培尖教育 2018 年学科竞赛夏令营物理模拟卷（十三）

考试时间：150 分钟　　总分 320 分

（参考答案）

T1

可想而知，平衡位置只可能是如下两种：

(i).有一个板垂直于地面，整个系统镜像对称

(ii).两个板均与地面成45°角，整个系统旋转对称

而这两个位置之间没有平衡位置.故这二者必然有一个稳定，有一个不稳定.现计算这两种情况的势能以确定哪一种是稳定的.

对于(i)，势能为$2mgR$，对于(ii)，势能为$\frac{2mgR}{\sqrt{2}}$.因此(ii)是稳定的.(8')

考虑系统做的一个小转动.设接触点相对于两个圆板的圆心转过的角度分别为θ,φ.

在系统的参考系中建立直角坐标系$Oxyz$，以两个圆板圆心的连线为x轴，连线中点为原点.

初始时，地面内的两个圆板的切线所在的直线方程容易写出，分别为

$$y = R, z = 0 \qquad (1') \tag{1}$$

$$y = 0, z = R \qquad (1') \tag{2}$$

滚动后，两条切线的单位矢量容易写出，即

$$\vec{n_A} = (\cos\theta, -\sin\theta, 0) \qquad (1') \tag{3}$$

$$\vec{n_B} = (\cos\varphi, 0, -\sin\varphi) \qquad (1') \tag{4}$$

而它们通过的点分别为

$$\vec{r_A} = (\frac{R}{2} + R\sin\theta, R\cos\theta, 0) \qquad (1') \tag{5}$$

$$\vec{r_B} = (-\frac{R}{2} + R\sin\varphi, 0, R\cos\varphi) \qquad (1') \tag{6}$$

因此两条切线构成的平面（也就是地面）的单位法向量为

$$\begin{aligned}\vec{n_{AB}} &= \frac{\vec{n_A}\times\vec{n_B}}{\sqrt{(\vec{n_A}\times\vec{n_B})^2}} \\ &= \frac{(\sin\theta\sin\varphi, \cos\theta\sin\varphi, \cos\varphi\sin\theta)}{\sqrt{\sin^2\theta\sin^2\varphi + \cos^2\theta\sin^2\varphi + \cos^2\varphi\sin^2\theta}} \qquad (3')\end{aligned} \tag{7}$$

要求两条切线共面，即

$$\vec{n_{AB}}\cdot(\vec{r_A} - \vec{r_B}) = 0 \qquad (2') \tag{8}$$

可推得

$$\sin\theta\sin\varphi + \sin\varphi - \sin\theta = 0 \qquad (2') \tag{9}$$

因此记

$$\sin\theta = \delta + \frac{\delta^2}{2} \qquad (1') \tag{10}$$

$$\sin\varphi = \delta - \frac{\delta^2}{2} \qquad (1') \tag{11}$$

可以使前式近似成立 故质心C到地面的距离为

$$\begin{aligned}h &= |\vec{n_{AB}}\cdot\frac{\vec{r_B}+\vec{r_A}}{2}| \\ &= \frac{1}{2}\frac{(\sin\theta+\sin\varphi)\sqrt{\sin\theta\sin\varphi} + \cos^2\theta\sqrt{\frac{\sin\varphi}{\sin\theta}} + \cos^2\varphi\sqrt{\frac{\sin\theta}{\sin\varphi}}}{\sqrt{\sin\theta\sin\varphi + \cos^2\theta\frac{\sin\varphi}{\sin\theta} + \cos^2\varphi\frac{\sin\theta}{\sin\varphi}}}R \qquad (3')\end{aligned} \tag{12}$$

(13)

