STA 104 Applied Nonparametric Statistics

Chapter 1: Introduction

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Binomial Problem

Setting

We observe the outcomes of n independent Bernoulli trials [1].

- The outcome of each trial can be classified as a success or a failure.
- The probability of a success, denoted by p, remains constant from trial to trial.
- The n trials are independent.

[1] Research problems possessing these assumptions underpinnings are common, and thus the binomial test procedures find frequent use. A particularly important special case occurs when we wish to test hypotheses about the unknown median of a population. The application of binomial test leads to the sign test.

Hypothesis

Two-Sided Test:

$$H_0 : p = p_0 \text{ versus } H_a : p \neq p_0$$

One-Sided Upper-Tail Test:

$$H_0 : p = p_0 \text{ versus } H_a : p > p_0$$

One-Sided Lower-Tail Test:

$$H_0 : p = p_0 \text{ versus } H_a : p < p_0$$

Motivation

$B = \sum_{i=1}^{n} Y_i$

$Y_i = \begin{cases} 1, & \text{if } i\text{th Bernoulli trial is success} \\ 0, & \text{if } i\text{th Bernoulli trial is failure} \end{cases}$

# of success out of n trials

$\underbrace{\hat{p} = \frac{B}{n}}$

proportion of success out of n trials

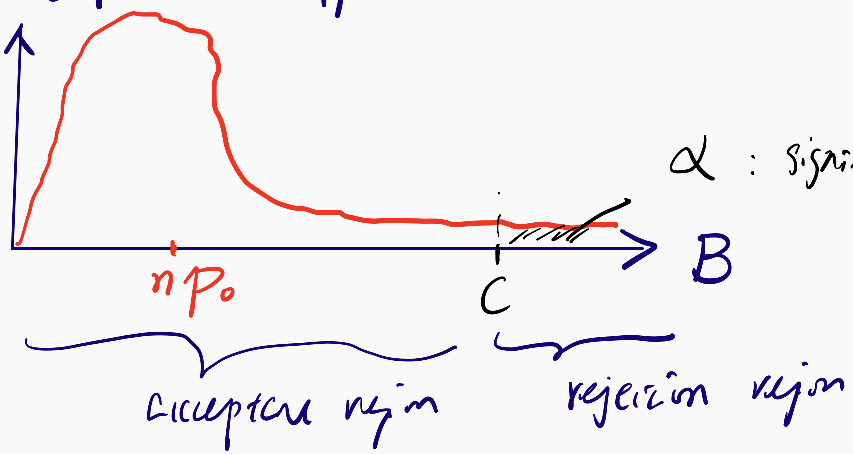
- upper-tail test: $H_0: p = p_0$ vs $H_1: p > p_0$

  when $p > p_0$:

  $\Rightarrow$ $\hat{p}$ will tend to be larger than p_0 $\Rightarrow$ B will large

  $\Rightarrow$ reject H_0 when large B

(1) critical value approach



α : significance level 0.05 / 0.1

$$\alpha = P(\text{Type I error}) = P(B \geq c \mid B \sim \text{Binomial}(n, p_0))$$

$$= \sum_{b \geq c}^{n} P(B = b) = \sum_{b \geq c}^{n} \binom{n}{b} p_0^{b} (1 - p_0)^{n-b}$$

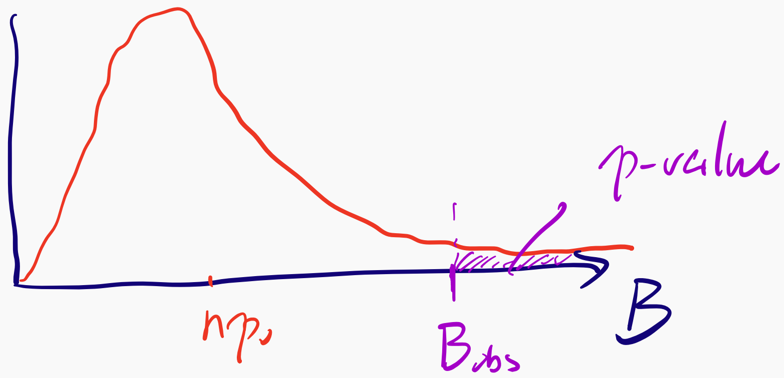
$$\Rightarrow \text{final critical value } c = \underbrace{b_\alpha}_{\text{upper } \alpha\text{th percentile of } \mathrm{Bin}(n, p_0)}$$

