Computer Science

CSC380: Principles of Data Science

Probability 3

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Review: "probability cheatsheet"

Additivity:

For any *finite* or *countably infinite* sequence of disjoint events $E_1, E_2, E_3, \ldots$, $P\left(\bigcup_{i\geq 1} E_i\right) = \sum_{i\geq 1} P(E_i)$

Inclusion-exclusion rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Law of total probability: For events $B_1, B_2, \ldots$ that partitions Ω, $P(A) = \sum_i P(A \cap B_i)$

Conditional probability: $P(A|B) \coloneqq \dfrac{P(A \cap B)}{P(B)}$ ($P(A|B) \neq P(B|A)$ in general)

Probability chain rule: $P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C)$

Law of total probability + Conditional probability: $P(A) = \sum_i P(A \cap B_i) = \sum_i P(B_i)P(A|B_i) = \sum_i P(A)P(B_i|A)$

Bayes' rule: $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(B|A)P(A)}{P(B)}$

Independence: (definition) A and B are independent if $P(A, B) = P(A)P(B)$

(property) A and B are independent if and only if $P(A|B) = P(A)$ (or $P(B|A) = P(B)$)

Outline

- Random variables
- Distribution functions
  - probability mass functions (PMF)
  - cumulative distribution function (CDF)

- Summarizing distributions: mean and variance
- Example discrete random variables

- Continuous random variables
  - Probability density functions (PDF)
  - Examples

Random Variables

Random variables (RVs)

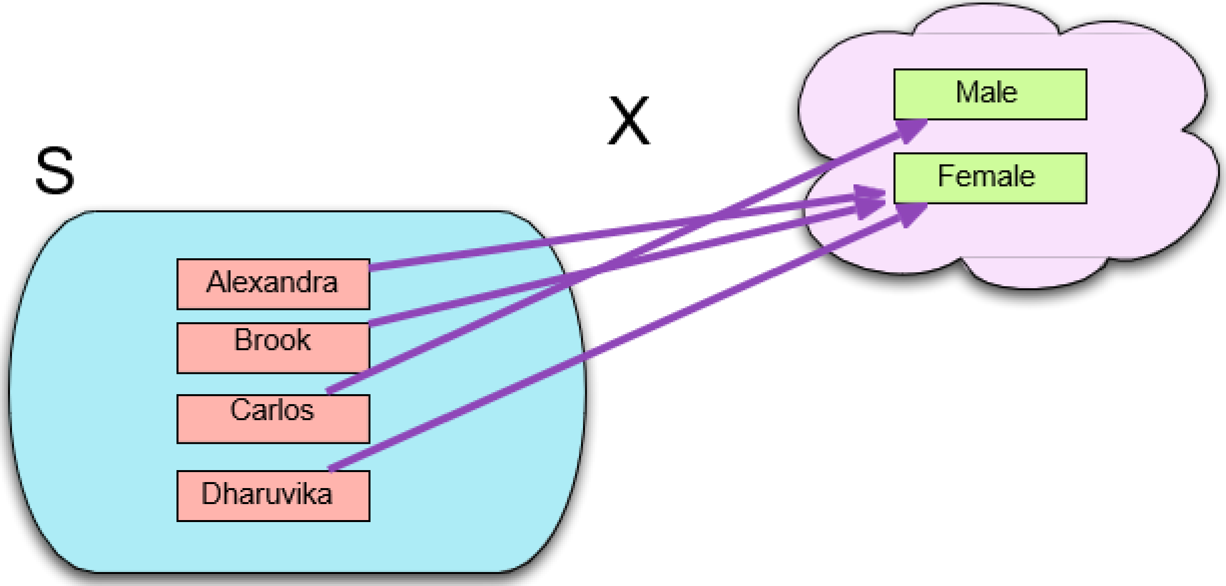
- A single random sample may have more than one characteristic that we can observe (i.e., it may be bi-/multivariate data).
- We can represent each characteristic (e.g., gender, weight, cancer status, etc.) using a separate random variable.

Random Variable

A **random variable** connects each possible outcome in the sample space to some property of interest.

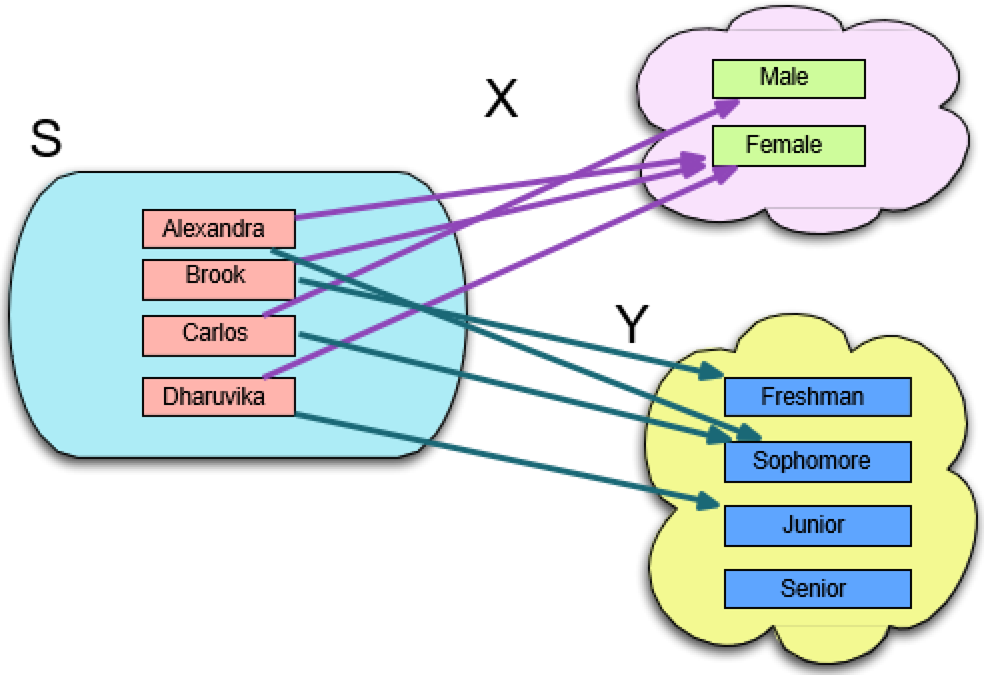
Each value of the random variable (e.g., male or female) has an associated probability.

