Infinitesimal G equivalence implies G-equivalence:

$$\frac{d}{dt}\phi e^{tX} = \phi X e^{tX} = X\phi e^{tX}$$

$$\frac{d}{dt}e^{tX}\phi = Xe^{tX}\phi$$

satisfy same ODE and initial value.

Finite-dimensional subspace is $\mathfrak{g}$ stable iff it is G stable: $e^{tX}v = \sum X^n/n!v$

1.a. Assume G is a connected Lie group. Show $I \subset \mathbb{C}[\mathfrak{g}^*]$ is Poisson ideal iff I is G-stable under the adjoint action.
(The Poisson structure is defined to be $\{f,g\} = \sum_{i,j,k} c_{ij}^k x_k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$ where $c_{i,j}^k$ are the structure constants.)

Proof.

$$\left.\frac{d}{dt}\right|_{t=0} f \circ Ad^*(e^{tX_j}) = \sum_i \frac{\partial f}{\partial x_i}(ad^* X_j)_i = \sum_{i,k} \frac{\partial f}{\partial x_i} c_{ij}^k x_k$$

To see that $(ad^* X_j)_i = \sum_k c_{ij}^k x_k$: $\langle X_k^*, [X_\alpha, X_j]\rangle = c_{\alpha,j}^k \Rightarrow ad^* X_j(X_k^*) = \sum_k c_{\alpha,j}^k X_\alpha^*$.

Hence $f \circ Ad^* g \in I, \forall f \in I \Rightarrow \{I, x_j\} \subset I, \forall j$. Since $\{,\}$ observes Leibniz rule, it follows that $\{I, \mathbb{C}[\mathfrak{g}^*]\} \subset I$.

On the other hand, $f \circ Ad^*(e^{tX}) = f(\sum ad(X)^n/n!) \in I$ when $X = X_j$ if $\{I, \mathbb{C}[\mathfrak{g}^*]\} \in I$. Since G is connected, it is generated by the image of exponential map as a group. □

b. $f \in \mathbb{C}[\mathfrak{g}^*]$ is fixed by G iff $\{f,h\} = 0, \forall h \in \mathbb{C}[\mathfrak{g}^*]$.

Proof. Same. □

c. The natrual Poisson sturcure in $\mathbb{C}[T^*G]$ is invariant under left and right translation by G. Thus, it gives a Poisson bracket on $\mathbb{C}[T^*G]^{\text{G-left invariant}} = \mathbb{C}[\mathfrak{g}^*]$, which agrees with the one considered in (a).

Proof. Invariance under left and right translation means the invariance of the canonical symplectic form on cotangent bundles.

Let $f, g \in \mathbb{C}[\mathfrak{g}^*] = \mathbb{C}[T^*G]^{\text{G-left invariant}} \subset \mathbb{C}[T^*G]$ (defined by $f(g, X^*) = f(e, L_{g^{-1}}X^*)$). Let ω be the canonical symplectic form on T^*G. Then there exists $\tilde{X}_f \in \mathfrak{X}$ such that $i_{\tilde{X}_f}\omega = -df$.

What we need to show is that $\omega(\tilde{X}_f, \tilde{X}_g) = \{f, g\}$ (defined in (a)).

By definition, at $(e, X_k^*) \in T^*G$, LHS= $\tilde{X}_f((X_k^L)^*(\pi_*\tilde{X}_g)) - \tilde{X}_g((X_k^L)^*(\pi_*\tilde{X}_f)) - X_k^*(\pi_*[\tilde{X}_f, \tilde{X}_g])$ where $\pi : T(T^*G) \to TG$ and X_k^L is the left invariant vector field valued X_k at e.

Given the Leibniz rule, it suffices to show that when $f = x_i, g = x_j$, LHS= c_{ij}^k.

But then we have $\pi_*\tilde{X}_{x_i} = X_i^L$ and $\pi_*\tilde{X}_{x_j} = X_j^L$. Given that $[X^R, Y^R] = -[X, Y]^R$, probably we need to replace L by R above. □

d.Assume G has a transitive Hamiltonian action on symplectic manifold X with moment map $\mu : X \to \mathfrak{g}^*$. Show the image is a coadjoint orbit and μ is a covering map.

Proof. First claim follows from that μ is G-equivariant. □

content...

Remark 0.1. Ad B acts by $B \otimes id + id \otimes (B^{-1})T$ on $M_d(\mathbb{C}) = \mathbb{C}^{d^2}$. By the definition of adjoint operator, Ad* B acts by $(B \otimes id + id \otimes (B^{-1})^T)^T = B^T \otimes id + id \otimes B^{-1}$, i.e acting by Ad B^T on $M_n(\mathbb{C}^d)^\vee \cong M_n(\mathbb{C}^d)$.