考虑到所期望的答案应为

$$h=\frac{1}{\sqrt{2}}(1+\alpha\delta^2)R \tag{14}$$

的形式，分母和分子均应保留至二阶项。故有

$$\begin{aligned}
h&=\frac{1}{2}\frac{2\delta^2+(1-\delta^2)\sqrt{\frac{1-\frac{\delta^2}{2}}{1+\frac{\delta^2}{2}}}+(1-\delta^2)\sqrt{\frac{1+\frac{\delta^2}{2}}{1-\frac{\delta^2}{2}}}}{\sqrt{\delta^2+(1-\delta^2)\frac{1-\frac{\delta^2}{2}}{1+\frac{\delta^2}{2}}+(1-\delta^2)\frac{1+\frac{\delta^2}{2}}{1-\frac{\delta^2}{2}}}}R\\
&=\frac{1}{2}\frac{2\delta^2+(1-\delta^2)\left[\left(1-\frac{\delta}{2}+\frac{\delta^2}{8}\right)+\left(1+\frac{\delta}{2}+\frac{\delta^2}{8}\right)\right]}{\sqrt{\delta^2+(1-\delta^2)\left[\left(1+\delta+2\frac{\delta^2}{4}\right)+\left(1-\delta+2\frac{\delta^2}{4}\right)\right]}}R\\
&=\frac{1}{\sqrt{2}}(1+\frac{\delta^2}{8})R \qquad (5^{'})
\end{aligned} \tag{15}$$

现计算δ与系统实际转过的角度ϕ的关系。n_{AB}与$(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$的夹角即为

$$\phi=\frac{\delta}{\sqrt{2}} \qquad (2^{'}) \tag{16}$$

故动能

$$T=\frac{1}{2}I\dot{\phi}^2 \tag{17}$$

势能

$$V=2mgh\sim 2mg\frac{\delta^2}{8\sqrt{2}}R \tag{18}$$

$$=\frac{\sqrt{2}}{4}\phi^2 mgR \tag{19}$$

故小振动周期

$$t=2\pi\sqrt{\frac{\sqrt{2}gR}{8I}} \qquad (3^{'}) \tag{20}$$

(2).角动量$L=I\omega$，ω的分量为

$$\omega_x=\frac{1}{\sqrt{3}}\omega \qquad (1^{'}) \tag{21}$$

$$\omega_y=\frac{1}{\sqrt{3}}\omega \qquad (1^{'}) \tag{22}$$

$$\omega_z=-\frac{1}{\sqrt{3}}\omega \qquad (1^{'}) \tag{23}$$

$$\tag{24}$$

一个圆盘绕垂直于盘面的对称轴的转动惯量为

$$I_0=\frac{1}{2}mR^2 \qquad (1^{'}) \tag{25}$$

由垂直轴定理，绕平行盘面的对称轴的转动惯量为

$$I_1 = \frac{1}{4}mR^2 \qquad (1^{'}) \tag{26}$$

于是绕三个坐标轴的转动惯量分别为

$$I_x = 2I_1 \qquad (1^{'}) \tag{27}$$

$$I_y = (I_0 + m(\frac{R}{2})^2) + (I_1 + m(\frac{R}{2})^2) \qquad (1^{'}) \tag{28}$$

$$I_z = (I_0 + m(\frac{R}{2})^2) + (I_1 + m(\frac{R}{2})^2) \qquad (1^{'}) \tag{29}$$

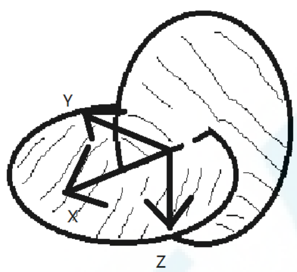
因此角动量沿实际转轴的分量为

$$L = 2m\frac{R}{\sqrt{2}}\omega R + I_x\omega_x\frac{1}{\sqrt{3}} + I_y\omega_y\frac{1}{\sqrt{3}} - I_z\omega_z\frac{1}{\sqrt{3}} \qquad (1^{'}) \tag{30}$$

故

$$I = \frac{1}{3}(I_x + I_y + I_z) + \sqrt{2}mR^2 \tag{31}$$

$$= (3+\sqrt{2})mR^2 \qquad (3^{'}) \tag{32}$$



T2

(1).洛伦兹变换:

$$y^{'}=\gamma(y-\beta ct) \tag{1}$$

$$t^{'}=\gamma(t-\frac{\beta y}{c}) \tag{2}$$

且有

$$y=a$$

得

$$y^{'}=\sqrt{1-\beta^2}a-\beta ct^{'} \tag{3}$$

在飞船系中，杆上z_0处t_0时刻发出的光在t_1时刻被收到，

$$c(t_1-t_0)=\sqrt{z^2+(\sqrt{1-\beta^2}a-\beta ct_0)^2} \tag{4}$$

那么人会认为这束光在

$$y^{'}=\sqrt{1-\beta^2}a-\beta ct_0 \tag{5}$$

$$z^{'}=z_0 \tag{6}$$

处发出.令$t_1=0$并消去t_0

$$(\sqrt{1-\beta^2}y-a)^2-\beta^2z^2=\beta^2a^2 \tag{7}$$

此即所求方程.