Random Variable: Example



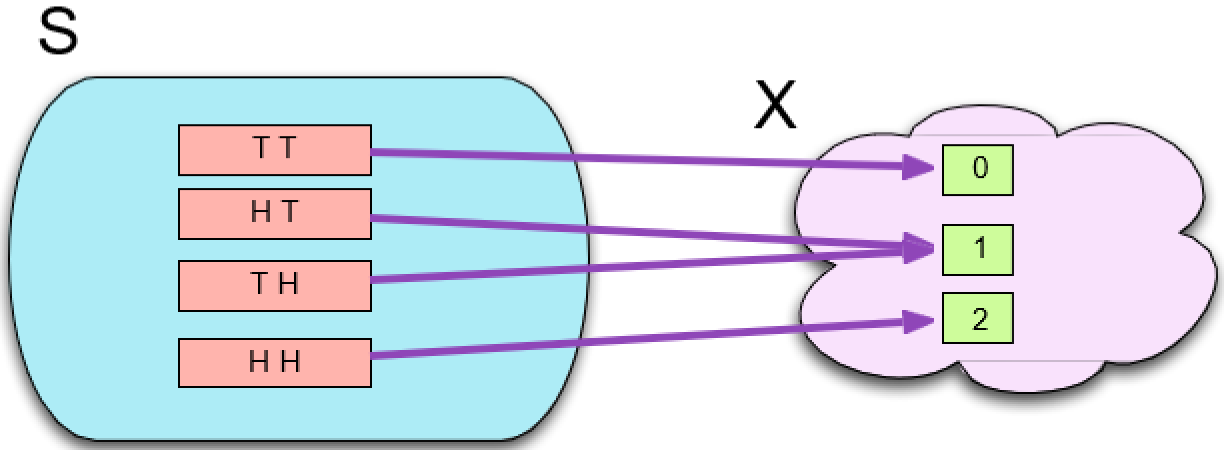
- X: people -> their genders

Random Variable: Example



- Y: people -> their class year

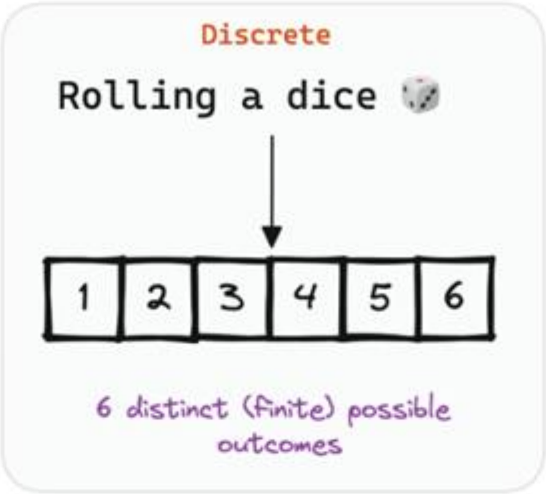
Random Variable: Example

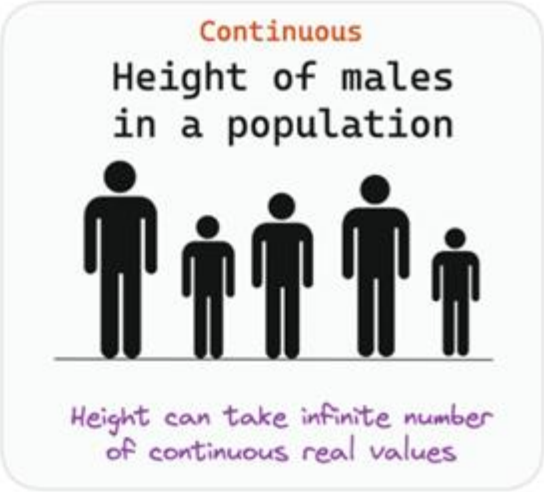


- X: sequence of coin flips -> Number of heads

Types of Random Variables

- Discrete random variable: takes a finite or countable number of distinct values.
- Continuous random variable: takes an infinite number of values within a specified range or interval.





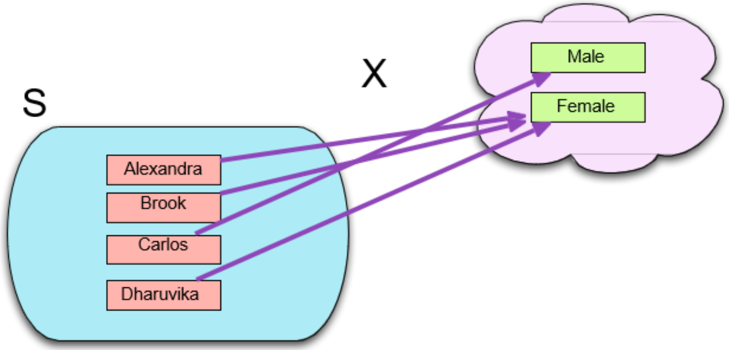
Distribution functions

Discrete distributions

- When a random variable is discrete, its *distribution* is characterized by the probabilities assigned to each distinct value.

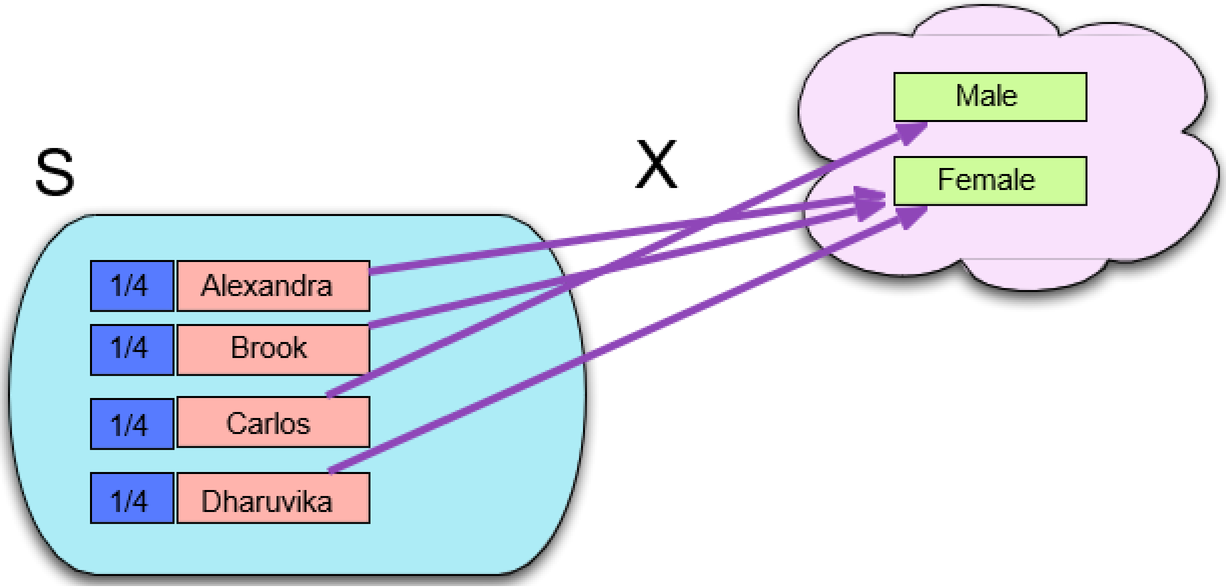
- The probability that the random variable takes a particular value comes from the probability associated with the set of individual outcomes that have that value.
    - This set is an event

- E.g. P(X = Female)



