Second Adjointness

Xinchun Ma

1 Setup

Let G be connected split reductive algebraic group over non-Archimedian local field $\mathbb{F}$. Let P be a parabolic subgroup in G with Levi decomposition MU where M is Levi subgroup and U the unipotent radical. Let V be a smooth representation of G and V' a smooth representation of M. We write $i_M^G V'$ as the parabolic induction of V' from M to G and $r_G^M V$ the restriction (or Jacquet module) of V from G to M.

Frobenious reciprocity tells us that Jacquet functor is left adoint to induction functor: $\mathrm{Hom}_G(V, i_M^G V') \cong \mathrm{Hom}_M(r_G^M V, V')$. Casselman found that if we require the representations to be admissilbe, the adjunction in another way around also holds:

$$\mathrm{Hom}_G(i_M^G V', V) \cong \mathrm{Hom}_M(V', \bar{r}_G^M V)$$

where $\bar{r}_G^M$ refers to induction with respect to the opposite parabolic. He proved it by Jacquet's Lemma. Bernstein generalized it to arbitrary smooth representations by generalized Jacquet's Lemma. Later, Bezrukavnikov and Kazhdan gave a geometric proof [BK15], which is what these notes are about.

By category theory, to show the functorial equivalence of $\mathrm{Hom}_G(i_M^G -, -)$ and $\mathrm{Hom}_M(-, \bar{r}_G^M -)$, it suffices to find unit/counit: $id \to \bar{r}i$, $i\bar{r} \to id$.

Let N, resp. M be Levi inside parabolic Q, resp. P. Let $\omega_1, ...\omega_k$ denote the double cosets $P \setminus G/Q$ and $\mathcal{O}_i = \omega_i Q$. We can order $\mathcal{O}_i$ such that $\mathcal{O}_1, \ldots, \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_k$ are open. By basic geometric lemma in Bernstein's notes [Ber87], we can approximate $r_G^N i_M^G$ by a finite filtration of subfunctors ordered by $\mathcal{O}_1, \ldots, \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_k$.

(More explicitly, we have inclusion $S_c(\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_r, \widetilde{E}) \hookrightarrow i_M^G(E)$ (S_c denotes compactly supported sections and E is M module). Then apply exact functor $J_{U'}$ (where $Q = NU'$) and we have the associated grading of the filtration is $J_{U'}(S_c(\mathcal{O}_r, \widetilde{E})) = S_c(\mathcal{O}_r/U', \mathcal{F})$ with $\mathcal{F} = r_M^{M'}(E)^{\sim}$ where $M' = M \cap wNw^{-1}$.)

When $Q = \overline{P}$, there is only one open orbit $\overline{P}$ with associated functor trivial.

Hence there is unit $\alpha : U \to \bar{r}iU$. We moreover need counit $\beta : i\bar{r} \to V$ such that

$$i \xrightarrow{i(\alpha)} i\bar{r}i \xrightarrow{\beta_i} i$$

$$\bar{r} \xrightarrow{\alpha_{\bar{r}}} \bar{r}i\bar{r} \xrightarrow{\bar{r}(\beta)} \bar{r}$$

are identities.

To simplify, we will denote smooth, compactly supported function on td-space X as $C(X)$. Let $\mathcal{H} = C(G)$. The first observation is that $i\bar{r}V \cong C(X) \otimes_{\mathcal{H}} V$ and $V \cong C(G) \otimes_{\mathcal{H}} V$, where $X = (G/\overline{U} \times G/U)/M$ (M acts diagnally). Indeed, $i\bar{r}V = C(G/U) \otimes_{C(M)} C(\overline{U} \setminus G) \otimes_{\mathcal{H}} V$ $= C((G/\overline{U} \times G/U)/M) \otimes_{\mathcal{H}} V$.

Hence we need to build appropriate $\beta : C(X) \to C(G)$. The construction is inspired by p-adic specialization (or nearby cycle) functor.

2 Wonderful Compactification

This section collects geometric facts about wonderful compactification. In this section, $\mathbb{F}$ can be any arbitrary field and we assume G is adjoint.

Let V_λ be an irreducible representation of G with highest weight λ such that $\langle \lambda, \alpha^\vee \rangle > 0$ for any positive coroot $\alpha^\vee$. Then the composition $G \to GL(V_\lambda) \hookrightarrow \mathrm{End}_{\mathbb{C}} V_\lambda \setminus \{0\} \twoheadrightarrow \mathbb{P}(\mathrm{End} V_\lambda)$ gives an embedding $G \hookrightarrow \mathbb{P}(\mathrm{End} V_\lambda)$.

