1 HK2

Denote $X := G/B$, $\tilde{X} := G/U$, $p : \tilde{G} \to G$, $q : \tilde{G} \to T$ as usual.

Let $\Delta \subset Y = (\tilde{X} \times \tilde{X})/T$ the preimage of the diagnal under the quotient map $Y \to X \times X$, which is isomorphic to $\mathcal{B} \times T$ under the map $(\overline{x}, t) \to (xt, x)$ for an arbitrary lift $x \in \tilde{X}$ of $\overline{x} \in \mathcal{B}$. Let $\pi : \mathcal{B} \times T \to T$ be the projection.

Define $f : G \times \mathcal{B} \to Y$ by $(g, \overline{x}) \to (gx, x)$ where again, x is an arbitrary lift. Then the observations are $f(\tilde{G}) = \Delta$ and $\pi \circ f = q$. We have commutative diagram

$$\begin{tikzcd}
G \times T & \tilde{G} \arrow[l, "p\times q"'] \arrow[r, "f"] \arrow[d, "p\times f = i\times q"] & \Delta \arrow[d, "diag"] \\
 & G \times X \times T \arrow[ul, "pr"] \arrow[r, "F"] & Y \times_{G/B} \Delta =: Z
\end{tikzcd}$$

where $F = f \times id$. Notice that $Z \cong Y \times T$

$$(p \times q)_{+}\mathcal{O}_{\tilde{G}} = pr_{+}F^{\dagger}\mathcal{B}|_{\Delta|Z}$$

Denote $r : \tilde{X} \to X$, $\delta : X \to Y$.Then from the diagram (by Z/T we mean modulo diagnal action of T)

$$\begin{tikzcd}
\Delta \arrow[r] \arrow[d, "diag"] & X \arrow[d, "diag"] \\
Z \arrow[r, "a"] & Z/T \cong Y
\end{tikzcd}$$

we have

$$\mathcal{B}|_{\Delta|Z} = a^{\dagger}(\delta.r.(D_{\tilde{X}})^{T} \otimes \Omega_{X}^{-1})$$

Following [HK2], we have computations:

$$pr_{+}F^{\dagger}M = Rpr.(D_{G\times T} \otimes_{D_{G\times X\times T}} D_{G\times X\times T \to Z} \otimes^{L}_{D_Z} M)$$

and from the diagram

$$\begin{tikzcd}
G \times X \times T \arrow[r, "\iota"] \arrow[dr, "\delta"] & G \times T \times Z \arrow[r, "pr"] \arrow[d, "\phi"] & G \times T \\
 & G \times T \times Z \arrow[ur, "pr_2"] &
\end{tikzcd}$$

$$
\begin{aligned}
&Rpr_\cdot(D_{G\times T}\otimes_{D_{G\times X\times T}} D_{G\times X\times T\to Z})\\
=&Rpr_\cdot(\mathcal{B}_{\iota(G\times X\times T)|G\times T\times Z}\otimes\Omega_Z)\\
=&R(pr_2)_\cdot\mathcal{O}_{G\times T}\otimes B_{X|Z}\otimes\Omega_Z\\
=&\mathcal{O}_{G\times T}\otimes R\Gamma(X,(r_\cdot D_{\tilde{X}})^T\otimes\Omega_X)
\end{aligned}
$$

Denote $R=\Gamma(\mathcal{B}\times T, D_{\mathcal{B}\times T})$. Compute (hope to have a description of HC D-module by comparing it with (3.5) of your paper)

$$
\begin{aligned}
p_\dagger q^\dagger M =&p_\dagger f^\dagger\pi^\dagger M\\
=&Rp_*(D_{G\leftarrow\tilde{G}}\otimes_{D_{\tilde{G}}} D_{\tilde{G}\to\Delta}\otimes_{D_\Delta} D_\Delta\otimes_R\Gamma(\Delta,\pi^\dagger M))\\
=&Rp_*(D_{G\leftarrow\tilde{G}}\otimes_{D_{\tilde{G}}} D_{\tilde{G}\to\Delta})\otimes_R\Gamma(\Delta,\pi^\dagger M)
\end{aligned}
$$