$\Rightarrow$ reject H_0 if $B \geq b_\alpha$

don't reject H_0 if $B < b_\alpha$

Motivation

(2) p-value approach



$$\text{p-value} = P\left(B \geq B_{obs} \mid B \sim \text{Binomial}(n, p_0)\right)$$

$$= \sum_{b \geq B_{obs}}^{n} P(B = b) = \sum_{b \geq B_{obs}}^{n} \binom{n}{b} p_0^{b} (1-p_0)^{n-b}$$

$\Rightarrow$ reject H_0 if p-value $< \alpha$

don't reject H_0 if p-value $\geq \alpha$

Exact test

Motivation

Asymptotic (large-sample) normal approximation:

$$\hat{p} = \frac{B}{n} = \frac{1}{n}\sum_{i=1}^{n} \underbrace{Y_i}_{\text{iid Bernoulli}(p) = \text{Bern}(p_0) \text{ if } H_0 \text{ is true}}$$

$\Rightarrow$ CLT:

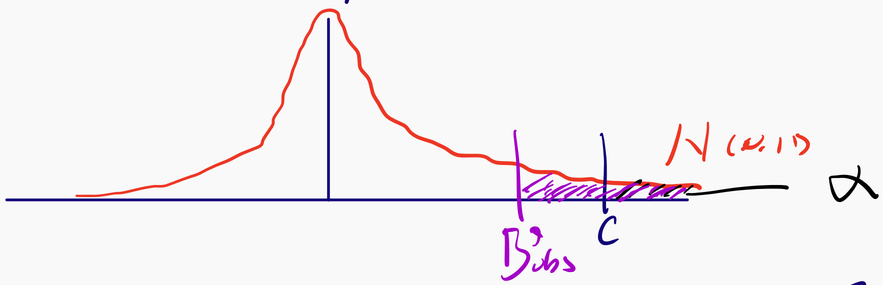
$$\frac{B}{n} \sim N\left(p_0, \frac{p_0(1-p_0)}{n}\right) \text{ for large } n$$

$$\Leftrightarrow B \sim N(np_0, np_0(1-p_0)) \text{ for large } n$$

standardize $B^* = \dfrac{B - E(B)}{\sqrt{Var(B)}} = \dfrac{B - np_0}{\sqrt{np_0(1-p_0)}} \sim N(0,1)$

Motivation

(1) critical value approach



$$\alpha = P(\text{Type 2 error}) = P\left(B^* \geq c \mid B^* \sim N(0,1) \right)$$

$$\Rightarrow \quad c \underbrace{= z_\alpha}_{\text{upper } \alpha \text{ th percentile of } N(0,1)}$$

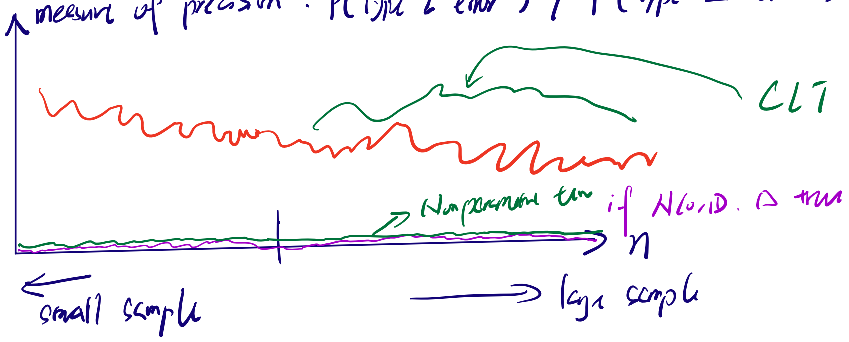
(2) p-value approach

$$\text{p-value} = P\left(B^* \geq B^*_{obs} \mid B^* \sim N(0,1) \right)$$

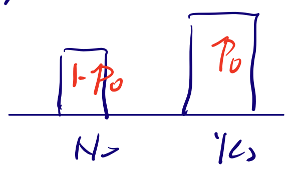
t-test : $t = \dfrac{\bar{X} - \mu_0}{\sqrt{\frac{s^2}{n}}} \sim$ t-dist. Assume $X_1 \cdots X_n \sim N(\mu, \sigma^2)$

CLT : $X_1 \cdots X_n \sim$ any dist.

measure of precision : P(Type 2 error) / P(Type II error)



Why Binomial problem is nonparametric ?



