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A New Approach to the Minkowski First Mixed Volume and the LYZ Conjecture

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Abstract
The variation of the functional U of a convex body in $\mathbb{R}^n$ introduced by Lutwak–Yang–Zhang is derived. It becomes the first mixed volume of Minkowski when the convex body is strictly convex. A Minkowski-type inequality for the variation of the of U is proved, which implies the LYZ conjecture for the functional U directly.

Keywords U-functional · Minkowski first mixed volume · LYZ conjecture

Mathematics Subject Classification 52A40

1 Introduction

The setting of this article is the n-dimensional Euclidean space, $\mathbb{R}^n$. A *convex body* (i.e., a compact convex subset with nonempty interior) K in $\mathbb{R}^n$ is uniquely determined by its *support function* $h_K : \mathbb{S}^{n-1} \to \mathbb{R}$, $h_K(u) = \max\{u \cdot x : x \in K\}$, where $\mathbb{S}^{n-1}$ is the unit sphere and $u \cdot x$ denotes the standard inner product of u and x. The *projection body* ΠK of K is defined as the convex body whose support function, for $u \in \mathbb{S}^{n-1}$, is given by $h_{\Pi K}(u) = \mathrm{vol}_{n-1}(K|u^\perp)$, where vol_{n-1} denotes the $(n-1)$-dimensional volume and $K|u^\perp$ denotes the image of the orthogonal projection of K onto the codimension 1 subspace orthogonal to u. The support function of ΠK can also be represented as

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$$h_{\Pi K}(u) = \frac{1}{2}\int_{\mathbb{S}^{n-1}} |u \cdot v|\, dS_K(v), \qquad u \in \mathbb{S}^{n-1}, \tag{1.1}$$

where S_K is the *surface area measure* of the convex body K. Formula (1.1) follows from the Cauchy projection formula. See, e.g., [19, p. 569] for details.

The projection body is one of the most important objects in convex geometry and has been intensively investigated during the past three decades. See, e.g., [1,6–8,13–15,23,24], etc. It is centro-affine invariant; that is, for $T \in \mathrm{SL}(n)$, $\Pi(TK) = T^{-t}(\Pi K)$, where T^{-t} denotes the inverse of the transpose of T. It is worth mentioning that there remains a celebrated unsolved problem regarding projection bodies, called the Schneider projection problem: as K ranges over the class of origin-symmetric convex bodies in $\mathbb{R}^n$, what is the least upper bound of the volume ratio

$$V(\Pi K)/V(K)^{n-1},$$

where $V(K)$ denotes the n-dimensional volume of K? See, e.g., [18,20] for details. The greatest lower bound for this volume ratio is also unknown, although Petty [17] conjectured that the minimum of this quantity is attained precisely by ellipsoids.

An effective tool to study Schneider's projection problem is the *cone-volume functional* U, which was introduced by Lutwak, Yang, and Zhang (LYZ) [16]: If P is a convex polytope in $\mathbb{R}^n$ that contains the origin o in its interior, then $U(P)$ is defined by

$$U(P)^n = \frac{1}{n^n} \sum_{u_{i_1}\wedge\cdots\wedge u_{i_n} \neq 0} h_{i_1}\ldots h_{i_n} a_{i_1}\ldots a_{i_n}, \tag{1.2}$$

where $u_1,\ldots,u_N$ are the outer normal unit vectors to the facets of P, $h_1,\ldots,h_N$ are the corresponding distances of the facets from the origin, and $a_1,\ldots,a_N$ are the corresponding areas of the facets.

Note that the functional U is centro-affine invariant, i.e., $U(TP) = U(P)$, for $T \in \mathrm{SL}(n)$. Let $V_i = a_i h_i/n,\ i = 1,\ldots,N$; then

$$U(P)^n = \sum_{u_{i_1}\wedge\cdots\wedge u_{i_n} \neq 0} V_{i_1}\cdots V_{i_n}.$$

Since $V(P) = \sum_{i=1}^N V_i$, it follows that $U(P) < V(P)$. It is interesting that, using this functional U, LYZ [16] presented an affirmative answer to the modified Schneider projection problem: If P is an origin-symmetric polytope in $\mathbb{R}^n$, then

$$\frac{V(\Pi P)}{U(P)^{n/2}\, V(P)^{n/2-1}} \leqslant 2^n \left(\frac{n^n}{n!}\right)^{1/2}, \tag{1.3}$$

with equality if and only if P is a parallelotope. Inequality (1.3) provides an asymptotically optimal bound for $V(\Pi K)/V(K)^{n-1}$: If K is a convex body in $\mathbb{R}^n$ that is symmetric about some point, then $V(\Pi K)/V(K)^{n-1} \leqslant 2^n\left(\frac{n^n}{n!}\right)^{1/2}$. See [16, Cor. 4.7] for details.

The finding of the lower bound of the functional U in terms of the volume V makes an interesting story. LYZ [16] conjectured that, for polytopes P with centroid at the origin, there holds

$$U(P) \geqslant \frac{(n!)^{1/n}}{n} V(P), \tag{1.4}$$

with equality if and only if P is a parallelotope.

It took more than a dozen years to completely settle this conjecture. In [10], He et al. proved (1.4) for origin-symmetric polytopes, including its equality condition. In [22], the third author of this article gave a simplified proof for symmetric polytopes and proved (1.4), including the equality case, for two- and three-dimensional polytopes with centroid at the origin. A complete and final solution to this conjecture is attributed to Henk and Linke [11].

In 2015, Böröczky and LYZ [4] extended the domain of the cone-volume functional U to $\mathcal{K}_o^n$, i.e., the set of convex bodies K in $\mathbb{R}^n$ with origin in their interiors, and defined

$$U(K)^n = \frac{1}{n^n} \int_{u_1 \wedge \cdots \wedge u_n \neq 0} h_K(u_1) \cdots h_K(u_n)\, dS_K(u_1) \ldots dS_K(u_n). \tag{1.5}$$

Since $V(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K dS_K$, it follows that $U(K) \leqslant V(K)$. $U(K)$ is still centro-affine invariant, i.e., $U(TK) = U(K)$, for $T \in \mathrm{SL}(n)$. In 2016, Böröczky and Henk [2] proved that the LYZ conjecture is also true for convex bodies with centroid at the origin.