The wonderful compatification $\overline{G}$ is the closure of the image of G inside $\mathbb{P}(\mathrm{End} V_\lambda)$.

Denote S as the set of simple roots. Given $\Sigma \subset S$ we have decomposition of parabolic subgroup associated to it: $P_\Sigma = M_\Sigma U_\Sigma$ where M_Σ is Levi and U_Σ is unipotent such that root vectors for Σ are in $\mathrm{Lie}(U_\Sigma)$. Denote $Z(M_\Sigma)^\circ$ as the the identity component of center of M_Σ.

Theorem 2.1. (DeConcini-Procesi)

1. $\overline{G}$ is smooth and independent of λ.
2. $\overline{G} = \bigsqcup_{\Sigma \subset S} G_\Sigma$ where G_Σ is a $G \times G$ orbit and $G_\Sigma = (G/\overline{U_\Sigma} \times G/U_\Sigma)/H_\Sigma$ where $H_\Sigma = \{(\bar{\ell}, \ell) \in M_\Sigma \times M_\Sigma | \bar{\ell} \cdot \ell^{-1} \in Z(M_\Sigma)^\circ\}$.

In particular, $\Sigma = 0 \Rightarrow G_\Sigma = G$ and $\Sigma = S \Rightarrow G_\Sigma = G/B \times G/\overline{B}$. The geometry is: the boundary $\overline{G} \setminus G = \bigcup_{\alpha \in S} \overline{G_\alpha}$ is a normal crossing smooth divisor consisting of hypersurfaces and $\overline{G_\Sigma} = \bigcap_{\alpha \in \Sigma} \overline{G_\alpha}$. Let $\overline{T}$ be the closure of T inside $\overline{G}$. Then it is a toric variety with fans the Weyl chambers.

2.1 Vinberg Semigroup

This subsection is a digression, which gives another nice construction of $\overline{G}$ via Vinberg semigroup.

Let $\widetilde{G}$ be the simply connected cover of G with maximal torus $\widetilde{T}$. Let $X_+ \subset X^*(\widetilde{T})$ be the dominant weights lying inside weight lattice. We have a natural partial order on $X^*(\widetilde{T})$: $\lambda \leq \mu$ if $\mu - \lambda$ is positive root.

Denote $A := \mathbb{C}[\widetilde{G}]$, coordinate ring of $\widetilde{G}$. Then $A = \bigoplus_{\mu \in X_+} V_\mu^\vee \otimes V_\mu$. We can define Rees algebra $R(A) := \bigoplus_{\lambda \in X_+} A_\lambda$ where $A_\lambda = \bigoplus_{\mu \leq \lambda} V_\mu^\vee \otimes V_\mu$.

One can embed $\mathbb{C}X^+ \hookrightarrow R(A)$ by $e^\lambda \mapsto 1 \in A_\lambda$ with associated morphism $\pi : Spec\ RA \to \mathbb{C}^r$ where r =rank(G). Then $\mathbb{C}[\pi^{-1}(0)] = gr(A)$.

By definition $\mathbb{C}[\widetilde{G}/U]_\lambda = \Gamma(\widetilde{G}/\widetilde{B}, \mathcal{O}_\lambda)$. Borel-Weil tells us that $\Gamma(\widetilde{G}/B, \mathcal{O}_\lambda) = V_\lambda$ if $\lambda \in x^+$ and 0 otherwise. Hence

Corollary 2.2. $\mathbb{C}[\widetilde{G}/U] = \bigoplus_{\mu \in X_+} V_\mu, \mathbb{C}[\widetilde{G}/\overline{U}] = \bigoplus_{\mu \in X_+} V_\mu^\vee$.

Corollary 2.3. $\mathbb{C}[(\widetilde{G}/\overline{U} \times \widetilde{G}/U)/T] = \bigoplus_{\mu \in X_+} V_\mu^\vee \otimes V_\mu = gr(A)$.

Define $\widetilde{G}_+ := Spec\ R(A)$.

Proposition 2.4.

1. $\widetilde{G}_+$ is a semigroup.
2. For any $\widetilde{G}$ representation V we have $\widetilde{G}_+ \to \mathrm{End}_{\mathbb{C}} V$.