(2).对于地面系中一个静止于(x,y,z)处的点，如(1)可算得其在$S^{'}$系中运动方程为

$$x^{'}=x$$

$$y^{'}=\sqrt{1-\beta^2}y-\beta ct$$

$$z^{'}=z$$

而$t^{'}=0$时其视觉形象应位于

$$x^{'}=x \tag{8}$$

$$y^{'}=\gamma(y+\beta\sqrt{y^2+z^2+x^2}) \tag{9}$$

$$z^{'}=z \tag{10}$$

故$d\vec{A}$的变换结果为

$$dx_A^{'}=dx_A \tag{11}$$

$$dy_A^{'}=\gamma[dy_A+\frac{\beta}{x^2+y^2+z^2}(dx_A+dy_A+dz_A)] \tag{12}$$

$$dz_A^{'}=dz_A \tag{13}$$

$d\vec{B}$的变换结果为

$$d{x_B}'=dx_B \tag{14}$$

$$dy_B^{'}=\gamma[dy_B+\frac{\beta}{x^2+y^2+z^2}(dx_B+dy_B+dz_B)] \tag{15}$$

$$dz_B^{'}=dz_B \tag{16}$$

$\vec{dl} = d\vec{A} \times d\vec{B}$的变换结果$d\vec{l}'$为

$$dx_l^{'} = dx_l \tag{17}$$
$$dy_l^{'} = \gamma[dy_l + \frac{\beta}{x^2+y^2+z^2}(dx_l + dy_l + dz_l)] \tag{18}$$
$$dz_l^{'} = dz_l \tag{19}$$

其中，

$$dx_l = dy_A dz_B - dy_B dz_A \tag{20}$$
$$dy_l = dz_A dx_B - dz_B dx_A \tag{21}$$
$$dz_l = dx_A dy_B - dy_A dx_B \tag{22}$$

再计算$d\vec{l}'' = d\vec{A}' \times d\vec{B}'$

$$dx_l^{''} = dy_A^{'} dz_B^{'} - dy_B^{'} dz_A^{'} \tag{23}$$
$$dy_l^{''} = dz_A^{'} dx_B^{'} - dz_B^{'} dx_A^{'} \tag{24}$$
$$dz_l^{''} = dx_A^{'} dy_B^{'} - dy_A^{'} dx_B^{'} \tag{25}$$

因此一般情况下$d\vec{l}'' \times d\vec{l}' \neq 0$,即一般不平行
(3)(4)4分，其余每式2分.$11 \sim 13, 14 \sim 16, 17 \sim 19$,$20 \sim 22, 23 \sim 25$每组2分，不平行的判断占剩下的分数

T3

(1). 当给一个匀质理想介质球加上均匀电场E_0时，设其极化强度为p,有

$$p=\chi_e\epsilon_0(E_0-\frac{p}{3\epsilon_0}) \tag{1}$$

∴

$$p=\frac{3\chi_e\epsilon_0}{3+\chi_e}E \tag{2}$$

Q产生的电场为

$$E_Q=\frac{Q}{4\pi\epsilon_0 r^2} \tag{3}$$

因此B的极化强度和偶极矩分别为

$$\begin{aligned} p_B&=\frac{3\chi_e\epsilon_0}{3+\chi_e}\frac{Q}{4\pi\epsilon_0 r^2}\\ P_B&=\frac{4\pi R^3}{3}\frac{3\chi_e\epsilon_0}{3+\chi_e}\frac{Q}{4\pi\epsilon_0 r^2} \end{aligned} \tag{4}$$

q产生的电场为

$$E_q=\frac{q}{4\pi\epsilon_0 r^2} \tag{5}$$

因此A的极化强度和偶极矩分别为

$$\begin{aligned} p_A&=\frac{3\chi_e\epsilon_0}{3+\chi_e}\frac{q}{4\pi\epsilon_0 r^2}\\ P_A&=\frac{4\pi R^3}{3}\frac{3\chi_e\epsilon_0}{3+\chi_e}\frac{q}{4\pi\epsilon_0 r^2} \end{aligned} \tag{6}$$