In light of the *volume* attribute of the cone-volume functional U, together with its strong applications, the main goal of this article is to calculate the variation of U. In Sect. 4, we first show that the limit

$$\frac{1}{n} \lim_{\lambda \to 0^+} \frac{U(K + \lambda L) - U(K)}{\lambda}$$

indeed exists. Naturally, we name the limit the *mixed cone-volume functional* $U_1(K, L)$.

It is striking that, when K is strictly convex, $U_1(K, L)$ becomes the classical *Minkowski first mixed volume* $V_1(K, L)$. Recall that $V_1(K, L)$ results from the variation of the volume functional V and is the *most* fundamental and important among all the mixed volumes of the convex bodies K and L. Both the volume and surface area are unified by $V_1(K, L)$ for the special cases of $K = L$ and when L is the unit ball. See Sect. 2 for details.

Observe that $U(K) = V(K)$ when K is strictly convex, but in general $U(K) < V(K)$, in particular when K is a polytope. So, in some sense, we provide a *new* approach to the Minkowski first mixed volume $V_1(K, L)$.

In view of the close resemblance between $U_1(K, L)$ and $V_1(K, L)$, and the significance of the extremal property of $\frac{U(K)}{V(K)}$, we aim to study the extremum of $\frac{U_1(K,L)}{V_1(K,L)}$ in this article. It is interesting that we obtain its sharp bounds in terms of $\frac{U(K)}{V(K)}$, as follows. Moreover, from the sharp lower bound, we can deduce the LYZ conjecture (1.4) *directly*.

Recall that a finite Borel measure μ on $\mathbb{S}^{n-1}$ is said to have *positive subspace mass* if $\mu(\xi \cap \mathbb{S}^{n-1}) > 0$, for some subspace ξ of codimension 1. Refer to [4, p. 409] for more details.

Theorem 1.1 *Suppose that K and L are convex bodies in $\mathbb{R}^n$ with the origin in their interiors. Then*

$$\frac{U_1(K,L)}{V_1(K,L)} \leqslant \left(\frac{V(K)}{U(K)}\right)^{n-1},$$

with equality if and only if V_K does not have positive subspace mass.

Theorem 1.2 *Suppose K and L are convex bodies in $\mathbb{R}^n$ with the origin in their interiors. If the centroid of K is at the origin, then*

$$\frac{U_1(K,L)}{V_1(K,L)} \geqslant \frac{n!}{n^n}\left(\frac{V(K)}{U(K)}\right)^{n-1},$$

with equality if and only if K is a parallelotope and $\operatorname{supp} S_1(K,L,\cdot) \subseteq \operatorname{supp} S_K$.

Letting $L = K$ in Theorem 1.2, we obtain the following LYZ conjecture [16], which was completely settled by Böröczky–Henk [2] in 2016:

Corollary 1.3 *Suppose K is a convex body in $\mathbb{R}^n$ with the origin in its interior. If the centroid of K is at the origin, then*

$$\frac{U(K)}{V(K)} \geqslant \left(\frac{n!}{n^n}\right)^{1/n},$$

with equality if and only if K is a parallelotope.

This article is organized as follows: For quick later reference, we collect some basic facts on convex bodies in Sect. 2. One can also refer to excellent books by Gardner [5], Gruber [9], Schneider [19], and Thompson [21]. In Sect. 3, we show some fundamental properties of *cone-volume measures* which are needed for later use. Then, the main results are proved in Sect. 4.

2 Preliminaries

Write $\mathcal{K}^n$, $\mathcal{K}^n_o$ for the set of convex bodies and the set of convex bodies with the origin in their interiors in $\mathbb{R}^n$, respectively. A *polytope* in $\mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$. Write V for the n-dimensional volume, i.e., n-dimensional Lebesgue measure. For $i < n$, the i-dimensional volume is denoted by vol_i. Let $\mathcal{H}^k$ be the k-dimensional Hausdorff measure.

For $u \in \mathbb{S}^{n-1}$, the *support set* of $K \in \mathcal{K}^n$ in the direction u is defined by

$$F_K(u) = \{x \in K : x \cdot u = h_K(u)\}.$$

K is *strictly convex* if, for each $u \in \mathbb{S}^{n-1}$, the support set $F_K(u)$ contains only one point.

The *Minkowski combination* of $K, L \in \mathcal{K}^n$ is defined by

$$tK + sL = \{tx + sy : x \in K, y \in L\}, \quad t, s \geqslant 0.$$

From the definition of support set, it follows that

$$F_{tK+sL}(u) = tF_K(u) + sF_L(u), \quad u \in \mathbb{S}^{n-1}. \tag{2.1}$$

Denote by $K|\xi$ the orthogonal projection of K onto a subspace ξ of $\mathbb{R}^n$. For any $u \in \mathbb{S}^{n-1} \cap \xi$, we have

$$h_{K|\xi}(u) = h_K(u). \tag{2.2}$$

The *surface area measure* S_K of $K \in \mathcal{K}^n$ is a finite Borel measure on $\mathbb{S}^{n-1}$, defined for the Borel set $\omega \subseteq \mathbb{S}^{n-1}$ by $S_K(\omega) = \mathcal{H}^{n-1}(\nu_K^{-1}(\omega))$, where $\nu_K : \partial' K \to \mathbb{S}^{n-1}$ is the Gauss map of K, defined on $\partial' K$, viz. the set of points of ∂K that have a unique outer unit normal. Recall that $\mathcal{H}^{n-1}(\partial K \setminus \partial' K) = 0$. See, e.g., [19, p. 84] for details.

By the definition of support set, it follows that $S_K(\{u\}) = \mathcal{H}^{n-1}(F_K(u))$, for $u \in \mathbb{S}^{n-1}$.

The *cone-volume measure* V_K of $K \in \mathcal{K}_o^n$ is a finite Borel measure on $\mathbb{S}^{n-1}$, defined for the Borel set $\omega \subseteq \mathbb{S}^{n-1}$ by

$$V_K(\omega) = \frac{1}{n} \int_\omega h_K(u)\, dS_K(u). \tag{2.3}$$

In particular,

$$V(K) = V_K(\mathbb{S}^{n-1}) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K(u)\, dS_K(u). \tag{2.4}$$

Let $K, L \in \mathcal{K}^n$ and L contain the origin. The *mixed cone-volume measure* $V_{K,L}$ of K, L, first defined by Hu and Xiong [12], is

$$V_{K,L}(\omega) = \frac{1}{n} \int_\omega h_L(u)\, dS_K(u), \quad \text{for a Borel set } \omega \subseteq \mathbb{S}^{n-1}. \tag{2.5}$$

Observe that $V_{K,B} = \frac{S_K}{n}$ and $V_{K,K} = V_K$. Thus, the mixed cone-volume measure contains two fundamental measures in geometry: the surface area measure S_K and the cone-volume measure V_K. For its properties and applications, refer to [12].

