Finite-dimensional representations of rational Cherednik algebras

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1 Category $\mathcal{O}(\mathrm{H}_c)$

Throughout, we work over $\mathbb{C}$ and focus on type A_{n-1}[1]: $W = S_n$, $\mathrm{rk}(\mathfrak{h}) = n-1$. Let $S \subset W$ be the subset consisting of reflections. For any $s \in S$, let α_s, resp. $\alpha_s^\vee$ denote the associated root, resp. coroot. Fix a parameter $c \in \mathbb{C}$.

Definition 1.1. The rational Cherednik algebra is defined to be

$$\mathrm{H}_c := \frac{\mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}[\mathfrak{h}^*] \ltimes W}{([y,x] - \langle x, y\rangle + c\sum_{s\in S}\langle x, \alpha_s^\vee\rangle\langle \alpha_s, y\rangle s,\ x \in \mathbb{C}[\mathfrak{h}], y \in \mathbb{C}[\mathfrak{h}^*])}$$

Similar to $U(\mathfrak{g})$, we have PBW-decomposition $\mathrm{H}_c \cong \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}[W]$, which inspires an analogous definition of category $\mathcal{O}(\mathrm{H}_c)$:

Definition 1.2. $\mathcal{O}(\mathrm{H}_c)$ is a full subcategory of H_c-mod whose objects are finitely generated over H_c such that the $\mathfrak{h}$ action is locally nilpotent.

As one can check, any $M \in \mathcal{O}(\mathrm{H}_c)$ is finitely generated over $\mathbb{C}[\mathfrak{h}]$. Hence $\mathcal{O}(\mathrm{H}_c)$ is an abelian category. It shares similar good properties with the BGG category $\mathcal{O}$. To name a few: [Gua03]

1. Every $M \in \mathcal{O}(\mathrm{H}_c)$ has finite length
2. enough projectives
3. BGG reciprocity
4. highest weight category
5. ...

As these spoilers suggest, we should have standard modules in this category, i.e, verma modules: for any W-representation τ, inflate it to $\mathbb{C}[\mathfrak{h}^*] \ltimes W$ and take

$$M_c(\tau) = \mathrm{H}_c \otimes_{\mathbb{C}[\mathfrak{h}^*]\ltimes W} \tau.$$

$M_c(\tau)$ has a maximal proper submodule with an irreducible quotient $L_c(\tau)$. $L_c(\tau)$'s give a complete list of irreducible objects in $\mathcal{O}(\mathrm{H}_c)$. For generic c, $M_c(\tau)$ is simple.

Let $\mathrm{e} := \frac{1}{|W|}\sum_{w\in W} w$ be the idempotent. Then we also have the spherical Cherednik algebra $\mathrm{eH}_c\mathrm{e}$.

Theorem 1.3 ([BEG03])**.** Only when $c = \frac{m}{n}$ with $m \in \mathbb{Z}$, $(m,n) = 1$, H_c has nontrivial finite dimensional representations. In this case, when $c > 0$, resp. $c < 0$ the only irreducible inite dimensional representation is $L_c(\mathrm{triv})$, resp. $L_c(\mathrm{sign})$.

[1]But one can deduce most of the results below for other types with more careful treatment.

2 Character of L_c

Consider the following three elements:

$$\mathbf{h} := \frac{1}{2}\sum(x_i y_i + y_i x_i), \mathrm{E} = -\frac{1}{2}\sum x_i^2, \mathrm{F} = \frac{1}{2}\sum y_i^2.$$

One can compute that $\mathbf{h}, \mathrm{E}, \mathrm{F}$ form a $\mathfrak{sl}_2$ triple and $\mathbf{h}$ commutes with W. Any $M \in \mathcal{O}(\mathrm{H}_c)$ has a generalized $\mathbf{h}$-weight space decomposition $M = \oplus M_a$ such that M_a has finite dimension, which allows us to define a $\mathbf{h}$-graded character of M by

$$\mathrm{Tr}_M(w \cdot t^{\mathbf{h}}) := \sum \mathrm{Tr}(w, M_a) t^a$$

Assume $c = \frac{m}{n} > 0$ with $m \in \mathbb{Z}$, $(m, n) = 1$. Write $\mathrm{L}_c = \mathrm{L}_c(\mathrm{triv})$ and $\mu := \frac{(m-1)(n-1)}{2}$. Then the $\mathbf{h}$-graded character of L_c can be expressed as follows. Note that here and below, det is taken with respect to a linear operator on $\mathfrak{h}$ if no otherwise specified.

