理论力学第二次作业

1900011413 吴熙楠

2020 年 6 月 3 日

1.

(a)

第一宇宙速度 $v_1 = \sqrt{\frac{GM_e}{R_e}}, v = \frac{1}{2}\sqrt{\frac{GM_e}{R_e}}$

$E = \frac{1}{2}mv^2 - \frac{GM_e m}{R_e} = -\frac{7GM_e m}{8R_e}, L = \frac{m}{4}\sqrt{2GM_e R_e}$

$m\ddot{r} = -\frac{GM_e m}{r^2} + \frac{L^2}{mr^3}$

令 $u = \frac{1}{r}, \dot{\theta} = \frac{Lu^2}{m}$

代入方程可得：$\frac{d^2u}{d\theta^2} + u = \frac{GM_e m^2}{L^2}$

代入数据可得：$\frac{d^2u}{d\theta^2} + u = \frac{8}{R_e}$

$u = Acos\theta + Bsin\theta + \frac{8}{R_e}$

带入初始条件可得：$A = -\frac{7}{R_e}, B = -\frac{1}{R_e}$

$r = \frac{R_e}{8-7cos\theta-sin\theta}$

(b)

轨道半长轴 $a = \frac{1}{2}R(1+0.7) = \frac{17}{20}R, E = -\frac{GM_s m}{2a} = -\frac{10GM_s m}{17R}$

在地球位置处的速度为 v_1 有能量守恒 $\frac{1}{2}mv_1^2 - \frac{GM_s m}{R} = -\frac{GM_s m}{2a} = -\frac{10GM_s m}{17R}$

$v_1 = \sqrt{\frac{14GM_s}{17R}}$

相对地球：$\triangle v = \sqrt{\frac{GM_s}{R}}(1 - \sqrt{\frac{14}{17}})$

在地球表面发射速度为 $v_0, \frac{1}{2}m\triangle v^2 = \frac{1}{2}mv_0^2 - \frac{GM_e m}{R_e}$

$v_0 = \sqrt{\frac{GM_s}{R}(\frac{31}{17} - 2\sqrt{\frac{14}{17}}) + \frac{2GM_e}{R_e}}$

(c)

以第三宇宙速度发射星体，脱离地球引力后相对地球速度为 $(\sqrt{2}-1)\sqrt{\frac{GM_s}{R}}$

所以在太阳系看来相对太阳速度是这速度与地球速度矢量叠加

同 (a) 的做法:$\frac{d^2u}{d\theta^2} + u = \frac{GM_e m^2}{L^2}, L = mR\sqrt{\frac{GM_s}{R}}$

带入初始条件可以解得：$u = \frac{\sqrt{2}-1}{R}sin\theta + \frac{1}{R}$

$r = \frac{R}{1+(\sqrt{2}-1)sin\theta}$

2.

(a)

$L = \frac{1}{2}m_1\dot{\vec{x}}_1^2 + \frac{1}{2}m_1\dot{\vec{x}}_1^2 - k|\vec{x}|^\beta$

引入相对位移和质心位矢后可得：$L = \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{1}{2}\frac{m_1m_2}{m_1+m_2}\dot{\vec{x}}^2 - k|\vec{x}|^\beta$ 则化为单体问题

而选择质心系后相对质心速度为 0，则拉格朗日函数只与相对位矢 $\vec{x}$ 有关，故可以化为一维问题

令系统角动量为 J，则有有效势能:$V_{eff} = kr^\beta + \frac{J^2}{2mr^2}$, 约化质量 $m = \frac{m_1m_2}{m_1+m_2}$

(b)

$V_{eff} = kr^\beta + \frac{J^2}{2mr^2}$

要使扰动为稳定的，则：$V_{eff}'' > 0$ 且 $V_{eff}'|_{r=r_0} = 0$

化简得：$k\beta(\beta+2) > 0$，由已知：$k\beta > 0$

$\beta > -2$ 且 $k\beta > 0$

$\therefore k > 0, \beta > 0 \quad or \quad k < 0, -2 < \beta < 0$

(c)

代入上式化简可得 $L = \frac{1}{2}m\dot{\eta}^2 - \frac{1}{2}\beta k(\beta+2)r_0^{\beta-2}$

可得 $\omega = \sqrt{\frac{k\beta(\beta+2)r_0^{\beta-2}}{m}}$,　其中 m 为约化质量

(d)

