Computer Science

CSC380: Principles of Data Science

Probability Primer 2

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Review

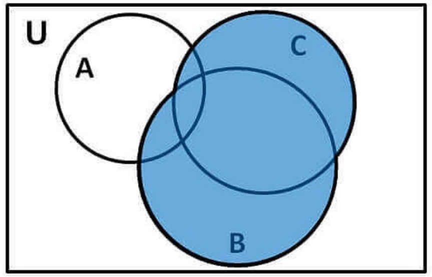
- What is probability?
- Axioms
- Event = set ⇒ use set theory!
- Set theory + axiom 3 is quite useful
- Draw diagrams
- Lots of jargons

- Make your own cheatsheet.

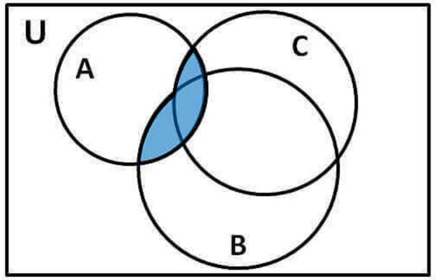
Review

- $A \cap (B \cup C) = (A \cap \mathrm{B}) \cup (A \cap \mathrm{C})$

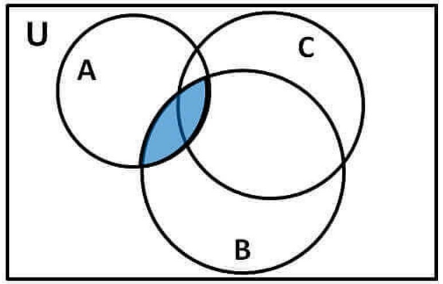
distributive law by Venn diagram



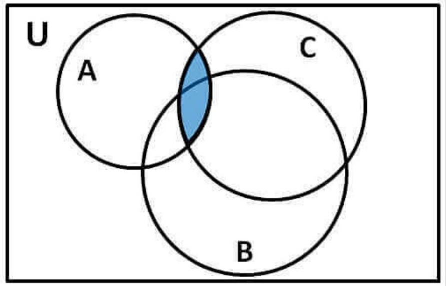
(B ∪ C)



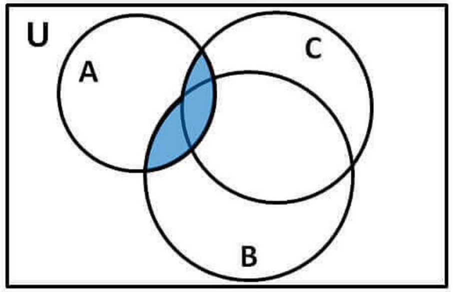
A ∩ (B ∪ C)



(A ∩ B)

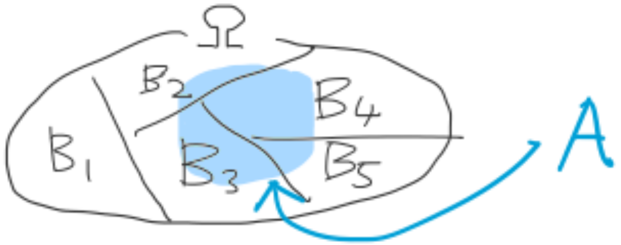


(A ∩ B)



(A ∩ B) ∪ (A ∩ C)

Review



- $A \cap (B \cup C) = (A \cap \mathrm{B}) \cup (A \cap \mathrm{C})$
- $A = A \cap \Omega = A \cap (\cup_i B_i) = \cup_i (A \cap B_i)$

$$= \mathrm{A} \cap (B_1 \cup B_2 \cup B_3 \ldots \cup B_n)$$

$$= (\mathrm{A} \cap B_1) \cup (\mathrm{A} \cap B_2) \cup (\mathrm{A} \cap B_3) \ldots \cup (\mathrm{A} \cap B_n)$$

Law of total probability*:* Let A be an event. For any events $\mathrm{B}_1, \mathrm{B}_2, \ldots$ that partitions Ω, we have

$$P(A) = \sum_i P(A \cap B_i)$$

Review

$$P(A) = \sum_i P(A \cap B_i)$$

A: the customer (100)

