Content of this lecture

1. Electromagnetic wave
   - Maxwell's equations
   - Boundary conditions
   - Constitutive relations
   - Wave equation
   - Time- and spatial-harmonic field
2. Dispersion of materials
   - What is dispersion?
   - k-ω dispersion relation
   - Phase and group velocities (in k-ω plot)
3. Microscopic and macroscopic theories of materials
   - Free and bound electrons
   - Band structures of materials
   - EM response of insulator/dielectric: **Lorentz model**
   - EM response of metal: **Drude model** (Lecture 5)
4. Example of engineering light-matter interaction with nanostructures – **form birefringence**

1. Electromagnetic wave

How to describe the wave property of light? – Maxwell's equations

$$\nabla \cdot \mathbf{D} = \rho_{\text{ext}}$$

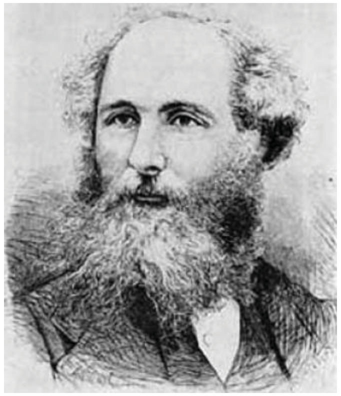
$$\nabla \cdot \mathbf{B} = 0$$

Divergence Eqs.

$$\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$$

$$\nabla \times \mathbf{H} = \partial \mathbf{D} / \partial t + \mathbf{J}_{\text{ext}}$$

Curl Eqs.



(1831 – 1879)

E – electric field vector

H – magnetic field vector

D – electric flux density

B – magnetic flux density

ρ_{ext} – external charge density

$\mathbf{J}_{\text{ext}}$ – external current density

If no external charges and currents:

$$\nabla \cdot \mathbf{D} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

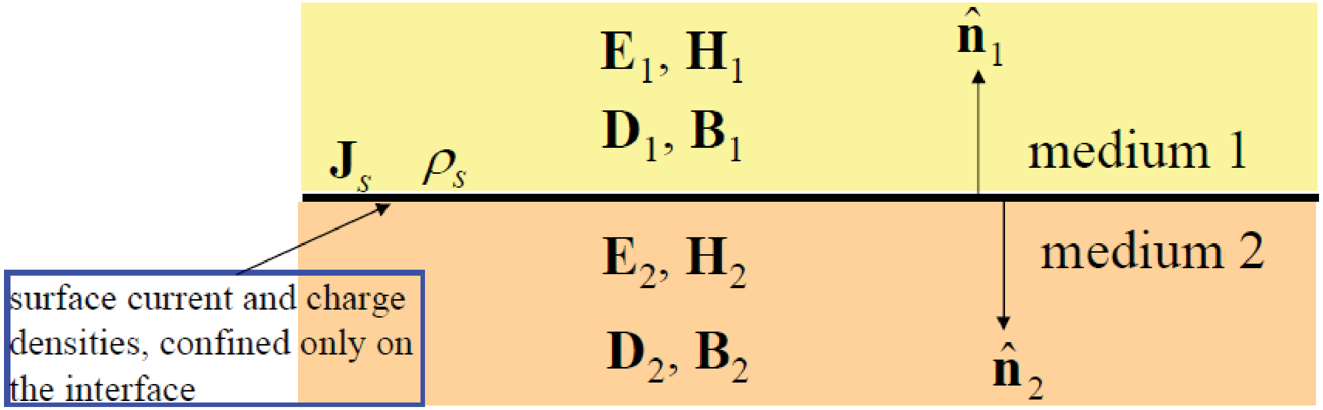
$$\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$$

$$\nabla \times \mathbf{H} = \partial \mathbf{D} / \partial t$$

linking four macroscopic fields **E, H, D, B**

divergence: 散度， curl: 旋度， macroscopic: 宏观的

Boundary conditions



For tangential components:

$$\boxed{\begin{aligned}\hat{\mathbf{n}}_1 \times (\mathbf{E}_1 - \mathbf{E}_2) &= 0 \\ \hat{\mathbf{n}}_1 \times (\mathbf{H}_1 - \mathbf{H}_2) &= \mathbf{J}_s\end{aligned}}$$

For normal components:

$$\boxed{\begin{aligned}\hat{\mathbf{n}}_1 \cdot (\mathbf{B}_1 - \mathbf{B}_2) &= 0 \\ \hat{\mathbf{n}}_1 \cdot (\mathbf{D}_1 - \mathbf{D}_2) &= \rho_s\end{aligned}}$$

If no external surface charges and currents:

$$E_{1t} = E_{2t},\ H_{1t} = H_{2t} \qquad B_{1n} = B_{2n},\ D_{1n} = D_{2n}$$

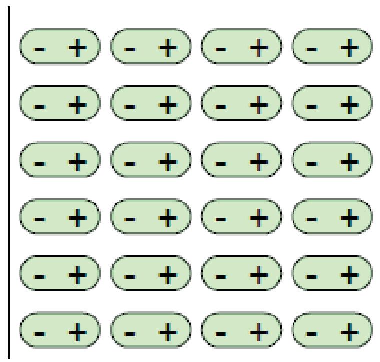
boundary condition: 边界条件，tangential: 切向，normal: 法向

Constitutive relations

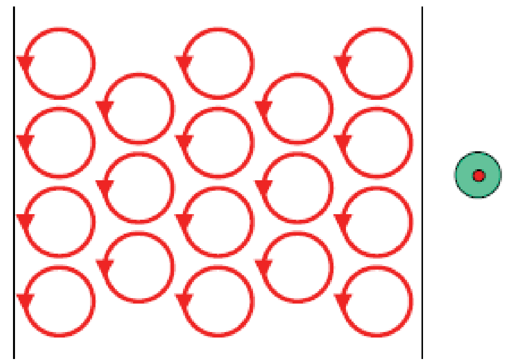
What are the relations between **E** and **D**, and **H** and **B**?

– determined by the EM response of materials

Polarization P



Magnetization M



relative permittivity

$$\mathbf{D} = \underline{\varepsilon_0}\mathbf{E} + \underline{\mathbf{P}} = \varepsilon_0 \underline{\varepsilon} \mathbf{E}$$

permittivity in vacuum

polarization

relative permeability

$$\mathbf{B} = \underline{\mu_0}\mathbf{H} + \mu_0 \underline{\mathbf{M}} = \mu_0 \underline{\mu} \mathbf{H}$$

permeability in vacuum

magnetization

Meaning: total electric/magnetic flux density = flux from external field + flux due to material polarization/magnetization

constitutive relation: 本构关系(物质方程)，polarization: 极化(矢量)，magnetization: 磁化(矢量)
permittivity: 介电常数，permeability: 磁导率， flux: 电/磁通量