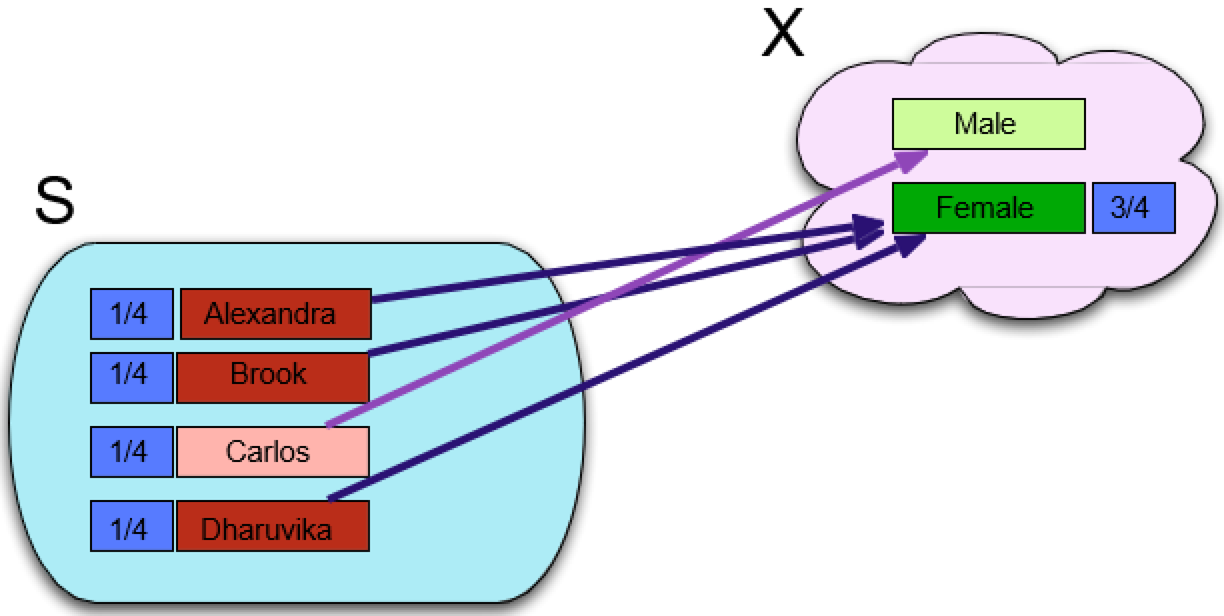
Discrete distributions

- How to find P(X = Female)?



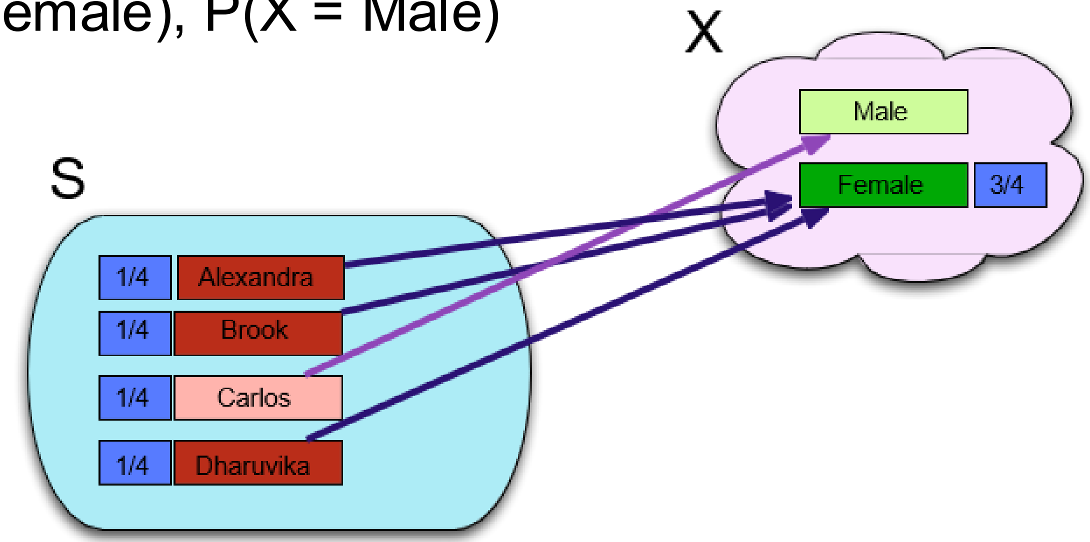
Discrete distributions

- How to find P(X = Female)?



Discrete distributions

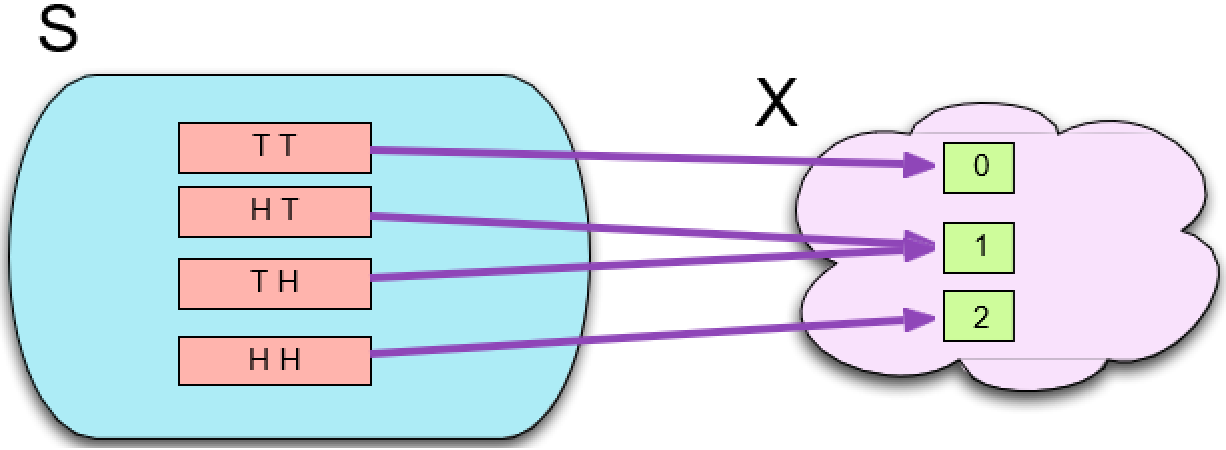
- What is the distribution of random variable X?
    - P(X = Female), P(X = Male)



x	Male	Female
$P(X = x)$	$1/4$	$3/4$

Discrete distributions

- What is the distribution of random variable X?



x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Properties of Discrete Distributions

- We can write P(X = x) to mean “The probability that the random variable X takes the value x”.

- What must be true of these probabilities?

Properties of Discrete Distributions

1. Each $P(X = x)$ is a probability, so must be between 0 and 1.
2. The $P(X = x)$ must sum to 1 over all possible x values.

