Computer Science

CSC380: Principles of Data Science

Probability Primer 4

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Review: Random Variable Examples

X_1, X_2: outcomes of two dice

- $R_1 = X_1 + X_2$
- $R_2 = \frac{(X_1 + X_2)}{2}$
- $R_3 = I\{X_1 = 1\}$

Random variable induces a partition of the outcome space!

$$\{R_3 = 1\} \Leftrightarrow \{(1,1), (1,2), \dots, (1,6)\}$$

$$\{R_3 = 0\} \Leftrightarrow \{(2,1), (2,2), \dots, (2,6), (3,1), (3,2), \dots, (3,6), \dots (6,1), (6,2), \dots, (6,6)\}$$

Q: what distribution does R_5 follow with what parameter?

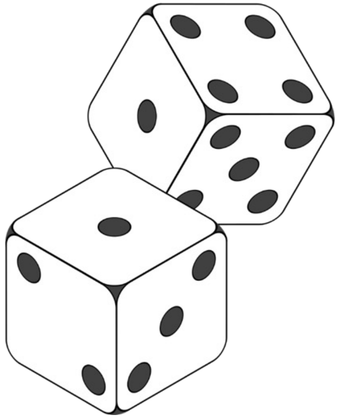
Bernoulli, PMF: $p(X = x) = \pi^x (1 - \pi)^{1-x}, \pi = \frac{1}{6}$

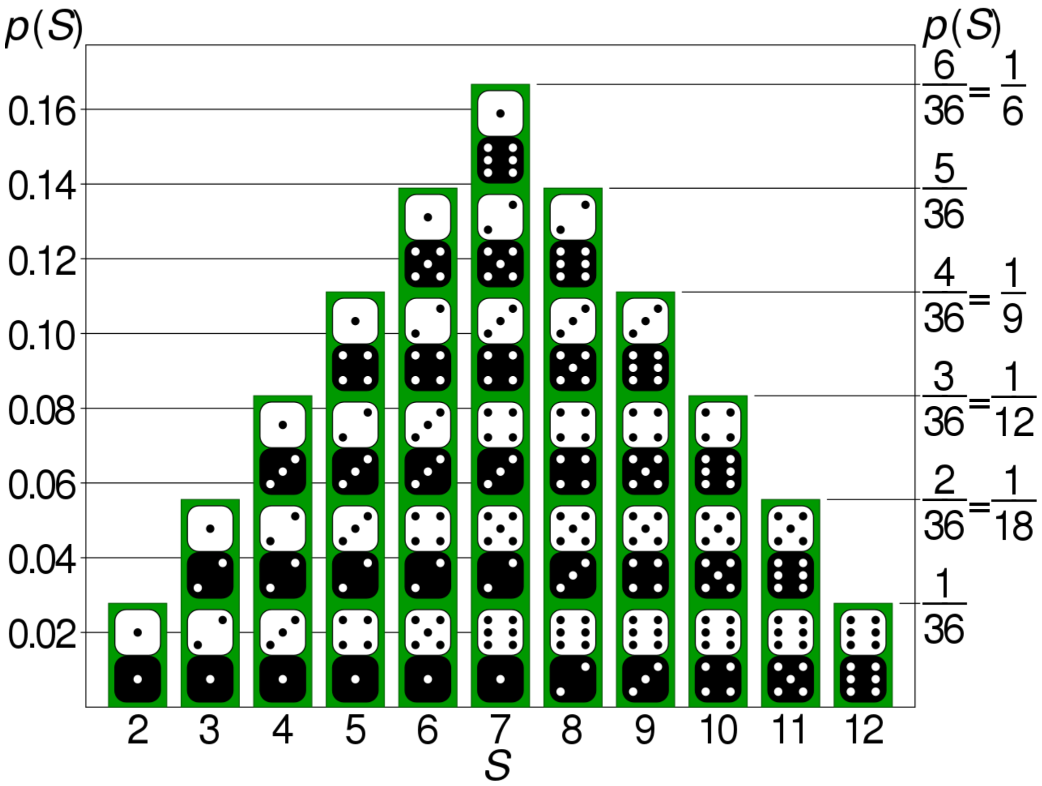
Review: Discrete Distribution

Another example.

- let S = sum of two dice;
- probability of S on different values:

$P(S = 2) = 1/36$
$P(S = 3) = 2/36$
$P(S = 4) = 3/36$
...
$P(S = 12) = 1/36$





PMF: $f_X(S) = \dfrac{\min(S-1, 13-S)}{36}$, for $S \in \{2,3,4,5,6,7,8,9,10,11,12\}$

Outline

- Continuous probability
- Continuous distribution
  - PDF
  - CDF
- Useful continuous distributions

Continuous Probability

(TV show spin the wheel)

Continuous Probability

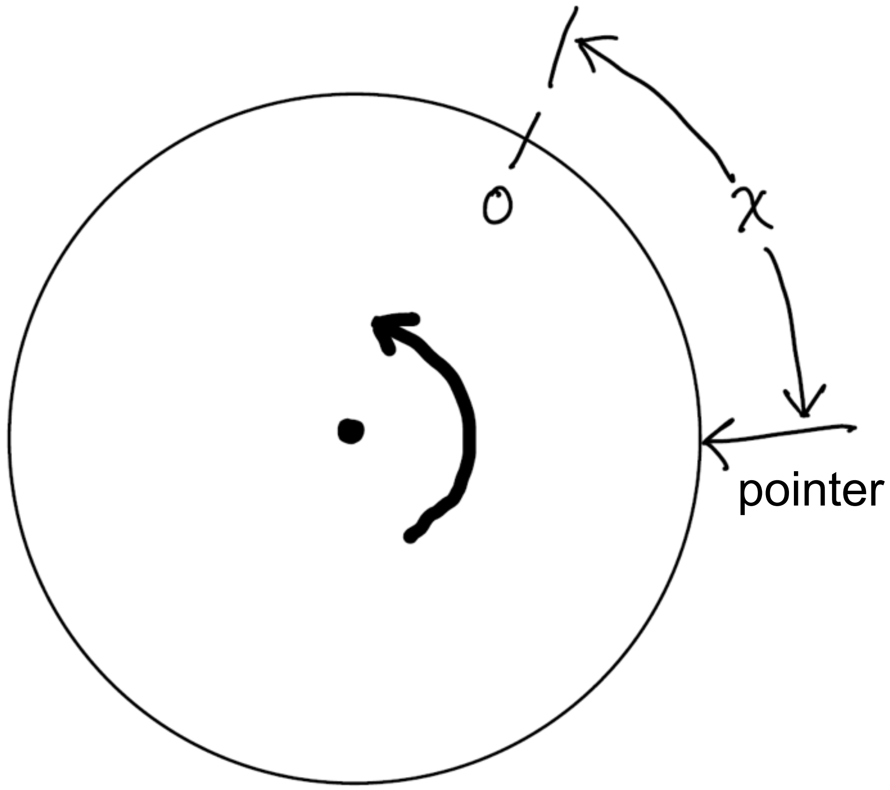
Experiment Spin continuous wheel and measure X displacement from 0

Say the circumference is 1.

Outcome space Ω is all points (real numbers) in $(0,1]$

Question Assuming uniform distribution,

what is $P(X = x)$?

