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Solution of a weakly two-dimensional melting problem by an approximate method

R.S. GUPTA and N.C. BANIK
Department of Mathematics, University of Roorkee, Roorkee-247667, India

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Abstract: A two-dimensional melting problem where the solid/liquid interface moves along the y-axis is solved by an approximate method. The numerical results from the present method are compared with the approximate methods due to earlier authors. The superiority of the present method is shown through a test problem whose analytical solution is known.

Keywords: Melting, moving boundary, approximate method, two dimensions, Stefan problem.

1. Introduction

A wide variety of practical problems in engineering and science are governed by parabolic equations whose domains of reference change shape and size with time. Examples of such problems are melting or solidification processes, ablation of space missiles due to aerodynamic heating, thawing of food stuffs, frost penetration into earth, diffusion of gas in an absorbing medium etc. The above problems are commonly known as moving boundary problems (MBPs) or Stefan problems. Since analytical solutions are limited to a very few particular cases, different techniques are adopted for solving them. A general survey of the various methods for dealing with MBPs may be found in [1,5,6,11,15]. Goodman [7] introduced an approximate analytical method called "Heat Balance Integral" (HBI) method for solving transient heat conduction (diffusion) problems including MBPs whose details may be found in [8]. Although there are no suitable checks to assess the accuracy or no systematic approach to improve them, the importance of the integral methods cannot be undermined due to their simplicity and their applicability to a wide variety of problems. Noble [12] suggests an improvement over the integral method by repeated spatial subdivision, using quadratic profiles, while Bell [2] makes equal subdivision of the dependent variable, i.e., temperature range instead of subdividing the space domain. As far as the application of the integral method to two-dimensional MBPs is concerned, only a few references may be cited. For example, Poots [13] assumes a two-dimensional surface profile in dealing with freezing of liquid inside a square prism. But the expression obtained in [13] for the two-dimensional profile is quite complicated. Rasmussen [14] solves a weakly two-dimensional problem choosing a linear profile for the temperature distribution in one of the directions at all

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times. Gupta and Kumar [10] suggested two methods—in one method a one-dimensional quadratic profile is chosen at preselected values of the other variable and in the other piecewise linear profiles are assumed.

The present authors [9] proposed an approximate method called "constrained integral method" (CIM) for solving MBPs in one dimension based on the method of moments originally suggested by Fujita (see [4]) for solving one-dimensional heat conduction problems in fixed domain. The method proceeds by assuming a temperature profile which consists of two unknowns and satisfies the boundary conditions. One of the unknowns is the position of the moving boundary and the other depends on the condition provided at the fixed end. That is, if temperature is prescribed at the fixed end, we assume the other unknown to be the flux at the fixed end, and if flux is provided there, the unknown is taken to be the temperature. Finally, two simultaneous first-order differential equations are obtained which give the position of the moving boundary and the value of the other unknown parameter in an implicit manner. In the present paper the method has been extended to solve weakly two-dimensional MBPs. We discuss the method in detail with respect to a problem given below.

2. The problem

Previously discussed by Gupta and Kumar [10] the problem in its nondimensional form may be defined by the following differential equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \text{in } \Omega(t), \ t > 0, \tag{2.1}$$

where $\Omega(t) \equiv \{x, y \,|\, 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant s(x, t)\}$ is occupied by water. The associated boundary conditions are

$$\frac{\partial u}{\partial x} = 0, \quad x = 0, \ x = 1, \tag{2.2}$$

$$u = 1, \quad y = 0, \tag{2.3}$$

$$u = 0, \quad y = s(x, t), \tag{2.4}$$

$$\frac{\partial s}{\partial t} = -\frac{\partial u}{\partial y} + \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial s}{\partial x}\right), \quad y = s(x, t), \tag{2.5}$$

and the initial temperature distribution

$$u(x, y, 0) = 1 - \frac{y}{g(x, 0)}, \tag{2.6}$$

where $u(x, y, t)$ denotes the temperature at a point (x, y) in a two-dimensional space domain $\Omega(t)$ at any time t, $y = s(x, t)$ defines the moving boundary and $g(x, 0)$ is a given function such that $s(x, 0) = g(x, 0)$. The region $\Omega'(t) \equiv [x, y \,|\, 0 \leqslant x \leqslant 1, \ y \geqslant s(x, t)]$ is assumed to be occupied by ice at $0\,°$C. It may be noted that the size of $\Omega(t)$ increases with time while $\Omega'(t)$ diminishes.

