Deformation of Poisson Schemes

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Introduction

These notes aim at conveying some down-to-earth understanding about the following theorem in [Kal].

Theorem 0.1. ([Kal] Theorem 1.4) Given a Poisson scheme X and a poisson line bundle L on X. If $H^2(X, \mathcal{O}_X) = 0$, then there exists a formal moment system $(\mathfrak{X}/S, \mathcal{L})$ extending the pair $\langle X, L \rangle$, where $S = \text{Spec } k[[t]]$. If furthermore $H^1(X, \mathcal{O}_X) = 0$, then such a moment system is unique up to an isomorphism.

Definition 0.2. By formal moment system, we mean a pair $(\mathfrak{X}, \mathcal{L})$ of a flat Poisson scheme $\mathfrak{X}$ (formal deformation) over S and line bundle $\mathcal{L}$ on $\mathfrak{X}$ such that (t should be think of as pullback from S to $\mathfrak{X}$)

1. $\{t, a\} = 0, \forall a \in \mathcal{O}_{\mathfrak{X}}$.
2. $\{t, e\} = e, \forall e \in \mathcal{L}$.

Similarly, one can define order-n moment system by a pair (X_n, L_{n-1}) of flat Poisson scheme X_n over $S_n = \text{Spec } k[[t]]/t^{n+1}$ (order-n deformation) and line bundle L_{n-1} on $X_{n-1} = X_n \times_{S_n} S_{n-1}$ with a $\mathcal{O}_{X_n}$ Poisson module structure such that 1,2 above are satisfied.

The proof in [Kal] shows that given a Poisson scheme X and an order-n moment system (X_n, L_{n-1}), the stack of groupoid classifying L_n extending L_{n-1} and the stack of groupoid classifying (X_{n+1}, L_n) extending (X_n, L_{n-1}), are equivalent by proving that: Affine locally, assume $X_n = \text{Spec } A_n$ and $L_{n-1} = A_{n-1}e$, where e is a trivialization of L_{n-1}. Then $\text{Ann}_e(A_n) := \{a \in A_n, \{a, e\} = 0\} \cong A_0$ and $A_n \cong \text{Ann}_e(A_n)[t]/t^{n+1}$.

Next by the vanishing of $H^2(X, \mathcal{O}_X)$, we can extend L_n to a line bundle L_{n+1} on X_{n+1}. Therefore the theorem follows by doing induction and then taking inverse limit.

In these notes, we hope to build X_n and L_n concretely, which is to say we can play with specific transition maps directly without any machinery.

1 Concrete construction

1.1 Symplectic case

Let us first do some elementary computation in symplectic case which turns out to perfectly coincide with the general Poisson setting as Kalendin's proof implies.

Firstly, the proof of proposition 2.5 in[Wie03] defines a map $\mathcal{O}_X^* \to \Omega_X^1 : f \to d \log f$ and the composition (ι_ω denotes contraction by symplectic form ω)

$$Pic(X) \cong H^1(X, \mathcal{O}_X^*) \xrightarrow{d} H^1(X, \Omega_X^1) \overset{\iota_\omega}{\cong} H^1(X, T_X) \tag{1}$$

sends any line bundle $[L] \in Pic(X)$ to $(U_{ij}, \xi_{ij}) \in H^1(X, T_X)$, which gives us a first-order deformation of X by:

$$\mathcal{O}_{U_{ij}} \otimes k[t] \to \mathcal{O}_{U_{ij}} \otimes k[t] : a + bt \to a + (b + \xi_{ij} a)t. \tag{2}$$

Moreover, we can deform 2-form by $\alpha + \beta t \to \alpha + (\beta + L_{\xi_{ij}} \alpha)t$ where $L_{\xi_{ij}}$ refers to Lie derivative. Especially, the symplectic form remains unchanged.

A second-order deformation can be built similarly. To simplify, we obmit subscripts if no confusion would be caused.

Set $D(a + bt + ct^2) = a + (b + \xi a)t + (c + \tau a + \xi b)t^2$. Then

$$D((a+bt+ct^2)(a'+b't+c't^2)) = aa' + [ab' + ba' + \xi(aa')]t + [ac' + ca' + bb' + \xi(ab' + ba') + \tau(aa')]t^2$$

and $D(a+bt+ct^2)D(a'+b't+c't^2) = aa' + [a(b'+\xi a') + (b+\xi a)a']t + (b+\xi a)(b'+\xi a')$

$$+(c + \xi b + \tau a)a' + a(c' + \xi b' + \tau a')t^2$$

being equal is equivalent to the condition $\tau(aa') = \xi a \xi a' + (\tau a)a' + a\tau a'$

On the other hand, cocycle condition

$$a + (b + \xi_{ik} a)t + (c + \tau_{ik} a + \xi_{ik} b)t^2 = a + (b + \xi_{ij} a + \xi_{jk} a)t + (c + \tau_{ij} a + \xi_{ij} b + \tau_{jk} a + \xi_{jk}(b + \xi_{ij} a))t^2$$

is equivalent to $\tau_{ik} a = \tau_{ij} a + \tau_{jk} a + \xi_{jk} \xi_{ij} a$.

