FROM CHEREDNIK ALGEBRAS TO KNOT HOMOLOGY VIA CUSPIDAL $\mathcal{D}$-MODULES

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Abstract. We show that the triply-graded Khovanov-Rozansky homology of the torus knot $T_{m,n}$ can be recovered from the finite-dimensional representation $\mathrm{L}_{\frac{m}{n}}$ of the rational Cherednik algebra at slope $\frac{m}{n}$, endowed with the Hodge filtration coming from the cuspidal character $\mathcal{D}$-module $\mathbf{N}_{\frac{m}{n}}$. Our approach involves expressing the associated graded of $\mathbf{N}_{\frac{m}{n}}$ in terms of a DG module closely related to the action of the elliptic Hall algebra on the equivariant K-theory of the Hilbert scheme of points on $\mathbb{C}^2$, thereby proving the rational master conjecture. As a corollary, we identify the Hodge filtration with the inductive and algebraic filtrations on $\mathrm{L}_{\frac{m}{n}}$.

Contents

1. Introduction

1.1. **Rational Cherednik algebras and link homology.** Recent years have seen extraordinary connections between seemingly unrelated mathematical objects across different fields, indexed by pairs of coprime natural numbers m, n. Topologically, there is the (m, n)-torus knot $T_{m,n}$, which winds m times around a circle in the interior of the torus, and n times around its axis of rotational symmetry. Surprisingly, the Khovanov-Rozansky(KhR) homology of $T_{m,n}$ is captured by the finite-dimensional simple representation of the rational Cherednik algebra.

We first briefly recall the definition of the KhR homology: for any element γ in the braid group, one can associate to it a Rouquier complex built on Soergel bimodules labeled by simple twists. For each i, take the i-th Hochschild homology of each term of the Rouquier complex to form a new complex. The sum over i of cohomology of these new complexes, usually denoted by $\mathrm{HHH}(\overline{\gamma})$, only depends on the link closure $\overline{\gamma}$ of the braid γ and defines the KhR homology of $\overline{\gamma}$ [Kho07]. Furthermore, under this definition, the KhR homology is naturally triply graded by the Hochschild homological grading, the internal grading of the Soergel bimodules and the cohomological grading.

On the representation theory side, for $n > 0$, we let $\mathfrak{h}$ denote the $(n-1)$-dimensional standard representation of the symmetric group S_n and $\mathcal{D}(\mathfrak{h})$ be the ring of differential operators on $\mathfrak{h}$. The rational Cherednik algebra (RCA) [EG02] $\mathrm{H}_c(S_n, \mathfrak{h})$, also known as the rational degeneration of the double affine Hecke algebra, is a deformation of $\mathcal{D}(\mathfrak{h}) \rtimes S_n$ at a complex parameter c. It turns out that only when $c = \frac{m}{n}$ for coprime m, n does $\mathrm{H}_c(S_n, \mathfrak{h})$ have finite-dimensional representations; in that case, there is an unique irreducible one [BEG03], which we denote by $\mathrm{L}_{\frac{m}{n}}$. There is a Euler grading on $\mathrm{L}_{\frac{m}{n}}$ induced by the action of the Euler field $\mathrm{h}_{\frac{m}{n}}$ (cf (16)). This grading induces a direct sum decomposition $\mathrm{L}_{\frac{m}{n}} = \bigoplus_{\ell\in\mathbb{Z}} \mathrm{L}_{\frac{m}{n}}(\ell)$.

Gorsky, Oblomkov, Rasmussen and Shende made the following remarkable observation:

Conjecture 1.1. *([GORS14, Conjecture 1.2]) For positive coprime integers m, n, there exists a filtration F on $\mathrm{L}_{\frac{m}{n}}$ compatible with the order filtration on $\mathrm{H}_{\frac{m}{n}}(S_n, \mathfrak{h})$ and the $\mathrm{h}_{\frac{m}{n}}$-grading such that there is an isomorphism of triply graded vector spaces:*

$$\bigoplus_{i,j,k} \mathrm{HHH}^{i,j,k}(T_{m,n}) \cong \bigoplus_{i} \mathrm{Hom}_{S_n}\big(\wedge^i \mathfrak{h}, \bigoplus_{j,k} \mathrm{gr}^F_j\big(\mathrm{L}_{\frac{m}{n}}(k)\big)\big). \tag{1}$$

The following gradings are matched:

$$\begin{aligned} \textit{Hochschild homological } a\textit{-grading} &\leftrightarrow \textit{degree of } \wedge^{\bullet}\mathfrak{h} \\ \textit{internal } q\textit{-grading} &\leftrightarrow \mathrm{h}_{\frac{m}{n}}\textit{-grading} \\ \textit{cohomological } t\textit{-grading} &\leftrightarrow \textit{filtration on } \mathrm{L}_{\frac{m}{n}} \end{aligned}$$

A bigraded isomorphism, disregarding the information of the t-grading, is established in [GORS14] by comparing the calculation results from both sides of (1), as obtained in [Jon87] and [BEG03].

For every coprime pair (m, n), $m > n > 0$, we define a Hodge filtration on $\mathrm{L}_{\frac{m}{n}}$ and show that:

Theorem A. *(See Theorem 6.7) When $m > n$ such that $(m, n) = 1$, Conjecture 1.1 holds with respect to the Kazhdan filtration associated to the Hodge filtration and h_c.*

For the cases when $0 < m < n$, see Corollary 1.3.

1.2. **Hilbert schemes and the rational master conjecture.** Write $c = \frac{m}{n}$ for positive coprime integers m, n. Our approach is to construct a coherent sheaf $\mathcal{G}_c$ on Hilb^n, the Hilbert scheme of n points on $\mathbb{C}^2$. This sheaf is used to compute the graded dimensions of the right hand side of isomorphism (1). More specifically, consider the $\mathbb{C}^* \times \mathbb{C}^*$-action on Hilb^n induced by the scaling action on $\mathbb{C}^2$. The equivariant K-theory of Hilb^n, denoted by $K^{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{Hilb}^n)$, is a module over $K^{\mathbb{C}^*\times\mathbb{C}^*}(pt) \cong \mathbb{C}[q^{\pm}, t^{\pm}]$. The generalized McKay correspondence, proved by Haiman, [Hai01] implies an isomorphism

$$\bigoplus_{n} \mathbb{C}(q,t) \otimes_{\mathbb{C}[q^{\pm},t^{\pm}]} K^{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{Hilb}^n) \cong \mathbb{C}(q,t)[z_1, z_2, \dots]^{S_\infty}. \tag{2}$$

The variables t, q turn out to correspond to the j, k gradings in (1) respectively. The i grading comes from the coefficient of the Schur polynomial labeled by the S_n-representation $\wedge^i\mathfrak{h}$, known as the hooks.

Beyond the hooks, Gorsky and Neguţ conjectured that the bigraded Frobenius character of $\mathrm{L}_{\frac{m}{n}}$ can be expressed via the action of a generator $P_{m,n}$ of the elliptic Hall algebra (EHA) on its polynomial representation, identified with (2).

Conjecture 1.2. [GN15, Conjecture 5.5] *For positive coprime integers m, n, the bigraded Frobenius character of L_c with respect to an appropriate filtration and the $\mathrm{h}_{\frac{m}{n}}$-grading is given by*

$$\mathrm{ch}_{S_n\times\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{L}_{\frac{m}{n}}) = (P_{m,n}\cdot 1)(q, q^{-1}t). \tag{3}$$

Theorem B. *(See Theorem 6.5) When $m > n$ such that $(m, n) = 1$, Conjecture 1.2 holds with respect to the Hodge filtration.*

For the cases when $0 < m < n$, also see Corollary 1.3. In the case of $m = n+1$, $\mathrm{L}_{\frac{n+1}{n}}$ is isomorphic to the *diagonal harmonics* and the EHA generator action can be identified with the action of the nabla operator [BG98]. Therefore, Conjecture 1.2 may be viewed as a rational generalization of the master conjecture [GH96] proved by Haiman [Hai02] via the geometry of the Procesi bundle. We explicitly compute the left hand side of (3) by computing the bigraded Frobenius characters of fibers of the sheaf $\mathcal{G}_c$ at fixed points and using localization formula. Comparing it with the computation by Neguţ [Neg15a] of the right hand side of (3) leads to our proof of the GN conjecture.

This result relates to the previous section in the following way. The computation of the Khovanov-Rozansky homology for torus links had long been a challenging open problem. It was finally addressed by Elias, Hogancamp, and others ([Hog17, EH19, Hog18] etc.) through recursive methods. The culmination of this work is a shuffle conjecture style formula of Mellit [Mel22] based on his work with Carlsson [CM18]. From this formula, it follows that the Euler characteristic of $\mathrm{HHH}(T_{m,n})$ equals the knot superpolynomial of $T_{m,n}$ defined in terms of $P_{m,n} \cdot 1$ [GN15]. Therefore, Theorem B implies Theorem A.

1.3. **DG flag commuting varieties.** To construct the coherent sheaf $\mathcal{G}_c$ on Hilb^n, we consider the cuspidal mirabolic D-module $\overline{\mathbf{N}}_c$ on $\mathfrak{sl}_n \times \mathbb{C}^n$ of central character c, closely related to the cuspidal character sheaf in the sense of Lusztig [Lus86]. A result of Calaque, Enriques and Etingof [CEE09] implies that $(\mathrm{L}_c)^{S_n}$ is the *quantum Hamiltonian reduction* of $\overline{\mathbf{N}}_c$. We take the associated graded of $\overline{\mathbf{N}}_c$ with respect to the Hodge filtration (Hodge associated graded for short henceforth) within the framwork of Saito's theory of mixed Hodge modules [Sai90]. The descent of $\mathrm{gr}(\overline{\mathbf{N}}_c)$ is the desired coherent sheaf on Hilb^n.

As noted by Neguţ in [Neg15a], the action of the generator $P_{m,n}$ on $K^{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{Hilb}^n(\mathbb{C}^2))$ is not visible using the classical Nakajima correspondence, and the notion of a flag Hilbert scheme turns out to be necessary. However, it is well-known that the naive flag Hilbert scheme is highly singular [Che98]. A remedy is to rather consider the DG flag Hilbert scheme $\mathrm{FHilb}^n_{\mathrm{dg}}(\mathbb{C}^2)$, defined independently in [Gin12] and [Neg15a]. In spite of its derived nature, $\mathrm{FHilb}^n_{\mathrm{dg}}(\mathbb{C}^2)$ is by definition a local complete intersection.

In [Gin12], Ginzburg studies the isospectral commuting variety by expressing the Hodge associated graded of the Harish-Chandra $\mathcal{D}$-module in terms of the DG flag commuting variety. In type A, this facilitates the study of the Procesi bundle from a different perspective, which leads to Gordon's new proof of the Macdonald positivity conjecture ([Gor12]).

Our setup parallels that of [Gin12] and we naturally obtain the *cuspidal DG module* $\mathcal{A}_c$ in the computation of $\mathrm{gr}(\overline{\mathbf{N}}_c)$. Motivated by the similar origins of $\mathcal{A}_0$ and the DG flag commuting variety, I expect that $\mathcal{A}_0$, whose definition is valid for all types, gives the correct definition for the DG nilpotent flag commuting scheme.

We also define a Catalan DG module, encoding the information of the q,t-Catalan number. We show the pushforwards of these two DG modules to the commuting variety correspond to the same equivariant K-theory classes and we state the sheaf-theoretic identification as a conjecture (Conjecture 5.2). While this paper was under editing, Gorsky and Neguţ generously shared a copy of their preprint [GN24]. Their Conjecture 2.2 is a non-derived version of our Conjecture 5.2.

1.4. **Filtrations and future directions.** Regarding Conjecture 1.1, the authors of [GORS14] have proposed several filtrations:

- The inductive filtration F^{ind} is defined inductively using the shift functor from $\mathrm{H}_{\frac{m}{n}}$-modules to $\mathrm{H}_{\frac{m}{n}+1}$-modules and the "flipping" isomorphism $(\mathrm{L}_{\frac{m}{n}})^{S_n} \cong (\mathrm{L}_{\frac{n}{m}})^{S_m}$.
- The algebraic filtration F^{alg} is defined using powers of Chern classes in the equivariant cohomology of the compactified Jacobian of the planar singular curve $y^m = x^n$.
- The geometric filtration $F^{\mathrm{geom}}$is defined in terms of the perverse filtration and the cohomological grading on the cohomology of a Hitchin fiber isomorphic to the compactified Jacobian as in (b).

The equality $F^{\mathrm{alg}} = F^{\mathrm{geom}}$ is shown in [OY16] and the equality $F^{\mathrm{ind}} = F^{\mathrm{alg}}$ is shown in [Ma24].

Theorem C. *(See Proposition 6.9)*

$$\begin{cases} F_i^{\mathrm{H}}\mathrm{L}_{\frac{m}{n}} = F_i^{\mathrm{ind}}\mathrm{L}_{\frac{m}{n}} & \forall i \in \mathbb{Z} \text{ when } m > n \\ F_i^{\mathrm{H}}(\mathrm{L}_{\frac{m}{n}})^{S_n} = F_i^{\mathrm{ind}}(\mathrm{L}_{\frac{m}{n}})^{S_n} & \forall i \in \mathbb{Z} \end{cases}$$

The inductive filtration and the Hodge filtrations are defined on L_c for $m > n$ and on $\mathrm{L}_c^{S_n}$ for all $m > 0$ while the algebraic filtration is well-defined on L_c for all $m > 0$.

As a corollary of Theorem C, we show that:

Corollary 1.3. *(See Proposition 6.10) For all integers $m > 0$ coprime to n, with respect to the algebraic filtration, Conjectures 1.1 and 1.2 hold.*

There are two natural generalizations of our setting: allowing m, n to be non-coprime or replacing $\mathrm{Hilb}^n(\mathbb{C}^2)$ by Gieseker varieties. The former is related to the study of RCA representations of minimal support and torus link homology ([EGL15]). The latter is related to the study of representations of a quantized Gieseker moduli algebra ([EKLS21]) and the study of higher Catalan numbers and a finite shuffle conjecture ([GSV23]). Despite the current absense of definitions of the inductive, algebraic and geometric filtrations in these settings, we believe that a notion of Hodge filtration is still available, and via a similar method, bigraded Frobenius characters can be computed and related to the corresponding link invariants or Catalan statistics.

On the other hand, it is conjectured that the stable envelopes on $\mathrm{Hilb}^n(\mathbb{C}^2)$ are closely related to Verma modules of Cherednik algebras [GN17, Conjecture 6.5]. In general, the connection between Verma modules over quantized symplectic resolutions and categorical stable envelopes is studied by Bezrukavnikov and Okounkov [BO] using positive characteristic technique. From a different perspective, suitable filtrations on Verma modules, satisfying properties such as compatibility with parabolic induction and restriction functors, can also validate such a program. We believe the Hodge filtration might be the correct candidate; however, it is neither clear nor has it been explored which explicit $\mathcal{D}$-modules correspond to the Verma modules. We plan to investigate these directions in future work.

1.5. **Structure of the paper.** Section 2 covers preliminaries. We introduce the category $\mathcal{O}$ of the RCA, mirabolic $D(\mathfrak{sl}_n \times \mathbb{C}^n)$-modules and coherent sheaves on Hilb^n. We define the quantum Hamiltonian reduction and the Gordon-Staffor functor connecting these objects.

In Section 3, we study Hodge filtrations on cuspidal mirabolic D-modules. We first restrict $\overline{\mathbf{N}}_c$ to a Borel subalgebra. There, the Hodge module is explicitly extended from a local system across a simple normal crossing divisor. We extend this structure back to $\overline{\mathbf{N}}_c$ via parabolic induction and D-module pushforward along the Springer resolution.

In Section 4, we express $\mathrm{gr}(\overline{\mathbf{N}}_c)$ as the pushforward of the cuspidal DG module $\mathcal{A}_c$. Using this, we compute the equivariant K-theory class on Hilb^n corresponding to $\mathrm{gr}(\overline{\mathbf{N}}_c)$.

In Section 5, we introduce the Catalan DG module $\mathcal{A}'_c$, analogous to $\mathcal{A}_c$. We show the equivariant K-theory classes on Hilb^n corresponding to $\mathcal{A}_c$ and $\mathcal{A}'_c$ are both given by the EHA generator action, namely $P_{m,n} \cdot 1$.

In Section 6, we conclude the proofs of the main theorems.

Appendix A is a digression on the invariance of $\overline{\mathbf{N}}_c$ under the Fourier transform.

Appendix B compiles examples on K-theory computations around $\mathcal{A}_c$ and $\mathcal{A}'_c$.

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2. Cherednik algebras, mirabolic D-modules, Hilbert schemes

Fix a positive integer n. We work over $\mathbb{C}$ throughout, and set up the following notations:
$\overline{G} = \mathrm{GL}_n$ with Lie algebra $\overline{\mathfrak{g}} := \mathfrak{gl}_n$;
$G = \mathrm{SL}_n$ with Lie algebra $\mathfrak{g} = \mathfrak{sl}_n$;
T: a maximal torus in G with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$, consisting of diagonal matrices;
$W = S_n$: the Weyl group;
$S \subset W$: the set of reflections.

2.1. **Rational Cherednik algebras.** For any $c \in \mathbb{C}$, we define the rational Cherednik algebra $\mathrm{H}_c := \mathrm{H}_c(W, \mathfrak{h})$ to be the $\mathbb{C}$-algebra generated by $\mathfrak{h}$, $\mathfrak{h}^*$ and W with relations

$$[x, x'] = [y, y'] = 0, \quad wxw^{-1} = w(x), \quad wyw^{-1} = w(y)$$
$$[y, x] = x(y) - \sum_{s \in S} c\langle \alpha_s, y\rangle\langle \alpha_s^\vee, x\rangle s$$

where $x, x' \in \mathfrak{h}^*$, $y, y' \in \mathfrak{h}$, $w \in W$. Here α_s, resp. $\alpha_s^\vee$, is the root, resp. coroot, associated to s and $(-,-)$ is the pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$.

For an irreducible W-representation τ, we regard it as a $S(\mathfrak{h}) \rtimes W$-representation by requiring $\mathbb{C}[\mathfrak{h}^*]$ to act trivially and define the Verma module

$$M_c(\tau) = \mathrm{H}_c \otimes_{S(\mathfrak{h}) \rtimes W} \tau.$$

The Verma module $M_c(\tau)$ has a unique nonzero irreducible quotient $\mathrm{L}_c(\tau)$.

Example 2.1. When $c = 0$, $\mathrm{H}_0 = \mathcal{D}(\mathfrak{h}) \rtimes W$ whose polynomial representation $\mathbb{C}[\mathfrak{h}] \cong M_0(\mathrm{triv})$ is irreducible.

Theorem 2.1. *([BEG03, Theorem 1.2]) When $c = \frac{m}{n}$ for positive integer m coprime to n, the unique irreducible finite-dimensional representation of H_c is $\mathrm{L}_c(\mathrm{triv})$. Moreover, only when $c = \frac{m}{n}$ for integer m coprime to n does H_c have finite-dimensional representations.*

We will simply write $\mathrm{L}_c := \mathrm{L}_c(\mathrm{triv})$. Let $\mathcal{O}(\mathrm{H}_c)$ denote the BGG category $\mathcal{O}$ of the RCA, which is a full subcategory of H_c-mod whose objects are finitely generated over H_c such that the $S(\mathfrak{h})$ action is locally nilpotent [GGOR03]. As τ varies over irreducible W-representations, the modules $\mathrm{L}_c(\tau)$ give a complete list of irreducible objects in $\mathcal{O}(\mathrm{H}_c)$.

Let $\mathrm{e} := \frac{1}{n!}\sum_{w \in W} w$ be the symmetrizing idempotent in H_c. Inside H_c, we have the isomorphisms $\mathrm{e}\mathbb{C}[\mathfrak{h}] \cong \mathbb{C}[\mathfrak{h}]^W$ and $\mathrm{e}S(\mathfrak{h}) \cong S(\mathfrak{h})^W$. The spherical Cherednik algebra is defined by $\mathrm{A}_c := \mathrm{e}\mathrm{H}_c\mathrm{e}$. By [BE09, Corollary 4.2], for all c satisfying

$$c \notin \{\frac{a}{b} \in (-1, 0) | a, b \in \mathbb{Z}, 2 \le b \le n\} \tag{4}$$

there is a Morita equivalence

$$\mathrm{H}_c-\mathrm{mod} \to \mathrm{A}_c-\mathrm{mod}, \quad M \mapsto \mathrm{e}M.$$

Denote the BGG category $\mathcal{O}$ of the spherical Cherednik algebra by $\mathcal{O}(\mathrm{A}_c)$.

Let R^+ be a chosen set of positive roots. Write $\delta = \prod_{\alpha \in R^+} \alpha$. Let $\mathfrak{h}_r = \{\delta \neq 0\}$ denote the regular locus of $\mathfrak{h}$, i.e., when the diagonals are pairwiese distinct. The action of H_c on $\mathbb{C}[\mathfrak{h}] \cong \mathrm{H}_c \otimes_{\mathbb{C}[\mathfrak{h}^*] \rtimes W} \mathbb{C}$ gives the Dunkl embedding of H_c into $\mathcal{D}(\mathfrak{h}_r) \rtimes W$.

Localized at $\delta := \prod_{\alpha \in R^+} \alpha$, we have that $(\mathrm{H}_c)_\delta \cong \mathcal{D}(\mathfrak{h}_r) \rtimes W$. Consider the symmetry on $(\mathrm{H}_c)_\delta$ defined by sending

$$x \mapsto x, \quad D_{y_i, c} \mapsto D_{y_i, -c}, \quad w \mapsto \mathrm{sign}(w)w$$

which restricts to an isomorphism $\mathrm{A}_d \to \delta^{-1}\mathrm{A}_{-d-1}\delta$ ([GGS09, 5.6, 5.8]). This induces an equivalence of categories, which we will use later:

$$\Omega_d = \Omega_{-d-1}^{-1} : \mathrm{A}_d\text{−mod} \cong \mathrm{A}_{-d-1}\text{−mod}.$$

2.2. **Quantum Hamiltonian reduction.** For any smooth algebraic variety X, we denote the sheaf of differential operators on X by $\mathcal{D}_X$ and write $\mathcal{D}(X) := \Gamma(X, \mathcal{D}_X)$. We say a filtration of a coherent $\mathcal{D}_X$-module M is good if the associated sheaf $\widetilde{\mathrm{gr}}(M)$ on T^*X is coherent. We denote the singular support of M by $SS(M)$.

Let $V = \mathbb{C}^n$ on which $\overline{G}$ act by left multiplication and $\mathfrak{G} = \mathfrak{g} \times V$. Let $\tau : \bar{\mathfrak{g}} \to \mathcal{D}(\mathfrak{G})$ denote the embedding induced by differentiating the diagonal action of $\overline{G}$ on $\mathfrak{G}$. Our convention is that for any $f \in \mathcal{O}(\mathfrak{G})$ and $g \in \overline{G}$,

$$(g \cdot f)(x) = f(g^{-1} \cdot x). \tag{5}$$

In particular, let $\mathbf{1} \in \bar{\mathfrak{g}}$ denote the identity matrix. Then $\tau(\mathbf{1}) = -\sum_{i=1}^n v_i \partial_{v_i}$.

