作业3（截止日期：11月30日）

1、在球坐标系下，$(x^0, x^1, x^2, x^3) = (t, r, \theta, \varphi)$，已知静态球对称度规的一般形式可以取为

$$ds^2 = -e^{A(r)}dt^2 + e^{B(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

求 $R_{\mu\nu}$ 所有独立的非零分量。

2、利用上题所得的结果，推导理想流体 $T^{\mu\nu} = (p+\rho)u^\mu u^\nu + pg^{\mu\nu}$ 在静态球对称情况下的 TOV 方程:

$$\frac{dm(r)}{dr} = 4\pi\rho(r)r^2$$
$$\frac{dp(r)}{dr} = -\frac{[\rho(r)+p(r)]\left[4\pi Gp(r)r^3 + Gm(r)\right]}{r\left[r - 2Gm(r)\right]}$$

3、利用 TOV 方程解析求解广义相对论中在均匀密度情况下静态球对称理想流体星的结构；即 $\rho(r)$、$p(r)$、$m(r)$ 的表达式。

4、Define a new radial coordinate in terms of the Schwarzschild r by $r = \bar{r}\left(1 + M/2\bar{r}\right)^2$. Notice that as $r \to \infty$, $\bar{r} \to r$, while at the horizon $r = 2M$, we have $\bar{r} = M/2$. (a) Show that the metric for spherical symmetry takes the form,

$$ds^2 = -\left[\frac{1 - M/2\bar{r}}{1 + M/2\bar{r}}\right]^2 dt^2 + \left[1 + \frac{M}{2\bar{r}}\right]^4 (d\bar{r}^2 + \bar{r}^2 d\Omega^2)$$

(b) Define quasi-Cartesian coordinates by the usual equations $x = \bar{r}\cos\varphi\sin\theta$, $y = \bar{r}\sin\varphi\sin\theta$, and $z = \bar{r}\cos\theta$, so that $d\bar{r}^2 + \bar{r}^2 d\Omega^2 = dx^2 + dy^2 + dz^2$. Thus, the metric has been converted into coordinates (x, y, z), which are called isotropic coordinates. Now take the limit as $\bar{r} \to \infty$, and show

$$ds^2 = -\left[1 - \frac{2M}{\bar{r}} + \mathcal{O}\left(\frac{1}{\bar{r}^2}\right)\right] dt^2 + \left[1 + \frac{2M}{\bar{r}} + \mathcal{O}\left(\frac{1}{\bar{r}^2}\right)\right] \left(dx^2 + dy^2 + dz^2\right)$$

5、在 Schwarzschild 几何中，一个有质量的粒子从 $r = 4GM$ 处沿径向向外发射。（a）要使粒子到达 $r = 10GM$ 时速度为零，它必须在发射处以多大的初速度 dr/dt 发射？（b）完成这一旅程粒子需要多少固有时间？