The *cone-volume functional* of $K \in \mathcal{K}_o^n$ is defined by

$$U(K)^n = \int_{u_1 \wedge \cdots \wedge u_n \neq 0} dV_K(u_1) \dots dV_K(u_n). \tag{2.6}$$

$U(K)$ is positively homogeneous of degree n and $\mathrm{SL}(n)$-invariant, that is,

$$U(tK) = t^n U(K), \quad t > 0, \quad \text{and} \quad U(TK) = U(K), \quad T \in \mathrm{SL}(n).$$

The *mixed area measure* $S(K_1, \dots, K_{n-1}, \cdot)$ of compact convex sets $K_1, \dots, K_{n-1}$ in $\mathbb{R}^n$ is defined by

$$S(K_1, \dots, K_{n-1}, \cdot) = \frac{1}{(n-1)!} \sum_{k=1}^{n-1} (-1)^{n+k-1} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n-1} S_{K_{i_1} + \cdots + K_{i_k}}(\cdot).$$

For simplicity, let

$$S_i(K, L, \cdot) = S(\underbrace{K, \dots, K}_{n-1-i}, \underbrace{L, \dots, L}_{i}, \cdot), \quad i = 0, \dots, n-1.$$

For $K, L \in \mathcal{K}^n$, there is the following Steiner-type formula for the surface area measure:

$$S_{K+\lambda L}(\cdot) = \sum_{j=0}^{n-1} \binom{n-1}{j} S_j(K, L, \cdot)\, \lambda^j, \quad \lambda \geqslant 0. \tag{2.7}$$

The *Minkowski first mixed volume* $V_1(K, L)$ of $K, L \in \mathcal{K}^n$ is defined by

$$V_1(K, L) = \frac{1}{n} \lim_{\lambda \to 0^+} \frac{V(K + \lambda L) - V(K)}{\lambda},$$

which can be represented as the following integral formula:

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L(u)\, dS_K(u) = \frac{1}{n} \int_{S^{n-1}} h_K(u)\, dS_1(K, L, u). \tag{2.8}$$

Observe that $V_1(K, L) = V_{K,L}(\mathbb{S}^{n-1})$ and $V_1(K, K) = V(K)$.

A finite Borel measure μ on $\mathbb{S}^{n-1}$ is said to have *positive subspace mass* if $\mu(\xi \cap \mathbb{S}^{n-1}) > 0$, for some subspace ξ of codimension 1; μ is said to satisfy the *subspace concentration inequality* if, for every subspace ξ of $\mathbb{R}^n$ such that $0 < \dim \xi < n$,

$$\frac{\mu(\xi \cap \mathbb{S}^{n-1})}{\mu(\mathbb{S}^{n-1})} \leqslant \frac{\dim \xi}{n}; \tag{2.9}$$

μ is said to satisfy the *subspace concentration condition* if, in addition to satisfying (2.9) whenever

$$\frac{\mu(\xi \cap \mathbb{S}^{n-1})}{\mu(\mathbb{S}^{n-1})} = \frac{\dim \xi}{n}$$

for some subspace ξ, there exists a subspace ξ' that is complementary to ξ in $\mathbb{R}^n$ such that also

$$\frac{\mu(\xi' \cap \mathbb{S}^{n-1})}{\mu(\mathbb{S}^{n-1})} = \frac{\dim \xi'}{n},$$

or equivalently such that μ is concentrated on $\mathbb{S}^{n-1} \cap (\xi \cup \xi')$. It is worth mentioning that Böröczky–LYZ [3] initially posed this subspace concentration condition and completely solved the existence of the solutions to the even logarithmic Minkowski problem in 2013.

3 Some Properties of Cone-Volume Measures

In this section, we prove some properties of cone-volume measures for later use.

Lemma 3.1 *Suppose that K is a strictly convex body in $\mathbb{R}^n$. Then S_K does not have positive subspace mass.*

Proof It suffices to show that, for any $u \in \mathbb{S}^{n-1}$, there holds $S_K(u^\perp \cap \mathbb{S}^{n-1}) = 0$.

Let $y \in \partial K$. Assume its outer normal vector $\nu_K(y) \in u^\perp$. From (2.2), it follows that

$$h_{K|u^\perp}(\nu_K(y)) = h_K(\nu_K(y)) = y \cdot \nu_K(y) = (y|u^\perp + y|l_u) \cdot \nu_K(y) = (y|u^\perp) \cdot \nu_K(y),$$

where $l_u = \mathrm{span}\{u\}$. So, $y|u^\perp \in \partial(K|u^\perp)$, and $\nu_{K|u^\perp}(y|u^\perp) = \nu_K(y)$.

Thus,

$$\begin{aligned} S_K(u^\perp \cap \mathbb{S}^{n-1}) &= \int_{\substack{y\in\partial K \\ \nu_K(y)\in u^\perp}} d\mathcal{H}^{n-1}(y) \\ &= \int_{\substack{x\in\partial(K|u^\perp) \\ \nu_{K|u^\perp}(x)\in u^\perp}} d\mathcal{H}^{n-2}(x) \int_{(x+tu)\in K} dt \\ &= \int_{\substack{x\in\partial(K|u^\perp) \\ \nu_{K|u^\perp}(x)\in u^\perp}} \mathrm{vol}_1((x+l_u)\cap K)\, d\mathcal{H}^{n-2}(x). \end{aligned}$$

For any $x \in \partial(K|u^\perp)$, since $(x + l_u) \cap K \subseteq F_K(\nu_{K|u^\perp}(x))$ and $F_K(\nu_{K|u^\perp}(x))$ contains only one point by the strict convexity of K, it follows that $\mathrm{vol}_1((x+l_u)\cap K) = 0$. Hence, $S_K(u^\perp \cap \mathbb{S}^{n-1}) = 0$. $\square$