Hence D defines transition maps iff $X := \tau - \frac{1}{2}\xi^2$ satisfy $X_{ik} - X_{ij} - X_{jk} = \frac{1}{2}[\xi_{jk}, \xi_{ij}]$

It does exist uniquely because: Locally assume $X = \iota_\omega dh$ so what we need is solution for

$$h_{ik} - h_{ij} - h_{jk} = \frac{1}{2}\{\log f_{jk}, \log f_{ij}\} \tag{3}$$

but view RHS as 2-cycle $c_{ijk} = \{f_{jk}, f_{ij}\}$ and then

$$dc_{ijkl} = \{\log\ f_{ij}, \log\ f_{jk}\} - \{\log\ f_{ik}, \log\ f_{kl}\} + \{\log\ f_{ij}, \log\ f_{jl}\} - \{\log\ f_{jk}, \log\ f_{kl}\} = 0$$

(using $f_{ij}f_{jk} = f_{ik}$). Hence under assumption $H^2(X, \mathcal{O}_X) = 0$, (1) has solution, which is unique given $H^1(X, \mathcal{O}_X) = 0$.

1.2 Deformation of Poisson schemes

1.2.1 Deformation of X

To simplify, sometimes we obmit subscripts if there would not cause confusion and use $\{\log\ e, -\}$ to stand for $\frac{1}{e}\{e, -\}$.

Suppose U_i form an affine cover of X such that $L(U_i) = \mathcal{O}_X e_i$ with transition map f_{ij}, i.e $e_i = f_{ij} e_j$ on U_{ij}.

Considering the Leibniz rule, the definition of moment system and that the deformed bracket has to coincide with the bracket on X modulo t^{n+1}, locally we can only define the bracket on $A_n := \mathcal{O}_X(U)[t]/(t^{n+1})$ by

$$\{\sum_{k=0}^{n} t^k \alpha_k, \sum_{k=0}^{n} t^k \alpha'_k\} = \sum_{i+j=0}^{n} t^{i+j}\{\alpha_i, \alpha'_j\}$$

We define a derivation ∂ on A_n by sending $a \mapsto \frac{\{e,a\}}{e}$. Denote $D = \sum_{k=0}^{n} \frac{1}{k!} t^k \partial^k$. Then

$$\begin{aligned}\{Da, e\} =& \{a, e\} + (\partial\alpha e + t\{\partial\alpha, e\}) + (t\partial^2\alpha e + \frac{1}{2}t^2\{\partial^2\alpha, e\}) + \cdots + \frac{1}{n!}(nt^{n-1}\partial^n\alpha e + t^n\{\partial^n a, e\}) \\ =& (\{a, e\} + \partial\alpha e) + (t\{\partial\alpha, e\} + t\partial^2\alpha e) + \cdots + \frac{1}{(n-1)!}(t^{n-1}\{\partial^{n-1}a, e\} + t^{n-1}\partial^n\alpha e) = 0\end{aligned}$$

Now consider $A_n \ni a = \sum_{k=0}^{n} t^k \alpha_k$ with $\alpha_k \in \mathcal{O}_X(U)$. Let $\beta_k(a) = \sum_{j=0}^{k} \frac{(-\partial)^j}{j!}\alpha_{k-j}$.

Then $a = \sum_{k=0}^{n} t^k D\beta_k(a)$. Indeed,

$$\begin{aligned}
\sum_{k=0}^{n} t^k D\beta_k(a) &= \sum_{k=0}^{n} t^k \sum_{i=0}^{n} \frac{1}{i!} t^i \partial^i \sum_{j=0}^{k} \frac{(-\partial)^j}{j!} \alpha_{k-j} \\
&= \sum_{i+j+m\leq n} (-1)^j \frac{1}{i!j!} \partial^{i+j} t^{i+j+m} \alpha_m \\
&= \sum_{m=0}^{n} \Big(\sum_{i+j=n-m} (-1)^j \frac{1}{i!j!} \partial^{i+j} t^{i+j} \Big) t^m \alpha_m \\
&= \sum_{m=0}^{n} t^m \alpha_m
\end{aligned}$$

