理论力学第三次作业

1900011413 吴熙楠

2020 年 6 月 17 日

1.

(1) 解: 由题意得：瞬心在 AO 连线上，设距离 A 为 x，则：

$v_1 = \omega x, v_2 = \omega(2a - x)$

$\therefore \omega = \frac{v_1+v_2}{2a}, x = \frac{2av_1}{v_1+v_2}$

$\therefore$ 距离 A 为 $x = \frac{2av_1}{v_1+v_2}$

(2) 解：因为 O 速度一定，其加速度为 0

O 相对瞬心加速度为 $\omega^2(a - x)$

$\because$A 相对瞬心加速度为 $\omega^2 x$

$\therefore$A 相对地面即相对 O 加速度为:$\omega^2 a = \frac{(v_1+v_2)^2}{4a}$

$\therefore a_A = \frac{(v_1+v_2)^2}{4a}$, 方向指向 O 点

2.

解：令 $\frac{I}{I_0} = x$, 求解本征值问题：

$$\frac{M}{I_0} = \begin{pmatrix} 8-x & -3 & -3 \\ -3 & 8-x & -3 \\ -3 & -3 & 8-x \end{pmatrix}$$

$\because det(M) = 0 \qquad \therefore x_1 = x_2 = 11, x_3 = 2$

本征矢量 $\eta_1 = [0, 1, -1]^T, \eta_2 = [1, 0, -1]^T, \eta_3 = [1, 1, 1]^T$

$\therefore$ 主转动惯量 $I_{xx} = I_{yy} = 11I_0, I_{zz} = 2I_0$

惯量主轴的方向余弦为:$ox' = (0, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}), oy' = (\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}), oz' = (\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$

3. 解：

在转动系内能量守恒：$\frac{1}{2}ma^2\dot{\theta}^2 - \frac{1}{2}m\omega^2 a^2(sin^2\theta - \frac{1}{2}) = mga(\frac{\sqrt{2}}{2} - cos\theta)$

化简可得:$a^2\dot{\theta}^2 = 2ga(\frac{\sqrt{2}}{2} - cos\theta) + \omega^2 a^2(sin^2\theta - \frac{1}{2})$

且 $\dot{\theta} = 0$ 时反向

$\therefore cos\theta_1 = \frac{\sqrt{2}}{2}, cos\theta_2 = -(\frac{\sqrt{2}}{2} + \frac{2g}{a\omega^2})$

① $\omega^2 < \frac{2(2+\sqrt{2})g}{a}$, 小球将在 $\theta = \frac{7\pi}{4}$ 处速度为 0，然后反向
② $\omega^2 = \frac{2(2+\sqrt{2})g}{a}$, 小球将在 $\theta = \pi$ 处速度为 0，然后停止运动
③ $\omega^2 > \frac{2(2+\sqrt{2})g}{a}, \theta = arccos(-(\frac{\sqrt{2}}{2} + \frac{2g}{a\omega^2})) = \pi - arccos(\frac{\sqrt{2}}{2} + \frac{2g}{a\omega^2})$

4.

(1) $\because p_i = \frac{\partial S}{\partial x_i}$

$\therefore \frac{\partial S}{\partial t} + \frac{1}{2m}[(\frac{\partial S}{\partial x_1})^2 + (\frac{\partial S}{\partial x_2})^2] + \frac{m}{2}\omega^2(x_1^2 + x_2^2) = 0$

(2) 分离变量，H 不显含时间 t, 令 $\frac{\partial S}{\partial t} = -E$, 则 $T = -Et$

$\frac{1}{2m}[(\frac{\partial S}{\partial x_1})^2 + (\frac{\partial S}{\partial x_2})^2] + \frac{m}{2}\omega^2(x_1^2 + x_2^2) = E$

$\frac{1}{2m}(\frac{dS_1}{dx_1})^2 + \frac{m}{2}\omega^2 x_1^2 = E - \frac{1}{2m}(\frac{dS_2}{dx_2})^2 - \frac{m}{2}\omega^2 x_2^2$

因为左边与右边变量分别独立，相等只可能是同时等于一个常数，令其等于 λ 则：

$\frac{1}{2m}(\frac{dS_1}{dx_1})^2 + \frac{m}{2}\omega^2 x_1^2 = E - \frac{1}{2m}(\frac{dS_2}{dx_2})^2 - \frac{m}{2}\omega^2 x_2^2 = \lambda$

$$\therefore \begin{cases} T = -Et \\ \frac{dS_1}{dx_1} = \sqrt{2m\lambda - m^2\omega^2 x_1^2} \\ \frac{dS_2}{dx_2} = \sqrt{2m(E-\lambda) - m^2\omega^2 x_2^2} \end{cases}$$