Proposition 3.2 *Suppose that K and L are convex bodies in $\mathbb{R}^n$ and $o \in L$. If K is strictly convex, then $V_{K,L}$ does not have positive subspace mass.*

Proof Let $c = \max_{u \in \mathbb{S}^{n-1}} h_L(u)$. From (2.5) and the strict convexity of K together with Lemma 3.1, it follows that, for each subspace ξ of codimension 1,

$$\begin{aligned} 0 \leqslant V_{K,L}(\xi \cap \mathbb{S}^{n-1}) &= \frac{1}{n} \int_{\xi \cap \mathbb{S}^{n-1}} h_L(u)\, dS_K(u) \leqslant \frac{c}{n} \int_{\xi \cap \mathbb{S}^{n-1}} dS_K(u) \\ &= \frac{c}{n} S_K(\xi \cap \mathbb{S}^{n-1}) = 0, \end{aligned}$$

as desired. □

Taking $L = K$ in Proposition 3.2, we obtain the following:

Corollary 3.3 *Suppose K is a convex body in $\mathbb{R}^n$ with the origin in its interior. If K is strictly convex, then V_K does not have positive subspace mass.*

Theorem 3.4 (Böröczky–Henk [2]) *Suppose K is a convex body in $\mathbb{R}^n$ with centroid at the origin. Then its cone-volume measure V_K satisfies the subspace concentration condition.*

Lemma 3.5 *Suppose P is an n-dimensional polytope in $\mathbb{R}^n$ with centroid at the origin. If, for any $u_1, \ldots, u_{n-1} \in \operatorname{supp} V_P$ with $u_1 \wedge \cdots \wedge u_{n-1} \neq 0$, the implication*

$$\frac{V_P(\operatorname{span}\{u_1, \ldots, u_{n-1}\} \cap \mathbb{S}^{n-1})}{V(P)} = \frac{n-1}{n}$$

holds, then P is a parallelotope.

For the proof of Theorem 3.4 and Lemma 3.5, refer to [2, Thm. 1.1] and [22, Lem. 2.3], respectively.

4 The Variational Formula for the Cone-Volume Functional

Lemma 4.1 *Suppose that K and L are convex bodies in $\mathbb{R}^n$ with the origin in their interiors. Then the limit*

$$\lim_{\lambda \to 0^+} \frac{U(K + \lambda L) - U(K)}{\lambda} \tag{4.1}$$

exists.

Proof Let $K_\lambda = K + \lambda L$, $\lambda \geqslant 0$. From (2.6), (2.3), and (2.7), it follows that

$$U(K_\lambda)^n = \int_{u_1 \wedge \cdots \wedge u_n \neq 0} dV_{K_\lambda}(u_1) \ldots dV_{K_\lambda}(u_n)$$

$$
\begin{aligned}
&= \frac{1}{n^n}\int_{u_1\wedge\cdots\wedge u_n\neq 0}\left(\prod_{i=1}^{n} h_{K_\lambda}(u_i)\right)\left(\prod_{i=1}^{n} dS_{K_\lambda}(u_i)\right)\\
&= \frac{1}{n^n}\int_{u_1\wedge\cdots\wedge u_n\neq 0}\left[\prod_{i=1}^{n}(h_K(u_i)+\lambda h_L(u_i))\right]\\
&\quad\times\left[\prod_{i=1}^{n}\left(\sum_{j=0}^{n-1}\binom{n-1}{j} dS_j(K,L,u_i)\,\lambda^j\right)\right]\\
&= \frac{1}{n^n}\int_{u_1\wedge\cdots\wedge u_n\neq 0} h_K(u_1)\cdots h_K(u_n)\, dS_K(u_1)\dots dS_K(u_n)\\
&\quad + \frac{\lambda}{n^n}\binom{n}{1}\Bigg[\int_{u_1\wedge\cdots\wedge u_n\neq 0} h_L(u_1)h_K(u_2)\cdots h_K(u_n)\, dS_K(u_1)\dots dS_K(u_n)\\
&\quad + \binom{n-1}{1}\int_{u_1\wedge\cdots\wedge u_n\neq 0} h_K(u_1)\cdots h_K(u_n)\\
&\qquad\qquad dS_1(K,L,u_1)\, dS_K(u_2)\dots dS_K(u_n)\Bigg] + \lambda^2 P(\lambda)\\
&= \frac{1}{n^n}\int_{u_1\wedge\cdots\wedge u_n\neq 0} h_K(u_1)\cdots h_K(u_n)\, dS_K(u_1)\dots dS_K(u_n)\\
&\quad + \frac{\lambda}{n^{n-1}}\Bigg(\int_{u_1\wedge\cdots\wedge u_n\neq 0} h_L(u_1)h_K(u_2)\cdots h_K(u_n)\, dS_K(u_1)\dots dS_K(u_n)\\
&\quad + (n-1)\int_{u_1\wedge\cdots\wedge u_n\neq 0} h_K(u_1)\cdots h_K(u_n)\\
&\qquad\qquad dS_1(K,L,u_1)\, dS_K(u_2)\dots dS_K(u_n)\Bigg) + \lambda^2 P(\lambda)\\
&= U(K)^n + \lambda\Bigg(\int_{u_1\wedge\cdots\wedge u_n\neq 0} h_L(u_1)\, dS_K(u_1)\, dV_K(u_2)\dots dV_K(u_n)\\
&\quad + (n-1)\int_{u_1\wedge\cdots\wedge u_n\neq 0} h_K(u_1)\, dS_1(K,L,u_1)\, dV_K(u_2)\dots dV_K(u_n)\Bigg)\\
&\quad + \lambda^2 P(\lambda),
\end{aligned}
$$

where $P(\lambda)$ is a polynomial of degree n^2-2. Thus, $\lim_{\lambda\to 0^+} U(K_\lambda) = U(K)$.

