# 理论力学第一次作业

1900011413 吴熙楠

2020 年 4 月 22 日

1. 解：以旋转圆环系为参考系，环心为重力势能零点

$\because T = \frac{1}{2}ma^2\dot{\theta}^2, V = -\frac{1}{2}m\omega^2a^2\sin^2\theta - mga\cos\theta$

$\therefore L = T - V = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}m\omega^2a^2\sin^2\theta + mga\cos\theta$

由欧拉拉格朗日方程：$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = \frac{\partial L}{\partial\theta}$ 代入可得：$\ddot{\theta} = -\frac{g}{a}\sin\theta + \omega^2\sin\theta\cos\theta$

又由于 L 不显含时间，故能量为初积分

$\therefore E = \dot{\theta}\frac{\partial L}{\partial\dot{\theta}} - L = \frac{1}{2}ma^2\dot{\theta}^2 - \frac{1}{2}m\omega^2a^2\sin^2\theta - mga\cos\theta$

$\ddot{\theta} = 0$

$\theta_1 = 0, \theta_2 = \arccos\frac{g}{\omega^2a}$

$\therefore$ 要使底部有一个解不存在，则：$\frac{g}{\omega^2a} > 1$

$\omega < \sqrt{\frac{g}{a}}$ 时，底部 $\theta_2$ 解不存在

2. 解：

$\because \vec{J} = m\vec{r}\times\vec{v}$

$\therefore U = V(\vec{r}) + \vec{\sigma}\cdot\vec{J}$

$= V(\vec{r}) + \vec{\sigma}\cdot(m\vec{r}\times\vec{v})$

$\because$ 广义力 $\vec{Q} = -\nabla U + \frac{d}{dt}\frac{\partial U}{\partial\vec{v}}$

$\therefore \vec{Q} = -\nabla V + 2m(\vec{\sigma}\times\vec{v})$

$\therefore$ 由欧拉拉格朗日方程 $\frac{d}{dt}\frac{\partial L}{\partial\vec{v}} = \frac{\partial L}{\partial\vec{r}}$ 可得：

$m\frac{d\vec{v}}{dt} = -\nabla U + \frac{d}{dt}\frac{\partial U}{\partial\vec{v}}$

$= -\nabla V + 2m(\vec{\sigma}\times\vec{v})$

3.

(a) 证明：

$\because \Box'^2 = \partial_\mu\partial^\mu = \Lambda^\gamma_\mu\partial\gamma\Lambda^\mu_\lambda\partial^\lambda$

$= \delta^\gamma_\lambda\partial_\gamma\partial^\lambda = \partial_\gamma\partial^\gamma$

$= \Box^2$

$\therefore$ $d$·Alembert 算符具有洛伦兹不变性

(b) 证明：

$\because$ 对于四阶反对称张量 $A^{\mu\gamma\alpha\beta} = \begin{cases} A & if\ (\mu,\gamma,\alpha,\beta)\ is\ an\ even\ permuation\ of\ (0,1,2,3) \\ -A & if\ (\mu,\gamma,\alpha,\beta)\ is\ an\ odd\ permuation\ of\ (0,1,2,3) \\ 0 & otherwise \end{cases}$

$\therefore$ 由定义可得：$A^{\mu\gamma\alpha\beta} = A\epsilon^{\mu\gamma\alpha\beta}$

$\therefore A'^{\mu\gamma\alpha\beta} = A\epsilon^{lkmn} = \Lambda^l_\mu\Lambda^k_\gamma\Lambda^m_\alpha\Lambda^n_\beta A^{\mu\gamma\alpha\beta} = A\Lambda^l_\mu\Lambda^k_\gamma\Lambda^m_\alpha\Lambda^n_\beta\epsilon^{\mu\gamma\alpha\beta}$

$\therefore \epsilon^{lkmn} = \Lambda^l_\mu\Lambda^k_\gamma\Lambda^m_\alpha\Lambda^n_\beta\epsilon^{\mu\gamma\alpha\beta}$

$\therefore \epsilon^{\mu\gamma\alpha\beta}$ 为洛伦兹变换下四阶反对称张量

$\epsilon^{\mu\gamma\alpha\beta}F_{\mu\gamma}F_{\alpha\beta} = \Lambda^\mu_l\Lambda^\gamma_k\Lambda^\alpha_m\Lambda^\beta_n\Lambda^x_\mu\Lambda^y_\gamma\Lambda^z_\alpha\Lambda^\lambda_\beta\epsilon^{lkmn}F_{xy}F_{z\lambda}$

$= \delta^x_l\delta^y_k\delta^z_m\delta^\lambda_n\epsilon^{lkmn}F_{xy}F_{z\lambda}$

$= \epsilon^{lkmn}F_{lk}F_{mn}$

$\therefore \epsilon^{\mu\gamma\alpha\beta}F_{\mu\gamma}F_{\alpha\beta}$ 为洛伦兹不变量

4. 解：

$A = \int 2\pi x\, ds = \int_{y_1}^{y_2} 2\pi x\sqrt{1+x'^2}\, dy,\ \ F = x\sqrt{1+x'^2}$

$\because \delta A = 0 \qquad \therefore 2\pi\int_{y_1}^{y_2}(\frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial x'}\delta x')\, dy = 0$

对第一项分部积分可得：$\frac{\partial F}{\partial x'}\delta x|_{y_1}^{y_2} + \int_{y_1}^{y_2}(\frac{\partial F}{\partial x} - \frac{d}{dy}\frac{\partial F}{\partial x'})\delta x\, dy = 0$

