Lecture 4: Two-Factor Studies with Equal Sample Sizes

STA 106: Analysis of Variance

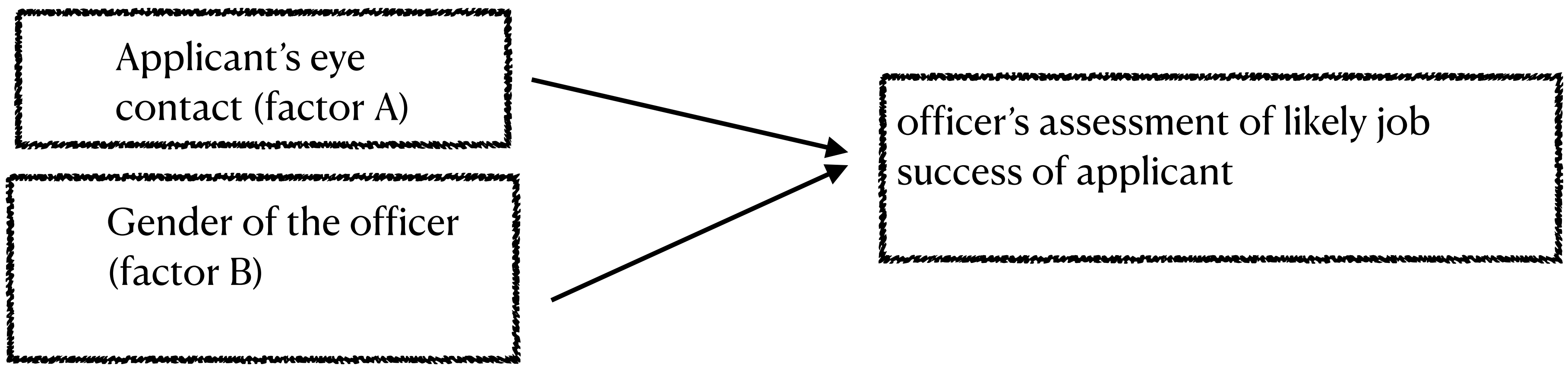
Suggested reading: ALSM Chapter 19

Xiner Zhou
Department of Statistics, University of California, Davis

Example

(The Eye Contact Study)

The objective of the study:



The study setup:

10 make and 10 female officers were chosen

Half of the officers in each gender group were chosen at random to receive a photograph of the applicant in which the application made make eye contact with the camera lens; the other half received a version in which there was no eye contact

Officers were asked to give a rating on a scale 0 (total failure) to 20 (outstanding success) on likely job success

		Factor *B* (gender of officer)	
Factor *A* (eye contact)		$j = 1$ Male	$j = 2$ Female
$i = 1$	Present	11	15
		7	12
		...	...
		10	16
$i = 2$	Absent	12	14
		16	17
		...	...
		14	18

☑ Randomized Block Design

Example

(The Kidney Failure Hospitalization Study)

The objective of the study:

What is the appropriate “dose” for effective dialysis treatment ?

Duration of treatment (factor A)

Weight gain between treatments as a result of fluid buildup (factor B)

the number of days hospitalized due to the disease during a year

The study setup:

A random sample of 10 patients per treatment group at dialysis facility were chosen

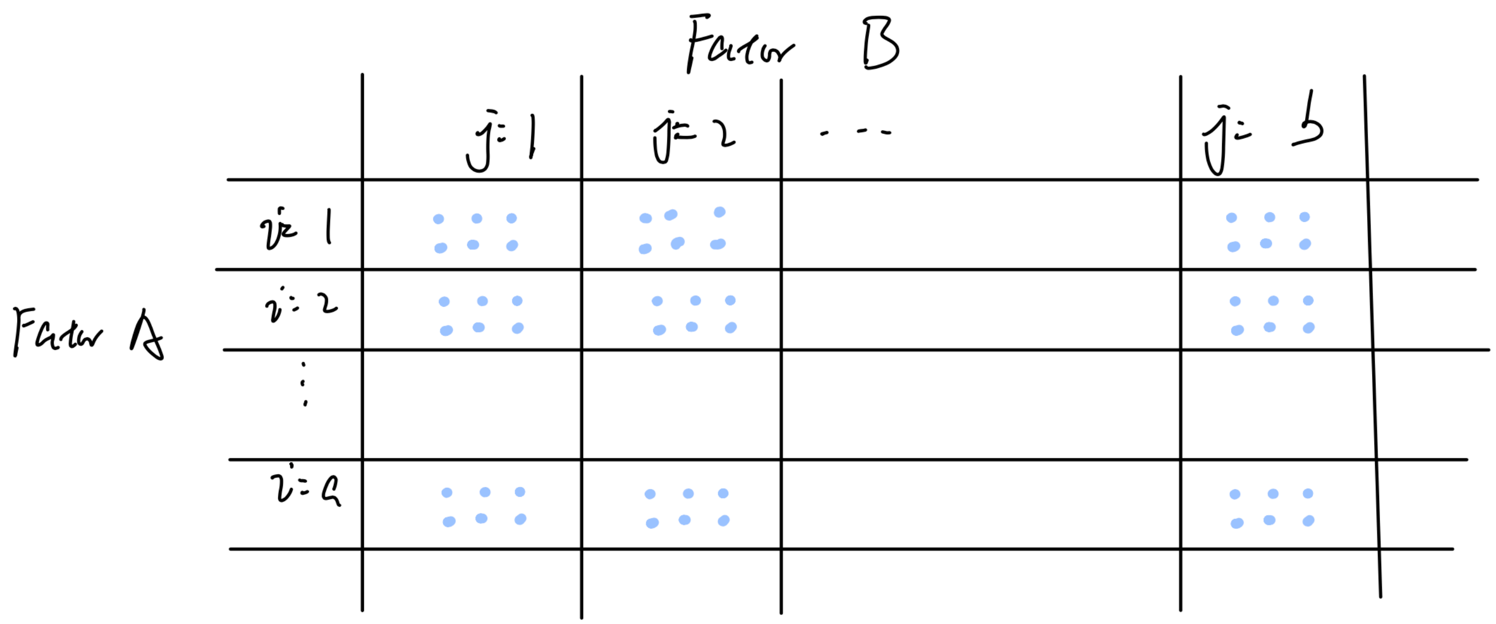
Factor A (duration)		Factor B (weight gain): $j=1$ Mild		$j=2$ Moderate		$j=3$ Substantial	
$i=1$	Short	0	2	2	4	15	16
		2	0	4	3	10	7
		...	...		...	...	...
		0	8	15	20	25	27
$i=2$	Long	0	2	5	1	10	15
		1	7	3	3	8	4
		..			...	.	
		4	3	1	9	7	1

☑ Observational study
Without randomly assigning factor levels

Two-Factor Studies with Equal Sample Sizes

- Two-Way ANOVA Model
- Analysis of Variance
- Tests for Interaction effects and main effects
- Analysis of Factor A and B Main Effects (When Factors Do Not Interact)
- Analysis of Treatment Means when Interactions Are Present

Two-Way ANOVA Model for Two-Factor Studies



Factor A is studied at a levels
Factor B is studies at b levels

All treatment sample sizes are equal with $n > 1$
Total sample size $n_T = abn$

kth observation ($k = 1 \ldots n$) for the treatment ($A = i, B = j$) is Y_{ijk}

Two-Way ANOVA Model for Two-Factor Studies

Assume: observed value of response variable is the sum of two components

Treatment mean μ_{ij} — fixed term

Error term ϵ_{ijk} — random term

$$Y_{ijk} = \mu_{ij} + \epsilon_{ijk}$$

- ϵ_{ijk} error term $\quad \epsilon_{ijk} \sim N(0,\sigma^2)$
- μ_{ij} Treatment means

$$E\left(Y_{ijk}\right) = E\left(\mu_{ij} + \varepsilon_{ijk}\right) = \mu_{ij} + E\left(\varepsilon_{ijk}\right) = \mu_{ij}$$

However, this way of parameterization in terms of treatment means is not enough, what's the complication for two-factor studies?

Two-Way ANOVA Model for Two-Factor Studies

We ultimately are interested in **simultaneous investigating the join effects of factor A and factor B**

That is, we want to parse out

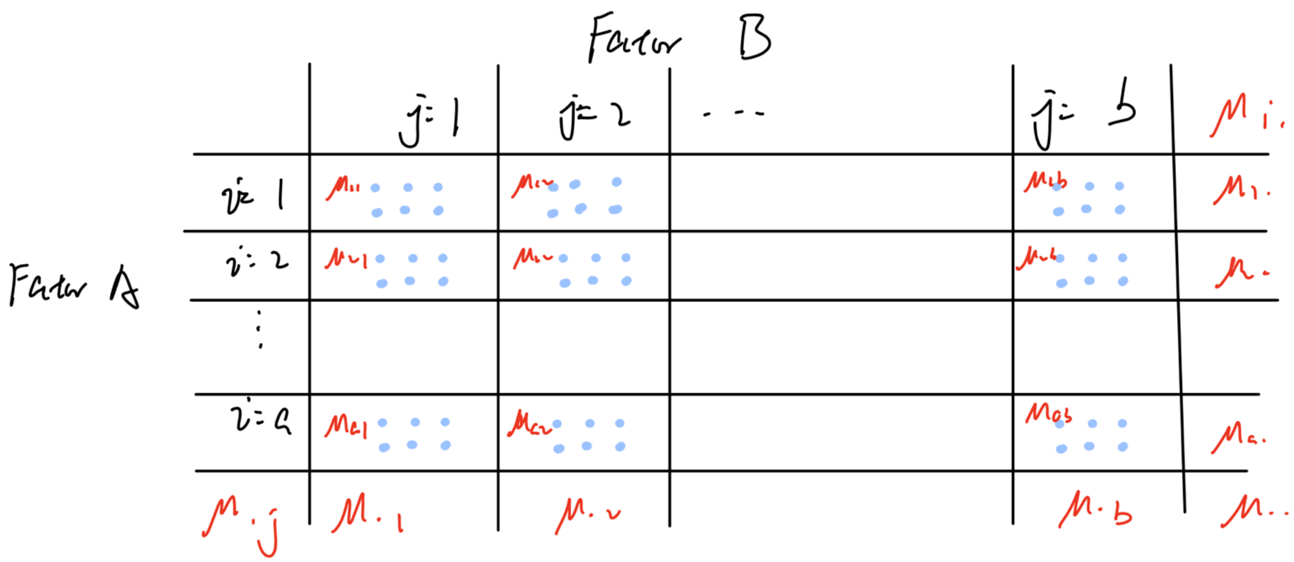
What is the effect due to factor A only?
What is the effect due to factor B only?
Whether there is some extra effects unique to certain combinations of factor A and factor B ?

It's not enough to just look at the treatment means μ_{ij}, as it tells only the difference between treatments, we care about parsing out difference due to factor A and factor B

We want to make statements such as:

" Factor A has beneficial / detrimental effects in general"
"Factor B has beneficial / detrimental effects in general"
" Factor A has beneficial / detrimental effects only when factor B is at certain levels"

Two-Way ANOVA Model for Two-Factor Studies



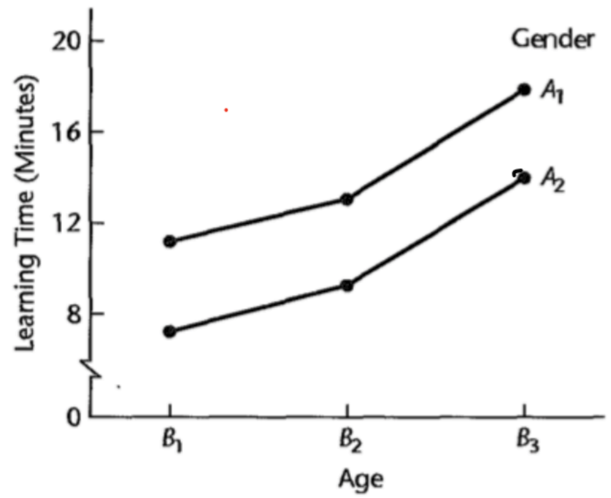
Factor level means: $$\mu_{\cdot j} = \frac{\sum_{i=1}^{a} \mu_{ij}}{a} \qquad \mu_{i\cdot} = \frac{\sum_{j=1}^{b} \mu_{ij}}{b}$$

Overall mean: $$\mu_{\cdot\cdot} = \frac{\sum_{i=1}^{a} \sum_{j=1}^{b} \mu_{ij}}{ab} = \frac{\sum_{i=1}^{a} \mu_{i\cdot}}{a} = \frac{\sum_{j=1}^{b} \mu_{\cdot j}}{b}$$

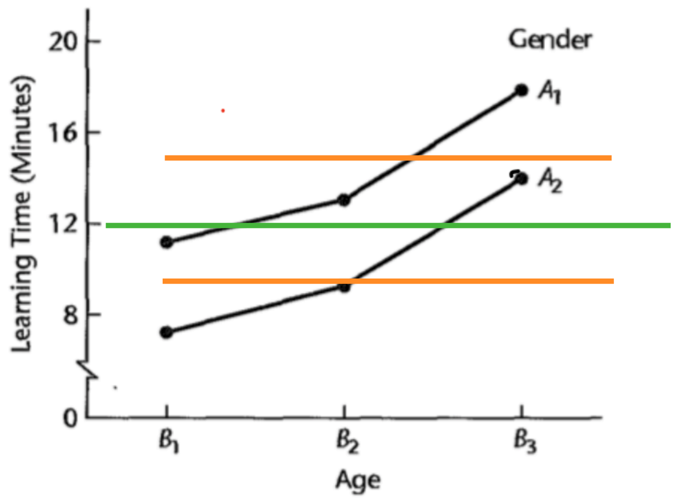
Case I: Additive Factor Effects

Suppose we know the true underlying model where μ_{ij} known

(a) Mean Learning Times (in minutes)				
	Factor B—Age			
Factor A—Gender	$j=1$ Young	$j=2$ Middle	$j=3$ Old	Row Average
$i=1$ Male	11 (μ_{11})	13 (μ_{12})	18 (μ_{13})	14 ($\mu_{1\cdot}$)
$i=2$ Female	7 (μ_{21})	9 (μ_{22})	14 (μ_{23})	10 ($\mu_{2\cdot}$)
Column average	9 ($\mu_{\cdot 1}$)	11 ($\mu_{\cdot 2}$)	16 ($\mu_{\cdot 3}$)	12 ($\mu_{\cdot\cdot}$)



Graphical representation: Treatment Means Plot or Interaction Plot



Graphical representation: Treatment Means Plot or Interaction Plot

What is the effect of factor A?