由题意得：系统圆运动角频率 $\omega_0 = \sqrt{\frac{k\beta r_0^{\beta-2}}{m}}$

$\frac{\omega}{\omega_0} = \sqrt{\beta+2}$

当 $\frac{\omega}{\omega_0}$ 为有理数时轨道闭合

①$\beta = 15/25$ 时，$\frac{\omega}{\omega_0}$ 不为有理数，故轨道不闭合

②$\beta = -2/9$ 时，$\frac{\omega}{\omega_0} = 4/3$ 故轨道闭合

如图：(其中蓝线表示微扰前的圆轨道，黄线表示微扰后的轨道)

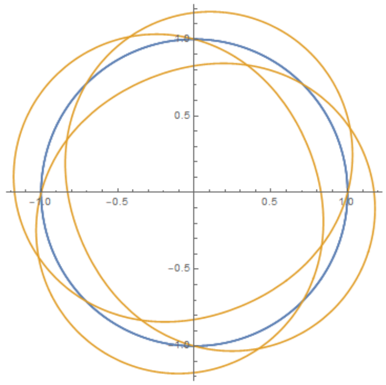

图 1: 2(d)

3.

粒子能量为：$E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V = 1.2V_0$

$J = \sqrt{2mE}s = mr^2\dot{\phi}$

$\phi = \int_{r_{min}}^{\infty} \frac{J/r^2}{\sqrt{2m(E-V)-\frac{J^2}{r^2}}} dr$

散射角 $\Theta = \pi - 2\phi$

微分截面 $\frac{d\sigma}{d\Omega} = \frac{sds}{sin\Theta d\Theta}$

其中最近距离满足：$\frac{1}{1+r_{min}} = \frac{6}{5}(1 - \frac{s^2}{r_{min}^2})$

将偏转角度代入进行数值积分可得如图图像：

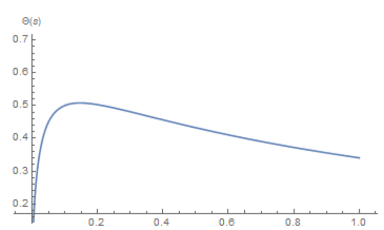

图 2: 3.1

由图 2 可得：该势能的散射角有最大值，即有最大散射角，大约在 s 为 0.15 时取得最大值为 0.5

将微分截面带入后可得如图：（未标识刻度，其中 Θ 轴范围为 0-0.5, 竖直轴范围为 0-1，故开始阶段斜率很大）

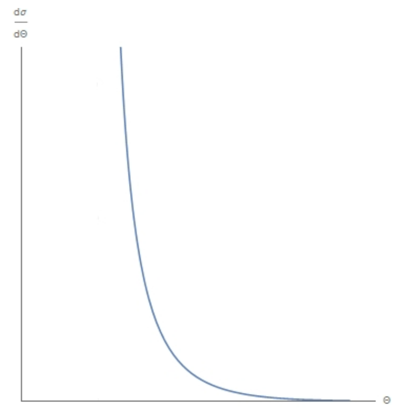

图 3: 3.2

我们观察此散射截面曲线可知，在散射角大于 0.5 时，散射截面降为 0，此散射有最大散射角，
故由定义有最大散射角的散射会出现彩虹散射可知：会出现彩虹散射
4.
(a)
$L = \frac{1}{2}m\dot{r}^2 - \frac{J^2}{2mr^2} + \alpha\frac{e^{-r/a}}{r}$J 为粒子角动量
由欧拉拉格朗日方程可得：$m\ddot{r} = \frac{J^2}{mr^3} - \alpha(1+\frac{r}{a})\frac{e^{-r/a}}{r^2}$
(b)
$V_{eff} = \frac{J^2}{2mr^2} - \alpha\frac{e^{-r/a}}{r}$
能量大于 0，粒子可以飞向无穷远，能量小于 0，则不能飞向无穷远；粒子角动量越大，则粒子的远心点距离越大
(c)
$V_{eff} = -\frac{\alpha e^{-\frac{r}{a}}}{r} + \frac{J^2}{2mr^2}$
将 V_{eff} 展开到二阶项 $V_{eff} = -\frac{\alpha e^{-\rho/a}}{\rho^3}(1+\frac{\rho}{a}+\frac{\rho^2}{2a^2})r^2 + \frac{3\alpha(1+\frac{\rho}{a})e^{-\frac{\rho}{a}}}{2\rho^3}r^2$
化简可得：$V_{eff} = \frac{\alpha(1+\frac{\rho}{a}-\frac{\rho^2}{a^2})e^{-\frac{\rho}{a}}}{2\rho^3}r^2$
$V''_{eff} = \frac{\alpha(1+\frac{\rho}{a}-\frac{\rho^2}{a^2})e^{-\frac{\rho}{a}}}{\rho^3}$
$\omega_r = \sqrt{\frac{V''_{eff}}{m}} = \sqrt{\frac{\alpha(1+\frac{\rho}{a}-\frac{\rho^2}{a^2})e^{-\rho/a}}{m\rho^3}}$
由 $m\omega_0^2\rho = \frac{\alpha e^{-\rho/a}(1+\frac{\rho}{a})}{\rho^2}$ 可得：圆运动角速度为 $\omega_0 = \sqrt{\frac{\alpha(1+\frac{\rho}{a})e^{-\rho/a}}{m\rho^3}}$
$\frac{\omega_r}{\omega_0} = \sqrt{1 - \frac{\rho^2}{a^2+a\rho}}$

