Computer Science

CSC380: Principles of Data Science

Probability 2

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Rules of probability

- To recap and summarize:

Rules of Probability

1. **Non-negativity:** All probabilities are between 0 and 1 (inclusive)
2. **Unity of the sample space:** $P(S) = 1$
3. **Complement Rule:** $P(E^C) = 1 - P(E)$
4. **Probability of Unions:**
   (a) *In general,* $P(E \cup F) = P(E) + P(F) - P(E \cap F)$
   (b) If E and F are disjoint, then $P(E \cup F) = P(E) + P(F)$

Summary: calculating probabilities

- If we know that all outcomes are equally likely, we can use

We will use combinatorics to do counting

$$P(E) = \frac{|E|}{|S|}$$

Number of elements in event set

Number of possible outcomes (e.g. 36)

- If |E| is hard to calculate directly, we can try
  - the rules of probability
  - the Law of Total Probability, using an appropriate partition of sample space S

Overview

- Conditional probability
- Probabilistic reasoning
  - contingency table
  - probability trees

Conditional Probability

Example: Seat Belts

		Child Buck.	Child Unbuck.	Marginal
Parent	Buck.	0.48	0.12	0.60
	Unbuck.	0.10	0.30	0.40
	Marginal	0.58	0.42	1.00

Table: Probability Estimates for Seat Belt Status

Suppose we pick a family from US at random:

- What is the probability of the event "Child is Buckled"?
- What should our new estimate be if we know that "Parent is Buckled"?

Example: blood types

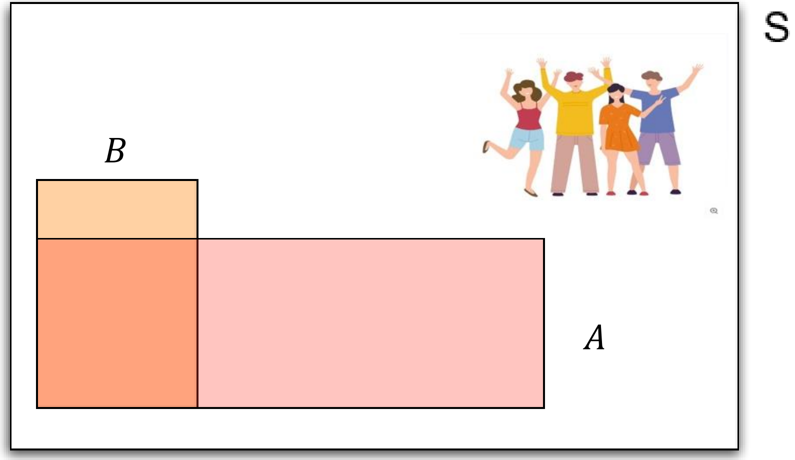
		Antigen B Absent	Antigen B Present	Marginal
Antigen A	Absent	0.44	0.10	0.54
	Present	0.42	0.04	0.46
	Marginal	0.86	0.14	1.00

Table: Probability Estimates for U.S. Blood Types

- A: "presence of antigen A", B: "presence of antigen B"
- Suppose someone of an unknown blood type gets a test that reveals the presence of antigen A. What is the chance that:
  - event A happens to them?
  - event B happens to them?

Relative area

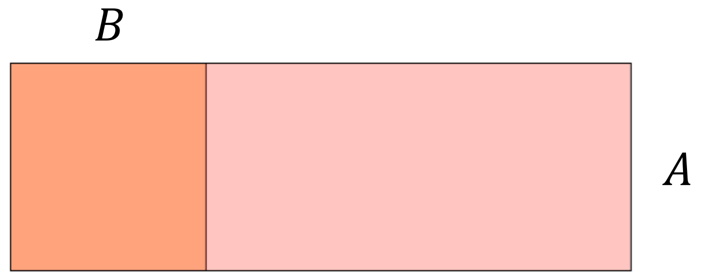
- A: antigen A present B: antigen B present
- Given that A happens, what is the chance of B happening?



- Restricted to people with antigen A present, what is the fraction of those people with antigen B?

Relative area

- Let’s zoom into people with antigen A present.