Combining the above with (2.8), it follows that

$$
\begin{aligned}
&\lim_{\lambda\to 0^+}\frac{U(K+\lambda L)-U(K)}{\lambda} = \lim_{\lambda\to 0^+}\frac{U(K_\lambda)-U(K)}{\lambda}\\
&= \frac{1}{n}U(K)^{1-n}\lim_{\lambda\to 0^+}\frac{U(K_\lambda)^n-U(K)^n}{\lambda}\\
&= U(K)^{1-n}\Bigg(\frac{1}{n}\int_{u_1\wedge\cdots\wedge u_n\neq 0} h_L(u_1)\, dS_K(u_1)\, dV_K(u_2)\dots dV_K(u_n)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{n-1}{n}\int_{u_1\wedge\cdots\wedge u_n\neq 0} h_K(u_1)\,dS_1(K,L,u_1)\,dV_K(u_2)\ldots dV_K(u_n)\Bigg)\\
&\leqslant U(K)^{1-n}\Bigg(\frac{1}{n}\int_{(\mathbb{S}^{n-1})^n} h_L(u_1)\,dS_K(u_1)\,dV_K(u_2)\ldots dV_K(u_n)\\
&+\frac{n-1}{n}\int_{(\mathbb{S}^{n-1})^n} h_K(u_1)\,dS_1(K,L,u_1)\,dV_K(u_2)\ldots dV_K(u_n)\Bigg)\\
&= nV_1(K,L)\left(\frac{V(K)}{U(K)}\right)^{n-1}. \qquad (4.2)
\end{aligned}
$$

This completes the proof. □

In light of the relation between the *volume* and the *Minkowski first mixed volume*, we introduce the following notion and naturally name it the *mixed cone-volume functional*.

Definition 4.2 Suppose that K and L are convex bodies in $\mathbb{R}^n$ with the origin in their interiors. The *mixed cone-volume functional* $U_1(K, L)$ of K and L is defined by

$$U_1(K,L)=\frac{1}{n}\lim_{\lambda\to 0^+}\frac{U(K+\lambda L)-U(K)}{\lambda}. \qquad (4.3)$$

If $L = K$, then $U_1(K, K) = U(K)$, which becomes the cone-volume functional U originally introduced by Böröczky and LYZ [4].

Proposition 4.3 *Suppose that K and L are convex bodies in $\mathbb{R}^n$ with the origin in their interiors. Then*

(1) $U_1(tK, sL) = st^{n-1}U_1(K, L)$, *for* $t, s > 0$.
(2) $U_1(TK, TL) = U_1(K, L)$, *for* $T \in \mathrm{SL}(n)$.

Proof By (4.3) and the affine invariance of U, it follows that

$$
\begin{aligned}
U_1(tK,sL)&=\lim_{\lambda\to 0^+}\frac{U(tK+\lambda sL)-U(tK)}{\lambda}\\
&=st^{n-1}\lim_{\lambda\to 0^+}\frac{U(K+\frac{\lambda s}{t}L)-U(K)}{\frac{\lambda s}{t}}=st^{n-1}U_1(K,L).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
U_1(TK,TL)&=\lim_{\lambda\to 0^+}\frac{U(TK+\lambda TL)-U(TK)}{\lambda}\\
&=\lim_{\lambda\to 0^+}\frac{U(K+\lambda L)-U(K)}{\lambda}=U_1(K,L).
\end{aligned}
$$

This completes the proof. □

Since the functional U is not always greater than the volume functional V, we are naturally interested in the size relation between $U_1(K, L)$ and $V_1(K, L)$. In the following, we provide two concrete examples.

Example 1 Let $K=[-1,1]^2,\ L=\mathrm{conv}\{\pm e_1,\pm e_2\}$. By (4.3) and (4.2), we have

$$\begin{aligned}2U_1(K,L)&=U(K)^{-1}\bigg(\frac{1}{2}\int_{u_1\wedge u_2\neq 0}h_L(u_1)\,dS_K(u_1)\,dV_K(u_2)\\&\qquad+\frac{1}{2}\int_{u_1\wedge u_2\neq 0}h_K(u_1)\,dS_L(u_1)\,dV_K(u_2)\bigg)\\&=(2\sqrt{2})^{-1}(8+16)=6\sqrt{2}.\end{aligned}$$

Since $V_1(K,L)=4$, it follows that

$$\frac{U_1(K,L)}{V_1(K,L)}=\frac{3\sqrt{2}}{4}>1.$$

Example 2 Let $K=[-1,1]^2,\ L=[-2,2]\times[-1,1]$. By (4.3) and (4.2), we have

$$2U_1(K,L)=(2\sqrt{2})^{-1}(12+12)=6\sqrt{2}.$$

Since $V_1(K,L)=6$, it follows that

$$\frac{U_1(K,L)}{V_1(K,L)}=\frac{3\sqrt{2}}{6}=\frac{\sqrt{2}}{2}<1.$$

Now, we prove Theorems 1.1 and 1.2, which heavily depend on the upper and lower bounds of $V(K)-V_K(\xi_k\cap\mathbb{S}^{n-1})$, respectively, where $\xi_k=\mathrm{span}\{u_1,\dots,u_k\}$, $u_1,\dots,u_k\in\mathbb{S}^{n-1}$ and $k=1,\dots,2$.

Theorem 4.4 *Suppose that K and L are convex bodies in $\mathbb{R}^n$ with the origin in their interiors. Then*

$$\frac{U_1(K,L)}{V_1(K,L)}\leqslant\left(\frac{V(K)}{U(K)}\right)^{n-1},\tag{4.4}$$

with equality if and only if V_K does not have positive subspace mass.

Proof From (4.3), (4.2), and (2.5), it follows that

$$\begin{aligned}U_1(K,L)=\frac{1}{n}U(K)^{1-n}\bigg(&\int_{u_1\wedge\cdots\wedge u_n\neq 0}dV_{K,L}(u_1)\,dV_K(u_2)\dots dV_K(u_n)\\&+(n-1)\int_{u_1\wedge\cdots\wedge u_n\neq 0}d\mu_{K,L}(u_1)\,dV_K(u_2)\dots dV_K(u_n)\bigg),\end{aligned}$$

where $d\mu_{K,L}=\frac{1}{n}h_K dS_1(K,L,\cdot)$.