B: fill gas

- B_1: unleaded (30)
- B_2: mid grade (30)
- B_3: premium (40)

30, 10   30, 15   40, 5

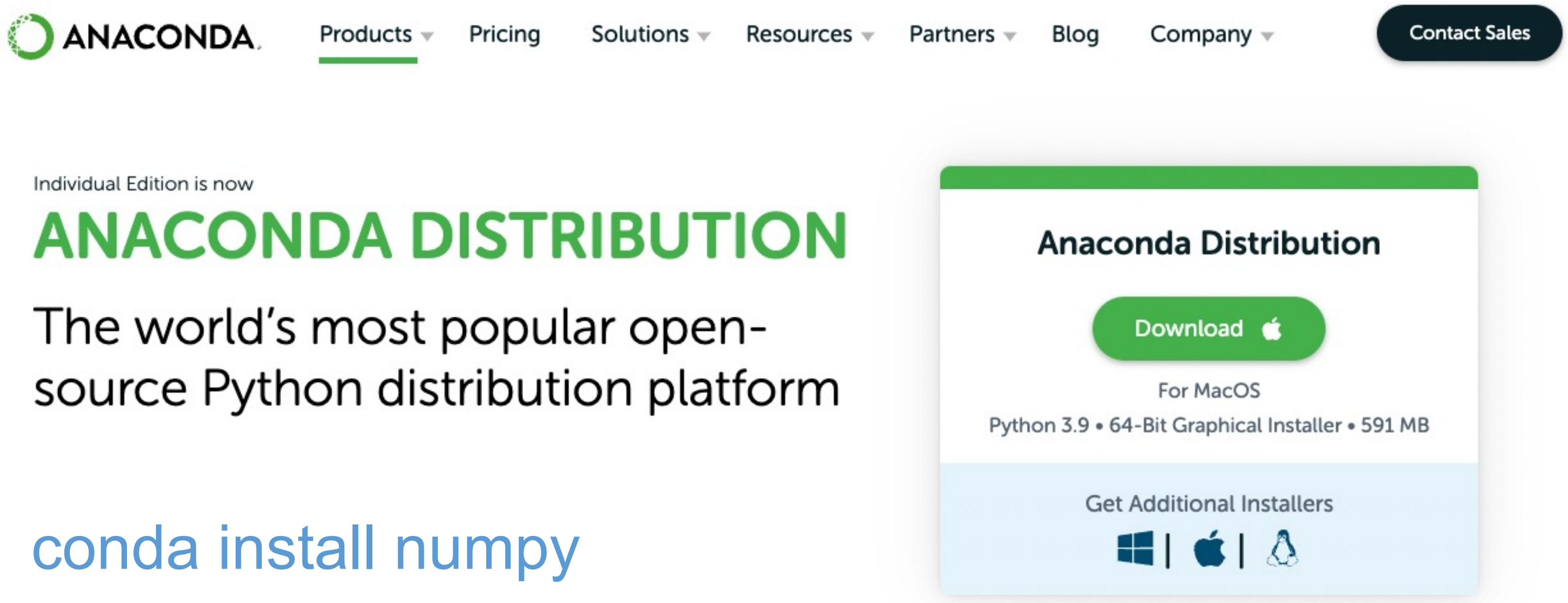
$$\mathrm{P}(\mathrm{A} = \mathrm{student})$$
$$= \mathrm{P}(\mathrm{A} = \mathrm{student}, \mathrm{B} = \mathrm{B}_1) + \mathrm{P}(\mathrm{A} = \mathrm{student}, \mathrm{B} = \mathrm{B}_2) + \mathrm{P}(\mathrm{A} = \mathrm{student}, \mathrm{B} = \mathrm{B}_3)$$
$$= \mathrm{P}(\mathrm{A} = \mathrm{student} | \mathrm{B} = \mathrm{B}_1)\mathrm{P}(\mathrm{B} = \mathrm{B}_1) + \mathrm{P}(\mathrm{A} = \mathrm{student} | \mathrm{B} = \mathrm{B}_2)\mathrm{P}(\mathrm{B} = \mathrm{B}_2) +$$
$$\mathrm{P}(\mathrm{A} = \mathrm{student} | \mathrm{B} = \mathrm{B}_3)P(\mathrm{B} = \mathrm{B}_3)$$