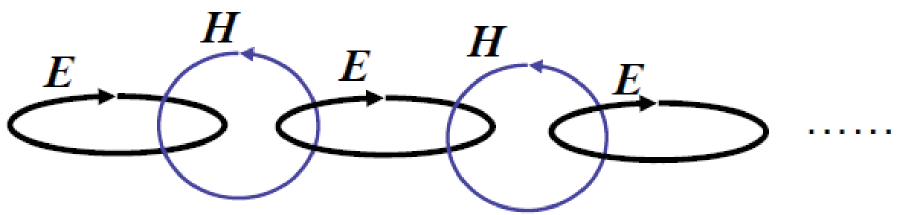
Wave equation

According to the curl equations:

$$\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$$

$$\nabla \times \mathbf{H} = \partial \mathbf{D} / \partial t$$

changing electric field results in changing magnetic field and vice versa → **electromagnetic wave**



In homogeneous (ε and μ are spatially independent) media, we have:

$$\nabla \times \nabla \times \mathbf{E} = -\mu_0 \mu \frac{\partial}{\partial t} \nabla \times \mathbf{H} = -\varepsilon_0 \mu_0 \varepsilon \mu \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\Downarrow$$

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{n^2}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Wave Equation

Light speed in vacuum:

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

Refractive index:

$$n = \sqrt{\varepsilon \mu}$$

wave equation: 波动方程， refractive index: 折射率

Harmonic field

Solution to wave equation:

$$\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}=-\frac{\varepsilon\mu}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}$$

Henceforth, we consider only nonmagnetic media ($\mathbf{M}$=0, μ=1)

→ time- and spatial-harmonic field:

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}_0\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)$$

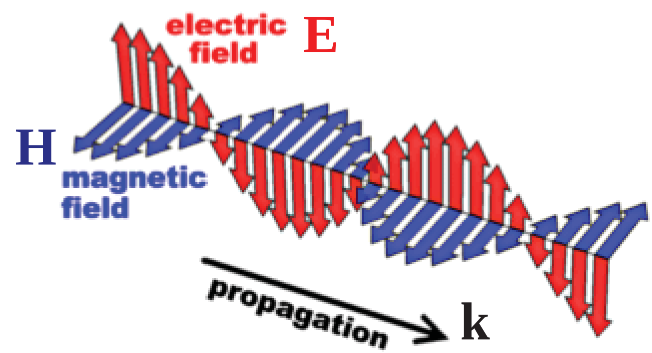
$\mathbf{k}$ – wave vector

(check this is a solution)

In this case, $\nabla \to i\mathbf{k},\ \ \partial/\partial t \to -i\omega$ (derive by yourself)

Therefore, the wave equation turns to

$$\mathbf{k}(\mathbf{k}\cdot\mathbf{E})-k^2\mathbf{E}=-\varepsilon\frac{\omega^2}{c^2}\mathbf{E}$$



- If transverse wave → $\mathbf{k}\cdot\mathbf{E}$ = 0 →

$$k=\sqrt{\varepsilon}\frac{\omega}{c}\equiv nk_0$$

- If longitudinal wave → $\mathbf{k}(\mathbf{k}\cdot\mathbf{E})$ = $k^2\mathbf{E}$ → ε = 0 ← more discussion in Lecture 5

harmonic field: 简谐场，transverse wave: 横波，longitudinal wave: 纵波

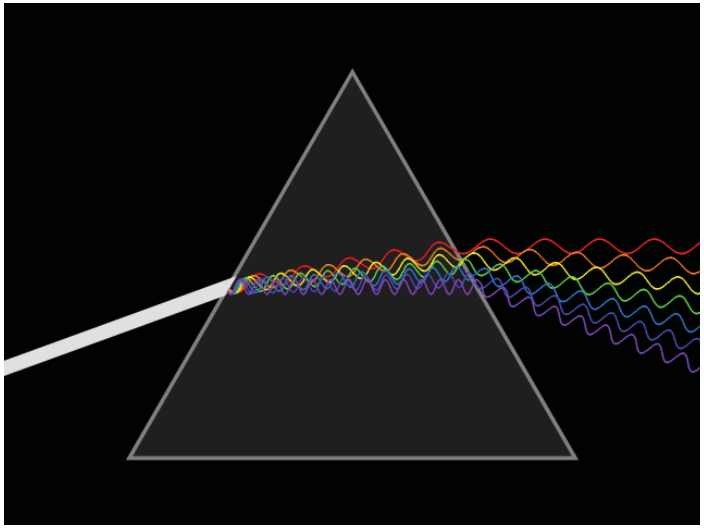
2. Dispersion of materials

- In most media, the permittivity ε is not a constant, which varies with respect to light frequency ω, i.e., $\varepsilon = \varepsilon(\omega)$.
- Therefore, $\mathbf{D} = \varepsilon(\omega)\mathbf{E}$ does not hold a linear relation; this frequency-dependent property is called ***dispersion***.
- Any real material has more or less dispersion.

Example:
Observation of dispersion in prism

$$n(\omega) = \sqrt{\varepsilon(\omega)}$$

refractive index is ω-dependent



dispersion: 色散， prism: 棱镜

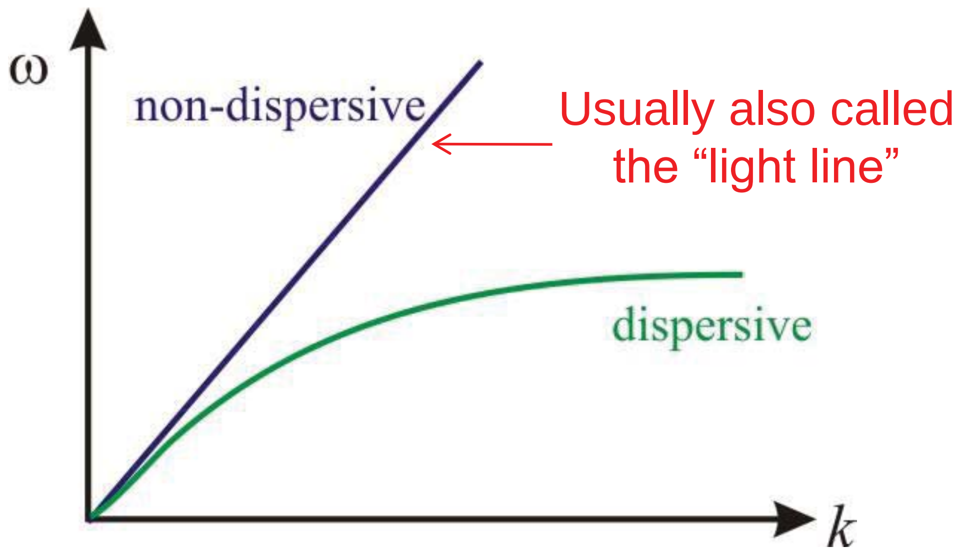
k-*ω* dispersion relation

For transverse EM wave: $k = \sqrt{\varepsilon}\,\dfrac{\omega}{c} \Rightarrow \omega = \dfrac{kc}{\sqrt{\varepsilon}}$

- For non-dispersive media (for example, in vacuum), ε is constant, k-ω relation is linear;
- Otherwise, for dispersive media, ε is ω-dependent so that the k-ω relation is nonlinear.

We can plot the k-ω dispersion relation:

Important!!!

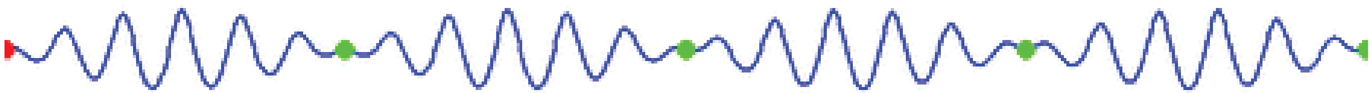


linear: 线性，nonlinear: 非线性

Phase and group velocities

Practical light cannot be ideally monochromatic (single ω).