Theorem 2.1 ([BEG03]).

$$\mathrm{Tr}_{\mathrm{L}_c}(w \cdot t^{\mathbf{h}}) = t^{-\mu}\frac{\det(1 - t^m \cdot w)}{\det(1 - t \cdot w)}.$$

Corollary 2.2 ([BEG03]).

$$\mathrm{Ch}_{\mathrm{eL}_c}(t) = t^{-\mu}\prod_{i=2}^{n}\frac{1 - t^{i+m-1}}{1 - t^i}$$

Proof. (of the corollary) Let $\Omega^\bullet(\mathfrak{h})^W$ denote the space of W-invariant differential forms on $\mathfrak{h}$. Then its double graded Hilbert series (x for rank of the form and t for homogeneous degree of the polynomial) is:

$$\frac{1}{|W|}\sum_{w \in W}\frac{\det(1 + wx)}{\det(1 - wt)}$$

Here we use the following exercises: for linear operator $A : V \to V$ on finite-dimensional vector space V,

$$\sum t^n \mathrm{Tr}(\mathrm{Sym}^n A) = \frac{1}{\det(1 - tA)}$$

and

$$\sum t^n \mathrm{Tr}(\wedge^n A) = \det(1 + tA)$$

Let $p_2, \ldots, p_n$ be the elementary symmetric polynomials. Then it is known that $\Omega^\bullet(\mathfrak{h})^W$ is generated over $\mathbb{C}[\mathfrak{h}]^W$ as a exterior algebra by $\mathrm{d}p_2, \ldots, \mathrm{d}p_n$.

Hence we have another ways to compute its double graded Hilbert series:

$$\prod_{i=2}^{n}\frac{1 + xt^{i-1}}{1 - t^i}$$

The corollary follows from plugging $x = -t^m$ into both expressions. $\square$

To prove the theorem, we need a BGG resolution of L_c which will be lent by the representation theory of the Hecke algebra and the bridge is the KZ functor.

2.1 Dunkl operators and KZ functor

Let $\mathfrak{h}_r = \{\delta \neq 0\}$ denote the regular locus of $\mathfrak{h}$. The action of H_c on $\mathbb{C}[\mathfrak{h}] \cong \mathrm{H}_c \otimes_{\mathbb{C}[\mathfrak{h}^*]\ltimes W} \mathbb{C}$ gives the Dunkl embedding of H_c into $D(\mathfrak{h}_r) \ltimes W$. In particular, the Dunkl operator associated to y_i is

$$\partial_{x_i} - c\sum_{s\in S} \frac{\alpha_s(x_i}{\alpha_s}(1-s). \tag{1}$$

Since any H_c-module M is also a $\mathbb{C}[\mathfrak{h}]$ module, we can take its localization with respect to $\delta := \prod_{\alpha\in\Delta^+}\alpha$, or equivalently, restrict M to $\mathfrak{h}_r = \{\delta \neq 0\}$. Since the Dunkl operators only have poles along the root hyperplanes, we have the identity

$$(\mathrm{H}_c)_\delta \cong D(\mathfrak{h}_r) \ltimes W.$$

Note that we have equivalence of categories: $\mathrm{Coh}(D(\mathfrak{h}_r) \ltimes W) \cong \mathrm{Coh}^W(D(\mathfrak{h}_r)) \cong \mathrm{Coh}(D(\mathfrak{h}_r/W))$ given by the functor $(-)^W$.

For the Verma $M_c(\tau)$, $(M_c(\tau)_\sigma)^W = \mathbb{C}[\mathfrak{h}_r] \otimes_W \tau$ is then a $D(\mathfrak{h}_r/W)$-module. It can also be viewed as a vector bundle of flat connection

$$\nabla_{KZ} := \mathrm{d} - c\sum_{s\in S}\frac{\mathrm{d}\alpha_s}{\alpha_s}(1-s). \tag{2}$$

which is the KZ (Knizhnik- Zamolodchikov) connection.[2]

Monodromy theorem tells us this flat vector bundle produces a representation of $B_W := \pi_1(\mathfrak{h}_r/W)$, which is the (type A) braid group. Let $T_s \in \pi_1(\mathfrak{h}_r/W)$ be the class corresponding to a small counterclockwise circle around the hyperplane $\{\alpha_s = 0\}$ modulo W.