Recall that in Kaledin's paper [Kal], we have a non-trivial trivialization:

$$A_n \cong \mathrm{Ann}_e(A_n)[t]/(t^{n+1}). \tag{4}$$

Consequently, we can define the transition map on $\mathcal{O}_X(U_{ij})[t]/(t^{n+1})$ to be the Poisson isomorphism

$$\phi_{ij} : \mathcal{O}_X(U_i)[t]/(t^{n+1})|_{U_{ij}} \to \mathcal{O}_X(U_j)[t]/(t^{n+1})|_{U_{ij}}, \quad a \to \sum_{k=0}^{n} t^k D_{f_{ij}} \beta_k(a)$$

where $D_{f_{ij}} = \sum_{k=0}^{n} \frac{1}{k!} t^k \partial_{ij}^k$ and $\partial_{f_{ij}} \in \mathrm{Der}(\mathcal{O}_X(U_{ij})[t]/(t^{n+1}))$ is defined by $\partial_{f_{ij}} a = \{\log\ f_{ij} e_i, a\}$. Then since (f_{ij}) satisfy cocycle condition, so are (ϕ_{ij}).

Notice that

$$\{Da, Db\} = \sum_{i+j=0}^{n} t^{i+j} \{\frac{\partial^i}{i!} a, \frac{\partial^j}{j!} b\} = \sum_{k=0}^{n} t^k \frac{\partial^k}{k!} \{a, b\} = D\{a, b\},$$

hence $\{D_f a, D_f b\} = D_f\{a, b\}$. Therefore the bracket defined before is compatible with the transition function thus well-defined given the Leibniz rule and the trivialization 4.

1.3 Extension of line bundles

To do Kalendin's induction , we need to deform the line bundle and the scheme at the same time.

1.3.1 First order

Denote $\xi_{ij} := \{-, \log\ f_{ij}\}$. Then the above 2 also gives a first order deformation.

The short exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{\iota} \mathcal{O}_{X_1}^* \to \mathcal{O}_X^* \to 0,$$

leads to long exact sequence

$$\cdots \to H^1(X, \mathcal{O}_{X_1}^*) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathcal{O}_X) \to \cdots$$

Recall that $[f_{ij}] \in H^1(X, \mathcal{O}_X^*)$ presents our line bundle L. Think of it as in $C^1(X, \mathcal{O}_{X_1}^*)$. Let $\widetilde{d}$ stands for differential in Cech complex $C^{\cdot}(X, \mathcal{O}_{X_1}^*)$. Formally, $(\widetilde{d}f)_{ijk} = f_{ik}^{-1} f_{ij} f_{jk}$.

Viewed as function on U_i, it is

$$\begin{aligned}
(\widetilde{d}f)_{ijk} =&(f_{ik} - \xi_{ij} f_{ik} t) f_{ik}^{-1} f_{ij} \\
=&1 - \xi_{ij} \log\ f_{jk} t \\
=&1 + \{\log\ f_{jk}, \log\ f_{ij}\} t
\end{aligned}$$

Consider the 2-cycle $\gamma_{ij} = \{\log\ f_{ij}, \log\ e_i\}$ with formally differential $d\gamma_{ij} = \{\log\ f_{ij}, \log\ e_i\} - \{\log\ f_{ik}, \log\ e_i\} + \{\log\ f_{jk}, \log\ e_j\}$ viewed as function on U_i equals to

$$\{\log\ f_{ij}, \log\ e_i\} - \{\log\ f_{ik}, \log\ e_i\} + \{\log\ f_{jk}, \log\ f_{ik} e_i\} = \{\log\ f_{jk}, \log\ f_{ij}\} \tag{5}$$

Hence $[f_{ij}(1 + \{\log\ f_{ij}, \log\ e_j\} t)^{-1}] \in H^1(X_1, \mathcal{O}_{X_1}^*)$ is a preimage of L.

More specifically, let

$$f_{ij}(1 + h_{ij} t) : \mathcal{O}_{X_1}(U_i)|_{U_{ij}} \cong L_1(U_i)|_{U_{ij}} = L_1(U_j)|_{U_{ij}} \cong \mathcal{O}_{X_1}(U_j)|_{U_{ij}}$$

be the transition map. To have it to be a $\mathcal{O}_{X_1}(U_{ij})$ module isormorphism, we need

$$f_{ij}(1 + h_{ij} t)(ka) = (1 + \xi_{ij}) k \cdot f_{ij}(1 + h_{ij} t) a, \quad \forall a, k \in \mathcal{O}_{X_1}(U_{ij})$$

which is equivalent to $g_{ij} := h_{ij} - \xi_{ij} \in Aut_{\mathcal{O}_X(U_{ij})}(\mathcal{O}_X(U_{ij})) = (\mathcal{O}_X(U_{ij}))^*$.