Multi-frequency components mixture → **wave package**



– **Phase velocity:** velocity of the oscillation or velocity of the equi-phase plane (red point)

$$v_p = \frac{1}{\sqrt{\varepsilon_0 \mu_0 \varepsilon \mu}} = \frac{c}{n} = \frac{\omega}{k}$$

$$\because \mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \exp(\underline{i\mathbf{k}\cdot\mathbf{r} - i\omega t})$$

Let it be constant

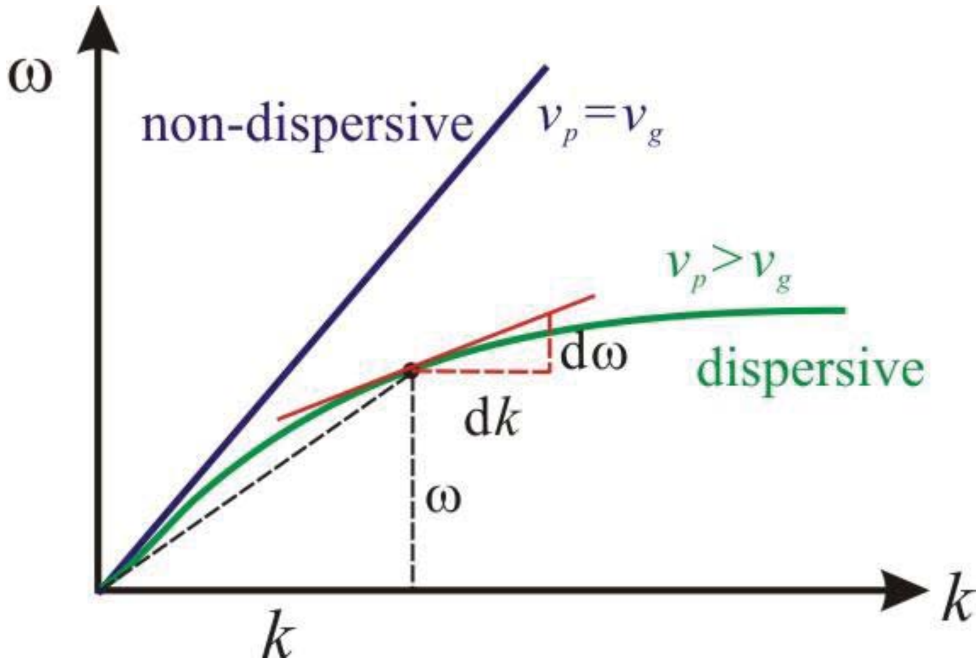
– **Group velocity:** velocity of the waveform envelope or velocity of the equi-amplitude plane (green points)

$$v_g = \frac{d\omega}{dk}$$

(derive by yourself. Hint: you may consider the simplest case of two-wave mixture.)

phase velocity: 相速度， group velocity: 群速度， monochromatic: 单色的， wave package: 波包
equi-phase plane: 等相面， equi-amplitude plane: 等幅面， waveform: 波形， envelope: 包络

Phase and group velocities in k-ω plot



$$v_p = \frac{\omega}{k}, \quad v_g = \frac{d\omega}{dk}$$

– In non-dispersive media:

$$v_p = v_g$$

– In dispersive media:

$$v_p \neq v_g$$

- v_g is the velocity of the transported signal or carried energy, therefore it must be $v_g < c$ according to Relativity!
- There is no such restriction on v_p, therefore it is possible that $v_p > c$

3. Micro- and macroscopic theories of materials

- So far, we have known that the EM properties of materials are characterized by several ***macroscopic*** parameters:
  - permittivity ε
  - permeability μ
  - conductivity σ ($\mathbf{J} = \sigma\mathbf{E}$, characterizes how easy charges can move; $\sigma = 0 \rightarrow$ insulator, $\sigma = \infty \rightarrow$ perfect conductor)

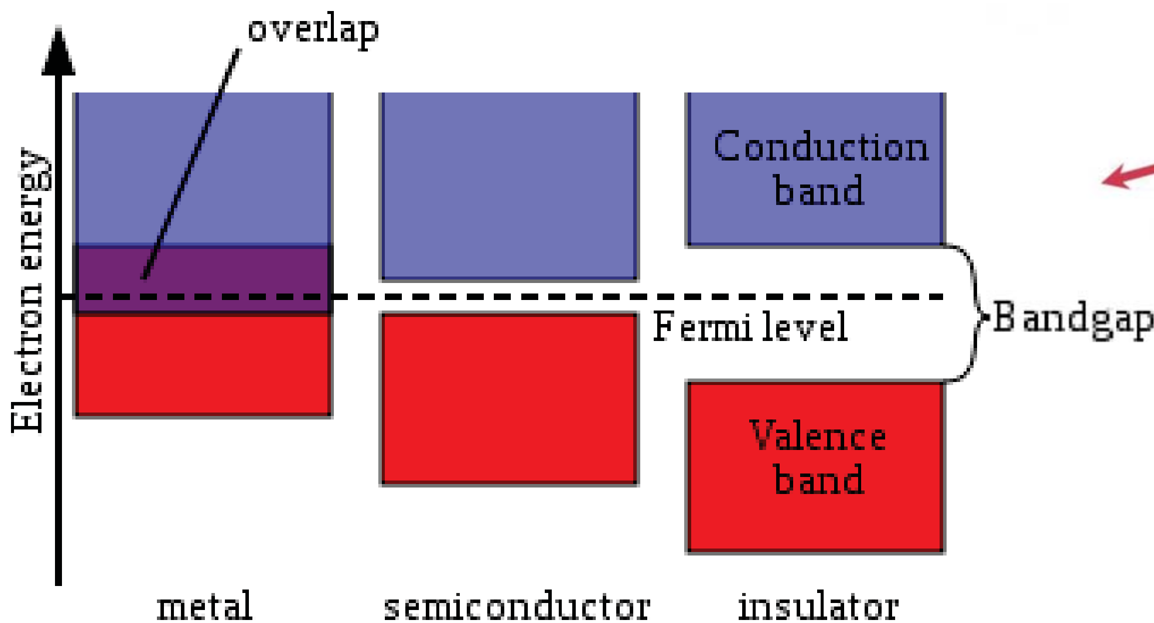
- In nanophotonics, we aim to ***engineer*** these ***macroscopic*** parameters with artificial nanostructure.