3. Method of solution

Instead of choosing a two-dimensional global profile, we assume the temperature profile for constant x as

$$u(x, y, t) = A\left(\frac{y}{s}\right)^2 + B\left(\frac{y}{s}\right) + 1 \tag{3.1}$$

in conformity with the boundary condition (2.3); A and B are unknown parameters—in general, functions of x and t. Here we have got a choice of expressing the polynomial (3.1) in terms of anyone of the unknown parameters A or B using the condition (2.4) at the moving boundary. We suggest that it be expressed in terms of B since the flux at the fixed end $y = 0$ can be given in terms of B as

$$-s\frac{\partial u(x, 0, t)}{\partial y} = -B. \tag{3.2}$$

Thus we can write (3.1) as

$$u(x, y, t) = -(1+B)\left(\frac{y}{s}\right)^2 + B\left(\frac{y}{s}\right) + 1, \tag{3.3}$$

where s and B are still to be found. It may be noted that flux at the fixed end (where temperature is prescribed) will be automatically evaluated as part of the solution in the determination of B.

In order to determine s and B we take the zeroth and the first moments of the basic heat conduction equation (2.1) with respect to y. They are respectively given by

$$\int_0^s \frac{\partial u}{\partial t} \mathrm{d}y = \int_0^s \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \mathrm{d}y \tag{3.4}$$

and

$$\int_0^s y\frac{\partial u}{\partial t} \mathrm{d}y = \int_0^s y\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \mathrm{d}y. \tag{3.5}$$

We can write (3.4) as

$$\frac{\partial}{\partial t}\int_0^s u \,\mathrm{d}y = \frac{\partial^2}{\partial x^2}\int_0^s u \,\mathrm{d}y - \left(\frac{\partial u}{\partial x}\right)_{y=s}\frac{\partial s}{\partial x} + \left.\frac{\partial u}{\partial y}\right|_{y=0}^{s}, \tag{3.6}$$

which, after using (2.5) and (3.3) and making some algebraic manipulation, becomes

$$s(10+B)\frac{\partial s}{\partial t} + s^2\frac{\partial B}{\partial t} = s(4+B)\frac{\partial^2 s}{\partial x^2} - 6B + s^2\frac{\partial^2 B}{\partial x^2} + 2s\frac{\partial s}{\partial x}\frac{\partial B}{\partial x}. \tag{3.7}$$

In the same manner, using (2.5) and (3.3) in (3.5) we get

$$s(9+B)\frac{\partial s}{\partial t} + \tfrac{1}{2}s^2\frac{\partial B}{\partial t} = (3+B)\left[s\frac{\partial^2 s}{\partial x^2} + \left(\frac{\partial s}{\partial x}\right)^2\right] + 6 + \tfrac{1}{2}s^2\frac{\partial^2 B}{\partial x^2} + 2s\frac{\partial s}{\partial x}\frac{\partial B}{\partial x}. \tag{3.8}$$

On solving (3.7) and (3.8) we finally obtain

$$\frac{\partial s}{\partial t} = \frac{2(3+B)\left(\frac{\partial s}{\partial x}\right)^2 + s(2+B)\frac{\partial^2 s}{\partial x^2} + 6(2+B) + 2s\frac{\partial s}{\partial x}\frac{\partial B}{\partial x}}{s(8+B)} \tag{3.9}$$

and

$$\frac{\partial B}{\partial t}=\frac{-2(30+13B+B^2)\left(\frac{\partial s}{\partial x}\right)^2+12s\frac{\partial^2 s}{\partial x^2}-12(B^2+10B+10)-4s\frac{\partial s}{\partial x}\frac{\partial B}{\partial x}}{s^2(8+B)}+\frac{\partial^2 B}{\partial x^2}. \tag{3.10}$$

4. Numerical results and discussion

In order to compare our results with those of Gupta and Kumar [10] we take the same cases of initial positions of the interface as considered by them, namely,

(a) $g(x)=0.5-0.2\cos\frac{1}{2}\pi x$,

(b) $g(x)=0.5-0.2\cos\pi x$,

(c) $g(x)=1.0-0.2\,\mathrm{e}^{-25x^2}$, $0\leqslant x\leqslant 1$, $t=0$.