$\delta\phi = 2\pi(1 - \sqrt{1 - \frac{\rho^2}{a^2+a\rho}})$

当 $a >> \rho$ 时，$\frac{\omega_r}{\omega_0} \doteq 1 - \frac{\rho^2}{2a^2}$

故进动角为：$\delta\phi = 2\pi \cdot \frac{\rho^2}{2a^2} = \pi\frac{\rho^2}{a^2}$

(d)

$\because$ 如果保留指数项，将得不出解析解 $\therefore$ 对汤川势展开到一阶：$V = -\frac{\alpha}{r} + \frac{\alpha}{a}$

偏转角 $\theta = \pi - 2\int_{r_{min}}^{\infty} \frac{\frac{J}{r^2}}{\sqrt{2m(\frac{1}{2}mv_\infty^2 - \frac{\alpha}{a} + \frac{\alpha}{r}) - \frac{J^2}{r^2}}} dr$

其中 $E = \frac{1}{2}mv_\infty^2, J = mv_\infty\rho, d\sigma = \frac{\rho d\rho}{sin\theta d\theta}d\Omega$

对比可知：此题只需令 $E' = E - \frac{\alpha}{a}$

即相当于将总能量改变一个常数，此题将等价于卢瑟福散射，但注意角动量不变

代入卢瑟福散射公式可得散射截面为:$d\sigma = (\frac{\alpha}{2mv_\infty^2})^2\frac{d\Omega}{(1-\frac{2\alpha}{amv_\infty^2})sin^4(\frac{\theta}{2})}$，其中 Ω 为立体角

5.

(a)

$E_k = \frac{1}{2}m(\dot{x_1}^2 + \dot{x_2}^2), V = \frac{1}{2}k(x_1^2 + x_2^2) + \frac{1}{2}k'(x_1 - x_2)^2, x_1\ x_2$ 分别为质点偏离平衡的量 $M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, K = \begin{pmatrix} 4k & -3k \\ -3k & 4k \end{pmatrix}$

$det(M\omega^2 - K) = 0$

$\omega_1 = \sqrt{\frac{k}{m}}, \omega_2 = \sqrt{\frac{7k}{m}}$

(b)

考虑在平衡点列方程可以直接以平衡位置为势能零点，设两物体平衡是距离为 l, 则：

平衡方程为：$\frac{q^2}{4\pi\varepsilon_0 l^2} = \frac{3}{2}k(l - l_0)$

$E_k = \frac{1}{2}m(\dot{x_1}^2 + \dot{x_2}^2), V = \frac{1}{2}k(x_1^2 + x_2^2) + \frac{1}{2}k'(x_1 - x_2)^2 + \frac{q^2}{4\pi\varepsilon_0(l+x_2-x_1)}$

小量近似后并考虑到只有二次项会影响频率

$V = k(x_1^2 + x_2^2 - x_1x_2) + \frac{q^2(x_1-x_2)^2}{4\pi\varepsilon_0 l^3}$

$M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, K = \begin{pmatrix} 2k + \frac{q^2}{2\pi\varepsilon_0 l^3} & -(k + \frac{q^2}{2\pi\varepsilon_0 l^3}) \\ -(k + \frac{q^2}{2\pi\varepsilon_0 l^3}) & 2k + \frac{q^2}{2\pi\varepsilon_0 l^3} \end{pmatrix}$

