Non-linear Fourier transform and very central sheaves

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0 History

Gamma function: $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ with $\Gamma(n) = (n-1)!$ and $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$.

$$\Gamma(z)\zeta(z) = \int_0^\infty \frac{t^{z-1}}{e^t - 1}dt = \mathcal{M}(f)(z).$$

where $f(t) = \frac{1}{e^t-1}$ and $\mathcal{M}(f)(z) = \int_0^\infty f(t)t^{z-1}dt$.
$\theta(t) := \sum e^{-n^2\pi t}$.

$$\frac{2}{\pi^z}\Gamma(z)\zeta(2z+1) = \mathcal{M}(\theta(t) - 1)(z)$$

Write $\mathcal{F}(f)(z) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i t z}dt$. Then

$$\mathcal{M}(f)(s) = \mathcal{F}(f(e^{-z}))(s).$$

The functional equation is deduced by Riemann from (actually equivalent to) the Poisson summation formula

$$\sum f(n) = \mathcal{F}(f)(n).$$

Example 0.1. Consider the Fourier expansion of a cusp form $f(z) = \sum_{n=1}^{\infty} a_n q^n$ with L-function $L(f,s) = \sum a_n n^{-s}$. Then

$$(2\pi)^{-s}\Gamma(s)L(f,s) = \int_0^\infty f(iy)y^s dy = \mathcal{M}(yf(iy))(s)$$

Given a certain type of function f on the adele ring of a number field F, resp. $\mathrm{M}_n(F)$ and a representation π of the idele group of F, resp. $\mathrm{GL}_n(F)$, one can define the associated zeta integral $\mathcal{Z}(f,\pi,s)$. The standard properties (meromorphic continuation, functional equation) of $\mathcal{Z}$ can be deduced from the Poisson summation formula involving the Fourier transform. One can view $\mathcal{Z}(-,\pi,s)$ as a distribution on the space of functions. The information of the poles of this distribution is encoded in the L-function $L(\pi,s)$.

1 Non-linear transform

1.1 Braverman-Kazhdan story

We work over $k = \overline{\mathbb{F}}_q$ only for this subsection. Write F for the Frobenious of k. Let G be a reductive group defined over $\mathbb{F}_q$.

Fact: $\widehat{G^F}$ is classified by the set of F-stable semisimple conjugacy classes of $G^\vee$, Lusztig series $LS(G)$.

Any representation $\rho : G^\vee \to GL(V)$ commuting with F induces a map

$$\tau : LS(G) \to LS(GL(V)).$$

For any character $\psi : \mathbb{F}_q \to \overline{Q}_\ell$, let $\mathrm{tr} : \mathrm{GL}(V_\rho)^F \to \mathbb{F}_q$ be the trace. First consider the Fourier kernel: $\phi' = \psi \circ \mathrm{tr}$ and the associated Fourier transform

$$\mathcal{F}'(f)(x) = \sum_{y \in GL(V_\rho)^F} \phi'(xy) f(y).$$

Let $\gamma'(\chi) := \sum_g \frac{\phi(g)\chi(g)}{\chi(1)}$. Then $\mathcal{F}'(\chi) = \gamma'(\chi)\chi^\vee$ and $\phi' = \sum \chi(1)\gamma'(\chi)\chi^\vee$.

Fact: γ' is constant on $LS(GL(V_\rho))$. As a result, we can define $\gamma = c\gamma' \circ \tau$ for some constant c.

Define $\phi = \sum \chi(1)\gamma(\chi)\chi^\vee$ and

$$\mathcal{F}(f)(x) = \sum_{y \in GL(V_\rho)^F} \phi(xy) f(y).$$

1.2 Geometrization of $\mathcal{F}$

1.3 Induction and restriction

- If $Av(\mathscr{G}')$ is supported on T, then $\mathrm{Ind}_T^G(\mathscr{G}) * \mathscr{G}' \cong \mathrm{Ind}_T^G(\mathscr{G} * \mathrm{Res}_T^G(\mathscr{G}'))$.
- $i^0 HC = Av$
- $\mathscr{G}' = \mathrm{Res}_T^G \mathscr{G}$ comes with a W-equivariant structure such that $\mathscr{G} = \mathrm{Ind}_T^G(\mathscr{G}')^W$.
- If $\mathscr{G}$ is W-equivariant, then $(\mathrm{Ind}_T^G \mathrm{Res}_T^G \mathscr{G})^W = \mathscr{G}$ as W-equivariant sheaves.

