fold wedge sum of 2-spheres. When $r = 3$, we show in Theorem 5.5 that $\mathcal{N}(\mathbb{I}_n, 3)$ is homotopy equivalent to a simplicial complex of dimension at most 4, and we show in Theorem 5.9 that for $n \geq 4$ the reduced homology of $\mathcal{N}(\mathbb{I}_n, 3)$ is nonzero in dimension 3, nonzero in dimension 4, and zero in all other dimensions. Finally, in Theorem 6.1 we show that for all $n \geq 1$ and $r \geq 0$, the inclusion $\mathcal{N}(\mathbb{I}_n, r) \hookrightarrow \mathcal{N}(\mathbb{I}_n, r+2)$ is null-homotopic, providing a bound on the length of bars in the Čech persistent homology of hypercube graphs.

We begin with preliminaries in Section 2, and a description of the simplicial complexes $\mathcal{N}(\mathbb{I}_n, r)$ in Section 3. In Section 4 we study the case $r = 2$, and in Section 5 we consider the case $r = 3$. We analyze the persistent homology of the filtration $\mathcal{N}(\mathbb{I}_n, -)$ in Section 6, and we conclude by sharing some open questions in Section 7.

2. Preliminaries

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover of a topological space X. The *nerve simplicial complex* $\mathbf{N}(\mathcal{U}) = \mathbf{N}(\{U_i\})$ has I as its vertex set, and has a finite subset $\sigma \subseteq I$ as a simplex if $\cap_{i\in\sigma} U_i \neq \emptyset$. If each U_i is contractible, and if each nonempty intersection $\cap_{i\in\sigma} U_i$ is contractible, then we say that the cover $\mathcal{U}$ is a *good cover*. The nerve theorem provides relatively mild point-set topology assumptions so that if $\mathcal{U}$ is a good cover of X, then the nerve $\mathbf{N}(\mathcal{U})$ is homotopy equivalent to the space X. This theorem applies, for example, if $\mathcal{U}$ is an open cover of a paracompact space, or if $\mathcal{U}$ is a cover of a simplicial complex by subcomplexes [11, 29, 9] (see Theorem 2.7).

Let $G = (V(G), E(G))$ be a finite simple connected graph. If two vertices v and w are adjacent, then we denote this by $v \sim w$. For $v \in V(G)$, the *(open) neighborhood* of v is $N_G(v) = \{w \mid w \sim v\}$. We equip the vertex set $V(G)$ with the shortest path metric $d \colon V(G) \times V(G) \to \mathbb{R}$. When the graph G is clear from context, we often simplify notation and write $N(v)$.

Definition 2.1. For a vertex $v \in V(G)$ and $r \geq 0$, the (closed) r-*neighborhood* about v is

$$N_{G,r}[v] = \{w \in V(G) \mid d(v, w) \leq r\}.$$

We let $N_G[v] := N_{G,1}[v]$ denote the closed 1-neighborhood. When the graph G is clear from context, we often simplify notation and write $N_r[v]$.

We emphasize that v is not an element of the open neighborhood $N_G(v)$, but v is an element of the closed neighborhood $N_r[v]$. Furthermore, $N_1[v] = N(v) \cup \{v\}$.

Definition 2.2. For G a finite simple connected graph and $r \geq 0$, the *Čech simplicial complex* $\mathcal{N}(G, r)$ is defined as follows. The vertex set of $\mathcal{N}(G, r)$ is $V(G)$. If r is even then the simplices of $\mathcal{N}(G, r)$ are generated by the sets $N_{\frac{r}{2}}[v]$ for $v \in V(G)$. If r is odd then the simplices of $\mathcal{N}(G, r)$ are generated by the sets $N_{\frac{r-1}{2}}[v] \cup N_{\frac{r-1}{2}}[w]$ for $(v, w) \in E(G)$.

As described in the introduction, the Čech simplicial complex can also be thought of as the nerve of all closed balls of radius $\frac{r}{2}$ centered at vertices of G, where these balls are drawn in the geometric realization of G equipped with the shortest path metric; these two definitions are equivalent. Indeed, if r is even, then a collection of such balls intersect at the vertex v if and only if their center vertices are all contained in the r-neighborhood $N_r[r]$. And if r is odd, then a collection of such balls intersect at the midpoint of the edge (v, u) (in the geometric realization of G) if and only if their center vertices are each contained in the union of r-neighborhoods $N_{\frac{r-1}{2}}[v] \cup N_{\frac{r-1}{2}}[w]$.

Lemma 2.3. *If ℓ is the length of the shortest cycle in the graph G, and if $1 \leq r < \frac{\ell}{2}$, then $\mathcal{N}(G, r)$ is homotopy equivalent to the geometric realization of G.*

Proof. We use the description of the Čech simplicial complex $\mathcal{N}(G, r)$ as the nerve of all closed balls of radius $\frac{r}{2}$ centered at vertices of G, where these balls are drawn in the geometric realization of G. If ℓ is the length of the shortest cycle in the graph G, and if $1 \leq r < \frac{\ell}{2}$, then any intersection of such balls of radius $\frac{r}{2}$ is either empty or contractible. Hence the nerve lemma applies to give that that $\mathcal{N}(G, r)$ is homotopy equivalent to the union of all such balls, which is equal to the geometric realization of G. □

The n-dimensional hypercube graph $\mathbb{I}_n$ has 2^n vertices, all binary strings of length n, with two vertices adjacent if their Hamming distance is one.

Definition 2.4. For a positive integer n, the *n-dimensional hypercube graph*, denoted by $\mathbb{I}_n$, is a graph whose vertex set is $V(\mathbb{I}_n) = \{x_1 \dots x_n : x_i \in \{0, 1\} \; \forall \; 1 \leq i \leq n\}$ and any two vertices $x_1 \dots x_n$ and $y_1 \dots y_n$ are adjacent if and only if $\sum_{i=1}^{n} |x_i - y_i| = 1$, *i.e.*, the strings corresponding to the two vertices differ in exactly one position.

Let G be a graph on vertex set V and edge set E. For $S \subseteq V$, the *induced subgraph* $G[S]$ is a subgraph of G, with vertex set S and edge set $E \cap \binom{S}{2}$.

Given two topological spaces X and Y, their wedge sum $X \vee Y$ is the space obtained by gluing X and Y together at a single point. The homotopy type of $X \vee Y$ is independent of this choice of points if X and Y are connected CW complexes. For $n \geq 1$, let $\vee_n X$ denote the n-fold wedge sum of X, namely $\vee_n X = X \vee \ldots \vee X$.

Simplicial complexes. A simplicial complex $\mathcal{K}$ on a vertex set V is a family of subsets of V, including all singletons, such that if $\sigma \in \mathcal{K}$ and $\tau \subseteq \sigma$, then $\tau \in \mathcal{K}$. We identify a simplicial complex with its geometric realization, which is a topological space. The *star* of a vertex $v \in V(\mathcal{K})$ is $\mathrm{st}_{\mathcal{K}}(v) = \{\sigma \in \mathcal{K} : \sigma \cup \{v\} \in \mathcal{K}\}$. Note that the star is contractible since it is a cone with apex v. The *link* of a vertex v is $\mathrm{lk}_{\mathcal{K}}(v) = \{\sigma \in \mathcal{K} : v \notin \sigma \text{ and } \sigma \cup \{v\} \in \mathcal{K}\}$. The *deletion* of a vertex v, denoted $\mathcal{K} \setminus v$, is the induced simplicial complex on vertex set $V \setminus v$; the simplices of $\mathcal{K} \setminus v$ are all those simplices $\sigma \in \mathcal{K}$ such that $v \notin \sigma$. Using the following result, which is well-known to computational topologists, the homotopy type of a complex $\mathcal{K}$ can be computed using the link and deletion of a vertex in $\mathcal{K}$ (under given conditions).

Lemma 2.5 ([3, Lemma 1]). *Let $\mathcal{K}$ be a simplicial complex, and let $v \in \mathcal{K}$ be a vertex such that the inclusion $\iota \colon \mathrm{lk}_{\mathcal{K}}(v) \hookrightarrow \mathcal{K} \setminus v$ is a null-homotopy. Then up to homotopy we have a splitting $\mathcal{K} \simeq (\mathcal{K} \setminus v) \vee \Sigma\, \mathrm{lk}_{\mathcal{K}}(v)$.*

More generally, we also have the following result that will be used in this article to determine the homotopy type of a simplicial complex.