For a constant $d \in \mathbb{C}$, define the shifted embedding τ_d by (later we will take d to be $-c$)

$$\tau_d : \bar{\mathfrak{g}} \to \mathcal{D}(\mathfrak{G}), \quad x \to \tau(x) - d \cdot \mathrm{tr}(x).$$

Throughout, we identify $\mathfrak{g} \cong \mathfrak{g}^*$ under the Killing form. Inside $T^*\mathfrak{G} \cong \mathfrak{g} \times \mathfrak{g} \times V \times V^*$, we define a Lagrangian subvariety:

$$\Lambda := \{(X, Y, i, j) \in \mathfrak{g} \times \mathscr{N} \times V \times V^* | [X, Y] + ij = 0\}.$$

Definition 2.1. ([FG10, GG06]) A finitely generated $\mathcal{D}_{\mathfrak{G}}$-module is **mirabolic** if its singular support is contained in Λ. We denote the category of mirabolic $\mathcal{D}_{\mathfrak{G}}$-modules by $\mathscr{C}(\mathfrak{G})$.

Fix a nonzero element $\mathrm{vol} \in \bigwedge^n V^*$. Consider the regular function on $\mathfrak{G}$ defined by [BFG06, (5.3.2)]:

$$s(x, v) = \langle \mathrm{vol}, v \wedge xv \wedge \cdots \wedge x^{n-1}v \rangle. \tag{6}$$

Let $\mathfrak{G}_{\mathrm{cyc}} := \{(x, v) \in \mathfrak{G} | s(x, v) \neq 0\}$, which is equivalently the locus when v is a cyclic vector for x, i.e., $\mathbb{C}[x]v = V$. By [BFG06, Lemma 5.3.3], the composition of the projection $\mathfrak{G}_{\mathrm{cyc}} \to \mathfrak{g}$ and the Chevalley map $\mathfrak{g} \to \mathfrak{h}/W$ is a principal $\overline{G}$-bundle, such that the diagonal $\overline{G}$-action on $\mathfrak{G}_{\mathrm{cyc}}$ acts freely on fibers. Therefore we have an isomorphism

$$\iota : \mathbb{C}[\mathfrak{G}_{\mathrm{cyc}}]^{\overline{G}} \cong \mathbb{C}[\mathfrak{h}]^W. \tag{7}$$

Note that $s^{-d} \in \mathbb{C}[\mathfrak{G}_{\mathrm{cyc}}]^{\tau_d(\bar{\mathfrak{g}})}$. In fact $\mathbb{C}[\mathfrak{G}_{\mathrm{cyc}}]^{\tau_d(\bar{\mathfrak{g}})} = \mathbb{C}[\mathfrak{G}_{\mathrm{cyc}}]^G s^{-d}$. Moreover,

Theorem 2.2. *([GG06, Theorem 1.3.1], [GGS09, Theorem 8.1] The radial part map*

$$\mathcal{D}(\mathfrak{G})^{\overline{G}} \to \mathcal{D}(\mathfrak{h}/W), \quad u \mapsto \Big[\mathbb{C}[\mathfrak{h}/W] \ni f \mapsto \iota\big(s^d u\big(s^{-d}\iota^{-1}(f)\big)\big)\Big]$$

defines an isomorphism

$$\mathscr{H}_d : (\mathcal{D}(\mathfrak{G})/\mathcal{D}(\mathfrak{G})\tau_d(\bar{\mathfrak{g}}))^{\overline{G}} \cong \mathrm{A}_{d-1}, \tag{8}$$

which induces the quantum Hamiltonian reduction functor

$$\mathbb{H}_d : \mathscr{C}(\mathfrak{G}) \to \mathcal{O}(\mathrm{A}_{d-1}), \quad \mathbf{M} \mapsto \Gamma(\mathfrak{G}, \mathbf{M})^{\tau_d(\bar{\mathfrak{g}})}$$

such that

$$\mathscr{C}(\mathfrak{G})/\mathrm{Ker}(\mathbb{H}_d) \cong \mathcal{O}(\mathrm{A}_{d-1}).$$

Remark 2.2. The "-1" factor in the subscript of A_{d-1} is consistent with the classical result of Harish-Chandra that

$$\mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h}/W), \quad u \mapsto \Big[\mathbb{C}[\mathfrak{h}/W] \ni f \mapsto \delta(u|_{\mathbb{C}[\mathfrak{h}/W]})\big(\delta^{-1} f\big)\Big]$$

induces an isomorphism $\big(\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\mathrm{ad}(\mathfrak{g})\big)^G \cong D(\mathfrak{h})^W$. Here δ is the product of all positive roots and the restriction $u|_{\mathbb{C}[\mathfrak{h}/W]}$ is taken with respect to the Chevalley isomorphism $C[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{h}]^W$.

2.3. **Cuspidal mirabolic $\mathcal{D}$-modules.**

2.3.1. *Functors on $\mathcal{D}$-modules.* Let $f : X \to Y$ be a morphism of smooth algebraic varieties. We write $f_\dagger, f_!, f^\dagger$ to denote the derived pushforward, proper pushforward and pullback functors of D-modules respectively.

When f is a locally closed embedding, we define the minimal extension functor $f_{!*}$ to be the image under the canonical morphism $f_! \to f_\dagger$. We say a local system on a locally closed subset $U \xrightarrow{i} X$ is clean if its minimal extension coincides with the extensions using $f_\dagger$ or $f_!$.

2.3.2. *Cuspidal local systems.* Let $\mathscr{N}$ be the nilpotent cone in $\mathfrak{g}$ consisting of nilpotent matrices. The zero fiber of the fibration $\mathfrak{G}_{\mathrm{cyc}} \to \mathfrak{h}/W$ is

$$U := \{(x, v) \in \mathscr{N} \times V | s(x, v) \neq 0\}.$$

The diagonal $\overline{G}$ action on U is transitive and free. Hence $\pi_1(U) = \mathbb{Z}$. We have that $g \cdot s = \det(g)^{-1} s$ for any $g \in \overline{G}$ and thus every simple local system on U of finite-order monodromy is spanned by the horizontal section s^a for some rational number a. We will denote the $\mathcal{D}_U$-module corresponding to such a local system by $\mathcal{E}_a$.

Let $\mathscr{N}_r \subset \mathscr{N}$ denote the regular nilpotent orbit. Then $\pi_1(\mathscr{N}_r) = \mathbb{Z}/n\mathbb{Z}$ and every simple local system on $\mathscr{N}_r$ of finite order monodromy corresponds to the representation of $\mathbb{Z}/n\mathbb{Z}$ defined by $e^{2\pi i b}$ for some $b \in \frac{1}{n}\mathbb{Z}$. We will denote the $\mathcal{D}_{\mathscr{N}_r}$-module corresponding to such a local system by $\mathcal{F}_b$. Both $\mathcal{E}_a$ and $\mathcal{F}_b$ are G-equivariant.

The projection $U \to \mathscr{N}$ has image inside $\mathscr{N}_r$ and the fibers of this projection are isomorphic to $\mathfrak{F} := \mathbb{C}^{n-1} \times \mathbb{C}^*$. The fibration

$$\begin{tikzcd} \mathfrak{F} \arrow[r] & U \arrow[d] \\ & \mathscr{N}_r \end{tikzcd}$$

induces an exact sequence

$$1 \to (\mathbb{Z} = \pi_1(\mathfrak{F})) \xrightarrow{n} (\mathbb{Z} = \pi_1(U)) \to (\pi_1(\mathscr{N}_r) = \mathbb{Z}/n\mathbb{Z}) \to 1.$$

It follows that $\mathcal{E}_a$ is the pullback of the local system $\mathcal{F}_a$ on $\mathscr{N}_r$ for $a \in \frac{\mathbb{Z}}{n}$.

2.3.3. *Cuspidal character/mirabolic $\mathcal{D}$-modules.* Let $c = \frac{m}{n}$ for positive integer m coprime to n.

Definition 2.2.
- The cuspidal character $\mathcal{D}_{\mathfrak{g}}$-module of parameter c is the minimal extension of $\mathcal{F}_c$ to $\mathfrak{g}$, which we denote by $\mathbf{N}_c$.
- The cuspidal mirabolic $\mathcal{D}_{\mathfrak{G}}$-module of parameter c is the minimal extension of $\mathcal{E}_c$ to $\mathfrak{G}$, which we denote by $\overline{\mathbf{N}}_c$.

By [Lus86], $\mathcal{F}_c$ is clean. We also have the following:

Lemma 2.3. *As G-equivariant $\mathcal{D}_{\mathfrak{G}}$-modules, $\overline{\mathbf{N}}_c \cong \mathbf{N}_c \boxtimes \mathcal{O}_V$. Moreover*

$$SS(\overline{\mathbf{N}}_c) = \{(x, y) \in \mathscr{N} \times \mathscr{N} | [x, y] = 0\}_{\mathrm{red}} \times V \tag{9}$$

where the subscript $_{\mathrm{red}}$ *refers to taking the reduced structure.*

Proof. The function s has degree n along the direction of V. Since $c \cdot n = m$, the local system $\mathcal{E}_c$, defined by the horizontal section s^c, has no monodromy along the V direction. Therefore the minimal extension of $\mathcal{E}_c$ to $\mathscr{N}_r \times V$ is $\mathcal{F}_c \boxtimes \mathcal{O}_V$ and the first statement follows from the cleanness of $\mathcal{F}_c$. On the other hand, the singular support of a cuspidal character $\mathcal{D}$-module is known to equal $\{(x,y) \in \mathscr{N} \times \mathscr{N} | [x,y] = 0\}_{\mathrm{red}}$ and hence the second statement of the lemma follows. ☐

Theorem 2.4. *([CEE09, Theorem 9.19])*

- *There is an* A_{-c-1}*-action on* $(\Gamma(\mathfrak{g}, \mathbf{N}_c) \otimes \mathrm{Sym}^m V)^G$.
- *Under this action,* $(\Gamma(\mathfrak{g}, \mathbf{N}_c) \otimes \mathrm{Sym}^m V)^G$ *is isomorphic to* $\Omega_c(\mathrm{eL}_c)$.

Corollary 2.5. *As* A_{-c-1}*-modules,* $\mathbb{H}_{-c}(\overline{\mathbf{N}}_c) \cong \Omega_c(\mathrm{eL}_c)$

Proof. Follows from Lemma 2.3, Theorem 2.4 and the fact that $\mathbb{C}[V]^{\tau_{-c}(\mathbf{1})} = \mathrm{Sym}^m V$. ☐

2.4. **Hilbert schemes of points.** Let $\overline{\mathrm{Hilb}}^n(\mathbb{C}^2)$ denote the moduli space of ideals of colength n in $\mathbb{C}[x,y]$. It is a smooth and quasi-projective variety of dimension $2n$.

The Hilbert-Chow map $\mathrm{Hilb}^n \to (\mathbb{C}^2)^n/S_n$, defined by sending a colength n ideal I to the subvariety defined by the quotient $\mathbb{C}[x,y]/I$, is a resolution of singularity. We let $\mathrm{Hilb}^n \subset \overline{\mathrm{Hilb}}^n(\mathbb{C}^2)$ denote the preimage of $\{(z_1, \cdots, z_n) \in (\mathbb{C}^2)^n, \sum z_i = (0,0)\}$ under the Hilbert-Chow map.

Define

$$\widetilde{\mathrm{Hilb}^n} := \{(X,Y,v) \in \mathfrak{g} \times \mathfrak{g} \times V | [X,Y] = 0, \mathbb{C}[X,Y]v = V\}_{\mathrm{red}}.$$

The diagonal G-action on $\widetilde{\mathrm{Hilb}}$ is free and the resulting GIT quotient is $\widetilde{\mathrm{Hilb}^n}/\!\!/G = \mathrm{Hilb}^n$([Nak99]).

Let $\widetilde{\mathrm{Hilb}_0^n} = (\mathscr{N} \times \mathscr{N} \times V) \cap \widetilde{\mathrm{Hilb}^n}$, which is an open subvariety of the singular support (9). The GIT quotient $\mathrm{Hilb}_0^n := \widetilde{\mathrm{Hilb}_0^n}/\!\!/G$ is the zero fiber of the Hilbert-Chow map, known as the punctual Hibert scheme.

2.5. **The Gordon-Stafford functor.** Consider the order filtration on H_c defined by $\deg y = 1$ and $\deg x = \deg w = 0$. We will use this filtration on all subspaces of H_c henceforth. With respect to this filtration, we have that $\mathrm{gr}(\mathrm{A}_c) = \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^W$. We are interested in A_c-modules with good filtrations, in the sense that the associated graded is finitely generated as $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^W$-modules.

In [GS05], Gordon and Stafford define a functor from the category of A_c-modules equipped with good filtrations to $\mathrm{Coh}^{\mathbb{C}^* \times \mathbb{C}^*}(\mathrm{Hilb}^n(\mathbb{C}^2))$, motivated by the following diagram.

$$\begin{tikzcd} ? \arrow[r, squiggly, "\mathrm{gr}"'] & \mathcal{O}_{\mathrm{Hilb}^n} \\ \mathrm{A}_c \arrow[u] \arrow[r, squiggly, "\mathrm{gr}"'] & \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^{S_n} \arrow[u, "\text{Hilbert-Chow}"'] \end{tikzcd}$$

The definition of their functor GS is based on the Proj construction of the Hilbert scheme due to Haiman [Hai98, Proposition 2.6]. Let $R_0 = \mathbb{C}[\mathfrak{h}]^W$ and $R_1 = \mathbb{C}[\mathfrak{h}]^{\mathrm{sgn}}$. The supscript $^{\mathrm{sgn}}$ indicates taking the sign-isotypic component. For each $k \geq 1$, let $R_k = (R_1)^k$ be the product of k copies of R_1 in $\mathbb{C}[\mathfrak{h}]$. Put $R = \oplus_{k\geq 0} R_k \delta^k$, where $\delta = \prod_{\alpha \in R^+} \alpha$. Then $\mathrm{Hilb}^n = \mathrm{Proj}(R)$. Therefore, any finitely generated graded R-module defines a coherent sheaf on Hilb^n.

Let $\mathrm{e} := \frac{1}{n!}\sum_{w\in W} \mathrm{sgn}(w) w$ be the skew-symmetrizing idempotent in H_c. We define two subspaces in $(\mathrm{H}_d)_\delta$:

$${}_{d+1}\mathrm{P}_d := \mathrm{eH}_d \delta \mathrm{e}_-, \quad {}_d\mathrm{Q}_{d+1} := \mathrm{e}_- \delta^{-1} \mathrm{H}_{d+1}\mathrm{e}.$$

Both ${}_{d+1}\mathrm{P}_d$ and ${}_d\mathrm{Q}_{d+1}$ inherit from $(\mathrm{H}_d)_\delta$ the order filtration. For any $d \in \mathbb{C}$, the isomorphism [BEG03, Proposition 4.6]

$$\mathrm{eH}_d\mathrm{e} \cong \mathrm{e}\delta^{-1}\mathrm{H}_{d+1}\delta\mathrm{e}$$

gives a $(\mathrm{A}_{d+1}, \mathrm{A}_d)$-module structure on ${}_{d+1}\mathrm{P}_d$ and a $(\mathrm{A}_d, \mathrm{A}_{d+1})$-module structure on ${}_d\mathrm{Q}_{d+1}$. Thus we can inductively define for any $k \in \mathbb{Z}_{>0}$,

$$ {}_{d+k}\mathrm{P}_d := {}_{d+k}\mathrm{P}_{d+k-1} \otimes_{\mathrm{A}_{d+k-1}} {}_{d+k-1}\mathrm{P}_d, $$

and similarly for ${}_{d-k}\mathrm{Q}_d$.

For two filtered modules M, N over a filtered ring R, the tensor product filtration on $M \otimes_R N$ is defined by

$$ F_k(M \otimes_R N) = \mathrm{Im}\big(\sum_{i+j=k} F_i M \otimes F_j N \to M \otimes_R N \big). $$

We equip ${}_{d+k}\mathrm{P}_d$ and ${}_{d-k}\mathrm{Q}_d$ with tensor product filtrations.

For any A_{-d-1}-module L, one has that [GGS09, Proposition 5.8]

$$ {}_{d+k}\mathrm{P}_d \otimes_{\mathrm{A}_d} \Omega_{-d-1} L \cong \Omega_{-(d+k)-1}\big({}_{-(d+k)-1}\mathrm{Q}_{-d-1} \otimes_{\mathrm{A}_{-d-1}} L \big). \tag{10} $$

Moreover, ${}_{c+k}\mathrm{P}_c$ is a $(\mathrm{A}_{c+k}, \mathrm{A}_c)$-module and hence defines a shift functor

$$ S_{c,k} : A_c\text{−mod} \to A_{c+k}\text{−mod}, \quad M \mapsto {}_{c+k}\mathrm{P}_c \otimes_{\mathrm{A}_c} M. $$

For a filtered module M, one equips $S_{c,k}(M)$ with the associated tensor product filtration.

Suppose M is an A_c-module equipped with a good filtration F, the Gordon-Stafford functor associates to $(M, F_\bullet)$ a coherent sheaf on Hilb^n defined by

$$ GS(M, F) = \mathrm{Proj}\Big(\mathrm{gr} \bigoplus_{k \geq 0} S_{c,k} M \Big). $$

2.6. **Relating the functors.** The parameter c in this subsection can be any complex number.

2.6.1. *Compatibility of filtrations.* Let $\mathbf{M} \in \mathscr{C}(\mathfrak{G})$ be a mirabolic $\mathcal{D}$-module with a good filtration F. Write $(M, F_\bullet) = \Gamma(\mathfrak{G}, (\mathbf{M}, F_\bullet))$. The filtration F restricts to a filtration on $M^{\tau_{-c-k}(\bar{\mathfrak{g}})}$ for any $k \in \mathbb{Z}_{\geq 0}$ and hence also on $L := \Omega_{-c-1} \circ \mathbb{H}_{-c}(\mathbf{M})$ as Ω_{-c-1} preserves the order filtration.

For any G-module E, write $E^{\det^{-k}} := \{f \in E | g \cdot f = \det(g)^{-k} f, \forall g \in G\}$. Also, define $\mathcal{D}_d(\mathfrak{G}) := \mathcal{D}(\mathfrak{G})/\mathcal{D}_d(\mathfrak{G})\tau_d(\bar{\mathfrak{g}})$. Then there is a homomorphism

$$ \phi_M^k : \mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}} \otimes_{\mathrm{A}_{-c-1}} M^{\tau_{-c}(\bar{\mathfrak{g}})} \to M^{\tau_{-c-k}(\bar{\mathfrak{g}})} \tag{11} $$

given by the left multiplication of $\mathcal{D}(\mathfrak{G})$ on M. By [BG15, Theorem 1.3.5], ϕ_M^k is an isomorphism of A_{-c-k}-modules.

Moreover, by [GGS09, Theorem 5.3(1)], when each of the rational numbers $-c-1, \cdots, -c-k-1$ satisfies the condition (4), there is an isomorphism

$$ \mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}} \cong {}_{-c-k-1}Q_{-c-1} \tag{12} $$

of filtered modules with respect to the order filtrations inherited from $\mathcal{D}(\mathfrak{G})$ and $(\mathrm{H}_c)_\delta$.

Using (10), we have that

$$ \Omega_{c+k}\big({}_{c+k}\mathrm{P}_c \otimes_{\mathrm{A}_c} L \big) = {}_{-(c+k)-1}\mathrm{Q}_{-c-1} \otimes_{\mathrm{A}_{-c-1}} \Omega_c(L). $$

By (12), we have a filtered isomorphism

$$ \Omega_{c+k}\big({}_{c+k}\mathrm{P}_c \otimes_{\mathrm{A}_c} L \big) \cong \mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}} \otimes_{\mathrm{A}_{-c-1}} M^{\tau_{-c}(\bar{\mathfrak{g}})}. \tag{13} $$

Consider the tensor product filtration on $\mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}} \otimes_{\mathrm{A}_{-c-1}} M^{\tau_{-c}(\mathfrak{g})}$ and the sub-filtration on $M^{\tau_{-c-k}(\mathfrak{g})}$. The homomorphism ϕ_M^k is filtered, i.e., for any $i \geq 0$,

$$ \phi_M^k F_i \left(\mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}} \otimes_{\mathrm{A}_{-c-1}} M^{\tau_{-c}(\bar{\mathfrak{g}})} \right) \subset F_i(M^{\tau_{-c-k}(\bar{\mathfrak{g}})}). $$

This is not an equality in general, i.e., ϕ_M^k is not necessarily a filtered isomorphism. Given (13), we see that ϕ_M^k is a filtered isomorphism if and only if

$$\operatorname{gr}(S_{c,k}L) \cong \operatorname{gr}\left(\Omega_{-c-k-1}M^{\tau_{-c-k}(\bar{\mathfrak{g}})}\right). \tag{14}$$

2.6.2. *A descent functor.* Let $F\mathscr{C}(\mathfrak{G})$ be the category whose objects are pairs $(\mathbf{M}, F_\bullet)$, where $\mathbf{M}$ is a mirabolic $\mathcal{D}$-module and F is a good filtration on $\mathbf{M}$. We define a descent functor

$$\begin{aligned}\Psi_c : F\mathscr{C}^G(\mathfrak{G}) \to& \operatorname{Coh}(\operatorname{Hilb}^n)\\ (\mathbf{M}, F_\bullet) \mapsto& \operatorname{Proj}\bigoplus_{\ell\geq 0}\Gamma(\widetilde{\operatorname{Hilb}}^n, \widetilde{\operatorname{gr}}^F\mathbf{M})^{\tau_{-c-\ell}(\bar{\mathfrak{g}})}\end{aligned} \tag{15}$$

whose essential image lands in $\operatorname{Coh}(\operatorname{Hilb}^n_1)$ where $\operatorname{Hilb}^n_1 \subset \operatorname{Hilb}^n$ is the preimage of $\{(x,0) \in (\mathbb{C}^n)^2\}/S_n$ under the Hilbert-Chow map.

2.6.3. *Gradings.* Let $((x_{ij})_{1\leq i,j\leq n}$ and $(\partial_{x_{ij}})_{1\leq i,j\leq n}$ be the dual bases of $\bar{\mathfrak{g}}^*$ and $\bar{\mathfrak{g}}$ respectively. Let $\{x_i\}$ and $\{y_i\}$ are dual bases of $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively.