Probability Mass function (PMF)

The Probability Mass Function

A discrete random variable, X, can be characterized by its **probability mass function**, f (might sometimes write f_X if it's not clear from context which random variable we're talking about).

The PMF takes in values of the variable, and returns probabilities:

$$f(x) \text{ is } \textit{defined} \text{ to be } P(X = x)$$

PMF is a table

- Think of the PMF as a lookup table.

x	Male	Female
$P(X = x)$	$1/4$	$3/4$

- Best way to think of discrete random variables: they take various values, and each value has a certain probability of happening.

Visualizing discrete distributions: spike plot

Flip two coins at the same time, probability distribution of number of heads:

- Often use the spike plot
- Like a bar plot, but with probabilities, instead of frequencies or proportions, on the y-axis.

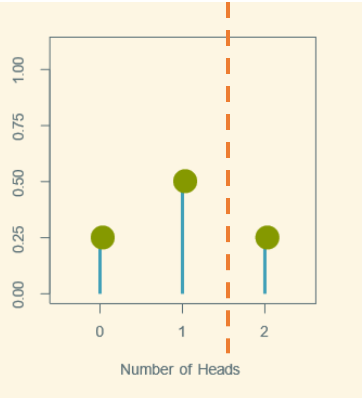
The cumulative distribution function (CDF)

- Often we are interested in the probability of falling in some range of values.
- We can use the cumulative distribution function (CDF), which gives the "accumulated probability" up to a particular value.

The Cumulative Distribution Function

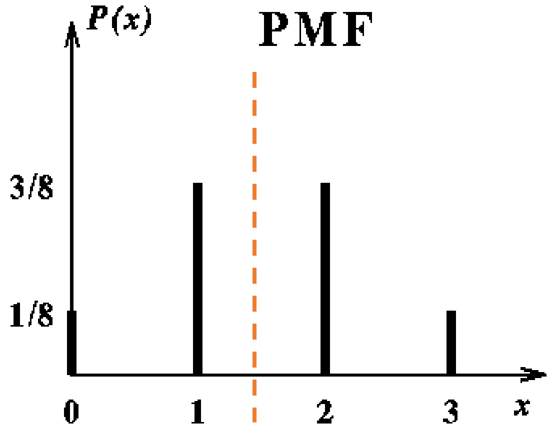
A random variable, X, can be characterized by its **cumulative distribution function**, F (or sometimes F_X if we need to be explicit), which takes values and returns *cumulative* probabilities:

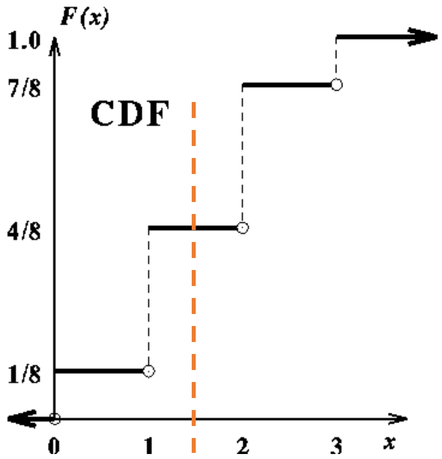
$$F(x) \text{ is defined to be } P(X \leq x)$$



Relating PMF to CDF

- How can we calculate F(x) from the PMF table f?
  - Add up all the probabilities up to and including f(x).
  - What is the value of $F(-0.1)$ (i.e., $P(X \leq -0.1)$)? F(1)?





- For discrete random variables, F(x) *jumps* at locations with nonzero probability mass

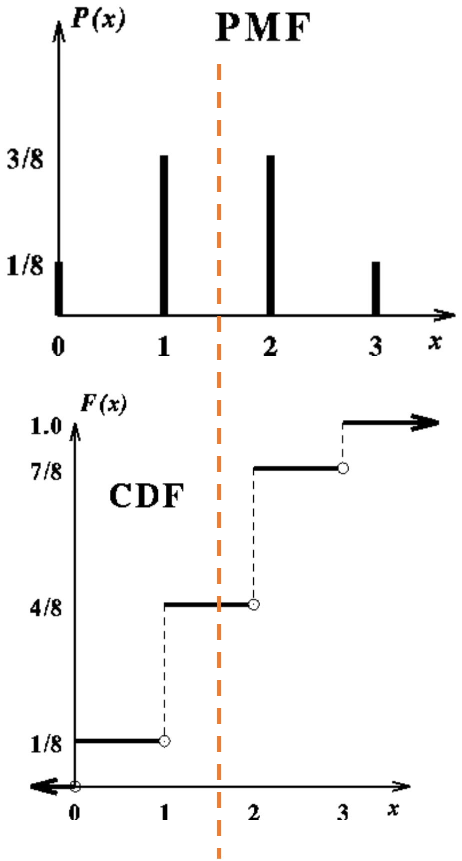
Relating PMF to CDF

- So the PMF of X is:

$$f(x) = \begin{cases} 1/8, & x = 0 \\ 3/8, & x = 1 \\ 3/8, & x = 2 \\ 1/8, & x = 3 \end{cases}$$

- We can write the CDF of X:

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{8}, & 0 \leq x < 1 \\ \frac{1}{2}, & 1 \leq x < 2 \\ \frac{7}{8}, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$



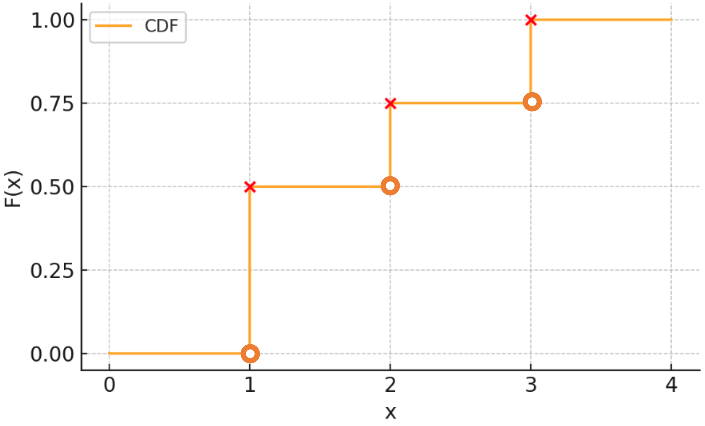
In-class activity

- Given by the PMF of X, find the CDF of X.

$$f(x) = \begin{cases} 1/2, & x = 1 \\ 1/4, & x = 2 \\ 1/4, & x = 3 \end{cases}$$

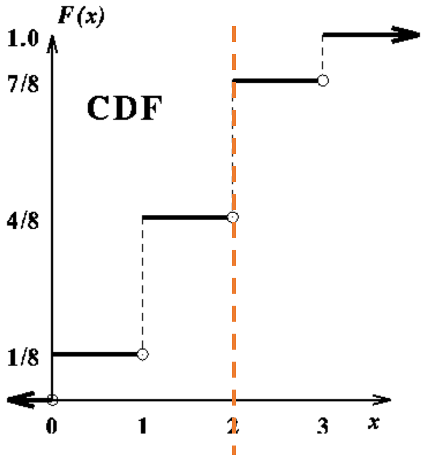
- Answer:

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{2}, & 1 \leq x < 2 \\ \frac{3}{4}, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$



Relating CDF to PMF

- How could we find f(x) from a cumulative distribution function F? e.g., f(2)?