Procedure

a. One-Sided Upper-Tail Test.

To test

$$H_0 : p = p_0$$

versus

$$H_1 : p > p_0$$

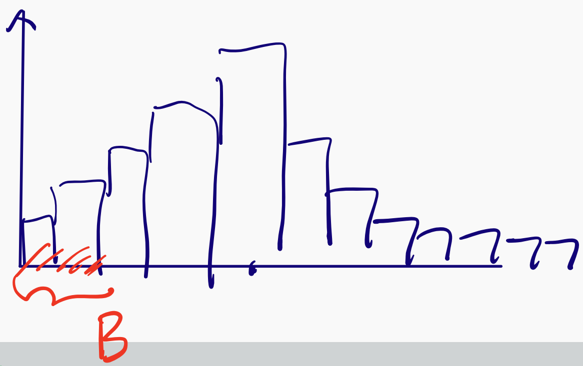
at the α level of significance,

Reject H_0 if $B \geq b_\alpha$; otherwise do not reject, where the constant b_α is chosen to make the type I error probability equal to α. The number b_α is the upper α percentile point of the binomial distribution with sample size n and success probability p_0.

The normal approximation:

Reject H_0 if $B^* \geq z_\alpha$; otherwise do not reject.

Procedure



if H_0 is true

b. One-Sided Lower-Tail Test.

To test

$$H_0 : p = p_0$$

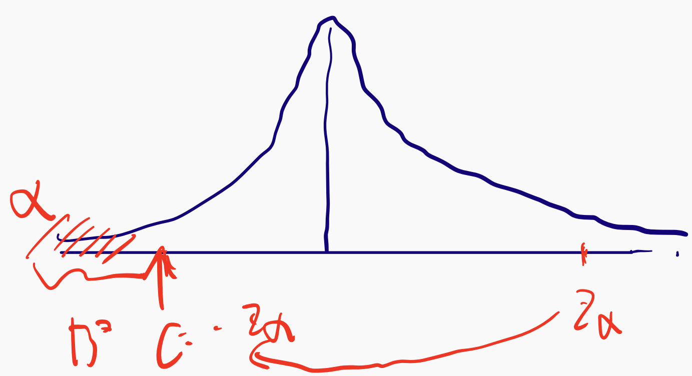
versus

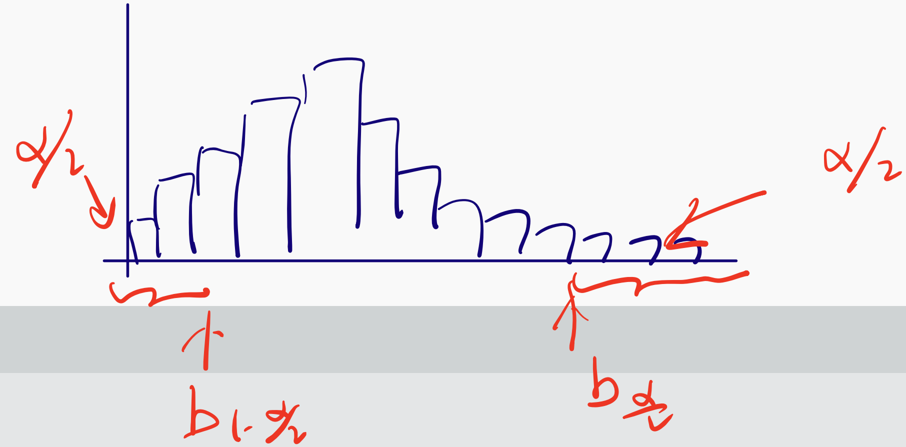
$$H_a : p < p_0$$

at the α level of significance, Reject H_0 if $B \leq b_{1-\alpha}$; otherwise do not reject.

The normal approximation:

Reject H_0 if $B^* \leq -z_\alpha$; otherwise do not reject.





c. Two-Sided Test.