- how does the mean response change with different levels of factor A, regardless of other factors (here factor B)?

Define: Main effects

Main effect of factor A at ith level $\alpha_i = \mu_{i\cdot} - \mu_{\cdot\cdot}$

How much ith level of factor A shift the factor level mean away from overall mean

$$\sum_{i=1}^{a} \alpha_i = 0$$ Sum of main effects is 0

Case I: Additive Factor Effects

Suppose we know the true underlying model where μ_{ij} known

(a) Mean Learning Times (in minutes)

Factor A—Gender	Factor B—Age: $j=1$ Young	$j=2$ Middle	$j=3$ Old	Row Average
$i=1$ Male	11 (μ_{11})	13 (μ_{12})	18 (μ_{13})	14 ($\mu_{1\cdot}$)
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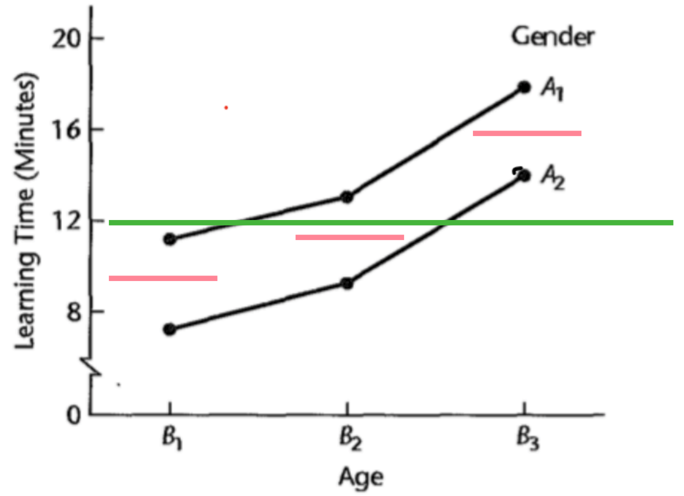
Column average	9 ($\mu_{\cdot 1}$)	11 ($\mu_{\cdot 2}$)	16 ($\mu_{\cdot 3}$)	12 ($\mu_{\cdot\cdot}$)



Graphical representation: Treatment Means Plot or Interaction Plot

What is the effect of factor B?

- how does the mean response change with different levels of factor B, regardless of other factors (here factor B)?

Define: Main effects

Main effect of factor B at jth level $\beta_j = \mu_{\cdot j} - \mu_{\cdot\cdot}$

How much jth level of factor B shift the factor level mean away from overall mean

$\sum_{j=1}^{b} \beta_j = 0$ Sum of main effects is 0

Case I: Additive Factor Effects

Suppose we know the true underlying model where μ_{ij} known

(a) Mean Learning Times (in minutes)

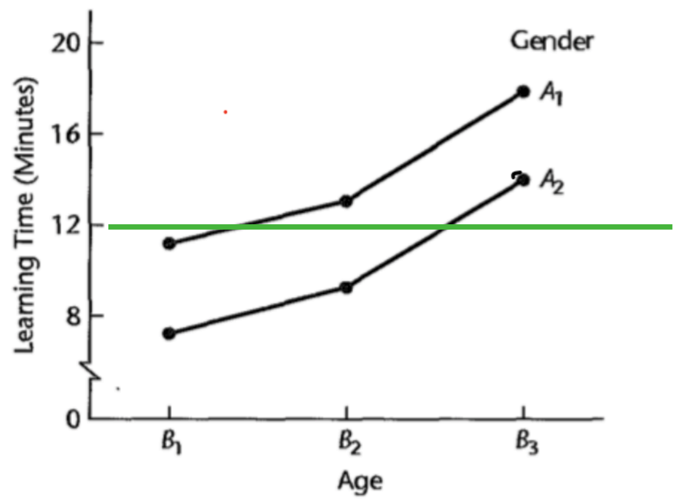
Factor A—Gender	Factor B—Age: $j=1$ Young	$j=2$ Middle	$j=3$ Old	Row Average
$i=1$ Male	11 (μ_{11})	13 (μ_{12})	18 (μ_{13})	14 ($\mu_{1\cdot}$)
$i=2$ Female	7 (μ_{21})	9 (μ_{22})	14 (μ_{23})	10 ($\mu_{2\cdot}$)
Column average	9 ($\mu_{\cdot 1}$)	11 ($\mu_{\cdot 2}$)	16 ($\mu_{\cdot 3}$)	12 ($\mu_{\cdot\cdot}$)

(b) Main Gender Effects (in minutes)

$\alpha_1 = \mu_{1\cdot} - \mu_{\cdot\cdot} = 14 - 12 = 2$
$\alpha_2 = \mu_{2\cdot} - \mu_{\cdot\cdot} = 10 - 12 = -2$

(c) Main Age Effects (in minutes)

$\beta_1 = \mu_{\cdot 1} - \mu_{\cdot\cdot} = 9 - 12 = -3$
$\beta_2 = \mu_{\cdot 2} - \mu_{\cdot\cdot} = 11 - 12 = -1$
$\beta_3 = \mu_{\cdot 3} - \mu_{\cdot\cdot} = 16 - 12 = 4$



$$\mu_{ij} = \mu_{\cdot\cdot} + \alpha_i + \beta_j$$

[mean response] = [overall mean] + [Factor A main effect] + [Factor B main effect]

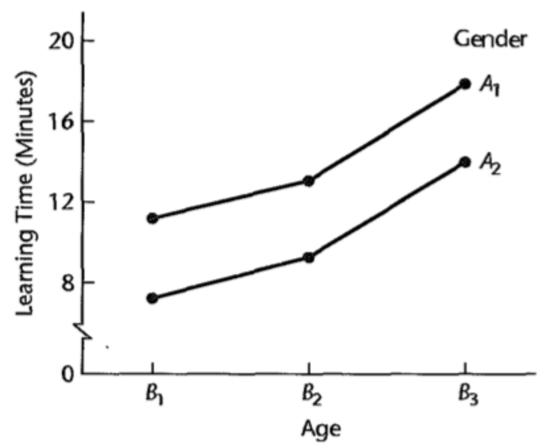
The joint effect of factor A and factor B, as measured by how much mean response deviate from overall mean is

Two main effects adding together : Additive Factor Effects or Factor Effects are Additive

$$\mu_{ij} - \mu_{\cdot\cdot} = \alpha_i + \beta_j$$

Case I: Additive Factor Effects

The significance of additive factor effects:



Additive factor effects <=> Parallelism in interaction plot

For any factor B level j : $\mu_{1j} - \mu_{2j} = \alpha_1 - \alpha_2$
—> curves in the interaction plot are all parallel

- Ease of interpretation:

  Effect of either factor (i.e. main effect of factor A or factor B) does not depend on the level of the other factor

  For example: No matter what is the level of factor A (gender), the effect of being young is reducing learning time by 3, no matter if we consider female and male separately or combined.

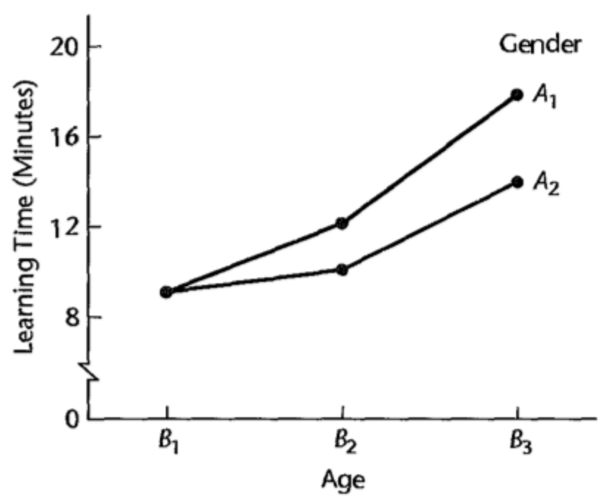
Case II: Interaction Effects

Case (II): Suppose we know the true underlying model where μ_{ij} known

(a) Mean Learning Times (in minutes)

Factor A—Gender	Factor B—Age $j=1$ Young	$j=2$ Middle	$j=3$ Old	Row Average	Main Gender Effect
$i=1$ Male	9 (μ_{11})	12 (μ_{12})	18 (μ_{13})	13 ($\mu_{1\cdot}$)	1 (α_1)
$i=2$ Female	9 (μ_{21})	10 (μ_{22})	14 (μ_{23})	11 ($\mu_{2\cdot}$)	−1 (α_2)
Column average	9 ($\mu_{\cdot 1}$)	11 ($\mu_{\cdot 2}$)	16 ($\mu_{\cdot 3}$)	12 ($\mu_{\cdot\cdot}$)	
Main age effect	−3 (β_1)	−1 (β_2)	4 (β_3)		

$$\mu_{ij} \neq \mu_{\cdot\cdot} + \alpha_i + \beta_j$$



Factor A has no effect on Y when factor B=1, but has substantial effect when B=2 and B=3
This differential influence of factor A, which depends on factor B, implies that factor A and factor B interact in their effect on Y

Case II: Interaction Effects

$$\mu_{ij} \neq \mu_{..} + \alpha_i + \beta_j$$

$$\Rightarrow \mu_{ij} - \left(\mu_{..} + \alpha_i + \beta_j\right) \neq 0$$

There exists some effect unique to this combination of factor levels with $A = i, B = j$, due to the interaction of ith level of factor A and $j$the level of factor B

Define: Interaction (or interaction effect) of ith level of factor A with $j$the level of factor B

$$\begin{aligned} \gamma_{ij} &= \mu_{ij} - \left(\mu_{..} + \alpha_i + \beta_j\right) \\ &= \mu_{ij} - \left(\mu_{..} + \mu_{i.} - \mu_{..} + \mu_{.j} \cdot \mu_{..}\right) \\ &= \mu_{ij} - \mu_{i.} - \mu_{.j} + \mu_{..} \end{aligned}$$

If two factors have no interaction = additive, then all interaction effects are 0

$$\gamma_{ij} = 0 \text{ for all } i, j$$

The interactions have two natural constraints by definition:

$$\sum_i \gamma_{ij} = 0 \quad j = 1, \ldots, b$$

$$\sum_j \gamma_{ij} = 0 \quad i = 1, \ldots, a$$

—> therefore, $\sum_i \sum_j \gamma_{ij} = 0$

Proof: $\sum_i \gamma_{ij} = \sum_{i=1}^{a} \left(\mu_{ij} - \mu_{..} - \alpha_i - \beta_j\right)$

$$= \sum_i \mu_{ij} - a\mu_{..} - \sum_i \alpha_i - a\beta_j = a\mu_{.j} - a\mu_{..} - a\left(\mu_{.j} - \mu_{..}\right) = 0$$

Case II: Interaction Effects

(a) Mean Learning Times (in minutes)

	Factor B—Age				
Factor A—Gender	**$j=1$ Young**	**$j=2$ Middle**	**$j=3$ Old**	**Row Average**	**Main Gender Effect**
$i=1$ Male	9 (μ_{11})	12 (μ_{12})	18 (μ_{13})	13 ($\mu_{1\cdot}$)	1 (α_1)
$i=2$ Female	9 (μ_{21})	10 (μ_{22})	14 (μ_{23})	11 ($\mu_{2\cdot}$)	−1 (α_2)
Column average	9 ($\mu_{\cdot 1}$)	11 ($\mu_{\cdot 2}$)	16 ($\mu_{\cdot 3}$)	12 ($\mu_{\cdot\cdot}$)	
Main age effect	−3 (β_1)	−1 (β_2)	4 (β_3)		

