A quantum Harish-Chandra type isomorphism

Xinchun Ma

Write $V = \mathbb{C}^n$, $Z := \{(X, u) \in \mathfrak{g} \times V | \mathbb{C}[X]u = V\} = \{s \neq 0\}$, where

$$s(X, u) = \langle \mathrm{vol}, u \wedge Xu \wedge \dots X^{n-1}v \rangle.$$

Then we have isomorphism

$$\eta^* : \mathbb{C}[Z]^G \cong \mathbb{C}[\mathfrak{h}]^W$$

where η^* is induced by the embedding map:

$$\eta : \mathfrak{h} \to Z, (\lambda_1, \dots, \lambda_n) \mapsto (\mathrm{diag}\{\lambda_1, \dots, \lambda_n\}, i_0)$$

where $i_0 := (1, \dots, 1)$. We also have isomorphism

$$\eta^* \mathrm{Fun}^G(Z, V) \cong \mathrm{Fun}^W(\mathfrak{h}, V) \cong \mathbb{C}[\mathfrak{h}]^{W_1}$$

by $\xi(F)(x) = F(\eta(x))j$ for any $x \in \mathfrak{h}$ where $j = (0, \dots, 0, 1)$. Let $f_i, 1 \leq i \leq n$ be the ith symmetric polynomial on $\mathfrak{h}$ and $F_i, 1 \leq i \leq n$ be the ith symmetric polynomial on $\mathfrak{g}$. Then $\mathbb{C}[\mathfrak{h}]^{W_1} = \mathbb{C}[\mathfrak{h}]^W[x_n]/(f_{x_n})$ where

$$f_{x_n}(x) = x^n - f_1 x^{n-1} + \dots \pm f_n$$

is the characteristic polynomial of x_n. Moreover $\eta^*(F_i) = f_i$ and $\xi^{-1} x_n^k = X^k u$, which matches the Cayley-Hamilton theorem that the characteristic polynomial of X is

$$F_{x_n}(x) = x^n - F_1 x^{n-1} + \dots \pm F_n$$

Also, fix $c \in \mathbb{C}$. Define $\mathbb{C}_c[Z] := \{f \in \mathbb{C}[Z], f(gx) = \det^c(g) f(x)\}$ and similarly $\mathrm{Fun}_c^G(Z, V) = \{F \in \mathrm{Fun}(Z, V), F(gx) = \det^c(g) F(x)\}$. Then we can build isomorphism

$$\xi_c : \mathrm{Fun}_c^G(Z, V) \cong \mathrm{Fun}^W(\mathfrak{h}, V)$$

by $m_{s^{-c}} \eta^*$. Similarly we can have $\eta_c^* : \mathbb{C}[Z]^G \cong \mathbb{C}[\mathfrak{h}]^W$.

Let $\mu : T^*(\mathfrak{g} \times \mathbb{C}) \to \mathfrak{g}$ be the moment map and

$$M := \mu^{-1}(0) = \{(X, Y, u, v) | [X, Y] + u \otimes v = 0\}.$$

Denote $\mathrm{D}(\mathfrak{g} \times V)$ by D. Then $\mathrm{gr}\ (\mathrm{D}/\mathrm{D}\mathfrak{g}_c)^G = \mathbb{C}[M]^G$, which is isomorphic to $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$.

Proposition 0.1. $\mathrm{e}_{n-1} \mathrm{H}_c \mathrm{e}_n \cong (\mathrm{D}/\mathrm{D}\mathfrak{g}_c \otimes V)^G$.

First of all, the isomorphism holds when we take associated graded:

$$\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{W_1} \cong \mathrm{Fun}^G(M, V).$$

In particular, LHS is generated as a $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^W$-module by $x_n^a y_n^b$, $0\le a,b\le n-1$. By Weyl's fundamental theorem on invariants, we see that RHS is generated as a $\mathbb{C}^G[M]$ module by X^aY^bu, $0\le a,b\le n-1$.

As a result, $\mathrm{e}_{n-1}\mathrm{H}_c\mathrm{e}_n$ is generated by $x_n^a y_n^b$ as a right $\mathrm{e}_n\mathrm{H}_c\mathrm{e}_n$-module and $(\mathrm{D}/\mathrm{D}\mathfrak{g}_c\otimes V)^G$ is generated by $u(X^T)^aY^b$, $0\le a,b\le n-1$ as a right $(\mathrm{D}/\mathrm{D}\mathfrak{g}_c)^G$-module.