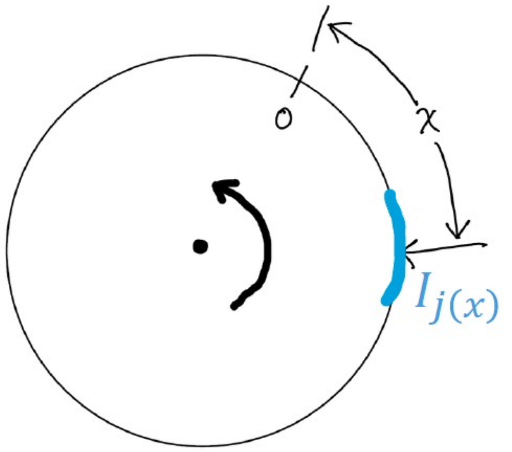


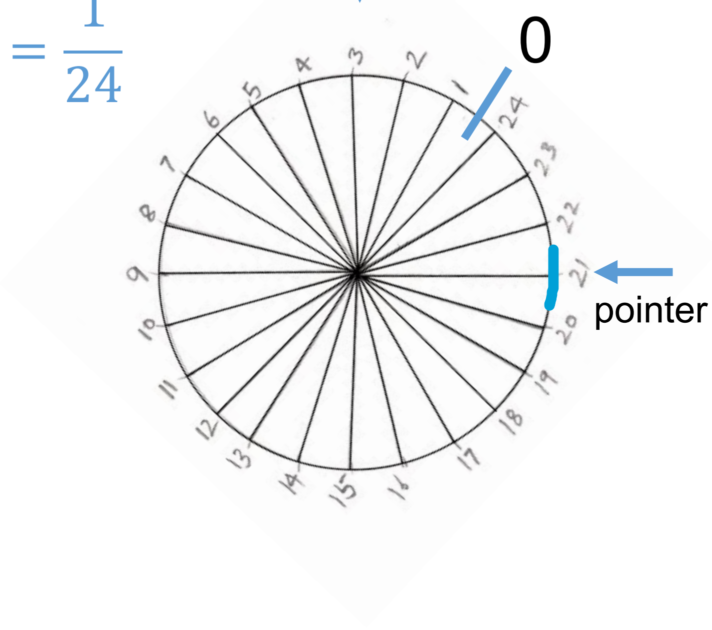
Proof

<u>Goal</u>: Show P(X=x) = 0

- Say the displacement X is in $(0,1]$
- Let N be a very large number. Q: how many such intervals?
- Let $I_k = (\frac{k-1}{N}, \frac{k}{N}]$ e.g., $I_1 = \left(\frac{1-1}{24}, \frac{1}{24}\right] = \left(0, \frac{1}{24}\right]$, $I_{21} = \left(\frac{21-1}{24}, \frac{21}{24}\right] = \left(\frac{20}{24}, \frac{21}{24}\right]$
- Let $j(x)$ be k such that $x \in I_k$
- $P(X = x) \leq P\left(X \in I_{j(x)}\right) = \frac{1}{N}$ e.g., $P(X = 21) \leq P\left(X \in (\frac{20}{24}, \frac{21}{24}]\right) = \frac{1}{24}$

- **We can make N as large as we want!**
- **⇒ P(X=x) must be 0.**





Continuous Probability

Maybe, it's not so weird.

- Q1: Probability that your house water usage tomorrow is 20.58 gallon?

- Q2: Probability that your house water usage tomorrow is 20.5891231285 gallon?

in reality, we never work with a precise real number.
we work with intervals!!

Continuous Probability

we could try to convince ourselves that it is sensible.

… or we could just accept this oddity…

Continuous Distributions

Fundamental Theorem of Calculus: example

- The area of any circular cylinder:

$$V = \pi \cdot r^2 \cdot x$$

- Think about slicing the cylinder into thin pieces, $r = 2, thickness = \Delta x$:

$$V_{slice} = \pi \cdot 2^2 \cdot \Delta x$$

- Letting $\Delta x \to 0$:

$$V = \int_0^3 \pi \cdot 2^2 \cdot dx = \int_0^3 4\pi \, dx$$

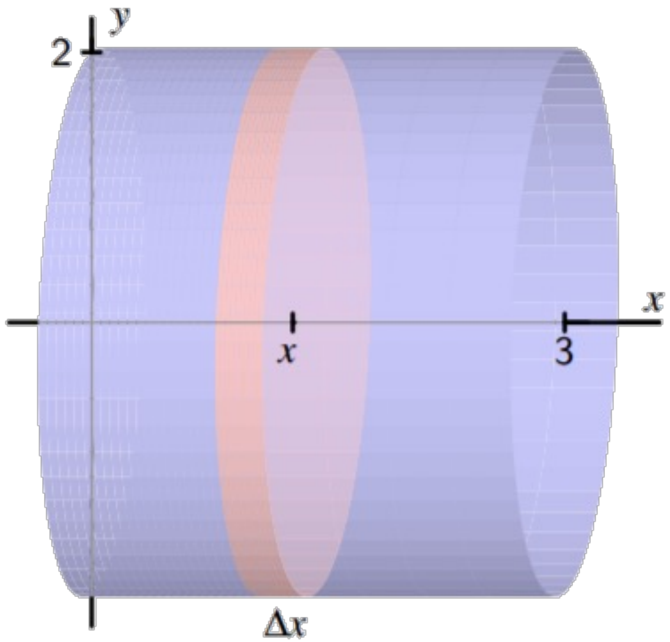
- Volume for each thin piece dv:

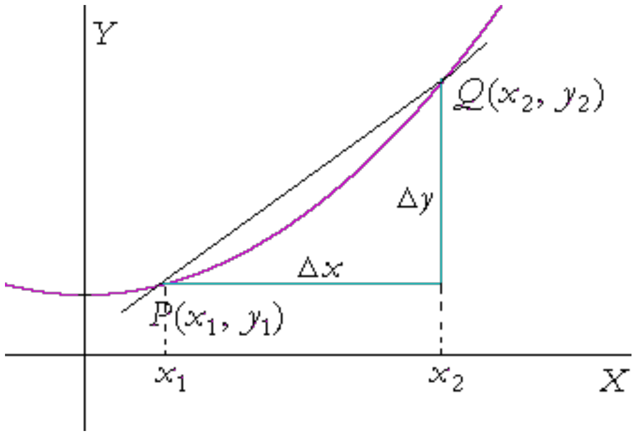
$$dv = 4\pi \, dx, \qquad \frac{dv}{dx} = 4\pi$$

- Get antiderivative:

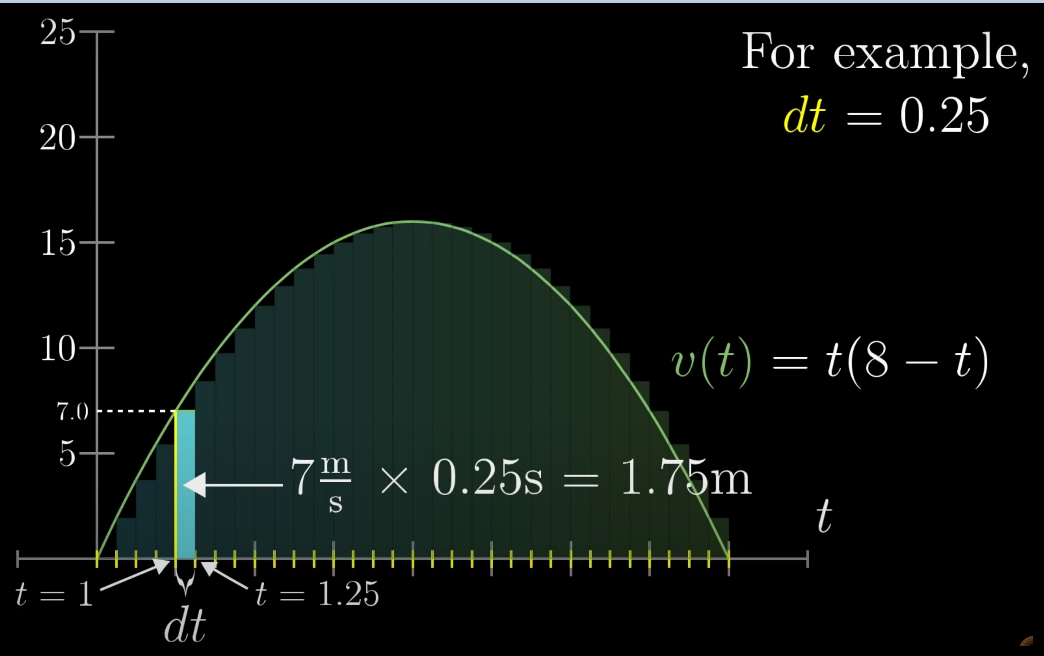
$$\mathrm{V} = 4\pi x$$

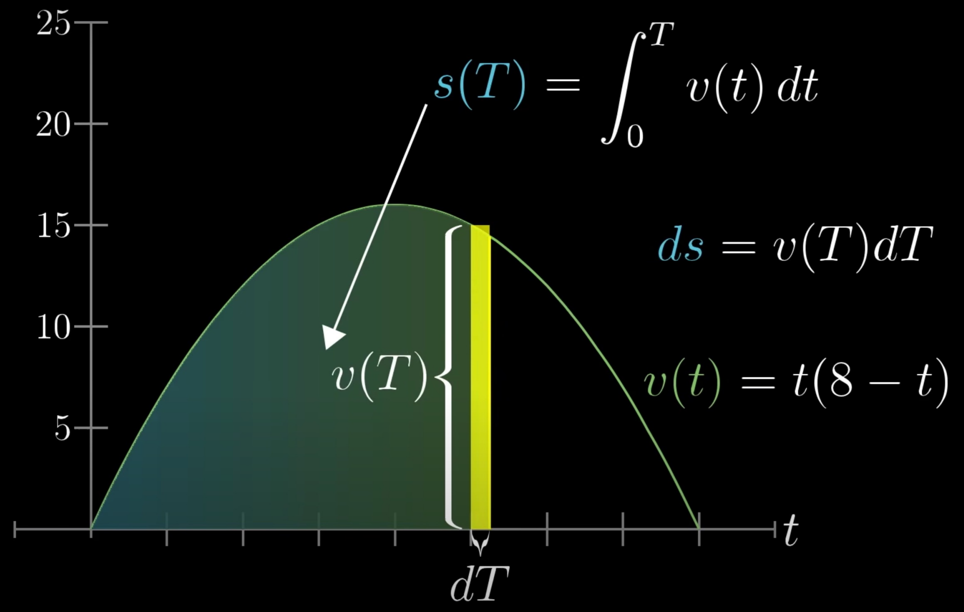
- $V = \int_0^3 4\pi \, dx = \mathrm{V}(3) - \mathrm{V}(0) = 4\pi \cdot 3 = 12\pi$