In the following, we separately estimate the two integrals. From (2.4) and (2.8), it follows that

$$\int_{u_1\wedge\cdots\wedge u_n\neq 0}dV_{K,L}(u_1)\,dV_K(u_2)\dots dV_K(u_n)$$

$$
\begin{aligned}
&= \int_{u_1\wedge\cdots\wedge u_{n-1}\neq 0} dV_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-1}) \int_{u_n\notin \xi_{n-1}} dV_K(u_n)\\
&= \int_{u_1\wedge\cdots\wedge u_{n-1}\neq 0} (V(K) - V_K(\xi_{n-1}\cap \mathbb{S}^{n-1}))\, dV_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-1})\\
&\leqslant V(K) \int_{u_1\wedge\cdots\wedge u_{n-1}\neq 0} dV_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-1})\\
&= V(K) \int_{u_1\wedge\cdots\wedge u_{n-2}\neq 0} dV_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-2}) \int_{u_{n-1}\notin \xi_{n-2}} dV_K(u_{n-1})\\
&= V(K) \int_{u_1\wedge\cdots\wedge u_{n-2}\neq 0} (V(K) - V_K(\xi_{n-2}\cap \mathbb{S}^{n-1}))\\
&\qquad\qquad dV_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-2})\\
&\leqslant V(K)^2 \int_{u_1\wedge\cdots\wedge u_{n-2}\neq 0} dV_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-2})\\
&\ldots\\
&\leqslant V(K)^{n-2} \int_{u_1\wedge u_2\neq 0} dV_{K,L}(u_1)\, dV_K(u_2)\\
&= V(K)^{n-2} \int_{u_1\neq 0} dV_{K,L}(u_1) \int_{u_2\notin \xi_1} dV_K(u_2)\\
&= V(K)^{n-2} \int_{u_1\in \mathbb{S}^{n-1}} (V(K) - V_K(\xi_1\cap \mathbb{S}^{n-1}))\, dV_{K,L}(u_1)\\
&\leqslant V(K)^{n-1} \int_{u_1\in \mathbb{S}^{n-1}} dV_{K,L}(u_1)\\
&= V(K)^{n-1} V_1(K,L).
\end{aligned}
\tag{4.5}
$$

Similarly, from (2.4) and (2.8), it follows that

$$
\begin{aligned}
&\int_{u_1\wedge\cdots\wedge u_n\neq 0} d\mu_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_n)\\
&= \int_{u_1\wedge\cdots\wedge u_{n-1}\neq 0} d\mu_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-1}) \int_{u_n\notin \xi_{n-1}} dV_K(u_n)\\
&= \int_{u_1\wedge\cdots\wedge u_{n-1}\neq 0} (V(K) - V_K(\xi_{n-1}\cap \mathbb{S}^{n-1}))\, d\mu_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-1})\\
&\leqslant V(K) \int_{u_1\wedge\cdots\wedge u_{n-1}\neq 0} d\mu_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-1})\\
&= V(K) \int_{u_1\wedge\cdots\wedge u_{n-2}\neq 0} d\mu_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-2}) \int_{u_{n-1}\notin \xi_{n-2}} dV_K(u_{n-1})\\
&= V(K) \int_{u_1\wedge\cdots\wedge u_{n-2}\neq 0} (V(K) - V_K(\xi_{n-2}\cap \mathbb{S}^{n-1}))\\
&\qquad\qquad d\mu_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-2})
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant V(K)^2 \int_{u_1\wedge\cdots\wedge u_{n-2}\neq 0} d\mu_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-2})\\
&\ldots\\
&\leqslant V(K)^{n-2} \int_{u_1\wedge u_2\neq 0} d\mu_{K,L}(u_1)\, dV_K(u_2)\\
&= V(K)^{n-2} \int_{u_1\neq 0} d\mu_{K,L}(u_1) \int_{u_2\notin\xi_1} dV_K(u_2)\\
&= V(K)^{n-2} \int_{u_1\in\mathbb{S}^{n-1}} (V(K) - V_K(\xi_1\cap\mathbb{S}^{n-1}))\, d\mu_{K,L}(u_1)\\
&\leqslant V(K)^{n-1} \int_{u_1\in\mathbb{S}^{n-1}} d\mu_{K,L}(u_1)\\
&= V(K)^{n-1} V_1(K,L).
\end{aligned}
\tag{4.6}
$$

Therefore,

$$
\begin{aligned}
U_1(K,L) &\leqslant \frac{1}{n} U(K)^{1-n}(V(K)^{n-1}V_1(K,L) + (n-1)V(K)^{n-1}V_1(K,L))\\
&= \left(\frac{V(K)}{U(K)}\right)^{n-1} V_1(K,L).
\end{aligned}
$$

That is,

$$
\frac{U_1(K,L)}{V_1(K,L)} \leqslant \left(\frac{V(K)}{U(K)}\right)^{n-1},
$$

as desired.

Finally, we consider the equality condition. Assume that the equality in (4.4) holds. Then, each equality in (4.5) has to hold. From the equality condition of the first inequality in (4.5), it follows that $V_K(\mathrm{span}\{u_1,\ldots,u_{n-1}\}\cap\mathbb{S}^{n-1}) = 0$ for any $u_1 \in \operatorname{supp} V_{K,L} = \operatorname{supp} V_K, u_2,\ldots,u_{n-1} \in \operatorname{supp} V_K, u_1\wedge\cdots\wedge u_{n-1} = 0$. Then, for each subspace ξ_{n-1} of dimension $n-1$, $V_K(\xi_{n-1}\cap\mathbb{S}^{n-1}) = 0$; that is, V_K does not have positive subspace mass.

Conversely, if V_K does not have positive subspace mass, then $V_K(\xi_{n-1}\cap\mathbb{S}^{n-1}) = 0$ for each subspace ξ_{n-1} of dimension $n-1$. It follows that $V_K(\xi_k\cap\mathbb{S}^{n-1}) = 0$ for each subspace ξ_k of dimension k, $k = 1,2,\ldots,n-1$. So, each equality in (4.5) and (4.6) holds. Therefore, the equality in (4.4) holds. □

Theorem 4.5 *Suppose that K and L are convex bodies in $\mathbb{R}^n$ with the origin in their interiors. If K is strictly convex, then $U_1(K,L) = V_1(K,L)$.*

Proof Letting $L = K$ in Theorem 4.4, it follows that

$$
\frac{U(K)}{V(K)} \leqslant 1, \tag{4.7}
$$

with equality if and only if V_K does not have positive subspace mass. From Theorem 4.4, the strict convexity of K, together with Corollary 3.3 and (4.7), it follows that

$$\frac{U_1(K,L)}{V_1(K,L)} = \left(\frac{V(K)}{U(K)}\right)^{n-1} = 1,$$

as desired. □

Remark If K is strictly convex, then $U(K) = V(K)$, while for a general convex body L and $\lambda > 0$, $U(K+\lambda L)$ is usually less than $V(K+\lambda L)$.