故B受力

$$\begin{aligned} F_B&=\frac{qQ}{4\pi\epsilon_0 r^2}-P_B\cdot\frac{dE_Q}{dr}-P_A\cdot\frac{dE_q}{dr}\\ &=\frac{qQ}{4\pi\epsilon_0 r^2}+\frac{4\pi R^3}{3}\frac{3\chi_e\epsilon_0}{3+\chi_e}\frac{2(q^2+Q^2)}{(4\pi\epsilon_0)^2 r^5} \end{aligned} \tag{7}$$

于是动力学方程和角动量守恒为

$$m(\ddot{r}-r\dot{\theta}^2)=-F_B \tag{8}$$

$$mr^2\dot{\theta}=L \tag{9}$$

将动力学方程写为

$$m\ddot{r}-\frac{L^2}{mr^3}=-\left(\frac{\alpha}{r^2}+\frac{\beta}{r^5}\right) \tag{10}$$

当不振动时，

$$\frac{L^2}{mr^3}=\frac{\alpha}{r^2}+\frac{\beta}{r^5} \tag{11}$$

小量展开，得

$$m\ddot{\delta}+\left[\frac{3L^2}{mr^4}-\frac{2\alpha}{r^3}-\frac{5\beta}{r^6}\right]\delta=0 \tag{12}$$

因此径向小振动的角频率

$$\begin{aligned}\Lambda&=\sqrt{\frac{1}{m}\left(\frac{3L^2}{mr^4}-\frac{2\alpha}{r^3}-\frac{5\beta}{r^6}\right)}\\&=\sqrt{\frac{1}{m}\left(\frac{\alpha}{r^3}-\frac{2\beta}{r^6}\right)}\end{aligned} \tag{13}$$

现计算对应的圆轨道半径以及角速度. 记没有极化时的原轨道的半径为$\frac{1}{a_0}$,而真实的圆轨道半径应为$\frac{1}{a_0+a_1}$, $a_1\ll a_0$

$$\frac{L^2}{m}a_0^3=\alpha a_0^2 \tag{14}$$

$$\frac{L^2}{m}(a_0+a_1)^3=\alpha(a_0+a_1)^2+\beta(a_0+a_1)^5$$

$$\left(\frac{L^2}{m}\cdot 3a_0^2-2\alpha a_0+5\beta a_0^4\right)a_1=\beta a_0^5 \tag{15}$$

$$a_1\approx\frac{a_0^4\beta}{\alpha} \tag{16}$$

$$\tag{17}$$

计算角速度ω:

$$\begin{aligned}m\omega^2&=\frac{\alpha}{r^3}+\frac{\beta}{r^6}\\&\approx(a_0+a_1)^3(\alpha+a_0^3\beta)\\&\approx a_0^3(\alpha+4a_0^3\beta)\end{aligned} \tag{18}$$

$\therefore$

$$\omega=\sqrt{\frac{a_0^3}{m}(\alpha+4\beta a_0^3)} \tag{19}$$

$\therefore$

$$\begin{aligned}\Omega&=\omega-\Lambda\\&\approx\sqrt{\frac{a_0^3}{m}(\alpha+4\beta a_0^3)}-\sqrt{\frac{a_0^3}{m}(\alpha+\beta a_0^3)}\\&\approx\sqrt{\frac{a_0^3\alpha}{m}}\frac{3}{2}\frac{\beta a_0^3}{\alpha}\\&=\frac{3\beta(\alpha m)^4}{2L^9}\end{aligned} \tag{20}$$