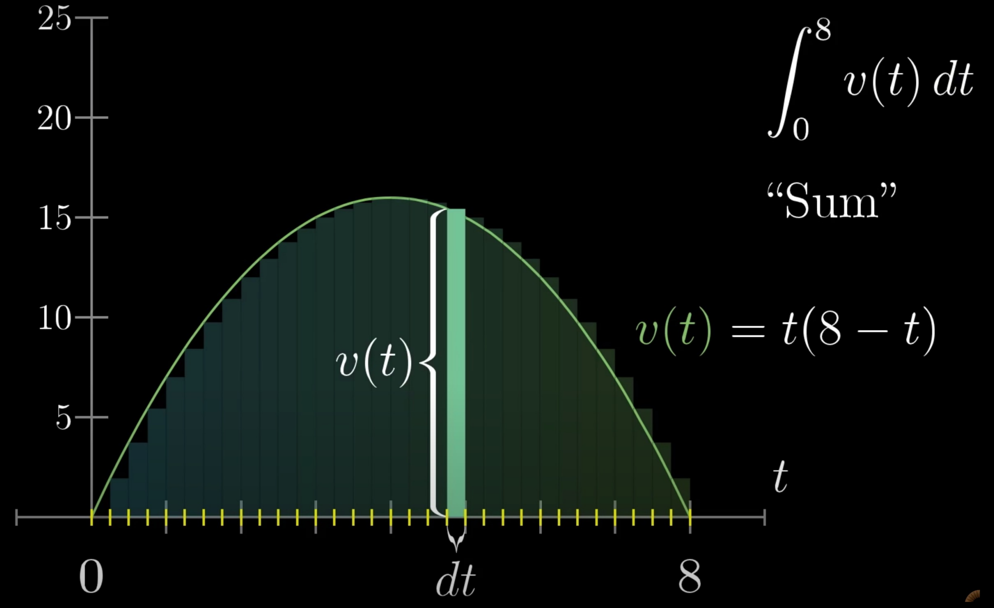


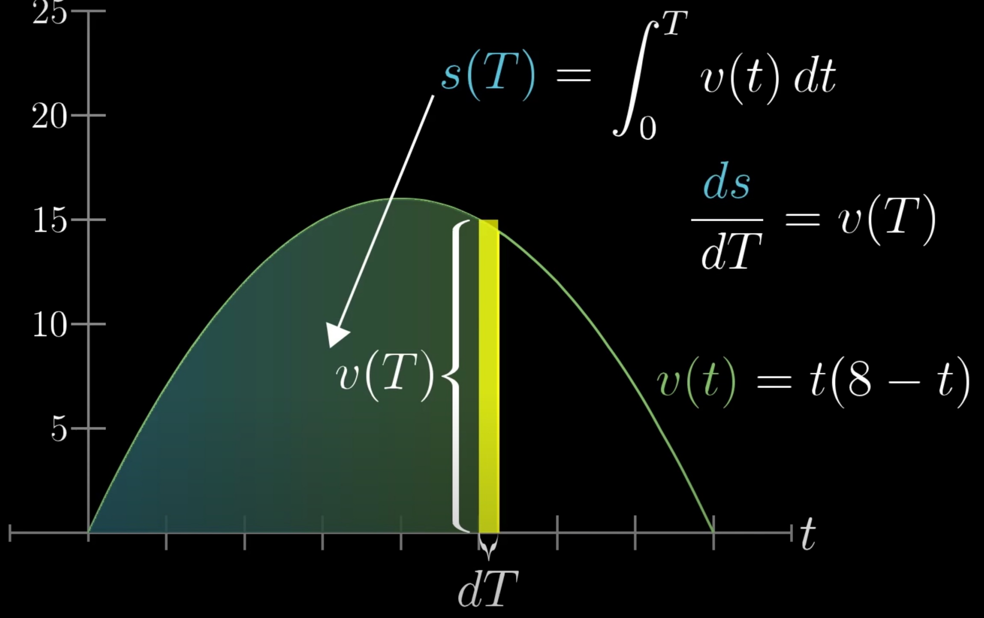


Fundamental Theorem of Calculus : example

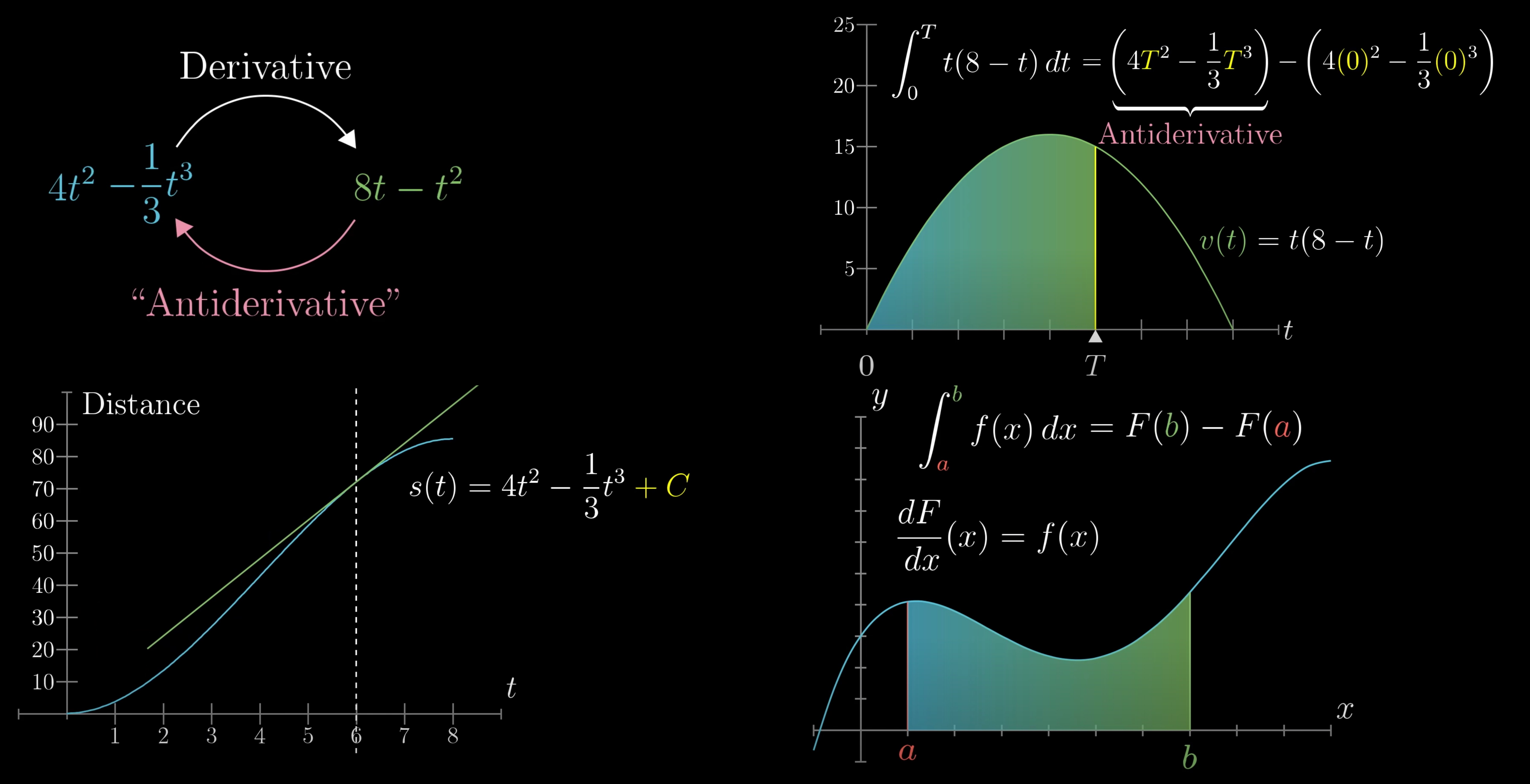




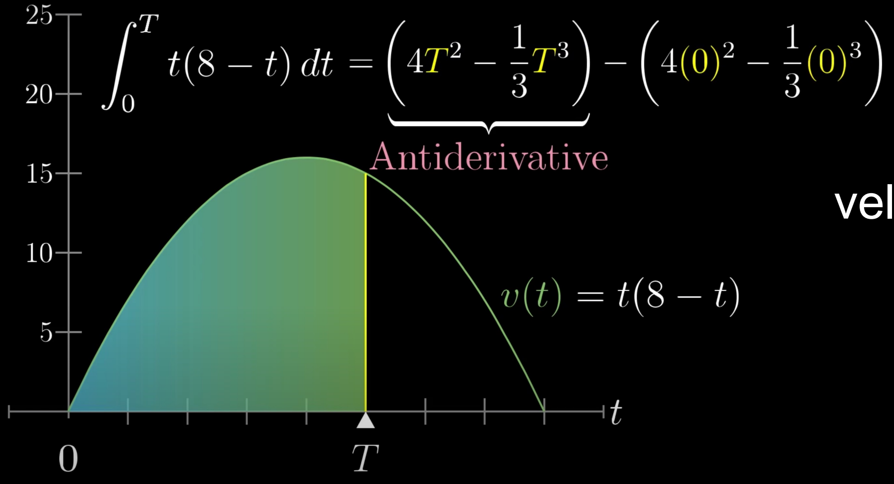


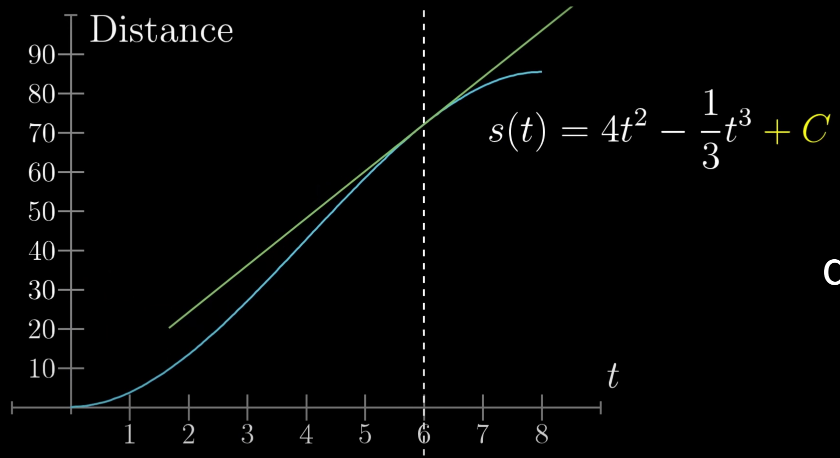


Fundamental Theorem of Calculus : example



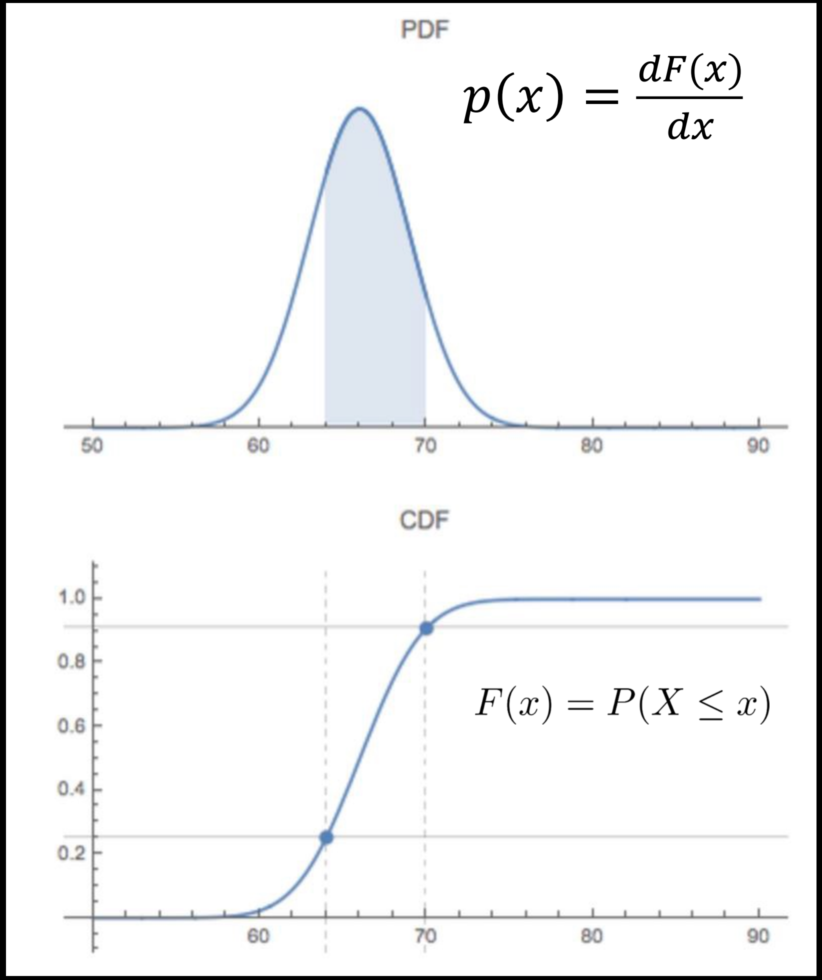
Mapping example to continuous probability





velocity → probability density

distance → cumulative distribution



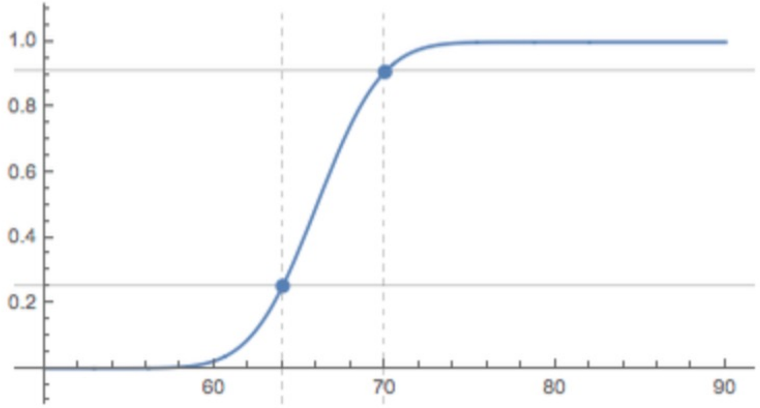
Continuous Probability Distributions

Definition *The cumulative distribution function (CDF) of a RV X is the function given by,*

$$F(x) = P(X \leq x)$$

Key properties:

F is monotonically increasing
$F(\mathrm{x})$ goes to 0/1 if x goes to $-\infty/+\infty$



➢ Can easily measure probability of closed intervals,

$$P(a < X \leq b) = F(b) - F(a)$$

e.g. $a = 64,\ \mathrm{b} = 70$

https://demonstrations.wolfram.com/ConnectingTheCDFAndThePDF/

Continuous Probability Distributions

- If $F(X)$ is differentiable then,

Fundamental Theorem of Calculus

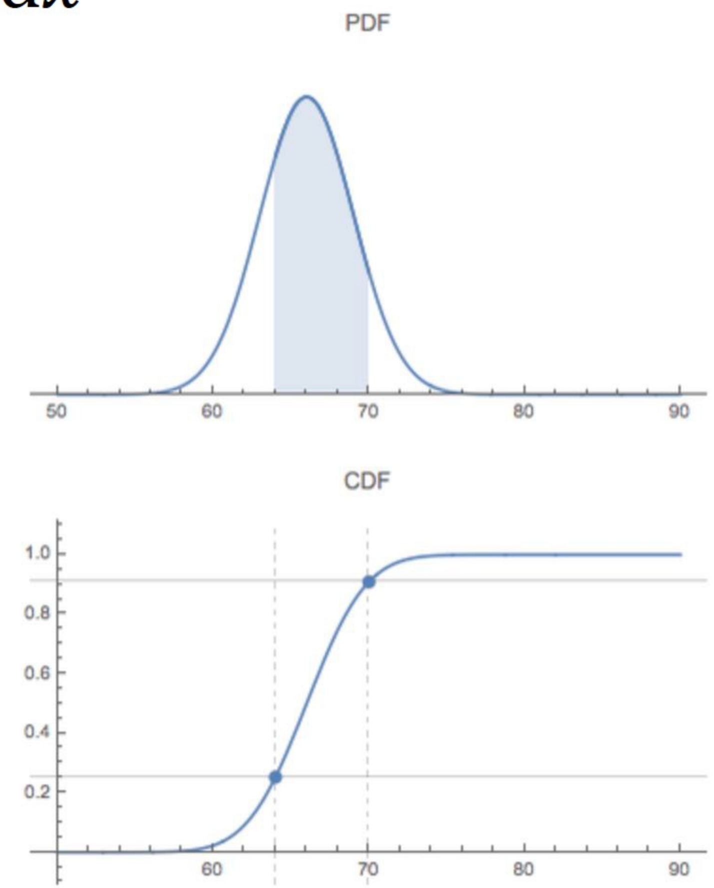
$$p(x) = \frac{dF(x)}{dx} \qquad \text{and} \qquad F(t) = \int_{-\infty}^{t} p(x)\,\mathrm{d}x$$

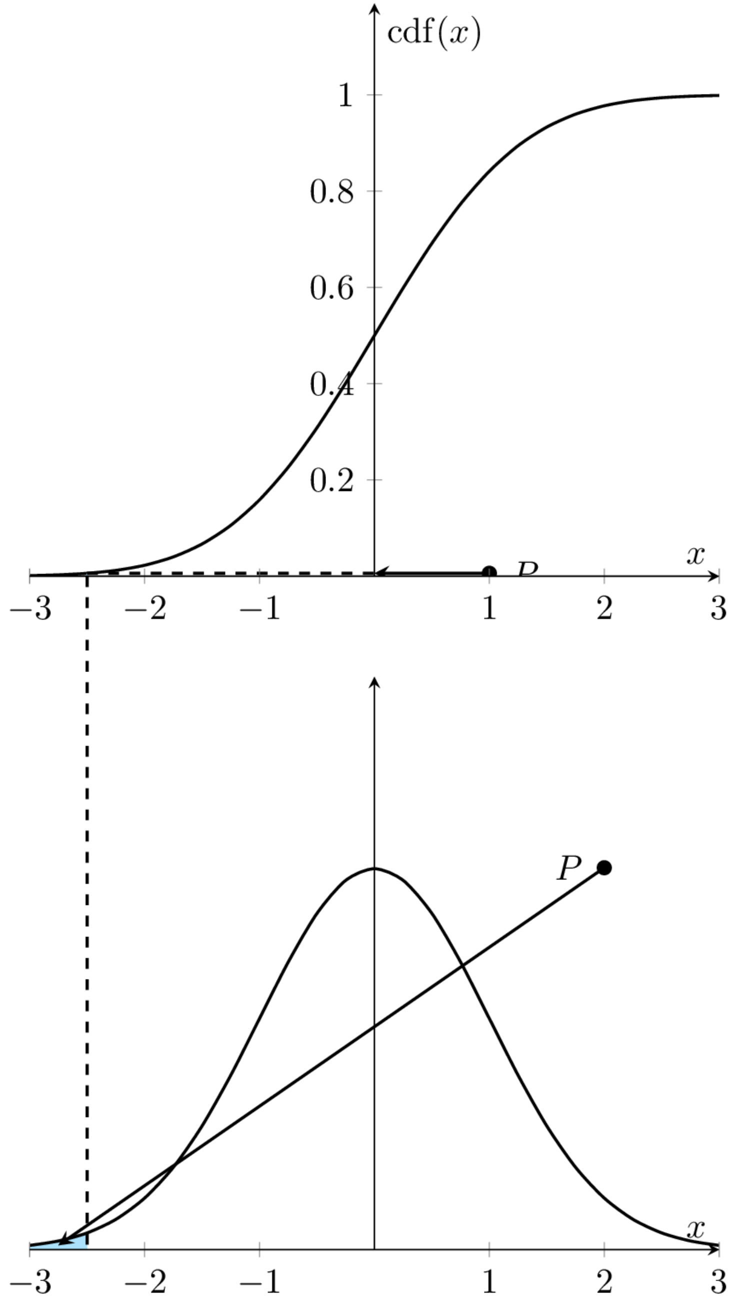
$p(x)$ is called X's ***probability density function* (PDF)**

$$\approx \frac{F(x)-F(x-\epsilon)}{x-(x-\epsilon)} = \frac{P(X \in (x-\epsilon, x])}{\epsilon} \text{ when } \epsilon \to 0$$

Intuition: $p(x)$ characterizes how likely X takes values in the neighborhood of x

- $p(x) \geq 0$ for all x
- $P(a < X \leq b) = F(b) - F(a) = \int_a^b p(x)\mathrm{d}x$
- $\int_{-\infty}^{+\infty} p(x)\mathrm{d}x = 1$