By $u(X^T)^aY^b$, first of all we view u as a row vector and $Y=(\partial_{X_{ij}})$ is a matrix of operators. For any $F\in\mathbb{C}[Z]$, $YF=(\partial_{X_{ij}}F)$ and Y^b acts by matrix multiplication. Finally, we take matrix multiplication $u(X^T)^a\cdot(Y^bF)$. For any $A\in G$, A acts on u by uA^T, acts on X by AXA^{-1} hence on X^T by $(A^T)^{-1}X^TA^T$ and acts on Y by $(A^T)^{-1}YA^T$ because Y is viewed as inside $\mathfrak{g}^*$. Therefore, $u(X^T)^aY^b$ is indeed G-equivariant.

Given η_c^* and ξ_c, we have isomorphism

$$\mathrm{DiffOp}(\mathbb{C}[\mathfrak{h}]^W,\mathrm{Fun}^W(\mathfrak{h},V))=\mathrm{DiffOp}(\mathbb{C}_c[Z]^W,\mathrm{Fun}_c^G(Z,V)).$$

Both sides lie in $\mathrm{DiffOp}(\mathbb{C}[\mathfrak{h}_r]^W,\mathrm{Fun}^W(\mathfrak{h}_r,V))$. [1]

Let us look at the image of uY: Write $F=\eta^*f$ for any $f\in\mathbb{C}[\mathfrak{h}_r]^W$. Then

$$\xi_c[uY(\eta_c^*f)](x)=s^{-c}uY(s^cF)(\eta(x))j=u(cFY(\log s)+YF)(\eta(x))j$$

$$uFY(\log s)(\eta(x))j=(\sum\partial_{X_{in}}\log s)(\eta(x))$$

Let v be the unit vector in the direction of $\sum_i X_{in}$. Then

$$\begin{aligned}&(\sum\partial_{X_{in}}\log s)(\eta(x))=(\partial_v\log\ s)(\eta(x))\\ =&\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\log s(\eta(x)+tv)\end{aligned}$$

Write

$$\eta(x)+tv=(\begin{pmatrix}\lambda_1&0&\dots&t\\0&\lambda_2&\dots&t\\\dots&\dots&\dots&\dots\\0&&\dots&\lambda_n+t\end{pmatrix},i_0)$$

$$\begin{pmatrix}\lambda_1&0&\dots&t\\0&\lambda_2&\dots&t\\\dots&\dots&\dots&\dots\\0&&\dots&\lambda_n+t\end{pmatrix}=B(t)\begin{pmatrix}\lambda_1&0&\dots&0\\0&\lambda_2&\dots&0\\\dots&\dots&\dots&\dots\\0&&\dots&\lambda_n+t\end{pmatrix}B(t)^{-1}$$

and $B(t)i_0=i_0$. where $B(t)=\begin{pmatrix}1-\frac{t}{\lambda_n-\lambda_1+t}&0&\dots&\frac{t}{\lambda_n-\lambda_1+t}\\0&1-\frac{t}{\lambda_n-\lambda_2+t}&\dots&\frac{t}{\lambda_n-\lambda_2+t}\\\dots&\dots&\dots&\dots\\0&&\dots&1\end{pmatrix}$.

[1] Recall that the Harish-Chandra radial map

$$(\mathrm{D}(\mathfrak{g})/(\mathfrak{g}))^G\cong\mathrm{D}(\mathfrak{h})^W$$

is $m_s\delta'm_{s^{-1}}$ where m_s is multiplication by s and δ' is the restriction map.

Hence $\log s(\eta(x)+tv) = \log\det(B(t)) + \log(\eta(x(t)))$ where $x = (\operatorname{diag}\{\lambda_1,\dots,\lambda_n+t\}, i_0)$ and

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\log s(\eta(x)+tv) =& \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\log\det(B(t)) + \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\log(\eta(x(t))) \\
=& \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\log(\prod_{i=1}^{n-1}\frac{\lambda_n-\lambda_i}{\lambda_n-\lambda_i+t}) + \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\log(\prod_{i=1}^{n-1}(\lambda_i-\lambda_n-t)\prod_{0<i<j<n}(\lambda_i-\lambda_j)) \\
=& -\sum_{i=1}^{n-1}\frac{1}{\lambda_n-\lambda_i} - \sum_{i=1}^{n-1}\frac{1}{\lambda_i-\lambda_n} = 0
\end{aligned}$$

Example 0.2. When $n=2$, we have $\overrightarrow{u}Y(s)=0$: Write

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \overrightarrow{u} = (u,v).$$

Then $X\overrightarrow{u}^T = (au+bv, cu+dv)^T$, $s = cu^2+(d-a)uv-bv^2$ and

$$Y(s) = \begin{pmatrix} -uv & -v^2 \\ u^2 & uv \end{pmatrix}$$

and so $\overrightarrow{u}Y(s)=0$.