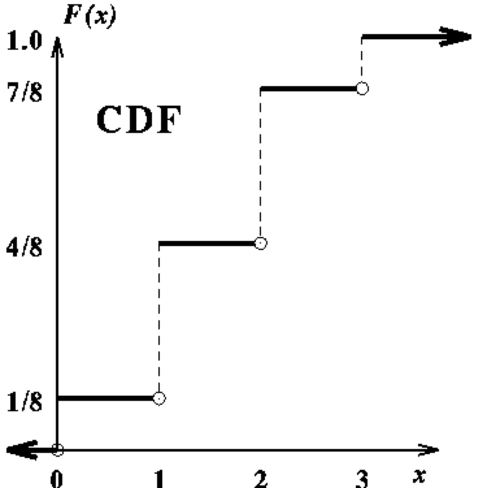
- Focus on "jumps": f(x) = F(x) – F(jump just below x)
  - $\mathrm{f}(2) = F(2) - F(1) = \frac{7}{8} - \frac{4}{8} = \frac{3}{8}$
  - $\mathrm{f}(2.1) = F(2.1) - F(2) = \frac{7}{8} - \frac{7}{8} = 0$
  - $\mathrm{f}(1.5) = F(1.5) - F(1) = \frac{4}{8} - \frac{4}{8} = 0$

Exercise: using CDF and PMF

Given the CDF F:

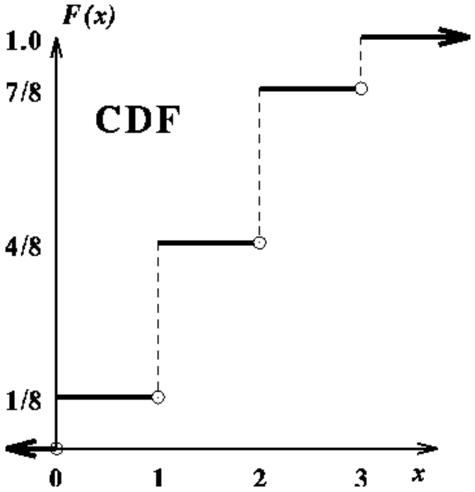
- How to calculate P(X > x)?
  - P(X > x) = 1 – P(X ≤ x) = 1 – F(x)

- How about P(X ≥ x)?
  - P(X ≥ x) = 1 – P(X < x) = 1 – (P(X ≤ x) – P(X=x))
  - 1 – F(x) + f(x)
  - f(x) can be 0 or nonzero, depending on whether x is a jump



Exercise: using CDF and PMF

- What is $P(X \geq 2)$?
  - P(X ≥ x) = 1 – F(x) + f(x)
  - f(x) can be 0 or nonzero, depending on whether x is a jump

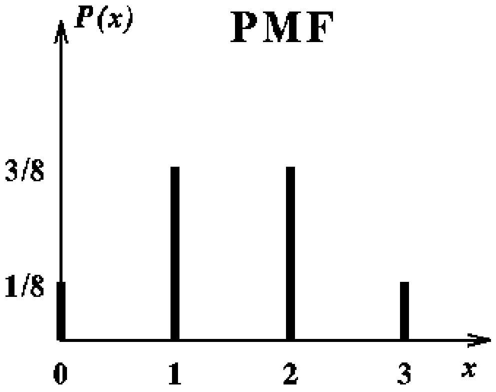


Using the formula:

- $P(X \geq 2) = 1 - F(2) + f(2) = 1 - \frac{7}{8} + \frac{3}{8} = \frac{1}{2}$

Another way:

- $P(X \geq 2) = P(X = 2) + P(X = 3) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$



Exercise: using CDF and PMF

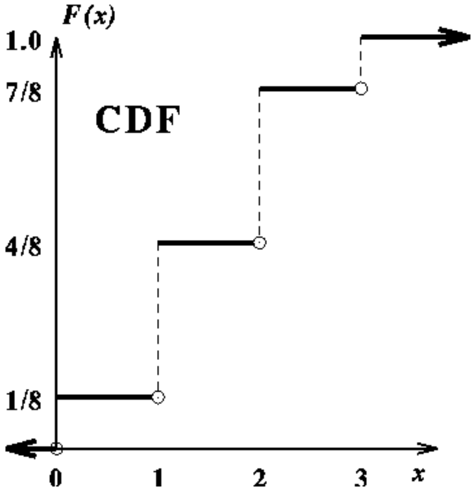
Given the CDF F:

- How to calculate $P(a < X \leq b)$?

  $= P(X \leq b) - P(X \leq a)$

  $= F(b) – F(a)$

- How to calculate $P(a < X < b)$?
  - (I'll leave this to you as an exercise..)



Transformations of random variables

- If X is a random variable, then $X + 5, 3X, X^2, \dots,$ are all random variables
- Given any transformation function f, $f(X)$ is a random variable
- How to find the PMF of $f(X)$ based on that of X?
  - First, find all values $f(X)$ can take
  - For each value c, try to find $\mathrm{P}(f(X) = c)$

Examples

- Suppose X has PMF

x	1	-1
$P(X = x)$	0.5	0.5

- What is the PMF of Y = X + 5?
  - Y can take values 6 and 4
  - P(Y = 6) = P(X = 1) = 0.5
  - P(Y = 4) = P(X = -1) = 0.5

y	6	4
$P(Y = y)$	0.5	0.5

Examples (cont'd)

- Suppose X has PMF

x	1	-1
$P(X = x)$	0.5	0.5

- What is the PMF of $Z = 3X$?

z	3	-3
$P(Z = z)$	0.5	0.5

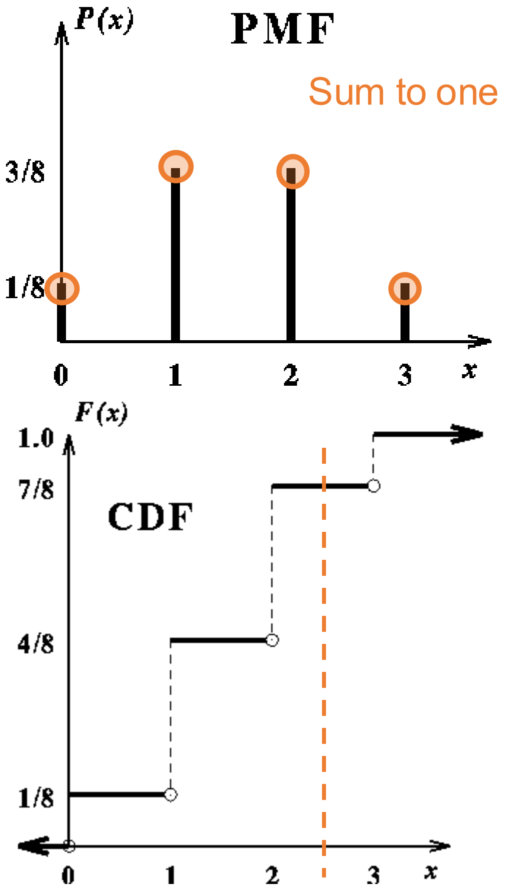
- What is the PMF of $W = X^2$?