Overview

- Numpy package
- Conditional probability
- Independence

Numpy Library

Package containing many useful numerical functions…



conda install numpy

If you use pip:
pip install numpy

…we are interested in numpy.random at the moment

numpy.random

numpy.random.randint

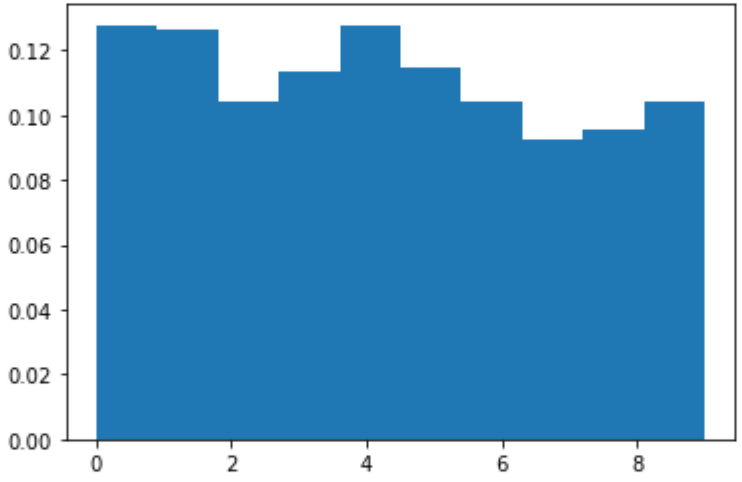
numpy.random.randint(*low, high=None, size=None, dtype='l'*)

Return random integers from *low* (inclusive) to *high* (exclusive).

Return random integers from the "discrete uniform" distribution of the specified dtype in the "half-open" interval [*low, high*). If *high* is None (the default), then results are from [0, *low*).

Sample a discrete uniform random variable,

```
import matplotlib.pyplot as plt
X = np.random.randint(0,10,1000)
count, bins, ignored = plt.hist(X, 10, density=True)
plt.show()
```



- **Caution** Interval is [low,high) and upper bound is **exclusive**
- `Size` argument accepts tuples for sampling ndarrays (multidimentional arrays)

numpy.random

Allows sampling from many common distributions

Set (global) random seed as,

```
import numpy as np

seed = 12345
np.random.seed(seed)
```

- ☺ easier to debug (otherwise, you may have ‘stochastic’ bug)
- ☹ can be risky

E.g., buy into the result based on a particular seed, publish a report.
… turns out, you get a widely different result if you use a different seed!

Recommendation: change the seed every now and then

Conditional Probability

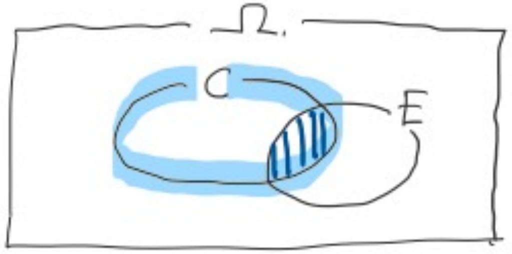
Conditional Probability

- Two fair dice example:
  - Suppose I roll two dice secretly and tell you that <u>one of the dice is 2</u>. C
  - <u>**In this situation**</u>, find the probability of two dice summing to 6. E

```
import numpy as np
for n in [10,100,1000,10_000,100_000, 1_000_000]:
    res_dice1 = np.random.randint(6,size=n) + 1
    res_dice2 = np.random.randint(6,size=n) + 1
    res = [(res_dice1[i], res_dice2[i]) for i in range(len(res_dice1))]

    conditioned = list(filter(lambda x: x[0] == 2 or x[1] == 2, res))
    n_eff = len(conditioned)

    cnt = len(list(filter(lambda x: x[0] + x[1] == 6, conditioned)))
    print("n=%9d, n_eff=%9d, result: %.4f " % (n, n_eff, cnt/n_eff))
```