Define

$$\begin{aligned}\widetilde{h} :=& \frac{1}{2}\sum_{1\leq i,j\leq n}(x_{ij}\partial_{x_{ij}} + x_{ij}\partial_{x_{ij}})\\ \mathrm{h}_c :=& \Omega_{-c-1}(\mathscr{H}_{-c}(h)) = \frac{1}{2}\sum_{i=1}^n(x_iy_i + y_ix_i).\end{aligned} \tag{16}$$

Let $F\mathcal{O}(\mathrm{A}_c)$ be the category whose objects are pairs $(L, F_\bullet)$ where L is an object in $\mathcal{O}(\mathrm{A}_c)$ and $F_\bullet$ is a good filtration on L. For any $(\mathbf{M}, F_\bullet) \in F\mathscr{C}^G(\mathfrak{G})$, resp. $(L, F_\bullet) \in F\mathcal{O}(\mathrm{A}_c)$, the action of $\widetilde{h}$, resp. h_c, is semisimple and induces a $\mathbb{Z}$-grading. This grading together with the filtration induces a $\mathbb{C}^* \times \mathbb{C}^*$-equivariant structure on $\Psi_c(\mathbf{M})$, resp. $GS(L)$.

On the other hand, the scalar action of $\mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{C}^2$ induces an $\mathbb{C}^* \times \mathbb{C}^*$-action on Hilb^n. We have the following correspondence:

$$\text{filtration grading} \leftrightarrow \left[C^* \hookrightarrow \mathbb{C}^* \times \mathbb{C}^* :, z \mapsto (1, z)\right] \tag{17}$$

$$\mathrm{h}_c\text{-grading} \leftrightarrow \left[C^* \hookrightarrow \mathbb{C}^* \times \mathbb{C}^* : z \mapsto (z, z^{-1})\right]. \tag{18}$$

2.6.4. *Commutativity of the diagram.* Consider the following diagram:

(19)
$$\begin{tikzcd} & F\mathscr{C}(\mathfrak{G}) \arrow[dl, "\Omega_{-c-1}\circ\mathbb{H}_{-c}"'] \arrow[dr, "\Psi_c"] & \\ F\mathcal{O}(\mathrm{A}_c) \arrow[rr, "GS"] & & \operatorname{Coh}^{\mathbb{C}^*\times\mathbb{C}^*}(\operatorname{Hilb}^n) \end{tikzcd}$$

Proposition 2.6. *Let $(\mathbf{M}, F_\bullet) \in F\mathscr{C}^G(\mathfrak{G})$ and $(M, F_\bullet) = \Gamma(\mathfrak{G}, (\mathbf{M}, F_\bullet))$. Suppose that $-c-k$ satisfies (4) for all $k \in \mathbb{N}$. Then the following are equivalent:*
(a) $\Psi_c(\mathbf{M}) \cong GS \circ \Omega_{-c-1} \circ \mathbb{H}_{-c}(\mathbf{M})$.
(b) ϕ_M^ℓ (defined in (11)) is a filtered isomorphism for all $\ell \gg 0$.
(c) ϕ_M^ℓ is a filtered isomorphism for all $\ell \geq 0$.

Proof. Let $L = \Omega_{-c-1} \circ \mathbb{H}_c(M)$. By definition,

$$GS(L) = \operatorname{Proj}(\bigoplus_{\ell\geq 0}\operatorname{gr}S_{c,\ell}L).$$

On the other hand,

$$\Psi_c(\mathbf{M}) = \operatorname{Proj}(\bigoplus_{\ell\geq 0}\Gamma(\widetilde{\operatorname{Hilb}}, \widetilde{\operatorname{gr}}^F\mathbf{M})^{\tau_{-c-\ell}(\bar{\mathfrak{g}})}).$$

Therefore, the equality $GS(L) = \Psi_c(\mathbf{M})$ is equivalent to

$$\Gamma(\widetilde{\mathrm{Hilb}}, \widetilde{\mathrm{gr}}^F\mathbf{M})^{\tau_{-c-\ell}(\bar{\mathfrak{g}})} \cong \mathrm{gr}^T S_{c,\ell}L, \quad \forall \ell \gg 0.$$

By [GGS09, Proposition 7.4], since $\widetilde{\mathrm{Hilb}}$ is the semistable locus with respect to the character det of $\overline{G}$, there is

$$\Gamma(\widetilde{\mathrm{Hilb}}, \widetilde{\mathrm{gr}}^F\mathbf{M})^{\tau_{-c-\ell}(\bar{\mathfrak{g}})} = \Gamma(T^*\mathfrak{G}, \widetilde{\mathrm{gr}}^F\mathbf{M})^{\tau_{-c-\ell}(\bar{\mathfrak{g}})} = \mathrm{gr}^F\Gamma(\mathfrak{G}, \mathbf{M})^{\tau_{-c-\ell}(\bar{\mathfrak{g}})}, \quad \text{for } \ell \gg 0.$$

Moreover, $\Omega_{-c-\ell-1}$ preserves filtration, i.e., $\mathrm{gr}^F\Omega_{-c-\ell-1}M^{\tau_{-c-k}(\bar{\mathfrak{g}})} \cong \mathrm{gr}^F M^{\tau_{-c-\ell}(\bar{\mathfrak{g}})}$. Given the discussion around (14), we conclude that (a) is equivalent to (b).

As for the implication (b) $\Rightarrow$ (c), for any $k \geq 0$, take $\ell \gg 0$ and consider

$$\begin{tikzcd}
\mathcal{D}_{-c}(\mathfrak{G})^{\det^{-\ell}} \otimes_{\mathrm{A}_{-c-k-1}} \mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}} \otimes_{\mathrm{A}_{-c-1}} M^{\tau_{-c}(\bar{\mathfrak{g}})} \arrow[r, "id\otimes\phi_M^k"] \arrow[d, "\mathrm{mul}\otimes id"] & \mathcal{D}_{-c}(\mathfrak{G})^{\det^{-\ell}} \otimes_{\mathrm{A}_{-c-k-1}} M^{\tau_{-c-k}(\bar{\mathfrak{g}})} \arrow[d, "\phi_M^\ell"] \\
\mathcal{D}_{-c}(\mathfrak{G})^{\det^{-\ell-k}} \otimes_{\mathrm{A}_{-c-1}} M^{\tau_{-c}(\bar{\mathfrak{g}})} \arrow[r, "\phi_M^{\ell+k}"] & M^{\tau_{-c-\ell-k}(\bar{\mathfrak{g}})}
\end{tikzcd}$$

Here the map $\mathrm{mul} : \mathcal{D}_{-c}(\mathfrak{G})^{\det^{-\ell}} \otimes_{\mathrm{A}_{-c-k-1}} \mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}} \to \mathcal{D}_{-c}(\mathfrak{G})^{\det^{-\ell-k}}$ is defined by multiplication. By [GGS09, Lemma 5.2(2)], mul is a filtered isomorphism. By assumption, $\phi_M^{\ell+k}$, ϕ_M^ℓ are filtered isomorphisms. As a result, ϕ_M^k is also a filtered isomorphism. This concludes the proof of the proposition. $\square$

3. Hodge filtrations on cuspidal $\mathcal{D}$-modules

From now on, $c = \frac{m}{n}$ for a positive integer m coprime to n.

The structure of a Hodge module [Sai90] consists the information of a triple $(M, F_\bullet, V_\mathbb{Q})$ where M is a D-module, $F_\bullet$ is its Hodge filtration and $V_\mathbb{Q}$ is a $\mathbb{Q}$-perverse sheaf such that $DR(M) = V_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{C}$. For an overview, we refer the readers to [Sch]. In this section, we will only adapt a naive notion of $\mathbb{C}$-Hodge modules, in the form of $(M, F_\bullet)$ by forgetting the data of $V_\mathbb{Q}$ in a Hodge module, as in [Sai22, Remark 2.6].

Below, we will have various filtrations but only one on each D-module. To ease notations, all of the filtrations will be denoted by F.

3.1. **Hodge filtrations.** Fix the Borel subgroup $B \subset G$ with Lie algebra $\mathfrak{b} \subset \mathfrak{g}$ of upper triangular matrices. Let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ be the nilpotent radical and $\mathfrak{n}_r = \mathfrak{n} \cap \mathscr{N}_r$. We give $\mathfrak{n} \times V$ coordinates by

$$\begin{pmatrix} 0 & x_1 & * & * & * & * \\ 0 & 0 & x_2 & * & * & * \\ & & & \cdots & * & * \\ 0 & 0 & 0 & \cdots & 0 & x_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ \\ \vdots \\ v_n \end{pmatrix} \tag{20}$$

Then the restriction of the function s to $\mathfrak{n} \times V$ equals $x_1 x_2^2 \cdots x_{n-1}^{n-1} v_n^n$ and $\mathfrak{n}_r = \mathfrak{n} \setminus D$ where D is a simple normal crossing (SNC) divisor defined by $D := \{x_1 x_2 \cdots x_{n-1} = 0\}$.

The restriction of the $\mathcal{D}_U$-module $\mathcal{E}_c$ to $U_0 := U \cap (\mathfrak{n} \times V)$ is generated by the G-invariant horizontal section

$$s_0^c := s^c|_{U_0} = x_c x_2^{2c} \cdots x_{n-1}^{(n-1)c} v_n^m. \tag{21}$$

So is the $\mathcal{D}_\mathfrak{n}$-module $\mathbf{L}_0 := \mathcal{F}_c|_{\mathfrak{n}_r}$. The extra factor v_n^m is necessary for s_0^c to be T-invariant: $(t_1, \cdots, t_n) \in T$ acts by

$$t_1^c t_2^{-c+2c} \cdots t_{n-1}^{(n-2)c-(n-1)c} t_n^{-(n-1)c+nc} = (t_1 \cdots t_n)^c.$$

Considering the order filtration F on $\mathbf{L}_0$, the pair $(\mathbf{L}_0, F)$ defines a variation of Hodge structure.

Write $i_0 : \mathfrak{n}_r \hookrightarrow \mathfrak{n}$. By [Sai90, Theorem 3.21], there exists a unique Hodge module structure on $\mathbf{L}_D := (i_0)_\dagger \mathbf{L}_0$. Following [Pop18, 4.4], Hodge filtrations across SNC divisors can be described explicitly as follows.

First of all, we have that

$$\Gamma(\mathfrak{n}, \mathbf{L}_D) = \mathcal{D}(\mathfrak{n})/\big(\sum_{i=1}^{n-1} \mathcal{D}(\mathfrak{n})(x_i\partial_{x_i} + ic) + \mathcal{D}(\mathfrak{n})S([\mathfrak{n},\mathfrak{n}])\big).$$

The $\mathcal{D}_\mathfrak{n}$-module $\mathbf{L}_D$ is a regular meromorphic extension of $\mathbf{L}_0$ across the SNC divisor D. Inside $\mathbf{L}_D$, we have Deligne's canonical extension $\mathbf{L}_0^{>-1}$ [Del70], which is a locally free $\mathcal{O}_\mathfrak{n}$-module extending $\mathbf{L}_0$ such that the residues (cf. [HTT08, 5.2.2]) of the meromorphic connection under this lattice along all the components of D lie in $(-1, 0]$ and satisfies

$$\mathbf{L}_D = \mathbf{L}_0^{>-1} \otimes_{\mathcal{O}_\mathfrak{n}} \mathcal{O}_\mathfrak{n}[D] = \mathcal{D}_\mathfrak{n} \cdot \mathbf{L}_0^{>-1}.$$

Here $\mathcal{O}_\mathfrak{n}[D]$ is the sheaf of rational functions on $\mathfrak{n}$ that are regular on U_0.

In our case, for

$$\lceil s_0^c \rceil := x_1^{\lceil c\rceil} x_2^{\lceil 2c\rceil} \dots x_{n-1}^{\lceil (n-1)c\rceil} v^m, \tag{22}$$

we compute that

$$s_0^c \lceil s_0^c\rceil^{-1} = x_1^{c-\lceil c\rceil} x_2^{2c-\lceil 2c\rceil} \dots x_{n-1}^{(n-1)c-\lceil (n-1)c\rceil}$$

and

$$x_i\partial_{x_i}(s_0^c\lceil s_0^c\rceil^{-1}) = ic - \lceil ic\rceil \in (-1, 0], \quad i = 1, \dots, n-1.$$

Hence $\mathbf{L}_0^{>-1} = \mathcal{O}_\mathfrak{n}\lceil s_0^c\rceil^{-1} s_0^c \subset \mathbf{L}_0$.

On $\mathbf{L}_0^{>-1}$ we have the filtration

$$F_k\mathbf{L}_0^{>-1} = \begin{cases} \mathbf{L}_0^{>-1} \cap (i_0)_* F_k \mathbf{L}_0 = \mathcal{O}_\mathfrak{n}\lceil s_0^c\rceil^{-1} & k \geq 0 \\ 0 & k < 0 \end{cases}$$

Let F^{ord} denote the order filtration. The induced filtration on $\mathcal{D}_\mathfrak{n} \cdot \mathbf{L}_0^{>-1}$ is

$$F_k(\mathcal{D}_\mathfrak{n} \cdot \mathbf{L}_0^{>-1}) = \sum F_i^{\mathrm{ord}}\mathcal{D}_\mathfrak{n} F_{k-i}\mathbf{L}_0^{>-1} = \begin{cases} (F_k^{\mathrm{ord}}\mathcal{D}_\mathfrak{n}) \cdot \lceil s_0^c\rceil^{-1} & k \geq 0 \\ 0 & k < 0 \end{cases}$$

Let $i_\mathfrak{n} : \mathfrak{n} \to \mathfrak{g}$ and $\mathcal{D}_{\mathfrak{g}\leftarrow\mathfrak{n}} := i_\mathfrak{n}^*\mathcal{D}_\mathfrak{g} \otimes_{\mathcal{O}_\mathfrak{n}} \omega_{\mathfrak{n}/\mathfrak{g}}$. The pushforward of the Hodge module $(\mathbf{L}_D, F_\bullet)$ has the underlying $\mathcal{D}_\mathfrak{g}$-module

$$\mathbf{L} = (i_\mathfrak{n})_\dagger \mathbf{L}_D = i_*\big(\mathcal{D}_{\mathfrak{g}\leftarrow\mathfrak{n}} \otimes_{\mathcal{D}_\mathfrak{n}} \mathbf{L}_D\big)$$

with the Hodge filtration defined by ([Pop18, 1.5])

$$F_k\mathbf{L} = \mathrm{Im}\Big(\Big(\sum_q F_q^{\mathrm{ord}}\mathcal{D}_{\mathfrak{g}\leftarrow\mathfrak{n}} \otimes F_{k-q}^D(\mathbf{L}_D)\Big) \to (i_\mathfrak{n})_\dagger \mathbf{L}_D\Big).$$

Let $\overline{\mathbf{L}}$ be the minimal extension of $\mathcal{E}_c|_{U_0}$ to $\mathfrak{G}$. One can run the same procedure to define the unique $\mathbb{C}$-Hodge module structure on $\overline{\mathbf{L}}$.

3.2. **Associated graded of L.** Recall the standard $\mathfrak{sl}_2$-triple in $\mathfrak{g}$:

$$\underline{e} = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \underline{f} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ n-1 & 0 & \cdots & 0 & 0 \\ 0 & 2(n-2) & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1-n & 0 \end{pmatrix} \quad \underline{h} = [E, F] \tag{23}$$

Define the Hessenberg space $\mathfrak{m} = \overline{B \cdot (\underline{f} + \mathrm{stab}_\mathfrak{g}(\underline{e}))}$, which is the sum of $\mathfrak{m}$ and the span of negative simple root spaces.

Remark 3.1. The affine space $\underline{f} + \mathrm{stab}_{\mathfrak{g}}(\underline{e})$ is the Kostant slice. The Whittaker reduction of $G \times^B \mathfrak{m}$ is the compactified regular centralizer [Bal23].

Under the coordinates (20) and

$$\begin{pmatrix} * & * & \cdots & * & * & * \\ y_1 & * & \cdots & * & * & * \\ 0 & y_2 & \cdots & * & * & * \\ & & \cdots & & & \\ 0 & 0 & \cdots & y_{n-2} & * & * \\ 0 & 0 & \cdots & 0 & y_{n-1} & * \end{pmatrix} \tag{24}$$

we define

$$\mathfrak{Y}_0 := \{(x, y) \in \mathfrak{n} \times \mathfrak{m} | x \in \mathfrak{n}, y \in \mathfrak{m}, x_i y_i = 0, 1 \leq i \leq n-1\}. \tag{25}$$

As a result of the identification:

$$\Gamma(\mathfrak{g}, \mathbf{L}) \cong \mathcal{D}(\mathfrak{g})/(\mathcal{D}(\mathfrak{g}) \cdot \mathcal{O}(\mathfrak{b}_-) + \sum_{i=1}^{n-1} \mathcal{D}(\mathfrak{g})(x_i \partial_{x_i} - ic) + \mathcal{D}(\mathfrak{g}) \cdot S([\mathfrak{n}, \mathfrak{n}]),$$

we have that $\widetilde{\mathrm{gr}}\ \mathbf{L} \cong (i_{\mathfrak{Y}_0})_* \mathcal{O}_{\mathfrak{Y}_0}$, where $i_{\mathfrak{Y}_0} : \mathfrak{Y}_0 \to T^*\mathfrak{g}$ denotes the closed embedding.

Although the D-modules $\mathbf{L}$ and $\overline{\mathbf{L}}$ are only B-equivariant but not $\overline{B}$-equivariant, the coherent sheaves $\widetilde{\mathrm{gr}}\ \mathbf{L}$ and $\widetilde{\mathrm{gr}}\ \overline{\mathbf{L}}$ are naturally $\overline{B}$-equivariant:

Construction 3.1. On $\widetilde{\mathrm{gr}}\ \mathbf{L}$ and $\widetilde{\mathrm{gr}}\ \overline{\mathbf{L}}$, pre-compose the $\overline{\mathfrak{b}}$-action with τ_c and integrate it into a $\overline{B}$-action.

This $\overline{B}$-equivariance is consistent with the definition of Ψ_c (15). For the rest of this subsection, we explicitly describe the $\overline{B}$-equivariance of $\widetilde{\mathrm{gr}}\ \mathbf{L}$ and $\widetilde{\mathrm{gr}}\ \overline{\mathbf{L}}$.

Let α be the weight associated to the relative canonical bundle $\omega_{\mathfrak{n}/\mathfrak{g}}$. Let $i_V : V \hookrightarrow T^*V$ be the zero section. Endow $\mathcal{O}_{\mathfrak{Y}_0}$, resp. $\mathcal{O}_{\mathfrak{Y}_0} \boxtimes \mathcal{O}_V$ with the trivial B, resp. $\overline{B}$-equivariant structure. Let $\mathbb{C}_\lambda$ be the 1-dimensional representation of $\overline{B}$ associated to the character λ. Denote the embedding $\mathfrak{Y}_0 \to \mathfrak{g} \times \mathfrak{g}$ by $i_{\mathfrak{Y}_0}$.

Moreover, we write down the important weights:

$$\lceil \mu_c \rceil = (\lceil \mu_c \rceil(1), \cdots, \lceil \mu_c \rceil(n)) := (\lceil c \rceil, \lceil 2c \rceil - \lceil c \rceil, \ldots, \lceil nc \rceil - \lceil (n-1)c \rceil). \tag{26}$$

and

$$\lfloor \mu_c \rfloor = (\lfloor \mu_c \rfloor(1), \cdots, \lfloor \mu_c \rfloor(n)) := (\lfloor c \rfloor, \lfloor 2c \rfloor - \lfloor c \rfloor, \ldots, \lfloor nc \rfloor - \lfloor (n-1)c \rfloor) \tag{27}$$

satisfying $\lfloor \mu_c \rfloor = w_0 \lceil \mu_c \rceil$ where w_0 is the longest element in W.

Lemma 3.1. *Under Construction 3.1, the following statements hold.*

- *As $\overline{B}$-equivariant $\mathcal{O}_{T^*\mathfrak{g}}$-modules, $\widetilde{\mathrm{gr}}^{\mathrm{H}}\mathbf{L} = (i_{\mathfrak{Y}_0})_* \mathcal{O}_{\mathfrak{Y}_0} \otimes \mathbb{C}_{\lceil \mu_c \rceil + \alpha}$.*
- *As $\overline{B}$-equivariant $\mathcal{O}_{T^*\mathfrak{G}}$-modules, $\widetilde{\mathrm{gr}}^{\mathrm{H}}\overline{\mathbf{L}} = \big((i_{\mathfrak{Y}_0})_* \mathcal{O}_{\mathfrak{Y}_0} \boxtimes (i_V)_* \mathcal{O}_V\big) \otimes \mathbb{C}_{\lceil \mu_c \rceil + \alpha}$.*

Proof. We have shown that $\widetilde{\mathrm{gr}}^{\mathrm{H}}\mathbf{L} \cong (i_{\mathfrak{Y}_0})_* \mathcal{O}_{\mathfrak{Y}_0}$. Moreover, the first non-vanishing filtered piece of $\mathrm{gr}^{\mathrm{H}}\mathbf{L}$ equals $\omega_{\mathfrak{n}/\mathfrak{g}} \otimes \mathbb{C}\lceil s_0^c \rceil^{-1}$. The lemma follows from that $\overline{B}$-acts on $\lceil s_0^c \rceil^{-1}$ (22) exactly by the character $\lceil \mu_c \rceil$ through $B \twoheadrightarrow T$. $\square$

3.3. **Functors on equivariant Hodge modules.** We state two important results about Hodge modules, which allow us to define the Hodge filtration and describe the associated graded of the cuspidal D-modules later.

Let G be an affine algebraic group and X be a smooth variety with a G-action. We denote the category of G-equivariant Hodge modules of weight ℓ on a smooth variety X by $\mathrm{HM}^G(X, \ell)$ and refer the readers to [Ach, Chapter 5] for a definition.

3.3.1. *Inductions.* Let H be a closed subgroup of G, which acts on the product $G\times X$ by $h\cdot(g,x) = (gh^{-1}, hx)$. Let $G\times^H X$ be the quotient of this action. Consider the diagram:

$$\begin{tikzcd} G\times X \arrow[r, "\pi"] \arrow[d, "pr"] & G\times^H X \arrow[d, "a"] \\ X & X \end{tikzcd}$$

where pr is the second projection, π is the quotient map and $a : \overline{(g,x)} \mapsto gx$.

For any H-equivariant $\mathcal{D}_X$-module $\mathcal{F}$, there is a unique G-equivariant $\mathcal{D}_{G\times^H X}$-module $\mathcal{E}$ such that $pr^\dagger\mathcal{F} \cong \pi^\dagger\mathcal{E}$ [BL94]. We denote $\mathcal{E} = \mathrm{Ind}_H^G\mathcal{F}$ and $\widetilde{\mathrm{Ind}}_H^G := a_\dagger \circ \mathrm{Ind}_H^G$.

Similarly, for any H-equivariant coherent $\mathcal{O}_X$-module $\mathcal{F}$, we write $\mathrm{ind}_H^G\mathcal{F}$ to denote the corresponding G-equivariant $\mathcal{O}_{G\times^H X}$-module.