1.4 Bessel sheaves

Let $\mathcal{L}_\psi$ be the Artin-Schreier sheaf on $\mathbb{G}_a$ associated to ψ coming from the the Lang map $1 \to \mathbb{G}_a^F \to \mathbb{G}_a \xrightarrow{L} \mathbb{G}_a \to 1$. ($\mathrm{Tr}(\mathcal{L}_\psi) = \psi$.) Assume that $V_\rho = \bigoplus_{i=1}^r V_{\lambda_i}$ for $\lambda_i \in X^*(T^\vee)$. Define

$$\Phi_T := pr_{\lambda,!} \mathrm{tr}^* \mathcal{L}_\psi[r]$$

where $pr_\lambda := \prod \lambda_i : \mathbb{G}_m^r \to T$. Φ_T carries a natural W-action.

The ρ-Bessel sheaf associated to ψ is defined to be

$$\Phi = \mathrm{Ind}_{T \subset B}^G(\Phi_T)^W,$$

which comes from a Weil structure $F^*\Phi \cong \Phi$. Then conjectured by Braverman-Kazhdan and proved by several groups of people in different generalities (most general by Tsao-Hsien Chen), one has

Theorem 1.1.

$$\mathrm{Tr}(\Phi) = \phi.$$

Definition 1.2. We call $\mathcal{G} \in D^G(G)$ very central if $\pi_!(\mathscr{G})$ is supported on $T = B/U$.

By [ref], the following statement implies theorem 1.1:

Theorem 1.3. Φ is very central.

Moreover, Φ defines an involution functor

$$\mathscr{F} := (-) * \Phi : D^b_c(G, \overline{Q}_\ell) \to D^b_c(G, \overline{Q}_\ell)$$

When $G = GL_n$, $\mathscr{F}$ is exactly the restriction of the linear Fourier-Deligne transform in that case. F is expected to be exact and commutes with the induction functor. The commutativity follows from theorem 1.3.

By certain universal local acyclicity argument, it turns out to be sufficient to work over $\mathbb{C}$. In particular, the counterpart of $\mathcal{L}_\psi$ is the exponential $\mathcal{D}$-module $\mathcal{L}_c := D/D(x\partial_x - c)$ for $c \in \mathbb{C}^\times$. We will work over $\mathbb{C}$ for the rest of these notes.

2 $\mathcal{E}_\theta$ and $\mathcal{M}_\theta$

For $\xi \in T^\vee \cong \widehat{\pi_1(T)}$ (identifying $\pi_1(T) \cong X_*(T) \subset \mathfrak{t}$), let $\mathcal{L}_\xi$ be the associated Kummer local system on T. Let $W_\xi := Stab_W(\xi)$, $S = S(\mathfrak{t})$ and $S_\xi = S/S_+^{W_\xi}$, the pullback of the skycraper sheaf at $0 \in \mathfrak{t}^\vee/W_\xi$. Then $W_\xi \ltimes \pi_1(T)$ acts on S_ξ by

$$\rho'_\xi(w, t)u = w(tu).$$

Take $\rho_\xi := \rho'_\xi \otimes \xi$. Let $\mathcal{E}_\xi$ be the W_ξ-equivariant local system on T corresponding to ρ_ξ. The induction $\mathrm{Ind}_{W_\xi \ltimes \pi_1(T)}^{W \ltimes \pi_1(T)} \rho_\xi$ only depends on $\theta = [\xi] \in T^\vee/W_\xi$. Let $\mathcal{E}_\theta$ denote the corresponding W-equivariant local system on T. Put $\mathcal{M}_\theta := (\mathrm{Ind}_T^G \mathcal{E}_\theta)^W$.

2.1 Drinfeld center

For $\xi \in \mathfrak{t}^\vee/\Lambda \cong T^\vee$. Let $M_{\xi,\xi^{-1}}$ be the category of (left) G-equivariant (right) $T \times T$-monodromic $D_{\widetilde{Y}}$ module with generalized monodromy (ξ, ξ^{-1}). Let M_ξ be the category of (left) G-equivariant (right) T-monodromic D_Y module with generalized monodromy ξ. Let H_ξ be the G-equivariant T-monodromic $D(\widetilde{\mathcal{B}})$-module with generalized monodromy ξ ($b(xU) = bxb^{-1}U$, $t(xU) = txU$).

Let $i : \widetilde{\mathcal{B}} \to Y$ be embedding to the second factor. There is a monoidal equivalence

$$i^0 := i^![\dim \widetilde{\mathcal{B}}] : (D(M_\xi), *) \xrightarrow{\sim} (D(H_\xi), *)$$

with inverse $\mathrm{Ind}_B^G i_\dagger[\dim \mathcal{B}]$.

Define $\widetilde{U} := U \otimes_Z S(\mathfrak{t})$ and $\widetilde{U}_{\hat{\lambda}}$ be the completion at $\lambda \in \mathfrak{t}^\vee$. Note that there is inclusion $\widetilde{U}-\mathrm{mod}_\lambda \subset \widetilde{U}_{\hat{\lambda}}-\mathrm{mod}$.