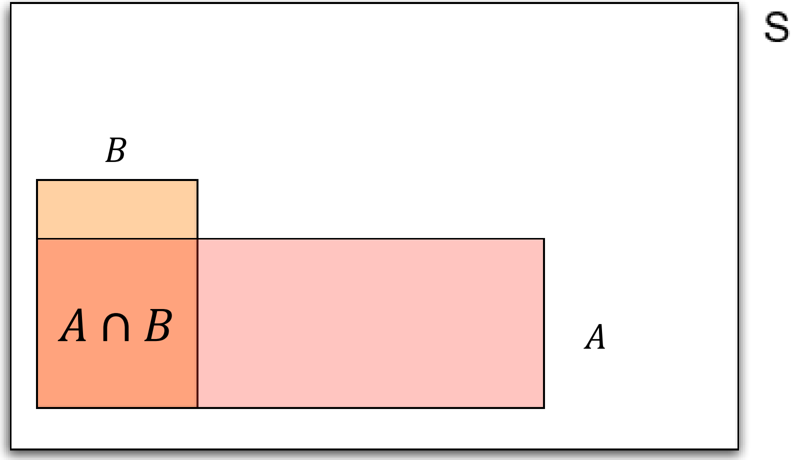


- It’s just as if the sample space had shrunk to include only A
- Now, probabilities correspond to proportions of A
- What does the orange square represent?
  - $A \cap B$
- How would we find the probability of B given A?

Conditional Probability

- To find the conditional probability of B given A, consider the ways B can occur in the context of A (i.e., $A \cap B$), out of all the ways A can occur:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$



Example:
A: currently inside a cafe
B: drinking coffee right now

Conditioning changes the sample space

- Before we knew anything, anything in sample space S could occur.
- After we know A happened, we are only choosing from within A.
- The set A becomes our new sample space
- Instead of asking "In what proportion of S is B true?", we now ask "In what proportion of A is B true?"

For example, rolling a fair die, define A: even numbers, B: get a 2.

- Before knew anything, P(B) is 1/6
- After knowing A, P(B) is (1/6) / (1/2) = 1/3

Every Probability is a Conditional Probability

- We can consider the original probabilities to be conditioned on the event S: at first what we know is that "something in S" occurs.

$$P(B) = P(B|S)$$

$$P(B \cap C) = P(B \cap C|S)$$

$$P(\,B \mid S\,) = \frac{P(B \cap S)}{P(S)} = P(B)$$

- $P(B|S)$ in words: what proportion of S does B happen?
- If we then learn that A occurs, A becomes our restricted sample space.
- $P(B|A)$ in words: what proportion of A does B happen?

Joint Probability and Conditional Probability

- We can rearrange $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$ and derive:

The "Chain Rule" of Probability

For any events, A and B, the joint probability $P(A \cap B)$ can be computed as

$$P(A \cap B) = P(B \mid A) \times P(A)$$

Or, since $P(A \cap B) = P(B \cap A)$

$$P(A \cap B) = P(A \mid B) \times P(B)$$

Terminology

When we have two events A and B…

- Conditional probability: $P(A|B), P(A^c|B), P(B|A)$ etc.

- Joint probability: $P(A, B)$ or $P(A^c, B)$ or ...

- Marginal probability: $P(A)$ or $P(A^c)$

Example revisited: blood types

		Antigen B Absent	Antigen B Present	Marginal
Antigen A	Absent	0.44	0.10	0.54
	Present	0.42	0.04	0.46
	Marginal	0.86	0.14	1.00

Table: Probability Estimates for U.S. Blood Types

- Suppose someone of an unknown blood type gets a test that reveals the presence of antigen A.
  - What is $P(\, A \mid A\,)$?

$$P(\, A \mid A\,) = \frac{P(A \cap A)}{P(A)} = 1$$

  - What is $P(\, B \mid A\,)$?

$$P(\, B \mid A\,) = \frac{P(A \cap B)}{P(A)} = \frac{0.04}{0.46} = 0.087$$

Example revisited: Seat Belts

A: parent is buckled
C: child is buckled

		Child Buck.	Child Unbuck.	Marginal
Parent	Buck.	0.48	0.12	0.60
	Unbuck.	0.10	0.30	0.40
	Marginal	0.58	0.42	1.00

Table: Probability Estimates for Seat Belt Status

Suppose we pick a family from US at random:

- What is the probability of the event "Child is Buckled"? $P(C)$
- What should our new estimate be if we know that ("given that") Parent is Buckled? $P(C \mid A)$