Proof. We imitate the comultiplication $\mathbb{C}[\widetilde{G}] \xrightarrow{\text{comult}} \mathbb{C}[\widetilde{G}] \otimes \mathbb{C}[\widetilde{G}]$ to get another comultiplication: $\mathbb{C}[V_\lambda] \to \mathbb{C}[\widetilde{G}] \otimes \mathbb{C}[V_\lambda]$. Since $\widetilde{G}_+ = Spec\ RA$ with $A_\mu = \bigotimes_{\lambda \leq \mu} V_\lambda^\vee \otimes V_\lambda$, we can naturally define comultiplications: $RA \to RA \otimes RA$ and $\mathbb{C}[V] \to RA \otimes \mathbb{C}[V]$. ☐

We have diagram

$$\begin{tikzcd}
\widetilde{G} \times \widetilde{T} \arrow[r, hook] \arrow[d] & Spec\ R(\mathbb{C}[\widetilde{G}]) \arrow[r, "="] \arrow[d] & \widetilde{G}_+ \arrow[d] & Spec\ gr\ \mathbb{C}[\widetilde{G}] \arrow[l, hook'] \arrow[d] \\
\widetilde{T} \arrow[r, hook] & Spec\ \mathbb{C}X_+ \arrow[r, "\cong"] & \mathbb{C}^r & \{0\} \arrow[l, hook']
\end{tikzcd}$$

and $\widetilde{G} \times \widetilde{T}$ action on V_λ by $(g,t)v = \lambda(t)g(v)$. Notice that $Z(\widetilde{G}) \hookrightarrow \widetilde{G} \times \widetilde{T} : z \mapsto (z, z^{-1})$ acts on V_λ trivially. We define Vinberg's semigroup as $G_+ := \widetilde{G}_+/Z(\widetilde{G})$ and set
$G_+^\circ := \{x \in G_+ |$ image of x in $\mathrm{End} V_\omega$ is nonzero for fundamental weights $\omega\}$,

which is basically stable locus inside G_+.

Theorem 2.5. (Vinberg)

1. G_+° is smooth.
2. $\widetilde{G} \times \widetilde{G} \times \widetilde{T}$ acts on G_+° with finitely many orbits: $G_{+,\Sigma}^\circ, \Sigma \subset S$.
3. $\widetilde{T}$ acts on G_+° freely. $G_+^\circ / \widetilde{T} = \overline{G}$ is smooth and the strata descend.

The final remark is that actually here one can replace $\mathbb{C}$ by any algebraic closed field.

Example 2.6. Let $G = PGL_2$. Then $\widetilde{G} = SL_2$. $\widetilde{G}_+ = M_2$. $G_+ = M_2 / \pm I_2$. $G_+^\circ = (M_2 - \{0\}) / \pm I_2$. $\overline{G} = \mathbb{P}^3$. $\overline{G} \setminus G = \mathbb{P}^1 \times \mathbb{P}^1 = G/B \times G/\overline{B}$.

2.2 View the space X via Normal bundle

Fix some $\Sigma \subset S$. Simply write $Y = \overline{G_\Sigma}$ and $N := N_Y \overline{G}$ as the normal bundle to Y_Σ in $\overline{G}$. We can decompose N into direct sum of line bundles: $N = \bigoplus_{\alpha \in \Sigma} N_Y \overline{G}_{\Sigma - \alpha}$. We can define an action of $Z(M)$ on N such that the action on $N_Y Y_{\Sigma-\alpha}$ is by α. Write $Z = Z(M)^\circ$ as the identity component in $Z(M)$. $\overline{Z}$ intersects $G_\Sigma^\circ := G_\Sigma \setminus \bigcup_{j \notin \Sigma} G_{\Sigma \cup \{j\}}$ transversely at one point, which we denote as y.

We have action of $G \times G$ on N which commutes with the action of Z. Let $N^\circ := N \setminus \bigcup_{\{\alpha\} \subsetneq \Sigma} N_Y \overline{G}_\alpha$. The action of Z is free. Thus $N^\circ \to G_\Sigma$ is a principal Z bundle. Recall that $G_\Sigma = (G/\overline{U_\Sigma} \times G/U_\Sigma)/H_\Sigma$ where $H_\Sigma = \{(\overline{\ell}, \ell) \in M_\Sigma \times M_\Sigma | \overline{\ell} \cdot \ell^{-1} \in Z\}$. Therefore $X \to G_\Sigma$ is also a principal Z bundle.

Actually, let $x \in N^\circ$ be a point lying in the fiber at y. Then $N^\circ \cong (G \times G)/H$ where $H = Stab_{G \times G}(x) = \{(\overline{p}, p) \in \overline{P} \times P | \overline{p} \bmod \overline{U} = w_0(p)\ mod\ U\}$ with w_0 the longest element in Weyl group. Hence $N^\circ \cong (G/\overline{U} \times G/U)/M \cong X$.

Moreover, $N \to G_\Sigma$ is isomorphic to the associated bundle $X \times_Z A \to G_\Sigma$ where A is the closure of Z inside Spec $\mathbb{F}[X^*(Z)]$.