Lemma 2.6 ([27, Remark 2.4]). *Let the simplicial complex $\mathcal{K} = K_1 \cup K_2$ be such that the inclusion maps $\iota \colon K_1 \cap K_2 \hookrightarrow K_1$ and $\iota \colon K_1 \cap K_2 \hookrightarrow K_2$ are null-homotopies. Then $\mathcal{K} \simeq K_1 \vee K_2 \vee \Sigma\,(K_1 \cap K_2)$.*

We note that Lemma 2.5 is a particular case of Lemma 2.6. Indeed, observe that $\mathcal{K} = (\mathcal{K} \setminus v) \cup \mathrm{st}_{\mathcal{K}}(v)$ and $(\mathcal{K} \setminus v) \cap \mathrm{st}_{\mathcal{K}}(v) = \mathrm{lk}_{\mathcal{K}}(v) \hookrightarrow \mathrm{st}_{\mathcal{K}}(v)$. Moreover, the fact that $\mathrm{st}_{\mathcal{K}}(v)$ is contractible implies that this inclusion is a null-homotopy.

A topological space X is said to be *k-connected* if every map from an m-dimensional sphere $S^m \to X$ can be extended to a map from the $(m+1)$-dimensional disk $D^{m+1} \to X$ for $m = 0, 1, \ldots, k$. If X is m-connected, then all its homotopy groups $\pi_i(X)$ are trivial for $0 \leq i \leq m$.

The following result, also known as a *nerve theorem*, can be used to compute the homotopy type of a simplicial complex $\mathcal{K}$ when $\mathcal{K}$ is union of two or more complexes.

Theorem 2.7. [9, Theorem 10.6] *Let Δ be a simplicial complex and let $(\Delta_i)_{i\in I}$ be a family of subcomplexes such that $\Delta = \bigcup_{i\in I} \Delta_i$.*

(i) *Suppose every nonempty finite intersection $\Delta_{i_1} \cap \ldots \cap \Delta_{i_t}$ for $i_j \in I, t \in \mathbb{N}$ is contractible, then Δ and the nerve $\mathbf{N}(\{\Delta_i\})$ are homotopy equivalent.*

(ii) *Suppose every nonempty finite intersection $\Delta_{i_1} \cap \ldots \cap \Delta_{i_t}$ is $(k-t+1)$-connected. Then Δ is k-connected if and only if $\mathbf{N}(\{\Delta_i\})$ is k-connected.*

3. Description of $\mathcal{N}(\mathbb{I}_n, r)$

Recall that $\mathbb{I}_n$ is the hypercube graph with 2^n vertices. The following lemma is a simple consequence of Definition 2.2, the definition of Čech complexes; see Figure 2.

Lemma 3.1. *For $n \geq 1$ and $r \geq 0$, let σ be a facet of the Čech complex $\mathcal{N}(\mathbb{I}_n, r)$. Then, there exists $u \in V(\mathbb{I}_n)$ or $(v, w) \in E(\mathbb{I}_n)$ such that*

$$\sigma = \begin{cases} N_{\frac{r}{2}}[u] & \text{if } r \text{ is even,} \\ N_{\frac{r-1}{2}}[v] \cup N_{\frac{r-1}{2}}[w] & \text{if } r \text{ is odd.} \end{cases}$$

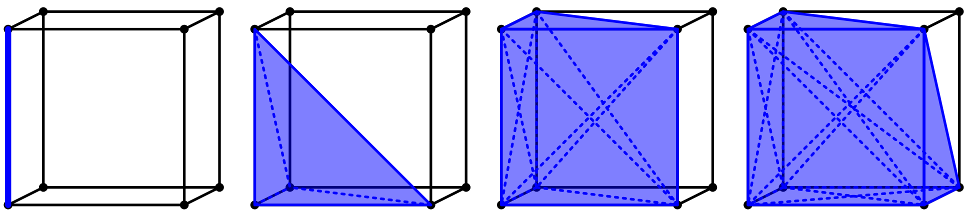

Figure 2. A maximal simplex in $\mathcal{N}(\mathbb{I}_n, r)$ for $r = 1, 2, 3, 4$. Complex $\mathcal{N}(\mathbb{I}_n, 1)$ has twelve maximal edges; $\mathcal{N}(\mathbb{I}_n, 2)$ has eight maximal tetrahedra; $\mathcal{N}(\mathbb{I}_n, 3)$ has twelve maximal 5-simplices; and $\mathcal{N}(\mathbb{I}_n, 4)$ has eight maximal 6-simplices. The homotopy types are $\mathcal{N}(\mathbb{I}_n, 1) \simeq \vee_5 S^1$; $\mathcal{N}(\mathbb{I}_n, 2) \simeq \vee_7 S^2$; $\mathcal{N}(\mathbb{I}_n, 3) \simeq \vee_3 S^4$; $\mathcal{N}(\mathbb{I}_n, 4) \simeq S^6$.

Table 1 shows the known homotopy types of $\mathcal{N}(\mathbb{I}_n, r)$. The row $r = 0$ follows since $\mathcal{N}(\mathbb{I}_n, 0)$ is a disjoint union of vertices. The row $r = 1$ follows from an Euler characteristic computation, since $\mathcal{N}(\mathbb{I}_n, 1)$ is a connected graph. The diagonal $r = 2n-2$ contains homeomorphisms $\mathcal{N}(\mathbb{I}_n, 2n-2) \cong S^{2^n-2}$ with (2^n-2)-dimensional spheres, since $\mathcal{N}(\mathbb{I}_n, 2n-2)$ is the boundary of the (2^n-1)-dimensional simplex with 2^n vertices. One can check that $\mathcal{N}(\mathbb{I}_n, 2n-3)$ is equal to $\Theta(\mathrm{Cube}(n, 1))$ from [18]; see the table in their Theorem 5, and see their Example 3 for a proof that $\mathcal{N}(\mathbb{I}_3, 3) = \Theta(\mathrm{Cube}(3, 1)) \simeq \vee_3 S^4$. The homotopy equivalences to wedge sums of 2-spheres in the row $r = 2$ are proven in Theorem 4.1. All of the other entries are not yet known up to homotopy type. When $(n, r) = (4, 3), (5, 3), (4, 4), (5, 4), (4, 6)$, computer computations using using Polymake [26] give the following reduced homology groups:

$$\begin{array}{lll}
\tilde{H}_i(\mathcal{N}(\mathbb{I}_4, 3)) = \mathbb{Z} \text{ for } i = 3 & \mathbb{Z}^{24} \text{ for } i = 4 & 0 \text{ for } i \neq 3, 4 \\
\tilde{H}_i(\mathcal{N}(\mathbb{I}_5, 3)) = \mathbb{Z}^9 \text{ for } i = 3 & \mathbb{Z}^{120} \text{ for } i = 4 & 0 \text{ for } i \neq 3, 4 \\
\tilde{H}_i(\mathcal{N}(\mathbb{I}_4, 4)) = \mathbb{Z} \text{ for } i = 4 & \mathbb{Z}^{10} \text{ for } i = 6 & 0 \text{ for } i \neq 4, 6 \\
\tilde{H}_i(\mathcal{N}(\mathbb{I}_5, 4)) = \mathbb{Z}^{11} \text{ for } i = 4 & \mathbb{Z}^{60} \text{ for } i = 6 & 0 \text{ for } i \neq 4, 6 \\
\tilde{H}_i(\mathcal{N}(\mathbb{I}_4, 6)) = \mathbb{Z}^7 \text{ for } i = 10 & & 0 \text{ for } i \neq 10
\end{array}$$

In Theorem 5.9, we show that for $n \geq 4$, the reduced homology group $\tilde{H}_i(\mathcal{N}(\mathbb{I}_n, 3); \mathbb{Z})$ is nonzero if and only if $i \in \{3, 4\}$.

$\mathcal{N}(\mathbb{I}_n, r)$	$n=1$	2	3	4	5	6	7	8
$r=0$	S^0	$\vee_3 S^0$	$\vee_7 S^0$	$\vee_{15} S^0$	$\vee_{31} S^0$	$\vee_{63} S^0$	$\vee_{127} S^0$	$\vee_{255} S^0$
1	$*$	S^1	$\vee_5 S^1$	$\vee_{17} S^1$	$\vee_{49} S^1$	$\vee_{129} S^1$	$\vee_{321} S^1$	$\vee_{769} S^1$
2	$*$	S^2	$\vee_7 S^2$	$\vee_{31} S^2$	$\vee_{111} S^2$	$\vee_{351} S^2$	$\vee_{1023} S^2$	$\vee_{2815} S^2$
3	$*$	$*$	$\vee_3 S^4$	$\beta_3 = 1; \beta_4 = 24$	$\beta_3 = 9; \beta_4 = 120$			
4	$*$	$*$	S^6	$\beta_4 = 1; \beta_6 = 10$	$\beta_4 = 11; \beta_6 = 60$			
5	$*$	$*$	$*$	$\beta_{10} = 7$				
6	$*$	$*$	$*$	S^{14}				
7	$*$	$*$	$*$	$*$				
8	$*$	$*$	$*$	$*$	S^{30}			

Table 1. The known homotopy types or Betti numbers of $\mathcal{N}(\mathbb{I}_n, r)$.