Example revisited: Seat Belts

A: parent is buckled
C: child is buckled

		Child		
		Buck.	Unbuck.	Marginal
Parent	Buck.	0.48	0.12	0.60
	Unbuck.	0.10	0.30	0.40
	Marginal	0.58	0.42	1.00

Table: Probability Estimates for Seat Belt Status

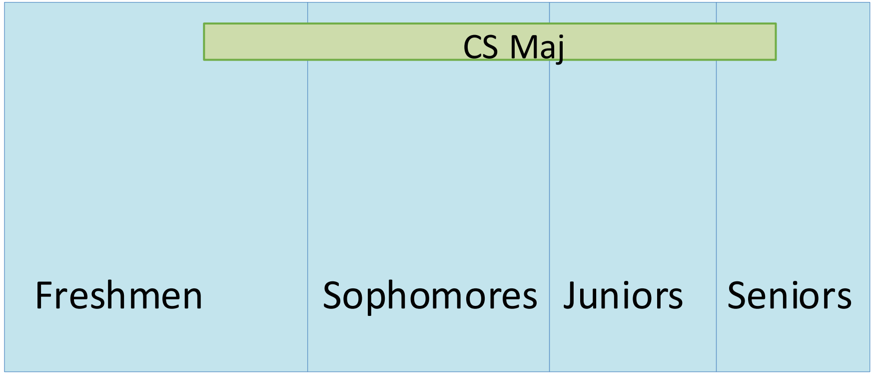
Suppose we pick a family from the US at random:

- $P(C) = 0.58$
- $P(\, C \mid A \,) = \frac{P(C \cap A)}{P(A)} = \frac{0.48}{0.60} = 0.8$ Larger than $P(C)$
- Suppose we see a buckled parent, it is much more likely that we see their child buckled

Law of Total Probability, revisited

Law of Total Probability Suppose $B_1, \ldots, B_n$ *form a partition* of the sample space S. Then,

$$P(A) = P(A, B_1) + \cdots + P(A, B_n)$$



Law of Total Probability, revisited

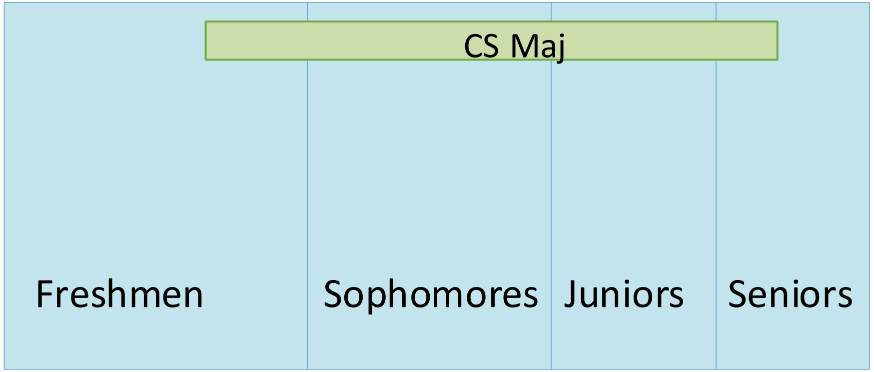
Expanding each $P(A, B_i) = P(A \mid B_i)P(B_i)$, we have:

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i)P(B_i)$$

A: student in CS major

B_i : student in class year i

$P(A \mid B_i)$ The fraction of CS major in class year i



Law of Total Probability, revisited

Example Suppose UA has an equal number of students in the 4 class years, and the fraction of CS major in these 4 class years are 10%, 10%, 20%, 80% respectively. What is fraction of CS majors?

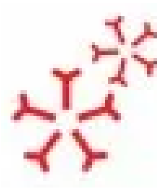
- $P(B_1) = P(B_2) = P(B_3) = P(B_4) = 0.25$
- $P(C \mid B_1) = 0.1, \ldots, P(C \mid B_4) = 0.8$
- Calculate $P(C)$ by:

$$P(C) = \sum_{i=1}^{4} P(C \mid B_i) P(B_i) = 30\%$$

Probabilistic reasoning

Probabilistic reasoning

- We have some prior belief of an event A happening
  - $P(A)$, prior probability
  - e.g. me infected by COVID



- We see some new evidence B
  - e.g. I test COVID positive

- How does seeing B affect our belief about A?
  - $P(A \mid B)$, posterior probability

Another example: detector

A store owner discovers that some of her employees have taken cash. She decides to use a detector to discover who they are.