其中，

$$\begin{aligned}\alpha&=\frac{Qq}{4\pi\epsilon_0}\\\beta&=\frac{2R^3\chi_e(q^2+Q^2)}{4\pi\epsilon_0(3+\chi_e)}\end{aligned}$$

每式2分

T4
由高斯公式，（每式$3'$）

$$\frac{1}{u}+\frac{1}{v_1}=\frac{1}{f} \tag{1}$$
$$\frac{1}{d-v_1}+\frac{n}{v_2}=\frac{n-1}{R} \tag{2}$$
$$\frac{1}{2R-v_2}+\frac{1}{v_3}=\frac{2}{R} \tag{3}$$
$$\frac{n}{2R-v_3}+\frac{1}{v_4}=\frac{n-1}{R} \tag{4}$$
$$\frac{1}{d-v_4}+\frac{1}{v}=\frac{1}{f} \tag{5}$$

得

$$v=\frac{(\frac{n}{2}+3)-30(6-n)}{\frac{n+2}{40}u-(\frac{n}{2}+3)} \qquad (10') \tag{6}$$

(i).$1<n<6$
$\therefore(2')$

$$(\frac{n}{2}+3)u>30(6-n) \tag{7}$$
$$\frac{n+2}{40}u>(\frac{n}{2}+3) \tag{8}$$

或者$(2')$

$$(\frac{n}{2}+3)u<30(6-n) \tag{9}$$
$$\frac{n+2}{40}u<(\frac{n}{2}+3) \tag{10}$$

打表可得，在$n>1$时，有

$$\frac{20(n+6)}{n+2}>\frac{60(6-n)}{n+6} \qquad (3') \tag{11}$$

故u的取值范围满足下面两式之一即可:

$$u>\frac{20(n+6)}{n+2} \qquad (2') \tag{12}$$
$$0<u<\frac{60(6-n)}{n+6} \qquad (2') \tag{13}$$

(ii).$n>6$
$\therefore$

$$\frac{n+2}{40}u>(\frac{n}{2}+3) \qquad (2') \tag{14}$$

$\therefore$

$$u>\frac{20(n+6)}{n+2} \qquad (2') \tag{15}$$

T5

(1).引入ε表示小虫完成爬行的比例$(0 \le \varepsilon \le 1)$，则

$$\varepsilon v + u = \frac{dx}{dt} \tag{1}$$

$$x = \varepsilon(\sqrt{2}a + vt) \tag{2}$$

将(2)微分然后整理得

$$\frac{dt}{\sqrt{2} + vt} = \frac{d\varepsilon}{u} \quad \Rightarrow \quad t = \frac{\sqrt{2}a}{v}(e^{\frac{v}{u}} - 1) \tag{3}$$

$$f = \mu mg$$

$$s = \sqrt{2}a + vt = \sqrt{2}ae^{\frac{v}{u}} \tag{4}$$

$$\therefore \quad W_f = f \cdot s = \mu mg\sqrt{2}ae^{\frac{v}{u}} \tag{5}$$

而末速$v_f = v + u \quad \Rightarrow$

$$\begin{aligned} \Delta E_k &= \frac{1}{2}v_f^2 - \frac{1}{2}mu^2 \\ &= \frac{1}{2}m(v^2 + 2vu) \end{aligned} \tag{6}$$

$\therefore$

$$A - Wf = \Delta E_k \tag{7}$$

小虫做功

$$A + \frac{1}{2}m(v^2 + 2uv) + \sqrt{2}\mu mgae^{\frac{v}{u}} \tag{8}$$

(2.1)物块受力

$$T = -k_x\vec{e_x} - k_y\vec{e_y} \tag{9}$$

摩擦力大小

$$f = \mu mg \tag{10}$$

记

$$\tan\varphi = \frac{y}{x}$$

$$\therefore f_x = -\mu mg\cos\varphi, \qquad f_y = -\mu mg\sin\varphi \tag{11}$$

$$ma_x = F_x = f_x + T_x, \qquad ma_y = F_y = f_y + T_y \tag{12}$$

由上，物块的加速度为

$$a_x = -\frac{kx}{m} - \mu g\frac{u_x - \frac{xv}{\sqrt{2}a+vt}}{\sqrt{(u_x - \frac{xv}{\sqrt{2}a+vt})^2 + (u_y - \frac{yv}{\sqrt{2}a+vt})^2}} \tag{13}$$

$$a_y = -\frac{ky}{m} - \mu g\frac{u_y - \frac{yv}{\sqrt{2}a+vt}}{\sqrt{(u_x - \frac{xv}{\sqrt{2}a+vt})^2 + (u_y - \frac{yv}{\sqrt{2}a+vt})^2}} \tag{14}$$