4. The case of $r = 2$

In this section, we characterize the homotopy type of $\mathcal{N}(\mathbb{I}_n, 2)$, which for convenience we restate below. It is easy to see that $\mathcal{N}(\mathbb{I}_1, 2)$ is contractible. For $n \geq 2$, the complex $\mathcal{N}(\mathbb{I}_n, 2)$ has 2^n maximal n-simplices, each of the form $N_1[v]$ as v varies over the vertices of $\mathbb{I}_n$.

We remark that in the seminal paper [34], Lovász considers simplicial complexes (called neighbourhood complex) which are generated by the open neighborhoods $N(v)$ instead of the closed neighborhoods $N_1[v]$ that we use here. Using topology of the neighbourhood complex of the graph G, Lovász gave a general lower bound for the chromatic number of G.

Theorem 4.1. *Let $n \geq 2$. Then the Čech complex $\mathcal{N}(\mathbb{I}_n, 2)$ is homotopy equivalent to a wedge sum of $2^{n-2}(n^2 - 3n + 4) - 1$ spheres of dimension 2, i.e.,*

$$\mathcal{N}(\mathbb{I}_n, 2) \simeq \bigvee_{2^{n-2}(n^2-3n+4)-1} \mathbb{S}^2.$$

Note that the sequence $2^{n-2}(n^2-3n+4)-1$ is listed as the Björner–Welker sequence $A055580(n-2)$ on the Online Encyclopedia of Integer Sequences (OEIS) [37, 10]. The idea of the proof here is along the same lines as the proof of [3, Theorem 1]. Indeed, to prove Theorem 4.1, we determine the homotopy type of $\mathcal{N}(G_m, 2)$ for a bigger class of metric spaces containing $\mathbb{I}_n$ for all $n \geq 1$. Recall that the metric space $\mathbb{I}_n$ is the set of all 2^n binary strings of length n, namely the numbers from 0 to $2^n - 1$ written in binary, equipped with the Hamming distance. We consider the metric spaces G_m, consisting of all numbers from 0 to $m-1$ written as binary strings, also equipped with the Hamming distance. Note that $G_{2^n} = \mathbb{I}_n$. To make our discussion simpler, we will treat G_m as the graph on m vertices in which two vertices are adjacent if and only if their binary representations differ at exactly one place.

Let $m \geq 2$ be a non-negative integer with the following binary representation for $m-1$:

$$m - 1 = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_k}, \text{ where } i_1 < i_2 < \cdots < i_k.$$

Define $\alpha(m-1) = (k-1)^2$. For $j \in [k] := \{1, 2, \ldots, k\}$, let $v^j = 2^{i_1} + \cdots + 2^{i_{j-1}} + 2^{i_{j+1}} + \cdots + 2^{i_k}$. Clearly, $N_{G_m}(m-1) = \{v^1, \ldots, v^k\}$. We now state a more general result, which will imply Theorem 4.1.

Theorem 4.2. *Let $m \geq 2$. Then, the complex $\mathcal{N}(G_m, 2)$ is homotopy equivalent to a wedge of 2-dimensional spheres. More precisely,*

$$\mathcal{N}(G_m, 2) \simeq \mathcal{N}(G_{m-1}, 2) \vee \bigvee_{\alpha(m-1)} \mathbb{S}^2. \tag{1}$$

Proof. The first part of Theorem 4.2 follows from induction and Equation (1). Therefore it is enough to prove Equation (1). We will prove this using Lemma 2.5 and by computing the link and deletion of vertex $m-1$ in $\mathcal{N}(G_m, 2)$.

Our base case is that $\mathcal{N}(G_2, 2)$ is contractible, *i.e.*, the 0-fold wedge sum of 2-spheres. This is the same for $\mathcal{N}(G_3, 2)$. From the induction hypothesis, we know that $\mathcal{N}(G_{m-1}, 2)$ is homotopy equivalent to a wedge of spheres. We first determine the homotopy type of the deletion complex, *i.e.*, $\mathcal{N}(G_m, 2) \setminus \{m-1\}$.

Claim 4.3. $\mathcal{N}(G_m, 2) \setminus \{m-1\} \simeq \mathcal{N}(G_{m-1}, 2) \vee \bigvee_{\binom{k-1}{2}} \mathbb{S}^2$.

Proof of Claim 4.3. It is easy to observe that

$$\mathcal{N}(G_m, 2)\setminus\{m-1\} = \mathcal{N}(G_{m-1}, 2) \cup \Delta^{N_{G_m}(m-1)} \text{ and } \mathcal{N}(G_{m-1}, 2) \cap \Delta^{N_{G_m}(m-1)} = (\Delta^{N_{G_m}(m-1)})^{(1)},$$

where Δ^S denotes the simplex on vertex set S and $(\Delta^S)^{(1)}$ denotes the 1-dimensional skeleton of this simplex. The 1-skeleton of a $(k-1)$-dimensional simplex is homotopy equivalent to a wedge of $\binom{k-1}{2}$ circles, since it is a connected graph with Euler characteristic $k - \binom{k}{2} = \binom{k-1}{2} - 1$. Hence

$$(\Delta^{N_{G_m}(m-1)})^{(1)} \simeq \bigvee_{\binom{k-1}{2}} \mathbb{S}^1.$$

From the induction hypothesis, we know that $\mathcal{N}(G_{m-1}, 2)$ is a wedge of 2-dimensional spheres and therefore the inclusion map $\iota\colon (\Delta^{N_{G_m}(m-1)})^{(1)} \hookrightarrow \mathcal{N}(G_{m-1}, 2)$ is null-homotopic. Hence, using Lemma 2.6, we get the following.

$$\begin{aligned}\mathcal{N}(G_m, 2) \setminus \{m-1\} &\simeq \mathcal{N}(G_{m-1}, 2) \vee \Delta^{N_{G_m}(m-1)} \vee \Sigma\left(\bigvee_{\binom{k-1}{2}} \mathbb{S}^1\right)\\ &\simeq \mathcal{N}(G_{m-1}, 2) \vee \bigvee_{\binom{k-1}{2}} \mathbb{S}^2.\end{aligned}$$

This completes the proof of Claim 4.3. □

We now determine the homotopy type of the complex $\mathrm{lk}_{\mathcal{N}(G_m,2)}(m-1)$. Observe that $\mathrm{lk}_{\mathcal{N}(G_m,2)}(m-1)$ is a simplicial complex whose facets are $N_{G_m}(m-1)$ and $N_{G_{m-1}}[v^j]$ for $j \in [k]$, *i.e.*,

$$\mathrm{lk}_{\mathcal{N}(G_m,2)}(m-1) = \Delta^{N_{G_m}(m-1)} \cup \Delta^{N_{G_{m-1}}[v^1]} \cup \Delta^{N_{G_{m-1}}[v^2]} \cup \cdots \cup \Delta^{N_{G_{m-1}}[v^k]}. \tag{2}$$

Recall that the v^j's are neighbors of vertex $m-1$ in the graph G_m, and that Δ^S denotes the simplex on vertex set S. For $1 \le r < s \le k$, let $v^{r,s}$ be the vertex of G_{m-1} whose binary representation is $2^{i_1} + \cdots + 2^{i_{r-1}} + 2^{i_{r+1}} + \cdots + 2^{i_{s-1}} + 2^{i_{s+1}} + \cdots + 2^{i_k}$. Clearly, $v^{r,s}$ is adjacent to both v^r and v^s in G_{m-1}. We now use Theorem 2.7 and Equation (2) to determine the homotopy type of $\mathrm{lk}_{\mathcal{N}(G_m,2)}(m-1)$. The following are some easy observations from the definition of G_m.

(1) Every member in the union of the right side of Equation (2) is a simplex and hence contractible.
(2) For any $1 \le r < s \le k$, $\Delta^{N_{G_{m-1}}[v^r]} \cap \Delta^{N_{G_{m-1}}[v^s]} = \Delta^{\{v^{r,s}\}}$, which is a point and therefore contractible.
(3) For any $1 \le i \le k$, $\Delta^{N_{G_m}(m-1)} \cap \Delta^{N_{G_{m-1}}[v^i]} = \Delta^{\{v^i\}}$, which is again a point and therefore contractible.
(4) The intersection of any three or more members from the union on the right side of Equation (2) is always empty.