Indeed, if L has an $(n-1)$-dimensional facet, then there exists a $u \in \mathbb{S}^{n-1}$ such that $\mathcal{H}^{n-1}(F_L(u)) > 0$. By (2.1), it follows that

$$\begin{aligned} S_{K+\lambda L}(\{u\}) &= \mathcal{H}^{n-1}(F_{K+\lambda L}(u)) = \mathcal{H}^{n-1}(F_K(u) + \lambda F_L(u)) \\ &\geqslant \mathcal{H}^{n-1}(\lambda F_L(u)) = \lambda^{n-1}\mathcal{H}^{n-1}(F_L(u)) > 0. \end{aligned}$$

So, $V_{K+\lambda L}(\{u\}) > 0$, and therefore $V_{K+\lambda L}$ has positive subspace mass. From the proof of Theorem 4.5, it follows that $U(K+\lambda L) < V(K+\lambda L)$.

However, if K is strictly convex, there still holds

$$\begin{aligned} U_1(K,L) &= \frac{1}{n} \lim_{\lambda\to 0^+} \frac{U(K+\lambda L) - U(K)}{\lambda} \\ &= \frac{1}{n} \lim_{\lambda\to 0^+} \frac{V(K+\lambda L) - V(K)}{\lambda} = V_1(K,L). \end{aligned}$$

Theorem 4.6 *Suppose that K and L are convex bodies in $\mathbb{R}^n$ with the origin in their interiors. If the centroid of K is at the origin, then*

$$\frac{U_1(K,L)}{V_1(K,L)} \geqslant \frac{n!}{n^n}\left(\frac{V(K)}{U(K)}\right)^{n-1}, \tag{4.8}$$

with equality if and only if K is a parallelotope and $\operatorname{supp} S_1(K,L,\cdot) \subseteq \operatorname{supp} S_K$.

Proof Recall that

$$\begin{aligned} U_1(K,L) = \frac{1}{n} U(K)^{1-n}\Bigg(\int_{u_1\wedge\cdots\wedge u_n\neq 0} & dV_{K,L}(u_1)\, dV_K(u_2)\dots dV_K(u_n) \\ + (n-1)\int_{u_1\wedge\cdots\wedge u_n\neq 0} & d\mu_{K,L}(u_1)\, dV_K(u_2)\dots dV_K(u_n)\Bigg), \end{aligned}$$

where $d\mu_{K,L} = \frac{1}{n} h_K dS_1(K,L,\cdot)$. In the following, we separately estimate the two integrals.

Since the centroid of K is at the origin, it follows that V_K satisfies the subspace concentration condition by Theorem 3.4. So,

$$V_K(\xi_k \cap \mathbb{S}^{n-1}) \leqslant \frac{k}{n} V(K). \tag{4.9}$$

From (2.4), (4.9), and (2.8), it follows that

$$\begin{aligned}
&\int_{u_1\wedge\cdots\wedge u_n\neq 0} dV_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_n)\\
&= \int_{u_1\wedge\cdots\wedge u_{n-1}\neq 0} dV_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-1}) \int_{u_n\notin\xi_{n-1}} dV_K(u_n)\\
&= \int_{u_1\wedge\cdots\wedge u_{n-1}\neq 0} (V(K) - V_K(\xi_{n-1}\cap\mathbb{S}^{n-1}))\, dV_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-1})\\
&\geqslant \frac{V(K)}{n} \int_{u_1\wedge\cdots\wedge u_{n-1}\neq 0} dV_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-1})\\
&= \frac{V(K)}{n} \int_{u_1\wedge\cdots\wedge u_{n-2}\neq 0} dV_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-2}) \int_{u_{n-1}\notin\xi_{n-2}} dV_K(u_{n-1})\\
&= \frac{V(K)}{n} \int_{u_1\wedge\cdots\wedge u_{n-2}\neq 0} (V(K) - V_K(\xi_{n-2}\cap\mathbb{S}^{n-1}))\\
&\qquad dV_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-2})\\
&\geqslant \frac{2!}{n^2} V(K)^2 \int_{u_1\wedge\cdots\wedge u_{n-2}\neq 0} dV_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-2})\\
&\ldots\\
&\geqslant \frac{(n-2)!}{n^{n-2}} V(K)^{n-2} \int_{u_1\wedge u_2\neq 0} dV_{K,L}(u_1)\, dV_K(u_2)\\
&= \frac{(n-2)!}{n^{n-2}} V(K)^{n-2} \int_{u_1\neq 0} dV_{K,L}(u_1) \int_{u_2\notin\xi_1} dV_K(u_2)\\
&= \frac{(n-2)!}{n^{n-2}} V(K)^{n-2} \int_{u_1\in\mathbb{S}^{n-1}} (V(K) - V_K(\xi_1\cap\mathbb{S}^{n-1}))\, dV_{K,L}(u_1)\\
&\geqslant \frac{(n-1)!}{n^{n-1}} V(K)^{n-1} \int_{u_1\in\mathbb{S}^{n-1}} dV_{K,L}(u_1)\\
&= \frac{n!}{n^n} V(K)^{n-1} V_1(K,L).
\end{aligned} \tag{4.10}$$