For any map $f : X \to Y$, we introduce the associated Lagrangian correspondence as indicated by (28): the map ρ_f is the co-differential of f and ϖ_f is the natural projection.

$$T^*X \xleftarrow{\rho_f} X\times_Y T^*Y \xrightarrow{\varpi_f} T^*Y. \tag{28}$$

Combining the cases when $f=\pi$ or $f = pr$, we obtain a commutative diagram:

$$\begin{tikzcd} & T^*(G\times X) & & G\times T^*X \arrow[ll, hook', "\rho_{pr}"'] \arrow[r, two heads, "\varpi_{pr}"] & T^*X \\ T^*(G\times^H X)\times_{G\times^H X}(G\times X) \arrow[ur, hook, "\rho_\pi"] \arrow[rr, two heads, "\varpi_\pi"] & & T^*(G\times^H X) & G\times^H T^*X \arrow[l, hook', "s"'] \end{tikzcd} \tag{29}$$

Proposition 3.2. *For any $\mathbf{M} \in \mathrm{HM}^G(G\times^H X)$, $SS(\mathbf{M}) \subset s(G\times^H T^*X)$ and the following diagram commutes, where $\widetilde{\mathrm{gr}}$ is taken with respect to Hodge filtration.*

$$\begin{tikzcd} \mathrm{HM}^H(X,\ell) \arrow[r, "\mathrm{Ind}_H^G", "\sim"'] \arrow[d, "\widetilde{\mathrm{gr}}"] & \mathrm{HM}^G(G\times^H X,\ell) \arrow[d, "\widetilde{\mathrm{gr}}"] \\ \mathrm{Coh}^H(T^*X) \arrow[r, "\mathrm{ind}_H^G", "\sim"'] & \mathrm{Coh}^G(G\times^H T^*X) \end{tikzcd}$$

Proof. That the functor Ind_H^G is an isomorphism in the level of Hodge modules can be found in [Ach, Theorem 6.2]. It remains to show the essential image of $\mathrm{HM}^G(G\times^H X)$ under $\widetilde{\mathrm{gr}}$ is as desired and the diagram is commutative.

Suppose $\mathbf{M} = \mathrm{Ind}_H^G\mathbf{L}$ for $\mathbf{L}\in\mathrm{HM}^H(X)$. By definition, $\pi^\dagger\mathbf{M} = pr^\dagger\mathbf{L}$ as G-equivariant $\mathcal{D}_{G\times X}$-modules. Taking associated graded of both sides and using [Kas03, Theorem 4.7], we obtain an isomorphism in $\mathrm{Coh}^G(T^*(G\times X))$

$$(\rho_\pi)_*\varpi_\pi^*\widetilde{\mathrm{gr}}\,\mathbf{M} \cong (\rho_{pr})_*\varpi_{pr}^*\widetilde{\mathrm{gr}}\,\mathbf{L}. \tag{30}$$

In particular, we see $SS(\mathbf{M}) \subset \varpi_\pi\rho_\pi^{-1}(G\times T^*X) \subset s(G\times^H T^*X)$.

Given the diagram (29), the identity (30) implies $\varpi_{pr}^*\widetilde{\mathrm{gr}}\,\mathbf{L} = q^*\widetilde{\mathrm{gr}}\,\mathbf{M}$. Therefore $\mathrm{ind}_B^G\widetilde{\mathrm{gr}}\,\mathbf{L} = \widetilde{\mathrm{gr}}\,\mathbf{M}$ and the diagram in the statement commutes. ☐

3.3.2. *Associated graded of a pushforward.* Let $p : X\to Y$ be a projective morphism between two smooth varieties and $M\in\mathrm{HM}(X,\ell)$. By the directed image theorem of Saito ([Sai90, Theorem 2.14]), one has that $R^ip_\dagger(M)\in\mathrm{HM}(Y,\ell+i)$. Let $\widetilde{\omega_{X/Y}}$ denote the pullback of the relative canonical bundle $\omega_{X/Y} = \omega_X\otimes p^*\omega_Y^{-1}$ to $X\times_Y T^*Y$.

Theorem 3.3. *([Lau83, 2.3.2], [Sch, 28]) The diagram below commutes.*

$$\begin{tikzcd} \mathrm{HM}^G(X,\ell) \arrow[r, "R^i p_\dagger"] \arrow[d, "\widetilde{\mathrm{gr}}"] & \mathrm{HM}^G(Y,\ell+i) \arrow[d, "\widetilde{\mathrm{gr}}"] \\ \mathrm{Coh}^G(T^*X) \arrow[r, "R^i(\varpi_p)_*(\widetilde{\omega_{X/Y}}\otimes L\rho_p^*(-))"] & \mathrm{Coh}^G(T^*Y) \end{tikzcd}$$

Here $\widetilde{\mathrm{gr}}$ is taken with respect to the Hodge filtration and ρ_p, ϖ_p are defined by (28).

Remark 3.2. Note that this result does not hold in general if $\widetilde{\mathrm{gr}}$ is taken with respect to an arbitrary good filtration.

3.4. **Springer cuspidal D-modules.** We apply the two results in the last section to study the cuspidal D-modules $\mathbf{N}_c$ and $\overline{\mathbf{N}}_c$.

Since every regular nilpotent x is contained in a unique Borel subalgebra, there are embeddings of $\mathscr{N}_r$ into $\mathcal{B}\times\mathfrak{g}$ and of U into $\mathcal{B}\times\mathfrak{G}$:

$$\begin{tikzcd} \mathscr{N}_r \arrow[r, hook] \arrow[dr, hook] & \mathfrak{g} \\ & \mathcal{B}\times\mathfrak{g} \arrow[u, "p"'] \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, hook] \arrow[dr, hook, "\widetilde{i}"'] & \mathfrak{G} \\ & \mathcal{B}\times\mathfrak{G} \arrow[u] \end{tikzcd}$$

Definition 3.2. The Springer cuspidal $\mathcal{D}_{\mathfrak{g}}$-module $\mathbf{M}_c$ is the minimal extension of $\mathcal{F}_c$ to $\mathcal{B}\times\mathfrak{g}$. The Springer cuspidal $\mathcal{D}_{\mathfrak{G}}$-module $\overline{\mathbf{M}}_c$ is the minimal extension of $\mathcal{E}_c$ to $\mathcal{B}\times\mathfrak{G}$.

The local system $\mathcal{F}_c$ is also clean with respect to the inclusion $\widetilde{i}$, i.e., $\mathbf{M}_c = \widetilde{i}_\dagger\mathcal{F}$. Therefore by functoriality, $p_\dagger\mathbf{M}_c = \mathbf{N}_c$. Similar to Lemma 2.3, we have that $\overline{\mathbf{M}}_c = \mathbf{M}_c \boxtimes \mathcal{O}_V$.

Lemma 3.4. *(1) The G-equivariant $\mathcal{D}_{\mathcal{B}\times\mathfrak{g}}$-module underlying $\mathrm{Ind}_B^G(\mathbf{L}, F_\bullet)$ is $\mathbf{M}_c$.*
(2) The G-equivariant $\mathcal{D}_{\mathcal{B}\times\mathfrak{G}}$-module underlying $\mathrm{Ind}_B^G(\overline{\mathbf{L}}, F_\bullet)$ is $\overline{\mathbf{M}}_c$.

Proof. We only show (1) and the same argument applies to (2) since $\overline{\mathbf{L}} \cong \mathbf{L}\boxtimes\mathcal{O}_V$. Consider the following cartesian diagram

$$\begin{tikzcd} \mathfrak{n}_r \arrow[d, "i_{0,r}"] & G\times\mathfrak{n}_r \arrow[l, "p_0"'] \arrow[r, "\pi_0"] \arrow[d, "i_r"] & G\times^B\mathfrak{n}_r \arrow[r, equal] & \mathscr{N}_r \arrow[d, "i"] \\ \mathfrak{g} & G\times\mathfrak{g} \arrow[l, "p"'] \arrow[rr, "\pi"] & & \mathcal{B}\times\mathfrak{g} \end{tikzcd}$$

Since $\mathcal{E}_c = \mathrm{Ind}_B^G\mathbf{L}_0$, using base change twice, we have

$$\pi^\dagger i_\dagger(\mathcal{E}_c) = (i_r)_\dagger\pi_0^\dagger(\mathcal{E}_c) = (i_r)_\dagger p_0^\dagger(\mathcal{F}_c) = p^\dagger(i_{0,r})_\dagger(\mathcal{F}_c)$$

which proves the lemma. □

Recall that $SS(\mathbf{L}) = \mathfrak{Y}_0$ (25). Consider $\mathfrak{Y} = G\times^B\mathfrak{Y}_0$ with embedding $i_{\mathfrak{Y}}:\mathfrak{Y}\to T^*(\mathcal{B}\times\mathfrak{g})$ such that the following diagram is Cartesian:

$$\begin{tikzcd} \mathfrak{Y}_0 \arrow[r] \arrow[d] & \mathfrak{g}\times\mathfrak{n}\times\mathfrak{g} \arrow[d] \\ G\times^B\mathfrak{Y}_0 \arrow[r, "i_{\mathfrak{Y}}"] & G\times^B(\mathfrak{g}\times\mathfrak{n}\times\mathfrak{g}) \end{tikzcd}$$

Let $\mathcal{L}_\lambda$ be the $\overline{G}$-equivariant line bundle on $\mathcal{B}$ associated to the weight λ and $\widetilde{\mathcal{L}}_\lambda$ be the pullback of $\mathcal{L}_\lambda$ to $T^*(\mathcal{B}\times\mathfrak{g})$. Endow $(i_{\mathfrak{Y}})_*\mathcal{O}_{\mathfrak{Y}}$, resp. $(i_{\mathfrak{Y}})_*\mathcal{O}_{\mathfrak{Y}}\boxtimes\mathcal{O}_V$, with the trivial G, resp. $\overline{G}$-equivariant structure.

We precompose the $\overline{\mathfrak{g}}$-action on $\widetilde{\mathrm{gr}}\ \overline{\mathbf{M}}_c$, resp. $\widetilde{\mathrm{gr}}\ \overline{\mathbf{N}}_c$, with τ_c and integrate it into a $\overline{G}$-action, as in Construction 3.1. In this way, we have the following descriptions:

Corollary 3.5. *The following $\overline{G}$-equivariant isomorphisms hold:*

- $\widetilde{\mathrm{gr}}^{\mathrm{H}}\mathbf{M}_c \cong \widetilde{\mathcal{L}}_{\lceil\mu_c\rceil+\alpha} \otimes (i_{\mathfrak{Y}})_*\mathcal{O}_{\mathfrak{Y}}$;
- $\widetilde{\mathrm{gr}}^{\mathrm{H}}\overline{\mathbf{M}}_c \cong \big(\widetilde{\mathcal{L}}_{\lceil\mu_c\rceil+\alpha} \otimes (i_{\mathfrak{Y}})_*\mathcal{O}_{\mathfrak{Y}}\big) \boxtimes \mathcal{O}_V$.

Proof. By Proposition 3.2 and Lemma 3.4, we have $\widetilde{\mathrm{gr}}\ \mathbf{M}_c \cong \mathrm{ind}_B^G \widetilde{\mathrm{gr}}\ \mathbf{L}$ and $\widetilde{\mathrm{gr}}\ \overline{\mathbf{M}}_c \cong \mathrm{ind}_B^G \widetilde{\mathrm{gr}}\ \overline{\mathbf{L}}$. Therefore the corollary follows from Lemma 3.1. $\square$

4. Cuspidal DG modules and bigraded characters

4.1. **Cuspidal DG modules.** We express the Hodge associated graded of the cuspidal D-modules as pushforward of certain DG modules.

Recall the coordinates from (20) and (24). When $x \in \mathfrak{n}$ and $y \in \mathfrak{m}$, we have $[x, y] \in \mathfrak{b}$. Moreover, the diagonals of $[x, y]$ equal

$$x_1y_1,\ x_2y_2 - x_1y_1, \ldots,\ -x_{n-1}y_{n-1}.$$

As a result, $x_iy_i = 0$, $1 \le i \le n-1$ if and only if $[x, y] = 0 \mod \mathfrak{n}$. That is to say, the following diagram is Cartesian:

(31)
$$\begin{tikzcd} \mathfrak{Y} \arrow[r, "q_{\mathfrak{n}}"] \arrow[d] & G \times^B \mathfrak{n} \arrow[d] \\ G \times^B (\mathfrak{n} \times \mathfrak{m}) \arrow[r, "q_{\mathfrak{b}}"] & G \times^B \mathfrak{b} \end{tikzcd}$$

with

$$q_{\mathfrak{b}} : G \times^B (\mathfrak{n} \times \mathfrak{m}) \to G \times^B \mathfrak{b}, \quad (g, x, y) \mapsto (g, [x, y])$$

and $q_{\mathfrak{n}}$ is the restriction of $q_{\mathfrak{b}}$ to $\mathfrak{Y}$.

On $\mathcal{B}$ we have the vector bundle $\underline{\mathfrak{b}^*}$ (resp. $\underline{\mathfrak{b}}$) whose total space equals $G \times^B \mathfrak{b}^*$ (resp. $G \times^B \mathfrak{b}$). Let $\pi_{\mathfrak{b}} : G \times^B \mathfrak{b} \to \mathcal{B}$ be the projection and $\iota_{\mathfrak{b}} : \mathcal{B} \to G \times^B \mathfrak{b}$ be the zero section. The Koszul complex $(\wedge^\bullet \pi_{\mathfrak{b}}^* \underline{\mathfrak{b}^*}, \partial_{\mathfrak{b}})$, with differential $\partial_{\mathfrak{b}}$ defined by contraction with the canonical section of $\pi_{\mathfrak{b}}^*\underline{\mathfrak{b}}$, is quasi-isomorphic to $(\iota_{\mathfrak{b}})_*\mathcal{O}_{\mathcal{B}}$.

One similarly defines $\pi_{\mathfrak{n}}$, $\iota_{\mathfrak{n}}$, $\underline{\mathfrak{n}^*}$, $\partial_{\mathfrak{n}}$, such that $(\wedge^\bullet \pi_{\mathfrak{n}}^* \underline{\mathfrak{n}^*}, \partial_{\mathfrak{n}})$ is quasi-isomorphic to $(\iota_{\mathfrak{n}})_*\mathcal{O}_{\mathcal{B}}$.

We define a DG algebra by

$$\mathcal{A}'' := ((\wedge^\bullet (\pi_{\mathfrak{n}} \circ q_{\mathfrak{n}})^* \underline{\mathfrak{n}^*}, q_{\mathfrak{n}}^* \partial_{\mathfrak{n}})).$$

By definition, the associated DG scheme $Spec(\mathcal{A}'')$ makes the following diagram Cartesian.

(32)
$$\begin{tikzcd} Spec(\mathcal{A}'') \arrow[r] \arrow[d] & \mathcal{B} \arrow[d, "\iota_{\mathfrak{n}}"] \\ \mathfrak{Y} \arrow[r, "q_{\mathfrak{n}}"] & G \times^B \mathfrak{n} \end{tikzcd}$$

Given the Cartesian diagram (31), we have that $(i_{\mathfrak{Y} \to G \times^B (\mathfrak{n} \times \mathfrak{m})})_* \mathcal{A}''$ is quasi-isomorphic to

$$\mathcal{A} := ((\wedge^\bullet q_{\mathfrak{b}}^* \pi_{\mathfrak{b}}^* \underline{\mathfrak{b}^*}, q_{\mathfrak{b}}^* \partial_{\mathfrak{b}})).$$

Because (32) and the diagram on the left of (33) of are Cartesian, diagram on the right of (33) is also Cartesian.

$$
\begin{tikzcd}
G \times^B T^*\mathfrak{g} \arrow[r] \arrow[d] & \mathcal{B} \arrow[d] & & Spec(\mathcal{A}'') \arrow[r] \arrow[d] & \mathcal{B} \times T^*\mathfrak{g} \arrow[d] \\
T^*(\mathcal{B} \times \mathfrak{g}) \arrow[r] & G \times^B \mathfrak{n} & & \mathfrak{Y} \arrow[r] & T^*(\mathcal{B} \times \mathfrak{g})
\end{tikzcd} \tag{33}
$$

Diagrams (31), (32) and (33) can be combined into Figure 1.

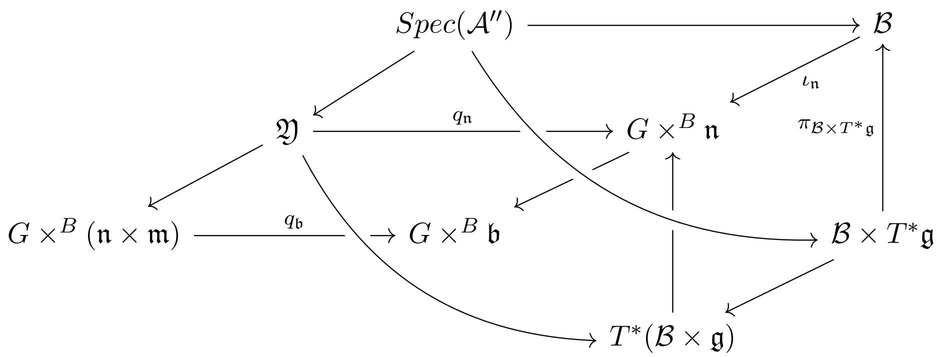


FIGURE 1. Cartesian diagrams

Below, we simply denote the projection from a variety X to $\mathcal{B}$ by π_X.

Definition 4.1. The cuspidal DG module of slope c is

$$\mathcal{A}_c := \mathcal{A} \otimes \pi^*_{G\times^B(\mathfrak{n}\times\mathfrak{m})}\mathcal{L}_{\lceil \mu_c \rceil} \tag{34}$$

Similarly, define $\mathcal{A}''_c := \mathcal{A}'' \otimes \pi^*_{\mathfrak{Y}}\mathcal{L}_{\lceil \mu_c \rceil}$.

Consider the following maps:

$$
\begin{tikzcd}
\mathfrak{Y} \arrow[r, "i_{\mathfrak{Y}}"] \arrow[dr, "p_{\mathfrak{Y}}"'] & T^*(\mathcal{B} \times \mathfrak{g}) \arrow[d, "p_{T^*(\mathcal{B}\times\mathfrak{g})}"] & \mathcal{B} \times T^*\mathfrak{g} \arrow[l, "i_{\mathcal{B}\times T^*\mathfrak{g}}"'] \arrow[dl, "p_{\mathcal{B}\times T^*\mathfrak{g}}"] \\
& T^*\mathfrak{g} &
\end{tikzcd}
$$

Here $p_{\mathfrak{Y}} : \mathfrak{Y} \to T^*\mathfrak{g}$ is the restriction of $p : G \times^B (\mathfrak{n} \times \mathfrak{m}) \to T^*\mathfrak{g} : (g, x, y) \mapsto (g \cdot x, g \cdot y)$.

Proposition 4.1. *There are $\overline{G}$-equivariant isomorphism:*

$$\widetilde{\mathrm{gr}}^{\mathrm{H}}\mathbf{N}_c = Rp_*\mathcal{A}_c, \qquad \widetilde{\mathrm{gr}}^{\mathrm{H}}\overline{\mathbf{N}}_c = Rp_*\mathcal{A}_c \boxtimes \mathcal{O}_V.$$

Proof. By Theorem 3.3, we have a G-equivariant isomorphism:

$$\widetilde{\mathrm{gr}}^{\mathrm{H}}\mathbf{N}_c = R(p_{\mathcal{B}\times T^*\mathfrak{g}})_* Li^*_{\mathcal{B}\times T^*\mathfrak{g}}(\widetilde{\mathrm{gr}}^{\mathrm{H}}\mathbf{M}_c).$$

The identification from Corollary 3.5: $\widetilde{\mathrm{gr}}^{\mathrm{H}}\overline{\mathbf{M}}_c = (\widetilde{\mathcal{L}}_{\lceil \mu_c \rceil + \alpha} \otimes (i_{\mathfrak{Y}})_*\mathcal{O}_{\mathfrak{Y}}) \boxtimes \mathcal{O}_V$ implies:

$$\widetilde{\mathrm{gr}}^{\mathrm{H}}\overline{\mathbf{N}}_c = R(p_{\mathcal{B}\times T^*\mathfrak{g}})_* \left(Li^*_{\mathcal{B}\times T^*\mathfrak{g}}\big(\widetilde{\mathcal{L}}_{\lceil \mu_c \rceil + \alpha} \otimes (i_{\mathfrak{Y}})_*\mathcal{O}_{\mathfrak{Y}}\big) \otimes \pi^*_{\mathcal{B}\times T^*\mathfrak{g}}\omega_{\mathcal{B}} \right) \boxtimes \mathcal{O}_V. \tag{35}$$

We will use the following result, which is a simple consequence of base changes: ([Gin12, Lemma 4.4.1]) *Suppose X is a smooth variety and $i_Y : Y \to X$, $i_Z : Z \to X$ are embeddings of closed subvarieties. Then:*

$$(i_Y)_* Li^*_Y (i_Z)_*\mathcal{O}_Z = (i_Y)_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} (i_Z)_*\mathcal{O}_Z = (i_Z)_* Li^*_Z (i_Y)_*\mathcal{O}_Y.$$

Applying this lemma to (35) in the setting of $Y=\mathcal{B}\times T^*\mathfrak{g}$ and $Z=\mathfrak{Y}$, we obtain:

$$
\begin{aligned}
&R(p_{\mathcal{B}\times T^*\mathfrak{g}})_*\Big(Li^*_{\mathcal{B}\times T^*\mathfrak{g}}(\widetilde{\mathcal{L}}_{\lceil\mu_c\rceil+\alpha}\otimes(i_{\mathfrak{Y}})_*\mathcal{O}_{\mathfrak{Y}})\otimes\pi^*_{\mathcal{B}\times T^*\mathfrak{g}}\omega_{\mathcal{B}}\Big)\\
=&\mathrm{R}(p_{T^*(\mathcal{B}\times\mathfrak{g})})_*\Big(\widetilde{\mathcal{L}}_{\lceil\mu_c\rceil}\otimes(i_{\mathfrak{Y}})_*\mathcal{O}_{\mathfrak{Y}}\otimes(i_{\mathcal{B}\times T^*\mathfrak{g}})_*\mathcal{O}_{\mathcal{B}\times T^*\mathfrak{g}}\Big)\\
=&R(p_{T^*(\mathcal{B}\times\mathfrak{g})})_*R(i_{\mathfrak{Y}})_*L(i_{\mathfrak{Y}})^*\Big(\widetilde{\mathcal{L}}_{\lceil\mu_c\rceil}\otimes(i_{\mathcal{B}\times T^*\mathfrak{g}})_*\mathcal{O}_{\mathcal{B}\times T^*\mathfrak{g}}\Big)\\
=&R(p_{\mathcal{B}\times T^*\mathfrak{g}})_*L(i_{\mathfrak{Y}})^*\Big(\widetilde{\mathcal{L}}_{\lceil\mu_c\rceil}\otimes(i_{\mathcal{B}\times T^*\mathfrak{g}})_*\mathcal{O}_{\mathcal{B}\times T^*\mathfrak{g}}\Big).
\end{aligned}
\tag{36}
$$

Since the diagram on the right of (33) is Cartesian, we have

$$
L(i_{\mathfrak{Y}})_*\Big(\widetilde{\mathcal{L}}_{\lceil\mu_c\rceil}\otimes(i_{\mathcal{B}\times T^*\mathfrak{g}})_*\mathcal{O}_{\mathcal{B}\times T^*\mathfrak{g}}\Big)=R(i_{\mathfrak{Y}})_*\mathcal{A}''_c.
$$

Therefore, (36) equals

$$
R(p_{\mathcal{B}\times T^*\mathfrak{g}})_*R(i_{\mathfrak{Y}})_*\mathcal{A}''_c=R(p_{\mathfrak{Y}})_*\mathcal{A}''_c=Rp_*\mathcal{A}_c.
$$

The matching on the $Z(\overline{G})$-equivariance is a result of Construction 3.1. The proposition follows. $\square$

To compute the bigraded character of the RCA-module L_c using $\widetilde{\mathrm{gr}}^{\mathrm{H}}\overline{\mathbf{N}}_c$, we work with the equivariant K-theory of Hilbert schemes utilize the localization theorem.