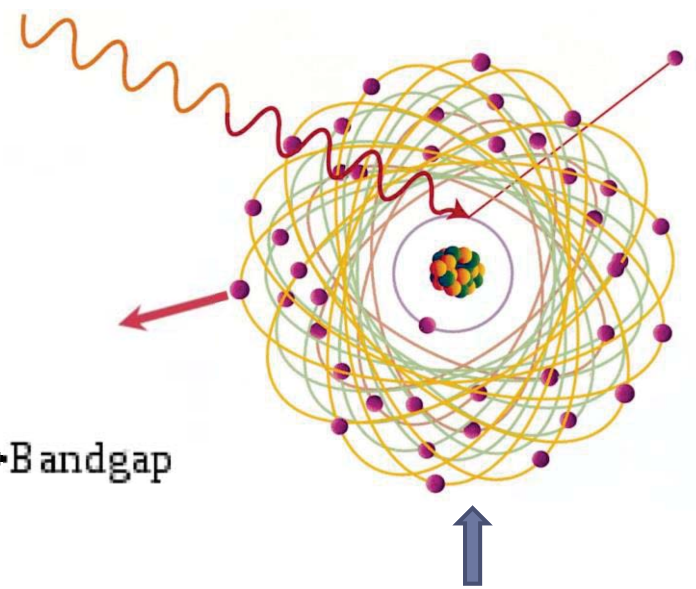
- For this, we should know the ***microscopic*** origin of these macroscopic parameters for different materials:
  - Insulator (dielectric)
  - metal

macroscopic: 宏观的，microscopic: 微观的，parameter: 参数，insulator: 绝缘体，dielectric: 电介质

Free and bound electrons in materials

- In conduction band – **free electrons**
- In valence band – **bound electrons**



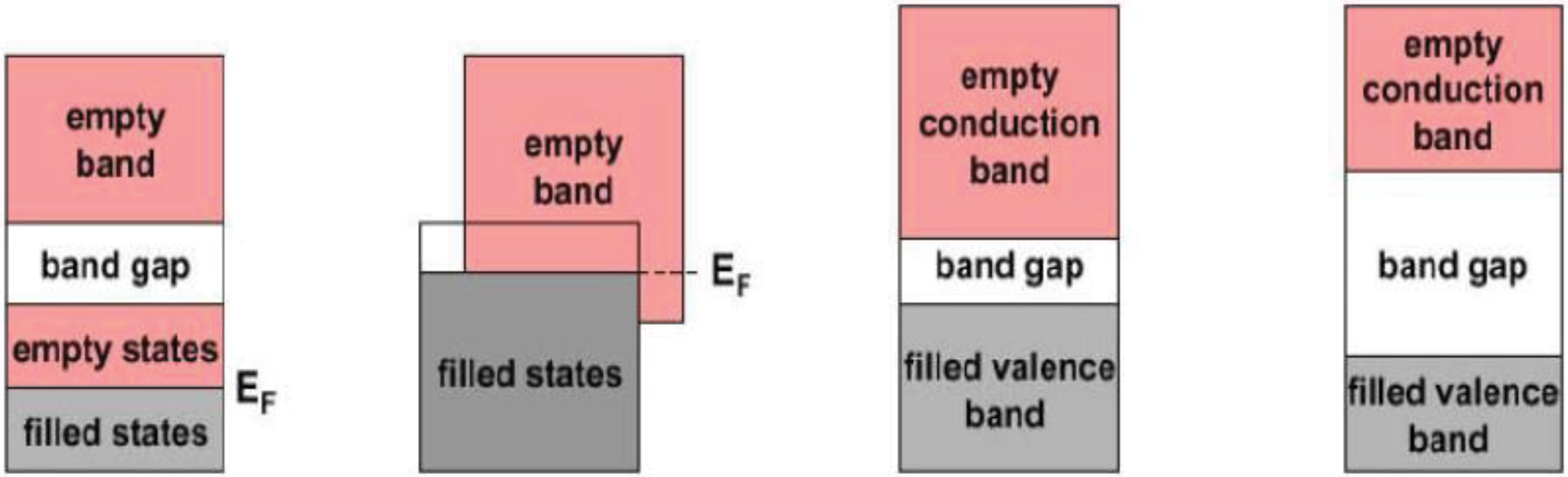


Electron cloud surrounding the atom cores

Energy needed for an electron to leap to higher energy level or the conduction band

free electron: 自由电子，bound electron: 束缚电子，conduction band: 导带，valence band: 价带，bandgap: 带隙

Typical band structures of materials



metals: available and filled states in the same band (Cu, Au, Ag)

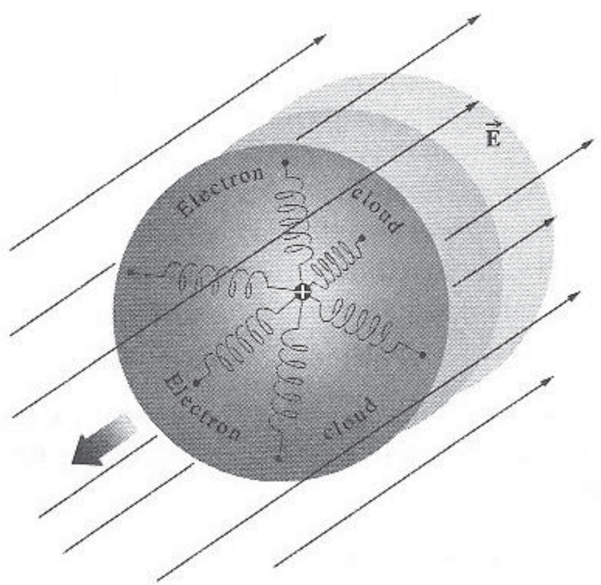
metals: overlap between filled valence band and empty conduction band (Al, Mg)

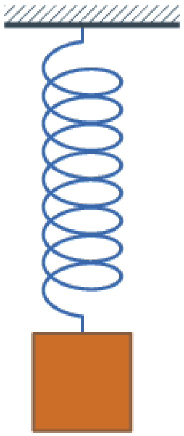
semiconductors: filled valence band separated from empty conduction band by a narrow band gap (< 2 eV)

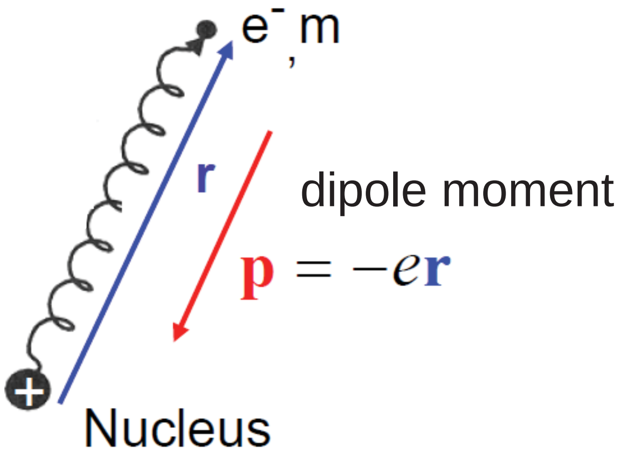
insulators: filled valence band separated from empty conduction band by a large band gap (> 2 eV)

Insulator – Lorentz model

- Insulator response is determined by the behavior of bound electrons.
- Under an external driving field $\mathbf{E}$, bound electrons can be treated as harmonic oscillators → Lorentz model





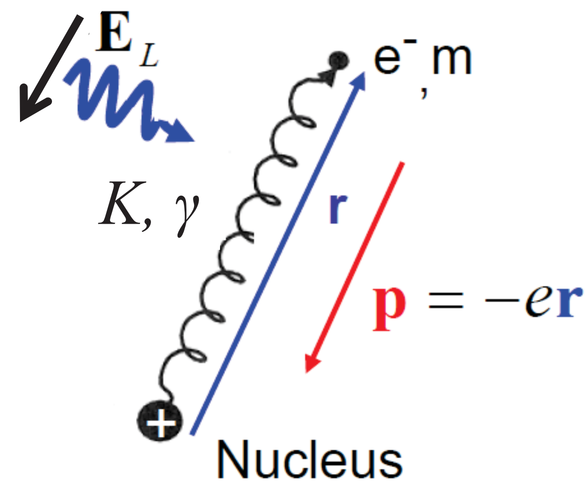


Lorentz model

harmonic oscillator: 简谐振子，bound electron: 束缚电子，Lorentz model: 洛仑兹模型

How many forces on the bound electron?

