On the Harish-Chandra D-module

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0 Introduction

To study characters, Harish-Chandra investigated a family of equivariant systems of partial differential equations on a semisimple Lie group and proved that any invariant eigendistribution on a semisimple group or Lie algebra is a locally integrable function and is analytic on the regular elements[HC63]. These systems were later called Harish-Chandra systems. Hotta and Kashiwara[HK84] formalized it in the language of D-modules and defined the Harish-Chandra D-module encoding the information of the family of Harish-Chandra systems. And they used these objects to study the Springer representation. Later, Ginzburg [Gin12] used the Harish- Chandra D-module to study the isospectral commuting variety, which led to a new proof of positivity of the Kostka-Macdonald polynomials. In [Gin19], he also used the Harish- Chandra D-module to prove the exactness of parabolic induction functor. This proposal will be around 1 the properties of the Harish-Chandra D-module as well as 2 its applications to the Springer representation and 3 parabolic induction and restriction.

Throughout the proposal, we fix a connected complex adjoint reductive group G with Lie algebra $\mathfrak{g}$, a Borel subgroup B with Lie algebra $\mathfrak{b}$ and a maximal torus T in B with Lie algebra $\mathfrak{h}$. We will identify $\mathfrak{g} \cong \mathfrak{g}^*$ and $\mathfrak{h} \cong \mathfrak{h}^*$ by an nondegenerated invariant bilinear form. We denote the flag variety G/B by $\mathcal{B}$. For any algebraic variety X, D_X stands for the sheaf of differential operators on X.

1 Properties of the Harish-Chandra D-module

Let $\tilde{\mathfrak{g}} := \{(\mathfrak{b}', x) \in \mathcal{B} \times \mathfrak{g}, x \in \mathfrak{b}'\}$ with two maps:

$$\mathfrak{g} \xleftarrow{p} \tilde{\mathfrak{g}} \xrightarrow{q} \mathfrak{h} \tag{1.1}$$

where p is the projection, which is called the Grothendieck-Springer resolution and $q : \mathcal{B} \times \mathfrak{g} \ni (g\mathfrak{b}g^{-1}, x) \mapsto g^{-1}xg \mod [\mathfrak{b}, \mathfrak{b}]$. Let $f = p \times q$. Use $\int_f$ to define D-module derived pushforward along f. Then our main object arises:

Definition 1.1. The Harish-Chandra D-module is defined to be $\boldsymbol{M} := \int_f \mathcal{O}_{\tilde{\mathfrak{g}}}$.

We will prove that $\boldsymbol{M}$ is a D-module rather than a complex. Let $\mathfrak{g}_{rs}$ be the set of semisimple regular elements of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}_{rs} = p^{-1}(\mathfrak{g}_{rs})$, both of which are open dense subsets. Then $\tilde{\mathfrak{g}}_{rs} \to f(\tilde{\mathfrak{g}}_{rs}) = \mathfrak{g}_{rs} \times_{\mathfrak{h}_r/W} \mathfrak{h}_r$ is an isomorphism. Therefore the higher cohomology sheaves of the restriction of $\boldsymbol{M}$ to $\mathfrak{g}_{rs} \times \mathfrak{h}_r$ vanish. To complete the proof, we need to look into how the pushforward of D-modules works, especially their characteristic cycles.

Recall that if $\pi : T^*X \to X$ is the natural projection, then the characteristic variety of a coherent D_X-module $\mathcal{F}$ is defined to be $SS(\mathcal{F}) = \mathrm{Supp}\pi^*\mathrm{gr}\,\mathcal{F}$. Here we take associated graded with respect to a good filtration. The characteristic cycle is a finer invariant taking the multiplicities of irreducible components into account. We denote it by $Ch(\mathcal{F})$.

It is clear that the image of f is the fiber product $\mathfrak{x} := \mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}$. One can analogously define $\mathfrak{C} \times_{\mathfrak{T}/W} \mathfrak{T}$ where $\mathfrak{C} = \{(x,y) \in \mathfrak{g} \times \mathfrak{g}, [x,y] = 0\}$ is the commuting variety and $\mathfrak{T} = \mathfrak{h} \times \mathfrak{h}$. Although $\mathfrak{x}$ is known to be reduced and $\mathfrak{X}_{rs} := \mathfrak{C}_{rs} \times_{\mathfrak{T}_r/W} \mathfrak{T}_r$ is reduced, $\mathfrak{C} \times_{\mathfrak{T}/W} \mathfrak{T}$ is not usually reduced. Take the reduced structure $\mathfrak{X} := (\mathfrak{C} \times_{\mathfrak{T}/W} \mathfrak{T})_{red}$ and call $\mathfrak{X}$ the isospectral commuting variety.