3 Construction of Functor β

We go back to our setting at the beginning.

3.1 Calculus on td-Space

Fix an open compact subgroup $K \subset G \times G$. Let U be a sufficiently small, K stable neighborhood of Y in $\overline{G}$.

Lemma 3.1. There exists analytic map $\phi : U \to N$ such that

- $\phi|_Y = Id$.
- $d\phi|_Y = Id$
- $\phi(G \cap U) \subset N^\circ$.

Lemma 3.2. Assume ϕ satisfies the conditions in 3.1. Let V a small neighborhood in N of $x \in Y$. For any $f_1, \ldots, f_k \in C(V)$ and $\tau_1, \ldots, \tau_m$ commuting automorphisms of V, there exists $(n_0^1, \ldots, n_0^m)$ such that $\forall n^i \geq n_0^i$, $f_i \circ \tau_j^{-n_j} \circ \phi = f_i \circ \tau_j^{-n_j}$.

Proof. Write $\phi(x) = x + o(x)$ and assume $|\tau(v)| = a|v|$. Firstly, let us assume $k = m = 1$. Because f_i are locally constant with compact support, there is $\epsilon > 0$ such that any $|v - v'| < \epsilon$, $f(v) = f(v')$. Choose $r > 0$ such that $\mathrm{Supp}(f, f \circ \phi) \subset B_r$.

Choose n such that $|v| < a^n r$. Then $|o(v)| < a^n \epsilon$. If $f(\tau^{-n}) \neq 0$ or $f(\phi \circ \tau^{-n}) \neq 0$, then $|\tau^{-n}(v)| < r \Rightarrow |v| < a^n r \Rightarrow |o(v)| < a^n \epsilon \Rightarrow |\tau^{-n} o(v)| < \epsilon$.

Hence $f(\tau^{-n}\phi(v)) = f(\tau^{-n} v + \tau^{-n} o(v)) = f(\tau^{-n} v)$. Similar for multiple f and τ's. One only need to choose sufficiently large n. □

Lemma 3.3. Assume ϕ satisfies the conditions in 3.1. There exists K stable open $V \subset N$ and $\mu = \mu(\phi) \in X^*(T)$ such that $\forall \lambda > \mu$:

1. $t^\lambda(V) \subset V$
2. $\forall f \in C(V)^K$, $\mathrm{Supp}(f \circ \tau^{-\lambda}, f \circ \tau^{-\lambda} \circ \phi) \subset U$.
3. $f \circ t^{-\lambda} = f \circ t^{-\lambda} \circ \phi$.

Proof. Since $[G(\mathcal{O}) \times G(\mathcal{O}) : K \cap G(\mathcal{O}) \times G(\mathcal{O})] < \infty$ and that $G(\mathcal{O}) \times G(\mathcal{O})$ acts on Y transitively by Iwasawa decomposition, Y is finite union of K orbits. Suppose $Y = \cup_{i=1}^n K y_i$. Then any K-invariant function on N is determined by its restrictions to $p^{-1}(y_i)$, where p is the projection: $N \to Y$.

For each y_i, pick a T_K stable open neighborhood of $0 \in p^{-1}(y_i)$ and call it V_i. Claim: $\cup V_i = \cup_{i=1}^k T_K X_{++} v_i$ for some $v_i \in V$ where $T_K = T \cap K$ and $X_{++} \subset X^*(T)$ are strictly dominant weights. Indeed, we may assume $n = 1$. T acts on $p^{-1}(y)$ with finitely many orbits.

Let $\tau_1 = t^{\lambda_1}, \ldots, \tau_m = t^{\lambda_m}$ be generators of X_{++}. Then $V \subset \cup_{i=1}^k T_K X_{++} v_i$. Hence $C(V)^{T_K} \subset \langle 1_{T_K t^{\lambda_i} v_j} = 1_{T_K v_i} \circ t^{-\lambda} \rangle_{ij}$ (linear expansion). Now we can apply lemma 3.2 and conclude. □

Corollary 3.4. 1. If $\widetilde{\phi}, \phi$ both satisfy the conditions in 3.1, then $\forall f \in C(N)^K$, there exists

λ such that $\forall \mu > \lambda$, $\mathrm{Supp}(f \circ t^{-\mu}) \subset \mathrm{Im}(\phi) \cap \mathrm{Im}(\widetilde{\phi})$ and $\phi^*(f \circ t^{-\mu}) = \widetilde{\phi}^*(f \circ t^{-\mu})$.