Let $\mathcal{HC}_{\hat{\lambda}}$ denote the category of finitely generated H-C bimodules over $\widetilde{U}_{\hat{\lambda}}$. The objects in $\mathcal{HC}_{\hat{\lambda}}$ are $\widetilde{U}_{\hat{\lambda}} \otimes \widetilde{U}_{\widehat{-\lambda-2\rho}}$-modules such that the diagonal $\mathfrak{g}$ action is locally finite.

Fact: for dominant regular λ with ξ as the image in $T^\vee$,

$$R\Gamma^{\hat{\lambda}, \widehat{-\lambda-2\rho}} : (D(M_{\xi,\xi^{-1}}), *) \xrightarrow{\sim} (D(\mathcal{HC}_{\hat{\lambda}}), \otimes^L)$$

Define $\mathcal{Z}_\lambda := \widetilde{U}_{\hat{\lambda}} \otimes_S S_\xi^\lambda \in \mathcal{HC}_{\hat{\lambda}}$. Then

Proposition 2.1. $\mathcal{Z}_\lambda$ defines an embedding

$$\mathcal{HC}_{\hat{\lambda}} \to Z(\mathcal{HC}_{\hat{\lambda}}, \otimes)$$

Proposition 2.2. $R\Gamma^{\hat{\lambda}, \widehat{-\lambda-2\rho}} \circ \pi^*(i^0)^{-1}(\mathcal{E}_\xi) = \mathcal{Z}_\lambda$.

Proof. First of all, $\mathcal{E}'_\xi := \mathrm{Ind}_B^G i_\dagger[\dim \mathcal{B}](\mathcal{E}_\xi) = a_\dagger(\mathcal{E}_\xi \boxtimes \Delta_\dagger \mathcal{O}_\mathcal{B})$ with

$$a : T \times Y \to Y, t(gU, g'U) = (gt^{-1}U, g'U).$$

Since

$$R\Gamma(\Delta_\dagger \mathcal{O}_\mathcal{B} \otimes p_2^* \omega_\mathcal{B}) \cong \widetilde{U}$$

we conclude

$$R\Gamma^{\hat{\lambda}, \widehat{-\lambda-2\rho}}(\pi^* \mathcal{E}'_\xi) = R\Gamma^{\hat{\lambda}}(a_\dagger(\mathcal{E}_\xi \boxtimes (\Delta_\dagger \mathcal{O}_\mathcal{B} \otimes p_2^* \omega_\mathcal{B}))) = (R\Gamma(\mathcal{E}_\xi) \otimes_{S(\mathfrak{t})} \widetilde{U})_{\hat{\lambda}}$$

□

Proposition 2.3. $Av(\mathcal{M}_\theta) = \mathcal{E}_\theta$.

Proof. Fact: $R\Gamma^{\hat{\lambda}, \widehat{-\lambda-2\rho}} \circ \pi^0 \circ HC : CS_\theta \to Z(\mathcal{HC}_{\hat{\lambda}}, \otimes)$. Therefore there exists $M' \in CS_\theta$ such that

$$R\Gamma^{\hat{\lambda}, \widehat{-\lambda-2\rho}} \circ \pi^0 \circ HC(M') = R\Gamma^{\hat{\lambda}, \widehat{-\lambda-2\rho}} \circ \pi^0(\mathcal{E}'_\xi).$$

Therefore $HC(M') \cong \mathcal{E}'_\theta$ and

$$\mathcal{E}_\theta = i^0 \mathcal{E}'_\theta = i^0 \circ HC(M') = Av(M') \cong \mathrm{Res}_T^G(M')$$

Now it remains to show the W-action on $\mathcal{E}_\theta$ such that $M' \cong \mathrm{Ind}_T^G(\mathcal{E}_\theta)^W$ coincides with the one we use to define $\mathcal{M}_\theta$. □

3 Mellin transform

Recall that $\mathcal{D}(T) = \mathbb{C}[x_i^{\pm 1}][v_i]/([v_i, x_j] = \delta_{ij} x_j)$ for $v_i = x_i \partial_{x_i}$. Write $\Lambda := X^*(T)$. The Mellin transform is defined to be

$$\mathscr{M} : \mathcal{D}_T\text{−mod} \to QCoh^\Lambda(\mathfrak{t}^\vee), \mathcal{M} \mapsto \Gamma(\mathcal{M})$$

which extends to

$$\mathscr{M}_{W_\xi} : \mathcal{D}_T\text{−mod}^{W_\xi} \to QCoh^{W_\xi \ltimes \Lambda}(\mathfrak{t}^\vee), \mathcal{M} \mapsto \Gamma(\mathcal{M})$$

where $x_i \in \Lambda$ acts in the obvious way. We have $\mathscr{M}(\mathscr{G} * \mathscr{G}') = \mathscr{M}(\mathscr{G}) \otimes_S \mathscr{M}(\mathscr{G}')$.

