作业4（截止日期：12月21日）

Note: 该作业使用了 $G=c=1$ 的单位制。

1、对于 Schwarzschild 时空中有质量的粒子的运动，（1）证明：可以用方程 $(dr/d\tau)^2+V(r)=E^2$ 描述，并求出 $V(r)$ 的表达式；（2）记 $L=r^2d\varphi/dt$，证明（a）for $L/m<2\sqrt{3}$，any incoming particle falls toward $r=2m$；（b）the most tightly bound, stable circular orbit is at $r=6m$ with $L/m=2\sqrt{3}$ and has fractional binding energy of $1-\sqrt{8/9}$；（c）any particle with $E\geq 1$ will be pulled into $r=2m$ if $2\sqrt{3}<L/m<4$。

2、Qualitatively discuss: what kind of orbits are possible outside a star of radius (a) $2.5M$, (b) $4M$, (c) $10M$?

3、It has become traditional to parameterize the static, spherically symmetric vacuum field independent of gravity theories. In “isotropic” coordinates one has

$$ds^2=-\left[1-2\frac{m}{r}+2\beta\left(\frac{m}{r}\right)^2+\cdots\right]dt^2+\left[1+2\gamma\frac{m}{r}+\cdots\right]\left(dx^2+dy^2+dz^2\right)\,,$$

where β and γ are the so-called Eddington-Robertson parameters. (1)Proof that in GR, one has $\beta=\gamma=1$ (compare it with Exercise 4 in hw3.pdf). (2) Derive the following generalizations for the deflection of light rays and the advance of the perihelion,

$$\Delta\theta=\frac{1}{2}(1+\gamma)\,\Delta\theta_{\rm GR}\,,$$
$$\dot\omega=\frac{1}{3}(2-\beta+2\gamma)\,\dot\omega_{\rm GR}\,.$$

4、A spaceship on a circular orbit at radius r in a Schwarzschild metric emits a photon with the rest-frame frequency ν_0 at an angle α outward from the tangential direction of the motion, in the plane of the orbit. Prove that the frequency of the photon as seen by a stationary observer at infty is

$$\nu_\infty=\gamma\nu_0(1+v\cos\alpha)\left(1-\frac{2M}{r}\right)^{1/2}$$

where $\gamma=(1-v^2)^{-1/2}$ with $v^2=(M/r)(1-2M/r)^{-1}$.