The observations above along with Theorem 2.7 imply that the complex $\mathrm{lk}_{\mathcal{N}(G_m,2)}(m-1)$ is homotopy equivalent to the nerve of $\{\Delta^{N_{G_m}(m-1)}, \Delta^{N_{G_{m-1}}[v^1]}, \Delta^{N_{G_{m-1}}[v^2]}, \ldots, \Delta^{N_{G_{m-1}}[v^k]}\}$, which is homotopy equivalent to the 1-dimensional skeleton of a k-simplex on $k+1$ vertices. Therefore,

$$\mathrm{lk}_{\mathcal{N}(G_m,2)}(m-1) \simeq (\Delta^k)^{(1)} \simeq \bigvee_{\binom{k}{2}} \mathbb{S}^1.$$

Thus, from induction and Claim 4.3, we get that the inclusion map $\iota : \mathrm{lk}_{\mathcal{N}(G_m,2)}(m-1) \hookrightarrow \mathcal{N}(G_m,2) \setminus \{m-1\}$ is a null-homotopy (since the latter is homotopy equivalent to a wedge of 2-dimensional spheres). Hence, Lemma 2.5 implies the following.

$$\begin{aligned}
\mathcal{N}(G_m,2) &\simeq (\mathcal{N}(G_m,2) \setminus \{m-1\}) \vee \Sigma\, \mathrm{lk}_{\mathcal{N}(G_m,2)}(m-1) \\
&\simeq \mathcal{N}(G_{m-1},2) \vee \left(\bigvee_{\binom{k-1}{2}} \mathbb{S}^2 \right) \vee \left(\bigvee_{\binom{k}{2}} \mathbb{S}^2 \right) \\
&= \mathcal{N}(G_{m-1},2) \vee \bigvee_{(k-1)^2} \mathbb{S}^2.
\end{aligned}$$

This completes the proof of Theorem 4.2. $\square$

The proof of Theorem 4.1 now follows as a special case.

Proof of Theorem 4.1. Theorem 4.2 implies that $\mathcal{N}(\mathbb{I}_n,2) \simeq \bigvee_{\beta(n)} \mathbb{S}^2$, where

$$\begin{aligned}
\beta(n) &= \sum_{m=2}^{2^n} \alpha(m-1) = \sum_{k=1}^{n} \binom{n}{k}(k-1)^2 = \sum_{k=1}^{n} \binom{n}{k} k^2 - 2\sum_{k=1}^{n} \binom{n}{k} k + \sum_{k=1}^{n} \binom{n}{k} \\
&= \sum_{k=0}^{n} \binom{n}{k} k^2 - 2\sum_{k=0}^{n} \binom{n}{k} k + \sum_{k=0}^{n} \binom{n}{k} - 1 = 2^{n-2}(n^2+n) - 2\cdot 2^{n-1} n + 2^n - 1 \\
&= 2^{n-2}(n^2 - 3n + 4) - 1.
\end{aligned}$$

$\square$

5. The case of $r = 3$

In this section, we prove two main results about $\mathcal{N}(\mathbb{I}_n,3)$ at scale $r=3$. First, we prove that $\mathcal{N}(\mathbb{I}_n,3)$ is homotopy equivalent to a simplicial complex of dimension at most 4 (Theorem 5.5). Second, for $n \geq 4$, we show that the reduced homology $\tilde{H}_i(\mathcal{N}(\mathbb{I}_n,3);\mathbb{Z})$ is nonzero if and only if $i \in \{3,4\}$ (Theorem 5.9). We remind the reader that the cases $n = 1,2,3$ are understood, since $\mathcal{N}(\mathbb{I}_1,3)$ and $\mathcal{N}(\mathbb{I}_2,3)$ are contractible, and since $\mathcal{N}(\mathbb{I}_3,3) \simeq \vee_3 S^4$ by [18].

For a positive integer n, we denote the set $\{1,\dots,n\}$ by $[n]$. Let $v = v_1 \dots v_n \in V(\mathbb{I}_n)$. For any $i \in [n]$, we let $v(i) = v_i$. For $\{i_1, i_2, \dots, i_k\} \subseteq [n]$, we let $v^{i_1,\dots,i_k} \in V(\mathbb{I}_n)$ be defined by

$$v^{i_1,\dots,i_k}(j) = \begin{cases} v(j) & \text{if } j \notin \{i_1,\dots,i_k\}, \\ \{0,1\} \setminus \{v(j)\} & \text{if } j \in \{i_1,\dots,i_k\}. \end{cases}$$

Observe that for any two vertices $v, w \in V(\mathbb{I}_n)$, $d(v,w) = \sum_{i=1}^{n} |v(i) - w(i)|$ and $d(v,w) = k$ if and only if $w = v^{i_1,\dots,i_k}$ for some $i_1,\dots,i_k \in [n]$. Clearly, $N_{\mathbb{I}_n}(v) = \{v^i : i \in [n]\}$.

For any $n \geq 1$, let $\mathbf{X}_n = \mathcal{N}(\mathbb{I}_n,3)$. From Lemma 3.1, the facets of $\mathbf{X}_n$ will be of the type $N_1[v] \cup N_1[w]$ for some $v, w \in V(\mathbb{I}_n)$, $v \sim w$. Since $v \sim w$, we remark that $v \in N(w)$ and $w \in N(v)$, and therefore $N_1[v] \cup N_1[w] = N(v) \cup N(w)$ when $v \sim w$.

For each $i \in [n]$ and $\epsilon \in \{0,1\}$, let $\mathbb{I}_n^{i,\epsilon}$ be the induced subgraph of $\mathbb{I}_n$ on the vertex set $\{v \in V(\mathbb{I}_n) : v(i) = \epsilon\}$. Observe that we have an isomorphism $\mathbb{I}_n^{i,\epsilon} \cong \mathbb{I}_{n-1}$.

5.1. **Collapsibility.** Let Δ be a finite simplicial complex. Let $\gamma \in \Delta$ be such that $|\gamma| \leq d$ and $\sigma \in \Delta$ is the only maximal simplex that contains γ. An *elementry d-collapse* of Δ is the simplicial complex Δ' obtained from Δ by removing all those simplices τ of Δ such that $\gamma \subseteq \tau \subseteq \sigma$, and we denote this elementary d-collapse by $\Delta \xrightarrow{\gamma} \Delta'$. If γ is not equal to σ, then the elementary d-collapse $\Delta \xrightarrow{\gamma} \Delta'$ is also a simplicial collapse.

The complex Δ is called *d-collapsible* if there exists a sequence of elementary d-collapses

$$\Delta = \Delta_1 \xrightarrow{\gamma_1} \Delta_2 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_{k-1}} \Delta_k = \emptyset$$

from Δ to the void complex $\emptyset$. Clearly, if Δ is d-collapsible and $d < c$, then Δ is c-collapsible. The *collapsibility number* of Δ is the minimal integer d such that Δ is d-collapsible.

A simple consequence of d-collapsibility is the following:

Proposition 5.1. [40] *If Δ is d-collapsible then it is homotopy equivalent to a simplicial complex of dimension smaller than d.*

Let Δ be a simplicial complex on vertex set $[n]$ and let $\prec: \sigma_1, \ldots, \sigma_m$ be a linear ordering of the maximal simplices of Δ. Given a $\sigma \in \Delta$, the *minimal exclusion sequence* $\mathrm{mes}_\prec(\sigma)$ is defined as follows. Let i denote the smallest index such that $\sigma \subseteq \sigma_i$. If $i = 1$, then $\mathrm{mes}_\prec(\sigma)$ is the null sequence. If $i \geq 2$, then $\mathrm{mes}_\prec(\sigma) = (v_1, \ldots, v_{i-1})$ is a finite sequence of length $i-1$ such that $v_1 = \min(\sigma \setminus \sigma_1) \in [n]$ and for each $k \in \{2, \ldots, i-1\}$,

$$v_k = \begin{cases} \min(\{v_1, \ldots, v_{k-1}\} \cap (\sigma \setminus \sigma_k)) & \text{if } \{v_1, \ldots, v_{k-1}\} \cap (\sigma \setminus \sigma_k) \neq \emptyset, \\ \min(\sigma \setminus \sigma_k) & \text{otherwise.} \end{cases}$$

Let $M_\prec(\sigma)$ denote the set of vertices appearing in $\mathrm{mes}_\prec(\sigma)$. Define

$$d_\prec(\Delta) := \max_{\sigma \in \Delta} |M_\prec(\sigma)|.$$

Proposition 5.2. [32, Theorem 6] *If $\prec$ is a linear ordering of the maximal simplices of Δ, then Δ is $d_\prec(\Delta)$-collapsible.*

We will use Proposition 5.2 to upper bound the collapsibility number of $\mathbf{X}_n = \mathcal{N}(\mathbb{I}_n, 3)$, and we will use the retracts in Lemma 5.3 to lower bound the collapsibility number.