Proof. First of all, id corresponds to tr under trace form. Let $V = \mathrm{End}(\mathbb{C}^d) \times \mathrm{Hom}(\mathbb{C}, \mathbb{C}^d)$. $B \in GL_d$ acts on $(f, i, g^*, j^*) \in T^*V$ by $B(f, i, g^*, j^*) = (AdB(f), Ai, Ad^*B(g^*), j^*A^{-1})$

We have

$$\mu : T^*(V) \to \mathfrak{g}^*$$
$$(f, i, g^*, j^*) \mapsto -g^* \circ ad(f) + j^*R_i$$

where R_i means right multiplication. Then

$$\begin{aligned}
\mu(B(f, i, g^*, j^*))(X) =& g^*(AdB^{-1}[X, AdB(f)]) + j^*B^{-1}XBi \\
=& g^*([AdB^{-1}X, f]) + j^*B^{-1}XBi \\
=& tr(B^{-1}XB) = tr(X)
\end{aligned}$$

Hence $\mu^{-1}(id)$ is stable under the action of GL_d.

Suppose $B \in GL_d$ fixes some $(f, i, g^*, j^*) \in T^*V$, i.e $Bf = fB, Ai = i, Ad^*B(g^*) = g^*$, $j^*A = j^*$.

Then $tr(B^n) = g^*([B^n, f]) + j^*B^ni = j^*i = tr(id) = d$ for all $n \in \mathbb{N}$.

Notice that T^*V gives a representation of GL_d. Look at the linear expansion of the orbit of (f, i, g^*, j^*), which is irreducible of dimension larger than 1. Since B acts trivially, it has to be scalar. As a result, $B = id$. □

b. $\dim M_d = 2d$.

Proof. By (a), $\dim M_d = 2d \Leftrightarrow \dim M = d^2 + 2d$(M is a submanifold) $\Rightarrow$ tr is a regular value $\Leftrightarrow$ $d\mu_p$ is surjective for any $p \in \mu^{-1}(tr)$, which follows from the following two (generally true)claims:

$$\ker d\mu_p = (T_p\mathcal{O}_p)^{\omega_p}.$$

$$\mathrm{Im}\, d\mu_p = (\mathfrak{g}_p)^{\perp}$$

where $(T_p\mathcal{O}_p)^{\omega}$ means the complement wrt ω and $\mathfrak{g}_p = Lie(Stab_p)$.

To see them, notice that $T_p\mathcal{O}_p = \{X^\#_p, X \in \mathfrak{g}\}$ where $X^\#$ is the vector field induced by the infinitesimal action of $\mathfrak{g}$ and $\omega_p(X^\#, v) = \langle d\mu_p(v), X\rangle$. Hence $\ker d\mu^p = (T_p\mathcal{O}_p)^{\omega_p}$ and $\mathrm{Im}\, d\mu_p \supset (\mathfrak{g}_p)^{\perp}$

Also we know some about the dimensions: $\dim(T_p\mathcal{O}_p)^{\omega_p} = \dim(T^*V) - \dim T_p\mathcal{O}_p$, $\dim(\mathfrak{g}_p)^{\perp} = \dim\mathfrak{g} - \dim\mathfrak{g}_p = \dim T_p\mathcal{O}_p$ and $\dim\ker d\mu^p + \dim \mathrm{Im}\, d\mu_p = \dim T^*V$. Thus the second claim follows. □

There are finitely many symplectic leaves in V/Γ where (V,ω) is a finitely dimensional vector space and $\Gamma \subset Sp(V,\omega)$.

Proof. Let $V_H = \{v \in V, \mathrm{stab}_v = H\}$. Claim that $V/\Gamma = \sqcup V_H/\Gamma$ where H runs over the conjugacy classes of subgroups of Γ (where we notice that $V_{gHg^{-1}}/\Gamma = gV/\Gamma = V_H/\Gamma$ for any $g \in \Gamma$).

We need to show that TV_H is a poisson ideal inside TV and ω is nondegenerated restricted to V_H (which is a finite covering of V_H/Γ, hence can descended to the quotient).

Notice that V_H is not necessarily a subspace of V. But it is an open subset of the subspace $V'_H = \{v \in V, \mathrm{stab}_v \geq H\}$. Notice that $V'_H/\Gamma = \overline{V_H/\Gamma}$. The original spaces?

