Recurrences revisited

A different view

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(2a) If $\lambda_1 \neq \lambda_2$, then $a_n = A\lambda_1^n + B\lambda_2^n$ for some constants A and B.

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Solutions in (2a) and (2b) are the general solution of the rec. relation.

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Initial conditions (values of a_0 and a_1) determine constants A, B or C, D.

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Initial conditions (values of a_0 and a_1) determine constants A, B or C, D.

A similar method works for higher degree. See pages 518-519.

Example

Let $f : \mathbb{N} \to \mathbb{Z}$ be a function defined recursively by $f(0) = -1$, $f(1) = 5$ and for all $n \geq 2$ by

$$f(n) = 10\ f(n-1) - 25\ f(n-2).$$

Fibonacci (again)

$F_0 = F_1 = 1$
$F_n = F_{n-1} + F_{n-2}$

Some recipes for solving recurrences

Consider a linear non-homogeneous recurrence relation of order 2:

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From before, the homogeneous equation $a_n = 10\ a_{n-1} - 25\ a_{n-2}$ has the general solution $a_n^{(h)} = C \cdot 5^n + D \cdot n5^n$.

Sums of squares

How many ways can we write an integer as a sum of two squares?

$$n = a^2 + b^2, \quad \text{where } a, b \in \mathbb{Z}$$

For $n = 13$, there are eight solutions:

$$\begin{aligned} 13 &= 3^2 + 2^2 = (-3)^2 + 2^2 = 3^2 + (-2)^2 = (-3)^2 + (-2)^2 \\ &= 2^2 + 3^2 = (-2)^2 + 3^2 = 2^2 + (-3)^2 = (-2)^2 + (-3)^2 \end{aligned}$$

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Define a function $f : \mathbb{N} \to \mathbb{N}$, where $f(n)$ denotes the number of solutions for n. So $f(13) = 8$, $f(16) = 4$, and $f(11) = 0$.

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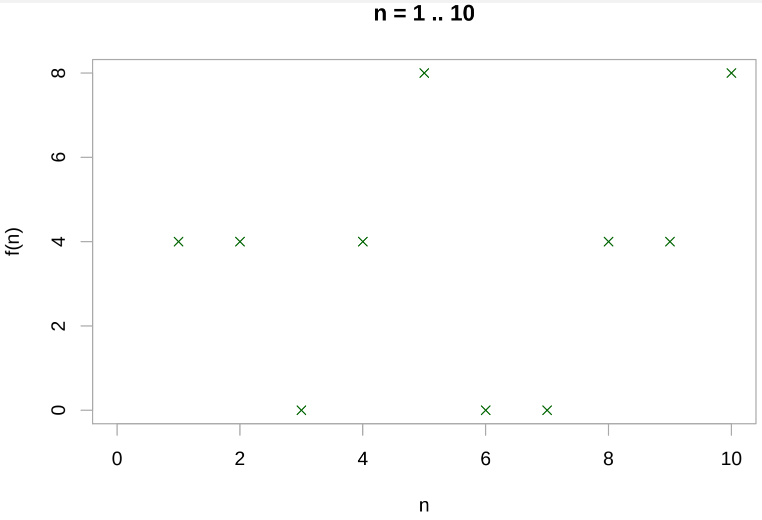
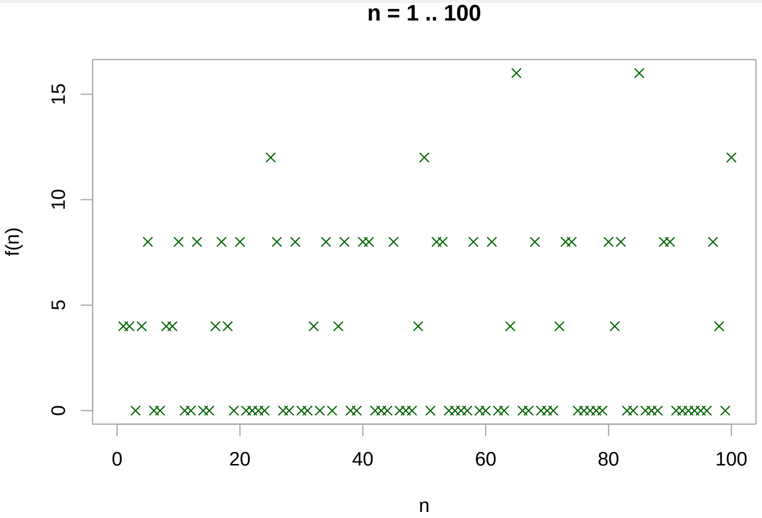
What do you think happens to $f(n)$ on average, as n grows large?