(3) 由题意得：$E = Q_1 + Q_2, \lambda = Q_1$

$T = -(Q_1 + Q_2)t$

$S_i = \int \sqrt{2mQ_i - m^2\omega^2 x_i^2}\, dx_i$

$\therefore S = -(Q_1 + Q_2)t + \int \sqrt{2mQ_1 - m^2\omega^2 x_1^2}\, dx_1 + \int \sqrt{2mQ_2 - m^2\omega^2 x_2^2}\, dx_2$

(4)$P_i = -\frac{\partial S}{\partial Q_i} = t - \int \frac{m}{\sqrt{2mQ_i - m^2\omega^2 x_i^2}}\, dx_i$

$\therefore P_i = t - \frac{1}{\omega} arccos(\frac{-x_i}{\sqrt{\frac{2Q_i}{m\omega^2}}})$

令 $P_i = t_i$, 则：$x_i = -\sqrt{\frac{2Q_i}{m\omega^2}} cos\omega(t - t_i)$

$\because P_i$ 表达式显含位移 x_i $\therefore P_i$ 与振动位相相关联

(5) 因为两个方向上的振动频率相同，所以经过一个周期粒子回到原来位置，轨道闭合

$\therefore \frac{m\omega^2 x_1^2}{2Q_1} + \frac{m\omega^2 x_2^2}{2Q_2} - \frac{m\omega^2 cos\omega(t_1 - t_2)}{\sqrt{Q_1 Q_2}} x_1 x_2 = sin^2\omega(t_1 - t_2)$

① 两位移相位相同时，轨道为直线 (可看作退化的椭圆)

② 两位移相位不同时，轨道为椭圆

(6) 令 $\frac{\partial S}{\partial t} = -E$, 代入方程化简可得：

$(\frac{d\Theta}{d\theta})^2 = 2mr^2E - m^2\omega^2 r^4 - r^2(\frac{dR}{dr})^2$

因为左边与右边变量分别独立，相等只可能是同时等于一个常数，令其等于 M^2 则：$\Theta = M\theta, \frac{dR}{dr} = \sqrt{2mE - m^2\omega^2 r^2 - \frac{M^2}{r^2}}$

$S=-Et+M\theta+\int\sqrt{2mE-m^2\omega^2r^2-\frac{M^2}{r^2}}\,dr$

(7) 令 $\frac{\partial S}{\partial M}=0$, 则：

$\theta=\int\frac{\frac{M}{r^2}}{\sqrt{2mE-m^2\omega^2r^2-\frac{M^2}{r^2}}}\,dr$

令 $u=\frac{1}{r^2}$, 则：$\theta=-\frac{M}{2}\int\frac{du}{\sqrt{2mEu-m^2\omega^2-M^2u^2}}$

$\theta=-\frac{1}{2}arccos(\frac{\frac{2mE}{M^2}-2u}{\sqrt{\frac{4m^2E^2}{M^4}-\frac{4m^2\omega^2}{M^2}}})+\theta_0$

$\therefore u=\frac{m}{M^2}[E-\sqrt{E^2-\omega^2M^2}cos2(\theta-\theta_0)]$

$\therefore r=\frac{M}{\sqrt{m[E-\sqrt{E^2-\omega^2M^2}cos2(\theta-\theta_0)]}}$

令 $x=rcos\theta, y=rsin\theta$, 取 $\theta_0=0$, 可得:

① $M=0$, 对于同一角度,r 的值不定，轨道形状为过力心的直线

② $M\neq0, a^2=\frac{E}{m\omega^2}(1+\sqrt{1-\frac{\omega^2M^2}{E^2}}), b^2=\frac{E}{m\omega^2}(1-\sqrt{1-\frac{\omega^2M^2}{E^2}})$

可得:$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \therefore$ 轨道为椭圆

5.