Lemma 5.3. *Let $n > m$ and let H be an m-dimensional cube subgraph of $\mathbb{I}_n$. Then there exists a retraction of $\mathbf{X}_n$ onto $\mathcal{N}(H, 3)$.*

Proof. It is enough to prove the result for $m = n-1$. There exist $i \in [n]$ and $\epsilon \in \{0, 1\}$ such that $H = \mathbb{I}_n^{i,\epsilon}$. Define $\phi : V(\mathbb{I}_n) \to V(H)$ as follows: for $v \in V(\mathbb{I}_n)$ and $t \in [n]$, let

$$\phi(v)(t) = \begin{cases} v(t) & \text{if } t \neq i, \\ \epsilon & \text{if } t = i. \end{cases}$$

We extend the map ϕ to $\tilde{\phi} : \mathbf{X}_n \to \mathcal{N}(H, 3)$ by $\tilde{\phi}(\sigma) := \{\phi(v) : v \in \sigma\}$ for all $\sigma \in \mathbf{X}_n$. Clearly, $\tilde{\phi}$ is surjective. Observe that for any $v \sim w$, if $w = v^i$ then $\phi(v) = \phi(w)$, and $w \neq v^i$ implies that $\phi(v) \sim \phi(w)$. Hence, for $\sigma \subseteq N(u_1) \cup N(u_2)$, where $u_1 \sim u_2$, we get that $\tilde{\phi}(\sigma) \subseteq N(\phi(u_1)) \cup N(\phi(u_2))$. Thus, $\tilde{\phi}$ is a simplicial map that doesn't move the simplices of $\mathcal{N}(H, 3)$, and hence a retraction. $\square$

Lemma 5.4. *Let $n \geq 5$ and $\sigma \in \mathbf{X}_n$ be a maximal simplex. If $|N(v) \cap \sigma| \geq 3$ for some v, then $N(v) \subseteq \sigma$.*

Proof. Without loss of generality assume that $\{v^1, v^2, v^3\} \subseteq \sigma$. Let $\sigma = N(u) \cup N(w)$, where $u \sim w$. First, suppose $\{u, w\} \cap \{v^1, v^2, v^3\} = \emptyset$. Since $v^1 \in \sigma$, either $v^1 \in N(u)$ or $v^1 \in N(w)$; without loss of generality assume that $v^1 \in N(u)$. So $u = v^{1,j_0}$ for some $j_0 \neq 1$. If $j_0 = 2$, then $v^3 \not\sim u$, and therefore $v^3 \sim w$. Then $w = v^{3,k_0}$ for some $k_0 \in [n]$. But then $d(u, w) \geq 2$, a contradiction. If $j_0 \neq 2$, then $v^2 \not\sim u$ and so $v^2 \sim w$. But then $d(u, w) \geq 2$, a contradiction. Hence $\{u, w\} \cap \{v^1, v^2, v^3\} \neq \emptyset$. Without loss of generality assume that $u \in \{v^1, v^2, v^3\}$ and $u = v^1$. Suppose $w \neq v$, then there exists $i_0 \neq 1$ such that $w = v^{1,i_0}$. Clearly $\{v^2, v^3\} \not\subseteq N(v^{1,i_0})$. Since $\{v^2, v^3\} \not\subseteq N(u)$, we see that $\{v^1, v^2, v^3\} \not\subseteq \sigma$, a contradiction. Thus we conclude that $w = v$, and hence $N(v) \subseteq \sigma$. $\square$

Theorem 5.5. *For $n \geq 4$, the collapsibility number of $\mathbf{X}_n$ is 5.*

Proof. Choose a linear order $\prec$ on the set of maximal simplices of $\mathbf{X}_n$, namely $\sigma_1 \prec \sigma_2 \prec \ldots \prec \sigma_q$. Let $\tau \in \mathbf{X}_n$. Let p be the smallest index such that $\tau \subseteq \sigma_p$. There exist $v, w \in V(\mathbb{I}_n)$ such that $v \sim w$ and $\sigma_p = N(v) \cup N(w)$. We first prove that $|M_\prec(\tau) \cap N(v)| \leq 3$.

Let $\text{mes}_\prec(\tau) = (a_1, \ldots, a_t)$. Suppose $|M_\prec(\tau) \cap N(v)| \geq 4$. Let l be the least integer such that $|\{a_1, \ldots, a_l\} \cap N(v)| = 3$. Clearly, $l < t$. Let $\{a_1, \ldots, a_l\} \cap N(v) = \{a_{i_1}, a_{i_2}, a_{i_3}\}$. Observe that $a_l \in \{a_{i_1}, a_{i_2}, a_{i_3}\}$. Let γ be a maximal simplex such that $\gamma \prec \sigma_p$. If $\{a_1, \ldots, a_l\} \cap (\tau \setminus \gamma) \neq \emptyset$, then $a_{l+1} \in \{a_1, \ldots, a_l\}$, and hence $\{a_1, \ldots, a_{l+1}\} \cap N(v) = \{a_{i_1}, a_{i_2}, a_{i_3}\}$. If $\{a_1, \ldots, a_l\} \cap (\tau \setminus \gamma) = \emptyset$, then $\{a_{i_1}, a_{i_2}, a_{i_3}\} \subseteq \gamma$. From Lemma 5.4, $N(v) \subseteq \gamma$. Thus $a_{l+1} \notin N(v)$, which implies that $\{a_1, \ldots, a_{l+1}\} \cap N(v) = \{a_{i_1}, a_{i_2}, a_{i_3}\}$. If $l+1 = t$, then we get a contradiction to the assumption that $|M_\prec(\tau) \cap N(v)| \geq 4$. Inductively assume that for all $l \leq k < t$, $\{a_1, \ldots, a_k\} \cap N(v) = \{a_{i_1}, a_{i_2}, a_{i_3}\}$. If $\{a_1, \ldots, a_{t-1}\} \cap (\tau \setminus \gamma) \neq \emptyset$, then $a_t \in \{a_1, \ldots, a_{t-1}\}$. Hence $\{a_1, \ldots, a_t\} \cap N(v) = \{a_{i_1}, a_{i_2}, a_{i_3}\}$. If $\{a_1, \ldots, a_{t-1}\} \cap (\tau \setminus \gamma) = \emptyset$, then $\{a_{i_1}, a_{i_2}, a_{i_3}\} \subseteq \gamma$. From Lemma 5.4, $N(v) \subseteq \gamma$. Thus $a_t \notin N(v)$. Hence $\{a_1, \ldots, a_t\} \cap N(v) = \{a_{i_1}, a_{i_2}, a_{i_3}\}$, which is a contradiction to the assumption that $|M_\prec(\tau) \cap N(v)| \geq 4$. Thus $|M_\prec(\tau) \cap N(v)| \leq 3$.

By using an argument similar to the above, $|M_\prec(\tau) \cap N(w)| \leq 3$. Thus $|M_\prec(\tau)| \leq 6$. Suppose $|M_\prec(\tau)| = 6$. Let $M_\prec(\tau) = \{z_1, z_2, z_3, z_4, z_5, z_6\}$, where $M_\prec(\tau) \cap N(v) = \{z_1, z_2, z_3\}$ and $M_\prec(\tau) \cap N(w) = \{z_4, z_5, z_6\}$. Without loss of generality we can assume that z_6 is appearing after $z_1, \ldots, z_5$ in $\text{mes}_\prec(\tau)$. Let i_0 be the first index such that $a_{i_0} = z_6$. From the definition of $\text{mes}(\tau)$, $i_0 < p$. Then $(\tau \setminus \sigma_{i_0}) \cap \{z_1, \ldots, z_5\} = \emptyset$, which implies that $N(v) \cup \{z_4, z_5\} \subseteq \sigma_{i_0}$. There exists j_0 such that $\sigma_{i_0} = N(v) \cup N(v^{j_0})$. Since $w \sim v$, $w = v^{k_0}$ for some $k_0 \in [n]$. Clearly $z_4, z_5 \sim v^{k_0}$. There exist l, s such that $z_4 = v^{k_0,l}$ and $z_5 = v^{k_0,s}$. Since z_4 and z_5 are not adjacent to v, we conclude that $z_4, z_5 \in N(v^{j_0})$. But this is possible only when $j_0 = k_0$, which implies that $\sigma_p = \sigma_{i_0}$, a contradiction as $i_0 < p$. Hence $|M_\prec(\tau)| \leq 5$. Since τ is an arbitrary simplex of $\mathbf{X}_n$, we have that $d_\prec(\mathbf{X}_n) \leq 5$. It therefore follows from Proposition 5.2 that $\mathbf{X}_n$ is 5-collapsible.