To compute $Rp_*(D_{G\leftarrow\tilde{G}}\otimes_{D_{\tilde{G}}} D_{\tilde{G}\to\Delta})$, first we use lemma 1 of [HK2] and get

$$D_{G\leftarrow\tilde{G}}\otimes_{D_{\tilde{G}}} D_{\tilde{G}\to\Delta}=B_{\tilde{G}|G\times\Delta}\otimes\Omega_\Delta.$$

And then we similarly draw the diagram ($\tilde{G}\times T\subset G\times\Delta\cong G\times G/B\times T$)

$$
\begin{tikzcd}
\tilde{G} \arrow[r, "\iota"] \arrow[dr, "\delta"'] & \tilde{G}\times T \arrow[r, "p"] \arrow[d, "\phi"] & G\\
& \tilde{G}\times T \arrow[ur, "p_2"'] &
\end{tikzcd}
$$

where $\iota = p\times f(= i\times q)$ and $\phi:(g,x,y)\mapsto(g,g^{-1}x,y)$. Hence δ is the restriction of $G\times\mathcal{B}\to G\times Y$, $(g,\overline{x})\to(g,x,x)$, i.e the imbedding $id\times pt:\tilde{G}\to\tilde{G}\times T$.

Then we have

$$R(p)_*(B_{\iota\tilde{G}|\tilde{G}\times T}\otimes\Omega_\Delta)=R(p_2)_*(B_{\delta\tilde{G}|\tilde{G}\times T}\otimes\Omega_\Delta)$$

$$B_{\delta\tilde{G}|\tilde{G}\times T}=\delta_\dagger\mathcal{O}_{\tilde{G}}=R\delta_*(\mathcal{O}_{\tilde{G}}[\partial_{t_i}]\otimes\Omega_T^{-1})=D_{\tilde{G}\times T}\,\delta(t-e)$$

where t_i are coordinates of T.

Actually, we have $B_{\tilde{G}|G\times\Delta}=D_{\tilde{G}\times T}\,\delta(t-q(\tilde{x}))$

Hence ($pt : \tilde{G} \to T$ the trivial map.)

$$
\begin{aligned}
&Rp_*(D_{G\leftarrow\tilde{G}} \otimes_{D_{\tilde{G}}} D_{\tilde{G}\to\Delta}) \\
=&R(p_2)_* R\delta_*(\mathcal{O}_{\tilde{G}}[\partial_{t_i}] \otimes \Omega_{\mathcal{B}}) \\
=&Rp_*(\mathcal{O}_{\tilde{G}}[\partial_{t_i}] \otimes \Omega_{\mathcal{B}}).
\end{aligned}
$$

1.1 semismall

Theorem 1.1. (Dimension formula) Let $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ be the Springer resolution and $\mathcal{B}_x$ be the Springer fiber at $x \in \mathcal{N}$. Then for any $x \in N$, $\dim Z_G(x) = rk(G) + 2\mathcal{B}_x$.

From the dimension formula, one can immediately deduce that the Springer resolution is semismall. Indeed, $\dim G = \dim \mathcal{N} + rk(G)$ hence

$$2\dim \mathcal{B}_x + \dim G \cdot x = 2\dim \mathcal{B}_x + \dim \mathcal{N} + rk(G) - \dim Z_G(x) = \dim \mathcal{N}.$$

Moreover, there are only finitely many unipotent orbits.

Proposition 1.2. The Grothendieck-Springer resolution $p : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is small.

Proof. First of all, for any $x \in \mathfrak{g}$ we have Jordan decomposition $x = x_s + x_n$ with x_s semisimple and x_n nilpotent with x_n lying inside the nilpotent radical of the lie algebra of H, the centralizer of x_s. Let $\mathcal{B}^H_{x_n}$ be the springer fiber at x_n and $\widetilde{\mathcal{B}}_x$ the Grothendieck-Springer fiber at x.

Claim: $\dim \mathcal{B}^H_{x_n} = \dim \widetilde{\mathcal{B}}_x$.

Now we are ready to construct the stratification of $\mathfrak{g}$ under which p is small. We first stratify the universal cartan $\mathfrak{H}$ by $\{\mathfrak{H}_{\underline{\alpha}}\}_{\underline{\alpha}\subset\Delta^+}$ where $\mathfrak{H}_{\underline{\alpha}} = \{\underline{\alpha} = 0\} \setminus (\cup_{\underline{\alpha}\subsetneq\underline{\beta}}\{\underline{\beta} = 0\})$. Then $x_s \in G \cdot \mathfrak{H}_{\underline{\alpha}}$ for some $\underline{\alpha} \subset \Delta^+$.