Say $x \in V_H$ lies in the radical of $\omega|_{V_H}$. Then for any $y \in V$,

$$\omega(x,y) = 1/|H| \sum_{h\in H} \omega(hx, hy) = \omega(x, 1/|H| \sum_{h\in H} hy) = 0.$$

Hence $x = 0$.

TV_H is generated by $i_\pi df$ where f runs over $\mathcal{O}_{V_H}$. We know that $\{i_\pi df, i_\pi dg\} = i_\pi d\{f,g\}$. So it remains to prove that $\mathcal{O}_{V'_H/\Gamma} = \mathcal{O}_{V'_H} \cap \mathcal{O}_{V/\Gamma}$ is a Poisson ideal inside $\mathcal{O}_{V/\Gamma}$.

Indeed, $\mathcal{O}_{V'_H/\Gamma} = (Ad^*hf - f, f \in \mathcal{O}_H, h \in H) \cap \mathcal{O}_{V/\Gamma}$. Let $(Ad^*hf - f)f' \in \mathcal{O}_{V'_H/\Gamma}$, $g \in \mathcal{O}_{V/\Gamma}$. Then $\{(Ad^*hf - f)f', g\} = \{Ad^*hf - f, g\}f' + (Ad^*hf - f)\{f', g\}$
$= Ad^*h\{f,g\}f' + (Ad^*hf - f)\{f', g\} \in \mathcal{O}_{V'_H/\Gamma}$. □

$\alpha = i_{\Pi}\text{vol}$. $[\Pi,\Pi] = 0$ iff $\alpha \wedge d\alpha = 0$.

Proof. Claim: $\text{vol}([\Pi,\Pi])\text{vol}= \pm\alpha \wedge d\alpha$. We may assume that $\Pi = \xi \wedge \eta$. Then $[\Pi,\Pi] = [\xi,\eta] \wedge \xi \wedge \eta$.

$$0 = i_{\Pi}^2(\text{vol} \wedge d\text{vol}) = i_{\Pi}(i_{\Pi}\text{vol} \wedge d\text{vol} - \text{vol} \wedge i_{\Pi}d\text{vol}) = 2i_{\Pi}\text{vol} \wedge i_{\Pi}d\text{vol} = 2i_{\Pi}\text{vol} \wedge (L_{\Pi} - di_{\Pi})\text{vol}$$

Hence $\alpha \wedge d\alpha = i_{\Pi}\text{vol} \wedge L_{\Pi}\text{vol} = -(L_{\xi}i_{\eta} - i_{\xi}L_{\eta})\text{vol} \wedge i_{\xi}i_{\eta}\text{vol} = -L_{\xi}i_{\eta}\text{vol} \wedge i_{\xi}i_{\eta}\text{vol}$

$= i_{[\xi,\eta]}\text{vol} \wedge i_{\xi}i_{\eta}\text{vol}$ where we use that $L_{\xi\wedge\eta} = [L_{\xi}, i_{\eta}]$, $i_{\xi}L_{\eta}\text{vol} \wedge i_{\xi}i_{\eta}\text{vol} = -i_{\xi}^2(L_{\eta}\text{vol} \wedge i_{\eta}\text{vol}) = 0$ and $[i_{\xi}, i_{\eta}] = 0$.

Finally, notice that $i_{[\xi,\eta]}\text{vol} \wedge i_{\xi}i_{\eta}\text{vol} = \pm\text{vol}([\xi,\eta] \wedge \xi \wedge \eta)\text{vol}$. □

$\{-,-\} : \mathbb{C}[x,y,z] \times \mathbb{C}[x,y,z] \to \mathbb{C}[x,y,z], \{f,g\}_{\alpha} = \dfrac{\alpha \wedge gf \wedge dg}{dx \wedge dy \wedge dz}$. When $\alpha = d\phi$, Poisson center is $\{f \in \mathbb{C}[x,y,z], \text{algebraic over } \mathbb{C}(\phi)\}$.

Let $\Gamma = \mathbb{Z}/n\mathbb{Z} \hookrightarrow Sp_2 = SL_2$ by $\eta \mapsto diag(\eta, \eta^{-1})$. SL_2 action preserves the standard symplectic form $du \wedge dv$ in $\mathbb{C}^2$ hence $\mathbb{C}[u,v]^{\Gamma}$ is a Poisson subalgebra of $\mathbb{C}[u,v]$. Construct a Poisson algebra isomorphism between $\mathbb{C}[u,v]^{\Gamma}$ and $\mathbb{C}[x,y,z]/(\phi)$.

Proof. Any f lying in the center has gradient parallel to ϕ.