4.1.1. *BKR equivalence for* Hilb^n. Let $A:=\mathbb{C}^*\times\mathbb{C}^*$.

Define the isospectral Hilbert scheme IHilb^n to be the *reduced* fibered product of the following diagram

$$
\begin{tikzcd}
\mathrm{IHilb}^n \arrow[r] \arrow[d, "\alpha"] & \mathbb{C}^{2n} \arrow[d] \\
\mathrm{Hilb}^n \arrow[r] & \mathbb{C}^{2n}/S_n
\end{tikzcd}
$$

A deep result of Haiman [Hai01] states that $\mathcal{P}:=\alpha_*\mathcal{O}_{\mathrm{IHilb}^n}$ is a vector bundle of rank $n!$, which is known since as the Procesi bundle $\mathcal{P}$. As a corollary, one has the following special case of Bridgeland-King-Reid equivalence between Grothendieck groups:

$$
\Gamma(\mathrm{Hilb}^n,\mathcal{P}\otimes(-)):K^A(\mathrm{Hilb}^n)\cong K^{S_n\times A}(\mathbb{C}^{2n}). \tag{37}
$$

Define the algebra of symmetric polynomials in infinite many variables:

$$
K:=\mathbb{C}(q,t)[z_1,z_2,\dots]^{S_\infty}. \tag{38}
$$

For every partition $\lambda\vdash n$, let V_λ be the associated irreducible representation of S_n. The Grothendieck group $K^{S_n\times A}(\mathbb{C}^{2n})$ is freely generated by $V_\lambda\otimes\mathbb{C}[\mathbb{C}^{2n}]$ over $\mathbb{C}[q^\pm,t^\pm]$. Let $s_\lambda\in K$ be the Schur function associated to $\lambda\vdash n$. The Frobenious character $\mathrm{ch}_{S_n}:\mathrm{Rep}(S_n)\to\mathbb{C}[z_1,z_2,\dots]^{S_\infty}$ is an additive map sending V_λ to s_λ. For a bigraded S_n representation $M=\bigoplus_{i,j}M_{i,j}$, one may further consider its bigraded Frobenius character valued in K

$$
\mathrm{ch}_{S_n\times A}(M)=\sum_{i,j}q^it^j\mathrm{ch}_{S_n}(M_{i,j}).
$$

Write $Z=z_1+\cdots+z_n$. For every $f\in K$, let $f[Z]$ the plethystic substitution of Z into f. The bigraded Frobenius character of $V_\lambda\otimes\mathbb{C}[\mathbb{C}^{2n}]$ is ([Hai03, Proposition 3.3.1])

$$
\mathrm{ch}_{S_n\times A}(V_\lambda\otimes\mathbb{C}[\mathbb{C}^{2n}])=s_\lambda\left[\frac{Z}{(1-q)(1-t)}\right].
$$

Therefore, composing (37) with $\mathrm{ch}_{S_n\times A}$ and taking a direct sum establish an isomorphism:

$$\kappa : \bigoplus_n K^A(\mathrm{Hilb}^n) \cong \{f \in K | f\,[(1-q)(1-t)Z] \text{ has coefficients in } \mathbb{C}[q^{\pm}, t^{\pm}]\}, \tag{39}$$

such that $\kappa(\mathcal{V}_\lambda) = s_\lambda\left[\frac{Z}{(1-q)(1-t)}\right]$ where $\mathcal{V}_\lambda = \mathrm{Hom}_{S_n}(V_\lambda, \mathcal{P})$. We thus have an identification:

$$\bigoplus_n \mathbb{C}(q,t) \otimes_{\mathbb{C}[q^{\pm},t^{\pm}]} K^A(\mathrm{Hilb}^n) \cong K. \tag{40}$$

4.1.2. *Macdonald polynomials and fixed points.* For $\lambda \vdash n$, let p_λ denote the associated power sum symmetric polynomial. On K, we have an inner product [Mac95]:

$$(p_\lambda, p_\mu) = \delta_{\lambda\mu} z_\lambda \prod_i \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}, \quad \text{where } z_{(1^{n_1}2^{n_2}\dots)} = \prod_i i^{n_i} n_i!$$

Let λ^t denote the transpose of the partition $\lambda \vdash n$. The modified Macdonald polynomials $\widetilde{H}_\lambda(z;q,t)$ are the unique q,t-symmetric polynomials satisfying

$$\begin{aligned}
&\widetilde{H}_\lambda\,[(1-q)z;q,t] \in \mathbb{C}(q,t)\langle s_\mu | \mu \geq \lambda\rangle;\\
&\widetilde{H}_\lambda\,[(1-t)z;q,t] \in \mathbb{C}(q,t)\langle s_\mu | \mu \geq \lambda^t\rangle;\\
&\widetilde{H}_\lambda[1;q,t] = (\widetilde{H}_\lambda, s_{(n)}) = 1.
\end{aligned} \tag{41}$$

The A-fixed points in Hilb^n are monomial ideals of co-length n, which are in bijection with partitions of n. For any $\lambda \vdash n$, let I_λ be the associated fixed point and $[I_\lambda]$ be the K-theory class corresponding to the skyscraper sheaf supported on I_λ.

Proposition 4.2. *([Hai03, Theorem 4.1.5 and Proposition 5.4.1]) The image of $[I_\lambda]$ under (39) is $\widetilde{H}_\lambda$.*

4.1.3. *Localization formula.* **Throughout, we may not distinguish between partitions and Young diagrams.** For a box x inside a Young diagram $\lambda \vdash n$, let a, ℓ, resp. a', ℓ', denote its arm and leg, resp. coarm and coleg (demonstrated in Figure (2)).

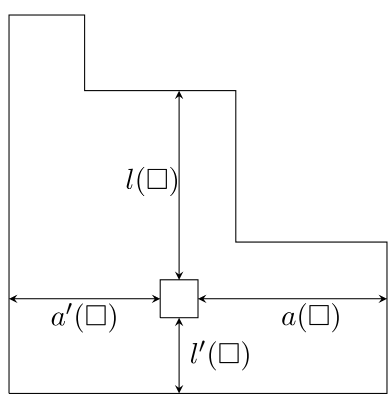


FIGURE 2. ([GN15, Fig.1])Arm, leg, co-arm and co-leg

For a representation of V of A, let $\mathrm{ch}_A(V) := \mathrm{tr}(q,t,V)$ denote the character. Equivalently, if $V = \bigoplus_{i,j} V_{i,j}$ is the weight space decomposition, then

$$\mathrm{ch}_A(V) = \sum q^i t^j \dim(V_{i,j}).$$

Define

$$\lambda(V) = \sum_{i=0}^{\dim(V)} (-1)^i \mathrm{ch}_A(\wedge^i V).$$

By [ESm88, Hai98],

$$\lambda(T^*_{I_\lambda}\mathrm{Hilb}^n) = \prod_{x\in\lambda}(1-q^{a(x)+1}t^{-\ell(x)})(1-q^{-a(x)}t^{\ell(x)+1}) =: g_\lambda. \tag{42}$$

For any $[\mathcal{F}] \in K^A(\mathrm{Hilb}^n)$, under the isomorphism (39), we have the localization formula [CG10, Remark 5.11.8]:

$$\kappa([\mathcal{F}]) = \sum_{\lambda\vdash n} \frac{\widetilde{H}_\lambda}{g_\lambda}\mathrm{ch}_A(\mathcal{F}|_{I_\lambda}). \tag{43}$$

The notation $\mathcal{F}|_{I_\lambda}$ denotes the **derived** pullback of $\mathcal{F}$ along the embedding $\{I_\lambda\} \hookrightarrow X$. **This convention will be adopted henceforth.**

To compute $\mathrm{ch}_{S_n\times A}(\mathrm{L}_c)$, we will apply the localization formula to $\Psi_c(\overline{\mathbf{N}}_c)$. For this goal, we will compute $\mathrm{ch}_A\left(\Psi_c(\overline{\mathbf{N}}_c)|_{I_c}\right)$ next.

4.2. **Principal nilpotent pairs.** The content in this section is valid for any semisimple Lie algebra $\mathfrak{g}$.

4.2.1. The commuting variety is defined by $\mathfrak{C} := \{(x,y) \in \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}, [x,y]=0\}_{\mathrm{red}}$. On $\mathfrak{C}$, there is a diagonal G-adjoint action. Following [Gin12], we define a variety $\mathfrak{IC}$ as a fibered product using the isomorphism $\mathbb{C}[\mathfrak{C}]^G \cong \mathbb{C}[\mathfrak{h}\times\mathfrak{h}]^W$:

$$\begin{tikzcd}
\mathfrak{IC} \arrow[r, "p_{\mathfrak{g}}"] \arrow[d] & \mathfrak{C} \arrow[d] \\
\mathfrak{h}^2 \arrow[r] & \mathfrak{h}^2/W
\end{tikzcd}$$

Define a sheaf on $\mathfrak{C}$ by $\mathcal{R} := p_*\mathcal{O}_{\mathfrak{IC}}$. We note that the isospectral commuting variety in *op. cit.* is defined via the normalization of $\mathfrak{C}$ but the variety $\mathfrak{IC}$ is sufficient for our purpose.

For any G-variety X, we write X_r to denote the regular locus, i.e., when the stablizer of the G-action is of the minimal possible dimension. The subvariety $\mathfrak{C}_r \subset \mathfrak{C}$ is smooth and open dense. The restriction $\mathcal{R}|_{\mathfrak{C}_r}$ is a vector bundle, with a W-action on each fiber making it a regular representation of W.

4.2.2.

Definition 4.2. ([Gin00]) An element $\mathbf{e} = (x_1,x_2)$ in $\mathfrak{C}_r$ is called a principal nilpotent pair if for any $(t_1,t_2) \in \mathbb{C}^*\times\mathbb{C}^*$, there exists some $g\in G$ such that $(t_1x_1, t_2x_2) = (\mathrm{Ad}(g)x_1, \mathrm{Ad}(g)x_2)$.

It is shown in [Gin00, Theorem 1.2] that for every principal nilpotent pair (e_1,e_2), there exists an associated semisimple pair $\mathbf{h} = (h_1,h_2)$ such that $\mathbf{h}$ is regular and

$$[h_i, e_j] = \delta_{ij}e_j, \quad i,j=1,2.$$

The adjoint action of (h_1,h_2) decomposes $\mathfrak{g}$ into weight spaces

$$\mathfrak{g} = \bigoplus_{a,b\in\mathbb{Z}^2} \mathfrak{g}_{a,b}, \quad \text{where } \mathrm{ad}(h_1)x = ax, \mathrm{ad}(h_2)x = bx, \forall x \in \mathfrak{g}_{a,b}.$$

For every fixed principal nilpotent pair $\mathbf{e}$ with an associated semisimple pair $\mathbf{h}$, let $\rho : A \to G$ be the 2-parameter subgroup with differential at the identity being $\mathbb{C}^2 \to \mathfrak{g}$: $(1,0) \mapsto h_1$, $(0,1) \mapsto h_2$. We define a A action on $\mathfrak{g}$ by $\mathrm{Ad}(\rho)$ and a A action $\mathfrak{g} \times \mathfrak{g}$ by

$$(t_1,t_2)(x,y) = (t_1^{-1}\mathrm{Ad}(\rho(t_1,t_2))x, t_2^{-1}\mathrm{Ad}(\rho(t_1,t_2))y) \tag{44}$$

such that $\mathbf{e}$ is a fixed point under this action. The weight space $(\mathfrak{g} \oplus \mathfrak{g})_{a,b}$ under the action (44) equals $\mathfrak{g}_{a+1,b} \oplus \mathfrak{g}_{a,b+1}$ under the diagonal $Ad(\rho)$-action.

4.3. **Principal nilpotent pairs for $\mathfrak{sl}_n$.** Take $(x,y,v) \in \widetilde{\mathrm{Hilb}^n}$. Having a cyclic vector ensures that $(x,y) \in \mathfrak{C}_r$. It is shown in [Gin12] that

$$(\mathcal{R}|_{\mathfrak{C}_r} \boxtimes \mathcal{O}_V)\,|_{\widetilde{\mathrm{Hilb}^n}} = \pi^*\mathcal{P} \quad \text{ where } \pi : \widetilde{\mathrm{Hilb}^n} \to \mathrm{Hilb}^n.$$

By Lemma 2.3, The support of $\widetilde{\mathrm{gr}}^{\mathrm{H}}\mathbf{N}_c$ is contained in $\mathfrak{C}_{\mathrm{red}}$. Therefore, there exists a coherent sheaf $\underline{\mathcal{F}}_c$ on $\mathfrak{C}_{\mathrm{reg}}$ such that $i_*\underline{\mathcal{F}}_c = \widetilde{\mathrm{gr}}^{\mathrm{H}}\mathbf{N}_c$ where $i : \mathfrak{C}_{\mathrm{red}} \hookrightarrow T^*\mathfrak{g}$. Let

$$\mathcal{F}_c := \underline{\mathcal{F}}_c|_{\mathfrak{C}_r}.$$

In the case of $\mathfrak{sl}_n$, conjugacy classes of principal nilpotent pairs are in bijection with both A-fixed points in Hilb^n and partitions of n. Indeed, for a partition λ, let e_1, resp. e_2 be the associated Jordan normal form associated to λ, resp. λ^t. Then $\mathbf{e}_\lambda := (e_1, e_2)$ defines a principal nilpotent pair and all principal nilpotent pairs up to conjugation can be constructed in this way.

Definition 4.3. We call a Young tableau an **almost** standard Young tableau (ASYT) if the labels increase rightwards on rows and upwards on columns, with the exception that the labels are allowed to decrease up to 1 going up.

For an example, see Appendix B.2.2 for a full list of all ASYT of three boxes. Recall that if the labels increase rightwards on rows and upwards on columns, we obtain a standard Young tableau. Let ASYT_λ, resp. SYT_λ, be the set of almost standard Young tableaux, resp. standard Young tableaux, of shape λ.

Remark 4.1. Almost standard Young tableaux appear in the discussion of the "eccentric correspondence" in [Neg15b, 4.5] and [GN24, 2.3].When specialized, the eccentric correspondence captures the geometry of the cuspidal DG algebra $\mathcal{A}_0$ at homological degree 0.

Every $\sigma \in \mathrm{SYT}_\lambda$ gives a semisimple pair associated to $\mathbf{e}_\lambda$ [Gin00, Section 5]: suppose for every label $k \in [1,n]$ the corresponding box $\square_k$ is at position (a_k, b_k), i.e, $a'(\square_k) = a_k$ and $\ell'(\square_k) = b_k$ (cf. Figure 2). Define diagonal matrices

$$h_1 = (a_n, \dots, a_1), \quad h_2 = (b_n, \dots, b_1). \tag{45}$$

The pair (h_1, h_2) satisfies the desired properties. For example, under the notations (23), when $(e_1, e_2) = (\underline{e}, 0)$, there is a unique associated semisimple pair given by $(h_1, h_2) = (\frac{1}{2}\underline{h}, 0)$.

Fix a principal nilpotent pair $\mathbf{e}_\lambda$ with an associated semisimple pair $\mathbf{h}_\lambda$. Since $\mathbf{h}_\lambda$ is regular, the Borel subalgebras containing both h_1 and h_2 are in bijection with the Weyl group. We fix such a bijection $w \leftrightarrow \mathfrak{b}_w$. Recall the following space from (31):

$$\mathfrak{Z} := G \times^B (\mathfrak{n} \times \mathfrak{m}).$$

For every $\mathfrak{b}_w$, we define associated $\mathfrak{n}_w$ and $\mathfrak{m}_w$. Similar to [BG13, Lemma 4.4.1], we have that:

Lemma 4.3. *There is a bijection:*

$$\{\mathfrak{b}_w, w \in W | e_1 \in \mathfrak{n}_w, e_2 \in \mathfrak{m}_w\} \leftrightarrow \mathrm{ASYT}_\lambda$$

We will write $w \in \mathrm{ASYT}_\lambda$ when $\mathfrak{b}_w$ satisfies the condition in Lemma 4.3.

Consider the fixed point set under the A-action defined by (44):

$$\mathfrak{Z}_r^A = \sqcup_{w\in W}\mathfrak{Z}_r^{w,A} \quad \text{ with } \mathfrak{Z}_r^{w,A} := \{\mathfrak{b}_w\} \times (\mathfrak{n}_w \oplus \mathfrak{m}_w) \cap (\mathfrak{g}\oplus\mathfrak{g})_r^A.$$

Though $[\mathfrak{n}_w, \mathfrak{m}_w] = \mathfrak{b}_w$, we will insist on writing $[\mathfrak{n}_w, \mathfrak{m}_w]$ to emphasize the A-action on it is induced by composing (44) with $[-,-]$. We fix an A-stable subspace $\mathfrak{R}_w \subset [\mathfrak{n}_w, \mathfrak{m}_w]$ such that

$$[\mathfrak{n}_w, \mathfrak{m}_w] = \mathfrak{R}_w \oplus [\mathfrak{n}_w, \mathfrak{m}_w]^A.$$

The following lemma resembles [BG13, Proposition 3.8.6] and the proof follows verbatim, via replacing their variety $G\times^B(\mathfrak{b}\times\mathfrak{b})$ by our $\mathfrak{Z}$.

Lemma 4.4. $\mathrm{ch}_A\left(\mathcal{F}_c|_{\mathbf{e}_\lambda}\right) = \lambda\left((T^*_{\mathfrak{C}_r^A}\mathfrak{C}_r)|_{\mathbf{e}_\lambda}\right)\cdot\sum_{w\in\mathrm{ASYT}_\lambda}\lambda\left((T^*_{\mathfrak{Z}_r^{w,A}}\mathfrak{Z}_r)|_{\mathbf{e}_\lambda}\right)^{-1}\lambda\left(\mathfrak{R}_w^*\otimes\mathbb{C}_{\lceil\mu_c\rceil}\right).$

We adopt the Ω-notation by setting

$$\Omega(\sum_{i,j} a_{i,j}q^it^j) = \prod(1-q^it^j)^{a_{i,j}}, \quad \Omega^0(F) = \Omega(F - a_{0,0}) \tag{46}$$

such that

$$\lambda(V) = \Omega(\mathrm{ch}_A(V)), \quad \lambda(V/V^A) = \Omega^0(\mathrm{ch}_A(V)).$$

For a Young tableau, define the weight of the box x labeled i by (cf. Figure (2))

$$\chi_i = q^{a'(x)}t^{l'(x)}.$$

We also write

$$\omega(x) = \frac{(1-x)(1-qtx)}{(1-qx)(1-tx)} = \Omega\left((1-q)(1-t)x\right).$$

4.3.1. The result below is an analogue to [BG13, Theorem 4.5.1].

Proposition 4.5. *The following identity holds*

$$\mathrm{ch}_A(\mathcal{F}_c|_{\mathbf{e}_\lambda}) = g_\lambda\frac{(1-qt)^{n-1}}{(1-t)^{n-1}(-t)^{n-1}}\sum_{\sigma\in\mathrm{ASYT}_\lambda}\frac{\Xi_\sigma\prod_{i=1}^n\chi_{n-i+1}^{\lfloor\mu_c\rfloor(i)}}{\prod_{i=1}^{n-1}(1-\frac{\chi_i}{t\chi_{i+1}})} \tag{47}$$

where

$$\Xi_\sigma := \hat{\prod_i}\frac{1}{(1-\chi_i^{-1})}\hat{\prod}_{1\le i<j\le n}\omega(\frac{\chi_i}{\chi_j}). \tag{48}$$

The "restricted" product $\hat{\prod}$ means we ignore all the zero linear denominators.

Proof. By Lemma 4.4 and (45),

$$\mathrm{ch}_A(\mathcal{F}_c|_{\mathbf{e}_\lambda}) = \lambda((T^*_{\mathfrak{C}_r^A}\mathfrak{C}_r)|_{\mathbf{e}_\lambda})\cdot\sum_{w\in\mathrm{ASYT}_\lambda}\lambda((T^*_{\mathfrak{Z}_r^{w,A}}\mathfrak{Z}_r)|_{\mathbf{e}_\lambda})^{-1}\lambda(\mathfrak{R}_w^*)\prod_{i=1}^n\chi_{n-i+1}^{\lceil\mu_c\rceil(i)}.$$

To ease notations, we will simply write $\mathbf{e} := \mathbf{e}_\lambda$ and $\mathrm{Stab}(\mathbf{e}) := \mathrm{Stab}_{\mathfrak{g}}(\mathbf{e})$ in the proof.