Let $X_{\underline{\alpha}}$ be the nilpotent orbit in $\mathfrak{h}$ containing x_n. Then $\{G \cdot (\mathfrak{H}_{\underline{\alpha}} \times X_{\underline{\alpha}})\}_{\underline{\alpha}\subset\Delta^+}$ gives the stratification of $\mathfrak{g}$ we are looking for:

$$\dim G \cdot (\mathfrak{H}_{\underline{\alpha}} \times X_{\underline{\alpha}}) = \dim \mathfrak{H}_{\underline{\alpha}} + \dim G - \dim Z_G(x) = \dim \mathfrak{H}_{\underline{\alpha}} + \dim G - \dim Z_H(x_n).$$

Therefore by $\dim Z_H(x_n) = rk(H) + 2\mathcal{B}^H_{x_n}$ (dimension formula), $rk(H) = rk(G)$ and $\dim \mathcal{B}^H_{x_n} = \dim \widetilde{\mathcal{B}}_x$ we can conclude that

$$\dim G \cdot (\mathfrak{H}_{\underline{\alpha}} \times X_{\underline{\alpha}}) + 2\mathcal{B}_x = \dim G + \dim \mathfrak{H}_{\underline{\alpha}} - rk(G) \leq \dim G$$

where the equality holds iff $\underline{\alpha} = \emptyset$. $\square$

2 HK1

$\rho : \tilde{\mathfrak{g}} \to \mathfrak{g}$. $\dim \widetilde{\mathcal{B}}_x = \dim \mathcal{B} - 1/2 \dim G \cdot x$.

$\mathcal{N} = \{P(x) = 0, P \in \mathbb{C}[\mathfrak{g}]^G_+\}$.

$\mathscr{H}^j \mathcal{M} = 0$ for $j \neq 0$, $\mathscr{H}^0 \mathcal{M} = 0$ is holonomic regular.

$$Ch(\mathcal{M}) = \{(x, y, t, s) \in \mathfrak{g}^2 \times \mathfrak{h}^2, [x, y] = 0, gx \equiv t, hy \equiv -s \ mod\ \mathfrak{n} \text{ for some } g, h \in G\}$$

$\mathscr{H}^0 \mathcal{M}$ is simple and isomorphic to $j_{!*}(B_{p\times q(\mathfrak{g})_{rs}|\mathfrak{g}_{rs}\times\mathfrak{h}})$ where $j : \mathfrak{g}_{rs} \times \mathfrak{h} \hookrightarrow \mathfrak{g} \times \mathfrak{h}$.

2.0.1 Build the map

We have maps

$$\mathfrak{h}^r \xleftarrow{q^r} G/T \times \mathfrak{h}^r (\cong G \times^B \mathfrak{b}^r \cong \tilde{\mathfrak{g}}^{rs}) \xrightarrow{p^r} \mathfrak{g}^{rs} \ (|W|\text{-cover map}).$$

$$T^r \xleftarrow{q^r} G/T \times T^r (\cong G \times^B B^r \cong \widetilde{G}^{rs}) \xrightarrow{p^r} G^{rs} \ (|W|\text{-cover map}).$$

Notice that $D^G(G/T \times T^r) = D^G(G/T) \times D(T^r) = (U(\mathfrak{g})/\mathfrak{h}U(\mathfrak{g}))^T \times D(T^r)$.

Define: $\delta = c_\sigma q^r_\dagger (p^r)^\dagger : D(G)^G \to D(T^r)^W$, where c_σ is defined by $c_\sigma(D)f = \sigma D \sigma^{-1} f$ for $D \in D(T^r)^W$ and $f \in \mathcal{O}_{T^r}$.

3 Computation on SL_2 solving some confusion

First of all, we have $Z(\mathfrak{g})$ is generated by $H^2 + 2EF + 2FE$. $k(\mathfrak{g})^G$ is generated by $h^2 + ef$. $D(\mathfrak{g})^G$ is generated by $h^2 + ef$ and $\partial_h^2 + 4\partial_e\partial_f$, which is the image of $H^2 + 2EF + 2FE$ after taking associated grading.