To test

$$H_0 : p = p_0$$

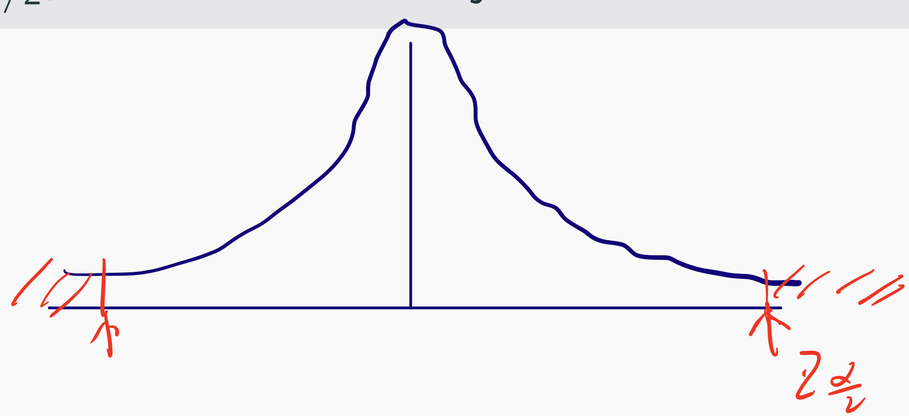
versus

$$H_a : p \neq p_0$$

at the α level of significance, Reject H_0 if $B \geq b_{\alpha/2}$ or $B \leq b_{1-\alpha/2}$; otherwise do not reject

The normal approximation:

Reject H_0 if $|B^*| \geq z_{\alpha/2}$; otherwise do not reject.



Example

Sensory Difference Tests

The triangle test is principally used to determine whether a sensory difference exists between two products. For example to test whether a change to a product's ingredient, process or packaging has had an impact on it's overall sensory properties.

To conduct the test; the assessors compare the test product against a control product. Each assessor is presented with three samples (two are the same and one is different) and asked to evaluate the samples from left to right, select the "different" sample and describe the difference perceived.

Out of 50 trials, there were 25 correct selections and 25 incorrect selections.

$B = 25$

B^x

Example

We consider the binomial test of $H_0 : p = \frac{1}{3}$ versus the one-sided alternative $p > \frac{1}{3}$. We set $\alpha = .05$ for purposes illustration.

Asymptotic approximation:

(1) critical value approach:

To find $z_{.05}$, the 95$^{\text{th}}$ upper quantile of the standard normal

```
> qnorm (.95, 0, 1)
[1] 1.644854
```

we find $z_{.05} = 1.645$. Thus Reject H_0 if $B^* \geq 1.645$; otherwise do not reject.

Example

From the data we have $n = 50$ and B (the number of correct identifications) $= 25$.

$$B^* = \frac{25 - 50\left(\frac{1}{3}\right)}{\left\{50\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\right\}^{1/2}} = 2.5.$$

The large sample approximation value $B^* = 2.5 > 1.645$ and thus we reject $H_0 : p = \frac{1}{3}$ in favor of $p > \frac{1}{3}$ at the approximate $\alpha = .05$ level.

Example

(2) p-value approach: To find the P-value corresponding to $B^* = 2.5$

```
> 1 -pnorm(2.5)
[1] 0.006209665
```

Example

Exact test: (1) critical value approach:

To find $b_{.05}$, the 95$^{\text{th}}$ upper quantile of the binomial distribution

```
> qbinom(0.05, size=50, prob=1/3, lower.tail =FALSE)
[1] 22
```

we find $b_{.05} = 22$. Thus Reject H_0 if $B \geq 22$; otherwise do not reject.

Example

From the data we have $n = 50$ and B (the number of correct identifications) $= 25$.

Thus we reject $H_0 : p = \frac{1}{3}$ in favor of $p > \frac{1}{3}$ at the approximate $\alpha = .05$ level.

Example

(2) p-value approach:

B = 25

To find the P-value corresponding to ~~$B^* = 2.5$~~

25

```
> pbinom(24,50,1 / 3, lower.tail =FALSE)
[1] 0.01082668
```

Conclusion: From both the exact test and large-sample test, there is statistically significant evidence that the food being tested tastes differently.

An estimator for the probability of success with uncertain quantification

Point estimate:

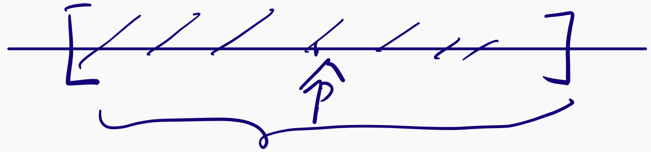
$$\hat{p} = \frac{B}{n}$$

Observed Relative Frequency of Success.

Standard errorof $\widehat{p}$

$$\widehat{sd}(\widehat{p}) = \sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}.$$

measures the variability of the estimate.



Confidence interval

A $(1-\alpha)100\%$ confidence interval:

$\alpha = 0.05$

95%

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$