- Suppose that 10% of employees stole.
- The detector buzzes 80% of the time that someone stole, and 20% of the time that someone not stole
- Is the detector reliable? In other words, if the detector buzzes, what’s the probability that the person did stole?

H: employee not stole

B: lie detector buzzes

Another example: detector

- Suppose that 10% of employees stole.

  H: employee did not stole $P(H) = 0.9$

- The detector buzzes 80% of the time that someone stoles, and 20% of the time that someone not stole.

  $$P(B \mid H^C) = 0.8$$

  B: lie detector buzzes

  $$P(B \mid H) = 0.2$$

- If the detector buzzes, what's the probability that the person stole?

$$P(H^C \mid B)$$

Detector analysis: Probability table

		Detector result		
		Pass (B^C)	Buzz (B)	Marginal
Employee	Not stole (H)			
	Stole (H^C)			
	Marginal			

$$P(H) = 0.9$$

$$P(B \mid H^C) = 0.8$$

$$P(B \mid H) = 0.2$$

Detector analysis: Probability table

$$P(H,B) = P(H) \cdot P(B \mid H) = 0.9 \times 0.2 = 0.18$$

		Detector result		
		Pass (B^C)	Buzz (B)	Marginal
Employee	Not stole (H)		0.18	0.9
	Stole (H^C)			0.1
	Marginal			

$$P(H) = 0.9$$

$$P(B \mid H^C) = 0.8$$

$$P(B \mid H) = 0.2$$

Detector analysis: Probability table

$$P(H) = P(H,B) + P(H,B^c) = 0.9$$

		Detector result		
		Pass (B^C)	Buzz (B)	Marginal
Employee	Not stole (H)	0.72	0.18	0.9
	Stole (H^C)			0.1
	Marginal			

$$P(H) = 0.9$$

$$P(B \mid H^C) = 0.8$$

$$P(B \mid H) = 0.2$$

Detector analysis: Probability table

		Detector result		
		Pass (B^C)	Buzz (B)	Marginal
Employee	Not stole (H)	0.72	0.18	0.9
	Stole (H^C)	0.02	0.08	0.1
	Marginal	0.74	0.26	1

$$P(H) = 0.9$$

$$P(B \mid H^C) = 0.8$$

$$P(B \mid H) = 0.2$$

Detector analysis: Probability table

		Detector result		
		Pass (B^C)	Buzz (B)	Marginal
Employee	Not stole (H)	0.72	0.18	0.9
	Stole (H^C)	0.02	0.08	0.1
	Marginal	0.74	0.26	1

- We have the full probability table. Can we calculate $P(H^C \mid B)$? Yes!