$det(M\omega^2 - K) = 0$

$\omega_1 = \sqrt{\frac{3k}{m} + \frac{q^2}{\pi m\varepsilon_0 l^3}}, \omega_2 = \sqrt{\frac{k}{m}}$

6. 未给出 x 方向势能表达式，参考 Goldstein 前面例题后可知 x 方向小球质量为 mMm, 两根弹簧弹性系数均为 k, 则：

由题目已知的 y,z 方向势能可知，线性三原子分子在 x,y,z 方向上的简振模

式相同且独立，则以 x 方向为例：

$E_k = \frac{1}{2}m_1\dot{x_1}^2 + \frac{1}{2}m_2\dot{x_2}^2 + \frac{1}{2}m_3\dot{x_3}^2$

$V = \frac{1}{2}k(x_1 - x_2)^2 + \frac{1}{2}k(x_3 - x_2)^2$

$M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, K = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}$

$det(M\omega^2 - K) = 0, m_1 = m_3 = m, m_2 = M$

$\omega_1 = 0, \omega_2 = \sqrt{\frac{k}{m}}, \omega_3 = \sqrt{\frac{k(M+2m)}{Mm}}$

可求出本征矢量：$\eta_1 = [1, 1, 1]^T; \eta_2 = [1, 0, -1]^T; \eta_3 = [1, -\frac{2m}{M}, 1]^T$

$q_1 = x_1 + x_2 + x_3; q_2 = x_1 - x_3; q_3 = x_1 - \dfrac{2m}{M}x_2 + x_3$

其中，第一种情况对应于分子做匀速直线运动；第二种情况对应于分子中间原子不动，两边原子做对称振动；第三种情况对应于原子相对于分子质心做振动，故可等效为约化质量单原子振动

此时只考虑了 x 方向上的振动，但对比可知 y,z 方向上和 x 方向的振动模式相同，故题目得解

7.

在以平衡长度为势能零点时可以抵消重力的影响，假设以板初始位置中心为原点竖直建系，左端点 $(-\frac{lcos\phi}{2} + x, y - \frac{l}{2}\phi)$，右端点 $(\frac{lcos\phi}{2} + x, y + \frac{l}{2}\phi)$

左弹簧 $l_1 = \sqrt{[x + bsin\theta_0 + \frac{l}{2}(1 - cos\phi)]^2 + (bcos\theta_0 - y + \frac{l}{2}\phi)^2}$

右弹簧 $l_2 = \sqrt{[-x + bsin\theta_0 + \frac{l}{2}(1 - cos\phi)]^2 + (bcos\theta_0 - y - \frac{l}{2}\phi)^2}$

$E_k = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} \cdot \frac{1}{12}ml^2\dot{\phi}^2$

$V = \frac{1}{2}k[(l_1 - b)^2 + (l_2 - b)^2]$

略去高阶项后可得：$V = \frac{1}{2}k(\frac{l^2\phi^2cos^2\theta_0}{2} + 2x^2sin^2\theta_0 + 2y^2cos^2\theta_0 + 2l\phi xsin\theta_0cos\theta_0)$

$M = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{ml^2}{12} \end{pmatrix}, K = \begin{pmatrix} 2ksin^2\theta_0 & 0 & klsin\theta_0cos\theta_0 \\ 0 & 2kcos^2\theta_0 & 0 \\ klsin\theta_0cos\theta_0 & 0 & \frac{kl^2cos^2\theta_0}{2} \end{pmatrix}$

$det(M\omega^2 - K) = 0$

$\omega_1 = \sqrt{\frac{2k}{m}}cos\theta_0, \omega_2 = 0, \omega_3 = \sqrt{\frac{k}{m} \cdot (2 + 4cos^2\theta_0)}$

以 $x, y, l\phi$ 为坐标，对于第一种情况，本征矢量：$\eta_1 = [0, 1, 0]^T$ 同理 $\eta_2 = [cos\theta_0, 0, -2sin\theta_0]^T, \eta_3 = [sin\theta_0, 0, 6cos\theta_0]^T$

模态矩阵为：$A = \frac{1}{\sqrt{m(1+2cos^2\theta_0)}}\begin{pmatrix} 0 & \sqrt{3}cos\theta_0 & sin\theta_0 \\ \sqrt{(1 + 2cos^2\theta_0)} & 0 & 0 \\ 0 & -2\sqrt{3}sin\theta_0 & 6cos\theta_0 \end{pmatrix}$