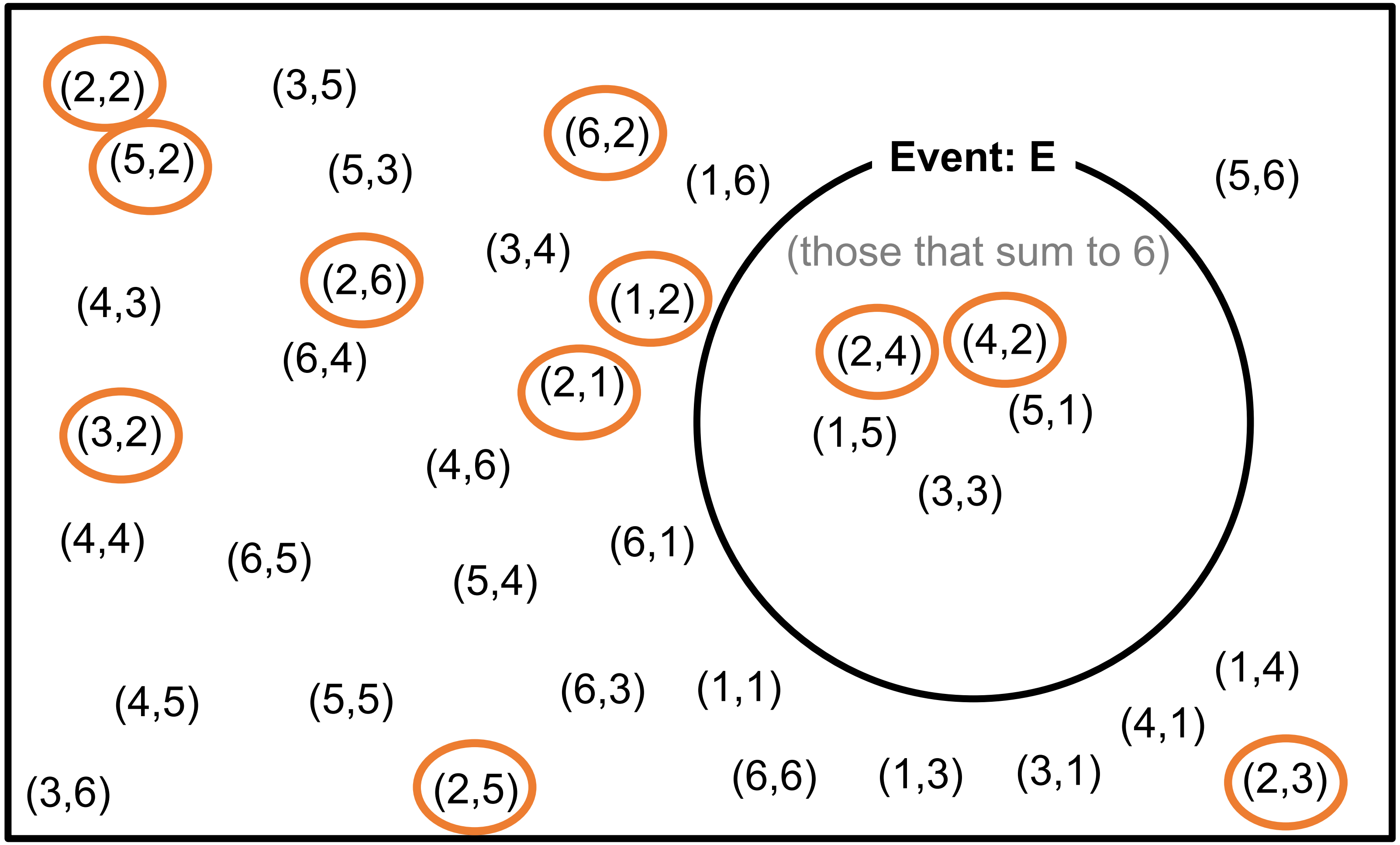
compare:
without conditioning,
it was 0.138.

```
n=        10, n_eff=         4, result: 0.0000
n=       100, n_eff=        32, result: 0.2500
n=      1000, n_eff=       300, result: 0.1733
n=     10000, n_eff=      3002, result: 0.1742
n=    100000, n_eff=     30590, result: 0.1823
n=   1000000, n_eff=    305616, result: 0.1818
```

```
n=        10, n_eff=         3, result: 0.3333
n=       100, n_eff=        32, result: 0.0625
n=      1000, n_eff=       343, result: 0.2245
n=     10000, n_eff=      3062, result: 0.1897
n=    100000, n_eff=     30651, result: 0.1811
n=   1000000, n_eff=    305580, result: 0.1808
```

Random Events and Probability

What is the probability of having two numbers sum to 6 given one of dice is 2?



