Computer Science

# CSC380: Principles of Data Science

## Probability 1

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# Outline

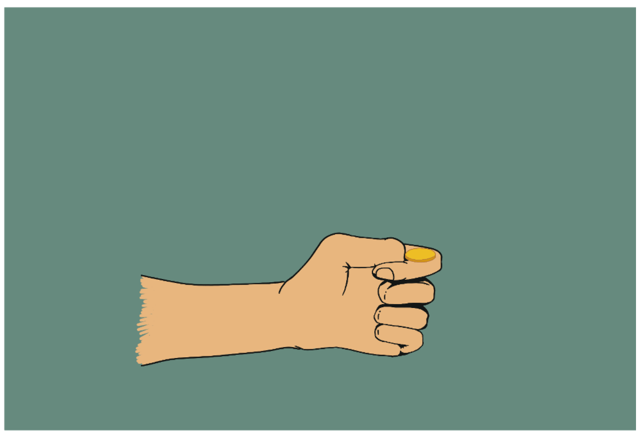


- What is probability?
- Events
- Calculating probabilities
- Set Theory
- Law of Total Probability

# What is probability?

# What is probability?

- Suppose I flip a coin, What is the probability it will come up heads? Most people say 50%, but why?



- “Nolan’s new movie is coming out next weekend! There’s a 100% chance you’re going to love it.”

# Interpreting probabilities

Basically two different ways to interpret:

- Objective probability
  - based on logical analysis or long-run frequency of an event. It's derived from known facts, symmetry, or repeated experiments.
- Subjective probability
  - based on personal belief, opinion, or information about how likely an event is, especially when there's uncertainty or limited data.

# Objective or Subjective?

r/AskReddit • 2 yr. ago

**What statistically improbable thing happened to you?**

• 2y ago

When I was a teenager I picked up a hitchhiker and then a few years later the same guy picked me up when I was walking after I ran out of gas. Never saw him before or after those two occasions.

• 2y ago

Got attacked by a robin in the morning, then attacked by a hawk 3 hours later. Weird day.

18K Award Share ...

It is a subjective probability:  a belief based on their perception of how rare or meaningful the coincidence is, not a calculation based on statistical data.

# Objective probability

- The probability of an event represents the long run proportion of the time the event occurs under repeated, controlled experimentation.
  - e.g. 0001110100111101000110
- Famous experiments in history on coin tosses

| Experimenter | # Tosses | # Heads | Half # Tosses |
|---|---|---|---|
| De Morgan | 4092 | 2048 | 2046 |
| Buffon | 4040 | 2048 | 2020 |
| Feller | 10000 | 4979 | 5000 |
| Pearson | 24000 | 12012 | 12000 |

# Subjective probability

- Probabilities aren’t in the world itself; they’re in our knowledge/beliefs about the world.
- Can assign a probability to the truth of any statement that I have a degree of belief about.

We will focus on objective probability in this class.

# Outcome, Event and Probability

# Outcome

- Outcome is a single result/observation of a random experiment.

- Example 1: You flip a coin once.
  - The outcomes are: "Heads" or "Tails”

- Example 2: You roll a 6-sided die.
  - The outcomes are: 1, or 2, or 3, or 4, or 5, or 6

- Example 3: You tap "shuffle play" on your favorite Spotify playlist with 100 songs and the first song played.
  - The outcomes are: the 1st song in the list, or the 2nd song, ….

# Random Events and Probability

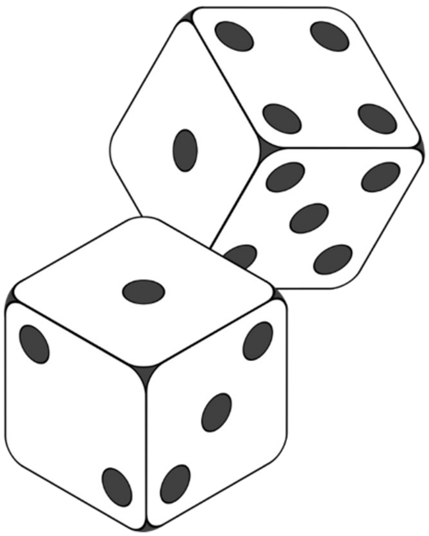
***Suppose we roll <u>two fair dice</u>…***

# Random Events and Probability



***Suppose we roll <u>two fair dice</u>…***

- What are the possible outcomes?
- What is the *probability* of rolling **even** numbers?
- What is the *probability* of having two numbers sum to 6?
- If one die rolls 1, then what is the probability of the second die also rolling 1?



*…this is a **random process**.*

How to formalize all these quantitatively?

# The Sample Space

- The set of all possible outcomes of a random experiment is called the sample space, written as S.

- In math, the standard notation for a set is to write the individual members in curly braces:
  - S = {Outcome1, Outcome2, . . . , }

- Useful to visualize the sample space with an actual space.