w	1
$P(W = w)$	1

Note: $\{W = 1\} = \{X = +1 \text{ or } X = -1\}$

Recap: RV, PMF and CDF

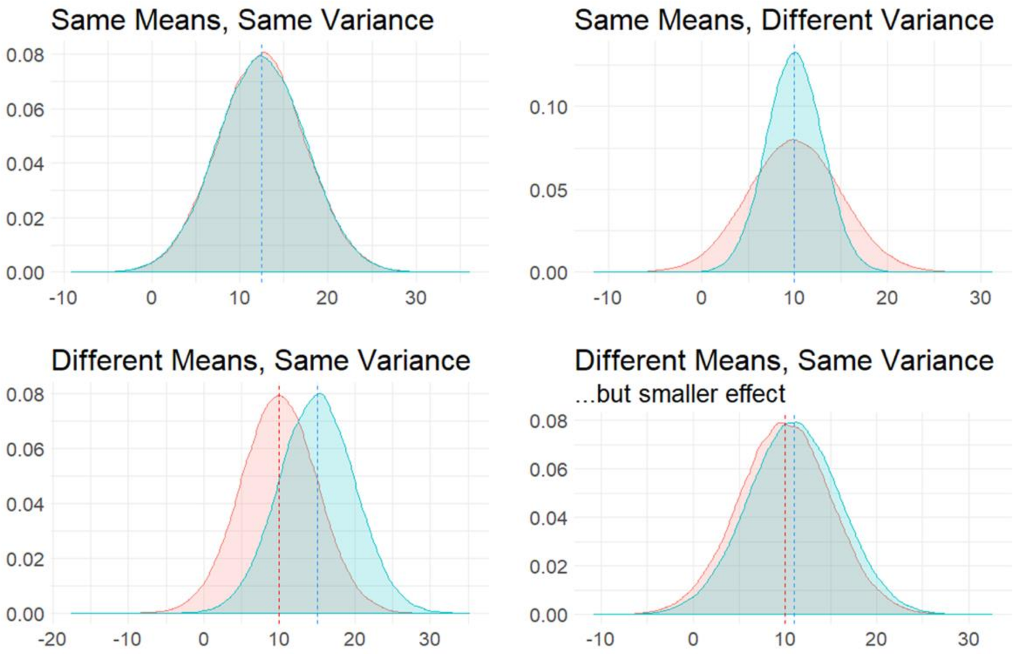
- RV: connects all outcomes to a property of interest
- A RV has a distribution, which assign a probability to each distinct value X can take
- For discrete RV X:
  - PMF: $\mathrm{f}(x)$ defined as $\mathrm{P}(\mathrm{X} = x)$
  - CDF: $\mathrm{F}(x)$ defined as $\mathrm{P}(\mathrm{X} \leq x)$
- Derive CDF from PMF, and vice versa
  - f(x) = F(x) – F(jump just below x)
  - F(x): the total of all jumps (PMF values) at points less than or equal to x
- PMF of $f(X)$
  - First, find all values $f(X)$ can take
  - For each value c, try to find $\mathrm{P}(f(X) = c)$



Mean and Variance

Summarizing random variables

- It is useful to characterize the *center* and *spread* of a probability distribution
  - “what value do we expect to occur?”, and
  - “how confident are we in our prediction?”



Mean (aka expectation, expected value)

- The mean of a random variable X is also called its *expected value*. Usually written as μ or $\mathrm{E}[X]$.

- As with a sample mean, it represents an average over the possible values; and the average is *weighted by the probabilities*.
    - (2 + 2 + 1 + 5)/4 = 2.5
    - 2 * ½ + 1 * ¼ + 5 * ¼ = 2.5

- Makes sense if you were to repeat the random process many times, the average of the observed values of X would approach $\mathrm{E}[X]$. It doesn't mean this value will be observed directly—it's a weighted average.

Example: expected winnings at Roulette

- 38 outcomes (18 red, 18 black, 2 green: 0, 00) equally likely

- Suppose we bet on black. Define X which takes the value 1($) for outcomes where we win, and −1($) for outcomes where we lose.

- Its probability mass function is given by

x	-1	1
$P(X = x)$	$20/38$	$18/38$

Example: expected winnings at Roulette

- X's PMF is

x	-1	1
$P(X = x)$	20/38	18/38

- Its expected value is

$$\mu = -1 \times P(X = -1) + 1 \times P(X = 1)$$
$$= -\frac{2}{38}$$

- expected value per spin is like saying, if I play this game thousands of times, what is my average profit/loss per spin?

Example: expected winnings at Roulette

- In general we have:

Expected Value of a Discrete Random Variable

$$\mu \text{ (aka } E(X)) := \sum_x xP(X = x)$$

Summation is over all values X can take

- Ex: find the mean of the random variable with PMF

x	0	1	2
$P(X = x)$	0.7	0.2	0.1

- Answer: 0 x 0.7 + 1 x 0.2 + 2 x 0.1 = 0.4

Expectation formula

- Given RV X and its PMF, how to find $\mathrm{E}[X+5]$, $\mathrm{E}[3X]$, etc?
- Idea 1: find the PMF of the transformed RV and use the definition of expectation
- Idea 2: use the following fact:

Expectation formula

$$\mathrm{E}[f(X)] = \sum_{x} f(x) \cdot P(X = x)$$

Expectation formula: example

- Suppose X has PMF
- Find: $\mathrm{E}[X+5], \mathrm{E}[X^2]$

x	1	-1
$P(X=x)$	0.5	0.5

Expectation formula

$$\mathrm{E}[f(X)] = \sum_x f(x) \cdot P(X=x)$$

- $\mathrm{E}[X+5] = (1+5) \times 0.5 + (-1+5) \times 0.5 = 5$
- $\mathrm{E}[X^2] = 1^2 \times 0.5 + (-1)^2 \times 0.5 = 1$

Variance

- The variance, written σ^2 or $\mathrm{Var}(X)$ or $\mathrm{E}[(X-\mu)^2]$ is the "expected squared deviation" from the mean.
- It is a weighted average of the squared deviations corresponding to the individual values.