2. Let h_i be a finite set in $\mathcal{H}$. $\exists V \subset N, \lambda \in X^*(T)$ such that $\forall \mu > \lambda, f \in C(N)^K$, $\phi^*(h_j(f \circ t^{-\mu})) = h_j \phi^*(f \circ t^{-\mu})$.

Proposition 3.5. $\mathcal{H}_K$ is noetherian and $C(N)^K$ is a finitely generated $\mathcal{H}_K$-module.

Let $\lambda \in X^*(T)$ be strictly dominant. Write $\tau = t^\lambda$, which can be regarded as an element in $\mathrm{Aut}_{\mathcal{H}_K} C(N)^K$.

Lemma 3.6. Let $M = C(N^\circ)^K$. If $L \subset M$ is a τ-stable $\mathbb{C}$-subspace such that $M = \cup_{n \geq 0} \tau^{-n} L$ then $M = \mathcal{H}L$.

3.2 Second Adjointness

Theorem 3.7. There exists a unique $\mathcal{H}_K$-module homomorphism $\beta_K : C(N^\circ)^K \to C(G)^K$ such that $\forall f \in C(N^\circ)^K$, $\exists n_f \gg 0$ such that $\forall n \geq n_f$, $\beta_K(f \circ \tau^{-n}) = \phi^*(f \circ \tau^{-n})(*)$.

Remark 3.8. (*) guarantees that the map is $\mathcal{H}_K$ equivariant and β_K can be glued together to our desired $\beta : C(X) \to C(G)$.

Proof. Let us simply write $C(N^\circ)^K$ as M.
Uniqueness: Let m_i' be a set of generators of M as $\mathcal{H}_K$-module. There exists $n \gg 0$ such that$\beta(m_i' \circ \tau^{-n}) = \phi^*(m_i' \circ \tau^{-n})(*)$ for all i. By lemma 3.6, $m_i' \circ \tau^{-n}$ is still a set of generators.

So $M = \sum \mathcal{H}_K m_i' \circ \tau^{-n}$. And β is determined by $\beta_K(\sum a_i'(m_i' \circ \tau^{-n})) = \sum a_i' \phi^*(m_i' \circ \tau^{-n})$.
Existence: Choose a presentation of M:

$$\mathcal{H}^{p'} \to \mathcal{H}^p \to M \to 0$$

The second map is $1_i \to m_i$ and the first map can be presented as a matrix $(h_{ij})_{p \times p'}$. Let $L = \{f \in M | \mathrm{Supp}\, f \subset V\}$. By corollary 3.4, there exists $n \gg 0$ such that

1. $m_i = m_i \circ \tau^{-n} \in L$
2. $h_{ij} m \in L$.
3. $\phi^*(h_{ij} m_i) = h_{ij} \phi^*(m_i)$.

By lemma 3.6, m_i generate M as $\mathcal{H}_K$ module.

Define β_K by $\beta(a_i m_i) = \sum a_i \phi^*(m_i) = \sum a_i m_i'$. It remains to show that it is well-defined: If $\sum a_i m_i = 0$ then $a_i = \sum b_j h_{ij}$ for some $b_j \in \mathcal{H}_K$. Thus

$$\sum a_i \phi^*(m_i) = \sum b_j h_{ij} \phi^*(m_i) = \sum b_j \phi^*(h_{ij} m_i) = 0. \qquad \square$$

Consider the composition $X \to G/P \times G/\overline{P} \to G/\overline{P}$. Let X° be the preimage of $U \cdot \overline{P}$. Then U acts on X° freely and $X^\circ/U \cong G/U$. Similarly, we can define ${}^\circ X$ and have ${}^\circ X/\overline{U} \cong G/\overline{U}$.

Proposition 3.9. The compositions

$$C(G/U) = C(X^\circ)_U \xhookrightarrow{\alpha} C(X)_U \xrightarrow{\beta} C(G)_U = C(G/U)$$

$$C(G/\overline{U}) = C({}^\circ X)_{\overline{U}} \to C(X)_{\overline{U}} \xrightarrow{\beta} C(G)_{\overline{U}} \to C(G/\overline{U})$$

are both identities. Hence $i \xrightarrow{i(\alpha)} i\overline{r}i \xrightarrow{\beta_i} i$ and $\overline{r} \xrightarrow{\alpha_{\overline{r}}} \overline{r}i\overline{r} \xrightarrow{\overline{r}(\beta)} \overline{r}$ are identities.

References

[Ber87] Joseph Bernstein, *Second adjointness for representations of p-adic reductive groups*, preprint (1987).

[BK15] Roman Bezrukavnikov and David Kazhdan, *Geometry of second adjointness for -adic groups*, Representation Theory of the American Mathematical Society **19** (2015), no. 14, 299–332.