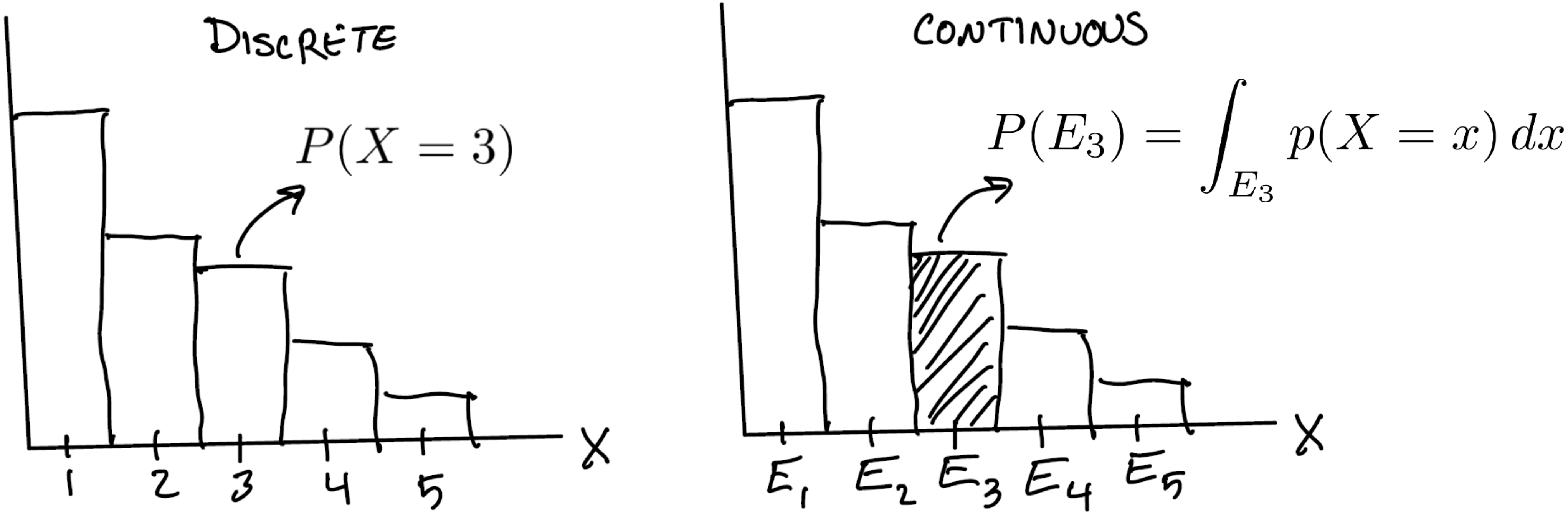


$$P(a < X \leq b) = F(b) - F(a)$$

$$F(t) = \int_{-\infty}^{t} p(x)\,\mathrm{d}x$$

$$p(x) = \frac{dF(x)}{dx}$$

Continuous Probability



- Events represented as intervals $a \leq X < b$ with probability,

$$P(a \leq X < b) = \int_a^b p(X = x)\, dx$$

- Specific outcomes have zero probability $P(X = x) = 0$
- But may have nonzero *probability density* $p(X = x) > 0$

Notation

- For continuous RV X, use $p(X = x), p(x), pX(x)$ to denote its PDF (probability density function)
  - Recall: $\mathrm{P}(X = x)$ is not its PDF value (in fact, always 0)

- For discrete RV X, use $p(X = x), p(x), pX(x)$ to denote its PMF (probability mass function)
  - In this case, $p(X = x) = P(X = x)$

- General suggestions for HW / exams: to be extra safe, you can explicitly declare "we use $p(X = x)$ to denote the PDF of continuous RV X"

Continuous Probability Distributions

Most definitions for discrete RVs hold, replacing sum with integral…

Law of Total Probability for continuous distributions,

$$p(x) = \int_y p(x, y)\, dy$$

Recall: for discrete X

$$P(X = x) = \sum_y P(Y = y, X = x)$$

All the rules apply when replacing PMF with PDF…

Conditional PDF:

$$p(X \mid Y) = \frac{p(X,Y)}{p(Y)} = \frac{p(X,Y)}{\int p(x,Y)\, dx}$$

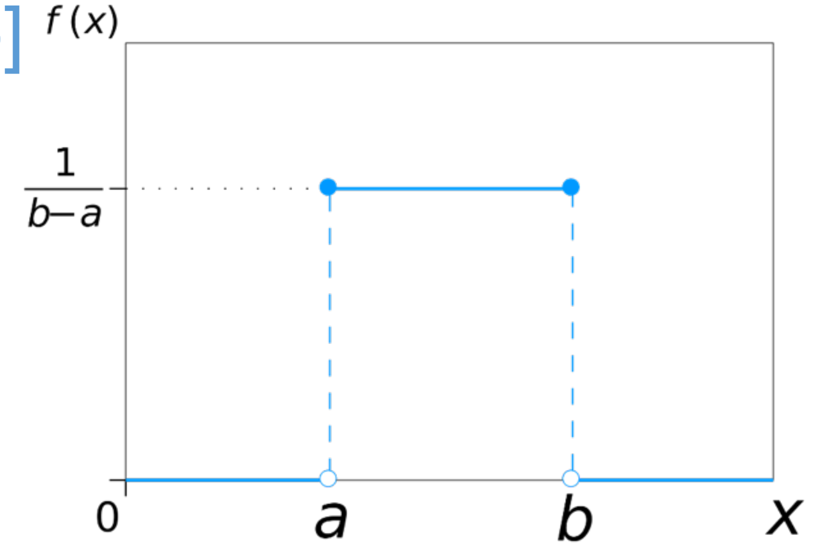
Probability Chain Rule:

$$p(X, Y) = p(Y)p(X \mid Y)$$

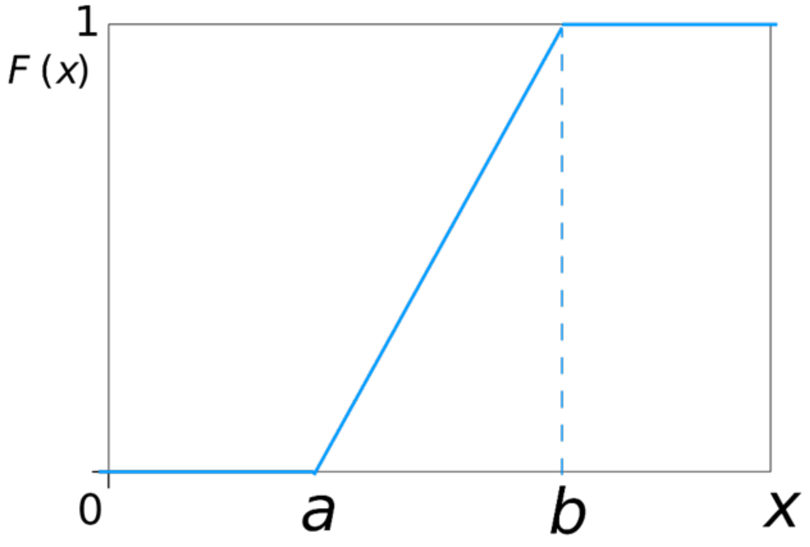
Uniform Continuous Distribution

Uniform distribution on interval $[a, b]$: Uniform[a,b]

$$p(x) = \begin{cases} 0 & \text{if } x \le a, \\ \frac{1}{b-a} & \text{if } a \le x \le b, \\ 0 & \text{if } b \le x \end{cases} \qquad P(X \le x) = \begin{cases} 0 & \text{if } x \le a, \\ \frac{x-a}{b-a} & \text{if } a \le x \le b, \\ 1 & \text{if } b \le x \end{cases}$$



$$\boxed{\mathrm{P}(X \le x) = \int_{-\infty}^{x} p(t)dt}$$

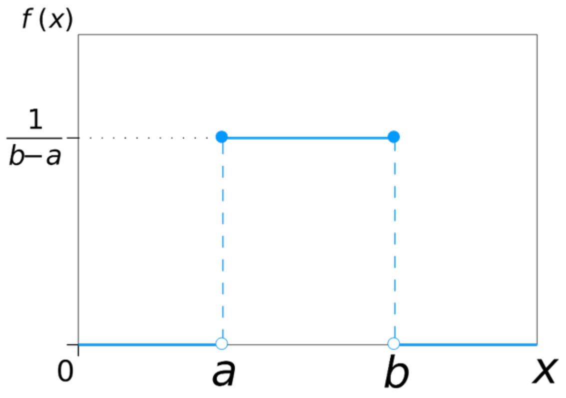


Notation:
$p(x)$ for the PDF function at location x
$\mathrm{P}(A)$ for the probability of event A

Again, PDF function $\neq$ probability

Uniform Continuous Distribution

Example: Let $X = \text{length}$ of an eight-week-old baby's smile ($X \sim \mathrm{U}(0, 23)$).
The probability density function is $\mathrm{p}(x) = \frac{1}{23-0} = \frac{1}{23}$ for $0 \leq \mathrm{X} \leq 23$.