Variance of a Discrete Random Variable

$$\sigma^2 \text{ (aka } Var(X)\text{, aka } E((X-\mu)^2)) = \sum_x (x-\mu)^2 P(X=x)$$

- $\mathrm{E}[(X-\mu)^2]$ – expectation of $(X-\mu)^2$, another RV

Example: Roulette

- X's PMF is

x	-1	1
$P(X = x)$	20/38	18/38

- Its expected value is $\mu = -\frac{2}{38}$

- Its variance is

$$\sigma^2 = (-1-\mu)^2 \cdot P(X=-1) + (1-\mu)^2 \cdot P(X=1)$$
$$= \left(-1 - \left(-\frac{2}{38}\right)\right)^2 \times \frac{20}{38} + \left(1 - \left(-\frac{2}{38}\right)\right)^2 \times \frac{18}{38}$$
$$= \ldots \approx 0.997$$

Standard deviation

- Just as with a sample, the standard deviation, σ, is the square root of the variance.

- E.g. in the roulette example, $\sigma = \sqrt{0.997} \approx 0.998$
  - In one spin, the "typical" variation of our balance is 0.998

Exercise

- Find the mean and variance for the random variable with PMF given by

x	0	1	2
$P(X = x)$	0.7	0.2	0.1

Ans:

- $\mu = 0 \times 0.7 + 1 \times 0.2 + 2 \times 0.1 = 0.4$
- $\sigma^2 = 0.4^2 \times 0.7 + 0.6^2 \times 0.2 + 1.6^2 \times 0.1$

  $= 0.44$
- For a random variable X, when is its σ^2 zero?

Properties of expectation

- What will happen to the roulette game if we bet \$2 instead of \$1?
- The new PMF becomes
- The new expected winnings are then

x	-2	2
$P(X = x)$	20/38	18/38

$$\mu = -2 \times P(X = -2) + 2 \times P(X = 2)$$
$$= -\frac{4}{38}$$

- What's the relationship between this value and the old expected value?
  - Doubling the individual values (w/o changing probs) doubles the expected value

Properties of expectation

- This works in general: if we change the values of a random variable by multiplying by a constant, the expectation gets multiplied by a constant.

- To see this, recall the expectation formula:

$$\mathrm{E}[f(X)] = \sum_x f(x) \cdot P(X = x)$$

$$E[aX] = \sum_x ax\, P(X = x) = a \sum_x x\, P(X = x) = aE[X]$$

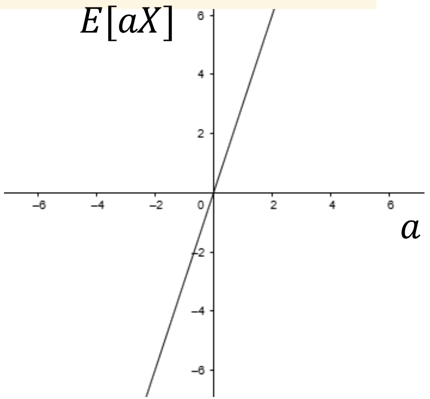
Properties of expectation

Property of Expectation

Multiplying a random variable by a constant scales the expected value by the same constant:

$$E(aX) = aE(X)$$

- Sometimes called "linearity of expectation"



Properties of Variance

- What will happen to the variance if we multiply every value of a random variable by a constant a?

- This is as if we increase our bet in the roulette game

x	-2	2
$P(X = x)$	$20/38$	$18/38$

- Variance = expected *squared* deviation
- All squared deviations are scaled by a^2, making variance also scaled by a^2

Properties of Variance

- Its old variance is

$$\sigma^2 = (-1-\mu)^2 \cdot P(X=-1) + (1-\mu)^2 \cdot P(X=1)$$
$$= \left(-1-\left(-\frac{2}{38}\right)\right)^2 \times \frac{20}{38} + \left(1-\left(-\frac{2}{38}\right)\right)^2 \times \frac{18}{38}$$
$$= \ \ldots \approx 0.997$$

- Its new variance is

$$\sigma^2 = (-2-2\mu)^2 \cdot P(X=-1) + (2-2\mu)^2 \cdot P(X=1)$$
$$= 4 \times \left(-1-\left(-\frac{2}{38}\right)\right)^2 \times \frac{20}{38} + 4 \times \left(1-\left(-\frac{2}{38}\right)\right)^2 \times \frac{18}{38}$$
$$= \ \ldots \approx 4 \times 0.997$$

Properties of Variance

Property of Variance

If the values of a random variable are multiplied by a constant, a, then the variance gets multiplied by a^2.

- In other words, $\mathrm{Var}(aX) = a^2\mathrm{Var}(X)$
- How would standard deviation change accordingly?
  - scaled by $|a|$ (!)

Properties of Variance

Alternative formula for finding variance

$$\mathrm{Var}(X) = \mathrm{E}[X^2] - (\mathrm{E}[X])^2$$

This sometimes simplifies calculations quite a bit

Example X has PMF

x	-1	1
$P(X = x)$	20/38	18/38

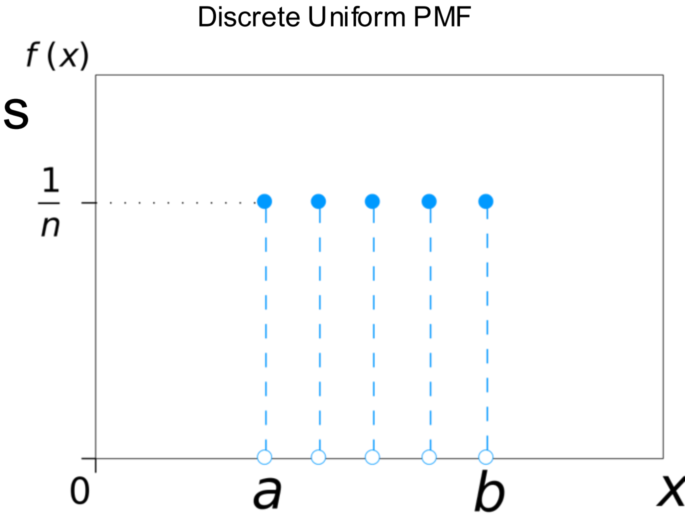
- $\mathrm{E}[X^2] = 1$
- $\mathrm{E}[X] = -\frac{2}{38}$
- $\Rightarrow \mathrm{Var}(X) = 1 - \left(\frac{2}{38}\right)^2 = 0.997$

Example Discrete Random Variables

Uniform distribution over a set

More generally, consider $S = \{v_1, v_2, \dots, v_N\}$; X is drawn from the uniform distribution of S, then

$$P(X = k) = \begin{cases} \frac{1}{N} & \text{if } k \in \{v_1, v_2, \dots, v_N\} \\ 0 & \text{otherwise} \end{cases}$$



We denote this by $X \sim \mathrm{Uniform}(S)$

- Selecting a student from a class
- Drawing a card from a shuffled deck
- Choosing a letter from the alphabet

numpy.random

To generate a sample from a uniform discrete distribution,

```
random.choice(a, size=None, replace=True, p=None)
```

Generates a random sample from a given 1-D array

numpy.random.choice([2,5,6])
Example output: 2

Binomial distribution

- Suppose we perform n repeated independent trials, each with success probability p, what is the distribution of the number of successes X?