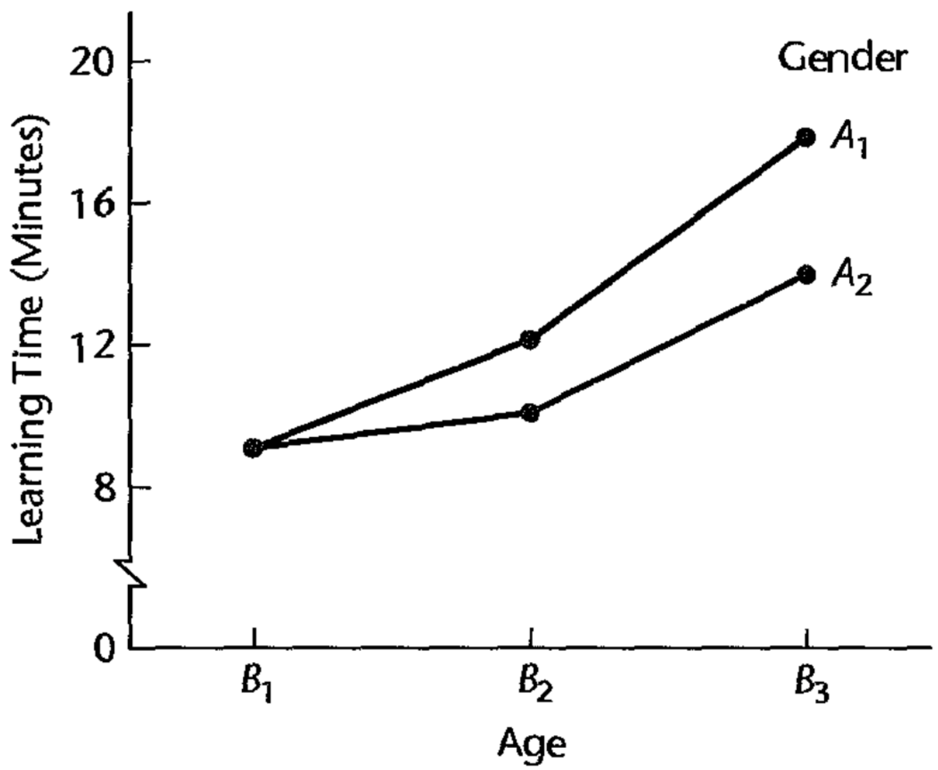
(b) Interactions (in minutes)

	$j=1$	$j=2$	$j=3$	Row Average
$i=1$	−1	0	1	0
$i=2$	1	0	−1	0
Column average	0	0	0	0

-1: 1 unit smaller than what is expected by an additive effects model where only main effects are considered

Case II: Interaction Effects

The significance of Interaction effects, or the lack of additive factor effects:



Presence of Interaction effects <=> Nonparallel in interaction plot

For factor B level j :

$$\mu_{1j} - \mu_{2j} = \left(\mu_{..} + \alpha_1 + \beta_j + \gamma_{1j}\right) - \left(\mu_{..} + \alpha_2 + \beta_j + \gamma_{2j}\right)$$

$$= \alpha_1 - \alpha_2 + \gamma_{1j} - \gamma_{2j}$$

Not same for different levels of j

—> curves in the interaction plot are not parallel

Distinction between Additive factor effects and Interaction effects:

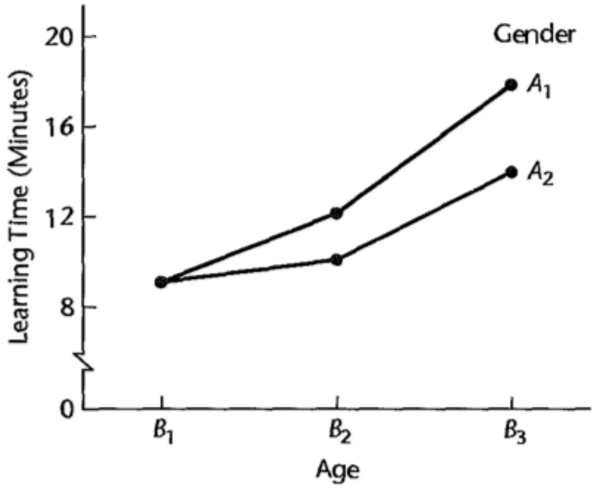
By examining whether the curves in interaction plot are parallel, nonparallel occurs if and only if the interaction effects are present.

Case II: Interaction Effects

What happens when interaction effects are present?

(a) Mean Learning Times (in minutes)

Factor A—Gender	Factor B—Age: $j=1$ Young	$j=2$ Middle	$j=3$ Old	Row Average	Main Gender Effect
$i=1$ Male	9 (μ_{11})	12 (μ_{12})	18 (μ_{13})	13 ($\mu_{1\cdot}$)	1 (α_1)
$i=2$ Female	9 (μ_{21})	10 (μ_{22})	14 (μ_{23})	11 ($\mu_{2\cdot}$)	−1 (α_2)
Column average	9 ($\mu_{\cdot 1}$)	11 ($\mu_{\cdot 2}$)	16 ($\mu_{\cdot 3}$)	12 ($\mu_{\cdot\cdot}$)	
Main age effect	−3 (β_1)	−1 (β_2)	4 (β_3)		



In additive factor effects case, the main effects is a meaningful measure of factor A (or B) effects.

interaction effects are not present, then it means factor A and B can be studied separately, since the effect of one factor does not depend on the other factor, i.e. they do not interact.

When interaction effects are present, the main effects alone no longer are meaningful measure of factor A (or B) effects, since they are intertwined.

more complicated interpretation

but if it is true, it means the factors A and B are in fact “interacting” in some way:

there are some combinations of A and B that may lead to better or worse outcome,

while some combinations may have no effect at all.

If interaction effects indeed are present, then it’s the unique strength of the two-factor studies

Case II: Interaction Effects

Important versus Unimportant Interactions:

Sometimes when two factors interact, the interaction effects are so small

Subject area specialist or researcher decide that such a small interaction effect can be considered unimportant interactions.

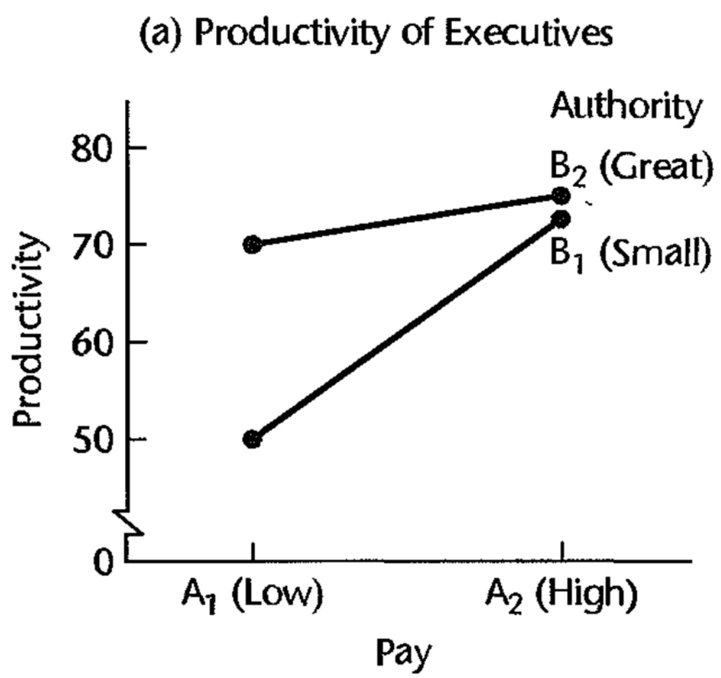
Reduce to additive factor effects, where no interaction exists and each factor can be studied separately based on factor level means $\mu_{i.}$ and $\mu_{.j}$

Case II: Interaction Effects

Interaction Patterns

(a) Productivity of Executives

Factor *A*—Pay	Factor *B*—Authority: Small	Factor *B*—Authority: Great
Low	50	72
High	74	75



For low-paid executives with small authority, raising the pay only or increasing the authority of low-paid executives alone, leads to substantial increased productivity.

However, combining both high pay and great authority has a smaller beneficial effect than either one alone.

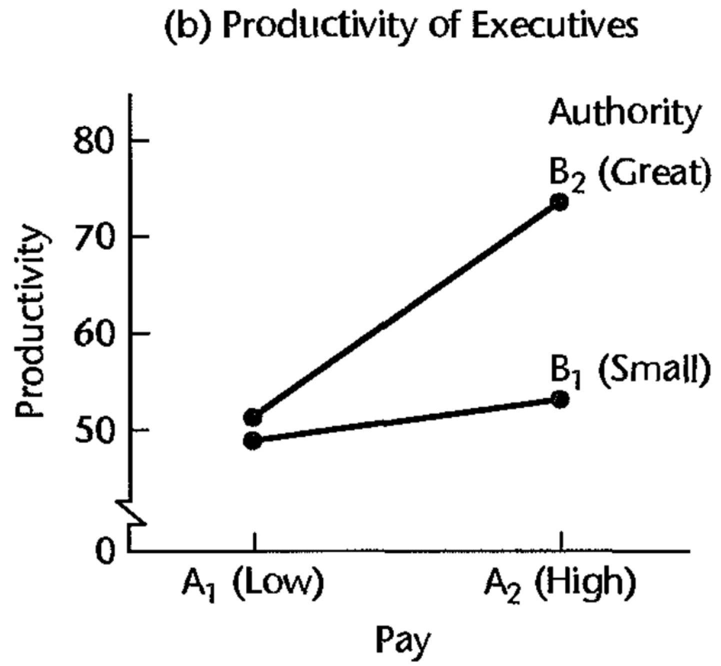
(Combined forces cancel each one out)

Case II: Interaction Effects

Interaction Patterns

(b) Productivity of Executives

	Factor *B*—Authority	
Factor *A*—Pay	**Small**	**Great**
Low	50	52
High	53	75



For low-paid executives with small authority, raising the pay or increasing the authority leads to almost negligible increased productivity.

Only when combining both high pay and great authority has a substantial beneficial effect.

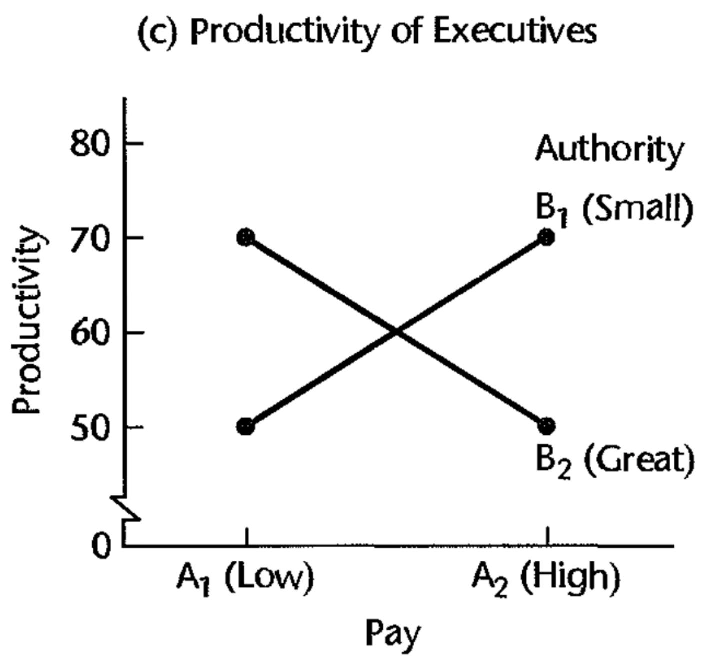
(Combined forces reinforce each other)

Case II: Interaction Effects

Interaction Patterns

(c) Productivity of Executives

	Factor B—Authority	
Factor A—Pay	Small	Great
Low	50	72
High	72	50



The main effects for both factors are 0: misleading if we assume no interaction effects present or run separate one-factor studies

There are indeed factor effects of A and B, but they would not be seen by main effects, due to interactions in opposite directions that balance out.

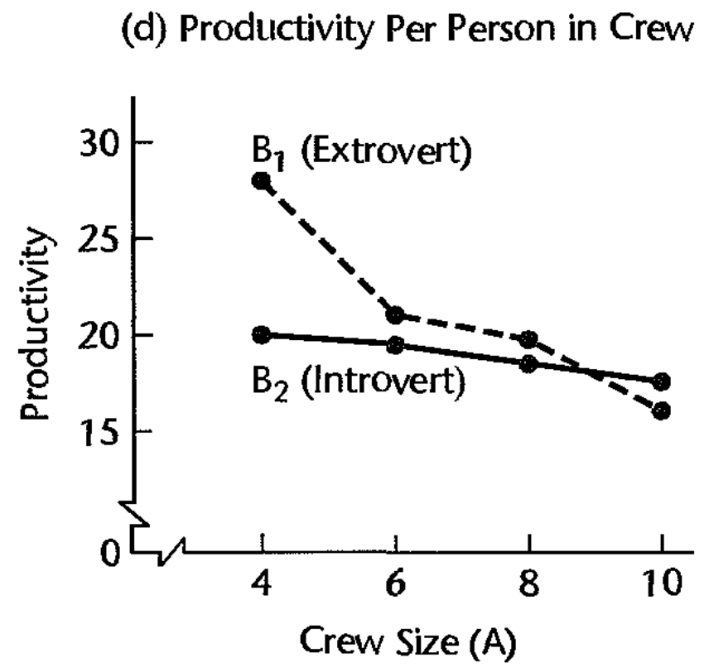
(Weird, rare, but possible situation)

Case II: Interaction Effects

Interaction Patterns

(d) Productivity per Person in Crew

Factor *A*—Crew Size	Factor *B*—Personality of Crew Chief: Extrovert	Introvert
4 persons	28	20
6 persons	22	20
8 persons	20	19
10 persons	17	18



size of crew and personality of crew chief interact in a complex way

Extrovert crew chief has a huge advantage over introvert, in a small crew.