(1) 解：因为这为开普勒运动，即平面运动，所以可用极坐标系

$\frac{d\vec{A}}{dt}=[\vec{A},H]+\frac{\partial\vec{A}}{\partial t}$

因为 A 不显含时间 t, 所以即证明 $[\vec{A},H]=0$

$\vec{A}=m(\dot{r}\hat{r}+r\dot{\theta}\hat{\theta})\times(mr^2\dot{\theta}\hat{z})-mk\hat{r}$

$A_r=\frac{p_\theta^2}{r}-mk, A_\theta=-p_rp_\theta$

$\vec{A}=A_r\hat{r}+A_\theta\hat{\theta}, H=\frac{1}{2m}(p_r^2+\frac{p_\theta^2}{r^2})-\frac{k}{r}$

$\because[\vec{A},H]=\frac{\partial\vec{A}}{\partial q_i}\frac{\partial H}{\partial p_i}-\frac{\partial\vec{A}}{\partial p_i}\frac{\partial H}{\partial q_i}$

其中 $\frac{\partial\hat{r}}{\partial\theta}=\hat{\theta}, \frac{\partial\hat{\theta}}{\partial\theta}=-\hat{r}$

$\therefore[\vec{A},H]=\frac{\partial\vec{A}}{\partial r}\frac{\partial H}{\partial p_r}-\frac{\partial\vec{A}}{\partial p_r}\frac{\partial H}{\partial r}+\frac{\partial\vec{A}}{\partial\theta}\frac{\partial H}{\partial p_\theta}-\frac{\partial\vec{A}}{\partial p_\theta}\frac{\partial H}{\partial\theta}$

$\therefore[\vec{A},H]=-\frac{p_\theta^2p_r}{mr^2}\hat{r}+\frac{kp_\theta}{r^2}\hat{\theta}-\frac{p_\theta^3}{mr^3}\hat{\theta}+\frac{p_\theta}{mr^2}[(\frac{p_\theta^2}{r}-mk)\hat{\theta}+p_rp_\theta\hat{r}]$

$\therefore[\vec{A},H]=0, \frac{d\vec{A}}{dt}=[\vec{A},H]+\frac{\partial\vec{A}}{\partial t}=0$

所以 $\vec{A}$ 是守恒量

(2) 解：令 i=1,2,3 分别代表 x,y,z 方向上的分量，则：

$A_1=xp_y^2-yp_xp_y-mk\frac{x}{\sqrt{x^2+y^2}}, A_2=yp_x^2-xp_xp_y-mk\frac{y}{\sqrt{x^2+y^2}}, A_3=0$

$L_1=L_2=0, L_3=xp_y-yp_x$

$\therefore$ 除了 $[A_1,L_3],[A_2,L_3]$ 外的所有 $[A_i,L_j]$ 的组合的值均为 0

$\because[A_1,L_3]=\frac{\partial A_1}{\partial x}\frac{\partial L_3}{\partial p_x}-\frac{\partial A_1}{\partial p_x}\frac{\partial L_3}{\partial x}+\frac{\partial A_1}{\partial y}\frac{\partial L_3}{\partial p_y}-\frac{\partial A_1}{\partial p_y}\frac{\partial L_3}{\partial y}$

$\therefore[A_1,L_3]=(p_y^2-mk\frac{y^2}{(x^2+y^2)^{1.5}})(-y)+yp_y^2+x(-p_xp_y+mk\frac{xy}{(x^2+y^2)^{1.5}})+p_x(2xp_y-yp_x)$

$=mk\frac{y}{\sqrt{x^2+y^2}}+xp_xp_y-yp_x^2=-A_2$

$\therefore[A_1,L_3]=\epsilon_{132}A_2=-A_2$

$\because [A_2, L_3] = \frac{\partial A_2}{\partial x}\frac{\partial L_3}{\partial p_x} - \frac{\partial A_2}{\partial p_x}\frac{\partial L_3}{\partial x} + \frac{\partial A_2}{\partial y}\frac{\partial L_3}{\partial p_y} - \frac{\partial A_2}{\partial p_y}\frac{\partial L_3}{\partial y}$
$\therefore [A_2, L_3] = (-p_x p_y + mk\frac{xy}{(x^2+y^2)^{1.5}})(-y) - p_y(2yp_x - xp_y) + x(p_x^2 - mk\frac{x^2}{(x^2+y^2)^{1.5}}) - xp_x^2$
$= xp_y^2 - yp_x p_y - mk\frac{x}{\sqrt{x^2+y^2}} = A_1$
$\therefore [A_2, L_3] = \epsilon_{231}A_1 = A_1$
而对于除开 $[A_1, L_3], [A_2, L_3]$ 外的所有 $[A_i, L_j]$ 的组合的值均为 0, 显然也满足 $[A_i, L_j] = \epsilon_{ijk}A_k$
$\therefore$ 综上所述:$[A_i, L_j] = \epsilon_{ijk}A_k$ 成立