Let $\mathfrak{g}_r$ be the set of regular elements of $\mathfrak{g}$ and $\mathfrak{x}_r := \mathfrak{g}_r \times_{\mathfrak{h}/W} \mathfrak{h}$. Use $N^*_{\mathfrak{x}_r}(\mathfrak{g} \times \mathfrak{h})$ to denote the normal bundle of $\mathfrak{x}_r$ inside $\mathfrak{g} \times \mathfrak{h}$ and take its closure inside $T(\mathfrak{g} \times \mathfrak{h})$.

Lemma 1.2 ([Gin12])**.** We have $\mathfrak{X}$ is irreducible and isomorphic to $\overline{N_{\mathfrak{x}_r}(\mathfrak{g} \times \mathfrak{h})}$ hence it is Lagrangian.

Let $\mathfrak{x}_{rs} := \mathfrak{g}_{rs} \times_{\mathfrak{h}_r/W} \mathfrak{h}_r \overset{i}{\hookrightarrow} \mathfrak{g}_{rs} \times \mathfrak{h}_r$ be the natural embeddings. Then

Proposition 1.3 ([HK84])**.** (1) $\mathscr{H}^k(\boldsymbol{M}) = 0, \forall k \neq 0$.
(2) $\mathscr{H}^0(\boldsymbol{M})$ is simple with $Ch(\boldsymbol{M}) = [\mathfrak{X}]$ and it is the minimal extension of $\int_i \mathcal{O}_{\mathfrak{x}_{rs}}$.

Proof. An upper bound for characteristic variety of the pushforward of a D-module is generally known ([KK81],[SKK73]), which gives $SS(\mathscr{H}^k(\boldsymbol{M})) \subset \mathfrak{X}$ for all k. As we discuss above, $\mathscr{H}^k(\boldsymbol{M}|_{\mathfrak{g}_{rs}\times\mathfrak{h}_r})$ vanishes for $k > 0$. Since the dimension of any irreducible component of the characteristic variety of a nonzero D-module is at least the dimension of the space and $(\mathfrak{g}_{rs} \times \mathfrak{h}) \cap \mathfrak{X}$ is open dense in the Lagrangian $\mathfrak{X}$, $\mathscr{H}^k(\boldsymbol{M})$ has to be zero for all $k > 0$. On the other hand, the dimension inequality for the characteristic variety and the inclusion $SS(\mathscr{H}^0(\boldsymbol{M})) \subset \mathfrak{X}$ imply $SS(\boldsymbol{M}) = \mathfrak{X}$. Since

$$Ch(\mathscr{H}^0(\boldsymbol{M}|_{\mathfrak{g}_{rs}\times\mathfrak{h}_r})) = Ch(\int_i \mathcal{O}_{\mathfrak{x}_{rs}}) = [\mathfrak{X}_{rs}]$$

which is reduced we conclude $Ch(\boldsymbol{M}) = [\mathfrak{X}]$. By the additivity of Ch, $\boldsymbol{M}$ is simple hence is the minimal extension of $\int_i \mathcal{O}_{\mathfrak{x}_{rs}}$. □

As its origin suggests, $\boldsymbol{M}$ can be described as solutions to a system of differential equations. To state and prove this result precisely, we need a deep isomorphism between spaces of invariant differential operators from Harish-Chandra. Let $\Delta := \prod_{\alpha \in R^+} \alpha$ where R^+ is the set of positive roots. Then $\mathfrak{h}_r = \{\Delta \neq 0\}$. Recall that we have isomorphism:

$\mathcal{O}(\mathfrak{g}_{rs}/G) \cong \mathcal{O}(\mathfrak{h}_r/W)$. This allows us to define a homomorphism $\delta' : D(\mathfrak{g})^G \to D(\mathfrak{h}_r)^W$. The differential operators in the image of δ' generally have poles along the root hyperplanes, i.e $\{\Delta = 0\}$. But those in the image of $\delta := \Delta\delta'\Delta^{-1}$ magically have no poles and so land in $D(\mathfrak{h})^W$. Moreover, $\mathfrak{g}$ can be embedded into $D(\mathfrak{g})$ by the differential of the adjoint action of G on $\mathfrak{g}$. Clearly the image of this embedding acts by 0 on $\mathcal{O}(\mathfrak{h}_r/W)$. Ultimately,

Theorem 1.4 ([HC65],[LS95],[LS96])**.** The radial parts map

$$\delta : (D(\mathfrak{g})/D(\mathfrak{g})\ \mathrm{ad}\mathfrak{g})^G \to D(\mathfrak{h})^W$$

is an algebra isomorphism.