By definition, $\lambda(\mathfrak{R}_w^*) = \lambda(([\mathfrak{n}_w,\mathfrak{m}_w]/[\mathfrak{n}_w,\mathfrak{m}_w]^A)^*)$. Moreover, by [BG13, Lemma 3.9.1],

$$\lambda\left((T^*_{\mathfrak{C}_r^A}\mathfrak{C}_r)|_{\mathbf{e}}\right) = g_\lambda\cdot\lambda\left(\left(\mathfrak{g}/(\mathrm{Stab}(\mathbf{e})\oplus\mathfrak{h})\right)^*\right);$$

$$\lambda\left((T^*_{\mathfrak{Z}_r^{w,A}}\mathfrak{Z}_r)|_{\mathbf{e}}\right) = \lambda(\mathfrak{n}_w)\cdot\lambda\left(\left((\mathfrak{n}_w\oplus\mathfrak{m}_w)/(\mathfrak{g}\oplus\mathfrak{g})^A\cap(\mathfrak{n}_w\oplus\mathfrak{m}_w)\right)^*\right).$$

Let R, resp. R^+, denote the set of all roots, resp. positive roots with respect to $\mathfrak{b}_w$ and $\Delta\subset R^+$ be the set of simple roots. Let us denote

$$\overline{\Sigma} := \sum_{1\le i<j\le n}\chi_i\chi_j^{-1}, \quad \underline{\Sigma} := \sum_{1\le i<j\le n}\chi_i^{-1}\chi_j.$$

We have identities:

$$\begin{aligned}
\mathrm{ch}_A(\mathfrak{h}^*) &= n-1\\
\mathrm{ch}_A(\mathfrak{g}^*) &= n-1+\sum_{\alpha_1,\alpha_2\in R} q^{\alpha_1(h_1)}t^{\alpha_2(h_2)} = n-1+\overline{\Sigma}+\underline{\Sigma}\\
\mathrm{ch}_A(\mathrm{Stab}(\mathbf{e})^*) &= \sum_{(a,b)\in\lambda} q^{-a}t^{-b} = \sum_{i=1}^{n}\chi_i^{-1}, \qquad (\because \mathrm{Stab}(\mathbf{e}) = \mathbb{C}[e_1,e_2])
\end{aligned}$$

which imply that

$$\lambda\left([\mathfrak{g}/(\mathrm{Stab}(\mathbf{e})\oplus\mathfrak{h})]^*\right) = \Omega(\overline{\Sigma}+\underline{\Sigma})\Omega(\sum_{i=1}^{n}\chi_i^{-1})^{-1}.$$

The identity

$$\mathrm{ch}_A(\mathfrak{n}_w) = \sum_{\alpha\in R^+} q^{\alpha(h_1)}t^{\alpha(h_2)} = \underline{\Sigma}$$

implies that $\lambda(\mathfrak{n}_w) = \Omega(\underline{\Sigma})$. The identities

$$\begin{aligned}
\mathrm{ch}_A(\mathfrak{n}_w^*\oplus\{0\}) &= \sum_{\alpha\in R^+} q^{1-\alpha_1(h_1)}t^{-\alpha_2(h_2)} = q\overline{\Sigma}\\
\mathrm{ch}_A(\{0\}\oplus\mathfrak{m}_w^*) &= \sum_{\alpha\in R^+} q^{-\alpha(h_1)}t^{1-\alpha(h_2)} + (n-1)t + \sum_{\alpha\in\Delta} q^{\alpha(h_1)}t^{1+\alpha(h_2)}\\
&=(n-1)t + t\overline{\Sigma} + t\sum_{i=1}^{n-1}\chi_{i+1}\chi_i^{-1}
\end{aligned}$$

imply

$$\lambda\Big(\big[(\mathfrak{n}_w\oplus\mathfrak{m}_w)/(\mathfrak{g}\oplus\mathfrak{g})^A\cap(\mathfrak{n}_w\oplus\mathfrak{m}_w)\big]^*\Big) = \Omega^0\Big((n-1)t+(q+t)\overline{\Sigma}+t\sum_{i=1}^{n-1}\chi_{i+1}\chi_i^{-1}\Big).$$

Finally,

$$\begin{aligned}
\mathrm{ch}_A([\mathfrak{n}_w,\mathfrak{m}_w]) &=(n-1)qt + \sum_{\alpha\in R^+} q^{1+\alpha_1(h_1)}t^{1+\alpha_2(h_2)}\\
&=(n-1)qt + qt\overline{\Sigma}.
\end{aligned}$$

implies

$$\lambda(([\mathfrak{n}_w,\mathfrak{m}_w]/[\mathfrak{n}_w,\mathfrak{m}_w]^A)^*) = \Omega^0((n-1)qt+qt\overline{\Sigma}).$$

Since $\lambda(V)=\Omega(\mathrm{ch}_A(V))$ and $\lambda(V/V^A)=\Omega^0(\mathrm{ch}_A(V))$, combining the identities above, we further deduce that:

$$
\begin{aligned}
&\frac{1}{g_\lambda}\lambda((T^*_{\mathfrak{C}_r^A}\mathfrak{C}_r)|_{\mathbf{e}})\cdot\lambda\left((T^*_{\mathfrak{Z}_r^{w,A}}\mathfrak{Z}_r)|_{\mathbf{e}}\right)^{-1}\cdot\lambda(\mathfrak{R}_w^*)\\
=&\lambda\left([\mathfrak{g}/(\mathrm{Stab}(\mathbf{e})\oplus\mathfrak{h})]^*\right)\cdot\left(\lambda(\mathfrak{n}_w)\lambda\Big(\big[(\mathfrak{n}_w\oplus\mathfrak{m}_w)/(\mathfrak{g}\oplus\mathfrak{g})^A\cap(\mathfrak{n}_w\oplus\mathfrak{m}_w)\big]^*\Big)\right)^{-1}\cdot\lambda\left(([\mathfrak{n}_w,\mathfrak{m}_w]/\mathfrak{g}_{1,1})^*\right)\\
=&\left[\Omega(\overline{\Sigma}+\underline{\Sigma})\Omega(\sum_{i=1}^n\chi_i^{-1})^{-1}\right]\cdot\left[\Omega(\underline{\Sigma})\Omega^0((n-1)t+(q+t)\overline{\Sigma}+t\sum_{i=1}^{n-1}\chi_{i+1}\chi_i^{-1})\right]^{-1}\cdot\Omega^0\big((n-1)qt+qt\overline{\Sigma}\big)\\
=&(\frac{1-qt}{1-t})^{n-1}\Omega^0((1+qt-q-t)\overline{\Sigma})\Omega(\sum_{i=1}^n\chi_i^{-1})^{-1}\Omega^0(t\sum_{i=1}^{n-1}\chi_{i+1}\chi_i^{-1})^{-1}\\
=&(\frac{1-qt}{1-t})^{n-1}\sum_{\sigma\in\mathrm{ASYT}_{\mathbf{e}}}\frac{\hat{\prod}_{1\le i<j\le n}\omega(\frac{\chi_i}{\chi_j})}{\hat{\prod}_i(1-\chi_i^{-1})\hat{\prod}_{i=1}^{n-1}(1-t\frac{\chi_{i+1}}{\chi_i})}\\
=&\frac{(1-qt)^{n-1}}{(1-t)^{n-1}(-t)^{n-1}}\sum_{\sigma\in\mathrm{ASYT}_{\mathbf{e}}}\frac{\hat{\prod}_{1\le i<j\le n}\omega(\frac{\chi_i}{\chi_j})}{\frac{\chi_n}{\chi_1}\hat{\prod}_i(1-\chi_i^{-1})\hat{\prod}_{i=1}^{n-1}(1-\frac{\chi_i}{t\chi_{i+1}})}
\end{aligned}
$$

The proposition now follows from the equality

$$([\mu_c\rceil(1),\cdots,\lceil\mu_c\rceil(n))=(\lfloor\mu_c\rfloor(1),\cdots,\lfloor\mu_c\rfloor(n))+(1,0,\cdots,0,-1).\qquad\square$$

5. Catalan DG modules and shuffle generators

5.1. **Catalan DG modules.** We define an analogue of the cuspidal DG module. On $\mathcal{B}$ we also have the vector bundle $\underline{[\mathfrak{n},\mathfrak{n}]^*}$ (resp. $\underline{[\mathfrak{n},\mathfrak{n}]}$) whose total space equals $G\times^B[\mathfrak{n},\mathfrak{n}]^*$ (resp. $G\times^B[\mathfrak{n},\mathfrak{n}]$). Let $\pi_{[\mathfrak{n},\mathfrak{n}]}:G\times^B[\mathfrak{n},\mathfrak{n}]\to\mathcal{B}$ be the projection and $\iota_{[\mathfrak{n},\mathfrak{n}]}:\mathcal{B}\to G\times^B[\mathfrak{n},\mathfrak{n}]$ be the zero section. The Koszul complex $(\wedge^\bullet\pi^*_{[\mathfrak{n},\mathfrak{n}]}\underline{[\mathfrak{n},\mathfrak{n}]^*},\partial_{[\mathfrak{n},\mathfrak{n}]})$, with differential given by contraction with the canonical section of $\pi^*_{[\mathfrak{n},\mathfrak{n}]}\underline{[\mathfrak{n},\mathfrak{n}]}$, is quasi-isomorphic to $(\iota_{[\mathfrak{n},\mathfrak{n}]})_*\mathcal{O}_{\mathcal{B}}$. Define

$$q_{[\mathfrak{n},\mathfrak{n}]}:G\times^B(\mathfrak{n}\times\mathfrak{n})\to G\times^B[\mathfrak{n},\mathfrak{n}],\quad(g,x,y)\mapsto(g,[x,y]).$$

Let

$$\mathcal{A}_c':=(\pi_{G\times^B(\mathfrak{n}\times\mathfrak{n})})^*\mathcal{L}_{\lfloor\mu_c\rfloor}\otimes\wedge^\bullet\Big(\big(\pi_{[\mathfrak{n},\mathfrak{n}]}\circ q_{[\mathfrak{n},\mathfrak{n}]}\big)^*\underline{[\mathfrak{n},\mathfrak{n}]^*},q^*_{[\mathfrak{n},\mathfrak{n}]}\partial_{[\mathfrak{n},\mathfrak{n}]}\Big).\tag{49}$$

and call it the Catalan DG module at slope c.

Warning: Note that the definitions (34) and (49) use different line bundles.

Write $p':G\times^B(\mathfrak{n}\times\mathfrak{n})\to\mathfrak{g}\times\mathfrak{g}:(g,x,y)\mapsto(g\cdot x,g\cdot y)$. The $\mathbb{C}^*\times\mathbb{C}^*$-actions in Section 4.2.2 induce a $\mathbb{C}^*\times\mathbb{C}^*$-action on $(\mathrm{R}p'_*\mathcal{A}_c')|_{\mathbf{e}}$. The support of $Rp'_*\mathcal{A}_c'$ is contained in $\mathfrak{C}$. Suppose $\underline{\mathcal{F}_c'}$ is a complex of coherent sheaves on $\mathfrak{C}$ such that $Rp'_*\mathcal{A}_c'=Ri'_*\underline{\mathcal{F}_c'}$ for $i':\mathfrak{C}\hookrightarrow T^*\mathfrak{g}$. Let $\mathcal{F}_c'=\underline{\mathcal{F}_c'}|_{\mathfrak{C}_r}$. Similar to Proposition 4.5, we have that

Proposition 5.1. *The folllowing identity holds:*

$$\mathrm{ch}_A(\mathcal{F}_c'|_{\mathbf{e}_\lambda})=g_\lambda\sum_{\sigma\in\mathrm{SYT}_\lambda}\frac{\Xi_\sigma\prod_{i=1}^n\chi_{n-i+1}^{\lfloor\mu_c\rfloor(i)}}{\hat{\prod}_{i=1}^{n-1}(1-qt\frac{\chi_i}{\chi_{i+1}})}\tag{50}$$

where Ξ_σ is as defined in (48).

Remark 5.1. [BG13, Theorem 4.5.1 and Proposition 4.8.2] state that

$$\mathrm{ch}_A(\mathcal{R}|_{\mathbf{e}_\lambda}) = \frac{g_\lambda}{(1-q)^n(1-t)^n} \sum_{\sigma\in \mathrm{SYT}_\lambda} \Xi_\sigma, \tag{51}$$

which coincides with the expression of $\mathrm{ch}_A(\mathcal{P}|_{I_\lambda})$ derived by Garsia and Haiman [GH98] using Pieri rule for Macdonald polynomials.

As a comparison, the DG algebra in [Gin12], whose pushforward equals $\mathcal{R}$, is defined by pulling $(\wedge^\bullet \pi_{\mathfrak{n}}^* \underline{\mathfrak{n}^*}, \partial_{\mathfrak{n}})$ back to $G\times^B(\mathfrak{b}\times\mathfrak{b})$.

Compare (51) and (50) when $\prod_{i=1}^n \chi_i^{\mu_c(n-i+1)} = 1$. The differences lie in the terms $(1-q)^n(1-t)^n$ and $\prod_{i=1}^{n-1}(1-qt\frac{\chi_i}{\chi_{i+1}})$. The term $(1-q)^n(1-t)^n$ arises from the distinction between the support being $\mathfrak{b}\times\mathfrak{b}$ versus $\mathfrak{n}\times\mathfrak{n}$ (in $\mathfrak{gl}_n$). The term $\prod_{i=1}^{n-1}(1-qt\frac{\chi_i}{\chi_{i+1}})$ results from the complex being defined by $\mathfrak{n}^*$ versus $[\mathfrak{n},\mathfrak{n}]^*$.

For any A-equivariant sheaf $\mathcal{F}$, we use the notation $q^at^b\mathcal{F}$ to shift the original A-action by the weight (a,b). We will prove in the next sections that the sheaves $\mathcal{F}_c$ and $\mathcal{F}'_c$ correspond to the same classes in $K^A(\mathrm{Hilb}^n)$ using shuffle algebra techniques. We conjecture that:

Conjecture 5.2. *There exists a $\overline{G}\times\mathbb{C}^*\times\mathbb{C}^*$-equivariant isomorphism:*

$$Rp'_*\mathcal{A}'_c \cong q^{1-n}Rp_*\mathcal{A}_c.$$

One should note that a priori it is not clear whether $Rp'_*\mathcal{A}'_c$ is concentrated in one degree. In contrast, the sheaf $Rp_*\mathcal{A}_c$ is automatically concentrated in one degree as it is the associated graded of $\mathbf{N}_c$.

5.2. **Cuspidal vs Catalan.**

5.2.1. *Shuffle algebras.* Recall the ring $K=\mathbb{C}(q,t)(z_1,z_2,\cdots)^{S_\infty}$. We endow K with a $\mathbb{C}(q,t)$-algebra structure via the shuffle product

$$f(z_1,\cdots,z_k)*g(z_1,\cdots,z_\ell) = \frac{1}{k!\ell!}\mathrm{Sym}\big[f(z_1,\cdots,z_k)g(z_{k+1},\cdots,z_{k+\ell})\prod_{i=1}^{k}\prod_{j=k+1}^{k+\ell}\omega(\frac{z_i}{z_j})\big].$$

Here Sym denotes symmetrization.

Definition 5.1. The shuffle algebra $\mathfrak{A}$ is defined as the subspace of K consisting of rational functions in the form of

$$F(z_1,\cdots,z_k) = \frac{f(z_1,\cdots,z_k)\prod_{1\le i<j\le k}(z_i-z_j)^2}{\prod_{1\le i\ne j\le k}(z_i-qz_j)(z_i-tz_j)}$$

such that f is a symmetric Laurent series satisfying the wheel conditions:

$$f(z_1,z_2,z_3,\dots)=0 \text{ if } \{\frac{z_1}{z_2},\frac{z_2}{z_3},\frac{z_3}{z_1}\} = \{q,t,\frac{1}{qt}\}.$$

It is shown in [SV13] that there is an isomorphism between $\mathfrak{A}$ and the positive half of the elliptic Hall algebra. Moreover,

Theorem 5.3. *([FT11, SV13]) There exists a geometric action of the algebra $\mathfrak{A}$ on the vector space K via the identification (40).*

5.2.2. *Shuffle generators.* Following [Neg22], we define[1]

$$P_{n,m} = \mathrm{Sym}\Big(\frac{\prod_{i=1}^n z_{n-i+1}^{\lfloor ic\rfloor - \lfloor (i-1)c\rfloor}}{\prod_{i=1}^{n-1}(1-qt\frac{z_i}{z_{i+1}})} \prod_{1\le i<j\le n} \omega(\frac{z_i}{z_j})\Big) \tag{52}$$

According to [BS12], $P_{n,m}$ with $n \geq 1$, $m \in \mathbb{Z}$ generate the shuffle algebra $\mathfrak{A}$.

By [Neg22, (2.34) and (2.35)], (52) equals

$$P_{n,m} = \Big(\frac{(1-qt)}{(1-t)(-qt)}\Big)^{n-1} \mathrm{Sym}\Big(\frac{\prod_{i=1}^n z_{n-i+1}^{\lfloor ic\rfloor - \lfloor (i-1)c\rfloor}}{\prod_{i=1}^{n-1}(1-\frac{z_i}{tz_{i+1}})} \prod_{1\le i<j\le n} \omega(\frac{z_i}{z_j})\Big). \tag{53}$$

Proposition 5.4. *([Neg15a, Proposition 5.5], [GN15, (49)])Under the action in Theorem 5.3,*

$$P_{n,m}\cdot 1 = \Big(\frac{(1-q)(1-t)}{(1-qt)}\Big)^n \sum_{\lambda\vdash n} \frac{\widetilde{H}_\lambda}{g_\lambda} \sum_{\sigma\in \mathrm{SYT}_\lambda} \frac{\prod_{i=1}^n \chi_{n-i+1}^{\lfloor ic\rfloor - \lfloor (i-1)c\rfloor}}{\hat{\prod}_{i=1}^{n-1}(1-qt\frac{\chi_i}{\chi_{i+1}})}\Theta_\sigma \tag{54}$$

$$= \frac{(1-q)^n(1-t)}{(1-qt)(-qt)^{n-1}} \sum_{\lambda\vdash n} \frac{\widetilde{H}_\lambda}{g_\lambda} \sum_{\sigma\in \mathrm{ASYT}_\lambda} \frac{\prod_{i=1}^n \chi_{n-i+1}^{\lfloor ic\rfloor - \lfloor (i-1)c\rfloor}}{\hat{\prod}_{i=1}^{n-1}(1-\frac{\chi_i}{t\chi_{i+1}})}\Theta_\sigma \tag{55}$$

where

$$\Theta_\sigma := \prod_{i=1}^n (1-qt\chi_i) \hat{\prod}_{1\le i<j\le n} \omega^{-1}(\frac{\chi_j}{\chi_i}). \tag{56}$$

Proof. The first identity is [Neg15a, Proposition 5.5] using the definition (52) of $P_{m,n}$. To illustrate how ASYT appears, we include a proof of the second equality closely following *loc. cit.* but using the definition (53) of $P_{m,n}$ instead.

The localization formula (43) and [Neg15a, Theorem 4.7] imply that $P_{n,m}\cdot 1$ equals

(57)
$$\Big(\frac{(1-qt)}{(1-t)(-qt)}\Big)^{n-1} \sum_{\lambda\vdash n} \frac{\widetilde{H}_\lambda}{g_\lambda} \int \frac{\prod_{i=1}^n z_{n-i+1}^{\lfloor ic\rfloor - \lfloor (i-1)c\rfloor}}{\prod_{i=1}^{n-1}(1-\frac{z_i}{tz_{i+1}})} \prod_{1\le i<j\le n} \omega(\frac{z_i}{z_j}) \prod_{\square\in\lambda}\prod_{i=1}^n \Big(\omega^{-1}(\frac{z_i}{\chi(\square)})(1-qtz_i)\frac{dz_i}{2\pi i z_i}\Big).$$

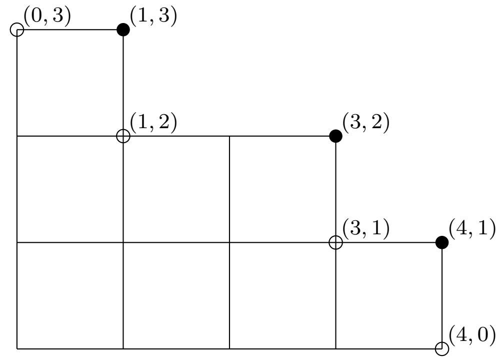
Each integral is taken along a contour separating poles of the function to be integrated.

For a partition $\lambda \vdash n$, [Neg15a, (5.4)] states (for a proof, cf. [BG13, Lemma 4.8.5])

$$\prod_{\square\in\lambda}\Big(\omega^{-1}(\frac{z}{\chi(\square)})(1-qtz)\Big) = \frac{\prod_{\circ \text{ inner corner of } \lambda}(1-\frac{qtz}{\chi(\circ)})}{\prod_{\bullet \text{ outer corner of } \lambda}(1-\frac{qtz}{\chi(\bullet)})}.$$

Notions of inner/outer corners are illustrated below. Hollow circles indicate the inner corners of the partition and solid circles indicate the outer corners. We use notations $\chi(\circ)$ and $\chi(\bullet)$ for weights of corners as well. For example, weights of outer corners in the figure below are qt^3, q^3t^2, q^4t.

[1]Note that the $P_{m,n}$ here and in [Neg22] is denoted by $\widetilde{P}_{m,n}$ in [GN15], as a certain modification of the $P_{m,n}$ in [Neg14].



[Neg15a, Figure 5.1]

Integrating over z_n results in a residue whenever qtz_n equals to the weight of some outer corner of the partition λ. Label the box adjacent to this corner by n. By change of variables, the residue is the weight of the box labeled by n. The integral in (57) thus becomes

$$\chi_n \int \left[\frac{\prod_{i=1}^n z_{n-i+1}^{\lfloor ic \rfloor - \lfloor (i-1)c \rfloor}}{\prod_{i=1}^{n-1}(1-\frac{z_i}{tz_{i+1}})} \hat{\prod}_{1\le i<j\le n} \omega(\frac{z_i}{z_j}) \hat{\prod}_{\square\in\lambda} \hat{\prod}_{i=1}^{n} \left(\omega^{-1}(\frac{z_i}{\chi(\square)})(1-qtz_i)\frac{dz_i}{2\pi i z_i} \right) \right] \Bigg|_{z_n=\chi_n}.$$

Notice the restriction products $\hat{\prod}$ here.

Next, when integrating over z_{n-1}, we obtain a residue whenever one of the following scenarios happens:

- qtz_{n-1} equals the weight of some outer corner of λ (i.e, z_{n-1} equals to the weight of some inner corner of λ).
- $q\frac{z_{n-1}}{\chi_n} = 1$ or $t\frac{z_{n-1}}{\chi_n} = 1$ (i.e, z_{n-1} equals to the weight of some box to the left or below the box labeled by n in the last step).
- $\frac{z_{n-1}}{t\chi_n} = 1$ (i.e, z_{n-1} equals to the weight of some box above the box labeled by n in the last step).

These three conditions together with the condition for z_n exactly define all the possible relative positions between the boxes labeled by n and $n-1$ in an ASYT such that the box labeled by n is in a corner. One can argue similarly starting from any label on a corner.