$$P(H^C \mid B) = \frac{P(H^C,B)}{P(B)} \qquad \frac{0.08}{0.26} = 0.307$$

It seems like the detector is not very reliable…

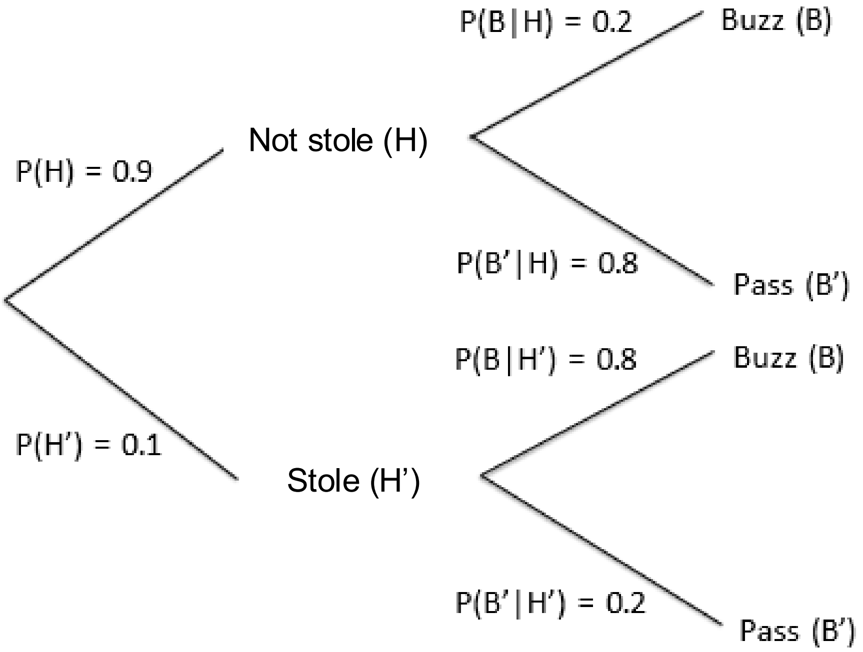
Recap

- Conditional probability: $P(\, B \mid A\,) = \frac{P(A \cap B)}{P(A)}$
- Law of total probability: $P(A) = \sum_{i=1}^{n} P(\, A \mid B_i\,)P(B_i)$
- If we know $\mathrm{P}(H), P(B|H^C), P(B|H)$:
  - $\mathrm{P}(H) \rightarrow P(H^C)$ Complement rule
  - $\mathrm{P}(H), P(B|H) \rightarrow P(B, H)$ Conditional probability
  - $\mathrm{P}(H^C), P(B|H^C) \rightarrow P(B, H^C)$ Conditional probability
  - $\mathrm{P(B)} \rightarrow P(B, H) + P(B, H^C)$ Law of total probability
  - $P(B), P(B, H) \rightarrow P(H|B)$ Conditional probability
  - $P(B), P(B, H^C) \rightarrow P(H^C|B)$ Conditional probability
- We can get $P(B), P(H|B), P(H^C|B)$

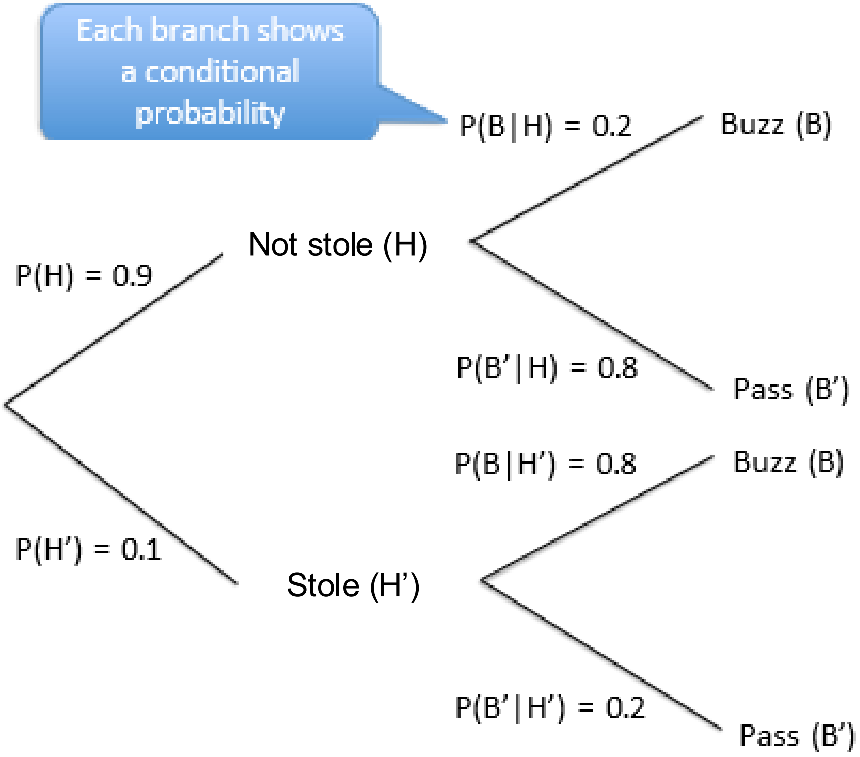
Today’s plan

- Another tool: probability trees
- Bayes rule
- Bayes rule and law of total probability