Now we can state and give a sketch of the proof to the following proposition:

Proposition 1.5 ([HK84])**.** $\boldsymbol{M} \cong D_{\mathfrak{g}\times\mathfrak{h}}/(D_{\mathfrak{g}\times\mathfrak{h}}\ \mathrm{ad}\mathfrak{g} + D_{\mathfrak{g}\times\mathfrak{h}}\{u\otimes 1 - 1\otimes\delta(u), u \in D(\mathfrak{g})^G\})$

Proof. ([Gin12]) Denote RHS by $\boldsymbol{M}_1$. We take order filtration on $D_{\mathfrak{g}\times\mathfrak{h}}$, which induces a filtration on $\boldsymbol{M}_1$. Now the composition of

$$\mathbb{C}[\mathfrak{C}] \otimes_{\mathbb{C}[\mathfrak{T}^W]} \mathbb{C}[\mathfrak{T}] \twoheadrightarrow \frac{\mathrm{gr}\ D(\mathfrak{g})}{\mathrm{gr}(D(\mathfrak{g})\mathrm{ad}\mathfrak{g})} \underset{\mathrm{gr}D(\mathfrak{h})^W}{\otimes} \mathrm{gr}D(\mathfrak{h}) \twoheadrightarrow \mathrm{gr}\left[\frac{D(\mathfrak{g})}{D(\mathfrak{g})\ \mathrm{ad}\mathfrak{g}} \underset{D(\mathfrak{h})^W}{\otimes} D(\mathfrak{h})\right]$$

gives a surjection $\mathbb{C}[\mathfrak{C}\times_{\mathfrak{T}^W}\mathfrak{T}] \twoheadrightarrow \mathrm{gr}\Gamma(\mathfrak{g}\times\mathfrak{h}, \boldsymbol{M}_1)$. Hence we have $SS(\boldsymbol{M}_1) \subset \mathfrak{C}\times_{\mathfrak{T}^W}\mathfrak{T}$.

Since $\mathfrak{X}$ is Lagragian and generically reduced, similar to the proof of proposition 1.3, we have that $Ch(\boldsymbol{M}_1) = 0$ or $[\mathfrak{X}]$. So $\boldsymbol{M}_1$ is either simple or 0. We shall construct a nonzero homomorphism from $\boldsymbol{M}_1$ to $\boldsymbol{M}$ to conclude the proof. First let dg, dt be bi-invariant volume forms on $\mathfrak{g}, \mathfrak{h}$ respective. Then there exists a volume form ω on $\tilde{\mathfrak{g}}$ such that $p^*dg = q^*\Delta\cdot\omega$. Set $s := (dgdt)^{-1}\otimes f_*\omega$. Then by definition of the pushforward of D-modules, $\boldsymbol{M} = D_{\mathfrak{g}\times\mathfrak{h}}s$. Now since s is G-invariant, it is killed by ad $\mathfrak{g} \subset D(\mathfrak{g})$ and from theorem 1.4, we conclude $(u\otimes 1)s = (\delta(u)\otimes 1)s, \forall u \in D(\mathfrak{g})^G$. As a result, we have a nonzero homomorphism $\boldsymbol{M}_1 \to \boldsymbol{M}$ by sending 1 to s. □

2 Interplay with Springer representation

In [BM81], Borho and MacPherson proved the Springer correspondance by applying decomposition theorem, with respect to the semismall map $\tilde{\mathcal{N}} \to \mathcal{N}$, to the Springer sheaf. Soon after, Hotta and Kashiwara gave an alternative proof via holonomic systems. To explain their approach, we need one more ingredient, the Fourier transform, which we will recall as below:

Given a n-dimensional vector space V over $\mathbb{C}$, we can build the Weyl algebra

$$D(V) = \mathbb{C}[v_1, \ldots, v_n, \partial_{v_1}, \ldots, \partial_{v_n}]/([v_i, v_j], [\partial_{v_i}, \partial_{v_j}], [\partial_{v_i}, v_j] - \delta_{ij}).$$

We have an isomorphism: $D(V) \cong D(V^*), v_i \mapsto \partial_{v_i}, \partial_{v_i} \mapsto -v_i$. This gives an equivalence $Coh(D_V) \cong Coh(D_{V^*})$. We call this the Fourier transform and denote the Fourier transform of $\mathscr{F}$ by $\mathscr{F}^F$.

Now we introduce another two holonomic systems, which will give the Springer representation. Let $pr : \mathfrak{g} \times \mathfrak{h} \to \mathfrak{g}$ be the projection, $\mathcal{N}$ the nilpotent cone and $\tilde{\mathcal{N}} := p^{-1}(\mathcal{N})$ the Springer resolution with $p' = p|_{\tilde{\mathcal{N}}}$ (recall 1.1). Define $\boldsymbol{N} := \int_p \mathcal{O}_{\tilde{\mathfrak{g}}}$ which by base change equals to $\int_{pr} \boldsymbol{M}$. Also by base change, $\int_{p'} \mathcal{O}_{\tilde{\mathcal{N}}} = j^\dagger \boldsymbol{M}$. Here $j : \mathfrak{g} \to \mathfrak{g} \times \mathfrak{h}$ is the zero section and $j^\dagger$ is the derived pullback without shift.