Each outcome is equally likely.
by the **independence**
(will learn this concept later)
=> 1/36

\# sum to 6:
=> 5

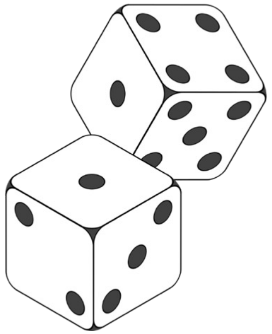
\# one of dice is 2:
=> 11

\# sum to 6 and one of dice is 2:
=> 2

answer:
2/11 = 0.181818….

Conditional Probability

Two fair dice example

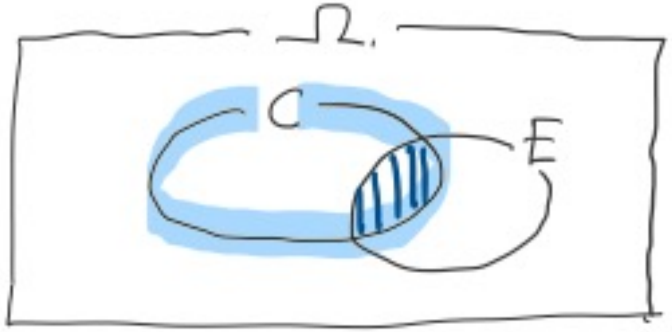


- Find the probability of one of the dice is 2 (event C) and two dice summing to 6 (E)

$$P(E \cap C)$$

- I secretly tell you one of the dice is 2, find the probability of two dice summing to 6.

$$\frac{P(E \cap C)}{P(C)}$$



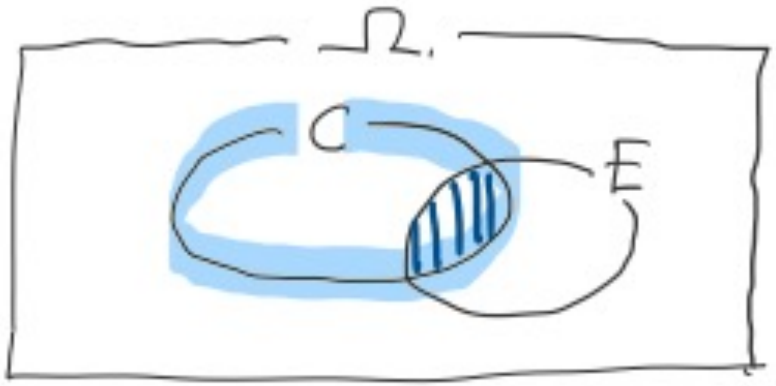
Conditional Probability

- Two fair dice example:
  - Suppose I roll two dice and secretly tell you that one of the dice is 2. C
  - **<u>In this situation</u>**, find the probability of two dice summing to 6. E

- Turns out, such a probability can be computed by $\frac{P(E \cap C)}{P(C)}$
- It's like "zooming in" to the condition.
- This happens a lot in practice, so let's give it a notation:

$$P(E|C) := \frac{P(E \cap C)}{P(C)}$$

Say: probability of "E given C", "E conditioned on C"



"it's the ratio"

Conditional Probability

Q: Conditional probability P(A|B) could be undefined. When?

- A: The denominator can be 0 already. In this case, numerator is also 0!

Note $P(A|B) \neq P(B|A)$ in general!

$$P(A|B) \coloneqq \frac{P(A \cap B)}{P(B)}$$

E.g., throw a fair die. X := outcome. A = {X=4}, B = {X is even}

Question: P(A | B) = P(B | A) ?