Repeating this procedure, we conclude that the integral in (57) equals

$$\sum_{\text{ASYT}} \prod_{i=1}^{n} \chi_i \left[\frac{\prod_{i=1}^n z_{n-i+1}^{\lfloor ic \rfloor - \lfloor (i-1)c \rfloor}}{\hat{\prod}_{i=1}^{n-1}(1-\frac{z_i}{tz_{i+1}})} \hat{\prod}_{1\le i<j\le n} \omega(\frac{z_i}{z_j}) \hat{\prod}_{\square\in\lambda} \hat{\prod}_{i=1}^{n} \left(\omega^{-1}(\frac{z_i}{\chi(\square)})(1-qtz_i)\frac{1}{z_i} \right) \right] \Bigg|_{z_i=\chi_i,\forall i}$$

$$= \left(\frac{(1-q)(1-t)}{(1-qt)} \right)^n \sum_{\text{ASYT}_\lambda} \left[\frac{\prod_{i=1}^n \chi_{n-i+1}^{\lfloor ic \rfloor - \lfloor (i-1)c \rfloor}}{\hat{\prod}_{i=1}^{n-1}(1-\frac{\chi_i}{t\chi_{i+1}})} \hat{\prod}_{1\le i<j\le n} \omega^{-1}(\frac{\chi_j}{\chi_i}) \prod_{i=1}^{n}(1-qt\chi_i) \right].$$

The factor $\left(\frac{(1-q)(1-t)}{(1-qt)}\right)^n$ comes from $\hat{\prod}_{i=1}^n \omega^{-1}(\frac{\chi_i}{\chi_i})$. The proposition thus follows. $\square$

5.2.3. *"Cuspidal = Catalan" via shuffle generators.*

Proposition 5.5.

$$(P_{n,m}\cdot 1) = \sum_{\lambda\vdash n} \mathrm{ch}_A(\mathcal{F}'_c|_{\mathbf{e}_\lambda}) \frac{\widetilde{H}_\lambda}{g_\lambda}$$
$$= q^{1-n} \sum_{\lambda\vdash n} \mathrm{ch}_A(\mathcal{F}_c|_{\mathbf{e}_\lambda}) \frac{\widetilde{H}_\lambda}{g_\lambda}$$

Proof. Compare formulae (54)/(55) with (50)/(47). To prove the proposition, it suffices to show that

$$\frac{\Theta_\sigma}{\Xi_\sigma} = g_\sigma \frac{(1-qt)^n}{(1-q)^n(1-t)^n} \tag{58}$$

for every Young tableau σ with n boxes. By definition (48) and (56),

$$\frac{\Theta_\sigma}{\Xi_\sigma} = \Omega^0\Bigg(qt\sum_i \chi_i + \sum_i \chi_i^{-1} - (1-q)(1-t)\sum_{1\le i<j\le n}\left(\frac{\chi_i}{\chi_j}+\frac{\chi_j}{\chi_i}\right)\Bigg),$$

which only depends on the shape of σ. So does the polynomial g_σ (42). Therefore, to show (58), we may assume that σ is a SYT.

For every positive integer $k \le n$, we let $\sigma(k)$ denote the Young sub-tableau of σ consisting of the first k labels. Then

$$\frac{\Theta_\sigma}{\Xi_\theta} = \prod_{j=1}^n \frac{\Theta_{\sigma(j)}}{\Xi_{\sigma(j)}}$$

with

$$\frac{\Theta_{\sigma(j)}}{\Xi_{\sigma(j)}} = \Omega^0\Bigg(qt\chi_j + \chi_j^{-1} - (1-q)(1-t)\sum_{1\le i<j}\left(\frac{\chi_i}{\chi_j}+\frac{\chi_j}{\chi_i}\right)\Bigg)$$

We let R_j (resp. C_j) denote the set of boxes of σ in the same row (resp. column) and to the left of (resp. below) the box labeled by j in the tableau $\sigma(k)$. For every box $x \in \sigma(k)$ we write $a_k(x), \ell_k(x)$ to indicate the arm and leg of x in $\sigma(k)$.

Then $g_\sigma = \prod_{j=1}^n \frac{g_{\sigma(j)}}{g_{\sigma(j-1)}}$ with $g_{\sigma(0)} = 1$ and

$$\frac{g_{\sigma(j)}}{g_{\sigma(j-1)}} =(1-q)(1-t)\prod\nolimits_{x\in C_j\cup R_j}\frac{1-q^{1+a_j(x)}t^{-\ell_j(x)}}{1-q^{1+a_{j-1}(x)}t^{-\ell_{j-1}(x)}}\cdot\frac{1-q^{-a_j(x)}t^{\ell_j(x)+1}}{1-q^{-a_{j-1}(x)}t^{\ell_{j-1}(x)+1}}. \tag{59}$$

To prove (58), it suffices to show that

$$\frac{\Theta_{\sigma(j)}}{\Xi_{\sigma(j)}} = \frac{g_{\sigma(j)}}{g_{\sigma(j-1)}}\frac{1-qt}{(1-q)(1-t)}, \quad \forall 1\le j\le n. \tag{60}$$

Without loss of generality, we argue for the case when $j = n$.

Suppose the shape of σ is given by the partition $(\lambda_1,\dots,\lambda_u)$ with transpose $(\lambda'_1,\dots,\lambda'_v)$. Here λ_j is the size of the j-th column and λ_i is the size of the i-th row. Assume the weight of the box labeled by n is $\chi_n = q^c t^r$. We have that:

$$\begin{aligned}
&\prod\nolimits_{x\in C_n}\frac{1-q^{1+a_n(x)}t^{-\ell_n(x)}}{1-q^{1+a_{n-1}(x)}t^{-\ell_{n-1}(x)}}\cdot\frac{1-q^{-a_n(x)}t^{\ell_n(x)+1}}{1-q^{-a_{n-1}(x)}t^{\ell_{n-1}(x)+1}}\\
=&\Omega\Bigg(\sum_{x\in C_n}\left(q^{1+a_n(x)}(1-t)t^{-\ell_n(x)}+q^{-a_n(x)}(t-1)t^{\ell_n(x)}\right)\Bigg)\\
=&\Omega\Bigg(\sum_{j=1}^{r-1}\left(q^{\lambda_{j+1}-c}(1-t)t^{j-r}+q^{c+1-\lambda_{i+1}}(1-t^{-1})t^{r+1-j}\right)\Bigg).
\end{aligned} \tag{61}$$

Similarly,

$$\begin{aligned}
&\prod\nolimits_{x\in R_n}\frac{1-q^{1+a_n(x)}t^{-\ell_n(x)}}{1-q^{1+a_{n-1}(x)}t^{-\ell_{n-1}(x)}}\cdot\frac{1-q^{-a_n(x)}t^{\ell_n(x)+1}}{1-q^{-a_{n-1}(x)}t^{\ell_{n-1}(x)+1}}\\
=&\Omega\Bigg(\sum_{i=1}^{c-1}\left(q^{c+1-i}(1-q^{-1})t^{r+1-\lambda'_{i+1}}+q^{i-c}(1-q)t^{\lambda'_{j+1}-r}\right)\Bigg).
\end{aligned} \tag{62}$$

It is not hard to see by labeling on a Young diagram (cf. [BG13, Lemma 4.8.4]) that

(63)
$$A(q,t) := \sum_{i=0}^{c-1} q^i(1-q)t^{\lambda'_{i+1}} + \sum_{j=0}^{r-1} q^{\lambda_{j+1}}(1-t)t^j = -(1-q)(1-t)\sum_{i=1}^{n-1}\chi_i + 1 - q^c t^c =: B(q,t).$$

The equality

$$q^{-c}t^{-r}A(q,t) + q^{c+1}t^{r+1}A(q^{-1},t^{-1}) = q^{-c}t^{-r}B(q,t) + q^{c+1}t^{r+1}B(q^{-1},t^{-1})$$

expands to

(64)
$$q^{-c}t^{-r}\left(\sum_{i=0}^{c-1} q^i(1-q)t^{\lambda'_{i+1}} + \sum_{j=0}^{r-1} q^{\lambda_{j+1}}(1-t)t^j\right) + q^{c+1}t^{r+1}\left(\sum_{i=0}^{c-1} q^{-i}(1-q^{-1})t^{-\lambda'_{i+1}} + \sum_{j=0}^{r-1} q^{-\lambda_{j+1}}(1-t^{-1})t^{-j}\right)$$

(65)
$$= -(1-q)(1-t)\sum_{i=1}^{n-1}\left(\frac{\chi_i}{\chi_n} + \frac{\chi_n}{\chi_i}\right) + q^{-c}t^{-r} - 1 + q^{c+1}t^{r+1} - qt.$$

By (61) and (62), $\frac{g_{\sigma(n)}}{g_{\sigma(n-1)}} = (1-q)(1-t)\Omega\big((64)\big)$. Moreover, $\frac{\Theta_{\sigma(n)}}{\Xi_{\sigma(n)}} = (1-qt)\Omega\big((65)\big)$. The proposition follows. □

Remark 5.2. A similar formula appears in [KT22, Lemma 5.13].

6. Proofs of the main theorems

6.1. **Hodge filtrations and shift functors.**

Proposition 6.1. *When $c > 1$ the following isomorphism is filtered:*

$$\mathrm{eL}_{c+k} \cong {}_{c+k}P_c \otimes_{\mathrm{A}_c} \mathrm{eL}_c.$$

Here eL_c and eL_c inherit Hodge filtrations from $\overline{\mathbf{N}}_c$ and the filtration on the right hand side is the tensor product filtration such that H_c is endowed with the order filtration.

Proof. To prove the proposition, we switch to working with $X := \mathfrak{g} \times \mathbb{P}^{n-1}$. By Theorem 2.4, we have that

$$\mathrm{eL}_c \cong \Gamma\left(X, \mathbf{N}_c \boxtimes \mathcal{O}_{\mathbb{P}^{n-1}}(m)\right)^G.$$

Let $\mathcal{O}_X(nk)$ be the pullback of $\mathcal{O}_{\mathbb{P}^{m-1}}$ to X. Consider the sheaf of twisted differential operators $\mathcal{D}_{-c,X}$ on X associated to the line bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(m)$. Define

$${}_{-c-k}\mathcal{D}_{-c,X} := \mathcal{O}_X(nk) \otimes_{\mathcal{O}_X} \mathcal{D}_{-c,X}.$$

It is shown in [GGS09, Lemma 6.7] that

$$\Gamma(X, {}_{-c-k}\mathcal{D}_{-c,X})^G \cong {}_{-c-k}Q_{-c}.$$

Write $\underline{\mathbf{N}}_c := \mathbf{N}_c \boxtimes \mathcal{O}_{\mathbb{P}^{n-1}}(m)$. We have a diagram

$$\begin{tikzcd}
\Gamma\left({}_{-c-k}\mathcal{D}_{-c,X} \otimes_{\mathcal{D}_{-c,X}} \underline{\mathbf{N}}_c\right)^G \arrow[r, "\sim"] \arrow[d, "\sim"] & \Gamma(X, \underline{\mathbf{N}}_{c+k})^G \arrow[d] \\
\Omega_{c+k}\left({}_{c+k}P_c \otimes_{\mathrm{eH}_c\mathrm{e}} \mathrm{eL}_c\right) \arrow[r, "\sim"] & \Omega_{c+k}\left(\mathrm{eL}_{c+k}\right)
\end{tikzcd}$$

By assumption, the right arrow is a filtered isomorphism. The left arrow is also a filtered isomorphism with respect to the tensor product filtrations. It remains to show that the top arrow is also a filtered isomorphism:

$$ {}_{-c-k}\mathcal{D}_{-c,X} \otimes_{\mathcal{D}_{-c,X}} \underline{\mathbf{N}}_c \cong \mathcal{O}_X(nk) \otimes_{\mathcal{O}_X} \underline{\mathbf{N}}_c \cong \underline{\mathbf{N}}_{c+k}. \qquad \square $$

Therefore, by Proposition 2.6, we conclude that:

Corollary 6.2. *With respect to the gradings defined by $\widetilde{h}$, resp. h_c, and gr^{H}, the isomorphism $\Psi_c(\overline{\mathbf{N}}_c) \cong GS(\mathrm{eL}_c)$ holds in $\mathrm{Coh}^A(\mathrm{Hilb}^n)$.*

As a corollary, we are able to recover the following result originally proved by Haiman [Hai98].

Corollary 6.3. *The punctual Hilbert scheme Hilb^n_0 is Cohen-Macaulay.*

Proof. By [GS05, Proposition 1.7], $GS(\mathrm{eL}_{\frac{1}{n}}) = i_*\mathcal{O}_{\mathrm{Hilb}^n_0}$ where i is the embedding $\mathrm{Hilb}^n_0 \hookrightarrow \mathrm{Hilb}^n$. Moreover, by [Sai88, Lemma 5.1.13], the restriction of the associated graded $\widetilde{\mathrm{gr}}(\overline{\mathbf{N}}_c)$ with respect to the Hodge filtration to its support, namely $\underline{\mathcal{F}}_c \boxtimes \mathcal{O}_V$, is Cohen-Macaulay. Hence the descent of $(\underline{\mathcal{F}}_c \boxtimes \mathcal{O}_V)|_{\widetilde{\mathrm{Hilb}^n}}$, or equivalently $\Psi_c(\overline{\mathbf{N}}_c)$, is also Cohen-Macaulay. $\square$

Proposition 6.1 also allows us to extend F^{H} from eL_c to L_c for all $c = \frac{m}{n} > 1$ by defining a tensor product filtration on L_c via [BEG03, Lemma 4.7]:

$$ \mathrm{L}_c \cong \mathrm{H}_c\delta\mathrm{e}_- \otimes_{\mathrm{A}_{c-1}} \mathrm{eL}_{c-1}. \tag{66} $$

Note that we do not use the tensor product filtration coming from the Morita equivalence $\mathrm{L}_c \cong \mathrm{H}_c\mathrm{e} \otimes_{\mathrm{A}_c} \mathrm{eL}_c$.

6.2. **Bigraded characters.** Sort the formula (54) as $P_{m,n} \cdot 1 = \sum_{\lambda \vdash n} c^\lambda_{m,n} \widetilde{H}_\lambda$, with

$$ c^\lambda_{m,n} = \left(\frac{(1-q)(1-t)}{(1-qt)} \right)^n \sum_{\sigma \in \mathrm{SYT}_\lambda} \frac{\prod_{i=1}^n \chi_{n-i+1}^{\lfloor ic \rfloor - \lfloor (i-1)c \rfloor}}{g_\lambda \prod_{i=1}^{n-1} (1 - qt\frac{\chi_i}{\chi_{i+1}})} \Theta_\sigma. $$

We will need the rational shuffle theorem. Since we will only use two simple consequences of it, we will not define the notions involved. Interested readers can refer to [GN15, 6.2] for definitions. We let DP be the set of (m,n)-Dyck paths and PF be the set of (m,n)-parking functions, which can be thought of as labeled (m,n)-Dyck paths. In particular, there is a surjection from PF to DP.

Theorem 6.4. [GN15, Conjectures 6.1, 6.3], [Mel21, Theorem 5.8]

$$ P_{m,n} \cdot 1 = \sum_{\pi \in PF} q^{\mathrm{area}(\pi)} t^{\mathrm{dinv}(\pi)} z_\pi \tag{67} $$

$$ \sum_{\lambda \vdash n} c^\lambda_{m,n} = \sum_{D \in DP} q^{\mathrm{area}(D)} t^{\mathrm{dinv}(D)} \tag{68} $$

Let $d = \frac{1}{2}(m-1)(n-1)$. Area and dinv are two combinatorial statistics associated to each dyck path with integer values in the interval $[0,d]$ and z_π is a monomial in the variables z_i's. For a Dyck path D, when $\mathrm{area}(D) = 0$, $\mathrm{dinv}(D) = d$ and when $\mathrm{dinv}(D) = 0$, $\mathrm{area}(D) = d$.

The expression on the right hand side of (67), resp. (68), is known as the q,t-parking function, resp. q,t-Catalan number.

Define a $\mathbb{C}(q,t)$-linear involution ι on K (cf. (38)) by sending s_λ to s_{λ^t}. For every $A \times S_n$-module M, one has that

$$ \mathrm{ch}_{A\times S_n}(\mathrm{sgn} \otimes M) = \iota(\mathrm{ch}_{A\times S_n}(M)). $$

The following theorem generalizes [BEG03, Conjecture 7.2, 7.3] proved in [GS06, Theorem 1.8].

Theorem 6.5. *Suppose $c = \frac{m}{n}$ for $m > 0$ coprime to n. With respect to the Hodge filtration and the Euler field h_c,*
(a) $\mathrm{ch}_{A\times S_n}(\mathrm{L}_{c+1}) = \mathrm{ch}_{A\times S_n}\left(\mathrm{sgn}\otimes\Gamma(\mathrm{Hilb}^n, \mathcal{P}\otimes\Psi_c(\overline{\mathbf{N}}_c))\right) = \iota\left((P_{m,n}\cdot 1)(q, q^{-1}t)\right).$
(b) $\mathrm{ch}_A(\mathrm{eL}_c) = \mathrm{ch}_A\Gamma(\mathrm{Hilb}^n, \Psi_c(\overline{\mathbf{N}}_c)) = \sum_{\lambda\vdash n} c^{\lambda}_{m,n}(q, q^{-1}t).$

Proof. By localization formula (43),

$$\mathrm{ch}_{A\times S_n}\Gamma(\mathrm{Hilb}^n, \mathcal{P}\otimes\Psi_c(\overline{\mathbf{N}}_c)) = \sum_{\lambda\vdash n}\frac{\widetilde{H}_\lambda}{g_\lambda}\mathrm{ch}_A(\Psi_c(\overline{\mathbf{N}}_c)|_{I_\lambda}).$$

Suppose $\pi : \widetilde{\mathrm{Hilb}^n} \to \mathrm{Hilb}^n$. Recall that

$$\pi^*\Psi_c(\overline{\mathbf{N}}_c) = (\mathcal{F}_c \boxtimes \mathcal{O}_V)|_{\widetilde{\mathrm{Hilb}^n}},$$

and thus there exists integers a, b such that $\mathrm{ch}_A(\Psi_c(\overline{\mathbf{N}}_c)|_{I_\lambda}) = q^a t^b \mathrm{ch}_A(\mathcal{F}_c|_{\mathbf{e}_\lambda})$ for all $\lambda \vdash n$. Therefore, by Proposition 5.5,

$$\mathrm{ch}_{A\times S_n}\left(\Gamma(\mathrm{Hilb}^n, \mathcal{P}\otimes\Psi_c(\overline{\mathbf{N}}_c))\right) = q^a t^b P_{m,n}\cdot 1(q, q^{-1}t). \tag{69}$$

The change of variables $(q, q^{-1}t)$ comes from that h_c acts by weight $(1, -1)$.

Next, we show that with respect to the order filtration,

$$\mathrm{gr}(\mathrm{H}_c\mathrm{e}_-) \cong \mathrm{sgn}\otimes\Gamma(\mathfrak{C}\times V, \mathcal{R}\boxtimes\mathcal{O}_V)^{\overline{G}} \tag{70}$$

as $A\times S_n$-modules. Indeed, $\Gamma(\mathfrak{C}\times V, \mathcal{R}\boxtimes\mathcal{O}_V)^{\overline{G}} = \mathbb{C}[\mathfrak{I}\mathfrak{C}\times V]^{\overline{G}}$ and we claim that

$$\mathbb{C}[\mathfrak{I}\mathfrak{C}\times V]^{\overline{G}} \cong \mathbb{C}[\mathfrak{h}\times\mathfrak{h}].$$

By the definition of $\mathfrak{I}\mathfrak{C}$, it suffices to show that $\mathbb{C}[\mathfrak{C}\times V]^{\overline{G}} \cong \mathbb{C}[\mathfrak{h}\times\mathfrak{h}]^W$. By [GG06, 2.8], there is a surjection $\mathbb{C}[\mathfrak{h}\times\mathfrak{h}]^W \twoheadrightarrow \mathbb{C}[\mathfrak{C}\times V]^{\overline{G}}$. On the other hand, we have an embedding $\mathbb{C}[\mathfrak{C}\times V]^{\overline{G}} \hookrightarrow \mathbb{C}[\widetilde{\mathrm{Hilb}^n}]^{\overline{G}} = \mathbb{C}[\mathrm{Hilb}^n] = \mathbb{C}[\mathfrak{h}\times\mathfrak{h}]^W$. Thus the claim holds. The identification (70) follows from the isomorphism that $\mathrm{gr}(\mathrm{H}_c\delta\mathrm{e}_-) \cong \mathbb{C}[\mathfrak{h}\times\mathfrak{h}]\otimes\mathrm{sgn}$.

Consider the following diagram:

$$\begin{tikzcd}
\mathrm{sgn}\otimes\Gamma\left(\mathfrak{C}\times V, (\mathcal{R}\otimes\underline{\mathcal{F}}_c)\boxtimes\mathcal{O}_V\right)^{\overline{G}} \arrow[r, "\sim"] \arrow[d, hook] & \mathrm{sgn}\otimes\Gamma(\mathfrak{C}\times V, \mathcal{R}\boxtimes\mathcal{O}_V)^{\overline{G}}\otimes_{\mathbb{C}[\mathfrak{C}\times V]^{\overline{G}}}\Gamma(\mathfrak{C}\times V, \underline{\mathcal{F}}_c\boxtimes\mathcal{O}_V)^{\overline{G}} \arrow[d, "\sim"] \\
\mathrm{sgn}\otimes\Gamma\left(\widetilde{\mathrm{Hilb}^n}, (\mathcal{R}\otimes\mathcal{F}_c)\boxtimes\mathcal{O}_V\right)^{\overline{G}} \arrow[d, "\sim"] & \mathrm{gr}(\mathrm{H}_{c+1}\mathrm{e}_-)\otimes_{\mathrm{gr}(\mathrm{A}_c)}\Gamma(\mathfrak{C}\times V, \widetilde{\mathrm{gr}}\,\overline{\mathbf{N}}_c)^{\overline{G}} \arrow[d, "\sim"] \\
\mathrm{sgn}\otimes\Gamma(\mathrm{Hilb}^n, \mathcal{P}\otimes\Psi_c(\overline{\mathbf{N}}_c)) \arrow[d, dotted] & \mathrm{gr}(\mathrm{H}_{c+1})\mathrm{e}_-\otimes_{\mathrm{gr}(\mathrm{A}_c)}\mathrm{gr}(\mathrm{eL}_c) \arrow[d, two heads] \\
\mathrm{gr}(\mathrm{L}_{c+1}) & \mathrm{gr}(\mathrm{H}_{c+1}\mathrm{e}_-\otimes_{\mathrm{A}_c}\mathrm{eL}_c) \arrow[l, "\sim"']
\end{tikzcd}$$

By (69),

$$\dim\Gamma(\mathrm{Hilb}^n, \mathcal{P}\otimes\Psi_c(\overline{\mathbf{N}}_c)) = (P_{m,n}\cdot 1)(z_i = 1, \forall\, i; q = 1, t = 1),$$

which equals the number of (m, n)-parking function by (67). On the other hand, $\dim(\mathrm{L}_c) = m^{n-1}$ [BEG03, Theorem 1.6], which is the number of (m, n)-parking functions. Therefore the injection and surjection in the diagram above have to be both isomorphisms and the first equality of (a) holds.

The first equality of (b) follows from Corollary 6.2. Moreover, by (66), $\mathrm{eL}_c = (\mathrm{L}_{c+1})^{\mathrm{sgn}}$. By (41),

$$(\iota(P_{m,n}\cdot 1), s_{(1^n)}) = (P_{m,n}\cdot 1, s_{(n)}) = \sum_{\lambda\vdash n} c^{\lambda}_{m,n}$$

and hence $\mathrm{ch}_A(\mathrm{eL}_c) = q^a t^b \sum_{\lambda\vdash n} c^{\lambda}_{m,n}(q, q^{-1}t)$.

It remains to show $q^a t^b = 1$. By [BEG03, Theorem 1.10], the highest, resp. lowest weight of eL_c under the action of h_c is d, resp. $-d$. The highest, resp. lowest, q-degree in $\sum_{\lambda\vdash n} c^{\lambda}_{m,n}(q, q^{-1}t)$ is d, resp. $-d$, as well. Therefore $q^a t^b = 1$ and the theorem follows. □

6.3. **Link homology.** We can now conclude Theorem A from the introduction.