Proposition 2.1 ([HK84])**.** (1) $\mathscr{H}^i(\boldsymbol{N}) = 0 = \mathscr{H}^i(\int_{p'} \mathcal{O}_{\tilde{\mathcal{N}}})$, for $i > 0$.

(2) $SS(\boldsymbol{N}) = \mathfrak{C} \cap (\mathfrak{g} \times \mathcal{N})$, $SS(\int_{p'} \mathcal{O}_{\tilde{\mathcal{N}}}) = \mathfrak{C} \cap (\mathcal{N} \times \mathfrak{g})$.

(3) $\boldsymbol{N}^F = \int_{p'} \mathcal{O}_{\tilde{\mathcal{N}}}$.

(4) $\boldsymbol{N}$ is the minimal extension of $(\int_{p|_{\tilde{\mathfrak{g}}_{rs}}} \mathcal{O}_{\tilde{\mathfrak{g}}_{rs}})$.

Proof. (1),(2) can be proved similarly as in proposition 1.3. Let $t_1, \ldots, t_r$ be coordinates of $\mathfrak{h}$. Then by definition,

$$\Gamma(\mathfrak{g}, \int_{pr} \boldsymbol{M}) = \Gamma(\mathfrak{g} \times \mathfrak{h}, \boldsymbol{M}) / \sum \frac{\partial}{\partial t_i} \Gamma(\mathfrak{g} \times \mathfrak{h}, \boldsymbol{M}) \tag{2.1}$$

$$\Gamma(\mathfrak{g}, j^\dagger \boldsymbol{M}) = \Gamma(\mathfrak{g} \times \mathfrak{h}, \boldsymbol{M}) / \sum t_i \Gamma(\mathfrak{g} \times \mathfrak{h}, \boldsymbol{M}) \tag{2.2}$$

Since $\boldsymbol{M}^F = \boldsymbol{M}$, we obtain (3). Use $\Gamma_{\mathfrak{g} \setminus \mathfrak{g}_{rs}}$ to denote taking sheaf of sections supported at $\mathfrak{g} \setminus \mathfrak{g}_{rs}$, which commutes with pushforward. Since pr is finite restricted to $\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}$, we have $\Gamma_{\mathfrak{g} \setminus \mathfrak{g}_{rs}}(\boldsymbol{N}) = \int_{pr} \Gamma_{\mathfrak{g} \setminus \mathfrak{g}_{rs}} \boldsymbol{M} = 0$. So $\boldsymbol{N}$ has no non-trivial submodule supported on $\mathfrak{g} \setminus \mathfrak{g}_{rs}$. On the other hand, $\mathcal{O}_{\tilde{\mathfrak{g}}}$ is self-dual hence $\mathcal{N}$ has no non-trivial quotient supported outside $\mathfrak{g}_{rs}$ as well and (4) follows. □

Remark 2.2. Note that $p : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is a small map, which also implies (1),(4) above.

Let W be the Weyl group. W acts on $\mathfrak{g} \times \mathfrak{h}$ by acting on $\mathfrak{h}$. Since $\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}$ is stable under this action, W acts on $\boldsymbol{M}$. Hence it also acts on the pushforward $\boldsymbol{N} = \int_{p'} \boldsymbol{M}$. Since $\mathfrak{g} \times \{0\}$ is stable under this action, W also acts on $j^\dagger \boldsymbol{M} = \int_{p'} \mathcal{O}_{\tilde{\mathcal{N}}}$. Taking these actions into account sharpens (3) of 2.1:

Proposition 2.3 ([HK84])**.** $\boldsymbol{N} = (\int_{p'} \mathcal{O}_{\tilde{\mathcal{N}}})^F \otimes sgn$ as W representations.

Proof. Similar to the proof of 1.5, $\boldsymbol{N}$ is generated by the section $dg^{-1} \otimes p_* \omega$, which is the image of $s = (dgdt)^{-1} \otimes f_* \omega$ under the isomorphism 2.1. Fix a $w \in W$. Since $w^* \omega = sgn(w) \omega$,

$$w \left(a(t) dg^{-1} \otimes p_* \omega \right) = sgn(w) a(w^{-1} t) dg^{-1} \otimes p_* \omega.$$

On the other hand, using2.2 and that $w^* dt = sgn(w) dt$,

$$w \Big(a(\partial_t) s \bmod \sum t_i \Gamma(\mathfrak{g} \times \mathfrak{h}, \boldsymbol{M}) \Big) = a(w^{-1} \partial_t) s \bmod \sum t_i \Gamma(\mathfrak{g} \times \mathfrak{h}, \boldsymbol{M}).$$