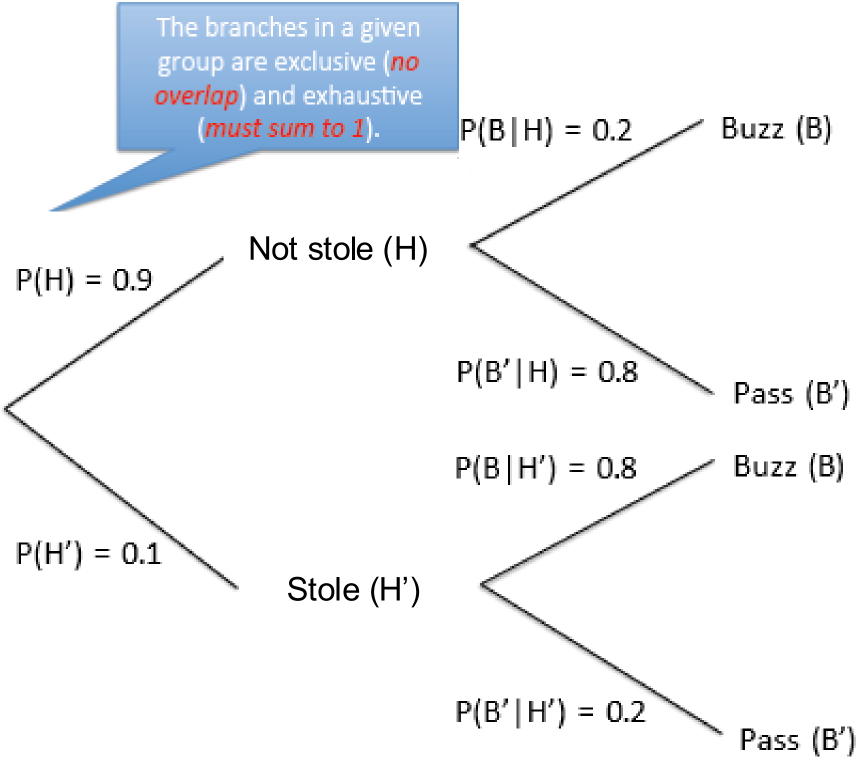
Probability trees: another useful tool



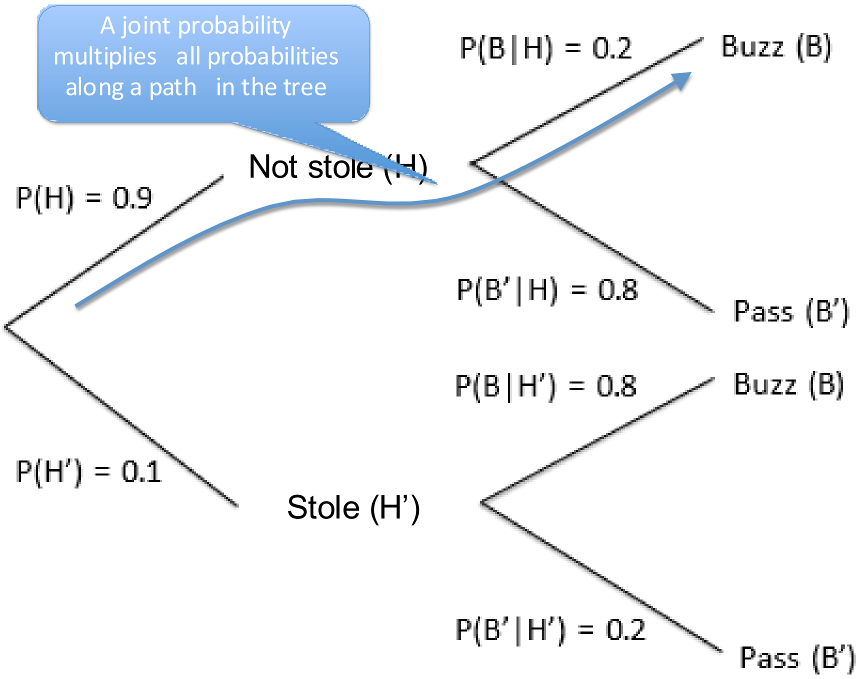
Probability trees: another useful tool



Probability trees: another useful tool



Probability trees: another useful tool



Conditional probability: additional note

- The rules of probability also applies to the rules of conditional probability
- Just replace $P(E), P(F)$ with $P(E|A), P(F|A)$
  - But, need to condition on the same A in the same equation