# The Sample Space

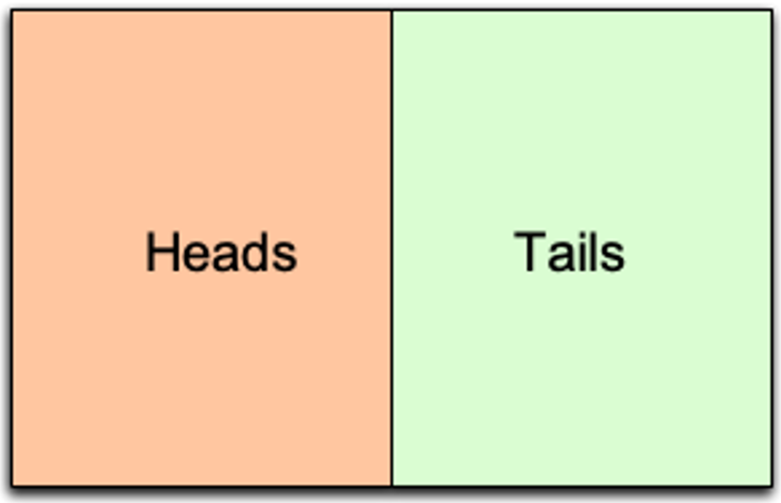
Probability very closely tied to area:


Figure: Visualization of a Sample Space

# Examples of Sample Spaces

What's the sample space for a single coin flip?

- S = {Heads, Tails}



# Examples of Sample Spaces

What is the sample space of rolling a die?

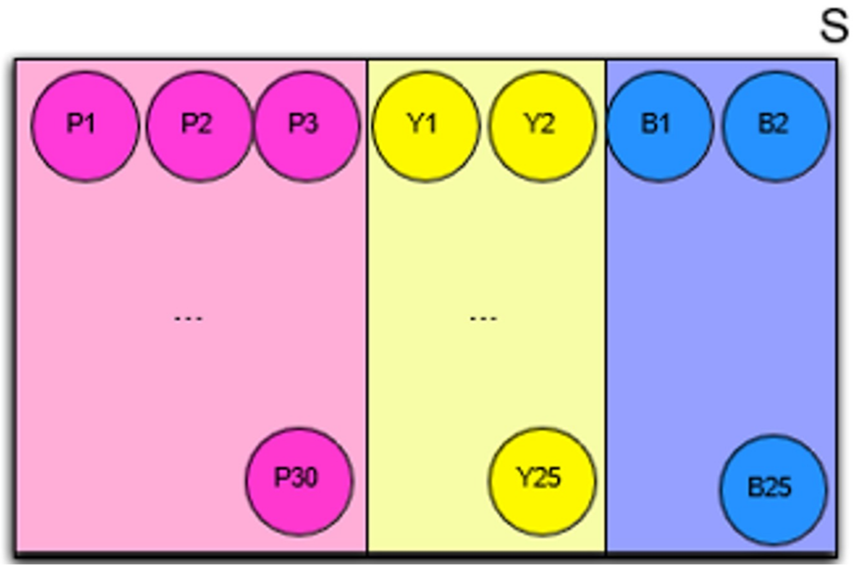
- S = {1, 2, 3, 4, 5, 6}

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |

# Examples of Sample Spaces

What is the sample space of drawing a ball out of a box containing 30 pink, 25 yellow, and 25 blue balls?

- S = {P1, P2, …, P30, Y1, …, Y25, B1, …, B25}



# Examples of Sample Spaces

What's the sample space for...

- Randomly choosing a student from UA?
  - S = {Aarhus, Amaral, Balkan, . . . , Yao, Zielinski}
- Flipping two different coins?
  - S = {HH, HT, TH, TT}
- Flipping one coin twice?
  - S = {HH, HT, TH, TT}
- Observing the number of earthquakes in San Francisco in a particular year?
  - S = {0, 1, 2, 3, . . . }