The proposition follows. □

Decompose $\int_{p'} \mathcal{O}_{\tilde{\mathcal{N}}}$ into isotypic components $\int_{p'} \mathcal{O}_{\tilde{\mathcal{N}}} = \bigoplus_{\chi \in \widehat{W}} L_\chi \otimes V_\chi$, where W acts on L_χ via the irreducible representation χ and $V_\chi = \mathrm{Hom}_W(L_\chi, \int_{p'} \mathcal{O}_{\tilde{\mathcal{N}}})$ is the multiplicity space. The Springer correspondence reads:

Theorem 2.4 ([HK84])**.** V_χ is a simple $D_{\mathfrak{g}}$-module supported on the closure of a nilpotent orbit. And V_χ's are not isomorphic to each other.

Proof. Applying Fourier tranform to the decomposition of $\boldsymbol{N}^F$, we get $\boldsymbol{N} = \bigoplus_{\chi \in \widehat{W}} L_\chi \otimes V^F_{\chi'}$, where $\chi' = \chi \otimes sgn$. For any algebraic variety X, let X_{an} denote the underlying complex manifold, $K_{X_{an}}$ be the canonical bundle on it and $DR = \mathcal{K}_{X_{an}} \otimes^L_{D_{X_{an}}} -$ be the de Rham functor. We have

$$\bigoplus_{\chi \in \widehat{W}} \chi \otimes DR(V^F_{\chi'}) = DR\boldsymbol{N} = p_* \mathbb{C}_{\tilde{\mathfrak{g}}}[\dim \mathfrak{g}].$$

Since $\tilde{\mathfrak{g}}_{rs} \to \mathfrak{g}_{rs}$ is a principal W-bundle, $p_* \mathbb{C}_{\tilde{\mathfrak{g}}_{rs}}$ is the local system corresponding to the regular representation of W. Write the isotypic decomposition

$$p_* \mathbb{C}_{\tilde{\mathfrak{g}}_{rs}} = \oplus_{\chi \in \widehat{W}} L_\chi \otimes \mathcal{L}^*_\chi$$

where $\mathcal{L}_\chi$ runs over all irreducible local systems with monodromy factoring through W. Note that the bi-W-action on $p_* \mathbb{C}_{\tilde{\mathfrak{g}}_{rs}}$ coincides with that on $DR(\boldsymbol{N})|_{\mathfrak{g}_{rs}}$. We thus have $DR(V^F_{\chi'})|_{\mathfrak{g}_{rs}} \cong L^*_\chi[\dim \mathfrak{g}]$ for all χ, which are simple and not isomorphic to each other. Since $\boldsymbol{N}$ is the minimal extension of its restrictions to $\mathfrak{g}_{rs}$ and the minimal extension functor is fully faithful, $V^F_{\chi'}$'s are simple and not isomorphic to each other as well. And so are V_χ's.

Now recall that $SS(\int_{p'} \mathcal{O}_{\tilde{\mathcal{N}}}) = \mathfrak{C} \cap (\mathcal{N} \times \mathfrak{g})$ and let $p_1 : \mathfrak{C} \cap (\mathcal{N} \times \mathfrak{g}) \to \mathcal{N}$ be the projection. By Dynkin-Kostant, the nilpotent cone $\mathcal{N}$ splits into finitely many G-orbits under the adjoint action. Let $x \in \mathcal{N}$, $G \cdot x$ be its orbit and $C(x) \subset G$ the centralizer of x. Then $\dim p_1^{-1}(G \cdot x) = \dim G \cdot x + \dim C(x) = \dim \mathfrak{g}$. Hence an irreducible component of $\mathfrak{C} \cap (\mathcal{N} \times \mathfrak{g})$ is the closures of $\dim p_1^{-1}(G \cdot x)$ for some $x \in \mathcal{N}$. By additivity of Ch, as a simple module, V_χ has to be supported on the closure of a nilpotent orbit. □

3 Interplay with Parabolic induction

Analogously, we define $\tilde{G} = \{(gB, x) \in \mathcal{B} \times G, x \in gBg^{-1}\}$ and two maps:

$$G \xleftarrow{p} \tilde{G} \xrightarrow{q} T \tag{3.1}$$

the projection p and $q : (gB, x) \mapsto g^{-1}xg \mod [B, B]$. We can define the Harish-Chandra D-module on G by $\boldsymbol{M} := \int_{p \times q} \mathcal{O}_{\tilde{G}}$ and prove proposition 1.3,1.5 and proposition 2.1 for $\boldsymbol{N} := \int_p \mathcal{O}_{\tilde{G}}$ similarly.

3.1 Different approach to the module N

One can also study $\boldsymbol{N}$ using a totally different approach. This approach strongly relies on the following special case of Beilinson-Bernstein localization theorem.