- $\mathrm{P(A)} = 1/6$
- $\mathrm{P(B)} = 1/2$
- $\mathrm{P\ (A \cap B)} = 1/6$
- Therefore, $\mathrm{P(A|B)} = 1/3$, $\mathrm{P(B|A)} = 1$

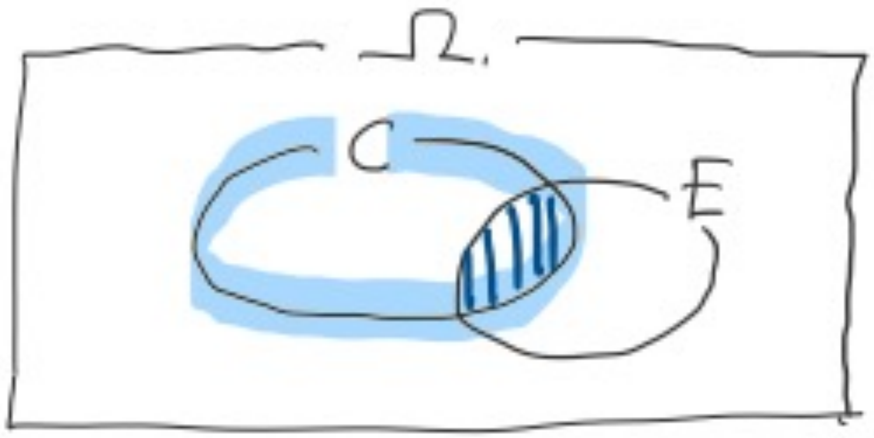
Conditional Probability

Chain rule

- $P(A \cap B) = P(A|B)P(B)$ ←just a rearrangement of definition: $P(A|B) \coloneqq \frac{P(A \cap B)}{P(B)}$
- $P(A \cap B \cap C) = P(A|B \cap C)P(B \cap C) = P(A|B \cap C)P(B|C)P(C)$
- $P(E_1 \cap E_2 \cap \cdots \cap E_n) = P(E_1)\prod_{i=2}^{n} P(E_i | \cap_{j=1}^{i-1} E_j)$ valid for any ordering!

Conditional Probability

- $P(E \cap C) = P(E|C)P(C) = P(C|E)P(E)$

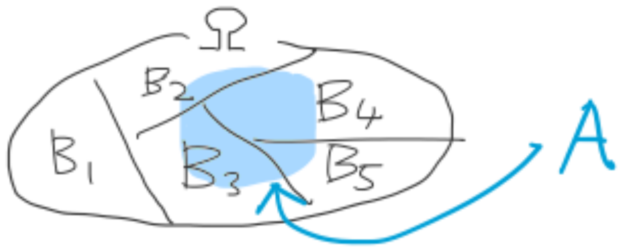


“it’s the ratio”

Conditional Probability

Recall: let A be an event. For events $\mathrm{B}_1, \mathrm{B}_2, \ldots$ that partitions Ω, we have

$$P(A) = \sum_i P(A \cap B_i)$$



$$A = A \cap \Omega = A \cap (\cup_i B_i) = \cup_i (A \cap B_i)$$

Check axiom 3 & distributive law!

$$\boxed{\begin{aligned} &P(E \cap C) \\ &= P(E|C)P(C) \\ &= P(C|E)P(E) \end{aligned}}$$

Law of total probability: If $A \in \mathcal{F}$ and $\{B_i \in \mathcal{F}\}_i$ partitions Ω, then

$$P(A) = \sum_i P(A, B_i) = \sum_i P(B_i)P(A|B_i)$$

$$= \sum_i P(A)P(B_i|A)$$

(by definition)

Shortcut:
P(A,B) := P(A ∩ B)

Review

- $P(A) = \sum_i P(A, B_i) = \sum_i P(B_i)P(A|B_i)$

$\mathrm{P(A = student)}$
$\mathrm{= P(A = student|B = B_1)P(B = B_1) + P(A = student|B = B_2)P(B = B_2) + P(A = student|B = B_3)}P\mathrm{(B = B_3)}$

$\mathrm{P(A = student)}$
$= 10/30 \times 30/100 + 15/30 \times 30/100 + 5/40 \times 40/100$

- $\sum_i P(B_i|A) = 1$

$\mathrm{P(B_1|A = student) + P(B_2|A = student) + P(B_3|A = student)}$
$= \frac{10}{10+15+5} + \frac{15}{10+15+5} + \frac{5}{10+15+5} = 1$


30, 10


30, 15


40, 5

Conditional Probability

The Public Health Department gives us the following information:

- A test for the disease yields a positive result (+) 90% of the time when the disease is present (Y)

  $P(+ \mid Y) = 0.9$

- A test for the disease yields a positive result 1% of the time when the disease is not present (N)

  $P(+ \mid N) = 0.01$

- One person in 1,000 has the disease. $P(Y) = 0.001$

Q: What is the probability that a person with positive test has the disease? $P(Y \mid +)$?