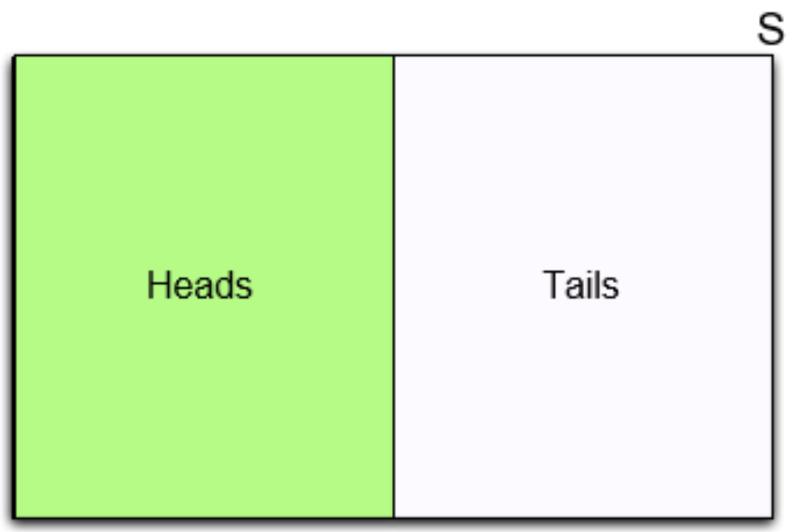
# Events

- An event E is a **subset** of the sample space.
- An event E is a **set** of outcomes
- When we make a particular observation, it is either “in” E or not. Helpful to think about events as propositions (TRUE/FALSE).
  - The proposition is TRUE when the outcome is among the elements of the event set, and FALSE otherwise.
  - Is 4 in event E = {2, 4, 6}? ✅ YES → the **proposition is TRUE**
  - Is 4 in event F = {1, 3, 5}? ❌ NO → the **proposition is FALSE**

# Examples of Events

What's the event set corresponding to the following propositions?

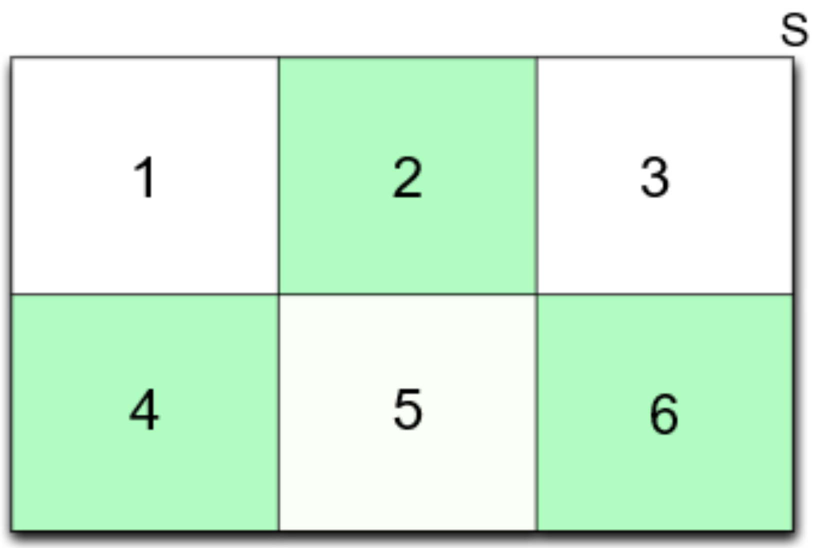
- "The coin comes up heads"
- E = {Heads}



# Examples of Events

What's the event set corresponding to the following propositions?

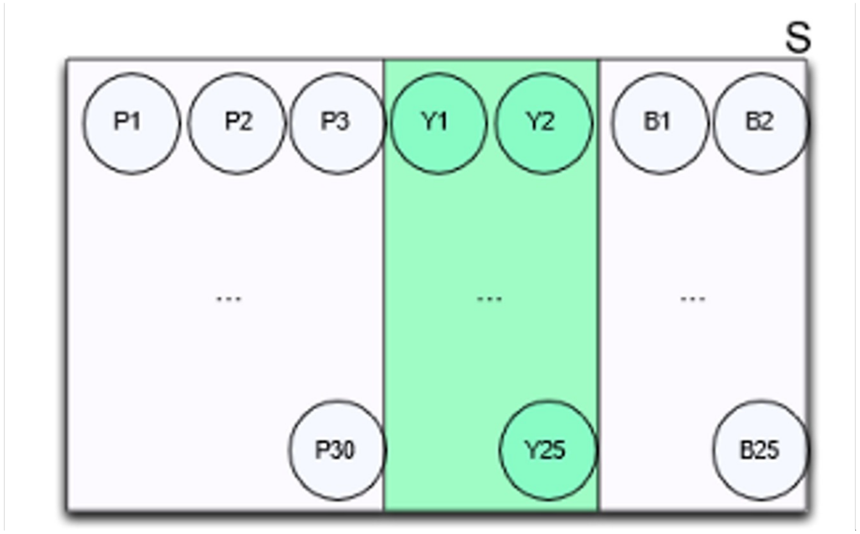
- "The die comes up an even number"
- E = {2, 4, 6}



# Examples of Events

What's the event set corresponding to the following propositions?