- electric field force $-e\mathbf{E}$
- damping force $m\gamma\mathbf{v}$ (γ: damping frequency)
- restoring force $K\mathbf{r}$ (K: restring-force constant)



Equation of motion:

$$m\ddot{\mathbf{r}} + m\gamma\dot{\mathbf{r}} + K\mathbf{r} = -e\mathbf{E}$$

For a time-harmonic stimulus $\mathbf{E}(t) = \mathbf{E}_0 \exp(-i\omega t)$, there is a time-harmonic solution $\mathbf{r}(t) = \mathbf{r}_0 \exp(-i\omega t)$, which is solved as:

$$\mathbf{r} = \frac{e/m}{\omega^2 + i\omega\gamma - \omega_0^2}\mathbf{E}$$

$$\omega_0 = \sqrt{K/m}$$

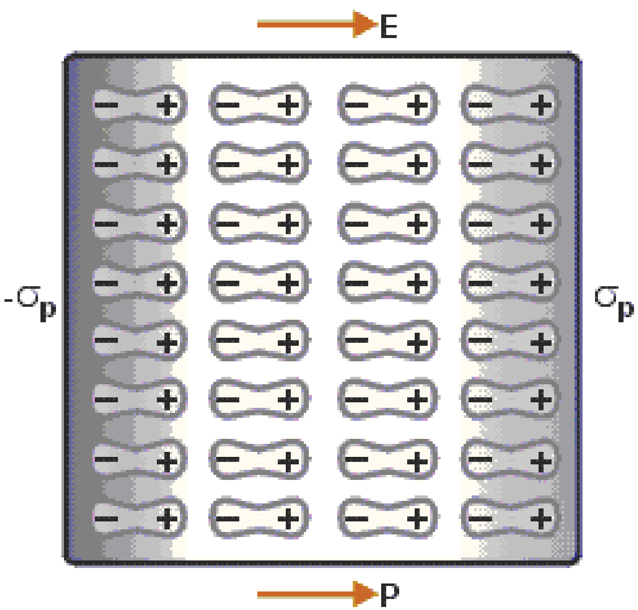
(natural frequency of bound electron)

Then the electric dipole moment is:

$$\mathbf{p} = -e\mathbf{r} = \frac{e^2/m}{\omega_0^2 - \omega^2 - i\omega\gamma}\mathbf{E}$$

damping force: 阻尼力， restoring force: 回复力
natural frequency: 固有频率， stimulus: 激励， electric dipole moment: 电偶极矩

Now we can connect the microscopic electric dipole moment $\mathbf{p}$ to the macroscopic polarization vector $\mathbf{P}$:



N – density of electrons

$$\mathbf{P} = \square\mathbf{p} = \frac{\square e^2 / m}{\omega_0^2 - \omega^2 - i\omega\gamma}\mathbf{E}$$

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_0\varepsilon\mathbf{E}$$

χ, electric susceptibility

$$\varepsilon(\omega) = 1 + \boxed{\frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\gamma}}$$

$$\omega_p = \sqrt{\frac{Ne^2}{\varepsilon_0 m}}$$ (plasma frequency)

What does it mean?

More discussion in Lecture 5

electric susceptibility: 电极化率，plasma frequency: 等离子体频率

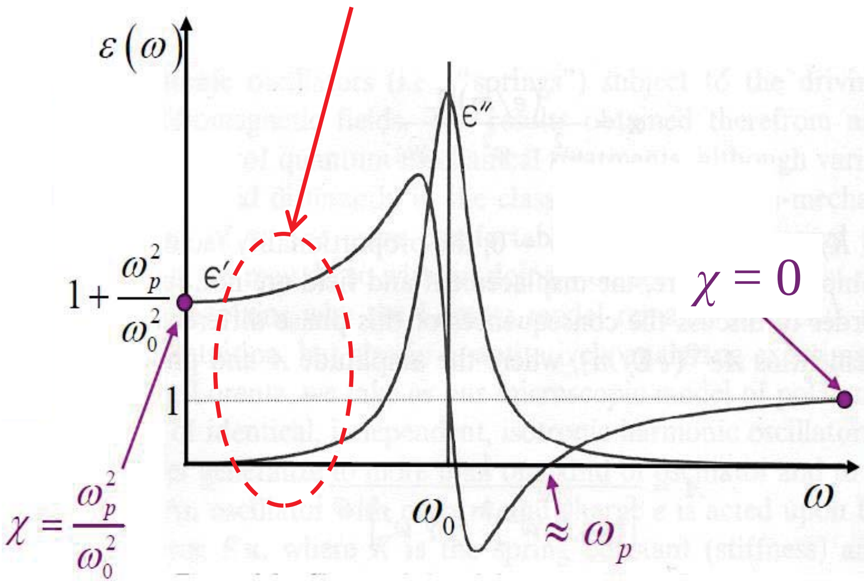
Now let's have a closer look at $\varepsilon(\omega)$:

If $\gamma \neq 0$, then ε is complex: $\varepsilon = \varepsilon' + i\varepsilon'' = 1 + \dfrac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\gamma}$

$$\varepsilon' = 1 + \chi' = 1 + \frac{\omega_p^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

$$\varepsilon'' = \chi'' = \frac{\omega_p^2\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

Optical frequency $\omega_0 >> \omega >> \gamma$



Several typical points:

- When $\omega \to 0$: $\varepsilon' \to 1 + \dfrac{\omega_p^2}{\omega_0^2}$, $\varepsilon'' \to 0$
- When $\omega = \omega_0$: $\varepsilon' = 1$, $\varepsilon'' \to$ maximum
- When $\omega \approx \omega_p$: $\varepsilon' = 0$
- When $\omega \to \infty$: $\varepsilon' \to 1$, $\varepsilon'' \to 0$