However, this advantage become smaller and smaller when crew size become larger, crew size of 10 with an introvert crew chief can even lead to a slightly higher productivity.

Two-Way ANOVA Model for Two-Factor Studies

$$Y_{ijk} = \mu_{ij} + \varepsilon_{ijk}$$ Treatment means parameterization

$$= \mu_{..} + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}$$ Factor effects parameterization

- $\mu_{..} = \dfrac{\sum_{i=1}^{a} \sum_{j=1}^{b} \mu_{ij}}{ab}$: overall mean
- $\alpha_i = \mu_{i.} - \mu_{..}$ main effect of factor A at ith level

  Subject to constraint $\sum \alpha_i = 0$
- $\beta_j = \mu_{.j} - \mu_{..}$ main effect of factor B at jth level

  Subject to constraint $\sum \beta_j = 0$
- $\gamma_{ij} = \mu_{ij} - \alpha_i - \beta_j = \mu_{ij} - \mu_{i.} - \mu_{.j} + \mu_{..}$ interaction effect of factor A at ith level with factor B at jth level

  Subject to a+b-1 constraints

  $$\sum_i \gamma_{ij} = 0 \; j = 1,\ldots,b$$
  $$\sum_j \gamma_{ij} = 0 \; i = 1,\ldots,a$$
- ε_{ijk} are independent $N\left(0,\sigma^2\right)$ for $i = 1,\ldots,a; j = 1,\ldots,b; k = 1,\ldots,n$

Fitting the Two-Way ANOVA Model

Least squares estimates for treatment means in treatment means parameterization:

$$Q = \sum_i \sum_j \sum_k \left(Y_{ijk} - \mu_{ij}\right)^2$$

$$\hat{\mu}_{ij} = \bar{Y}_{ij\cdot} \text{ for } i = 1\ldots a, b = 1\ldots b$$

Least squares estimates for parameters in factor effects parameterization:

$$\hat{\mu}_{\cdot\cdot} = \frac{\sum_i \sum_j \hat{\mu}_{ij}}{ab} = \frac{\sum_i \sum_j \bar{Y}_{ij\cdot}}{ab} = \bar{Y}_{\ldots}$$

$$\hat{\alpha}_i = \hat{\mu}_{i\cdot} - \hat{\mu}_{\cdot\cdot} = \frac{\sum_j \bar{Y}_{ij\cdot}}{b} - \bar{Y}_{\ldots} = \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\ldots}$$

$$\hat{\beta}_j = \hat{\mu}_{\cdot j} - \hat{\mu}_{\cdot\cdot} = \frac{\sum_i \bar{Y}_{ij\cdot}}{a} - \bar{Y}_{\ldots} = \bar{Y}_{\cdot j\cdot} - \bar{Y}_{\ldots}$$

$$\hat{\gamma}_{ij} = \hat{\mu}_{ij} - \hat{\mu}_{i\cdot} - \hat{\mu}_{\cdot j} + \hat{\mu}_{\cdot\cdot} = \bar{Y}_{ij\cdot} - \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot j\cdot} + \bar{Y}_{\ldots}$$

Where:

$$\bar{Y}_{\ldots} = \frac{\sum_i \sum_j \sum_k Y_{ijk}}{abn}$$

$$\bar{Y}_{i\cdot\cdot} = \frac{\sum_j \sum_k Y_{ijk}}{bn}$$

$$\bar{Y}_{\cdot j\cdot} = \frac{\sum_i \sum_k Y_{ijk}}{an}$$

Fitting the Two-Way ANOVA Model

- fitted value for an observation Y_{ij}

  ANOVA model's "best guess" or "best prediction" for Y_{ijk}

  $\hat{Y}_{ijk} = \hat{\mu}_{ij} = \overline{Y_{ij\cdot}}$

- residual e_{ij} corresponds to observation Y_{ij} is

$$e_{ijk} = Y_{ijk} - \hat{Y}_{ijk} = Y_{ijk} - \bar{Y}_{ij\cdot}$$

Difference between observed value and fitted value which is estimated factor level mean

Example

(Kidney failure hospitalization)

Kidney failure patients are commonly treated on dialysis machines that filter toxic substances from the blood.
The appropriate "dose" for effective treatment depends, among other things, on duration of treatment and weight gain between treatments as a result of fluid buildup.
To study the effects of these two factors on the number of days hospitalized (attributable to the disease) during a year, a random sample of '10 patients per group who had undergone treatment at a large dialysis facility was obtained.

Treatment duration (factor A) was categorized into two groups: short duration (average dialysis time for the year under four hours) and long duration (average dialysis time for the year equal to or greater than four hours).
Average weight gain between treatments (factor B) during the year was categorized into three groups: slight, moderate, and substantial.

Two-way ANOVA Model:

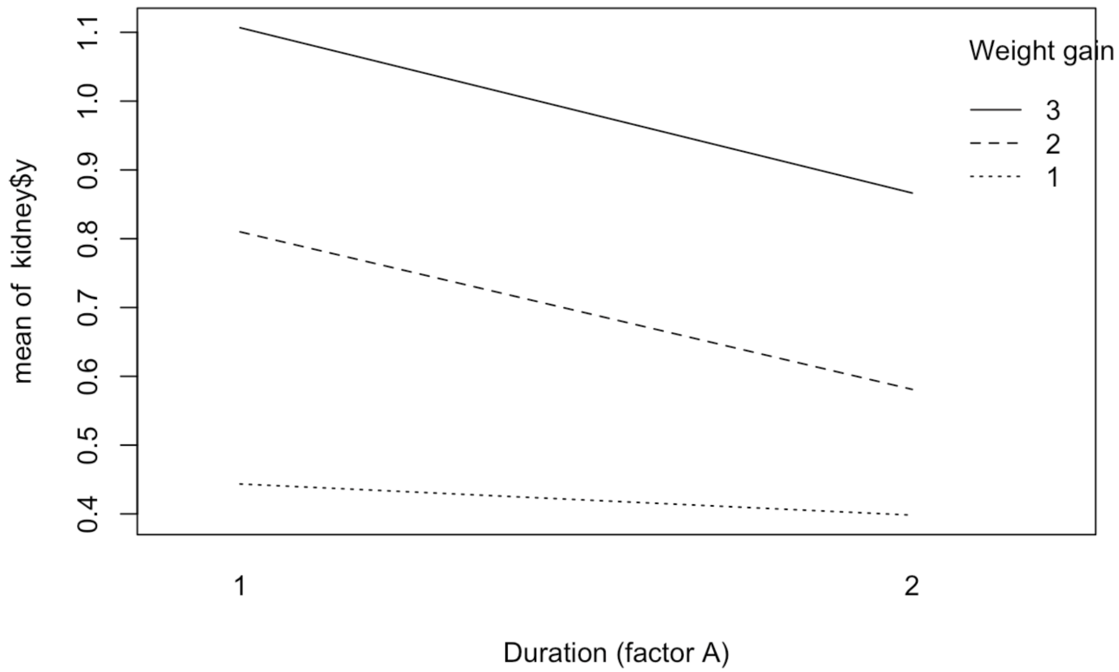
$Y_{ijk} = \mu_{ij} + \varepsilon_{ijk}$ Treatment means parameterization

$= \mu_{..} + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}$ Factor effects parameterization

$i = 1,2; \;\; j = 1,2,3; k = 1\ldots10$

Example

Draw an interaction plot based on the estimated treatment means. Comment on the plot in terms of interaction effects, factor A and B main effects.



The nonparallel lines suggests that the two factors interact, that is, the effect of factor B depends on the levels of factor A and vice versa.

Main effects:

Long treatment duration tends to reduce days of hospitalization, regardless of weight gain.
Larger weight gain tends to increase days of hospitalization, regardless of treatment duration.

The interaction pattern:

Larger weight gain prolongs the hospitalization days substantially for patients with short treatment duration, but this prolonged effect is smaller for patients with long treatment duration.

Longer treatment duration has negligible effect for patients with small weight gain, but has larger effect for patients with moderate or substantial weight gains.

Example

Least squares estimates for the parameters in the factor effects parameterization.

Overall mean:

$$\hat{\mu}_{..} = \frac{\sum_i \sum_j \hat{\mu}_{ij}}{ab} = \frac{\sum_i \sum_j \bar{Y}_{ij}}{ab} = \bar{Y}_{...} = 0.7$$

Factor A main effects:

$$\hat{\alpha}_1 = 0.09 \qquad \hat{\alpha}_2 = -0.09$$

Factor B main effects:

$$\hat{\beta}_1 = -0.28 \qquad \hat{\beta}_2 = -0.01 \qquad \hat{\beta}_3 = 0.29$$

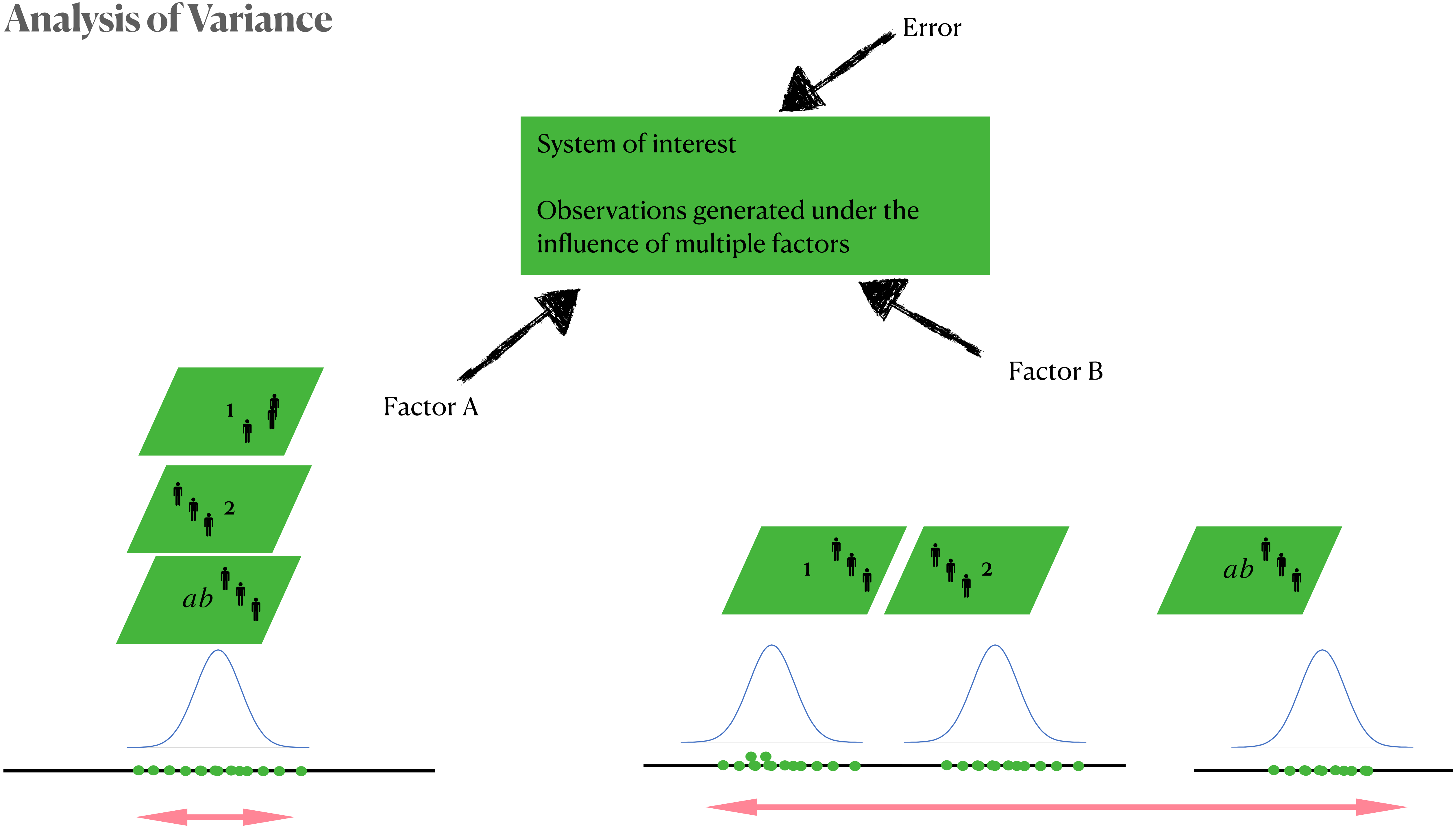
Interaction effects $\hat{\gamma}_{ij}$:

	j=1 Mild	j=2 Moderate	j=3 Substantial
i=1 Short	-0.06	0.03	0.03
i=2 Long	0.06	-0.03	-0.03

Two-Factor Studies with Equal Sample Sizes

- Two-Way ANOVA Model
- Analysis of Variance
- Tests for Interaction effects and main effects
- Analysis of Factor A and B Main Effects (When Factors Do Not Interact)
- Analysis of Treatment Means when Interactions Are Present

Analysis of Variance



Without factors A and B, the observations have some natural variation due to other extraneous factors, i.e. “error variance”

If some combinations of factor A and B indeed has some effects on the system, then we would expect more volatility .