- "A yellow ball is chosen"
- E = {Y1, Y2, . . . , Y25}



# Special events

- The sample space (S) includes all possible outcomes.
  - If an event E = S, then no matter what outcome occurs it's always in E.
  - e.g., E = {Heads, Tails}

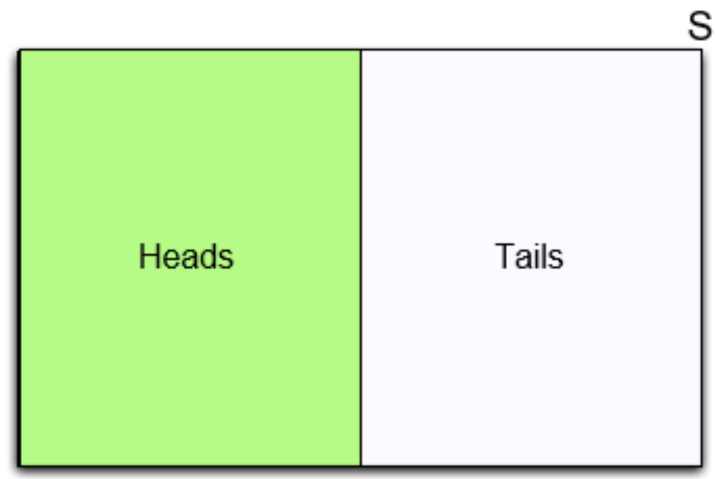
- The empty set ∅ is also an event
  - It is an event that never happens
  - e.g. “the die comes up 7”, E = {7}

# Calculating Probabilities

# Calculating probability

- We can think of the probability of an event E as its area, where S (sample space) always has a total area of 1.0

- So, the probability of E is the fraction of S that it takes up.

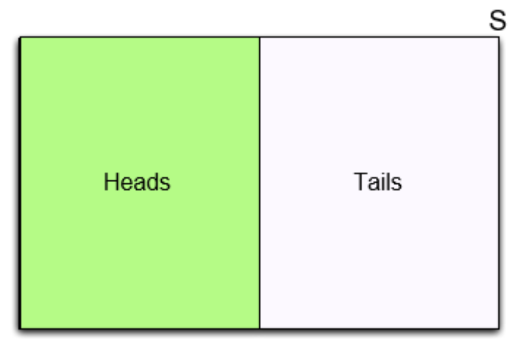


# Calculating probability using symmetry

- If we have a sample space for which every outcome is equally likely, then we can find event probabilities easily.

- Classic probability model: if every outcome has the same "area", we can just count:

$$P(E) = \frac{\#\text{outcomes in } E}{\#\text{outcomes in } S}$$



# Probability as Area

What is the probability of

$$P(E) = \frac{\#\text{outcomes in } E}{\#\text{outcomes in } S}$$

- Rolling a fair die and see an even number?
- E = {2,4,6}
- $P(E) = \frac{\#\{2,4,6\}}{\#\{1,2,3,4,5,6\}} = \frac{3}{6} = \frac{1}{2}$

| | | |
|---|---|---|
| 1 | 2<br>P(2) = 1/6 | 3 |
| 4<br>P(4) = 1/6 | 5 | 6<br>P(6) = 1/6 |

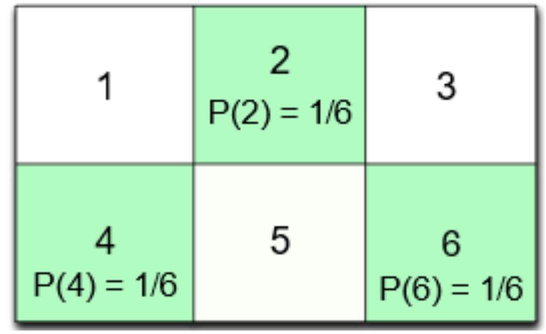
P(S) = 1

P(Even) = P(2) + P(4) + P(6) = 3/6

# Elementary events

- Notice that we can find the total probability of an event by breaking it into pieces and adding up the probabilities of the pieces:

  $$P(\text{Even}) = P(2) + P(4) + P(6)$$

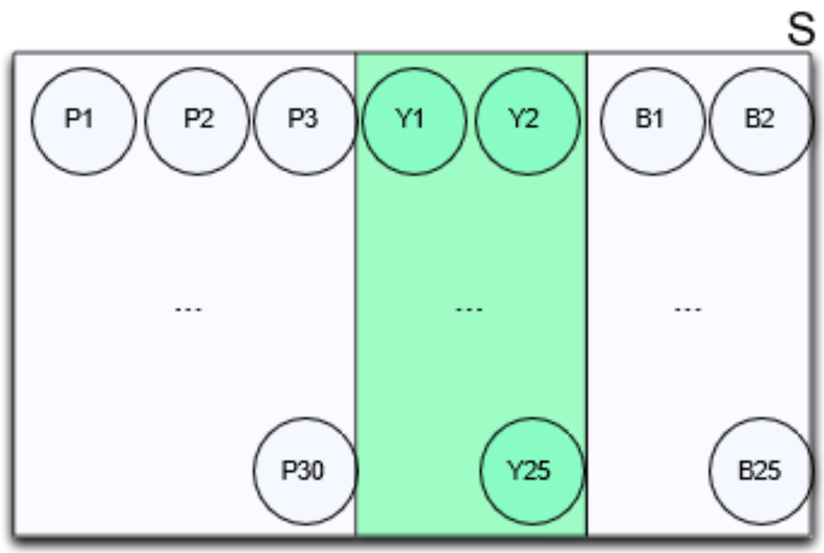



- These pieces are called ‘elementary events’: Events that correspond to exactly one outcome