Q: find the probability that a random eight-week-old baby smiles more than 12 seconds knowing the baby smiles more than 8 seconds.

Method 1 (write a new PDF):

$X \sim U(8, 23)$

$$\mathrm{p}(x) = \frac{1}{23-8} = \frac{1}{15}$$

$$\mathrm{P}(23 > x > 12\,) = \frac{(23-12)}{15} \approx 0.7333$$

Method 2 (bayes rule):

$$\mathrm{P}(x > 12 \mid x > 8\,) = \frac{P(x > 12 \text{ and } x > 8)}{P(x > 8)} = \frac{P(x > 12)}{P(x > 8)} = \frac{(23-12)\times\frac{1}{23}}{(23-8)\times\frac{1}{23}} \approx 0.7333$$

Uniform Continuous Distribution

numpy.random.uniform

numpy.random.uniform(*low=0.0, high=1.0, size=None*)

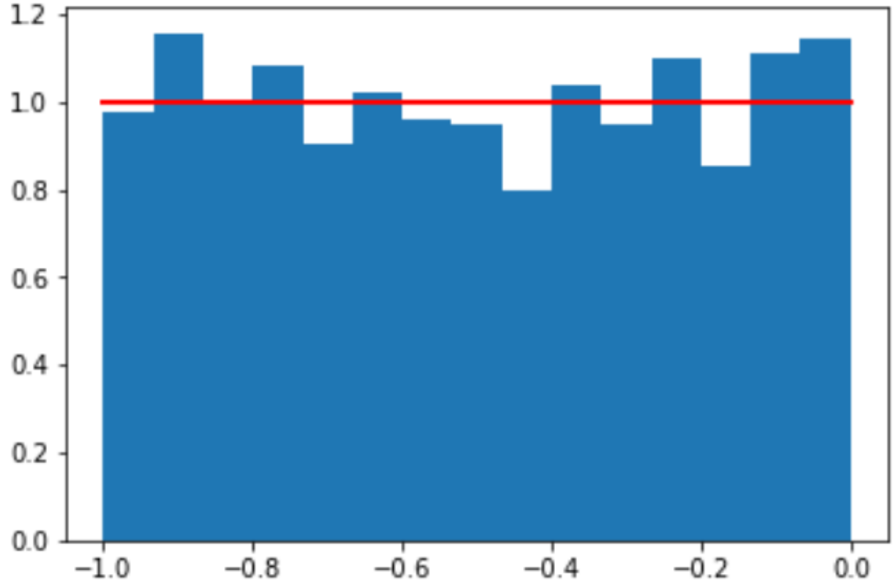
Draw samples from a uniform distribution.

Samples are uniformly distributed over the half-open interval `[low, high)` (includes low, but excludes high). In other words, any value within the given interval is equally likely to be drawn by **uniform**.

Example Draw 1,000 samples from a uniform on [-1,0),

```
a = -1
b = 0
N = 1000
X = np.random.uniform(a,b,N)
count, bins, ignored = plt.hist(X, 15, density=True)
plt.plot(bins, np.ones_like(bins), linewidth=2, color='r')
plt.show()
```

redline: PDF of uniform distr.

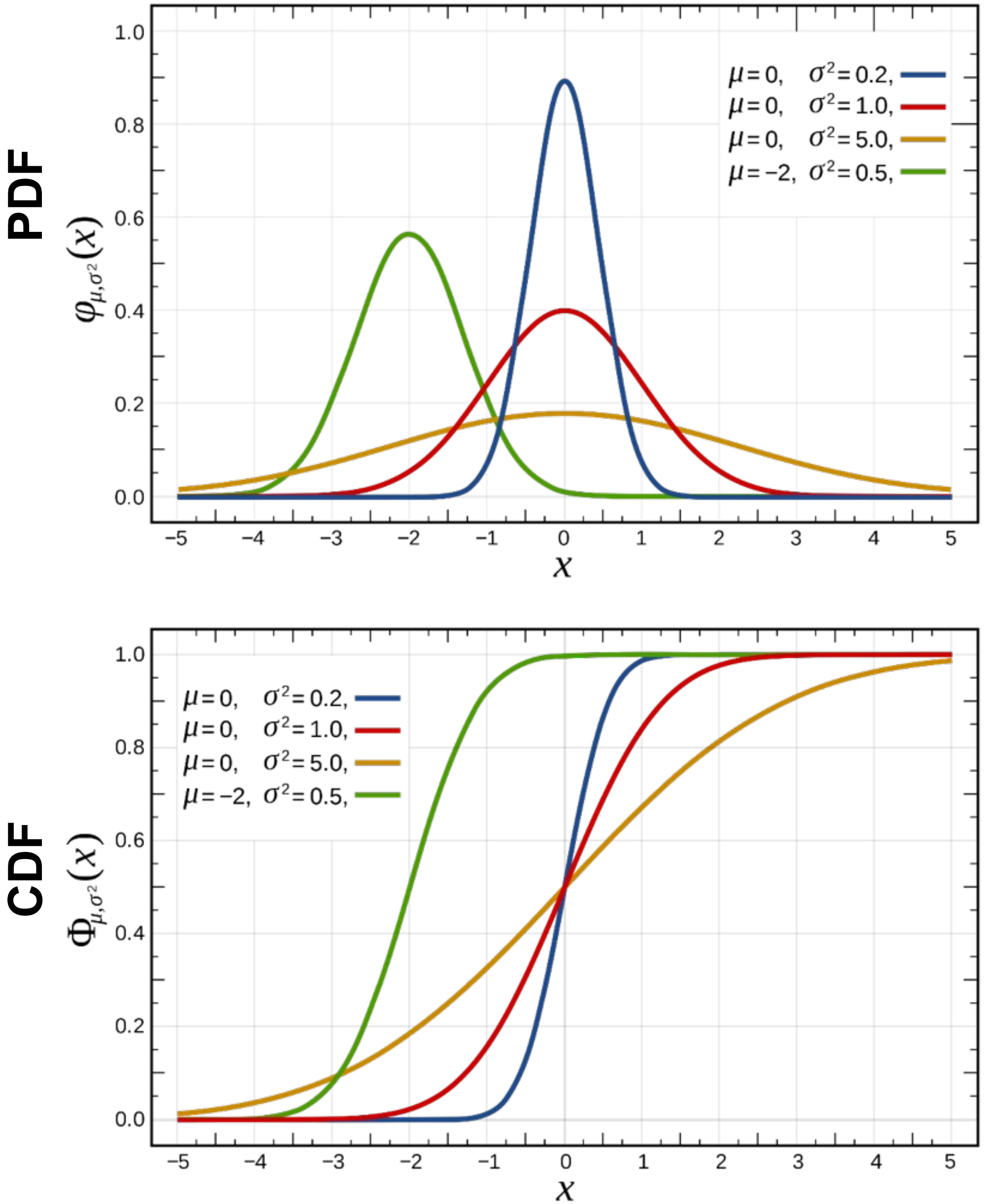


Gaussian/Normal Distribution

Gaussian (a.k.a. Normal) distribution with mean mean (location) μ and variance (scale) σ^2 parameters,