Then the first statement comes from some fact in Galois theory: let K/k be an infinite field extension. Then there exists $D \in Der_{\mathbb{Z}}(K)$ restricted to k being zero.

For the second statement, notice that $\mathbb{C}[u,v]^{\Gamma} = \mathbb{C}[u^n, v^n, uv]$. Define the map to be

$$u^n \to -\sqrt{\frac{ni}{2}}(x+iy), v^n \to \sqrt{\frac{ni}{2}}(-x+iy), uv \to \sqrt{\frac{ni}{2}}z$$

□

Let T act on V with weight vectors $v_1, \ldots, v_n$ and weight $\mu_1, \ldots, \mu_n \in X^*(T)$. Let $x = v_1 + \cdots + v_n$ and $\sigma(V) = \sum \mathbb{R}_{\geq 0}\mu_i$. Then $\sigma(V) = \mathfrak{t}_{\mathbb{R}}^* := \mathbb{R} \otimes_{\mathbb{Z}} X^*(T)$ iff Tx is closed.

Proof. Let $v = \dim T$. Clearly, $\sigma(V) = \mathfrak{t}_{\mathbb{R}}^*$ iff $n \geq v$ and for any μ_i, $-\mu_i \in \sum \mathbb{R}_{\geq 0}\mu_j$ for some j's.

Also, Tx is not closed iff there exists some sequence z_k inside T such that $\mu_i(\gamma(z_k))$ goes to zero for $i \in I \subset [1,n]$ (because $\frac{\mu_i(\gamma(z_k))}{|\mu_i(\gamma(z_k))|}$ is always convergent) and $\mu_i(\gamma(z_k))$ converges to nonzero limit for $i \in [1,n] \setminus I$.

If Tx is not closed, and let $i \in I \subset [1, n]$ such that $\mu_i(\gamma(z_k))$ goes to zero. Suppose $\sigma(V) = \mathfrak{t}^*_{\mathbb{R}}$. Then there exists some $\sum a_j \mu_j$ with $a_j \in \mathbb{R}_{\geq 0}$ such that $\sum a_j \mu_j(\gamma(z_k)) \to \infty$ so there is some $\mu_j(\gamma(z_k)) \to \infty$. A contradiction.

On the other hand, suppose $\sigma(V) \neq \mathfrak{t}^*_{\mathbb{R}}$. We may assume μ_i are distinct from each other. Firstly, we may assume $\mu_1 = -\mu_2, \ldots, \mu_{2k-1} = -\mu_{2k}$ and $-\mu_j \notin \sum \mathbb{R}_{\geq 0}\mu_i, 2k+1 \leq j \leq n$. Claim that any μ_i lies in the linear expansion of the other weights. Otherwise we may take $(a_1, \ldots, a_v)$ such that $\sum a_\ell m_\ell^{(j)} = 0$ (where $\mu_j = (m_\ell^{(j)})$) for all $j \neq i$ but $\sum a_\ell m_\ell^{(i)} \neq 0$. Then $\{(z^{a_1}, z^{a_2}, \ldots, z^{a_v})x, z \in \mathbb{C}^*\}$ is not closed. Then we do induction on $n - 2k$. $n - 2k$ can not be 1 because otherwise $-\mu_n$ as a linear combination of the rest weights lies in $\sum \mathbb{R}_{\geq 0}\mu_i$.

When $(n - 2k) = 2$, not hard.

When $(n - 2k) > 2$, we write $\mu_n = a_1\mu_1 + \cdots + a_{n-1}\mu_{n-1}$ with the least a_i being nonzero. Then at least one of $a_{2k+1}, \ldots, a_{n-1}$ being positive. We may assume it is a_{n-1}. Then there exists $(b_1, \ldots, b_v)$ such that $\sum b_\ell m_\ell^{(j)} = 0, j = 1, \ldots, 2k$ but $\sum b_\ell m_\ell^{(n)} \neq 0$. For $\{\mu_1, \ldots, \mu_{n-1}\}$, by induction hypothesis, we have $z_i \in \mathbb{C}^*$ such that

If for any μ_i, $-\mu_i \in \sum \mathbb{R}_{\geq 0}\mu_j$ for some j's, then $n < v$. WLOG, we may assume that $(\mu_1, \ldots, \mu_{n/2}) = \begin{bmatrix} 1 & 0 \ldots & 0 & * & \ldots & * \\ 0 & 1 \ldots & 0 & * & \ldots & * \\ & & \ldots & & & \\ 0 & 0 \ldots & 1 & * & \ldots & * \end{bmatrix}$ and $\mu_{n/2+i} = -\mu_i$. It seems to be fine. Take $T = (S^1)^2$ acting on $\mathbb{C}^2$ by $(x, y) \to (z_1 x, y/z_1)$. Then it is closed but $\sigma(V) \neq \mathfrak{t}^*_{\mathbb{R}}$.