# Probability as Area

What is the probability of

- Selecting a yellow ball?
- E = {Y1, Y2, . . . , Y25}
- $P(E) = \frac{\# E}{\# S} = \frac{25}{30+25+25} = \frac{5}{16}$



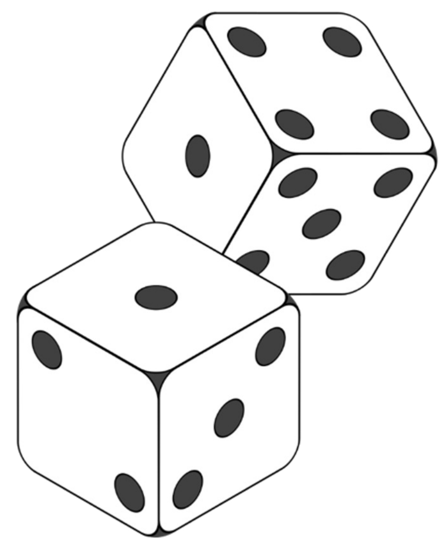

P(Yellow) = P(Y1) + ... + P(Y25) = 25 * (1/80)

# Random Events and Probability



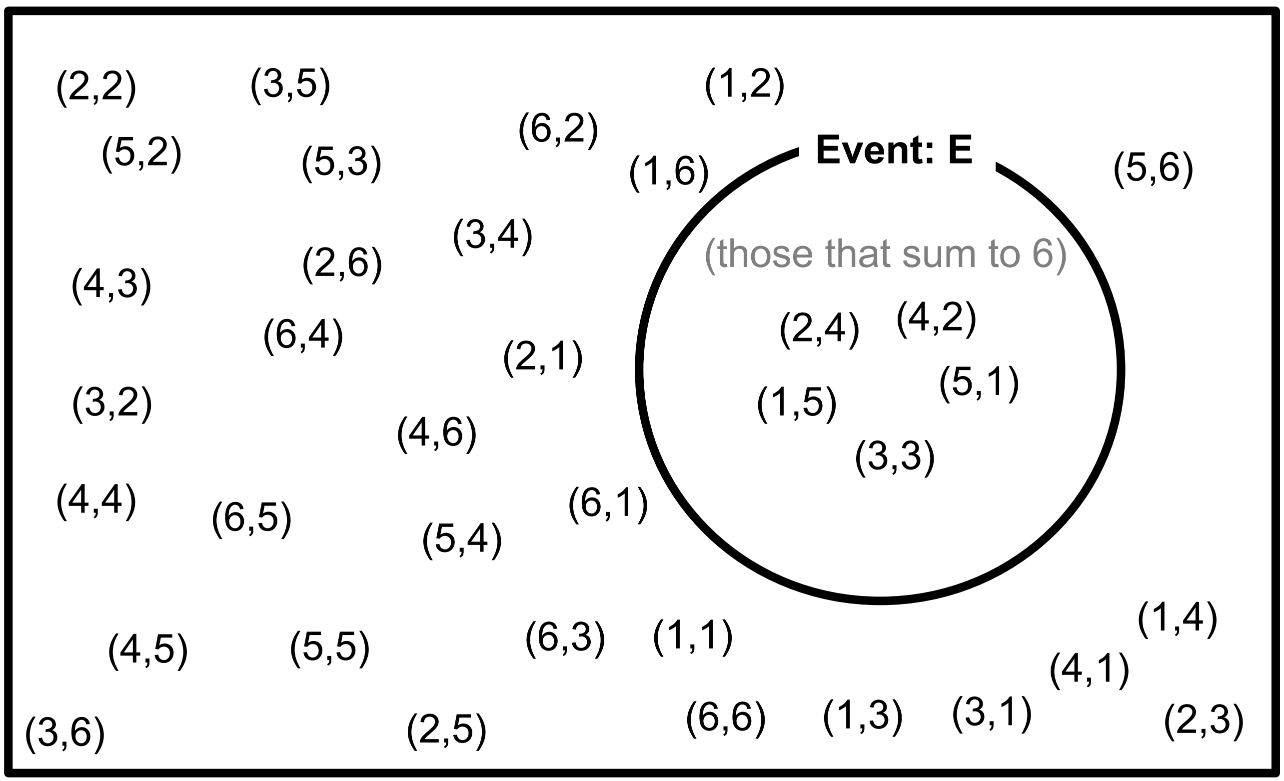
- Suppose we throw two fair dice
  - What is the sample space S (space of all possible outcomes)?

  Event E: the two dice's outcomes sum to 6
  - What is the size of E?
  - What is the probability of E?