- What values can X take?

$$m = 0,1,\ldots,n$$

- We have seen that $P(X = m) = \binom{n}{m} \cdot p^m (1-p)^{n-m}$

- In this case, X is said to be drawn from a *binomial distribution,* denoted by

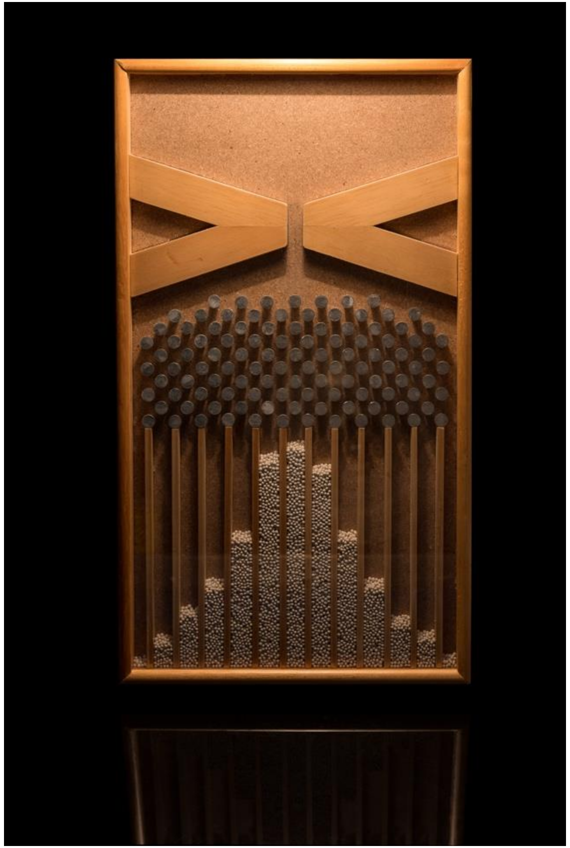
$$X \sim \mathrm{Bin}(n, p)$$

Galton Boards

- Illustration of binomial distribution
- Bead has 10 chances hitting pegs (10 rows of pegs)
- each time a peg is hit, bead randomly bounces to the left or the right with equal probabilities

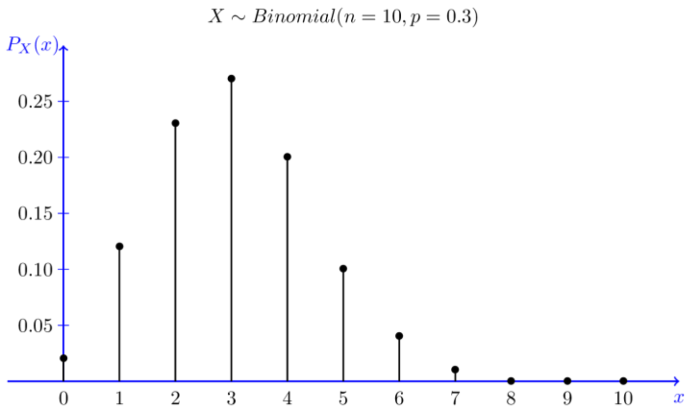
- Number of times it bounces to the left:

$$X \sim \mathrm{Bin}(10, 0.5)$$



Binomial distribution

- $X \sim \mathrm{Bin}(n, p)$
- X 's PMF is "Bell-shaped"



Facts:

- $\mathrm{E}[X] = E[n \cdot X_i] = n \cdot E[X_i] = np$

- $\mathrm{Var}[X] = np(1 - p)$
  - Small when p is close to 0 or 1

Bernoulli distribution

- What does $X \sim \mathrm{Bin}(1, p)$ mean?

x	0	1
$P(X = x)$	1-p	p

- This is called the Bernoulli distribution with parameter p, abbreviated as $\mathrm{Bernoulli}(p)$
- $\mathrm{E}[X] = 0 \cdot (1 - p) + 1 \cdot p = p$

Geometric distribution

- Suppose we perform repeated independent trials with success probability p. What is the distribution of X, the number of trials needed to get a success? (related to Q4 in HW3)

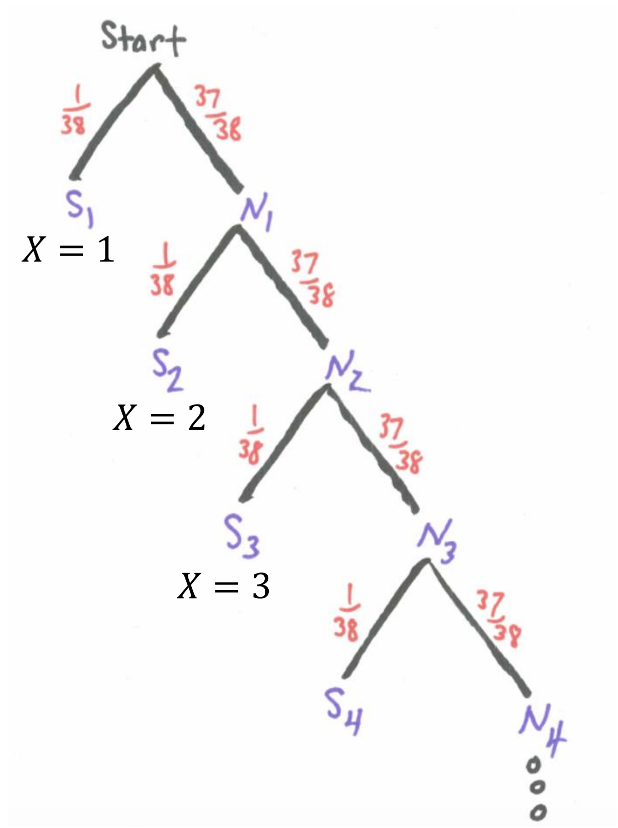
- Applications:
  - Call center: # calls before encountering first dissatisfied customer
  - Basketball: # shots before scoring the first
  - Networking: # attempts before a successful transmission
  - Gambling: # plays before first win

Geometric distribution

- How to find $\mathrm{P}(X = x)$?
- Let's draw a probability tree..

- Example: $p = \frac{1}{38}$ (roulette)

- $\mathrm{P}(X = 1) = p$
- $\mathrm{P}(X = 2) = (1 - p)\,p$
- $\mathrm{P}(X = 3) = (1 - p)^2\,p$
- …



https://randombooks.org/geometric-distribution.html

Geometric distribution

- In conclusion,

$$\mathrm{P}(X = x) = p\,(1 - p)^{x-1}$$

for $x = 1,2,\ldots$

Fact:

- $\mathrm{E}[X] = \frac{1}{p}$
- $\mathrm{Var}[X] = \frac{1-p}{p^2}$
  - Smaller when p closes to 1