Similarly, from (2.4) and (2.8), it follows that

$$\begin{aligned}
&\int_{u_1\wedge\cdots\wedge u_n\neq 0} d\mu_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_n)\\
&= \int_{u_1\wedge\cdots\wedge u_{n-1}\neq 0} d\mu_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-1}) \int_{u_n\notin\xi_{n-1}} dV_K(u_n)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{u_1\wedge\cdots\wedge u_{n-1}\neq 0} (V(K) - V_K(\xi_{n-1}\cap\mathbb{S}^{n-1}))\, d\mu_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-1})\\
&\geqslant \frac{V(K)}{n}\int_{u_1\wedge\cdots\wedge u_{n-1}\neq 0} d\mu_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-1})\\
&= \frac{V(K)}{n}\int_{u_1\wedge\cdots\wedge u_{n-2}\neq 0} d\mu_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-2})\\
&\quad\times\int_{u_{n-1}\notin\xi_{n-2}} dV_K(u_{n-1})\\
&= \frac{V(K)}{n}\int_{u_1\wedge\cdots u_{n-2}\neq 0} (V(K) - V_K(\xi_{n-2}\cap\mathbb{S}^{n-1}))\\
&\qquad\qquad d\mu_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-2})\\
&\geqslant \frac{2!}{n^2}\, V(K)^2\int_{u_1\wedge\cdots\wedge u_{n-2}\neq 0} d\mu_{K,L}(u_1)\, dV_K(u_2)\ldots dV_K(u_{n-2})\\
&\ldots\\
&\geqslant \frac{(n-2)!}{n^{n-2}}\, V(K)^{n-2}\int_{u_1\wedge u_2\neq 0} d\mu_{K,L}(u_1)\, dV_K(u_2)\\
&= \frac{(n-2)!}{n^{n-2}}\, V(K)^{n-2}\int_{u_1\neq 0} d\mu_{K,L}(u_1)\int_{u_2\notin\xi_1} dV_K(u_2)\\
&= \frac{(n-2)!}{n^{n-2}}\, V(K)^{n-2}\int_{u_1\in\mathbb{S}^{n-1}} (V(K) - V_K(\xi_1\cap\mathbb{S}^{n-1}))\, d\mu_{K,L}(u_1)\\
&\geqslant \frac{(n-1)!}{n^{n-1}}\, V(K)^{n-1}\int_{u_1\in\mathbb{S}^{n-1}} d\mu_{K,L}(u_1)\\
&= \frac{n!}{n^n}\, V(K)^{n-1} V_1(K,L).
\end{aligned}
\tag{4.11}
$$

Therefore,

$$
\begin{aligned}
U_1(K,L) &\geqslant \frac{1}{n}\, U(K)^{1-n}\left(\frac{n!}{n^n}\, V(K)^{n-1}V_1(K,L) + (n-1)\,\frac{n!}{n^n}\, V(K)^{n-1}V_1(K,L)\right)\\
&= \frac{n!}{n^n}\left(\frac{V(K)}{U(K)}\right)^{n-1} V_1(K,L).
\end{aligned}
$$

That is,

$$
\frac{U_1(K,L)}{V_1(K,L)} \geqslant \frac{n!}{n^n}\left(\frac{V(K)}{U(K)}\right)^{n-1},
$$

as desired.

Finally, we consider the equality condition. Assume the equality in (4.8) holds. Then each equality in (4.10) has to hold. From the equality condition of the last inequality in (4.10), it follows that $V_K(\mathrm{span}\{u\}\cap\mathbb{S}^{n-1}) = \frac{1}{n}\, V(K)$ for any $u \in \operatorname{supp} V_{K,L} = \operatorname{supp} V_K$. So, $\operatorname{supp} V_K$ contains at most $2n$ unit vectors, and hence K is a polytope. From the equality condition of the first inequality in (4.10), it follows that $V_K(\mathrm{span}\{u_1, \ldots, u_{n-1}\} \cap \mathbb{S}^{n-1}) = \frac{n-1}{n} V(K)$ for any $u_1, \ldots, u_{n-1} \in \mathrm{supp}\, V_K$ with $u_1 \wedge \cdots \wedge u_{n-1} \neq 0$. From Lemma 3.5, it follows that K is a parallelotope. Moreover, each equality in (4.11) has to hold. From the equality condition of the last inequality in (4.11), it follows that $V_K(\mathrm{span}\{u\}) = \frac{1}{n} V(K)$ for any $u \in \mathrm{supp}\, S_1(K, L, \cdot)$. Therefore, $\mathrm{supp}\, S_1(K, L, \cdot) \subseteq \mathrm{supp}\, V_K = \mathrm{supp}\, S_K$.

Conversely, suppose K is a parallelotope and $\mathrm{supp}\, S_1(K, L, \cdot) \subseteq \mathrm{supp}\, S_K = \mathrm{supp}\, V_K$. Then $V_K(\mathrm{span}\{u_1, \ldots, u_k\} \cap \mathbb{S}^{n-1}) = \frac{k}{n} V(K)$ for any $u_1, \ldots, u_k \in \mathrm{supp}\, V_K$ with $u_1 \wedge \cdots \wedge u_k \neq 0$ or for any $u_1 \in \mathrm{supp}\, S_1(K, L) \subseteq \mathrm{supp}\, V_K$ and $u_2, \ldots, u_k \in \mathrm{supp}\, V_K$ with $u_1 \wedge \cdots \wedge u_k \neq 0$. It follows that each equality in (4.10) and (4.11) holds. Therefore, the equality in (4.8) holds. □

Remark (1) If $n = 2$, i.e., in $\mathbb{R}^2$, the equality condition in (4.8) becomes "K and L are parallel parallelograms." Indeed, since $S_1(K, L, \cdot) = S_L(\cdot)$, it follows that $\mathrm{supp}\, S_L \subseteq \mathrm{supp}\, S_K = \{\pm u_1, \pm u_2\}$, where $u_1, u_2 \in \mathbb{S}^1$ and $u_1 \neq \pm u_2$. Therefore, K and L are parallel parallelograms.

(2) Let $K = [-1, 1]^3$ and L be the same cube with a corner missing. By (5.22) and [19, Thm. 5.1.8], it follows that $\mathrm{supp}\, S_1(K, L, \cdot) = \mathrm{supp}\, S_K(\cdot) = \{\pm e_1, \pm e_2, \pm e_3\}$. So, in general, the fact that $\mathrm{supp}\, S_1(K, L, \cdot) \subseteq \mathrm{supp}\, S_K(\cdot)$ does not imply that K and L are parallel parallelotopes.

Letting $L = K$ in Theorem 4.6, we immediately obtain the LYZ conjecture on the functional U.

Corollary 4.7 *Suppose that K is a convex body in $\mathbb{R}^n$ with its centroid at the origin. Then*

$$\frac{U(K)}{V(K)} \geqslant \left(\frac{n!}{n^n}\right)^{1/n}, \tag{4.12}$$

with equality if and only if K is a parallelotope.

If the centroid of K is at the origin, combining (4.8), (4.7), (4.4), and (4.12), then

$$\frac{n!}{n^n} \leqslant \frac{U_1(K, L)}{V_1(K, L)} \leqslant \left(\frac{n^n}{n!}\right)^{(n-1)/n}.$$

Problem 4.8 *Suppose that K and L are convex bodies in $\mathbb{R}^n$ with their centroids at the origin. What are the sharp bounds for $U_1(K, L)/V_1(K, L)$?*

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