# Random Events and Probability

*What is the probability of having two numbers sum to 6?*




$$P(E) = \frac{\#\text{outcomes in } E}{\#\text{outcomes in } S}$$

S = $\{(a, b): a, b \in \{1, \ldots, 6\}\}$

Each outcome is equally likely

# of outcomes that sum to 6:
5

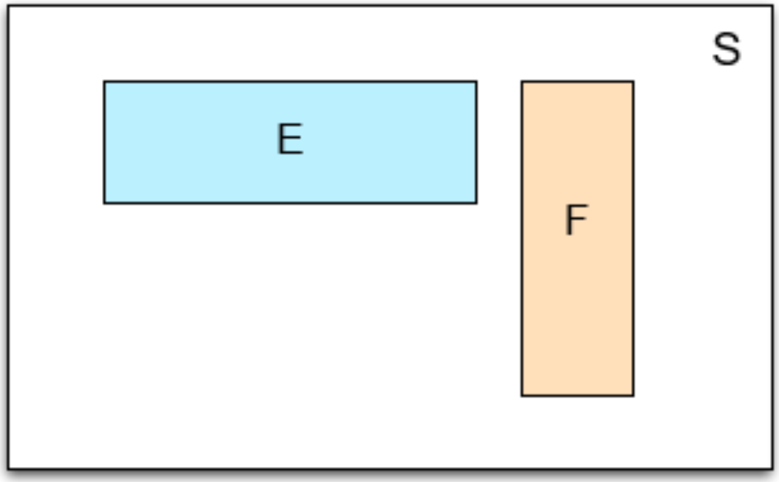
answer:
5/36 = 0.13888…

# Disjoint events

- In general, breaking an event into *disjoint events* preserves the total probability
- **Disjoint***:* $E$ and $F$ are *disjoint* if they cannot happen simultaneously,
- e.g. E = {2, 4}, F={1, 3, 5}

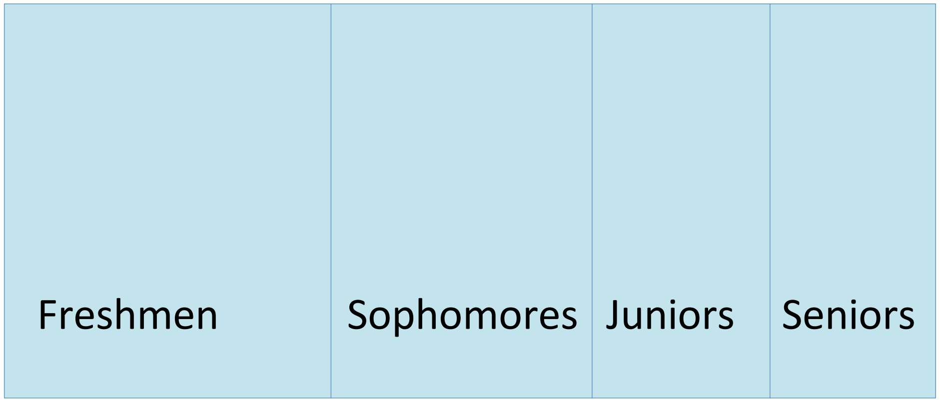
- In such cases,

$$P(E \text{ or } F) = P(E) + P(F)$$



# Partition

- We say that disjoint events $E_1, \ldots, E_n$ form a *partition* of $E$ if any outcome in $E$ lies in exactly one $E_i$
- e.g.
  - {Fr.} $E_1$, {Soph.} $E_2$ form a partition of {Lower division} $E$
  - {Fr.}, {Soph.} {J.} {Sen.} form a partition of $S$ (sample space)



# Probability as Area

For disjoint events $E, F$:

$$P(E \text{ or } F) = P(E) + P(F)$$

If disjoint events $E_1, \dots, E_n$ forms a partition of $E$ :

$$P(E) = P(E_1) + P(E_2) + \cdots + P(E_n)$$

If disjoint events $E_1, \dots, E_n$ forms a partition of $S$ , for event $F$(law of total probability):