Pick a person uniformly at random from the population. Apply the test. When test=+, what is the probability of this person having the disease (Y) ?

Conditional Probability

What we know:

P(+ | Y) = 0.9
P(+ | N) = 0.01
P(Y) = 0.001

$\Rightarrow$

P(- | Y) = 0.1
P(- | N) = 0.99
P(N) = 0.999

Question: P(Y | +)

$$= \frac{P(Y,+)}{P(+)}$$

$$P(+) = P(+,Y) + P(+,N)$$

$$P(+,Y) = P(+|\,Y)P(Y)$$

$$P(+,N) = P(+|\,N)P(N)$$

Law of total probability

$$P(A) = \sum_i P(A,B_i) = \sum_i P(B_i)P(A|B_i)$$

The answer is 0.0826…

Terminology

When we have two events A and B…

- Conditional probability: $P(A|B), P(A^c|B), P(B|A)$ etc.
- Joint probability: $P(A, B)$ or $P(A^c, B)$ or ...
- Marginal probability: $P(A)$ or $P(A^c)$

Conditional Probability

Tip: Make a table of **joint probabilities**

P(+ | Y) = 0.9
P(+ | N) = 0.01
P(Y) = 0.001

Each cell is P(column event ∩ row event) = P(T=t ∩ D=d) = P(T= t | D= d) P(D=d)

	Test = +	**Test = -**	
Disease=Y	$0.9 \cdot 0.001 = 0.0009$	$0.1 \cdot 0.001 = 0.0001$	0.001
Disease=N	$0.01 \cdot 0.999 = 0.00999$	$0.99 \cdot 0.999 = 0.98901$	0.999
	0.01089	0.98911	

$P(test = +)$

Workflow:

- make a table, then fill in the cells.
- write down the target P(A|B) all in terms of joint probabilities and marginal probabilities.

Conditional Probability

We can directly calculate:

$$P(Y|+) = \frac{P(Y,+)}{P(+)} = \frac{P(+|Y)P(Y)}{P(+)}$$

Bayes rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

proof: definition and definition!

$\Longrightarrow$ particularly useful in practice: infer P(A|B) given P(B|A)!

P(A): **prior** probability

P(A|B): **posterior** probability

e.g., A='dice sum to 6', B='one of the die is 2'

e.g., A='disease=Y', B='test=+'

Independence

Independence

- Informally, given two events A and B, they are **independent** if the probability of A is not affected by whether B is true or false (and vice versa)
  - E.g., A = "die1=1" and B="die2=1" are independent.

    $\Longrightarrow$ we know that the probability of die1 being 1 would not be changed just because die2=1.

- Mathematically, this can be written as $P(A|B) = P(A)$ or $P(B|A) = P(B)$.

- E.g., A = "die1=6" and B="two dice sum to 6" are **not independent**.

  quiz candidate

  $\because$ intuitively, when B is true, A can never happen! So, P(A|B)=0 but P(A) = 1/6.

- E.g., A = "die1=1" and B="two dice sum to 6" are **not independent**.

  quiz candidate

  $\because$ P(A) = 1/6 = 0.166... . However, P(A|B) = 1/5 = 0.2

$$\boxed{P(A|B) = \frac{P(A,B)}{P(B)}}$$

Independence

[Def] Two events A and B are **independent** if

$$P(A,B) = P(A)P(B)$$

"joint probability is product of two marginal probabilities"

$A \perp B$ means A and B are independent

=> note: symmetric!