(2.2)最终物块要么与场源保持距离$\frac{\mu mg}{k}$(5')，要么跟场源处于同一点(3')， 受微扰后保持距离$\frac{\mu mg}{k}$
(13)(14)每个4分，其余每式2分

T6

(1).计算系统动能:

$$v_{xi} = \frac{\dot{x_E}}{2} \tag{1}$$

$$v_{y_i} = \frac{\dot{x_E}}{2}\cot\theta(2i+1)^2 \tag{2}$$

故动能

$$\begin{aligned} T &= \sum_0^{\infty}\frac{1}{2}\cdot m_i\cdot {v_x}_i^2 + \frac{1}{2}\cdot m_i\cdot {v_y}_i^2 \\ &= \frac{1}{2}2m\frac{\dot{x_E}^2}{4} + \frac{1}{2}34m\frac{\dot{x_E}^2}{4}\cot^2\theta \\ &= \frac{1}{2}(2m + 34m\cot^2\theta)\frac{\dot{x_E}^2}{4} \end{aligned} \tag{3}$$

(4)

势能

$$\begin{aligned} V &= \sum_0^{\infty} m_i g(2i+1)\sqrt{l^2 - (\frac{x}{2})^2} \\ &= 6mg\sqrt{l^2 - (\frac{x}{2})^2} \end{aligned} \tag{5}$$

故

$$\ddot{x_E} = \frac{3\sqrt{3}g}{26} \tag{6}$$

T7

(1).对于初态，令$\alpha=e$

$$
\begin{aligned}
C\frac{dT_{a_1}}{dt} &= -(\alpha-1)kT_0 && (1)\\
C\frac{dT_{a_2}}{dt} &= (\alpha-1)kT_0\cdot\frac{1}{\alpha}-(1-\frac{1}{\alpha})kT_0 && (2)\\
C\frac{dT_{a_3}}{dt} &= (1-\frac{1}{\alpha})kT_0\cdot\frac{1}{\alpha} && (3)\\
\frac{dW_1}{dt} &= (\alpha-1)kT_0(1-\frac{1}{\alpha}) && (4)\\
\frac{dW_2}{dt} &= (1-\frac{1}{\alpha})kT_0(1-\frac{1}{\alpha}) && (5)\\
C\frac{dT_{b_1}}{dt} &= -\frac{T_0}{2T_0-T_0}\frac{dW_1}{dt}\\
&= -(\alpha-1)kT_0(1-\frac{1}{\alpha})\frac{T_0}{2T_0-T_0} && (6)\\
C\frac{dT_{b_2}}{dt} &= \frac{2T_0}{2T_0-T_0}\frac{dW_1}{dt}-\frac{2T_0}{3T_0-2T_0}\frac{dW_2}{dt}\\
&= (\alpha-1)kT_0(1-\frac{1}{\alpha})\frac{2T_0}{2T_0-T_0}-(1-\frac{1}{\alpha})kT_0(1-\frac{1}{\alpha})\frac{2T_0}{3T_0-2T_0} && (7)\\
C\frac{dT_{b_3}}{dt} &= \frac{3T_0}{3T_0-2T_0}\frac{dW_2}{dt}\\
&= (1-\frac{1}{\alpha})kT_0(1-\frac{1}{\alpha})\frac{3T_0}{3T_0-2T_0} && (8)\\
& && (9)
\end{aligned}
$$