$$P(F) = P(E_1, F) + P(E_2, F) + \cdots + P(E_n, F)$$

Notation: $P(A, B)$ is a shorthand for $P(A \text{ and } B)$

# Probability as Area

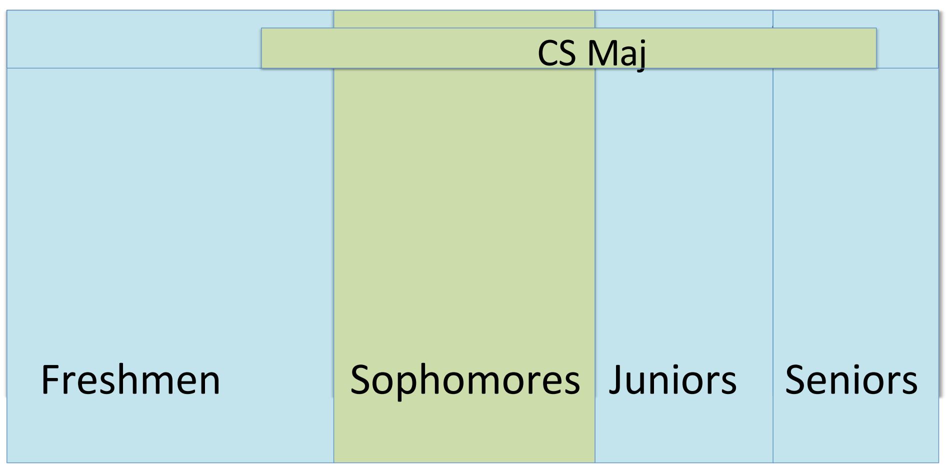
Examples

- $P(\text{CS}) = P(\text{Fr.}, \text{CS}) + P(\text{Soph.}, \text{CS}) + P(\text{J.}, \text{CS}) + P(\text{Sen.}, \text{CS})$

Notation: $P(A, B)$ is a shorthand for $P(A \text{ and } B)$

- $P(\text{Soph.}) = P(\text{CS}, \ \text{Soph.}) + P(\text{nonCS}, \ \text{Soph.})$

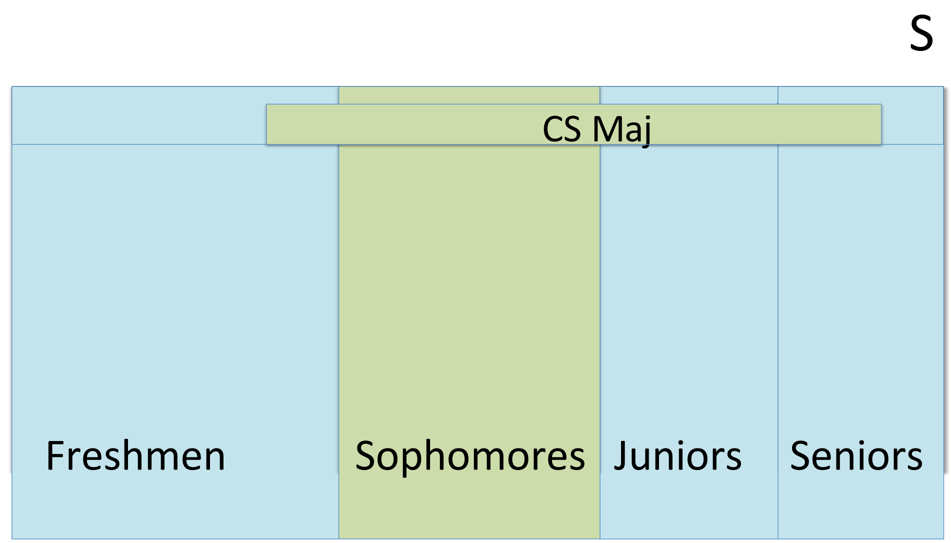
S



# Probability as Area

If events $E, F$ are non-disjoint, what is $P(E \text{ or } F)$?

- $P(\text{Soph.}\, or \text{ CS})$?
- Note: events "Sophomore" and "CS Major" may overlap