Let X be the Čech complex of a 4-dimensional cube subgraph of $\mathbb{I}_n$ at scale $r = 3$, *i.e.*, $X \cong \mathbf{X}_4$. Then using Lemma 5.3, there exists a retraction $r : \mathbf{X}_n \to X$. Since $X \cong \mathbf{X}_4$ and since we know $\tilde{H}_4(\mathbf{X}_4; \mathbb{Z}) \neq 0$ from a homology computation (Table 1), we see that $\tilde{H}_4(X; \mathbb{Z}) \neq 0$. Further, since the homomorphism $r_* : \tilde{H}_4(\mathbf{X}_n; \mathbb{Z}) \to \tilde{H}_4(X; \mathbb{Z})$ induced by r is surjective, $\tilde{H}_4(\mathbf{X}_n; \mathbb{Z}) \neq 0$. Using Proposition 5.1, we conclude that the collapsibility number of $\mathbf{X}_n$ is 5. $\square$

5.2. **Homology.** In this subsection we will prove Theorem 5.9, which states that for $n \geq 4$, $\tilde{H}_i(\mathcal{N}(\mathbb{I}_n, 3); \mathbb{Z}) \neq 0$ if and only if $i \in \{3, 4\}$. The proof will proceed as follows. Theorem 5.5 and Proposition 5.1 imply that $\tilde{H}_i(\mathcal{N}(\mathbb{I}_n, 3); \mathbb{Z}) = 0$ for $i \geq 5$. Table 1 displays a homology computation on a computer that $\tilde{H}_i(\mathcal{N}(\mathbb{I}_4, 3); \mathbb{Z}) \neq 0$ for $i \in \{3, 4\}$, and so using the retractions in Lemma 5.3, we get that $\tilde{H}_i(\mathcal{N}(\mathbb{I}_n, 3); \mathbb{Z}) \neq 0$ for $i \in \{3, 4\}$. It therefore suffices to show that $\tilde{H}_i(\mathcal{N}(\mathbb{I}_n, 3); \mathbb{Z}) = 0$ for $0 \leq i \leq 2$, and we now build up the lemmas and the machinery in order to do this.

Recall that for each $i \in [n]$ and $\epsilon \in \{0, 1\}$, $\mathbb{I}_n^{i,\epsilon}$ is the induced subgraph of $\mathbb{I}_n$ on the vertex set $\{v \in V(\mathbb{I}_n) : v(i) = \epsilon\}$. For $1 \leq i \leq n$ and $\epsilon \in \{0, 1\}$, let $\mathbf{X}_n^{i,\epsilon} = \mathcal{N}(\mathbb{I}_n^{i,\epsilon}, 3)$ and let

$$\partial(\mathbf{X}_n) = \bigcup_{i \in [n], \epsilon \in \{0,1\}} \mathbf{X}_n^{i,\epsilon}.$$

We say that a simplex $\sigma \in \mathbf{X}_n$ *covers all places* if for each $i \in [n]$, there exist vertices $v, w \in \sigma$ such that $v(i) = 1$ and $w(i) = 0$.

The following lemma plays a key role in the proof of Theorem 5.9.

Lemma 5.6. *Let $n \geq 4$ and let $l \leq n - 2$. Then any l-cycle z in $\mathbf{X}_n$ is homologous to an l-cycle $\tilde{z}$ in $\partial(\mathbf{X}_n)$.*

Proof. For any chain $c = \sum c_i \sigma_i$ in $\mathbf{X}_n$, if $c_i \neq 0$, then we say that $\sigma_i \in c$. For a cycle c in $\mathbf{X}_n$, let $\Theta(c) = \{\sigma \in c : \sigma \notin \partial(\mathbf{X}_n)\}$. Let z be an l-cycle in $\mathbf{X}_n$. If $\Theta(z) = \emptyset$, then z is an l-cycle in $\partial(\mathbf{X}_n)$. Suppose $\Theta(z) \neq \emptyset$. We show that z is homologous to an l-cycle z_1 such that $|\Theta(z_1)| < |\Theta(z)|$. Let

$\sigma \in z$ be such that $\sigma \notin \partial(\mathbf{X}_n)$, *i.e.*, σ covers all places. Let γ be a maximal simplex such that $\sigma \subseteq \gamma$. Let $\gamma = N(v) \cup N(w)$, where $v \sim w$.

For any $k \in [n]$ and $\tau \in \mathbf{X}_n$, we say that τ *covers k places* if there exist distinct indices $i_1, \ldots, i_k \in [n]$ such for each $1 \leq t \leq k$, we have vertices $a, b \in \tau$ with $a(i_t) = 0$ and $b(i_t) = 1$.

Observe that if the l-simplex σ satisfies $\sigma \subseteq N(v)$ or $\sigma \subseteq N(w)$, then σ can cover at most $l+1$ places. Since $n > l+1$, this contradicts the assumption that σ covers all places. Hence $N(v) \cap \sigma \neq \emptyset$ and $N(w) \cap \sigma \neq \emptyset$.

Since $w \sim v$, $w = v^p$ for some $p \in [n]$. Suppose $v, w \in \sigma$. If $N(w) \cap \sigma = \{v\}$, then $\sigma = \{v, w, v^{i_1}, \ldots, v^{i_{l-1}}\}$ for distinct $i_1, i_2, \ldots, i_{l-1} \in [n] \setminus \{p\}$. Observe that σ covers only $l < n$ places, namely $i_1, \ldots, i_{l-1}, p$, a contradiction to the assumption that σ covers all places. Hence $|N(w) \cap \sigma| \geq 2$. Then $\sigma = \{v, w, v^{i_1}, \ldots, v^{i_s}, v^{p,j_1}, \ldots, v^{p,j_t}\}$, for some $i_1, \ldots, i_s, j_1 \ldots, j_t \in [n]$, where $s + t = l - 1$. Here σ can cover at most l places, namely $i_1, \ldots, i_s, j_1, \ldots, j_t, p$ (and furthermore σ covers l places only if $\{i_1, \ldots, i_s\} \cap \{j_1, \ldots, j_t\} = \emptyset$). Since $l < n$, σ does not cover all places. Hence $\{v, w\} \not\subseteq \sigma$.

Suppose $v \in \sigma$. Then $w \notin \sigma$. If $N(w) \cap \sigma = \{v\}$, then $\sigma = \{v, v^{i_1}, \ldots, v^{i_l}\}$ for some $i_1, i_2, \ldots, i_l \in [n]$. Observe that σ covers only l places, namely $i_1, \ldots, i_l$. Hence $|N(w) \cap \sigma| \geq 2$. Let $\sigma = \{v, v^{i_1}, \ldots, v^{i_s}, v^{p,j_1}, \ldots, v^{p,j_t}\}$, where $w = v^p$, where $i_1, \ldots, i_s, j_1 \ldots, j_t \in [n]$, and where $s + t = l$. Here σ can cover at most $l + 1 < n$ places, namely $i_1, \ldots, i_s, j_1, \ldots, j_t, p$ (and furthermore σ covers $l+1$ places only if $\{i_1, \ldots, i_s\} \cap \{j_1, \ldots, j_t\} = \emptyset, p \notin \{i_1, \ldots, i_s\}$). Thus, we conclude that $v \notin \sigma$. By a similar argument, $w \notin \sigma$.

For a simplex δ, let $Bd(\delta)$ denote the simplicial boundary of δ. Let the coefficient of σ in z be $(-1)^s z_\sigma$, and let the coefficient of σ in $Bd(\sigma \cup \{v\})$ be $(-1)^t$. Define an l-cycle z_1 as follows:

$$z_1 = \begin{cases} z - z_\sigma Bd(\sigma \cup \{v\}) & \text{if } s \text{ and } t \text{ are of same parity, and} \\ z + z_\sigma Bd(\sigma \cup \{v\}) & \text{if } s \text{ and } t \text{ are of opposite parity.} \end{cases}$$

Clearly, z is homologous to z_1. Let $\tau \subseteq \sigma \cup \{v\}$ be such that $|\tau| = l+1$ and $\tau \neq \sigma$. Then $v \in \tau$, and so from the argument in the paragraphs above we conclude that $\tau \in \partial(\mathbf{X}_n)$. Hence $|\Theta(z_1)| < |\Theta(z)|$. Since $|\Theta(z)|$ is finite, by repeating the above argument a finite number of times, we get a cycle z_k such that z is homologous to z_k and $|\Theta(z_k)| = 0$. We take $\tilde{z} = z_k$.

This completes the proof. $\square$

We first show that $\mathbf{X}_n$ is simply connected, and then we show it is 2-connected.