化简上述表达式并将$\alpha=e$代入，得

$$
\begin{aligned}
\dot{T_{a_1}} &= -(e-1)\frac{kT_0}{C} && (10)\\
\dot{T_{a_2}} &= 0 && (11)\\
\dot{T_{a_3}} &= \frac{(e-1)}{e^2}\frac{kT_0}{C} && (12)\\
\dot{T_{b_1}} &= -\frac{(e-1)^2}{e}\frac{kT_0}{C} && (13)\\
\dot{T_{b_2}} &= 2\frac{(e-1)^3}{e^2}\frac{kT_0}{C} && (14)\\
\dot{T_{b_3}} &= 3\frac{(e-1)^2}{e^2}\frac{kT_0}{C} && (15)\\
& && (16)
\end{aligned}
$$

(2).显然$\Delta S=0(1^{'})$

(3).观察到$\dot{T_{a_2}}=0$，因此猜测

$$T_{a_1}T_{a_3}=T_{a_2}^2\qquad(1^{'})\tag{17}$$

发现三个方程相容.因此有

$$\dot{\alpha}=-(\alpha-1)\frac{k}{C}\qquad(1^{'})\tag{18}$$

得

$$\alpha(t) = 1 + (e-1)e^{-\frac{k}{C}t} \qquad (1^{'}) \tag{19}$$

$$\begin{aligned} T_{a_1} &= \alpha T_0 \\ &= \left[1 + (e-1)e^{-\frac{k}{C}t}\right] T_0 \end{aligned} \tag{20}$$

$$T_{a_2} = T_0 \tag{21}$$

$$\begin{aligned} T_{a_3} &= \alpha^{-1} T_0 \\ &= \left[1 + (e-1)e^{-\frac{k}{C}t}\right]^{-1} T_0 \end{aligned} \tag{22}$$

除单独标注分值的式子外，所有式子均为$2^{'}$

T8
(1).

$$p_i = n_0RT = \rho g\Delta h_0 \tag{1}$$
$$\therefore \qquad \Delta h_0 = \frac{n_0RT}{\rho g} \tag{2}$$

(2).首先计算溶液刚开始沉淀时活塞高度，

$$n_0SH = cSh_0 \quad \Rightarrow \quad h_0 = \frac{n_0}{c}H = aH \tag{3}$$

(i).开始沉淀前，设活塞高度x,则

$$n_0SH = nSx \quad \Rightarrow \quad n = \frac{n_0H}{x} \tag{4}$$
$$p_0 + \frac{F}{S} + \rho gx = nRT + \rho g(H - \Delta h_0) + p_0 \tag{5}$$
$$\Rightarrow \quad F = S[n_0RT(\frac{H}{x} - 1) + \rho g(H - x)] \tag{6}$$

故做功

$$\begin{aligned} W_1 &= \int_{h_0}^{H} F dx \\ &= \int_{h_0}^{H} S[n_0RT(\frac{H}{x} - 1) + \rho g(H - x)]dx \\ &= n_0SRTH\ln\frac{H}{h_0} + S(\rho gH - n_0RT)(H - h_0) - \rho gS\frac{H^2 - h_0^2}{2} \\ &= -n_0SRTH\ln a + SH(\rho gH - n_0RT)(1 - a) - \rho gSH^2\frac{1 - a^2}{2} \\ &= \rho gSH^2a(\frac{a}{2} - 1) - n_0RTSH(\ln a + 1 - a) \end{aligned} \tag{7}$$

(ii).开始沉淀后，当完成沉淀时，

$$cSh_0 - cSh_f = N \quad \Rightarrow \quad h_f = a(1 - b)H \tag{8}$$
$$\therefore p_0 + \frac{F}{S} + \rho gx = cRT + \rho g(H - \Delta h_0) + p_0 \tag{9}$$
$$\Rightarrow \quad F = S[(c - n_0)RT + \rho g(H - x)] \tag{10}$$

做功

$$\begin{aligned} W_2 &= \int_{h_f}^{h_0} S[(c - n_0)RT + \rho g(H - x)]dx \\ &= SRTHc(1 - a)ab + \rho gSH^2\frac{ab^2}{2} \end{aligned} \tag{11}$$

所作总功为

$$W = W_1 + W_2 = \frac{a\rho gSH^2}{2}(b^2 + a - 2) - acSHRT[(1 - a)(1 - b) - \ln a] \tag{12}$$

(2)(7)(11)(12)每个4分，剩下式子每个3分