Looking at $f(n)$



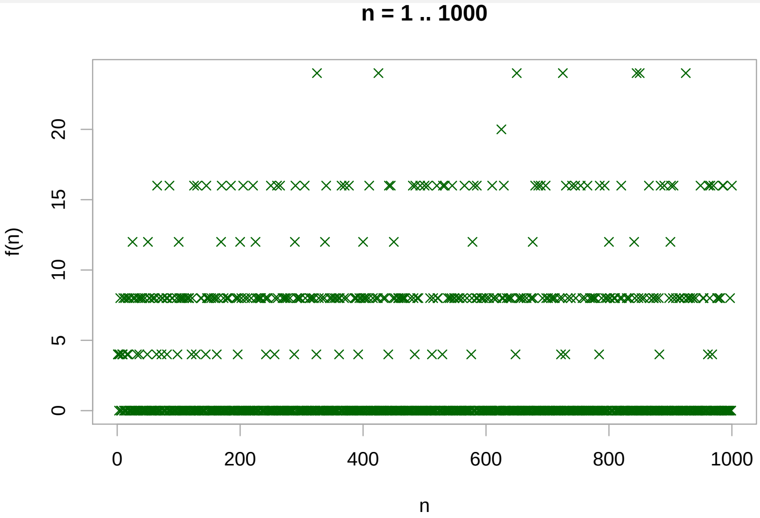
Average $f(n) \simeq 3.6$

Looking at $f(n)$



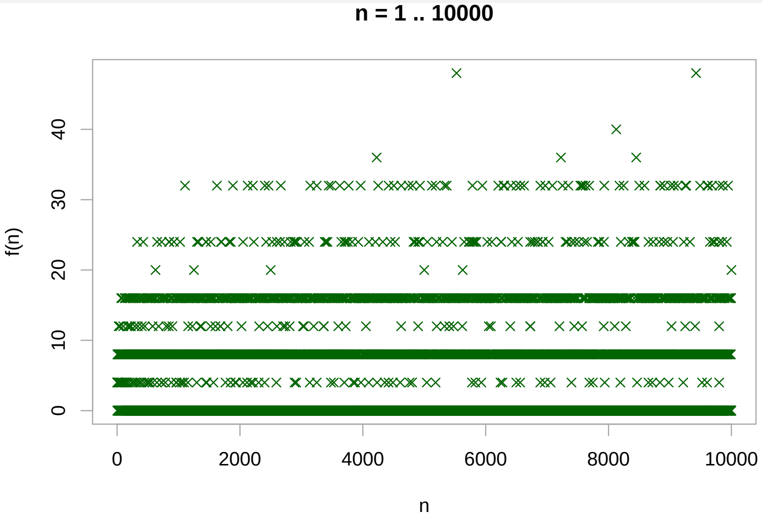
Average $f(n) \simeq 3.16$

Looking at $f(n)$



Average $f(n) \simeq 3.148$

Looking at $f(n)$



Average $f(n) \simeq 3.1416$

Theorem

As n grows, the average of $f(1), \ldots, f(n)$ approaches π.

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Proof: Consider pairs of integers a, b as lattice points.