Let $\sigma^\vee(V) = \{\gamma \in \mathfrak{t}_{\mathbb{R}}, \langle \gamma, \mu \rangle \geq 0, \forall \mu \in \sigma(V)\}$, Claim that $\sigma(V) \neq \mathfrak{t}^*_{\mathbb{R}}$ then $\sigma^\vee(V) \neq 0$. If not, then any $(a_1, \ldots, a_v) \in \mathfrak{t}$, there exists $(x_i)^+, (x_i)^- \in \sigma(V)$ such that $\langle x^+, a \rangle > 0$ and $\langle x^-, a \rangle < 0$. As a result $\sigma(V) = \mathfrak{t}^*_{\mathbb{R}}$ □

Let G be a semisimple group with (e, f, g) an $\mathfrak{sl}_2$ triple. The $ad(h)$ action on $\mathfrak{g}$ gives a $\mathbb{Z}$-grading $\mathfrak{g} = \oplus \mathfrak{g}_i$. Let P be the subgroup corresponding to $\mathfrak{p} := \mathfrak{g}_{\geq 0}$. Set $X := G \times_P \mathfrak{g}_{\geq 2}$ with map $x \to \mathfrak{g}$, $(g, X) \to Ad(g)x$ which is G equivariant under the action $g(h, x) = (gh, x)$ on X.

This map is proper with image $\overline{Ad(G)e}$ and $\pi : \pi^{-1}(Ad(G)e) \to Ad(G)e$ is injective. Furthermore, let ω be the canonical symplectic form on $AdG(e)$. Then $\pi^*\omega$ can be extended to the whole X.

Proof. First of all, denote $\mathcal{P} = Ad(G)\mathfrak{p} \cong G/P$? $X \cong \tilde{\mathfrak{g}}_p := \{(x, \mathfrak{p}) \in \mathfrak{g} \times \mathcal{P}, x \in \mathfrak{p}\}$ with map given by $(g, x) \mapsto (Adg(x), gP/P)$, which is G-equivariant. Hence the projection is the same as the projection $\tilde{\mathfrak{g}}_p \to \mathfrak{g}$ which is clearly a projective map hence proper.

To show that $Im(\pi) = \overline{Ad(G)e}$, first we claim that if G is connected, then for any G representation V, if $\mathfrak{g} \cdot v = V$ then $G \cdot v$ is open dense in V. Notice that the image of

$G \to V : g \to g \cdot v$ is locally closed (i.e, open inside some closed subvariety of V) and connected. Since the differential surjective, the image contains some open neighborhood in usual topology. Hence $G \cdot v$ cannot be contained in any proper closed subvariety of V. Hence $G \cdot v$ is open inside V which is irreducible thus dense.

Then recall that $[\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_{i+j}$ and any grading can be written as direct sum $Im(e) \oplus Ker(f)$ (and symmetrically, $Im(f) \oplus Ker(e)$) (hence $[\mathfrak{g}_0, e] = \mathfrak{g}_2$. So $Ad(Z_G(h))e$ is open dense in $\mathfrak{g}_2$.) hence $[\mathfrak{p}, e] = \mathfrak{g}_{\geq 2}$ so $Ad(P)e$ is open dense in $\mathfrak{g}_{\geq 2}$ hence $Ad(G)e$ is open dense in $\pi(X)$.

To show that $\pi : \pi^{-1}(Ad(G)e) \to Ad(G)e$ is injective: it suffices to show that if $\mathfrak{g}_{\geq 2} \ni e' = Ad(h)e$ then $h \in P$. Actually, $Stab_{\mathfrak{g}}(\mathfrak{g}_{\geq 2}) = \mathfrak{p}$ implies that $Stab_G(\mathfrak{g}_{\geq 2}) = P$.

Notice that $T_{(id,e)}X = (\mathfrak{g} \oplus \mathfrak{g}_{\geq 2})/\mathfrak{p}$ where $\mathfrak{p}$ imbeds by $(-Z, [e, Z])$. Set ω' on X by $\omega'((X, Y), (X', Y')) = \kappa(Y', X) - \kappa(Y, X') + \kappa(e, [X, X'])$. Then it is a well-defined nondegenerate 2-form on X and equals to $\pi^*\omega$ when restricted to $\pi^{-1}Ad(G)e$. $\square$