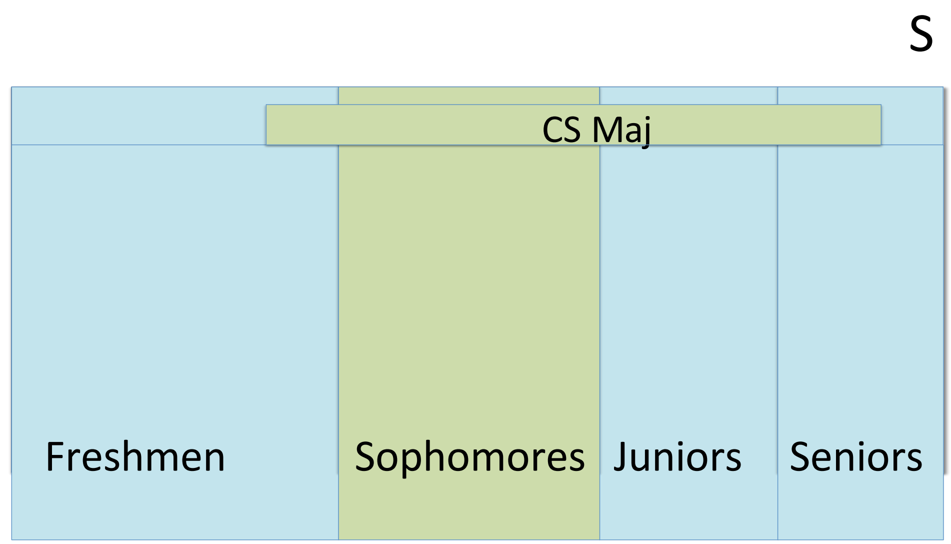


# Probability as Area

- E = { Soph OR CS }

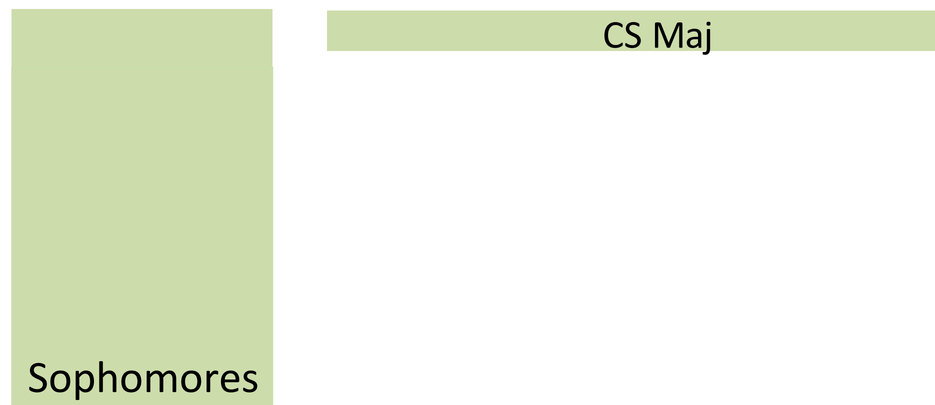
- Is P(E) = P(Soph) + P(CS)?
  - No

- Which one is larger?
  - Let's see..

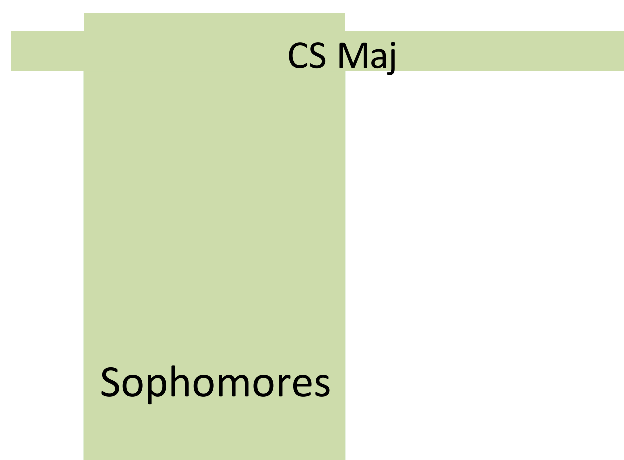


# Probability as Area

P(Soph) + P(CS) =




P(Soph or CS) =



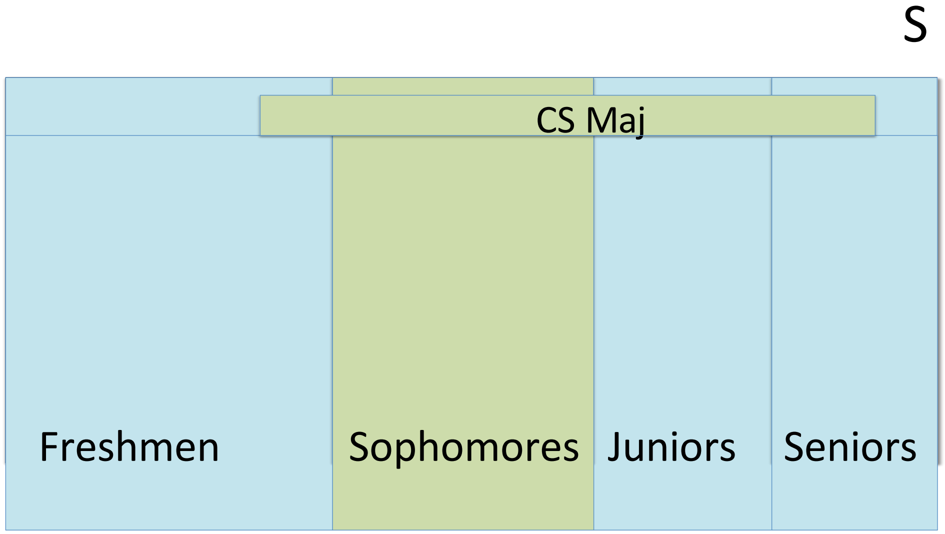
# Probability as Area

P(Soph) + P(CS)

= P(CS , Soph.) + P(Non-CS , Soph.) +

P(Fr. CS) + P(Soph. CS) + P(J. CS) + P(Sen. CS)

Soph. CS is counted twice




So, P(Soph OR CS)

= P(Soph) + P(CS) - P(Soph. CS)

# Inclusion-Exclusion Principle

**Inclusion-Exclusion Principle** For any events $E$ and $F$,

$$P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$$

Accounting for overlap between $E$ and $F$