Analysis of Variance

Partition of Total Sum of Squares

$$Y_{ijk} - \bar{Y}_{\dots} = \bar{Y}_{ij\cdot} - \bar{Y}_{\dots} + Y_{ijk} - \bar{Y}_{ij\cdot}$$

Total deviation

Deviation of estimated treatment mean around overall mean

Deviation around estimated treatment mean

$$\underbrace{\sum_i \sum_j \sum_k \left(Y_{ijk} - \bar{Y}_{\dots}\right)^2}_{\text{Total variation}} = \sum_i \sum_j \sum_k \left(\bar{Y}_{ij\cdot} - \bar{Y}_{\dots}\right)^2 + \sum_i \sum_j \sum_k \left(Y_{ijk} - \bar{Y}_{ij\cdot}\right)^2$$

$$= \underbrace{n \sum_i \sum_j \left(\bar{Y}_{ij\cdot} - \bar{Y}_{\dots}\right)^2}_{\text{Variation due to factor A and B}} + \underbrace{\sum_i \sum_j \sum_k \left(Y_{ijk} - \bar{Y}_{ij\cdot}\right)^2}_{\text{Variation due to extraneous factors}}$$

Let $SSTO = \sum_i \sum_j \sum_k \left(Y_{ijk} - \bar{Y}_{\dots}\right)^2$

$$SSTR = n \sum_i \sum_j \left(\bar{Y}_{ij} - \bar{Y}_{\dots}\right)^2$$

$$SSE = \sum_i \sum_j \sum_k \left(Y_{ijk} - \bar{Y}_{ij\cdot}\right)^2 = \sum_i \sum_j \sum_k e_{ijk}^2$$

$$\longrightarrow \boxed{\text{SSTO} \quad = \quad \text{SSTR} \quad + \quad \text{SSE}}$$

Analysis of Variance

Partition of Treatment Sum of Squares.

$$\underbrace{\bar{Y}_{ij} - \bar{Y}_{\ldots}}_{\text{Deviation of estimated treatment mean around overall mean}} = \underbrace{\bar{Y}_{i..} - \bar{Y}_{...}}_{\text{A main effect}} + \underbrace{\bar{Y}_{.j.} - \bar{Y}_{...}}_{\text{B main effect}} + \underbrace{\bar{Y}_{ij} - \bar{Y}_{i..} - \bar{Y}_{j.} + \bar{Y}_{...}}_{\text{AB interaction effect}}$$

$$\underbrace{\sum_i \sum_j \sum_k \left(\bar{Y}_{ij} - \bar{Y}_{\ldots}\right)^2}_{\text{SSTR : treatment sum of squares}} = bn \underbrace{\sum_i \left(\bar{Y}_{i..} - \bar{Y}_{...}\right)^2}_{\text{SSA: factor A sum of squares}} + an \underbrace{\sum_j \left(\bar{Y}_{.j.} - \bar{Y}_{...}\right)^2}_{\text{SSB: factor B sum of squares}} + \underbrace{\sum_i \sum_j \sum_k \left(\bar{Y}_{ij} - \bar{Y}_{i..} - \bar{Y}_{j.} + \bar{Y}_{...}\right)^2}_{\text{AB interaction sum of squares}}$$

$$\longrightarrow \quad \boxed{\text{SSTR} = \text{SSA} + \text{SSB} + \text{SSAB}}$$

Analysis of Variance

Combined Partition of Total Sum of Squares.

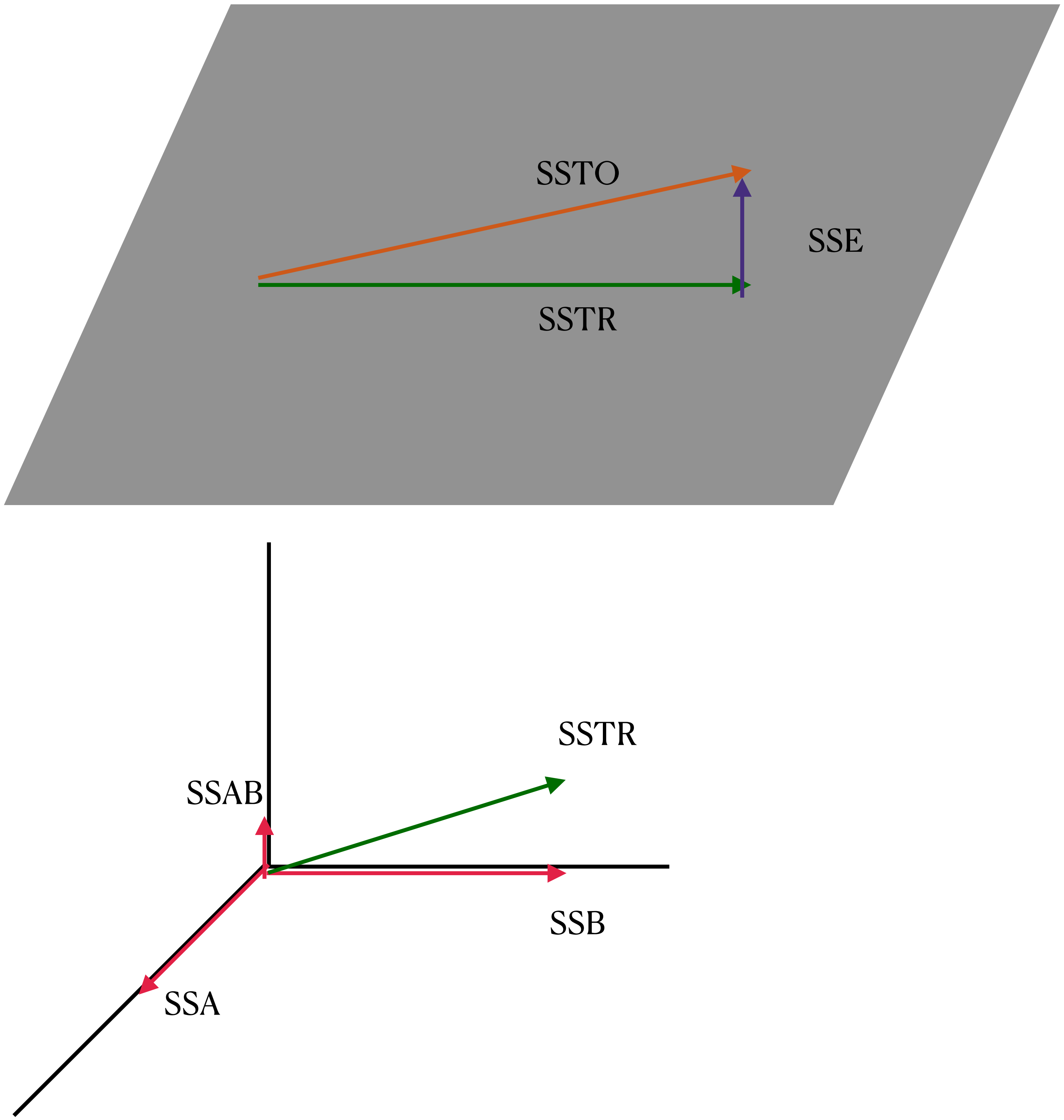
$$\underbrace{\sum_i \sum_j \sum_k \left(Y_{ijk} - \bar{Y}_{\dots}\right)^2}_{\text{SSTO}} = \sum_i \sum_j \sum_k \left(\bar{Y}_{ij\cdot} - \bar{Y}_{\dots}\right)^2 + \sum_i \sum_j \sum_k \left(Y_{ijk} - \bar{Y}_{ij\cdot}\right)^2$$

$$= \underbrace{n \sum_i \sum_j \left(\bar{Y}_{ij\cdot} - \bar{Y}_{\dots}\right)^2}_{\text{SSTR}} + \underbrace{\sum_i \sum_j \sum_k \left(Y_{ijk} - \bar{Y}_{ij\cdot}\right)^2}_{\text{SSE}}$$

$$= \underbrace{bn \sum_i \left(\bar{Y}_{i\cdot\cdot} - \bar{Y}_{\dots}\right)^2}_{\text{SSA: factor A sum of squares}} + \underbrace{an \sum_j \left(\bar{Y}_{\cdot j\cdot} - \bar{Y}_{\dots}\right)^2}_{\text{SSB: factor B sum of squares}} + \underbrace{\sum_i \sum_j \sum_k \left(\bar{Y}_{ij} - \bar{Y}_{i\cdot\cdot} - \bar{Y}_{j\cdot} + \bar{Y}_{\dots}\right)^2}_{\text{SSAB: AB interaction sum of squares}} + \underbrace{\sum_i \sum_j \sum_k \left(Y_{ijk} - \bar{Y}_{ij\cdot}\right)^2}_{\text{SSE}}$$

$$\longrightarrow \boxed{\text{SSTO} = \text{SSA + SSB + SSAB + SSE}}$$

Geometry of Decomposition of Variance:



Degrees of Freedom

Think of: dimensions of the space where an estimator lives in and allows to run free

$$\underbrace{\sum_i \sum_j \sum_k \left(Y_{ijk} - \bar{Y}_{...}\right)^2}_{\text{SSTO}} = \underbrace{bn \sum_i \left(\bar{Y}_{i..} - \bar{Y}_{...}\right)^2}_{\text{SSA: factor A sum of squares}} + \underbrace{an \sum_j \left(\bar{Y}_{.j.} - \bar{Y}_{...}\right)^2}_{\text{SSB: factor B sum of squares}} + \underbrace{\sum_i \sum_j \sum_k \left(\bar{Y}_{ij} - \bar{Y}_{i..} - \bar{Y}_{j.} + \bar{Y}_{...}\right)^2}_{\text{SSAB: AB interaction sum of squares}} + \underbrace{\sum_i \sum_j \sum_k \left(Y_{ijk} - \bar{Y}_{ij.}\right)^2}_{\text{SSE}}$$

How many independence pieces of information go into each quantity?

$$Y_{ijk} - \overline{Y_{...}}$$

abn pieces

But $\sum_i \sum_j \sum_k (Y_{ijk} - \overline{Y_{...}}) = 0$

$$df(SSTO) = abn - 1$$

$$\overline{Y_{i..}} - \overline{Y_{...}}$$

a pieces

But $\sum_i (\overline{Y_{i..}} - \overline{Y_{...}}) = 0$

$$df(SSA) = a - 1$$

$$\overline{Y_{.j.}} - \overline{Y_{...}}$$

b pieces

But $\sum_j (\overline{Y_{.j.}} - \overline{Y_{...}}) = 0$

$$df(SSB) = b - 1$$

$$\bar{Y}_{ij} - \bar{Y}_{i..} - \bar{Y}_{j.} + \bar{Y}_{...} = \hat{\gamma}_{ij}$$

ab pieces

But $\sum_i \hat{\gamma}_{ij} = 0 \; j = 1,\ldots,b$

$\sum_j \hat{\gamma}_{ij} = 0 \; i = 1,\ldots,a$

$$Y_{ijk} - \overline{Y_{ij.}}$$

abn pieces

But $\sum_k (Y_{ijk} - \overline{Y_{ij.}}) = 0$ for $i = 1\ldots a, j = 1\ldots b$

$$df(SSE) = abn - ab = ab(n-1)$$

$$df(SSAB) = ab - (a + b - 1) = (a-1)(b-1)$$

$$MSA = \frac{SSA}{a-1} \qquad MSB = \frac{SSB}{b-1} \qquad MSAB = \frac{SSAB}{(a-1)(b-1)} \qquad MSE = \frac{SSE}{ab(n-1)}$$

What's expected values of Mean Squares?

$$E\{MSE\} = \sigma^2$$

$$E\{MSA\} = \sigma^2 + nb\frac{\sum \alpha_i^2}{a-1} = \sigma^2 + nb\frac{\sum \left(\mu_{i\cdot} - \mu_{\cdot\cdot}\right)^2}{a-1}$$

$$E\{MSB\} = \sigma^2 + na\frac{\sum \beta_j^2}{b-1} = \sigma^2 + na\frac{\sum \left(\mu_{\cdot j} - \mu_{\cdot\cdot}\right)^2}{b-1}$$

$$E\{MSAB\} = \sigma^2 + n\frac{\sum \sum \gamma_{ij}^2}{(a-1)(b-1)}$$

$$= \sigma^2 + n\frac{\sum \sum \left(\mu_{ij} - \mu_{i\cdot} - \mu_{\cdot j} + \mu_{\cdot\cdot}\right)^2}{(a-1)(b-1)}$$

ANOVA Table for Two-factor Studies (Two-Way ANOVA Table)