Let $\mathfrak{g} = \mathfrak{gl}_n$, $G = \mathrm{GL}_n$, $X = (X_{ij})$, $Y = (\partial_{X_{ij}})$. Let F_k be the k-th symmetric polynomial in $\mathbb{C}[\mathfrak{g}]^G$, which means the characteristic polynoimial of X is

$$f_X(t) = t^n - F_1t^{n-1} + F_2t^{n-1} + \cdots \pm F_n.$$

In particular, $F_1 = \operatorname{tr}(X)$ and $F_n = \det(X)$ and we assume $F_0 = 1$.

We use notation YF for the matrix $(\partial_{X_{ij}}F)$.

Proposition 0.3. $YF_k = \sum_{i=0}^{k-1}(-X^T)^iF_{k-1-i}$.

Proof. Clearly the identity holds for $k=1$. To do induction, it suffices to prove that for all $k\geq 1$,

$$YF_{k+1} = F_k - X^TYF_k.$$

Now, recall that F_k is the sum of all principal k-minors and there are $\binom{n}{k}$ many k-minors. Morever, $\partial_{X_{ii}}F_{k+1}$ would be the sum of all principal k-minors of the $(n-1)\times(n-1)$ matrix when we cross out i-th row and i-th column of X. There are $\binom{n-1}{k}$ many such k-minors.

On the other hand, the i,i entry of X^TYF_k is

$$\sum_{\ell=1}^{n} X_{\ell i}\partial_{X_{\ell i}}F_k \tag{1}$$

Observe that $X_{\ell i}\partial_{X_{\ell i}}F_k$ exactly selects the term in F_k containing $X_{\ell i}$ so the i,i entry of X^TYF_k selects the term in F_k containing entries from the i-th column, or equivalently, i-th row since there are principal minors. Hence we see the equation

$$(YF_{k+1})_{ii} = (F_k)_{ii} - (X^TYF_k)_{ii}$$

is the same as the combinatorical identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Moreover, we can count the terms of 1: as discussed above, the number of terms in $X_{\ell i}\partial_{X_{\ell i}}F_k$ is

$$\#\{k\text{-minors containing } X_{\ell i}\}.$$

When $\ell = i$, this number is $\binom{n-1}{k-1}$ and when $\ell \neq i$, this number becomes $\binom{n-2}{k-2}$. But when we talk about these terms in $X_{\ell i}\partial_{X_{\ell i}}F_k$, we are actually counting the $k-1$-minors and when we do cofactor expansion of a k-minor, we get a sum giving by k of $k-1$-minors. Now observe that

$$\frac{1}{k}(\binom{n-1}{k-1} + (n-1)\binom{n-2}{k-2}) = \binom{n-1}{k-1}$$

which matches the discussion above.

Now we look at i, j entry when $i \neq j$. Let us use the symbol

$$\binom{i_1, i_2, \ldots, i_k}{j_1, j_2, \ldots, j_k}$$

to indicate the k-minor given by the entries in the intersections of $i_1, i_2, \ldots, i_k$-th rows and $j_1, j_2, \ldots, j_k$-th columns.

See that $\partial_{X_{ij}}F_{k+1}$ is the sum of all nonzero k-minors in the form

$$\binom{j, i_2, \ldots, i_k}{i, i_2, \ldots, i_k} \tag{2}$$

with sign $(-1)^{\alpha+1}$ where $\alpha = \#\{q \in \{i_2, \ldots, i_k\}, \text{lies between } i \text{ and } j\}$. On the other hand

$$(X^T Y F_k)_{ij} = \sum_{\ell=1}^{n} X_{\ell i}\partial_{X_{\ell j}}F_k$$

This time $X_{\ell i}\partial_{X_{\ell i}}F_k$ selects the principal k-minors in F_k containing elements from the j-th column but replaces them by the according elements (from the same rows) from the i-th column. Let

$$\binom{j, i_2, \ldots, i_k}{j, i_2, \ldots, i_k}$$

be such a k-minor. Then the described manipulation changes it to exactly 2 with sign $(-1)^{\alpha}$. $\square$