Lemma 5.7. *For $n \geq 2$, $\mathbf{X}_n$ is simply connected.*

Proof. Let γ be a closed path in $\mathbf{X}_n$. Since $\mathbf{X}_n$ is a simplicial complex, γ is homotopic to a closed path $\delta = v_1, v_2 \ldots v_m v_1$, where $\{v_m, v_1\}$ and $\{v_i, v_{i+1}\}$ are edges in $\mathbf{X}_n$ for each $1 \leq i \leq m-1$. Without loss of generality, we can assume that successive vertices in δ are distinct. Otherwise, if $v_i = v_{i+1}$ for some i, we could delete v_i and obtain a homotopic path. We show that δ is homotopic to a constant path. If for some i, $d(v_i, v_{i+1}) = 3$, then there exist u_i, w_i such that $d(v_i, w_i) = d(w_i, u_i) = d(u_i, v_{i+1}) = 1$ and $\{v_i, w_i, u_i, v_{i+1}\} \in \mathbf{X}_n$. Clearly, the path $\delta_1 = v_1 \ldots v_i w_i u_i v_{i+1} \ldots v_m v_1$ is homotopic to δ. Similarly, if for some i, $d(v_i, v_{i+1}) = 2$, then there exists a_i such that $d(v_i, a_i) = d(a_i, v_{i+1}) = 1$ and $\{v_i, a_i, v_{i+1}\} \in \mathbf{X}_n$. Clearly, the path $\delta_2 = v_1 \ldots v_i a_i v_{i+1} \ldots v_m v_1$ is homotopic to δ. Hence, by inserting new vertices between each such pair of the vertices of distance 2 and 3, we can assume that $d(v_m, v_1) = 1 = d(v_i, v_{i+1})$ for all $1 \leq i \leq m-1$. If for some i, $v_i = v_{i+2 (\text{mod } m)}$, then we could delete v_{i+1} and obtain a homotopic path. Hence we can also assume that $v_i \neq v_{i+2 (\text{mod } m)}$ for all i.

Our proof is by induction on n. If $n = 2$, then clearly $\mathbf{X}_2$ is contractible and therefore any path is homotopic to a constant path. Now let $n \geq 3$ and assume that $\mathbf{X}_r$ is simply connected for all $2 \leq r < n$. We will show that the closed path δ in $\mathbf{X}_n$ is homotopic to a closed path which lies in $\mathbf{X}_{n-1}$. Let l be the least integer such that $v_l(n) \neq v_1(n)$. Clearly, $l \geq 2$. Since $d(v_{l-1}, v_l) = 1$, we have $v_{l-1} = v_l^n$. Let us first assume that $v_{l+1 (\text{mod } n)} \neq v_1$, *i.e.*, $l \neq n$. Since $v_{l+1} \neq v_{l-1}$, $d(v_{l-1}, v_{l+1}) = 2$ and therefore there exists a vertex w such that $\{v_{l-1}, v_l, v_{l+1}, w\}$ are vertices of a square in $\mathbb{I}_n$. Here, $d(v_{l-1}, w) = 1$ and $d(v_l, w) = 2$. Since $\{v_{l-1}, v_l, v_{l+1}, w\}$ are vertices of a square, $w = v_l^{n,i_o}$ for some

$i_o \neq n$. Observe that $w(n) = v_{l-1}(n) = v_1(n)$. Clearly, the path $\delta_1 = v_1 \dots v_{l-1} w v_{l+1} \dots v_n v_1$ is homotopic to δ. By repeating this process, after a finite number of steps we get a path $\delta_k = u_1 \dots u_q$ which is homotopic to δ and which satisfies $u_i(n) = u_1(n)$ for all $1 \leq i \leq q$. Hence δ_k is a path in $\mathcal{N}(\mathbb{I}_n^{n,v_1(n)}) \simeq \mathbf{X}_{n-1}$. By the induction hypothesis, δ_k is homotopic to a constant path, and therefore δ is also homotopic to a constant path. □

Lemma 5.8. *For any $n \geq 2$, $\mathbf{X}_n$ is 2-connected.*

Proof. Since $\mathbf{X}_n$ is simply connected by Lemma 5.7, by the Hurewicz theorem ([29, Theorem 4.32]) it is enough to show that $\tilde{H}_2(\mathbf{X}_n) = 0$. Since $\mathbf{X}_3 \simeq \vee_3 S^4$ by [18, Example 3], clearly $\tilde{H}_2(\mathbf{X}_3) = 0$. So fix $n \geq 4$, and inductively assume that $\tilde{H}_2(\mathbf{X}_m) = 0$ for all $2 \leq m < n$. From Lemma 5.6, any 2-cycle in $\mathbf{X}_n$ is homologous to a 2-cycle in $\partial(\mathbf{X}_n)$. Hence it is sufficient to show that $\tilde{H}_2(\partial(\mathbf{X}_n)) = 0$. For each $i \in [n]$ and $\epsilon \in \{0,1\}$, recall $\mathbf{X}_n^{i,\epsilon} = \mathcal{N}(\mathbb{I}_n^{i,\epsilon}, 3)$. Note $\partial(\mathbf{X}_n) = \bigcup\limits_{i \in [n], \epsilon \in \{0,1\}} \mathbf{X}_n^{i,\epsilon}$. Observe that each non-empty intersection $\mathbf{X}_n^{i_1,\epsilon_1} \cap \dots \cap \mathbf{X}_n^{i_t,\epsilon_t}$ is homeomorphic to the Čech complex of some cube subgraph of dimension less than n and therefore it is 2-connected by the induction hypothesis. Hence by Theorem 2.7 (ii), $\partial(\mathbf{X}_n)$ is 2-connected if and only if $\mathbf{N}(\{\mathbf{X}_n^{i,\epsilon}\})$ is 2-connected. We now show that $\mathbf{N}(\{\mathbf{X}_n^{i,\epsilon}\})$ is 2-connected. For any $i, j \in [n]$ and $\epsilon, \delta \in \{0,1\}$, let $\overline{\{(i,\epsilon),(j,\delta)\}}$ be a simplicial complex on vertex set $\{(i,\epsilon),(j,\delta)\}$, which is isomorphic to S^0. It is easy to check that

$$\mathbf{N}(\{\mathbf{X}_n^{i,\epsilon}\}) \cong \overline{\{(1,0),(1,1)\}} * \overline{\{(2,0),(2,1)\}} * \dots * \overline{\{(n,0),(n,1)\}}$$

is the join of n copies of S^0. Hence $\mathbf{N}(\{\mathbf{X}_n^{i,\epsilon}\}) \simeq S^{n-1}$. Since $n \geq 4$, we see that $\mathbf{N}(\{\mathbf{X}_n^{i,\epsilon}\})$ is 2-connected. □

Theorem 5.9. *Let $n \geq 4$. Then $\tilde{H}_i(\mathcal{N}(\mathbb{I}_n, 3); \mathbb{Z}) \neq 0$ if and only if $i \in \{3, 4\}$.*

Proof. From Lemma 5.8, we have $\tilde{H}_i(\mathcal{N}(\mathbb{I}_n, 3); \mathbb{Z}) = 0$ for $0 \leq i \leq 2$. Since $\tilde{H}_i(\mathcal{N}(\mathbb{I}_4, 3); \mathbb{Z}) \neq 0$ for $i \in \{3, 4\}$ from a computer computation (Table 1), using the retractions in Lemma 5.3 we get that $\tilde{H}_i(\mathcal{N}(\mathbb{I}_n, 3); \mathbb{Z}) \neq 0$ for $i \in \{3, 4\}$. Theorem 5.5 and Proposition 5.1 imply that $\tilde{H}_i(\mathcal{N}(\mathbb{I}_n, 3); \mathbb{Z}) = 0$ for $i \geq 5$. □

6. Persistence

In this section, we show that in the persistent (reduced) homology of Čech complexes of hypercube graphs, no bars in the persistence barcode have length longer than two filtration steps. Indeed, we will show that for any integers n and r, the inclusion $\mathcal{N}(\mathbb{I}_n, r) \hookrightarrow \mathcal{N}(\mathbb{I}_n, r+2)$ is null-homotopic. The proof technique was shown to us by Žiga Virk, who recently proved that Vietoris–Rips complexes of hypercube graphs have no persistence intervals of length longer than one filtration step (see the forthcoming paper [6]).

Theorem 6.1. *Let $n \geq 1$ and $r \geq 0$. Then the inclusion $\iota \colon \mathcal{N}(\mathbb{I}_n, r) \hookrightarrow \mathcal{N}(\mathbb{I}_n, r+2)$ is homotopic to a constant map.*

Proof. For $1 \leq i \leq n$, let $p_i \colon \{0,1\}^n \to \{0,1\}^n$ be the map defined by

$$p_i(x_1 \dots x_i x_{i+1} \dots x_n) = (0 \dots 0 x_{i+1} \dots x_n).$$

First, we claim that each p_i induces a well-defined map $\tilde{p}_i \colon \mathcal{N}(\mathbb{I}_n, r) \to \mathcal{N}(\mathbb{I}_n, r+2)$ defined on vertices by $\tilde{p}_i(v) = p_i(v)$, and extended linearly to simplices via $\tilde{p}_i(\{v_0, \dots, v_k\}) = \{p_i(v_0), \dots, p_i(v_k)\}$. To see that $\tilde{p}_i$ is well-defined, note that $\tilde{p}_i(N_r[v]) \subseteq N_r[p_i(v)]$ for all $v \in V(G)$. (We remark that $\tilde{p}_i$ would still be well-defined even if its codomain were the smaller simplicial complex $\mathcal{N}(\mathbb{I}_n, r+1)$.) Define $\tilde{p}_0 \colon \mathcal{N}(\mathbb{I}_n, r) \to \mathcal{N}(\mathbb{I}_n, r+2)$ to be the inclusion map $\tilde{p}_0 = \iota$.