We solve the simultaneous differential equations (3.9) and (3.10) in a step-by-step manner employing central difference in the space direction and forward difference in the time direction and by taking the initial value of B from (3.2), i.e.,

$$B(x,0)=s(x,0)\frac{\partial u(x,0,0)}{\partial y}=s(x,0)\times\frac{-1}{s(x,0)}=-1.$$

The time and space steps are taken to be $\Delta t=0.001$ and $\Delta x=0.1$, respectively. Gupta and Kumar [10] have solved these problems by the method of Rasmussen [14] also. In Table 1 we show the positions of the moving boundary at different times, from our results and the corresponding numerical values due to them. It may be mentioned that Gupta and Kumar [10] obtain one of the parameters in the quadratic profile using an extra condition derived at the interface, while in the present method we have used the first moment as an additional condition. We observe that our results are closer to those obtained from the piecewise linear profile of Gupta and Kumar [10] rather than their quadratic profile. Now a question arises which results are better and which are worse. Although it may not be possible to prove theoretically the superiority of the results obtained from the present method in comparison to others, we have tested various methods on a simple one-dimensional melting problem whose analytical solution is known (see [3]). The problem concerns melting of a semi-infinite sheet of ice initially at its fusion temperature and fixed surface being subjected to a constant temperature. In nondimensional form the position of the solid/liquid interface by various methods is given by

$$s(t)=\begin{cases}1.2401\,t^{1/2} & \text{analytical method [13]},\\ 1.3200\,t^{1/2} & \text{HBI},\\ 1.2100\,t^{1/2} & \text{Gupta and Kumar [10]},\\ 1.2346\,t^{1/2} & \text{CIM (present method)}.\end{cases} \tag{4.1}$$

As may be seen from (4.1), the expression corresponding to the present method is much closer to the analytical one in comparison to the other two. We have also used a cubic profile taking the

Table 1
Comparison of position of the moving boundary ($10^4 \cdot s$) along $x = 0$ and $x = 1$ at various times; (A) $s(x, 0) = 0.5 - 0.2 \cos \frac{1}{2}\pi x$; (B) $s(x, 0) = 0.5 - 0.2 \cos \pi x$; (C) $s(x, 0) = 1 - 0.2\, e^{-25x^2}$

t	$x = 0$				$x = 1$			
	Rasmussen [14]	Gupta and Kumar [10]			Rasmussen [14]	Gupta and Kumar [10]		
		Method I	Method II	Present		Method I	Method II	Present
(A) 0.05	4022	4084	4144	4137	5252	5460	5533	5492
0.10	4832	4938	5016	5009	5721	5990	6085	6033
0.15	5523	5664	5756	5750	6196	6509	6623	6564
0.20	6135	6308	6410	6405	6657	7003	7135	7070
0.25	6689	6892	7002	6998	7099	7472	7621	7552
0.30	7197	7429	7547	7543	7523	7918	8083	8011
0.35	7668	7928	8055	8050	7929	8344	8524	8450
0.40	8110	8396	8532	8526	8320	8752	8945	8869
(B) 0.05	4197	4244	4251	4250	7179	7371	7459	7416
0.10	5111	5218	5193	5200	7409	7745	7861	7785
0.15	5873	6034	5983	5996	7674	8116	8251	8147
0.20	6534	6741	6674	6693	7961	8481	8634	8506
0.25	7122	7369	7296	7318	8263	8838	9009	8862
0.30	7654	7938	7864	7888	8573	9188	9375	9214
0.35	8142	8459	8390	8414	8885	9530	9732	9560
0.40	8594	8942	8881	8905	9198	9865	10081	9901
(C) 0.05	9133	9324	8913	8988	10328	10360	10421	10408
0.10	9742	9940	9593	9669	10639	10707	10801	10781
0.15	10211	10421	10147	10209	10931	11040	11158	11131
0.20	10616	10840	10629	10679	11205	11362	11497	11464
0.25	10984	11223	11064	11104	11470	11672	11824	11784
0.30	11327	11582	11466	11498	11729	11972	12140	12093
0.35	11651	11923	11843	11868	11984	12263	12445	12393
0.40	11960	12249	12199	12219	12236	12547	12744	12686

zeroth and the first moments as in the present method along with a derived condition at the moving boundary, giving $s(t) = 1.2389\, t^{1/2}$ which is still closer to the analytical expression.

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