Note: usually $\omega_p >> \omega_0 >> \gamma$, $\gamma \sim 100\text{THz}$

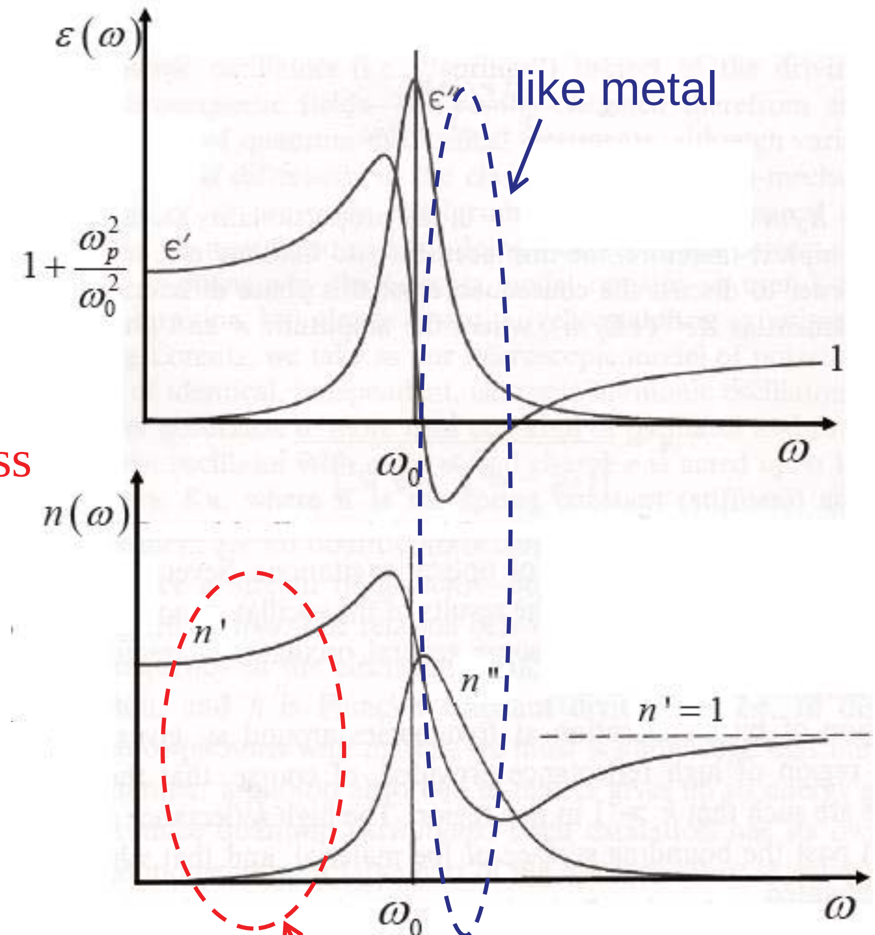
Refractive index □(ω) – characterizing EM wave propagation

For nonmagnetic media:

$$n = n' + in'' = \sqrt{\varepsilon}$$

$$n' = \sqrt{\frac{\sqrt{\varepsilon'^2 + \varepsilon''^2} + \varepsilon'}{2}} \qquad n'' = \sqrt{\frac{\sqrt{\varepsilon'^2 + \varepsilon''^2} - \varepsilon'}{2}}$$

- When $\omega << \omega_0$:
  high $n' \rightarrow$ low $v_p = c/n'$, $n'' \approx 0 \rightarrow$ lossless
  ↑ **This is the so-called insulator!**
- When $\omega \sim \omega_0$:
  rapidly varying $n' \rightarrow$ strong dispersion
  n'' high $\rightarrow$ large absorption
  ↑ **like metal**
- When $\omega >> \omega_0$:
  $n' \approx 1 \rightarrow v_p \approx c$, $n'' = 0 \rightarrow$ lossless
  ↑ **like vacuum or air**



Optical frequency $\omega_0 >> \omega >> \gamma$

Refractive indices of some typical insulators (at optical frequency)

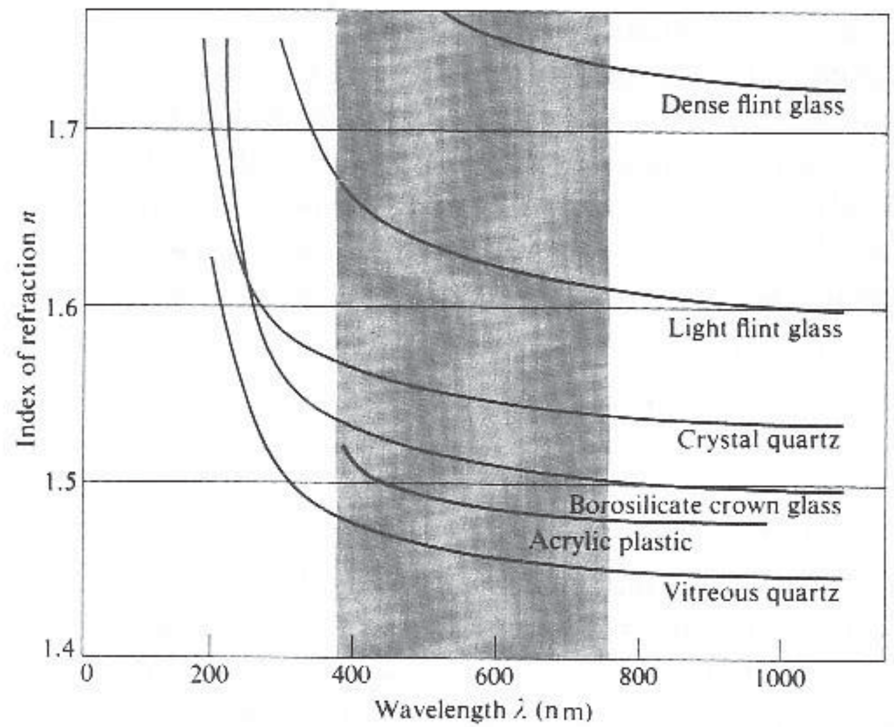


Figure 3.40 The wavelength dependence of the index of refraction for various materials.