Let $Z_+ := Z \cap \mathfrak{g}U(\mathfrak{g})$ and $R := U(\mathfrak{g})/(Z_+)$. Then

Theorem 3.1. *$R \cong \Gamma(\mathcal{B}, D_{\mathcal{B}})$ and for any $D_{\mathcal{B}}$ module $\mathcal{F}$, $R^i\Gamma(\mathcal{F}) = 0, \forall i > 0$. Hence there is an equivalence between categories*

$$\Gamma : D_{\mathcal{B}}-\text{mod} \leftrightarrows R-\text{mod} : D_{\mathcal{B}} \otimes_R -$$

Our goal is to show

Proposition 3.2 ([HK85])**.** *$N \cong D_G/(D_G \operatorname{ad}\mathfrak{g} + D_G Z_+)$.*

We prove it as a corollary of a more general formula. Consider the following maps (by abuse of notations)

$$G \xleftarrow{p} G \times \mathcal{B} \xrightarrow{f} \mathcal{B} \times \mathcal{B}$$

where p is the projection and $f : (g, x) \mapsto (gx, x)$. Let $pr_2 : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be the projection to the second factor and $\mathcal{K}_{\mathcal{B}}$ the canonical bundle on $\mathcal{B}$. Also note that $D_G/(Z_+) = D_G \otimes_{U(\mathfrak{g})} R$ has two right R-module structure - as left or right invariant differential operators.

Proposition 3.3 ([HK85])**.** *For any coherent $D_{\mathcal{B}\times\mathcal{B}}$ module $\mathcal{F}$, we have*

$$\int_p f^\dagger \mathcal{F} \cong D_G/(Z_+) \otimes^L_{R\otimes R} R\Gamma(\mathcal{B} \times \mathcal{B}, pr_2^* \mathcal{K}_{\mathcal{B}} \otimes_{\mathcal{O}_{\mathcal{B}\times\mathcal{B}}} \mathcal{F}).$$

To simplify exposition, tensor product over structure sheaf will be denoted by $\otimes$ and we also drop all the f^{-1}, p_2^{-1}, etc. if there is no confusion.

Proof. By definition, $\int_p f^\dagger \mathcal{F} = Rp_*(D_{G\leftarrow G\times\mathcal{B}} \otimes^L_{D_{G\times\mathcal{B}}} D_{G\times\mathcal{B}\to\mathcal{B}\times\mathcal{B}} \otimes^L_{D_{\mathcal{B}\times\mathcal{B}}} \mathcal{F})$. To further expand it, we will use a general result: for any closed subvariety $i' : V \hookrightarrow X \times Y$, one has $D_{X\leftarrow V} \otimes^L_{D_Z} D_{V\to Y} \cong \int_{i'} \mathcal{O}_V \otimes \mathcal{K}_Y$ as (D_X, D_Y)-modules.

Plug in $V = G \times \mathcal{B}$, $X = G$, $Y = \mathcal{B} \times \mathcal{B}$ and we have

$$D_{G\leftarrow G\times\mathcal{B}} \otimes^L_{D_{G\times\mathcal{B}}} D_{G\times\mathcal{B}\to\mathcal{B}\times\mathcal{B}} = (\int_{p\times f} \mathcal{O}_{G\times\mathcal{B}}) \otimes \mathcal{K}_{\mathcal{B}\times\mathcal{B}}. \tag{3.2}$$

Now consider commutative diagram

$$\begin{tikzcd} G \times \mathcal{B} \arrow[r, "p\times f"] \arrow[dr, "\sigma"'] & G \times \mathcal{B} \times \mathcal{B} \arrow[r, "p"] \arrow[d, "\phi"] & G \\ & G \times \mathcal{B} \times \mathcal{B} \arrow[ur, "p_2"'] & \end{tikzcd}$$

where $\phi(g, x, y) = (g, g^{-1}x, y)$. Then $\sigma(g, x) = (g, x, x)$ and

$$\phi_* \left((\int_{p\times f} \mathcal{O}_{G\times\mathcal{B}}) \otimes \mathcal{K}_{\mathcal{B}\times\mathcal{B}} \right) = (\int_\sigma \mathcal{O}_{G\times\mathcal{B}}) \otimes \mathcal{K}_{\mathcal{B}\times\mathcal{B}} = (\mathcal{O}_G \boxtimes \int_{\sigma'} \mathcal{O}_{\mathcal{B}}) \otimes \mathcal{K}_{\mathcal{B}\times\mathcal{B}}$$

where $\sigma' : \mathcal{B} \hookrightarrow \mathcal{B} \times \mathcal{B}$ is the diagonal embedding. Notice that

$$\sigma'_* D_{\mathcal{B}} = \int_{\sigma'} \mathcal{O}_{\mathcal{B}} \otimes pr_2^* \mathcal{K}_{\mathcal{B}}. \tag{3.3}$$