To show that $h^2 + ef$ is in $k(\mathfrak{g})^G$: we need to compute, for example, $(\mathrm{Ad}e^{tE})^*(h^2 + ef) = h^2 + ef$. Notice that, $\mathrm{Ad}E^* h = f$, $\mathrm{Ad}E^* f = 0$, $\mathrm{Ad}E^* e = -2h$, $1/2\ \mathrm{Ad}E^* e = -f$. Therefore we have $(\mathrm{Ad}e^E)^*(h^2 + ef) = (h + f)^2 + (e - 2h - f)f = h^2 + ef$. Actually just determinant of the matrix.....

Chevalley restriction theorem $\mathcal{O}(\mathfrak{g})^G \cong \mathcal{O}(\mathfrak{h})^W$ is simply by restriction functions to the subset, which looks like just projection: $h^2 + ef \mapsto h^2$.

Harish-Chandra isomorphism $Z(\mathfrak{g})^G \cong S(\mathfrak{h})^W$ is given by projection and then translation by ρ: $H^2 + 2EF + 2FE \to H^2 \pm 2H \to H^2 - 1$.

The restriction $\partial_h^2 + 4\partial_e\partial_f$ to $\mathfrak{h}$ is $\partial_h^2 + 2\frac{1}{h}\partial_h$ and $m_h(\partial_h^2 + \frac{2}{h}\partial_h)m_h^{-1} = \partial_h^2$, which actually coincides with restriction theorem. It makes sense because these principal symbols commute with each other, i.e, we are actually looking at $S(\mathfrak{g})^G = S(\mathfrak{h})^W$.m

3.1 Group case

We give SL_2 coordinates $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | ad - bc = 1\}$ and $k[T] = k[t, t^{-1}]$. Then $k(G)^G$ is generated by $tr = a + d$, which is mapped to $t + t^{-1}$ under the isomorphism $k(G)^G = k[T]^W$.

Computation gives $\partial_H tr = -a + d$, $\partial_E tr = -c$, $\partial_F tr = -b$, $\partial_E b = -d$, $\partial_F c = -a$, where we differential the action of left translation (multiplication by inverse). As a result, $(\partial_H = -t\partial_t)$

$$(\partial_H^2 + 2\partial_E\partial_F + 2\partial_F\partial_E)\phi(tr) = \phi''(tr)(tr^2 - 4) + 3\phi'(tr)tr = (\partial_H^2 - 2\frac{t + t^{-1}}{t - t^{-1}}\partial_H)\phi(t + t^{-1}).$$

Claim: $m_{t-t^{-1}}(\partial_H^2 - 2\frac{t + t^{-1}}{t - t^{-1}}\partial_H)m_{t-t^{-1}}^{-1} = \partial_H^2 - 1$.

Indeed,

$$\begin{aligned}
&(\partial_H^2 - 2\frac{t + t^{-1}}{t - t^{-1}}\partial_H)(t - t^{-1})^{-1} f \\
=&\frac{t(f''(t^6 - 2t^4 + t^2) + f'(t^5 - 2t^3 + t) + f(-t^4 + 2t^2 - 1))}{(t - 1)^3(t + 1)^3} \\
=&\frac{t}{t^2 - 1}(t^2 f'' + tf' - f) \\
=&(t - t^{-1})^{-1}(\partial_H^2 - 1)f
\end{aligned}$$

From here one sees that in group case, the restriction to $Z(\mathfrak{g}) \subset D(G)^G$ gives exactly the famous Harish-Chandra homomorphism $Z(\mathfrak{g}) \to U(\mathfrak{h})^W$, which is the composition of projection and translation by ρ. It can't be seen in Lie algebra case, where $D(\mathfrak{g}) = S(\mathfrak{g}^*) \ltimes S(\mathfrak{g})$ and we only have $S(\mathfrak{g})^G = S(\mathfrak{h})^W$.

$$\begin{tikzcd}
\widetilde{G} \arrow[r, "p"] \arrow[d, "q"] & G \arrow[d, "\chi"] \\
T \arrow[r, "\pi"] & T/W
\end{tikzcd}$$

Proposition 3.1. (HC)For any $f \in \mathcal{O}(G)^G$ and $u \in D(G)^G$,

$$(uf)|_T = \Delta^{-1}\delta(u)(\Delta f)|_T.$$

Proposition 3.2. $\mathcal{M} := D_{G\times T}/(D_{G\times T}(\mathrm{ad}\mathfrak{g}\otimes 1 + D_{G\times T}\{u\otimes 1 - 1\otimes\delta(u), u\in\mathcal{D}(G)^G\}))\cong f_\dagger\mathcal{O}_{\tilde{G}}$.