$\because \delta x$ 的任意性且边界项为 0 $\therefore \frac{\partial F}{\partial x} = \frac{d}{dy}\frac{\partial F}{\partial x'}$

化简可得：$xx'' = 1 + x'^2$

积分后可得：$1 + x'^2 = Cx^2$ C 为常数

5. 解：

$\because S = -mc\int ds = -mc\int(g_{\mu\nu}dx^\mu dx^\nu)^{\frac{1}{2}}$

$\therefore \delta S = -\frac{mc}{2}\int\frac{g_{\mu\nu}(dx^\mu d\delta x^\nu + dx^\nu d\delta x^\mu) + dx^\mu dx^\nu\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\delta x^\lambda}{ds}$

对其前两项进行分部积分，且 $\because$ 边界项为 0$\therefore$ 舍掉边界项可得：

$\delta S = \frac{mc}{2}\int[\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\delta x^\lambda - \frac{d}{ds}(g_{\mu\nu}\frac{dx^\nu}{ds})\delta x^\mu - \frac{d}{ds}(g_{\mu\nu}\frac{dx^\mu}{ds})\delta x^\nu]\, ds$

将后两项的哑标 $\mu$ 和 $\nu$ 替换后可得：

$\delta S = \frac{mc}{2}\int[\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{\partial g_{\mu\nu}}{\partial x^\lambda} - 2\frac{d}{ds}(g_{\lambda\nu}\frac{dx^\nu}{ds})]\delta x^\lambda\, ds$

$\because \delta S = 0$

$\therefore$ 由 $\delta x^\lambda$ 的独立性可得：

$\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{\partial g_{\mu\nu}}{\partial x^\lambda} - 2\frac{d}{ds}(g_{\lambda\nu}\frac{dx^\nu}{ds}) = 0$

$\therefore$ 运动方程为：$\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{\partial g_{\mu\nu}}{\partial x^\lambda} - 2\frac{d}{ds}(g_{\lambda\nu}\frac{dx^\nu}{ds}) = 0$

6. (a) 解：

$\because S = \int\mathcal{L}\, d^4x$

$\therefore \delta S = \int (\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi)\, d^4x = 0$

对第一项分部积分可得：$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi|_{x_1}^{x_2} + \int (\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)})\delta\phi\, d^4x = 0$

$\therefore$ 由 $\delta\phi$ 的任意性且边界项为 0，化简可得：

$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$

代入 $\mathcal{L}$ 可得：$\partial_\mu \partial^\mu \phi + m^2\phi = 0$

$\therefore (\Box^2 + m^2)\phi = 0$

(b) 解：

$\because \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}, \frac{\partial \mathcal{L}}{\partial \phi^*} = \partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^*)}$

$\therefore$ 代入 $\mathcal{L}$ 化简可得：$\begin{cases} (\Box^2 + m^2)\phi = 0 \\ (\Box^2 + m^2)\phi^* = 0 \end{cases}$

(c) 证明：

$\because \partial^\mu j_\mu = \partial^\mu \phi^* \partial_\mu \phi + (\Box^2 \phi)\phi^* - \partial^\mu \phi \partial_\mu \phi^* - (\Box^2 \phi^*)\phi$

将（b）计算结果代入可得：

$\partial^\mu j_\mu = (\partial^\mu \phi^* \partial_\mu \phi - m^2 \phi\phi^*) - (\partial^\mu \phi \partial_\mu \phi^* - m^2 \phi\phi^*)$

$= 2\mathcal{L} - 2\mathcal{L}$

$= 0$

$\therefore \partial^\mu j_\mu = 0$

$\because \int_{all\ \ space} \partial_\mu j^\mu\, d^3x$

$= \int_{all\ \ space} \partial_0 j^0\, d^3x + \oiint_S j^i\, dS \qquad i = 1, 2, 3$

$= 0$

$\because$ 对于无穷大全空间而言，第二项面积分趋于 0

$\therefore \int_{all\ \ space} \partial_0 j^0\, d^3x = 0$

$\because Q = \int_{all\ \ space} j^0\, d^3x$

$\therefore \frac{d}{dt}Q = c\int_{all\ \ space} \partial_0 j^0\, d^3x = 0$

$\therefore Q$ 为一个守恒荷

(d) 解：

$\because \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}, \frac{\partial \mathcal{L}}{\partial \phi^*} = \partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^*)}$

$\therefore$ 代入 $\mathcal{L}$ 化简可得：$\begin{cases} (\Box^2 + m^2)\phi = 2[\partial_\mu(A^\mu \phi) + A^\mu(\partial_\mu \phi)] \\ (\Box^2 + m^2)\phi^* = -2[\partial_\mu(A^\mu \phi^*) + A^\mu(\partial_\mu \phi^*)] \end{cases}$

$\therefore \partial^\mu j_\mu = \partial^\mu \phi^* \partial_\mu \phi + (\Box^2 \phi)\phi^* - \partial^\mu \phi \partial_\mu \phi^* - (\Box^2 \phi^*)\phi$

$= (\partial^\mu \phi^* \partial_\mu \phi - m^2\phi\phi^*) - (\partial^\mu \phi \partial_\mu \phi^* - m^2 \phi\phi^*) + 2[\partial_\mu(A^\mu \phi) + A^\mu(\partial_\mu \phi) + \partial_\mu(A^\mu \phi^*) +$
$A^\mu(\partial_\mu \phi^*)]$

$$= 4\partial_\mu (A^\mu \phi \phi^*)$$

$$\therefore \partial^\mu j_\mu = 4\partial_\mu (A^\mu \phi \phi^*)$$