Source of Variation	SS	df	MS	E{MS}
Factor A	$SSA = nb\sum(\bar{Y}_{i..} - \bar{Y}_{...})^2$	$a-1$	$MSA = \frac{SSA}{a-1}$	$\sigma^2 + bn\frac{\sum(\mu_{i.} - \mu_{..})^2}{a-1}$
Factor B	$SSB = na\sum(\bar{Y}_{.j.} - \bar{Y}_{...})^2$	$b-1$	$MSB = \frac{SSB}{b-1}$	$\sigma^2 + an\frac{\sum(\mu_{.j} - \mu_{..})^2}{b-1}$
AB interactions	$SSAB = n\sum\sum(\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...})^2$	$(a-1)(b-1)$	$MSAB = \frac{SSAB}{(a-1)(b-1)}$	$\sigma^2 + n\frac{\sum\sum(\mu_{ij} - \mu_{i.} - \mu_{.j} + \mu_{..})^2}{(a-1)(b-1)}$
Error	$SSE = \sum\sum\sum(Y_{ijk} - \bar{Y}_{ij.})^2$	$ab(n-1)$	$MSE = \frac{SSE}{ab(n-1)}$	σ^2
Total	$SSTO = \sum\sum\sum(Y_{ijk} - \bar{Y}_{...})^2$	$nab-1$		

Compare E(MSAB) vs E(MSE)

- If there are no interactions, i.e. the additive factor effects model holds
  - MSAB has same expiation as MSE (they tend to be similar in numeric values)
- Otherwise, when interaction exists
  - MSAB tend to be larger than MSE

Analysis of Variance

Compare E(MSA) vs E(MSE)

- If there are no factor A main effects
  - MSA has same expiation as MSE (they tend to be similar in numeric values)
- Otherwise
  - MSA tend to be larger than MSE

Compare E(MSB) vs E(MSE)

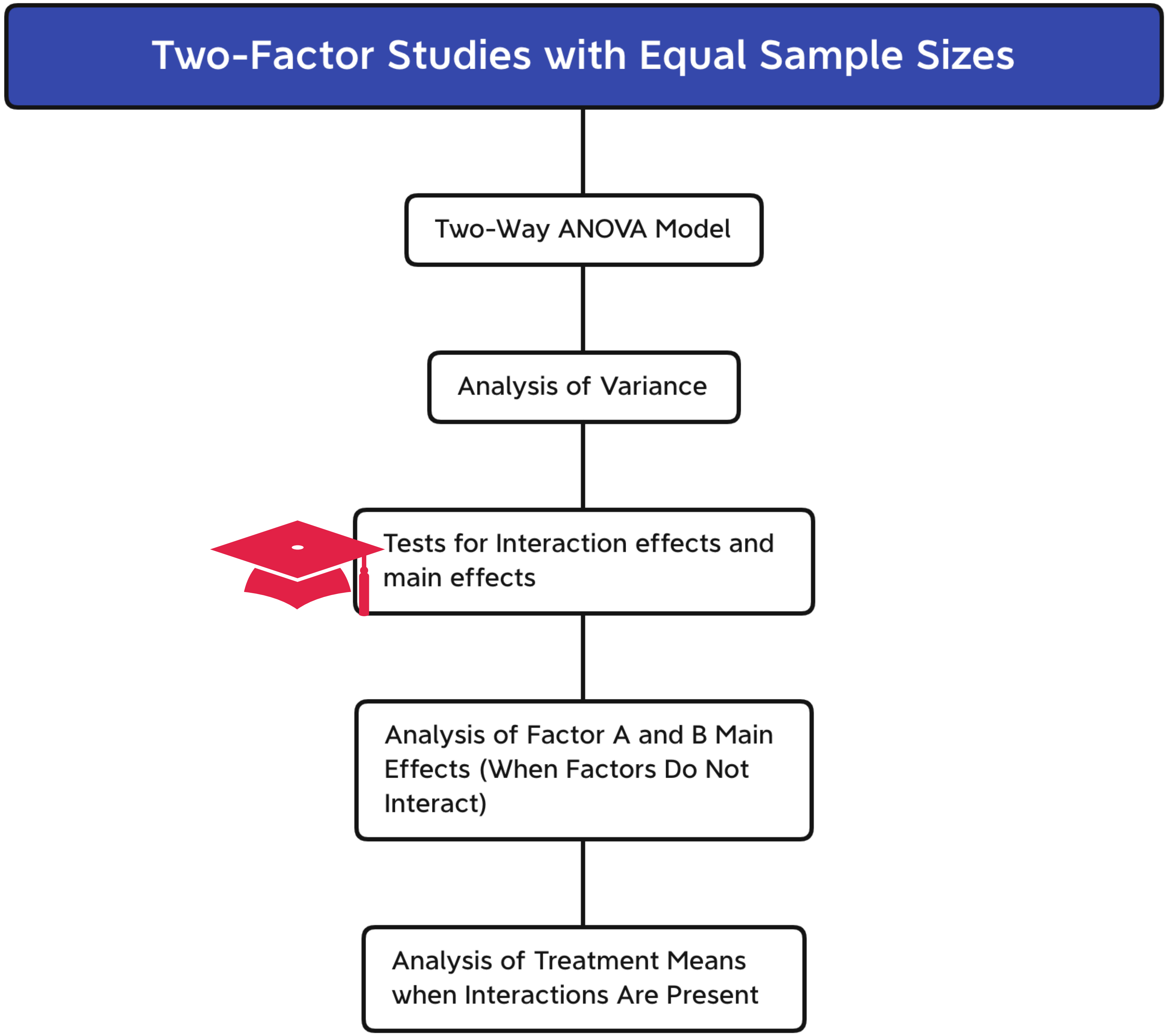
- If there are no factor B main effects
  - MSB has same expiation as MSE (they tend to be similar in numeric values)
- Otherwise
  - MSB tend to be larger than MSE

They suggest that ratios of Mean Squares provide evidence about the main effects and interactions, which will be the basis for F tests

Example

ANOVA Table

	SS	df	MS
factor A	0.441293435727881	1	0.441293435727881
factor B	3.20098397061084	2	1.60049198530542
interaction AB	0.11989261439729	2	0.0599463071986451
Error	5.46770181944887	54	0.101253737397201
Totoal	9.22987184018488	59	•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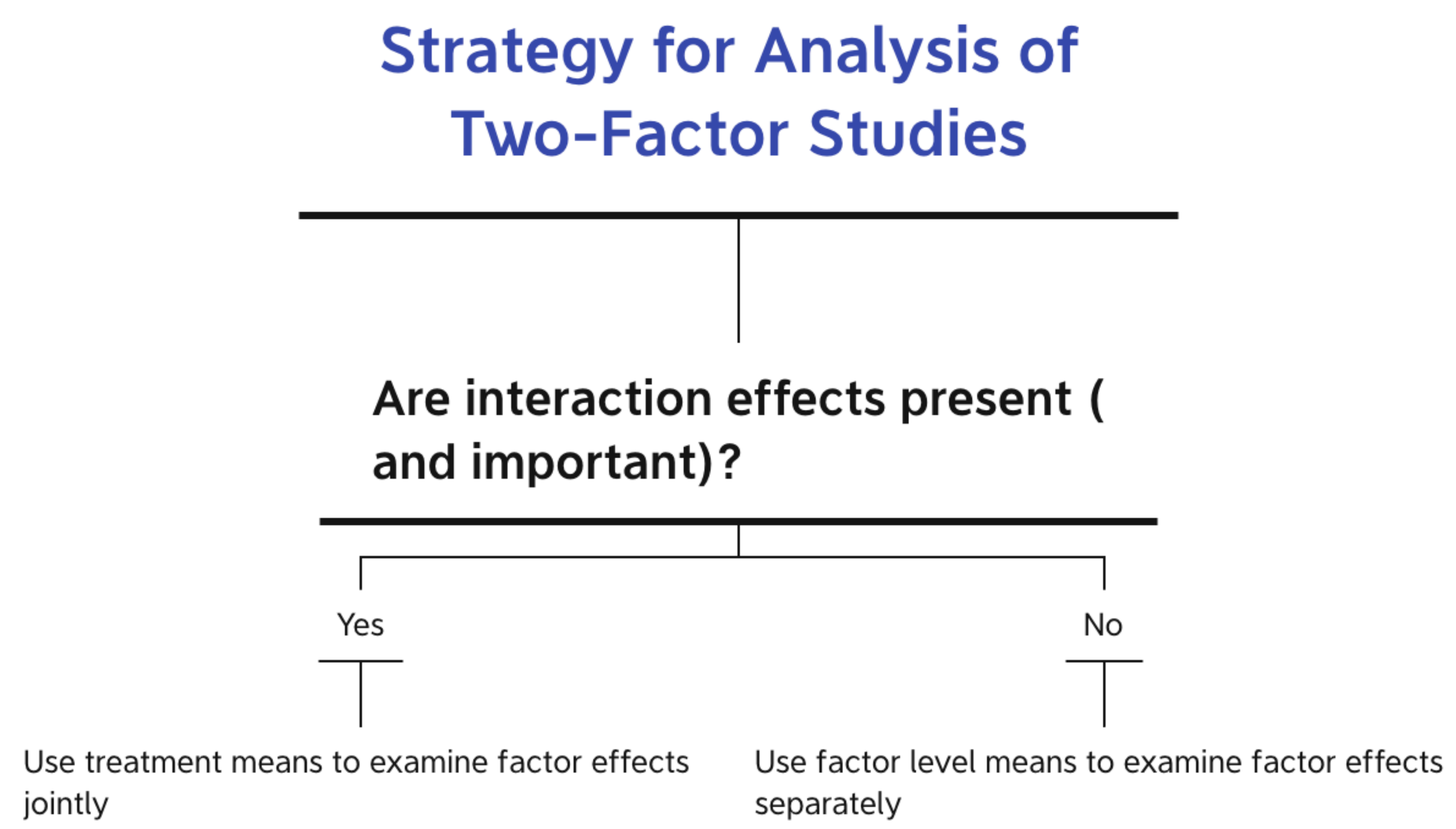
Strategy of Analysis of Two-Factor Studies

Scientific inquiry is guided by the principle:

simple, parsimonious explanations of observed phenomena tend to be the most effective

Additive factor effects : much simpler explanation of factor effects

Interaction effects complicates the explanation



Test for Interactions

To test whether or not the factor level means are the same:

H_0 : all $\gamma_{ij} = 0$

H_a : not all $\gamma_{ij} = 0$

Test statistic: $F^* = \dfrac{MSAB}{MSE}$

Large value of F^* support H_a

Small value, when $F^* \approx 1$ support H_0

—> We reject H_0 for large value of F^*, i.e. $F^* \geq c$

$$\longrightarrow \quad F^* = \frac{MSAB}{MSE} = \frac{\frac{\frac{((a-1)(b-1))MSAB}{\sigma^2}}{(a-1)(b-1)}}{\frac{\frac{ab(n-1)MSE}{\sigma^2}}{ab(n-1)}} \sim \frac{\frac{\chi^2_{df=(a-1)(b-1)}}{(a-1)(b-1)}}{\frac{\chi^2_{df=ab(n-1)}}{ab(n-1)}} \sim F((a-1)(b-1), ab(n-1))$$

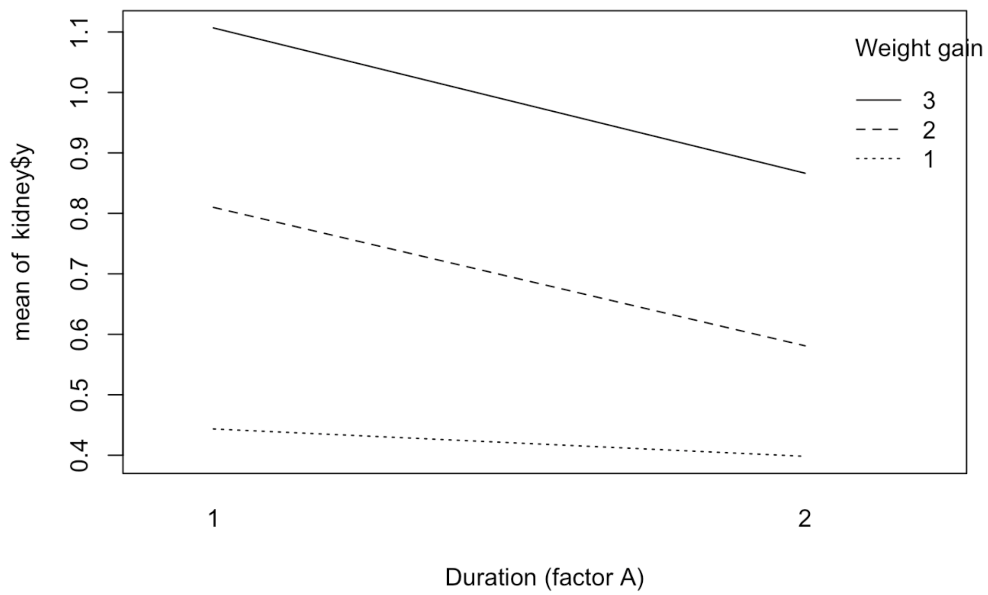
$\longrightarrow$ Decision rule:

If $F^* \leq F_{1-\alpha}((a-1)(b-1), ab(n-1))$, then conclude H_0

If $F^* > F_{1-\alpha}((a-1)(b-1), ab(n-1))$, then conclude H_a

Example

Test whether or not interaction effects are present at 0.05 significance level.