(其中以 $l\phi$ 为第三个坐标)

$A^{-1} = \frac{1}{\sqrt{14(1+2cos^2\theta_0)}}\begin{pmatrix} 0 & \sqrt{(1+cos^2\theta_0)} & 0 \\ \sqrt{3}cos\theta_0 & 0 & -2\sqrt{3}sin\theta_0 \\ sin\theta_0 & 0 & 6cos\theta_0 \end{pmatrix}$

$Q^T = A^{-1} \cdot X$

故可得简振模式为：$q_1 = y; q_2 = cos\theta_0 x - 2sin\theta_0 l\phi; q_3 = sin\theta_0 x + 6cos\theta_0 l\phi$

8.

(a)

$X(t) = R\theta(t), x = X(t) + \frac{R}{2}sin\phi, y = R - \frac{R}{2}cos\phi$

(b)

$\dot{X} = R\dot{\theta}, \dot{(x)} = R\dot{\theta} + \frac{R}{2}cos\phi\dot{\phi}, \dot{y} = \frac{R}{2}sin\phi\dot{\phi}$

(c)

平动 $E_{k1} = \frac{1}{2}MR^2\dot{\theta}^2$, 转动 $E_{k2} = \frac{1}{2}MR^2\dot{\theta}^2$

(d)

$E_{k3} = \frac{1}{8}mR^2(4\dot{\theta}^2 + \dot{\phi}^2 + 4cos\phi\dot{\phi}\dot{\theta})$

(e)

由纯滚动条件：$R\dot{\theta} = -\frac{R}{2}\dot{\phi} + \omega\frac{R}{2}$ 得到 $\omega = \dot{\phi} + 2\dot{\theta}$

$E_{k4} = \frac{mR^2}{16}(\dot{\phi}^2 + 4\dot{\theta}^2 + 4\dot{\phi}\dot{\theta})$

(f)

取圆筒中心为重力势能零点 $V = -\frac{1}{2}mgRcos\phi$

$L = \sum E_k - V = MR^2\dot{\theta}^2 + mR^2(\frac{3}{4}\dot{\theta}^2 + \frac{3}{16}\dot{\phi}^2 + \frac{1}{2}cos\phi\dot{\phi}\dot{\theta} + \frac{1}{4}\dot{\phi}\dot{\theta}) + \frac{1}{2}mgRcos\phi$

(g)

$\frac{\partial L}{\partial\dot{\theta}} = 2MR^2\dot{\theta} + mR^2(\frac{3}{2}\dot{\theta} + \frac{1}{2}cos\phi\dot{\phi} + \frac{1}{4}\dot{\phi})$

$\frac{\partial L}{\partial\dot{\phi}} = mR^2(\frac{3}{8}\dot{\phi} + \frac{1}{2}cos\phi\dot{\theta} + \frac{1}{4}\dot{\theta})$

由欧拉拉格朗日方程可以导出：$\begin{cases} (\frac{4M}{m} + 3)\ddot{\theta} + \frac{1}{2}(1 + 2cos\phi)\ddot{\phi} = sin\phi\dot{\phi}^2 \\ \frac{1}{2}(1 + 2cos\phi)\ddot{\theta} + \frac{3}{4}\ddot{\phi} = -\frac{g}{R}sin\phi \end{cases}$

$(h)\phi << 1$ 时 , $sin\phi \doteq \phi, cos\phi \doteq 1$ 略去高阶小量后：$\begin{cases} (\frac{4M}{m} + 3)\ddot{\theta} + \frac{3}{2}\ddot{\phi} = 0 \\ \frac{3}{2}\ddot{\theta} + \frac{3}{4}\ddot{\phi} + \frac{g}{R}\phi = 0 \end{cases}$

令 $\theta, \phi \propto e^{i\omega t}$, 则：$\begin{cases} (\frac{4M}{m} + 3)\theta + \frac{3}{2}\phi = 0 \\ -\frac{3}{2}\omega^2\theta + (-\frac{3}{4}\omega^2 + \frac{g}{R})\phi = 0 \end{cases}$

方程有非零解的条件为：$(\frac{4M}{m} + 3)(-\frac{3}{4}\omega^2 + \frac{g}{R}) + \frac{9}{4}\omega^2 = 0$

解之得：$\omega = \sqrt{\frac{g}{R} \cdot \frac{4M+3m}{3M}}$