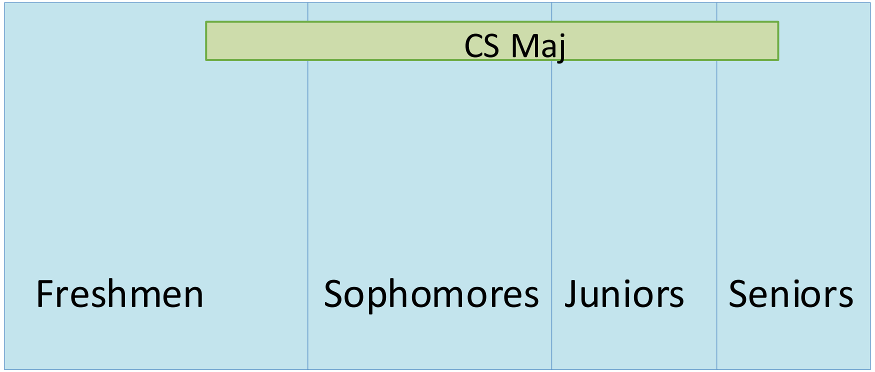
Rules of Probability

1. **Non-negativity:** All probabilities are between 0 and 1 (inclusive)
2. **Unity of the sample space:** $P(S) = 1$
3. **Complement Rule:** $P(E^C) = 1 - P(E)$
4. **Probability of Unions:**
   (a) *In general,* $P(E \cup F) = P(E) + P(F) - P(E \cap F)$
   (b) If E and F are disjoint, then $P(E \cup F) = P(E) + P(F)$

Some examples

A: CS major

- $P(S|A) = 1$
- $P(E|A) + P(E^C|A) = 1$
- $P(E|A) + P(F|A) = P(E \cup F|A)$ for disjoint E and F



Bayes rule

Reversing conditional probabilities

- Is $P(A \mid B) = P(B \mid A)$ in general?
- Let's see..

$$P(A,B) = P(A \mid B) \cdot P(B) = P(B \mid A) \cdot P(A)$$

  - Equal only when $P(A)$ and $P(B)$ are equal

- Let's take a look at a real-world example when they are unequal…

Bayes rule

Bayes rule For events A, B,

$$P(A \mid B) = \frac{P(A) \cdot P(B \mid A)}{P(B)}$$

- Very easy to derive from the chain rule, so remember that first.
- Named after Thomas Bayes (1701-1761), English philosopher & pastor

Bayes rule

Bayes rule For events A, B,

Prior probability Support of evidence

$$P(A \mid B) = \frac{P(A) \cdot P(B \mid A)}{P(B)}$$

Posterior probability Probability of evidence

Examples:

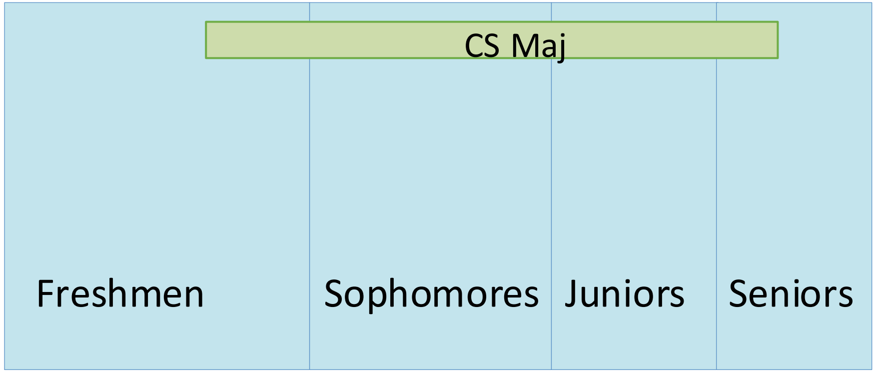
- A: I have COVID, B: my test shows positive
- A: employee stole B: the detector buzzes
- A: student is CS major B: student is a senior

Bayes rule and Law of Total Probability

Bayes rule (equivalent form) For event A and $B_1, \dots, B_n$ forming a partition of S,

$$P(B_i \mid A) = \frac{P(A \mid B_i) \cdot P(B_i)}{\sum_{j=1}^{n} P(A \mid B_j) \cdot P(B_j)}$$

$P(A)$



....

Extension: chain rule for conditional probability

- If we deal with more than 3 events happening together, we can apply the chain rule of probability repeatedly:

Treat (B, C) as a single event

$$P(A, B, C) = P(\, A \mid B, C \,)\, P(B, C)$$

$$= P(\, A \mid B, C \,)\, P(\, B \mid C \,)\, P(C)$$