Also, a set of events $\{A_i\}_{i=1}^{n}$ (n can be ∞) are **mutually independent** if

for every J ⊆ $\{1, \ldots, n\}$, we have $P\left(\cap_{i \in J} A_i\right) = \prod_{i \in J} P(A_i)$

Independence

- Ex) recall two fair dice
  - We took it for granted that P((1,1)) is 1/36.
  - But why is it true, really?
  - To be rigorous,

$$P(die1 = 1, die2 = 1) = P(die1 = 1)P(die2 = 1) = \frac{1}{6} \cdot \frac{1}{6}$$

  due to independence.

or, …= P(die1=1 | die2=1) * P(die2=1) = P(die1=1) * P(die2=1)

- E.g., two biased coin **C1** and **C2**. Suppose P(C1=H) = 0.3 and P(C2=H) = 0.4. Compute the probability of P(C1=H,C2=T).

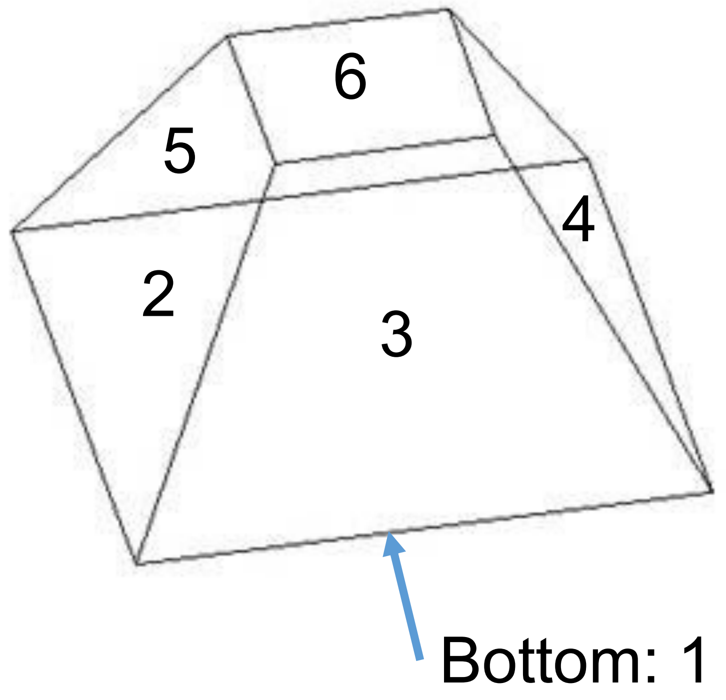
0.3·0.6 = 0.18

quiz candidate

Independence

- Ex) Unfair die
  - Let A be the outcome of a single throw.
  - P(A=1) << P(A=2) = … = P(A=5) << P(A=6)

say, 0.1 0.15 0.15. 0.3



- Throw this die twice. What's the probability of observing (1,3)?
  - P((1,3)) = 0.1*0.15 = 0.015
- Similarly, P((6,3)) = 0.3*0.15 = 0.045

Example: Dependent Coin Flips

- First coin (X1): fair coin
- Second coin (X2):
  - if X1=H, throw a **fair** coin.
  - If X1=T, throw an **unfair** coin P(H) = 0.2, P(T) = 0.8

- Q: Are X1=H and X2=H independent or not?

P(X1=H) = ____ 0.5

P(X2=H) = ____ = P(X2=H,X1=H) + P(X2=H,X1=T) = 0.25 + 0.1 = 0.35

P(X1=H, X2=H) = ____ 0.25

P(X1=H)*P(X2=H) = 0.175

Quiz candidate

Review

Axiom 3:

For any *finite* or *countably infinite* sequence of disjoint events $E_1, E_2, E_3, \ldots,$ $\quad P\left(\bigcup_{i\geq 1} E_i\right) = \sum_{i\geq 1} P(E_i)$

Inclusion-exclusion rule: $\quad P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

Law of total probability: For events $\mathrm{B}_1, \mathrm{B}_2, \ldots$ that partitions Ω, $\quad P(A) = \sum_i P(A \cap B_i)$

Conditional probability: $\quad P(E|C) \coloneqq \dfrac{P(E \cap C)}{P(C)} \quad$ ($P(A|B) \neq P(B|A)$ in general)

Probability chain rule: $\quad P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C)$

Bayes' rule: $\quad P(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$

Independence: (definition) A and B are independent if $P(A, B) = P(A)P(B)$

(property) A and B are independent if and only if $P(A|B) = P(A)$ (or $P(B|A) = P(B)$)