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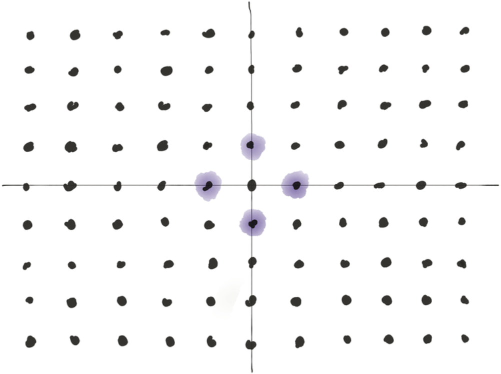
Marking the solutions to $a^2 + b^2 = 1$:



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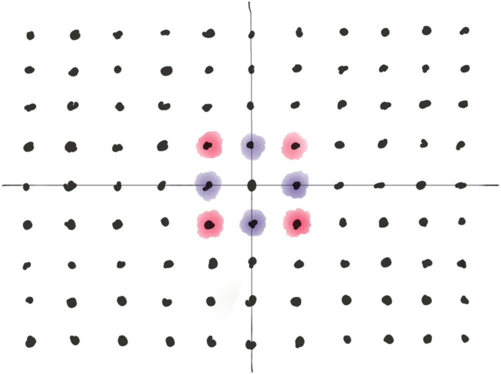
Marking the solutions to $a^2 + b^2 = 2$:



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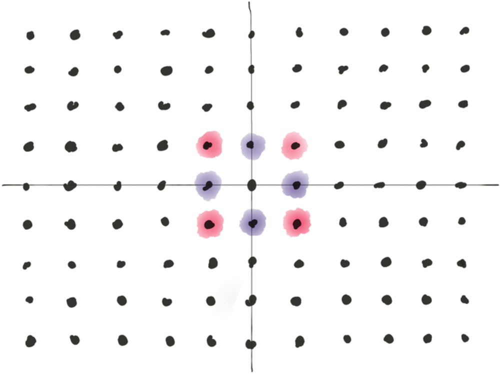
Marking the solutions to $a^2 + b^2 = 3$:



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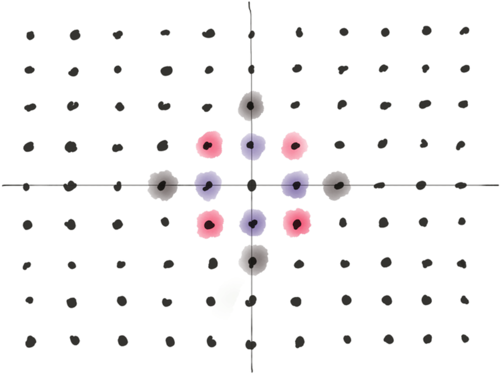
Marking the solutions to $a^2 + b^2 = 4$:



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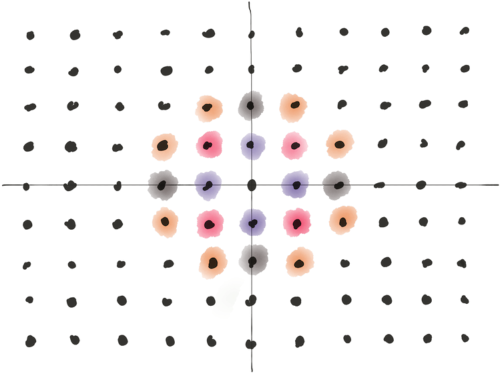
Marking the solutions to $a^2 + b^2 = 5$:



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As n grows, the average of $f(1), \ldots, f(n)$ approaches π.