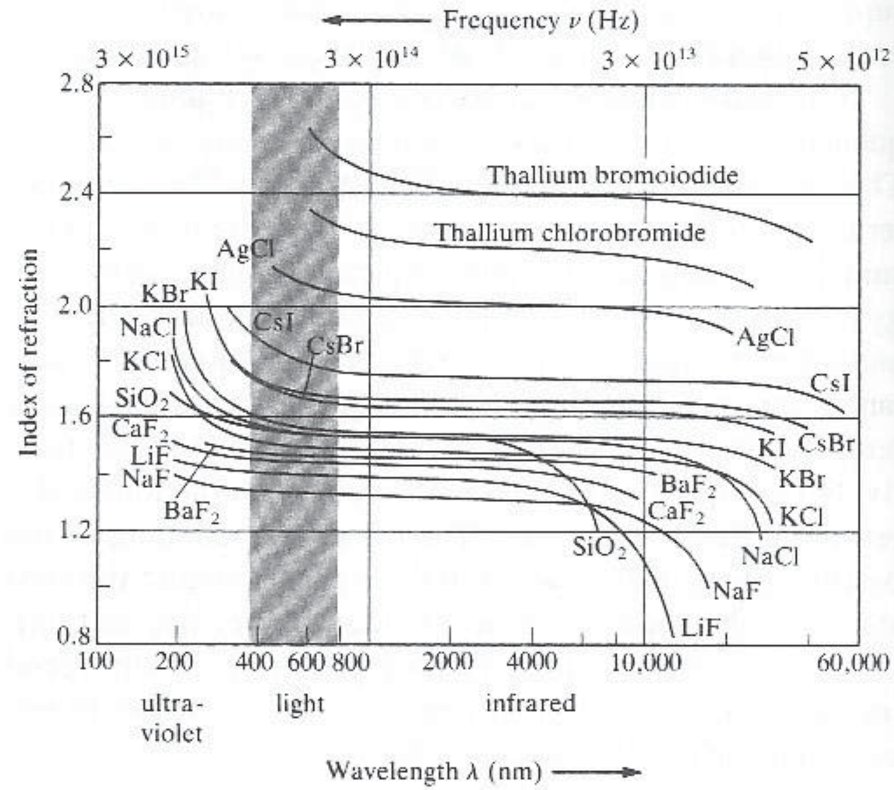


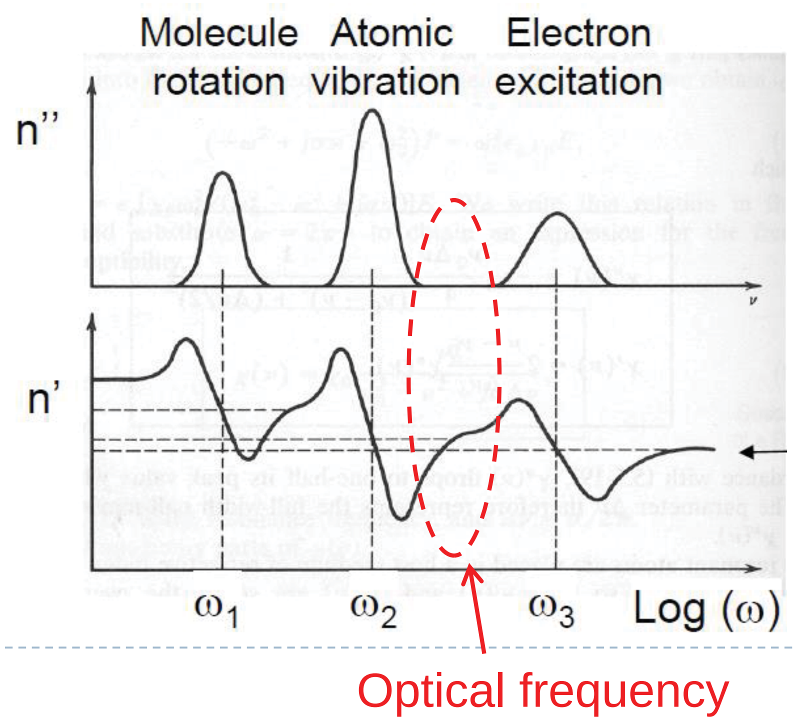
Figure 3.42 Index of refraction versus wavelength and frequency for several important optical crystals. (Adapted from data published by The Harshaw Chemical Co.)

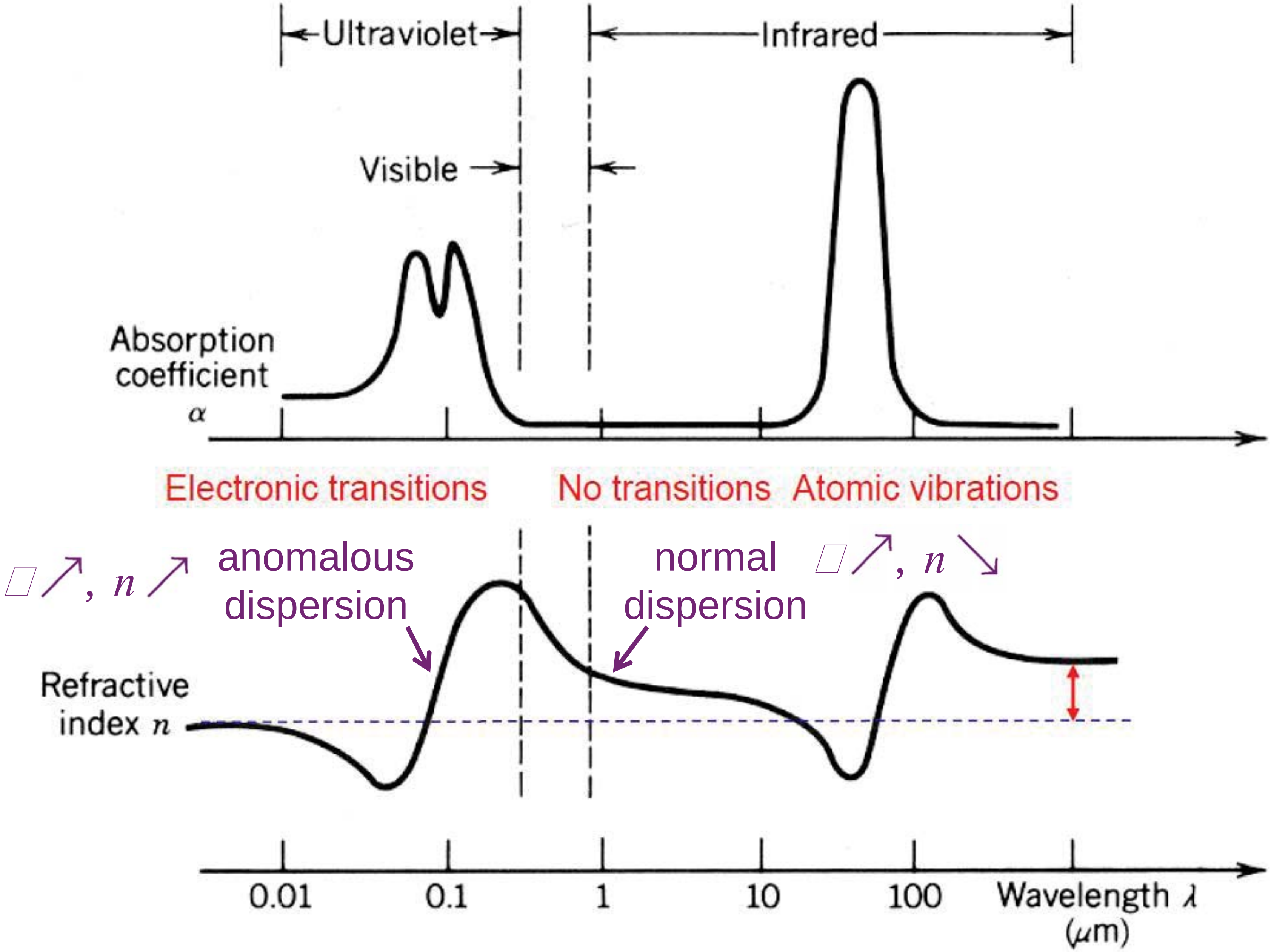
But, realistic media usually have multiple resonances:

- Resonances occur due to motion of molecules (very low ω_0), atoms (low ω_0), or electron transitions (high ω_0)
- Therefore, the Lorentz model should be modified by taking into account multiple resonance terms:

$$\varepsilon = 1 + \sum_{j} \frac{\omega_{jp}^2}{\omega_{j0}^2 - \omega^2 - i\omega\gamma_j} \qquad \left(\omega_{jp}^2 = \frac{N_j e^2}{\varepsilon_0 m},\ \omega_{j0}^2 = \frac{K_j}{m} \right)$$

Schematic of realistic n' and n'' with multiple resonances





normal dispersion: 正常色散，anomalous dispersion: 反常色散

4. Example of engineering light-matter interaction with nanostructures – **form birefringence**

- Having understood the EM response of materials from both microscopic and macroscopic aspects, we can try to **engineer (alter/modify/tailor)** the light-matter interaction with artificial nanostructures.