Next, we show that for all $1 \leq i \leq n$, the two maps $\tilde{p}_{i-1}, \tilde{p}_i \colon \mathcal{N}(\mathbb{I}_n, r) \to \mathcal{N}(\mathbb{I}_n, r+2)$ are contiguous. Two simplicial maps $f, g \colon K \to L$ are *contiguous* if for each simplex $\sigma \in K$, the union $f(\sigma) \cup g(\sigma)$ is a simplex in L. When f and g are contiguous, they induce homotopy equivalent maps on geometric realizations. We split the verification that $\tilde{p}_{i-1}$ and $\tilde{p}_i$ are contiguous into two cases. When r is even, for any vertex $v \in V$, we have both

- $\tilde{p}_i(N_{\frac{r}{2}}[v]) \subseteq N_{\frac{r}{2}}[p_i(v)] \subseteq N_{\frac{r+2}{2}}[p_{i-1}(v)]$ since $d(p_{i-1}(v), p_i(v)) \leq 1$, and
- $\tilde{p}_{i-1}(N_{\frac{r}{2}}[v]) \subseteq N_r[p_{i-1}(v)] \subseteq N_{\frac{r+2}{2}}[p_{i-1}(v)]$.

This shows that the maps $\tilde{p}_{i-1}, \tilde{p}_i \colon \mathcal{N}(\mathbb{I}_n, r) \to \mathcal{N}(\mathbb{I}_n, r+2)$ are contiguous when r is even. Similarly, when r is odd, for any edge $(v, w) \in E(G)$, we have both

- $\tilde{p}_i(N_{\frac{r-1}{2}}[v] \cup N_{\frac{r-1}{2}}[w]) \subseteq N_{\frac{r-1}{2}}[p_i(v)] \cup N_{\frac{r-1}{2}}[p_i(w)] \subseteq N_{\frac{r+1}{2}}[p_{i-1}(v)] \cup N_{\frac{r+1}{2}}[p_{i-1}(w)]$ since $d(p_{i-1}(v), p_i(v)) \leq 1$ and $d(p_{i-1}(w), p_i(w)) \leq 1$, and
- $\tilde{p}_{i-1}(N_{\frac{r-1}{2}}[v] \cup N_{\frac{r-1}{2}}[w]) \subseteq N_{\frac{r-1}{2}}[p_{i-1}(v)] \cup N_{\frac{r-1}{2}}[p_{i-1}(w)] \subseteq N_{\frac{r+1}{2}}[p_{i-1}(v)] \cup N_{\frac{r+1}{2}}[p_{i-1}(w)]$.

Hence the maps $\tilde{p}_{i-1}, \tilde{p}_i \colon \mathcal{N}(\mathbb{I}_n, r) \to \mathcal{N}(\mathbb{I}_n, r+2)$ are contiguous.

So for all $1 \leq i \leq n$, the maps $\tilde{p}_{i-1}, \tilde{p}_i$ are contiguous and hence homotopic. We have the chain of homotopy equivalences $\iota = \tilde{p}_0 \simeq \tilde{p}_1 \simeq \ldots \simeq \tilde{p}_{i-1} \simeq \tilde{p}_n$. Since $\tilde{p}_n$ is a constant map, this shows that the inclusion $\iota \colon \mathcal{N}(\mathbb{I}_n, r) \hookrightarrow \mathcal{N}(\mathbb{I}_n, r+2)$ is a null-homotopy, as claimed. □

Theorem 6.1 implies that in the persistent (reduced) homology of Čech complexes of hypercube graphs, no bars in the persistence barcode have length longer than two filtration steps.

We remark that our proof strategy above will not work with the inclusion $\iota \colon \mathcal{N}(\mathbb{I}_n, r) \hookrightarrow \mathcal{N}(\mathbb{I}_n, r+1)$, when $r+2$ in the codomain is replaced by $r+1$. Indeed, consider the 2-dimensional hypercube $\mathbb{I}_2$ with $n = 2$, and let $r = 1$ be odd. The complex $\mathcal{N}(\mathbb{I}_2, 1)$ consists of four vertices and four edges, arranged in a square. Let $v = 11$ and $w = 10$. Note that

$$N_{\frac{r-1}{2}}[v] \cup N_{\frac{r-1}{2}}[w] = N_0[11] \cup N_0[10] = \{11, 10\}$$

and so

$$\iota\left(N_{\frac{r-1}{2}}[v] \cup N_{\frac{r-1}{2}}[w]\right) \cup \tilde{p}_1\left(N_{\frac{r-1}{2}}[v] \cup N_{\frac{r-1}{2}}[w]\right) = \{11, 10, 01, 00\}.$$

There is no vertex $v \in \mathbb{I}_2$ such that $N_{\frac{r+1}{2}}[v] = N_1[v]$ contains this set, as the complex $\mathcal{N}(\mathbb{I}_2, 2)$ is the *boundary* of a tetrahedron. Therefore the maps $\iota, \tilde{p}_1 \colon \mathcal{N}(\mathbb{I}_n, r) \hookrightarrow \mathcal{N}(\mathbb{I}_n, r+1)$ are not contiguous.

7. Conclusion and open questions

We end with some open questions. Though determining all of the homotopy types of Čech complexes of hypercubes (*i.e.* all of the homotopy types in Table 1) may be a difficult task, we hope these open questions will inspire further progress.

Since $\mathcal{N}(\mathbb{I}_n, 1)$ is isomorphic to the graph $\mathbb{I}_n$ itself, the complex $\mathcal{N}(\mathbb{I}_n, 1)$ is a wedge of circles. Moreover, from Theorem 4.2, we know that $\mathcal{N}(\mathbb{I}_n, 2)$ is also homotopy equivalent to a wedge of spheres. Thus the following is a very natural question.

Question 7.1. For any $n \geq 1$ and $r \geq 0$, is the Čech complex $\mathcal{N}(\mathbb{I}_n, r)$ of the hypercube graph always homotopy equivalent to a wedge of spheres?

From Theorem 4.2, $\tilde{H}_i(\mathcal{N}(\mathbb{I}_n, 2)) \neq 0$ only for $i = 2$. For $r = 3$, in Theorem 5.9, we showed that $\tilde{H}_i(\mathcal{N}(\mathbb{I}_n, r)) \neq 0$ only if $i \in \{3, 4\}$. Therefore, in view of the Table 1, we propose the following question.

Question 7.2. When is $\tilde{H}_i(\mathcal{N}(\mathbb{I}_n, r))$ nonzero?

- For $r = 2k$ even and for $n \geq k+1$, is $\tilde{H}_i(\mathcal{N}(\mathbb{I}_n), r) \neq 0$ if and only if $i \in \{r, 2(2^k - 1)\}$?
- For $r = 2k+1$ odd and for $n \geq k+2$, is $\tilde{H}_i(\mathcal{N}(\mathbb{I}_n), r) \neq 0$ if and only if $i \in \{r, 3 \cdot 2^k - 2\}$?

Another interesting problem would be to find the collapsibility number of $\mathcal{N}(\mathbb{I}_n, r)$. Since $\mathcal{N}(\mathbb{I}_n, 1)$ is isomorphic to the graph $\mathbb{I}_n$ itself, the collapsibility number of $\mathcal{N}(\mathbb{I}_n, 1)$ is 2.

Lemma 7.3. *For $n \geq 2$, the collapsibility number of $\mathcal{N}(\mathbb{I}_n, 2)$ is 3.*

Proof. It is easy to see that, for any two vertices $v, w \in \mathbb{I}_n$, $N_1[v] \cap N_1[w]$ has at most 2 elements. Therefore, for any two maximal simplices $\tau, \sigma \in \mathcal{N}(\mathbb{I}_n, 2)$, if $|\tau \cap \sigma| \geq 3$, then $\sigma = \tau$. Hence, by using Proposition 5.2, we conclude that $\mathcal{N}(\mathbb{I}_n, 2)$ is 3-collapsible. The result then follows from Theorem 4.2 and Proposition 5.1. □

From Theorem 5.5, we know that $\mathcal{N}(\mathbb{I}_n, 3)$ is 5 collapsible. In this direction, we ask the following.

Question 7.4. What is the collapsibility number of $\mathcal{N}(\mathbb{I}_n, r)$?

- For $r = 2k$ even and for $n \geq k+1$, is the collapsibility number of $\mathcal{N}(\mathbb{I}_n, 2k)$ equal to $2^{k+1}-1$?
- For $r = 2k+1$ odd and for $n \geq k+2$, is the collapsibility number of $\mathcal{N}(\mathbb{I}_n, 2k+1)$ equal to $3 \cdot 2^k - 1$?

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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