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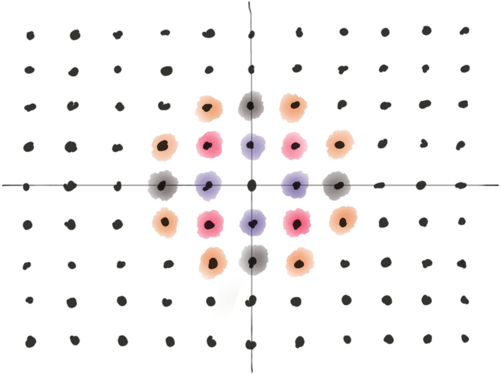
Marking the solutions to $a^2 + b^2 = 6$:

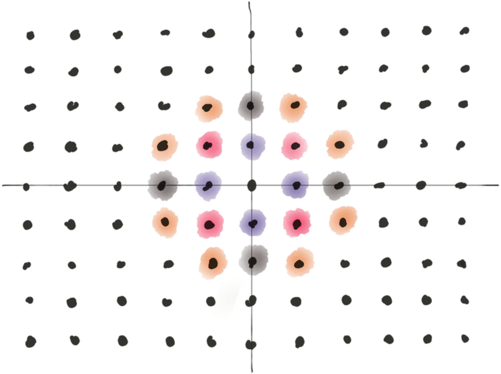


Theorem

As n grows, the average of $f(1), \ldots, f(n)$ approaches π.

Proof: Consider pairs of integers a, b as lattice points.

Marking the solutions to $a^2 + b^2 = 7$:

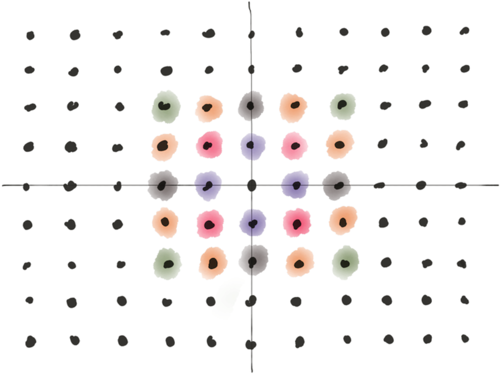


Theorem

As n grows, the average of $f(1), \ldots, f(n)$ approaches π.

Proof: Consider pairs of integers a, b as lattice points.

Marking the solutions to $a^2 + b^2 = 8$:



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As n grows, the average of $f(1), \ldots, f(n)$ approaches π.

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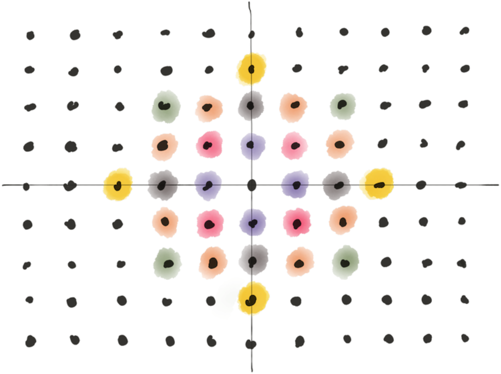
Marking the solutions to $a^2 + b^2 = 9$:



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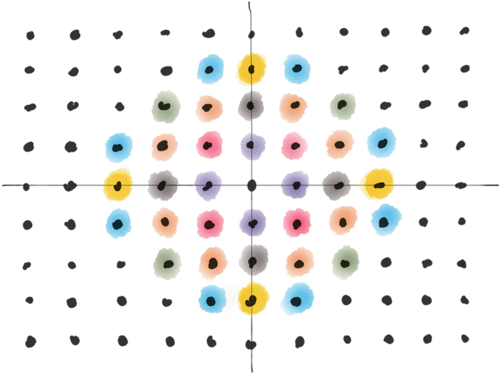
Marking the solutions to $a^2 + b^2 = 10$:

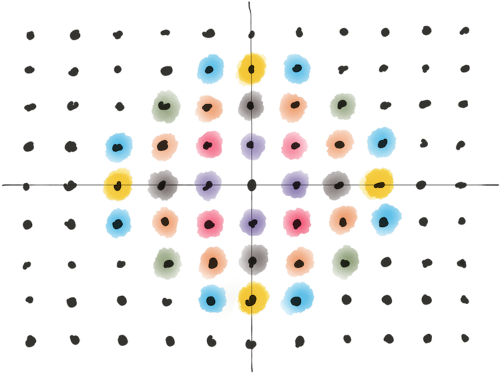


Theorem

As n grows, the average of $f(1), \ldots, f(n)$ approaches π.