Define $S_\xi^\mu := \tau_\mu^* S_\xi$. Then $S_\xi^\mu = S_\xi$ as vector spaces and for any $s \in S, m \in S_\xi^\mu$, $s \cdot m = a_\lambda(s)m$ where $a_\lambda(s)(x) = s(x + \lambda)$. The isomorphisms $\tau_\lambda(S_\xi^\mu) = S_\xi^{\mu+\lambda}$ and $u_w^\mu : S_\xi^\mu = w^* S_\xi^{w(\mu)}$ define a $W_\xi \ltimes \Lambda$-equivariant structure on $\bigoplus_{\mu \in \xi + \Lambda} S_\xi^\mu$.

$$\mathscr{M}_{W_\xi}(\mathcal{E}_\xi) = \bigoplus_{\mu \in \xi + \Lambda} S_\xi^\mu$$

On the other hand,

$$\mathscr{M}(\Phi) = \mathbb{C}[x_i^{\pm 1}] e^{\sum c x_i} \otimes_{\mathbb{C}[v]} S$$

$\mathbb{C}[v]$ acts on S via $d\lambda$.

Proposition 3.1. $\Phi_T * \mathcal{E}_\xi = \mathcal{E}_\xi$ as W_ξ-equivariant local system on T.

Proof. Notice that $\Phi_T = \Phi_1 * \Phi_2 * \cdots * \Phi_r$. For simplification, we will write $\lambda := \lambda_i$ $x = x_i$, $v = v_i$ below.

$\mathscr{M}(\Phi_i * \mathcal{E}_\xi) = \mathscr{M}(\Phi_i) \otimes_S \mathscr{M}(\mathcal{E}_\xi) \cong \bigoplus_{\mu\in[\xi]} \mathbb{C}[x^{\pm 1}]e^{cx} \otimes_{\mathbb{C}[v]} S_\xi^\mu$. $\mathbb{C}[v]$ acts on S_ξ^μ via $d\lambda$.

Take $n_i = [\lambda(\mu)]$ and define

$$E_{i,\mu} = \mathbb{C}[v]x^n e^{cx} \subset \mathbb{C}[x^{\pm 1}]e^{cx}$$

Then

$$\mathscr{M}(\Phi_i * \mathcal{E}_\xi) = \bigoplus_{\mu\in[\xi]} E_{i,\mu} \otimes_{\mathbb{C}[v]} S_\xi^\mu = \bigoplus_{\mu\in[\xi]} S_\xi^\mu = \mathscr{M}(\mathcal{E}_\xi)$$

Hence $\Phi_i * \mathcal{E}_\xi = \mathcal{E}_\xi$.

Consider

$$E_\mu := \mathbb{C}[v_i] \prod x_i^{n_i} e^{c\sum x_i} \subset \mathbb{C}[x_i^{\pm 1}]e^{c\sum x_i}$$

Then

$$\mathscr{M}(\Phi * \mathcal{E}_\xi) = \bigoplus_{\mu\in[\xi]} E_\mu \otimes_{\mathbb{C}[v]} S_\xi^\mu = \bigoplus_{\mu\in[\xi]} S_\xi^\mu$$

where the last isomorphism is defined by $\prod x_i^{n_i} \otimes s \mapsto s$.

It remains to show this isomorphism is W-equivariant. □

Corollary 3.2. $\Phi * \mathcal{M}_\theta = \mathcal{M}_\theta$.

Proof.

$$\Phi * \mathcal{M}_\theta = (\mathrm{Ind}_T^G(\Phi) * \mathcal{M}_\theta)^W = \mathrm{Ind}_T^G(\Phi * \mathcal{E}_\xi)^{W_\xi} = \mathrm{Ind}_T^G(\mathcal{E}_\xi)^{W_\xi} = \mathcal{M}_\theta$$

□

Proof. (of theorem 1.3) By proposition 3.1, proposition 3.1 and corollary 3.2,

$$Av(\Phi) * \mathcal{E}_\theta = Av(\Phi * \mathcal{M}_\theta) = Av(\mathcal{M}_\theta) = \mathcal{E}_\theta = \Phi_T * \mathcal{E}_\theta = \mathrm{Res}_T^G(\Phi) * \mathcal{E}_\theta,$$

which is to say $cone(\alpha) * \mathcal{E}_\theta = 0$ for the natural map $\alpha : \mathrm{Res}_T^G(\Phi) \to Av(\Phi)$. Note that $\mathcal{E}_\theta = \bigoplus_{\xi\in\theta} \mathcal{E}_\xi$ where $\mathcal{E}_\xi$ is a local system of generalized monodromy ξ. Therefore $cone(\alpha) * \mathcal{E}_\xi = 0$ for all $\xi \in T^\vee$, which implies $cone(\alpha) = 0$ by standard argument. □