The Hecke algebra of (W, S) with parameter $q : S \to \mathbb{C}$ is defined to be

$$\mathcal{H}_W(q) := \frac{\mathbb{C}[B_W]}{((T_s-1)(T_s-q_s), s\in S)}.$$

Since s has eigenvalues $1, -1$, the monodromy around the hyperplane $\{\alpha_s = 0\}$ given by $(M_c(\tau), \nabla_{KZ})$ is diagonalizable with eigenvalues 1 and $e^{2\pi i c}$. Thus we see that the monodromy representation given by $(M_c(\tau), \nabla_{KZ})$ factors through $\mathcal{H}_W(e^{2\pi i c})$. In general, we have the KZ functor (GGOR):

$$KZ : \mathcal{O}(\mathrm{H}_c) \to \mathcal{H}_W(e^{2\pi i c}).$$

Moreover, let $\mathcal{O}(\mathrm{H}_c)_{tor}$ be the Serre subcategory of modules supported on $\{\delta = 0\}$. Then

Theorem 2.3 ([GGOR03])**.** For all $c \in \mathbb{C}$, KZ induces an equivalence

$$\mathcal{O}(\mathrm{H}_c)/\mathcal{O}(\mathrm{H}_c)_{tor} \to \mathcal{H}_W(e^{2\pi i c}).$$

People know the representation theory of the Hecke algebra well:

For any partition λ of n, we have the associated Specht module $S^\lambda \subset \mathcal{H}_W(q)$ flat with respect to q. Let τ_λ be the associated representation of S_n. Assume $q = e^{2\pi i c}$ and $c = \frac{m}{n}$ with $(m, n) = 1$. Then

Theorem 2.4 ([GGOR03])**.** $KZ(M(\tau_\lambda)) = S^\lambda$.

[2] It is also immediate that ∇_{KZ} is flat since $\frac{\mathrm{d}\alpha_s}{\alpha_s} = \mathrm{d}\log\alpha_s$ is closed.

Let λ^i be the hook corresponding to $\wedge^i\mathfrak{h}$. Then S^λ is irreducible iff $\lambda \neq \lambda^i$, $\forall i = 1, 2, ...n-2$. When $0 < i < n-1$, S^{λ^i} has nonzero composition factors D^{λ^i} and $D^{\lambda^{i-1}}$ both with multiplicity 1. These give a complete list of irreducible representations of $\mathcal{H}_W(q)$. Moreover, as we will see in the proof of theorem 2.1, when $j > i$, $\kappa(c, \wedge^j\mathfrak{h}) > \kappa(c, \wedge^i\mathfrak{h})$ and so $\mathrm{Hom}(M(\wedge^j\mathfrak{h}), M(\wedge^i\mathfrak{h}) = 0$. Hence $\mathrm{Hom}(M(\lambda), M(\mu)) \neq 0$ is only possible when $\lambda = \wedge^i\mathfrak{h}$ and $\mu = \wedge^{i-1}\mathfrak{h}$. This is also a sufficient condition:

Proposition 2.5 ([BEG03])**.** $\dim \mathrm{Hom}(M(\wedge^i\mathfrak{h}), M(\wedge^{i-1}\mathfrak{h})) = 1$, $i = 1, 2, .., n-1$.

Proof. First of all, ∇_{KZ} has regular singularity since it only has first order poles. As a result, when $0 < i < n-1$, S^{λ^i} is reducible, so is $M(\wedge^i\mathfrak{h})$ by the classical Riemann-Hilbert correspondence. Therefore, there is a nontrivial map from another standard module to $M(\wedge^i\mathfrak{h})$, which has to be $M(\wedge^{i+1}\mathfrak{h})$ given the discussion above. This suggests that for $0 < i < n-1$ we have short exact sequence

$$0 \to D^{\lambda^i} \to S^{\lambda^i} \to D^{\lambda^{i-1}} \to 0. \tag{3}$$

As $D^{\lambda^0} = S^{\lambda^0}$, we bave a nonzero map $M(\mathfrak{h})|_{\mathfrak{h}_r} \to M(\mathrm{triv})|_{\mathfrak{h}_r}$. Let N be the intersection of the preimage of this map and $M(\mathfrak{h})$. Then there exists some verma $M(\lambda)$ mapping to N nontrivially. By composition, we have maps $M(\lambda) \to M(\mathfrak{h})$ and $M(\lambda) \to M(\mathrm{triv})$. By the discussion above, λ has to be $\mathfrak{h}$ and the proposition is proved. □

Hence we have a complex with nonzero differential

$$0 \to M_c(\wedge^{n-1}\mathfrak{h}) \to \cdots \to M_c(\mathfrak{h}) \to M_c(\mathrm{triv}) \to \mathrm{L}_c \to 0.$$

To show it is exact, it suffices to prove that

Proposition 2.6. For $0 \leq i < n-1$, we have short exact sequence

$$0 \to \mathrm{L}_c(\wedge^{i+1}\mathfrak{h}) \to M(\wedge^i\mathfrak{h}) \to \mathrm{L}_c(\wedge^i\mathfrak{h}) \to 0$$