Proof: Consider pairs of integers a, b as lattice points.

Marking the solutions to $a^2 + b^2 = 11$:



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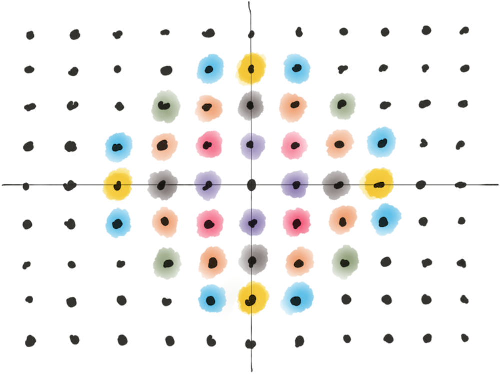
Marking the solutions to $a^2 + b^2 = 12$:



Theorem

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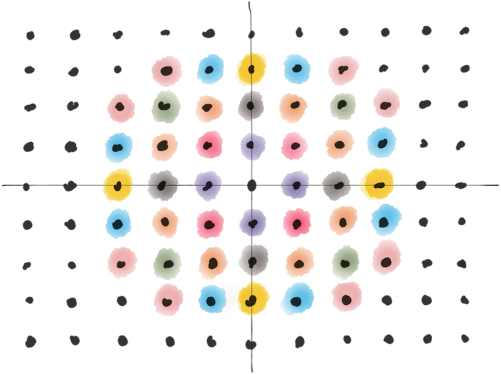
Marking the solutions to $a^2 + b^2 = 13$:



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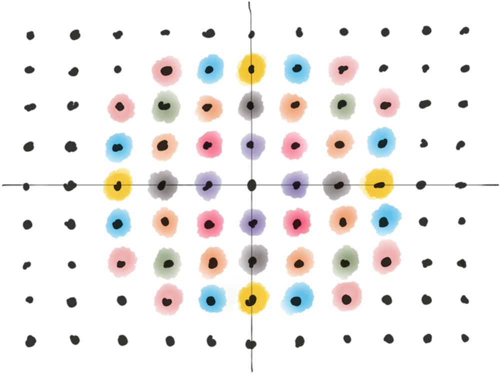
Marking the solutions to $a^2 + b^2 = 14$:



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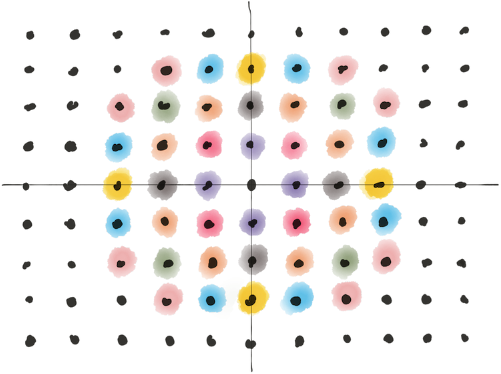
Marking the solutions to $a^2 + b^2 = 15$:



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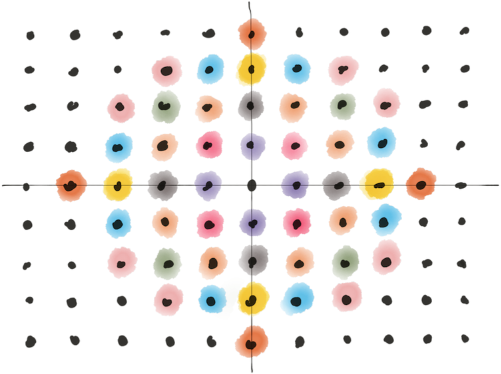