Hence the above equals to $\mathcal{O}_G \boxtimes (\sigma'_* D_{\mathcal{B}} \otimes pr_1^* \mathcal{K}_{\mathcal{B}})$. Applying $R(p_2)_*$, we then have

$$Rp_* \left[(\int_{p\times f} \mathcal{O}_{G\times\mathcal{B}}) \otimes \mathcal{K}_{\mathcal{B}\times\mathcal{B}} \right] = \mathcal{O}_G \otimes R\Gamma(\mathcal{B}, D_{\mathcal{B}} \otimes \mathcal{K}_{\mathcal{B}}). \tag{3.4}$$

We apply the localization theorem to $\mathcal{F}$:

$$- \otimes^L_{D_{\mathcal{B}\times\mathcal{B}}} \mathcal{F} = - \otimes^L_{D_{\mathcal{B}\times\mathcal{B}}} D_{\mathcal{B}\times\mathcal{B}} \otimes_{R\times R} \Gamma(\mathcal{B}\times\mathcal{B}, \mathcal{F}) = - \otimes^L_{R\times R} \Gamma(\mathcal{B}\times\mathcal{B}, \mathcal{F}). \tag{3.5}$$

On the other hand, it is clear

$$\mathcal{O}_G \otimes^L - = D_G \otimes^L_{U(\mathfrak{g})} - = D_G \otimes_{U(\mathfrak{g})} R \otimes^L_R - = D_G/(Z_+) \otimes^L_R -. \tag{3.6}$$

We can now put all the pieces together, using (3.2,3.4,3.5,3.6) and localization theorem:

$$\begin{aligned}
\int_p f^\dagger \mathcal{F} &= \left[D_G/(Z_+) \otimes^L_R R\Gamma(\mathcal{B}, D_{\mathcal{B}} \otimes \mathcal{K}_{\mathcal{B}}) \right] \otimes^L_{R\times R} \Gamma(\mathcal{B}\times\mathcal{B}, \mathcal{F}) \\
&= \left[D_G/(Z_+) \otimes^L_{R\otimes R} R \otimes_{\mathbb{C}} R\Gamma(\mathcal{B}, D_{\mathcal{B}} \otimes \mathcal{K}_{\mathcal{B}}) \right] \otimes^L_{R\times R} \Gamma(\mathcal{B}\times\mathcal{B}, \mathcal{F}) \\
&= D_G/(Z_+) \otimes^L_{R\otimes R} R\Gamma(\mathcal{B}\times\mathcal{B}, pr_2^*\mathcal{K}_{\mathcal{B}} \otimes D_{\mathcal{B}\times\mathcal{B}} \otimes_{D_{\mathcal{B}\times\mathcal{B}}} \mathcal{F}) \\
&= D_G/(Z_+) \otimes^L_{R\otimes R} R\Gamma(\mathcal{B}\times\mathcal{B}, p_2^*\mathcal{K}_{\mathcal{B}} \otimes \mathcal{F}). \qquad \square
\end{aligned}$$

Let $i : \tilde{G} \hookrightarrow G \times \mathcal{B}$ be the embedding. Plug $\int_i \mathcal{O}_{\tilde{G}} = f^\dagger \int_{\sigma'} \mathcal{O}_{\mathcal{B}}$ into the formula and we obtain proposition 3.2.

A similar formula to 3.3 is proved in [BFO12, 3.2]. The functor $CH := \int_p f^\dagger = p_! f^\dagger$ is called character functor. It is left adjoint to the Harish-Chandra functor $HC := \int_f p^![-\dim \mathcal{B}] = \int_f p^\dagger$. Here we have used that p is proper and q is smooth. Let X be an algebraic variety with G action $a : G \times X \to X$. For any D_G module A, D_X module E, we define the convolution $A * E = \int_a (A \boxtimes E)$. Then

Lemma 3.4. [BFO12, lemma 2.1]

$$R\Gamma(X, A * E) \cong \Gamma(G, A) \otimes^L_{U(\mathfrak{g})} R\Gamma(X, E).$$

Pluging in $X = \mathcal{B} \times \mathcal{B}$ (we will use X for $\mathcal{B}\times\mathcal{B}$ below) and a the G-action on the second factor, we deduce from the lemma:

$$R\Gamma(X, HC(A)) = R\Gamma(X, a * \int_{\sigma'} \mathcal{O}_{\mathcal{B}}) = \Gamma(G, A) \otimes^L_{U(\mathfrak{g})} R\Gamma(X, \int_{\sigma'} \mathcal{O}_{\mathcal{B}})$$

which equals to $\Gamma(G, A) \otimes^L_{U(\mathfrak{g})} R\Gamma(\mathcal{B}, D_{\mathcal{B}} \otimes^L \mathcal{K}_{\mathcal{B}}^{-1})$ given 3.3.

