problem1: ① scaling theorem: $F.T.\{f(t)\} = F(w)$

if we want to calulate $F.T.\{f(at)\}$, we can set $z = at, w' = \frac{w}{a}$

$\because F(w) = F.T.\{f(t)\} = \int_{-\infty}^{+\infty} f(t) e^{jwt} dt \Rightarrow F.T.\{f(at)\} = \int_{-\infty}^{+\infty} f(at) e^{jwt} dt$

$\therefore \int_{-\infty}^{+\infty} f(at) e^{jwt} dt = \frac{1}{a} \int_{-\infty}^{+\infty} f(z) e^{jw'z} dz = \frac{1}{a} F(w') = \frac{1}{a} F(\frac{w}{a})$

$\Rightarrow$ if $h(t) = f(at)$, then $H(w) = \frac{1}{a} F(\frac{w}{a})$, $H(w) = F.T.\{h(t)\}$

② Time-decay theorem: we can firstly set $t' = t - \tau$, $h(t) = f(t-\tau)$

so $dt' = dt$, $F.T.\{f(t-\tau)\} = \int_{-\infty}^{+\infty} f(t-\tau) e^{jwt} dt = \int_{-\infty}^{+\infty} f(t') e^{jw(t'+\tau)} dt'$

$\Rightarrow F.T.\{f(t-\tau)\} = e^{jw\tau} \int_{-\infty}^{+\infty} f(t') e^{jwt'} dt' = F(w) \cdot e^{jw\tau}$

$\Rightarrow H(w) = F.T.\{h(t)\} = e^{jw\tau} \cdot F(w)$

③ Frequency-offset theorem

if $h(t) = f(t) \cdot e^{-jw_0 t} \Rightarrow F.T.\{h(t)\} = \int_{-\infty}^{+\infty} f(t) e^{j(w-w_0)t} dt$

we can set $w' = w - w_0 \Rightarrow \int_{-\infty}^{+\infty} f(t) e^{jw't} dt = F(w') = F(w-w_0)$

$\Rightarrow H(w) = F.T.\{f(t)\} = F(w-w_0)$

④ convolution theorem: $\because f(t) \otimes g(t) = \int_{-\infty}^{+\infty} f(\tau) g(t-\tau) d\tau$

$\Rightarrow F.T.\{f(t) \otimes g(t)\} = \int_{-\infty}^{+\infty} [\int_{-\infty}^{+\infty} f(\tau) g(t-\tau) d\tau] e^{jwt} dt$

$= \int_{-\infty}^{+\infty} f(\tau) [\int_{-\infty}^{+\infty} g(t-\tau) e^{jwt} dt] d\tau = \int_{-\infty}^{+\infty} f(\tau) e^{jw\tau} G(w) d\tau$

$= F(w) \cdot G(w)$

so $F.T.\{f(t) \otimes g(t)\} = F(w) \cdot G(w)$

problem2:

① $F.T.\{\frac{\partial f(t)}{\partial t}\} = \int_{-\infty}^{+\infty} \frac{\partial f(t)}{\partial t} e^{jwt} dt$

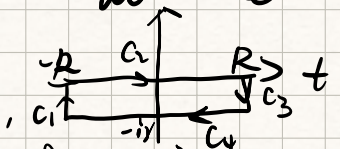
$\Rightarrow \int_{-\infty}^{+\infty} \frac{\partial f(t)}{\partial t} e^{j\omega t}\, dt = f(t)\, e^{j\omega t}\Big|_{t=-\infty}^{t=+\infty} - \int_{-\infty}^{+\infty} f(t)\, d(e^{j\omega t})$

$= -j\omega \int_{-\infty}^{+\infty} f(t)\, e^{j\omega t}\, dt = -j\omega F(\omega)$

$\Rightarrow F.T.\{\frac{\partial f(t)}{\partial t}\} = -j\omega F(\omega)$

② $F.T.\{e^{-\pi t^2}\} = \int_{-\infty}^{+\infty} e^{-\pi t^2} \cdot e^{j\omega t}\, dt = \int_{-\infty}^{+\infty} e^{-\pi t^2 + 2\pi j \gamma t}\, dt$

$= \int_{-\infty}^{+\infty} e^{-\pi (t-j\gamma)^2 - \pi\gamma^2}\, dt = e^{-\pi\gamma^2} \int_{-\infty}^{+\infty} e^{-\pi (t-j\gamma)^2}\, dt$, set $t' = t - j\gamma$

$\because \oint_C e^{-\pi z^2}\, dz = 0$, 

$\Rightarrow (\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4})\, e^{-\pi z^2}\, dz = 0$

$\lim_{R\to\infty} \int_{C_1} e^{-\pi z^2}\, dz = \lim_{R\to\infty} \int_{C_2} e^{-\pi z^2}\, dz = 0 \Rightarrow \lim_{R\to\infty} \int_{C_2} e^{-\pi z^2}\, dz = -\lim_{R\to\infty} \int_{C_4} e^{-\pi z^2}\, dz$

$\Rightarrow \int_{-\infty}^{+\infty} e^{-\pi (t-j\gamma)^2}\, dt = \int_{-\infty}^{+\infty} e^{-\pi t^2}\, dt = 1$

$\Rightarrow F.T.\{e^{-\pi t^2}\} = e^{-\pi \gamma^2}$

Problem 3: For a Gaussian beam, $E(z) = E_0 \frac{W_0}{W(z)} \exp(-\frac{r^2}{W^2(z)})$

$\Rightarrow I(z,r) = \frac{2P}{\pi W^2(z)} \exp(\frac{-2r^2}{W^2(z)}) \Rightarrow W(z) = 5\,\mu m$

$\Rightarrow I_{(r=0)} = \frac{2P}{\pi r^2} = 2.546 \times 10^{15}\ W/m^2$

$\because I = \frac{1}{2} \varepsilon_0 E^2(z)\, c \Rightarrow E(z) = \sqrt{\frac{2 I(z)}{\varepsilon_0 c}} = 1.385 \times 10^{9}\ V/m$