Let $\mathcal{V}$ be the tautological vector bundle on Hilb^n of rank n characterized by $\mathcal{V}|_I = \mathbb{C}[x,y]/I$ for any $I \in \mathrm{Hilb}^n$. Denote by $\mathcal{V}_{\mathrm{st}}$ the direct summand of $\mathcal{V}$ such that $\mathcal{V} := \mathcal{O}\oplus\mathcal{V}_{\mathrm{st}}$. In $\bigoplus_{n\geq 0} K^A(\mathrm{Hilb}^n)\otimes_{\mathbb{C}[q^{\pm},t^{\pm}]}\mathbb{C}(a,q,t)$, define $\Lambda(\mathcal{V}_{\mathrm{st}}, a) = \bigoplus_{i=0}^{n-1} a^i(\wedge^i\mathcal{V}_{\mathrm{st}})$.

Theorem 6.6. *([Mel22, Corollary 3.4]) Up to a constant factor, the triply graded Euler characteristic* $\mathrm{ch}_{a,q,t}(\mathrm{HHH}(T_{m,n}))$ *equals the matrix coefficient* $\langle\Lambda(\mathcal{V}_{\mathrm{st}}, a)|P_{m,n}|1\rangle(a, qt, q^{-1}t)$.

Definition 6.1. The Kazhdan filtration associated to F^{H} and h_c is defined by

$$K_i\mathrm{L}_c = \sum_{2j+k\leq i} F^{\mathrm{H}}_j\mathrm{L}_c(k)$$

As a corollary of Theorem 6.5 and Theorem 6.6, we conclude Theorem A.

Theorem 6.7. *For $m > n$ and $(m,n) = 1$, there is a triply graded isomorphism when $m > n$:*

$$\bigoplus_{i,j,k}\mathrm{HHH}^{i,j,k}(T_{m,n}) \cong \bigoplus_i \mathrm{Hom}_{S_n}\big(\wedge^i\mathfrak{h}, \bigoplus_{j,k}\mathrm{gr}^K_j\big(\mathrm{L}_{\frac{m}{n}}(k)\big)\big). \tag{71}$$

6.4. **Filtrations.**

6.4.1. *Inductive filtrations.* Since the left hand side of (71) is m,n-symmetric, we see that

Corollary 6.8. *Hodge filtrations on* $\mathrm{eL}_{\frac{m}{n}}$ *and* $\mathrm{eL}_{\frac{n}{m}}$ *are compatible with the isomorphism* $\mathrm{eL}_{\frac{m}{n}} \cong \mathrm{eL}_{\frac{n}{m}}$.

Consider a partial order on the positive rational numbers in the following way: for coprime pairs (m,n), $\frac{m}{n}\prec\frac{m+n}{n}$; if $n < m$, then $\frac{m}{n}\prec\frac{n}{m}$.

On can always go from $c = \frac{m}{n}$ to $\frac{1}{n'}$ for some integer $n' > 1$ through a chain of rational numbers decreasing under the order $(\mathbb{Q}_{>0}, \prec)$.

Definition 6.2. [GORS14, Theorem 4.1] We define a filtration F^{ind} inductively as follows:

$$0 = F^{\mathrm{ind}}_{-1}\mathrm{eL}_{\frac{1}{n}} \subset F^{\mathrm{ind}}_0\mathrm{eL}_{\frac{1}{n}} = \mathrm{eL}_{\frac{1}{n}} = \mathrm{L}_{\frac{1}{n}}.$$

Next, F^{ind} is defined inductively on L_c under the order $(\mathbb{Q}_{>0}, \prec)$ using the isomorphisms:

when $m, n > 1$,	$\mathrm{e}_n\mathrm{L}_{\frac{m}{n}} \cong \mathrm{e}_m\mathrm{L}_{\frac{n}{m}}$	[CEE09, 8.2]
when $c > 1$,	$\mathrm{L}_c \cong \mathrm{H}_c\delta\mathrm{e}_-\otimes_{A_{c-1}}\mathrm{eL}_{c-1}$	[BEG03, Lemma 4.7]

Combining Proposition 6.1 and Corollary 6.8, we obtain that

Proposition 6.9. $F^{\mathrm{H}}_j\mathrm{L}_c = F^{\mathrm{ind}}_j\mathrm{L}_c$ *when $c > 1$ and* $F^{\mathrm{H}}_j\mathrm{eL}_c = F^{\mathrm{ind}}_j\mathrm{eL}_c$ *when $c > 0$.*

6.4.2. *Algebraic filtrations.* On H_c we have the Fourier transform defined by

$$\Phi_c(x_i) = y_i, \quad \Phi_c(y_i) = -x_i, \quad \Phi_c(w) = w \tag{72}$$

which defines the Dunkl bilinear form

$$(-,-)_c : \mathbb{C}[\mathfrak{h}] \times \mathbb{C}[\mathfrak{h}] \to \mathbb{C}, \quad (f,g)_c = [\Phi_c(f)g]|_{x_i=0}.$$

When $c = \frac{m}{n} > 0$, with m,n coprime, $(-,-)_c$ has a nonzero kernel I_c and the resulting quotient $\mathbb{C}[\mathfrak{h}]/I_c$ is exactly isomorphic to L_c ([DO03, Proposition 2.34]). Inside $\mathbb{C}[\mathfrak{h}]$, take the ideal $\mathfrak{a} := (\mathbb{C}[\mathfrak{h}]^W_+)$. Let β_c be a nonzero highest weight vector in L_c and let $^{\perp_c}$ denote orthogonal complement with respect to $(-,-)_c$.

Definition 6.3. (1) The $\mathfrak{a}$-filtration is defined by $F^{\mathfrak{a}}_i \mathrm{L}_c = \Phi_c[(\mathfrak{a}^{i+1})^{\perp_c}]\beta_c$.

(2) The algebraic filtration is the Kazhdan filtration associated to $F^{\mathfrak{a}}$ and h_c: $F^{\mathrm{alg}}_i(\mathrm{L}_c) = \sum_{2j+k\leq i} F^{\mathfrak{a}}_j \mathrm{L}_c(k)$.

Proposition 6.10. *For $m > 0$ and $(m,n) = 1$, there is a triply graded isomorphism:*

$$\bigoplus_{i,j,k} \mathrm{HHH}^{i,j,k}(T_{m,n}) \cong \bigoplus_i \mathrm{Hom}_{S_n}\big(\wedge^i \mathfrak{h}, \bigoplus_{j,k} \mathrm{gr}^{\mathrm{alg}}_j\big(\mathrm{L}_{\frac{m}{n}}(k)\big)\big). \tag{73}$$

Proof. It is shown in [Ma24] that $F^{\mathrm{ind}} = F^{\mathfrak{a}}$. This plus Proposition 6.9 and Corollary 6.7 implies that the proposition holds when $m > n$. Denote the right hand side of (73) by $\mathfrak{H}_{m,n}$, it remains to show a triply graded isomorphism $\mathfrak{H}_{m,n} \cong \mathfrak{H}_{n,m}$.

By [Gor13, Corollary 1.1], for all $k \geq 0$ there is an isomorphism

$$\mathrm{Hom}_{S_n}(\wedge^k \mathbb{C}^{n-1}, \mathrm{L}_{\frac{m}{n}}) \cong \mathrm{Hom}_{S_m}(\wedge^k \mathbb{C}^{m-1}, \mathrm{L}_{\frac{n}{m}}). \tag{74}$$

via identifications with spaces of differential forms on a zero-dimensional moduli space associated with the plane curve singularity $x^m = y^n$. By [GORS14, Proposition 1.5], (74) is a bigraded isomorphism with respect to the algebraic filtration and the Euler field h_c. This finishes the proof of the proposition. $\square$

Appendix A. Fourier transforms of cuspidal $\mathcal{D}$-modules

We prove an auxilary result about the Fourier transform of the cuspidal character $\mathcal{D}$-module. Although unrealted to the main results of the paper, this may be of independent interest. In particular, it implies the q,t-symmetry: $(P_{m,n}\cdot 1)(q,t) = (P_{m,n}\cdot 1)(t,q)$.

A.1. **Invariance under the Fourier transform.** The map $\mathfrak{g}\times\mathfrak{g}^* \to \mathfrak{g}^*\times\mathfrak{g}$ sending $(x,x^*)\mapsto (x^*,-x)$ induces an isomorphism $\sigma_1 : \mathcal{D}(\mathfrak{g}) \cong \mathcal{D}(\mathfrak{g}^*)$.

Further identitifying $\sigma_2 : \mathcal{D}(\mathfrak{g}^*) \cong \mathcal{D}(\mathfrak{g})$ via a non-degenerate bilinear form $\mathfrak{g}\cong\mathfrak{g}^*$, we have obtain the Fourier transform induced by $\sigma := \sigma_2\circ\sigma_1$:

$$\mathbb{F} : \mathcal{D}_{\mathfrak{g}}\text{−mod} \to \mathcal{D}_{\mathfrak{g}}\text{−mod}.$$

Proposition A.1. *As $\mathcal{D}_{\mathfrak{g}}$-modules, $\mathbb{F}\mathbf{N}_c \cong \mathbf{N}_c$.*

Proof. Put $\iota : T^*\mathfrak{g} \to T^*\mathfrak{g}, (x,x^*)\mapsto (x^*,-x)$. Then

$$SS(\mathbb{F}(\mathbf{N}_c)) = \iota(SS(\mathbf{N}_c)) \subset \mathscr{N}\times\mathscr{N}.$$

By [Lus87], $\mathbb{F}(\mathbf{N}_c)$ is again a cupidal character $\mathcal{D}$-module and is determined by the monodromy of its restriction to $\mathscr{N}_r$. Therefore the proposition follows from the following claim:

Claim: There is an isomorphism $\iota^\dagger(\mathbb{F}(\mathbf{N}_c)) \cong \mathcal{F}_c$ where $\iota : \mathscr{N}_r \hookrightarrow \mathfrak{g}$.

Recall that the cuspidal character $\mathcal{D}$-module $\mathbf{N}_c$ can be expressed as $\widetilde{\mathrm{Ind}}_B^G\mathbf{L}$. Therefore $\mathbb{F}(\mathbf{N}_c) = \widetilde{\mathrm{Ind}}_B^G\mathbb{F}(\mathbf{L})$.

Since

$$\Gamma(\mathfrak{g},\mathbf{L}) = \mathcal{D}(\mathfrak{g})/(\mathcal{D}(\mathfrak{g})\cdot O(\mathfrak{b}_-) + \sum_i \mathcal{D}(\mathfrak{g})(x_i\partial_{x_i} - ic) + \mathcal{D}(\mathfrak{g})\cdot S([\mathfrak{n},\mathfrak{n}])) \tag{75}$$

we see that

$$\Gamma(\mathfrak{g},\mathbb{F}(\mathbf{L})) = \mathcal{D}(\mathfrak{g})/(\mathcal{D}(\mathfrak{g})\cdot S(\mathfrak{b}) + \sum_i \mathcal{D}(\mathfrak{g})(\partial_{y_i}y_i + ic) + \mathcal{D}(\mathfrak{g})\cdot O([\mathfrak{n}_-,\mathfrak{n}_-])). \tag{76}$$

Consider the standard $\mathfrak{sl}_2$-triple E,F,H (defined by (23)). Then (75) and (76) tells us that $\mathbf{L}$ is the minimal extension of a local system supported on $B\cdot E$ and $\mathbb{F}(\mathbf{L})$ is the minimal extension of a local system supported on $B\cdot F$ defined by a horizontal section

$$y_1^{-c}y_2^{-2c}\dots y_{n-1}^{-(n-1)c}. \tag{77}$$

Moreover, $\mathbf{L}|_{T\cdot E}$ is a local system on $T\cdot E$ such that

$$\mathcal{F}_c = \widetilde{\mathrm{Ind}}_T^G i^E_\dagger (i^E)^\dagger \iota^\dagger\mathbf{L}$$

with $i^E : T\cdot E\to\mathscr{N}_r$. Similarly, $\mathbb{F}(\mathbf{N}_c)|_{T\cdot F}$ is a local system on $T\cdot F$ such that

$$(\mathbb{F}(\mathbf{N}_c))|_{\mathscr{N}_r} = \widetilde{\mathrm{Ind}}_T^G i^F_\dagger (i^F)^\dagger \iota^\dagger\mathbb{F}(\mathbf{L})$$

with $i^F : T\cdot F\to\mathscr{N}_r$.

It suffices to show that $(i^F)^\dagger\mathcal{F}_c = (i^F)^\dagger\iota^\dagger\mathbb{F}(\mathbf{L})$. Recall that the pullback of $\mathcal{F}_c$ along the fibration $q : U\to\mathscr{N}_r$ is $\mathcal{E}_c$ and $\mathcal{E}_c$ is defined by the horizontal section s^c (eq. (6)).

$$s^c|_{T\cdot F} = v_1^{nc}y_1^{(n-1)c}y_2^{(n-2)c}\cdots y_{n-1}^c.$$

Therefore, $(i^F)^\dagger\mathcal{F}_c$ is defined by the horizontal section

$$y_1^{(n-1)c}y_2^{(n-2)c}\cdots y_{n-1}^c.$$

Finally, the lemma follows from the observation that the functions $y_1^{(n-1)c}y_2^{(n-2)c}\cdots y_{n-1}^c$ and $y_1^{-c-1}y_2^{-2c-1}\dots y_{n-1}^{-(n-1)c-1}$ (eq. (77)) define the same local system on $T\cdot F$ as

$$((n-1)c,(n-2)c,\cdots,c) - (m,m,\cdots,m) = (-c,-2c,\cdots,-(n-1)c). \qquad \square$$

A.2. **An explicit description of the Fourier transform.** This subsection comes from an observation of V. Ginzburg.

Let $x = (x_{ij})$ be the standard coordinates of $\mathfrak{gl}_n$ and $(\partial) = (\partial_{x_{ij}})_{1\leq i,j\leq n}$. Take

$$e := \frac{1}{2}\mathrm{tr}(x^2), \quad f := -\frac{1}{2}\mathrm{tr}(\partial^2), \quad h = \sum_{1\leq i,j\leq n} x_{ij}\partial_{x_{ij}} + (n^2-1)/2. \tag{78}$$

Clearly, $[e,f] = h$, $[h,e] = 2e$ and $[h,f] = -2f$. Notice that $[e,-],[f,-],[h,-]$ all preserve the homogeneous components of $\mathcal{D}(\mathfrak{g})$ (with $\deg(x_{ij})$=$\deg(\partial_{x_{ij}})$=1). In other words, this $\mathfrak{sl}_2$-action on $\mathcal{D}(\mathfrak{g})$ is locally finite and thus integrable. Moreover, $e-f$ acts on $\mathcal{D}(\mathfrak{g})$ via

$$[e-f, x_{ij}] = [-f, x_{ij}] = \partial_{x_{ij}}, \quad [e-f, \partial_{x_{ij}}] = [e, \partial_{x_{ij}}] = -x_{ij}. \tag{79}$$

Hence its exponential Ad $e^{\frac{i\pi}{2}(e-f)}$ gives exactly the Fourier transform σ.

On the other hand, any $\mathcal{D}(\mathfrak{g})$-module inherits such an $\mathfrak{sl}_2$-action. By [CEE09, example 63], the action of $\{e,f,h\}$ on $\mathbf{N}_c$ is locally finite and thus also integrable. Denote the action of $e^{\frac{i\pi}{2}(e-f)}$ on $\mathbf{N}_c$ by Φ. Then (79) implies that

$$\Phi(x_{ij}a) = \partial_{x_{ij}}\Phi(a), \quad \Phi(\partial_{x_{ij}}a) = -x_{ij}\Phi(a), \quad \forall a \in \mathbf{N}_c, 1\leq i,j\leq n.$$

Therefore, Φ gives an explicit isomorphism between $\mathbf{N}_c$ and $\mathbb{F}(\mathbf{N}_c)$.

Appendix B. Examples

Here are some calculation results of $\sum_{\lambda\vdash n}\frac{\mathrm{ch}_{q,t}((\mathrm{R}p_*\mathcal{A}_c)|_{\mathbf{e}_\lambda})}{g_\lambda}$ and $\sum_{\lambda\vdash n}\frac{\mathrm{ch}_{q,t}((\mathrm{R}p'_*\mathcal{A}'_c)|_{\mathbf{e}_\lambda})}{g_\lambda}$. In view of [KT22], setting $q\to 1$ these statistics are also Shalika germs in the corresponding cases.

B.1. $n=2$.

B.1.1. *Catalan* $\mathcal{A}'_{k+\frac{1}{2}}$.

$$\overbrace{\frac{q^k}{1-\frac{t}{q}}}^{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}} + \overbrace{\frac{t^k}{1-\frac{q}{t}}}^{\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}} = \sum_{i=0}^{k} q^i t^{k-i}$$

B.1.2. *Cuspdial* $\mathcal{A}_{k+\frac{1}{2}}$.

$$\overbrace{\frac{q^{k+1}}{1-\frac{t}{q}}}^{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}} + \overbrace{\frac{t^{k+1}(1-q)(1-qt)}{(1-t)(1-t^2)(1-\frac{q}{t})}}^{\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}} + \overbrace{\frac{t^k(1-qt^2)}{(1-t^2)(1-\frac{1}{t})}}^{\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}} = q(\sum_{i=0}^{k} q^i t^{k-i})$$

B.2. $n = 3$.

B.2.1. *Catalan* $\mathcal{A}'_{\frac{2}{3}}$.

$$\overbrace{\frac{q}{\left(1-\frac{t}{q}\right)\left(1-\frac{q^2}{t}\right)}}^{\begin{array}{|c|c}\cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}} + \overbrace{\frac{t}{(1-\frac{q}{t})\left(1-\frac{t^2}{q}\right)}}^{\begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}} + \overbrace{\frac{q}{(1-\frac{t}{q})\left(1-\frac{t}{q^2}\right)}}^{\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}} + \overbrace{\frac{t}{(1-\frac{q}{t})\,(1-\frac{q}{t^2})}}^{\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}}$$

$$= q + t$$

B.2.2. *Cuspidal* $\mathcal{A}_{\frac{2}{3}}$.

$$\overbrace{\frac{q^3}{(1-\frac{t}{q})\left(1-\frac{t}{q^2}\right)}}^{\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}}$$

$$+ \overbrace{\frac{q\left(1-q^2\right)t(1-qt)}{(1-t)\left(1-\frac{q^2}{t}\right)\left(1-\frac{t}{q}\right)\left(1-\frac{t^2}{q}\right)}}^{\begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}} + \overbrace{\frac{qt(1-qt)^2}{(1-t)^2\left(1-\frac{q}{t}\right)\left(1-\frac{t^2}{q}\right)}}^{\begin{array}{|c|c}\cline{1-1} 1 & \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline\end{array}} + \overbrace{\frac{q\left(1-qt^2\right)}{\left(1-\frac{1}{t}\right)(1-t)\left(1-\frac{t^2}{q}\right)}}^{\begin{array}{|c|c}\cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}}$$

$$+ \overbrace{\frac{(1-q)^2t^3(1-qt)^2}{(1-t)^2\,(1-t^2)^2\left(1-\frac{q}{t^2}\right)\left(1-\frac{q}{t}\right)}}^{\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}} + \overbrace{\frac{(1-q)t^3(1-qt)\left(1-qt^2\right)}{\left(1-\frac{1}{t}\right)(1-t)\,(1-t^2)\,(1-t^3)\left(1-\frac{q}{t^2}\right)}}^{\begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline 2 \\ \hline\end{array}}$$

$$+ \overbrace{\frac{(1-q)t^2(1-qt)\left(1-qt^2\right)}{\left(1-\frac{1}{t}\right)(1-t)\,(1-t^2)\,(1-t^3)\left(1-\frac{q}{t^2}\right)}}^{\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 1 \\ \hline\end{array}} + \overbrace{\frac{t\left(1-qt^2\right)\left(1-qt^3\right)}{\left(1-\frac{1}{t^2}\right)\left(1-\frac{1}{t}\right)(1-t^2)\,(1-t^3)}}^{\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}}$$

$$= q^2(q + t)$$

B.3. $n = 4$.

B.3.1. *Catalan* $\mathcal{A}'_{\frac{3}{4}}$.

$$
\begin{aligned}
&\overbrace{\frac{qt(1-t)}{\left(1-\frac{q^2}{t}\right)\left(1-\frac{t}{q}\right)^2\left(1-\frac{t^2}{q^2}\right)}}^{\begin{array}{|c|c|c|}\hline 3 \\ \cline{1-1} 1 & 2 & 4 \\ \hline \end{array}}
+\overbrace{\frac{qt}{\left(1-\frac{q}{t}\right)\left(1-\frac{t}{q}\right)\left(1-\frac{t^2}{q^2}\right)}}^{\begin{array}{|c|c|c|}\hline 2 \\ \cline{1-1} 1 & 3 & 4 \\ \hline \end{array}} \\
&+\overbrace{\frac{(1-q)qt}{\left(1-\frac{q}{t}\right)\left(1-\frac{q^2}{t}\right)\left(1-\frac{t}{q}\right)^2}}^{\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}}
+\overbrace{\frac{q^3}{\left(1-\frac{t}{q^3}\right)\left(1-\frac{t}{q^2}\right)\left(1-\frac{t}{q}\right)}}^{\begin{array}{|c|c|c|}\hline 4 \\ \cline{1-1} 1 & 2 & 3 \\ \hline \end{array}}
+\overbrace{\frac{q^3}{\left(1-\frac{q^3}{t}\right)\left(1-\frac{t}{q^2}\right)\left(1-\frac{t}{q}\right)}}^{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline \end{array}} \\
&+\overbrace{\frac{qt(1-t)}{\left(1-\frac{q^2}{t^2}\right)\left(1-\frac{q}{t}\right)^2\left(1-\frac{t^2}{q}\right)}}^{\begin{array}{|c|c|}\hline 4 \\ \cline{1-1} 2 \\ \cline{1-1} 1 & 3 \\ \hline \end{array}}
+\overbrace{\frac{qt}{\left(1-\frac{q^2}{t^2}\right)\left(1-\frac{q}{t}\right)\left(1-\frac{t}{q}\right)}}^{\begin{array}{|c|c|}\hline 4 \\ \cline{1-1} 3 \\ \cline{1-1} 1 & 2 \\ \hline \end{array}} \\
&+\overbrace{\frac{qt(1-t)}{\left(1-\frac{q}{t}\right)^2\left(1-\frac{t}{q}\right)\left(1-\frac{t^2}{q}\right)}}^{\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}}
+\overbrace{\frac{t^3}{\left(1-\frac{q}{t^2}\right)\left(1-\frac{q}{t}\right)\left(1-\frac{q}{t^3}\right)}}^{\begin{array}{|c|c|}\hline 3 \\ \cline{1-1} 2 \\ \cline{1-1} 1 & 4 \\ \hline \end{array}}
+\overbrace{\frac{t^3}{\left(1-\frac{q}{t^3}\right)\left(1-\frac{q}{t^2}\right)\left(1-\frac{q}{t}\right)}}^{\begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}} \\
&=q^3+q^2t+qt+qt^2+t^3
\end{aligned}
$$

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