On the other hand, cocycle condition can be written as

$$f_{jk}(1 + \xi_{jk} t + g_{jk} t) f_{ij}(1 + \xi_{ij} t + g_{ij} t) = f_{ik}(1 + \xi_{ik} t + g_{ik} t).$$

which is equivalent to $g_{ik} - g_{ij} - g_{jk} = f_{ij}^{-1} \xi_{jk} f_{ij} = \{\log\ f_{jk}, \log\ f_{ij}\}$

Notice that this computation shows that we can extend the line bundle to the deformation induced from itself without assuming the vanishing of $H^2(X, \mathcal{O}_X)$.

Now assume we have obtained $X_1, ..., X_{n-1}, L_1, ...L_{n-1}$. Replace $D_{f_{ij}}$ by $D_{f_{ij}^{n-1}}$ in ϕ_{ij} where f_{ij}^{n-1} is in the form of $f_{ij}\prod_{k=1}^{n-1}(1+g_{ij}^k t^k)^{-1}$ such that $(f_{ij}\prod_{k=1}^{m}(1+g_{ij}^k t^k)^{-1})$ satisfy cocycle condition on X_m for all $m < n$. Then it defines the n-th order deformation X_n given by L_{n-1}.

1.4 Examples

Example 1.1. When $n = 1$, let $a \in \mathcal{O}_X(U_{ij})$, then

$$\alpha_i^1 := \{\log\ e_i, a\} = \{\log\ f_{ij}e_j, a\} = \{\log\ e_j, a\} + \{\log\ f_{ij}, a\} = \alpha_j^1 + \xi_{ij}a.$$

We deform the bracket locally trivially by $\{a + bt, a' + b't\} = \{a, a'\} + (\{a, b'\} + \{b, a'\})t$. It is well-defined because:

$$\begin{aligned}
&\{a + \{a, \log\ e\}t, b + \{b, \log\ e\}t\} \\
=&\{a, b\} + \{a, \{b, \log\ e\}t + \{\{a, \log\ e\}, b\}\} \\
=&\{a, b\} + \{\{a, b\}, \log\ e\}
\end{aligned}$$

$$\begin{aligned}
\text{and } &\{a + \{a, \log\ fe\}t, b + \{b, \log\ fe\}t\} = \{a, b\} + \{\{a, b\}, \log\ fe\} \\
=&\{a, b\} + \{\{a, b\}, \log\ e\} + \{\{a, b\}, \log\ f\}
\end{aligned}$$

These coincide with the discussion in 1.1 cited from [Wie03].

Example 1.2. When $n = 2$, $D_{f_{ij}}a = (1 + \xi_{ij}t + (\frac{1}{2}\xi_{ij}^2 + X_{ij})t^2)Da$. where $\xi_{ij}a = \{\log\ f_{ij}, a\}$ and

$$X_{ij}a = \frac{1}{2}\{\log\ e_i, \{\log\ f_{ij}, a\}\} - \frac{1}{2}\{\log\ f_{ij}, \{\log\ e_i, a\}\} = \frac{1}{2}\{\{\log e_i, \log\ f_{ij}\}, a\}.$$

which coincides with 3 given 5.

Example 1.3. Instead, if one wants to get second order deformation from the extended line bundle L_1, then f_{ij} would be replaced by $f_{ij}^1 = f_{ij}(1 + g_{ij}t)$ where $g_{ij} = \xi_{ij} - \{\log\ f_{ij}, \log\ e_i\}$. Short computation shows that first order deformation coincides with example 1.1. For second order, (again, we obmit subscripts if no confusion would be caused.)

$$\begin{aligned}
D_{f^1}a =&[\{\log(fe), \{\log(fe), a\}\} + \{\log\frac{(1+gt)e}{e}, \{\log\frac{(1+gt)e}{e}, a\}\} \\
&- \{\log(fe), \{\log\frac{(1+gt)e}{e}, a\}\} - \{\frac{(1+gt)e}{e}, \{\log(fe), a\}\}]\frac{t^2}{2} \\
&+(\{\log(fe), a\} - \{\log\frac{(1+gt)e}{e}, a\})t + a.
\end{aligned}$$

Notice that $\log \frac{(1+g)e}{e} t = \frac{ge}{e} t + O(t^2)$ while $\frac{ge}{e} = \frac{\xi e}{e} - \{\log f, \log e\} = \{\log f, \log e\} - \{\log f, \log e\} = 0$.

Hence $D_{f^1} a = (\{\log(fe), \{\log(fe), a\}\} = D_f a$, coinciding with example 1.2 as well.

References

[Kal] D Kaledin, *On the projective coordinate ring of a poisson scheme*, arXiv preprint math.AG/0312134.

[Wie03] Jan Wierzba, *Contractions of symplectic varieties*, Journal of Algebraic Geometry **12** (2003), no. 3, 507–534.