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

Compactly, $X \sim \mathcal{N}(\mu, \sigma^2)$

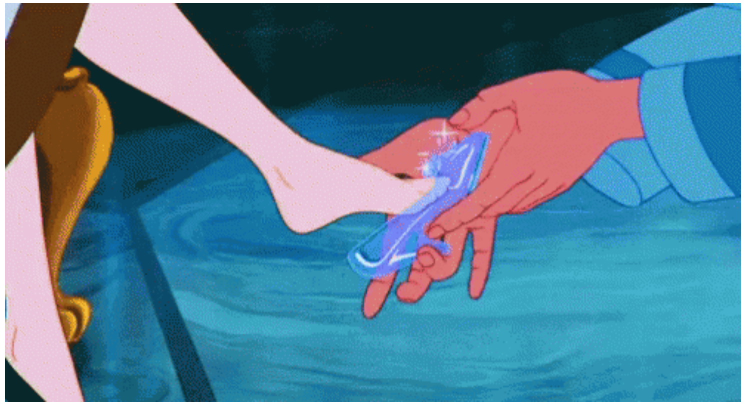


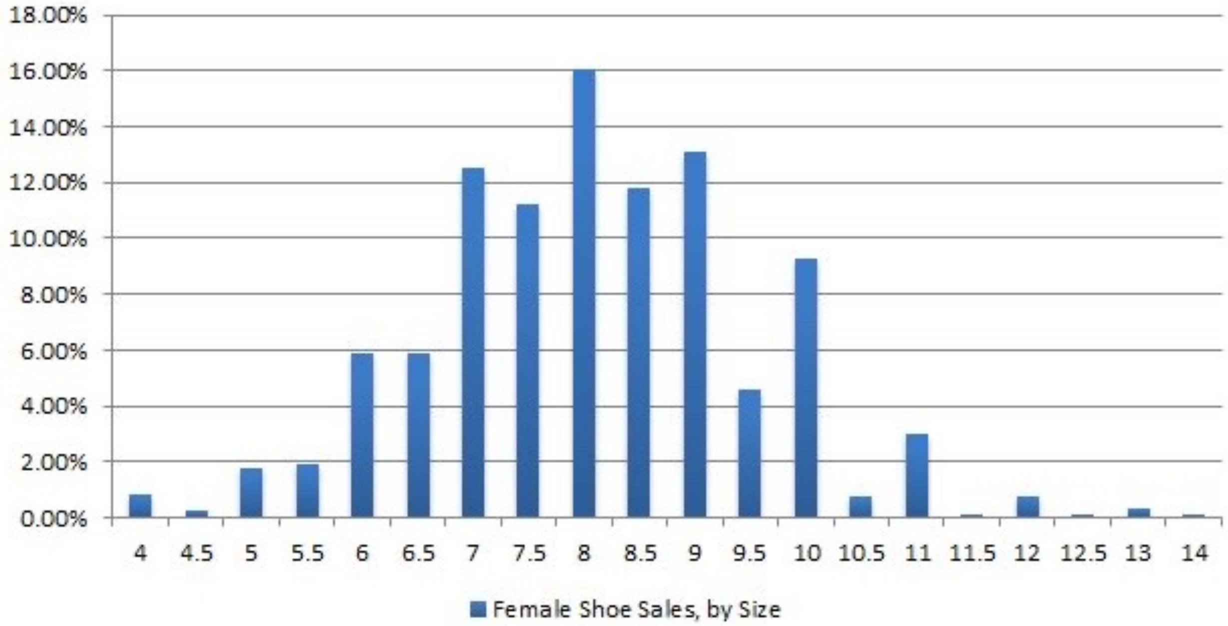
Observations:

- Larger σ^2 : $p(x)$ more “spread out”
- Larger μ : $p(x)$ ’s center shifts to the right more

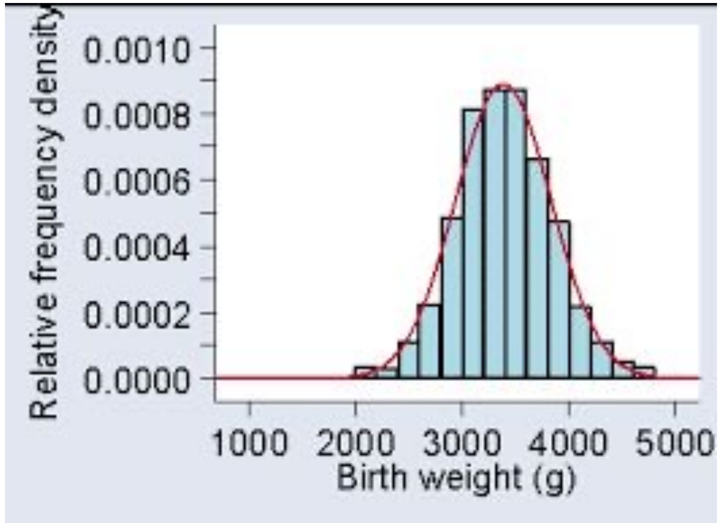
Things that follow Gaussian

Female shoe size





Birth Weight



(From https://studiousguy.com/real-life-examples-normal-distribution/)

numpy.random

numpy.random.normal

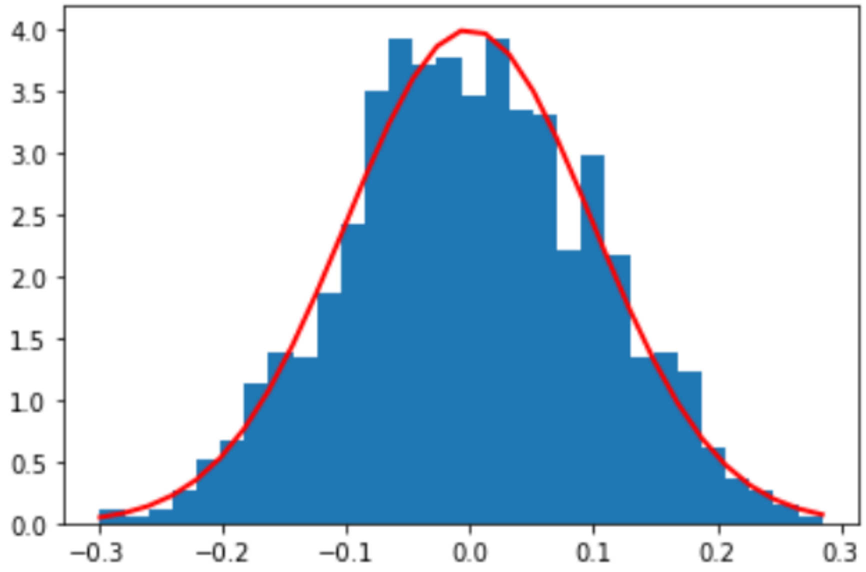
scale = $\sqrt{\sigma^2}$

numpy.random.normal(*loc=0.0, scale=1.0, size=None*)

Draw random samples from a normal (Gaussian) distribution.

Example Sample zero-mean gaussian with scale 0.1,

```
mu, sigma = 0, 0.1 # mean and standard deviation
X = np.random.normal(mu, sigma, 1000)
count, bins, ignored = plt.hist(X, 30, density=True)
plt.plot(bins, 1/(sigma * np.sqrt(2 * np.pi)) *
         np.exp( - (bins - mu)**2 / (2 * sigma**2) ),
         linewidth=2, color='r')
plt.show()
```



`bins`: length 31, consisting of boundary points

Recap

Useful discrete distributions

- Bernoulli → "Coinflip Distribution"
- Binomial → Multiple Bernoulli draws

Continuous probability

- P(X=x) = 0 does not mean you won't see x
- Probabilities assigned to *intervals* via CDF P(X > x)
- PDF measures probability *density* of single points p(X=x) >= 0

Useful continuous distributions

- Exponential → waiting time.
- Univariate / Multivariate Gaussian → Probably most ubiquitous distribution
- There are a lot more we will touch on later in the course…