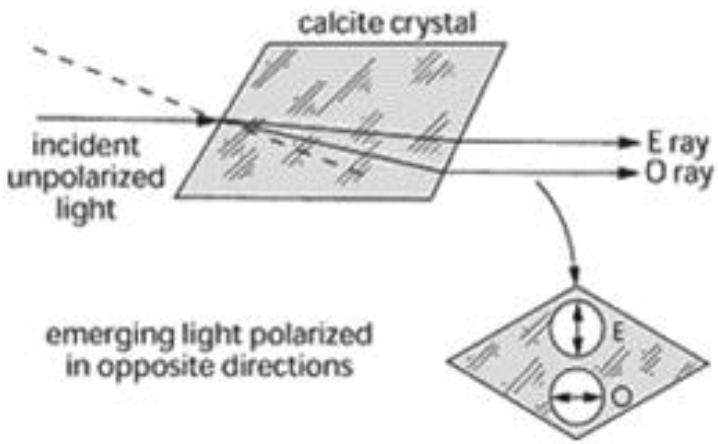
- In this course, we are mostly talking about this kind of things …

- Here, let's see a typical example – **form birefringence**

- More will be discussed in the succedent lectures

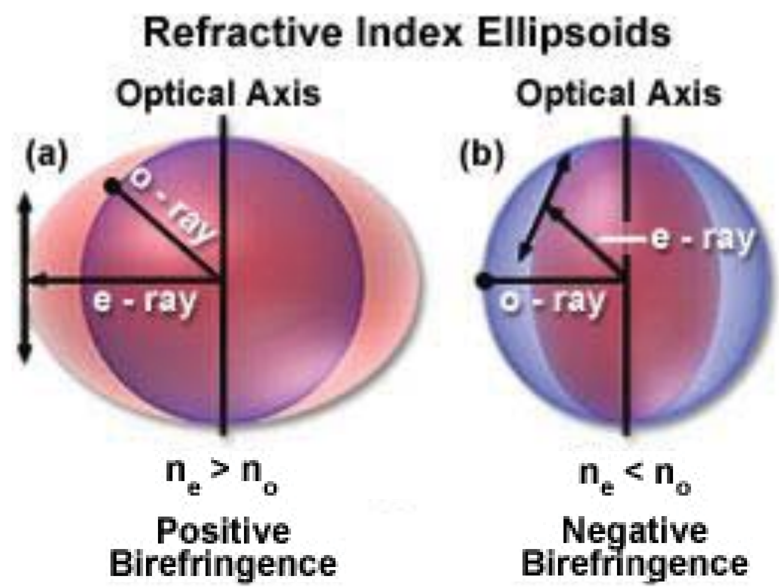
form birefringence: 形序双折射

Birefringence (also called **double refraction**): the splitting of a light ray into two rays (an **ordinary** one and an **extraordinary** one) in an optically anisotropic medium such as calcite.



Reason: refractive index is directionally dependent

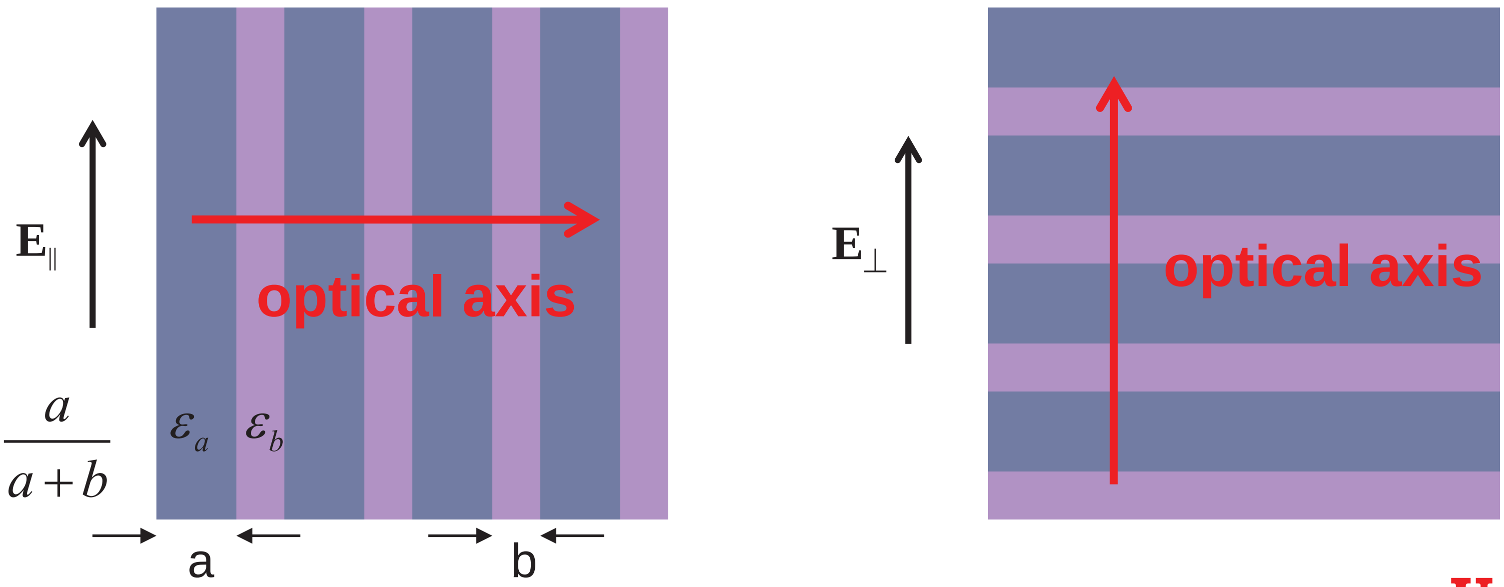
$$\Delta n = n_e - n_o$$



birefringence: 双折射，anisotropic: 各向异性的，calcite: 方解石，optical axis: 光轴，ellipsoid: 椭球

What happens for stacked isotropic dielectric nano-plates?

molecule size << a, b << λ (can be treated as effective media)



$$f = \frac{a}{a+b}$$

The effective permittivity can be derived as:

Uniaxial crystal!

o-ray $\longrightarrow$ $\varepsilon_{\parallel} = f\varepsilon_a + (1-f)\varepsilon_b$

e-ray $\longrightarrow$ $\varepsilon_{\perp} = \dfrac{\varepsilon_a \varepsilon_b}{f\varepsilon_b + (1-f)\varepsilon_a}$

(Homework: derive these expressions and prove that it is a negative uniaxial crystal)

Form birefringence – birefringence/anisotropy induced by the structural arrangement rather than the composing materials themselves.

isotropic: 各向同性的，uniaxial crystal: 单轴晶体

Summary

- Focus of this lecture: understanding light-matter interaction
- Electromagnetic wave:
  Review of Maxwell's equations, boundary conditions, constitutive relations, wave equation, time- and spatial-harmonic field
- Dispersion of materials:
  dispersion (frequency-dependent effect), understand *k*-*ω* plot, physical meaning of phase and group velocities
- Microscopic and macroscopic theories of materials:
  EM response of insulator: Lorentz model (bound electrons)
- Example of engineering light-matter interaction with artificial nanostructures: form birefringence