H_0 : all $\gamma_{ij} = 0$

H_a : not all $\gamma_{ij} = 0$

Test statistic: $F^* = \dfrac{MSAB}{MSE} = 0.592$

Critical value $F(1 - \alpha, 2, 54) = 3.168$

Since the test statistic is less than the critical value, we cannot reject the null hypothesis at 0.05 significance level and conclude that the interaction effect is not statistically significant.

Test for Factor A and Factor B Main Effects

To test whether or not factor A main effects are present:

$$H_0 : \alpha_1 = \ldots = \alpha_a = 0$$

$$H_a : \text{not all } \alpha_i = 0$$

Test statistic: $F^* = \dfrac{MSA}{MSE}$

Decision rule:

If $F^* \leq F_{1-\alpha}(a-1, ab(n-1))$, then conclude H_0

If $F^* > F_{1-\alpha}(a-1, abn-1))$, then conclude H_a

To test whether or not factor B main effects are present:

$$H_0 : \beta_1 = \ldots = \beta_b = 0$$

$$H_a : \text{not all } \beta_j = 0$$

Test statistic: $F^* = \dfrac{MSB}{MSE}$

Decision rule:

If $F^* \leq F_{1-\alpha}(b-1, ab(n-1))$, then conclude H_0

If $F^* > F_{1-\alpha}(b-1, ab(n-1))$, then conclude H_a

Example

Test whether or not main effects for duration and weight gain are present. Use $\alpha = .05$ in each case and state the alternatives, decision rule, and conclusion. Is it meaningful here to test for main factor effects? Explain.

It is meaningful to test main effects, since the interaction effects are not statistically significant, one can examine the effects of two factors separately using main effects only, as one factor's effects no longer depends on the levels of another factor.

To test the significance of Factor A main effect

$$H_0 : \alpha_i = 0, i = 1,2,3 \text{ vs } H_a : \text{ not all } \alpha_i \text{'s are } 0$$

Test statistic: $F^* = \dfrac{MSA}{MSE} = 4.36$

Critical value $F(0.95,2,54) = 4.02$

Since the F test statistics of factor A is larger than the critical value, we reject the null hypothesis at 0.05 significance level and conclude that factor A treatment duration has significant effects on the days hospitalized.

To test the significance of Factor B main effect

$$H_0 : \beta_j = 0, j = 1,2,3 \text{ vs } H_a : \text{ not all } \beta_j' \text{s are } 0$$

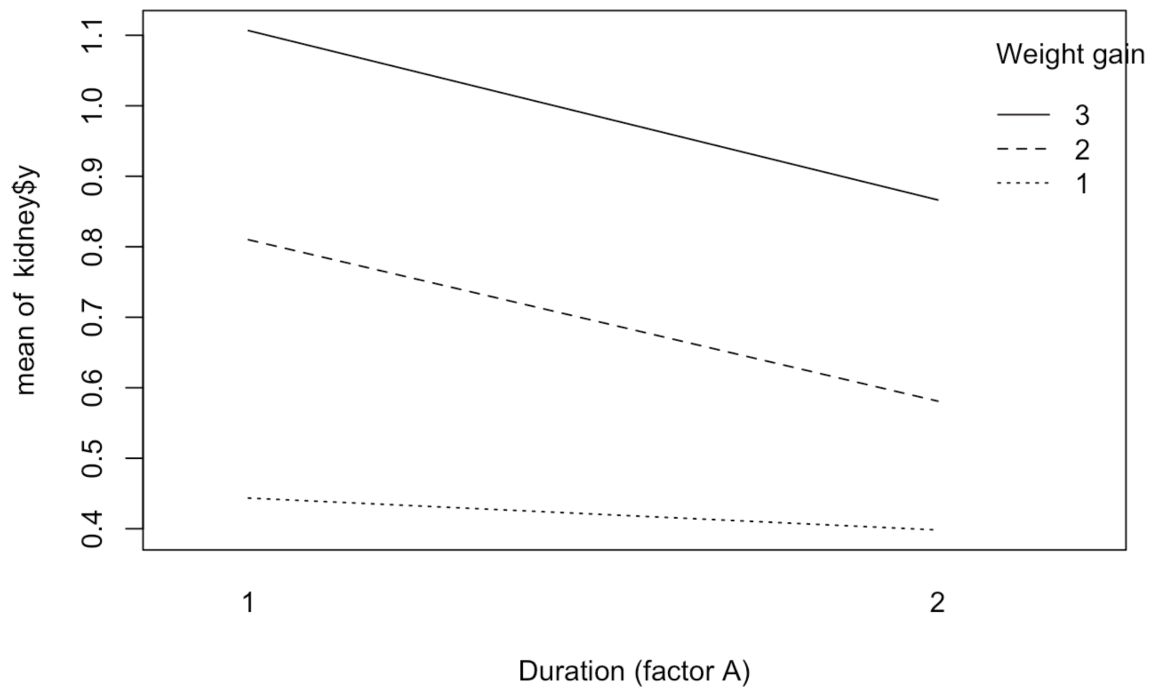
Test statistic: $F^* = \dfrac{MSB}{MSE} = 15.8$

Critical value $F(0.95,2,54) = 4.02$

Since the F test statistic of factor B is larger than the critical value, we reject the null hypothesis at 0.05 significance level and conclude that factor B weight gain as a result of fluid builup has significant effects on the days hosptalized.

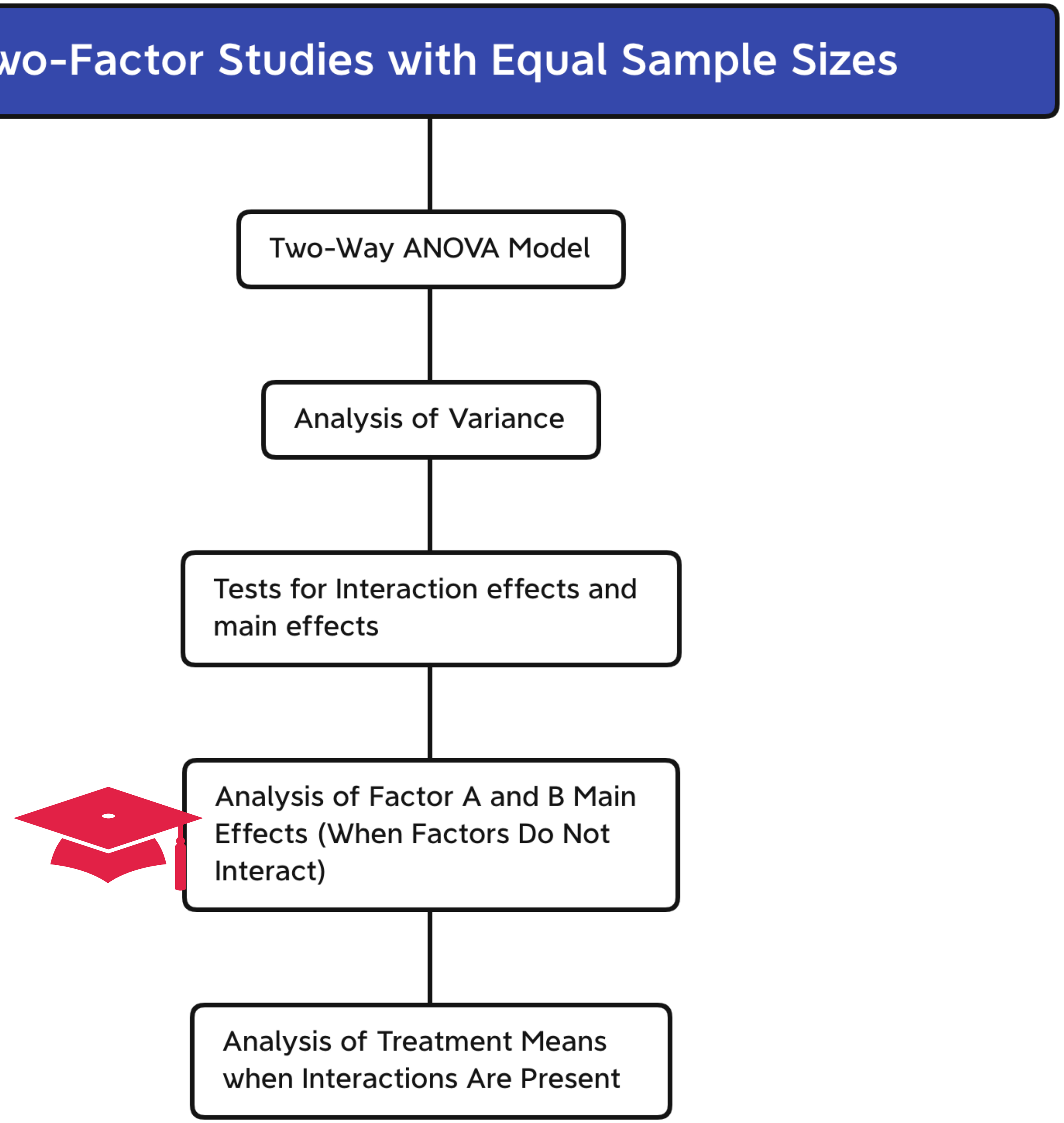
Example

Do the results confirm your graphic analysis?



It confirms that the interaction effects we see in the graph might not be systematic and could just due to random chance along, so that the two factors do not interact.

But the factor A and B have significant effects on the response, separately.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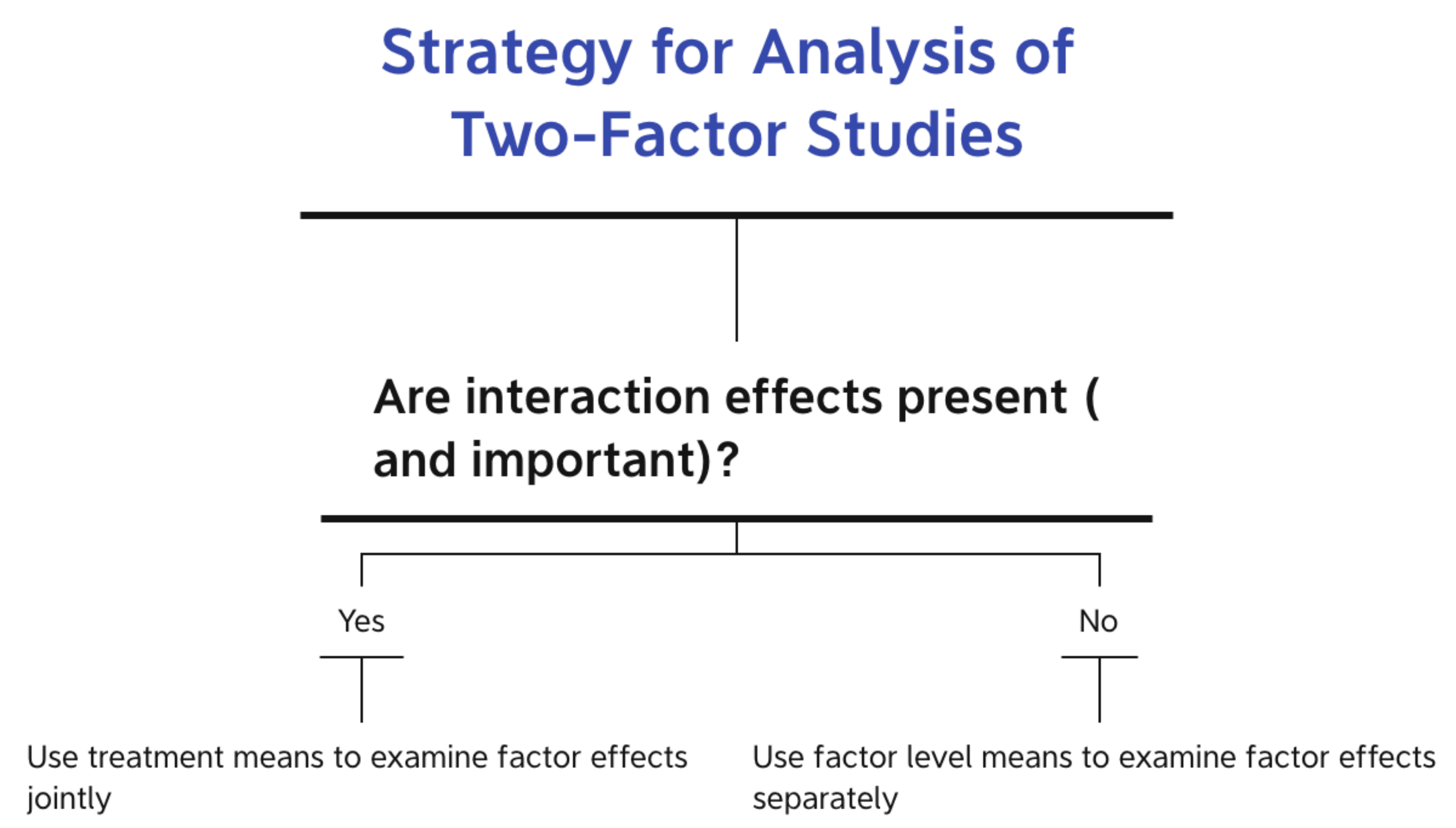


All the analytic techniques stay the same as in One-factor Studies!