Proof. Decompose f by $\tilde{G}\xrightarrow{id_{\tilde{G}}\times q}\tilde{G}\times T\xrightarrow{p\times id_T} G\times T$. $(id_{\tilde{G}}\times q)_\dagger\mathcal{O}_{\tilde{G}}\cong D_{\tilde{G}\times T}\delta(t-q(\tilde{x}))$. Therefore

$$((p\times q)_\dagger\mathcal{O}_{\tilde{G}})_{an} = (p\times id_T)_\dagger(D_{\tilde{G}\times T}\,\delta(t-q(\tilde{x})))_{an}\cong(\mathcal{O}_{G\times T}\delta(q'(x)-t))_{an}^{|W|}$$

on a neighborhood of $(t_0,t_1)\in G_{rs}\times T$, where $p' = qp^{-1}(x)$, clearly well-defined. This isomorphism is given by

$$(dg)^{-1}\otimes\omega\otimes\delta(t-q'(x))\mapsto\phi(x,wt).$$

where $\Phi(x,t)$ is a G invariant distribution defined on a neighborhood of $t_0\times T_{rs}$ given by $\Phi(x,t)|_{T\times T} = \Delta(p'(x))^{-1}\delta(q'(x)-t)$.

Using proposition 3.1, for any $u\in D(G)^G$, $f\in\mathcal{O}(G)^G\times\mathcal{O}(T)^W$ we have (restricted to $T_r\times T_r$)

$$\begin{aligned}
&\langle f, u(\Delta^{-1}\delta(t'-t))\rangle\\
=&\langle\Delta^{-1}u(f),\delta(t'-t)\rangle\\
(\star)\ =&\langle u(\Delta^{-1}f),\delta(t'-t)\rangle\\
=&\int_{t'=t}u(\Delta^{-1}f)(t',t)dt\\
=&\int_{t'=t}\Delta^{-1}\delta(u)(f(t',t))dt\\
=&\Delta^{-1}\delta(u)(f),\delta(t'-t)\rangle\\
=&\langle f,\delta(u)(\Delta^{-1}\delta(t'-t))\rangle
\end{aligned}$$

To see $(\star)$, by Leibniz rule, we have

$$u(\Delta^{-1}f) = fu(\Delta^{-1}) + \sum(\partial_{x_i}f)u_{\partial_{x_i}}(\Delta^{-1}) + \sum(\partial_{x_{i_1}}\partial_{x_{i_2}}f)u_{\partial_{x_{i_1}}\partial_{x_{i_2}}}(\Delta^{-1}) + \cdots =: v(\Delta^{-1})$$

where notations like $u_{\partial_{x_{i_1}}\partial_{x_{i_2}}}$ stand for partial derivatives of the principal symbol of u along $\partial_{x_{i_1}}$ and then $\partial_{x_{i_2}}$. Notice that v is still in $D(G)^G$. Hence we can apply proposition 3.1 and have

$$v(\Delta^{-1}) = \Delta^{-1}v(0) = \Delta^{-1}\sum(\partial_{x_{i_1}}\partial_{x_{i_2}}\ldots\partial_{x_{i_j}}f)u_{\partial_{x_{i_1}}\partial_{x_{i_2}}\ldots\partial_{x_{i_j}}}(0)$$

where we plug 0 into ∂_x. This is exactly the Taylor expansion of $\Delta^{-1}u(f)$.

Hence $(u-\delta(u))\Phi = 0$ holds on $T_r\times T_r$. Since it is G-invariant, it also holds on $G_{rs}\times T_r$. Hence we have $(u-\delta(u))s = 0$ on $G_{rs}\times T_r$. Since $(p\times q)_\dagger\mathcal{O}_{\tilde{G}}$ is simple, it is true on the whole $G \times T$. $\square$