Let E be a $R \times R$-module, F be a D_G-module with Z_+ acting on it locally finitely and $\Delta = D_X \otimes_{R\times R} -$. Then by a series of standard adjunctions, we have

$$
\begin{aligned}
&\mathrm{Hom}_{D_G}(CH\circ\Delta(E),F)=\mathrm{Hom}_{R\times R}(E,R\Gamma(X,HC(F))\\
&=\mathrm{Hom}_{R\times R}(E,\Gamma(G,F)\otimes^L_{U(\mathfrak{g})}R\Gamma(\mathcal{B},D_{\mathcal{B}}\otimes^L\mathcal{K}_{\mathcal{B}}^{-1}))\\
&=\mathrm{Hom}_{R\times R}(E,\Gamma(G,F)\otimes^L_{U(\mathfrak{g})}R\otimes^L_{R\times R}R\otimes R\Gamma(\mathcal{B},D_{\mathcal{B}}\otimes^L\mathcal{K}_{\mathcal{B}}^{-1}))\\
&=\mathrm{Hom}_{D_X}(\Delta E,\Delta(\Gamma(G,F)\otimes^L_{U(\mathfrak{g})}R)\otimes^L(pr_2^*\mathcal{K}_{\mathcal{B}})^{-1})\\
&=\mathrm{Hom}_{D_X}(pr_2^*\mathcal{K}_{\mathcal{B}}\otimes\Delta E,\Delta(\Gamma(G,F)\otimes^L_{U(\mathfrak{g})}R)))\\
(*)\ &=\mathrm{Hom}_{R\times R}(R\Gamma(X,pr_2^*\mathcal{K}_{\mathcal{B}}\otimes\Delta E),R\mathrm{Hom}_{U(\mathfrak{g})}(R,\Gamma(G,F))\\
&=\mathrm{Hom}_{R\times R}(R\Gamma(X,pr_2^*\mathcal{K}_{\mathcal{B}}\otimes\Delta E),R\mathrm{Hom}_{D_G}(D_G/Z_+,F))\\
&=\mathrm{Hom}_{D_G}(D_G/Z_+\otimes^L_{R\otimes R}R\Gamma(X,pr_2^*\mathcal{K}_{\mathcal{B}}\otimes\Delta E),F)
\end{aligned}
$$

where $(*)$ uses that Z_+ action is locally finite on F. We obtain proposition 3.3 again.

3.2 Parabolic induction and restriction

From the diagram 3.1, one can define the parabolic induction functor $pind := \int_p q^\dagger$ and its left adjoint, the parabolic restriction functor $pres := \int_q p^\dagger[\dim T - \dim G]$. We denote $\mathcal{M} := \Gamma(G\times T, \boldsymbol{M})$.

Theorem 3.5 ([Gin19])**.** (a) $R\Gamma(G, pind(-)) \cong \mathcal{M}\otimes^L_{D(T)} R\Gamma(T,-)$ and $pind$ is exact.
(b) We have the following commutative diagram of functors:

$$
\begin{tikzcd}
Coh^G(D_G) \arrow[r, shift left, "\mathscr{H}^0 pres"] \arrow[d, "\Gamma"] & Coh^W(D_T) \arrow[l, shift left, "\mathscr{H}^0 pind"] \arrow[d, "\Gamma"] \\
Coh^G(D(G)) \arrow[r, shift left, "\mathrm{Hom}_{D(G)}(\boldsymbol{M},-)"] & Coh^W(D(T)) \arrow[l, shift left, "\mathbf{M}\otimes_{D(T)}-"]
\end{tikzcd}
$$

The key step of the proof is that by diagram chasing, one reduces the proof of exactness to the flatness of $\boldsymbol{M}$ as a D_T module. This is proved by showing that $\boldsymbol{M}$ is noncharacteristic with respect to any embedding of subvariety $i : G\times V \subset G\times T$, where V is a closed subvariety of T, and then applying [HTT07, theorem 2.4.6], which says that the high cohomology sheaves of a noncharacteristic pullback vanish.

The group version radial parts map $\delta : (D(G)/D(G)\ \mathfrak{g})^G \to D(T)^W$ was defined in [HC63]. But there seems to be no proof in the literature showing it is an isomorphism. Assuming the isomorphism is true, it follows that $\mathcal{M}\otimes^L_{D(T)} - = (D(G)/D(G)\mathfrak{g})\otimes^L_{D(T)^W} -$. Moreover, under this assumption one can write $\mathrm{Hom}_{D(G)}(\boldsymbol{M},-) = D(T)\otimes_{D(T)^W}(-)^G$ hence conclude that $\mathscr{H}^0 pres : Coh^G(D_G)\to Coh^W(D_T)$ is exact.

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