Proof. Let $J_c(\wedge^i\mathfrak{h})$ be the maximal proper submodule of $M(\wedge^i\mathfrak{h})$. Then 3 tells us that $KZ(J_c(\wedge^i\mathfrak{h})) = D^{\lambda^i}$ and $KZ(\mathrm{L}_c(\wedge^i\mathfrak{h})) = D^{\lambda^{i-1}}$. Hence $J_c(\wedge^i\mathfrak{h})$ has composition factor $\mathrm{L}_c(\wedge^{i+1}\mathfrak{h})$ and possibly $\mathrm{L}_c(\mathrm{triv})$. But $\kappa(c, \mathrm{L}_c(\mathrm{triv})$ is smaller than the lowest weight in $J_c(\wedge^i\mathfrak{h})$. As a result, $J_c(\wedge^i\mathfrak{h}) \cong \mathrm{L}_c(\wedge^{i+1}\mathfrak{h})$. □

Now we can prove theorem 2.1

Proof. (of theorem 2.1)Let $\kappa(c, \tau)$ be the lowest $\mathbf{h}$-weight of $M(\tau)$. Then since

$$\mathbf{h} = \sum x_i y_i + \frac{\dim \mathfrak{h}}{2} - c \sum s,$$

the lowest weight $\kappa(c, \tau) = \frac{n-1}{2} - c\sum_{s\in S} s$ where $\sum_{s\in S} s$ is the scalar of this central element acting on the representation τ. In particular

$$\kappa(c, \wedge^i\mathfrak{h}) = \frac{n-1}{2} - c|S|\left(\frac{\binom{n-2}{i} - \binom{n-2}{i-1}}{\binom{n-1}{i}}\right) = \frac{n-1}{2} - c(\frac{n(n-1)}{2} - in) = -\mu + im.$$

By BGG resolution, we compute

$$\begin{aligned}\mathrm{Tr}_{\mathrm{L}_c}(w\cdot t^{\mathbf{h}}) &= \sum_{i=0}^{n-1}(-1)^i\mathrm{Tr}_{M_c(\wedge^i\mathfrak{h})}(w\cdot t^{\mathbf{h}})\\ &= \sum_{i=0}^{n-1}(-1)^i\frac{\chi_{\wedge^i\mathfrak{h}}(w)\kappa(c,\tau)}{\det(1-t\cdot w)}\end{aligned}$$

Now theorem 2.1 follows from

$$\sum_{i=0}^{n-1}(-1)^i\chi_{\wedge^i\mathfrak{h}}(w)t^{im} = \det(1-t^m\cdot w). \qquad \square$$

3 Relation with Hilbert schemes

Taking limit when $t\to 1$ for the character formulas, we get

$$\dim \mathrm{L}_c = m^{n-1}, \quad \dim \mathrm{eL}_c = \frac{1}{m}\binom{m+n-1}{n}.$$

In particular, $\dim \mathrm{L}_{1+\frac{1}{n}} = (n+1)^{n-1} = \dim \mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]/(\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*])_+$ and the later is the two-parameter Catalan number $C(m,n)$.

We also see these numbers when we work with Hilbert schemes: Let Hilb_0 denote the zero fiber of the Hilbert-Chow map and P the Procisi bundle to Hilb_0. Then

$$\dim(\mathrm{H}^0(\mathrm{Hilb}_0,\mathcal{O}(k))) = C(nk+1,n)$$

and

$$\dim(\mathrm{H}^0(\mathrm{Hilb}_0,P\otimes\mathcal{O}(k))) = (kn+1)^{n-1}$$

Recall that we have a $\mathbb{C}^*\times\mathbb{C}^*$ action on Hilb, which gives a bigrading on $\mathrm{H}^0(\mathrm{Hilb}_0,\mathcal{O}(k))$, resp. $\mathrm{H}^0(\mathrm{Hilb}_0,P\otimes\mathcal{O}(k))$.

On the other hand, we have order filtration on L_c and eL_c and the $\mathbf{h}$-action endow the associated graded with an extra grading.

Proposition 3.1 ([GS06])**.** When $c = k+\frac{1}{n}$, we have canonical bigraded isomorphisms

$$\begin{aligned}\mathrm{gr}\ \mathrm{L}_c &\cong \mathrm{H}^0(\mathrm{Hilb}_0,P\otimes\mathcal{O}(k)),\\ \mathrm{gr}\ \mathrm{eL}_c &\cong \mathrm{H}^0(\mathrm{Hilb}_0,\mathcal{O}(k)).\end{aligned}$$

References

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