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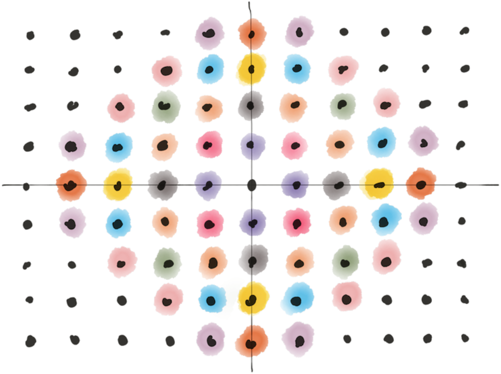
Marking the solutions to $a^2 + b^2 = 17$:



Theorem

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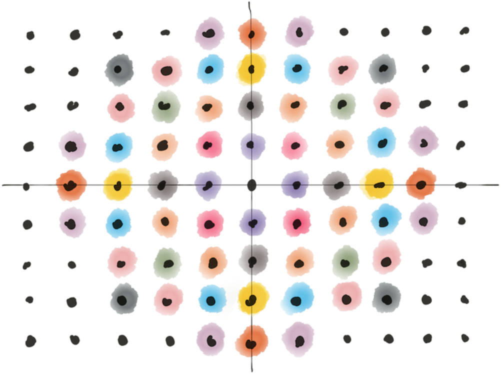
Proof: Consider pairs of integers a, b as lattice points.

Marking the solutions to $a^2 + b^2 = 18$:



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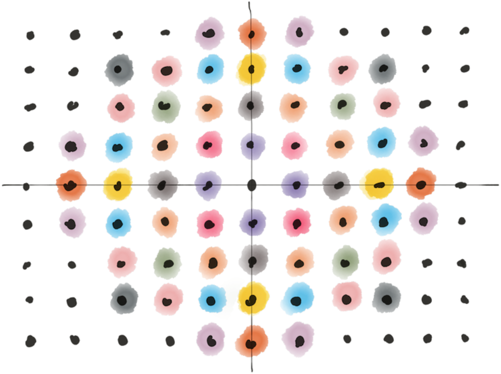
Proof: Consider pairs of integers a, b as lattice points.

Marking the solutions to $a^2 + b^2 = 19$:



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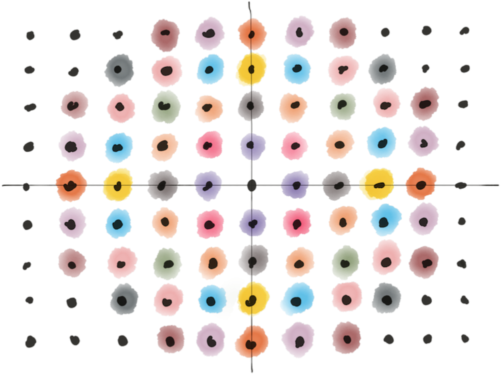
Marking the solutions to $a^2 + b^2 = 20$:

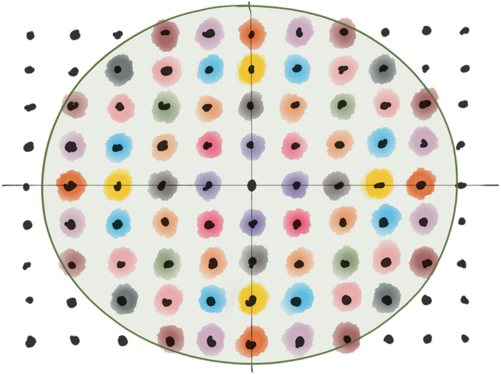


Theorem

As n grows, the average of $f(1), \ldots, f(n)$ approaches π.

Proof: Consider pairs of integers a, b as lattice points.

The solutions to $a^2 + b^2 \leq n$ fill a circle of radius $\sqrt{n}$.



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This is the number of lattice points inside the circle of radius $\sqrt{n}$ (excluding the origin),

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This is the number of lattice points inside the circle of radius $\sqrt{n}$ (excluding the origin), which is roughly the area of this circle.

The average of $f(1), \ldots, f(n)$ is

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This completes the proof. □