Analysis of Factor A (and B) Main Effects (When Factors Do Not Interact)

Multiple Comparison Procedure: Bonferroni

Suppose we're interested in making infernece about multiple quantities,

that are linear combinations of factor A level means,

i.e., a family containing g linear combinations of factor level means

$$\mathscr{L} = \{L_1 = \sum_{i=1}^{r} c_{1i}\mu_{i\cdot}, \ldots, L_g = \sum_{i=1}^{r} c_{gi}\mu_{i\cdot}\}$$

$$\hat{L} = \sum_i c_i \bar{Y}_{i\cdot\cdot} \qquad s^2(\hat{L}) = \frac{MSE}{bn}\sum_i c_i^2$$

Bonferroni's idea:

One very easy and conservative way to control family-wise error rate at α is to control individual test's significance level at $\alpha_0 = \dfrac{\alpha}{g}$

This procedure includes any inference about a single quantity as special case, just take g=1.

Analysis of Factor A (and B) Main Effects (When Factors Do Not Interact)
Multiple Comparison Procedure: Bonferroni

$(1-\alpha)100\,\%$ confidence interval for individual quantity in this family:

$$\hat{L}_i \pm Bs\left(\hat{L}_i\right) \text{ for } i = 1\ldots g$$

$$B = t\left(1-\frac{\alpha}{2g}; ab(n-1)\right)$$

Guarantee:
family-wise confidence coefficient is at least $(1-\alpha)100\,\%$

Meaning:
in at least $(1-\alpha)100\,\%$ of repetition of experiments, all the intervals in the family cover the true corresponding $L_i's$ $\alpha\ \%$ of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

Hypothesis testing (t-test) for individual quantity in this family:

$$H_0^i : L_i = 0 \ H_a^i : L_i \neq 0$$

$$t^* = \frac{\hat{L}_i}{s(\hat{L}_i)} \sim t_{n_T-r} \text{if } H_0 \text{ is true}$$

If $|\,t^*\,| \leq B$, conclude H_0

If $|\,t^*\,| > B$, conclude H_a

Guarantee:
family-wise Type I error is at most α

Meaning:
in at most $\alpha\ \%$ of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

Analysis of Factor A (and B) Main Effects (When Factors Do Not Interact)

Multiple Comparison Procedure: Sheffe

Suppose we're interested in making inference about <u>all possible contracts of factor A level means</u>

i.e., a family containing all possible contracts of factor A level means

$$\mathscr{L} = \{L = \sum_{i=1}^{r} c_i \mu_{i.} \text{ where } \sum_{i=1}^{r} c_i = 0\}$$

Infinitely many claims or quantities

$$\hat{L} = \sum_i c_i \bar{Y}_{i..} \qquad s^2(\hat{L}) = \frac{MSE}{bn} \sum_i c_i^2$$

Analysis of Factor A (and B) Main Effects (When Factors Do Not Interact)

Multiple Comparison Procedure: Sheffe

$(1-\alpha)100\,\%$ confidence interval for individual quantity in this family:

$$\hat{L}_i \pm Ss\left(\hat{L}_i\right)$$

$$S = \sqrt{(a-1)F(1-\alpha; a-1, ab(n-1))}$$

Guarantee:
family-wise confidence coefficient is at least $(1-\alpha)100\,\%$

Meaning:
in at least $(1-\alpha)100\,\%$ of repetition of experiments, all the intervals in the family cover the true corresponding $L_i's$ $\alpha\ \%$ of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

Hypothesis testing (t-test) for individual quantity in this family:

$$H_0^i : L_i = 0 \ H_a^i : L_i \neq 0$$

$$t^* = \frac{\hat{L}_i}{s(\hat{L}_i)}$$

If $|t^*| \leq S$, conclude H_0

If $|t^*| > S$, conclude H_a

Guarantee:
family-wise Type I error is at most α

Meaning:
in at most $\alpha\ \%$ of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

Analysis of Factor A (and B) Main Effects (When Factors Do Not Interact)

Multiple Comparison Procedure: Tukey

Suppose we're interested in making inference about all pairwise comparisons of factor level means
i.e., a family containing <u>all pairwise comparisons of factor level means</u>

$$\mathscr{L} = \{D_{ii'} = \mu_{i.} - \mu_{i'.} \text{ for } i \neq i'\}$$

$\frac{a(a-1)}{2}$ Pairwise comparisons

$$\hat{D}_{ii'} = \bar{Y}_{i..} - \bar{Y}_{i'..} \qquad s^2(\hat{D}_{ii'}) = MSE\,\frac{2}{bn}$$

Analysis of Factor A (and B) Main Effects (When Factors Do Not Interact)

Multiple Comparison Procedure: Tukey

$(1-\alpha)100\,\%$ confidence interval for individual quantity in this family:

$$\hat{D}_{ii'} \pm Ts\left(\hat{D}_{ii'}\right)$$

$$T = \frac{1}{\sqrt{2}} q(1-\alpha; a, ab(n-1))$$

Guarantee:
family-wise confidence coefficient is at least $(1-\alpha)100\,\%$

Meaning:
in at least $(1-\alpha)100\,\%$ of repetition of experiments, all the intervals in the family cover the true corresponding $L_i's\ \alpha\ \%$ of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

Hypothesis testing (t-test) for individual quantity in this family:

$$H_0^i : D_{ii'} = 0 \;\; H_a^i : D_{ii'} = \neq 0$$

$$q^* = \frac{\hat{D}_{ii'}}{s(\hat{D}_{ii'})}$$

If $|q^*| \leq T$, conclude H_0

If $|q^*| > T$, conclude H_a

Guarantee:
family-wise Type I error is at most α

Meaning:
in at most $\alpha\ \%$ of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

Analysis of Factor A (and B) Main Effects (When Factors Do Not Interact)

Combined Factor A and Factor B Family ?

When we are interested in composite statement involving both factor A and factor B mean effects

Bonferroni method

Sheffe method:

If contrasts among factor A main effects and among factor B main effects are interested:

contrasts for factor A using Sheffe method with a family confidence coefficient of .975,

contrasts for factor B using Sheffe method with a family confidence coefficient of .975,

Then, by Bonferroni, combine the two families together, we get a family confidence coefficient of .95 for the "bigger family"

Tukey method:

If pairwise comparisons among factor A main effects and among factor B main effects are interested:

Pairwise comparisons for factor A using Tukey method with a family confidence coefficient of .975,

Pairwise comparisons for factor B using Tukey method with a family confidence coefficient of .975,

Then, by Bonferroni, combine the two families together, we get a family confidence coefficient of .95 for the "bigger family"

Compare: Bonferroni vs Scheffe vs Tukey

Compare 3 Simultaneous Inference Procedure

- **Bonferroni**
  - most general: quantities of interest can be mixture of any type: factor level mean, difference between factor level mean, contrast, or linear combination
  - can't handle data snooping, since need to specify the number of "data snooping" beforehand
- **Sheffe**
  - all possible contrasts
  - allow for data snooping as long as they are all contrasts
  - since applications never involve all contrasts, the actual family confidence coefficient is larger than stated, and the actual significance is smaller than stated
  - if only contrasts are of interest, Sheffe should be preferred, especially when the number of contrasts are large. However, when the number of contrasts is not that large, Bonferron might be better at times. (narrower C.I.s and smaller Type I error)
- **Tukey**
  - all possible pairwise differences
  - allow for data snooping as long as they are all differences
  - since applications never involve all differences, the actual family confidence coefficient is larger than stated, and the actual significance is smaller than stated
  - if only pairwise differences are of interest, Tukey should be preferred, especially when the number of differences are large. However, when the number of differences is not that large, Bonferroni might be better at times. (narrower C.I.s and smaller Type I error)

Compare: Bonferroni vs Scheffe vs Tukey

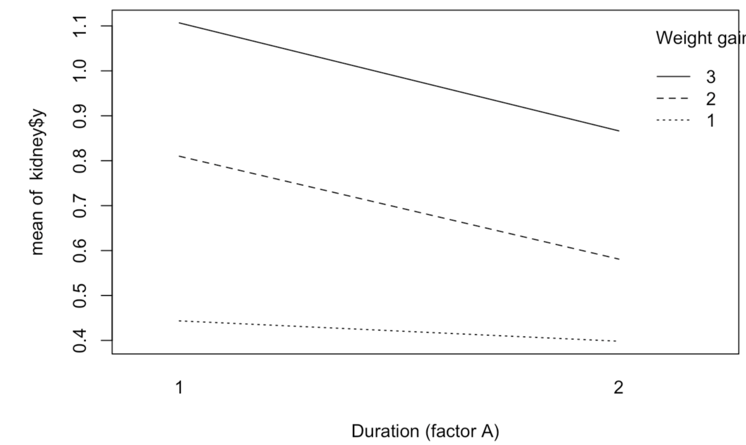
All three procedures are of the form "estimator $\pm$ multiplier $\times$SE."

The only difference among the three procedures is the multiplier.

In any given problem, one may compute the Bonferroni multiple as well as the Scheffé multiple and, when appropriate, the Tukey multiple, and select the one that is smallest.

Example

The researcher wishes to study the main effects of each of the two factors by making all pairwise comparisons of factor level means with a 90 percent family confidence coefficient for the entire set of comparisons. Which multiple comparison procedure is most efficient here?



There are 1 pairwise comparison for factor A and 3 pairwise comparisons for factor B, 4 in total.

Bonferroni method: $B = t(1-\alpha/8,(n-1)ab) = 2.3$

Tukey method:

Pairwise comparisons for factor A using Tukey method with a family confidence coefficient of .95, with Tukey multiple:

$$T = \frac{1}{\sqrt{2}} q(1-\alpha/2; a, ab(n-1)) = 2.83$$

Pairwise comparisons for factor B using Tukey method with a family confidence coefficient of .95, with Tukey multiple:

$$T = \frac{1}{\sqrt{2}} q(1-\alpha/2; b, ab(n-1)) = 3.4$$

Then, combine the two families together, we get a family confidence coefficient of .90 for all pairwise comparisons of factor level means.

Sheffe method:

Pairwise comparisons for factor A using Sheffe method with a family confidence coefficient of .95, with Tukey multiple:

$$S = \sqrt{(a-1)F(1-\alpha/2; a-1, ab(n-1))} = 2$$

Pairwise comparisons for factor B using Sheffe method with a family confidence coefficient of .95, with Tukey multiple:

$$S = \sqrt{(a-1)F(1-\alpha/2; b-1, ab(n-1))} = 2.5$$

Then, combine the two families together, we get a family confidence coefficient of .90 for all pairwise comparisons of factor level means.

The Bonferroni procedure is the most efficient overall.

Example

Using the most efficient procedure, make all pairwise comparisons. State your findings.

For factor A, $\hat{D}_{12} = \bar{Y}_{1..} - \bar{Y}_{2..}$

$$\bar{Y}_{1..} - \bar{Y}_{2..} \pm B\sqrt{\frac{2MSE}{bn}}$$

Code

```
## [1] -0.01790859  0.36095122
```

For factor B, $\hat{D}_{12} = \bar{Y}_{.1.} - \bar{Y}_{.2.}$

$$\bar{Y}_{.1.} - \bar{Y}_{.2.} \pm B\sqrt{\frac{2MSE}{an}}$$

Code

```
## [1] -0.5066143 -0.0426077
```

$\hat{D}_{13} = \bar{Y}_{.1.} - \bar{Y}_{.3.}$

$$\bar{Y}_{.1.} - \bar{Y}_{.3.} \pm B\sqrt{\frac{2MSE}{an}}$$

Code

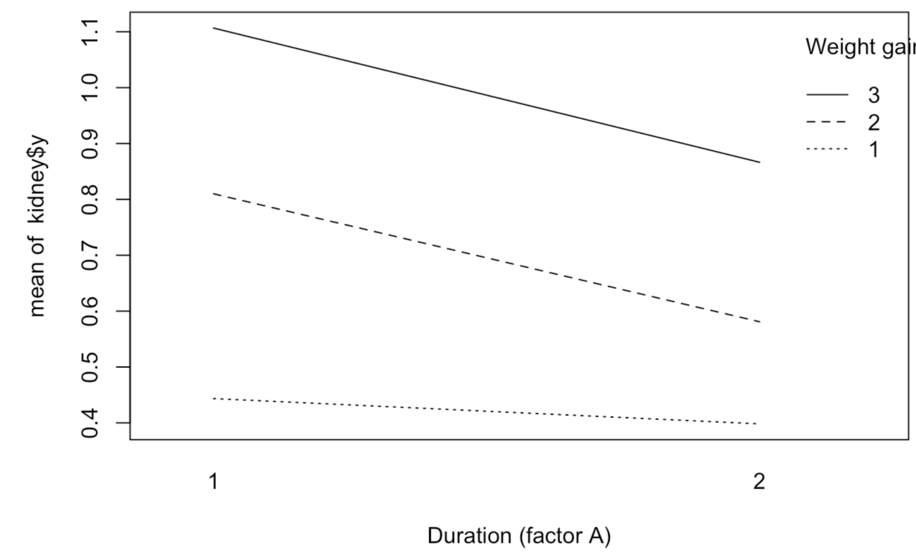
```
## [1] -0.7976958 -0.3336892
```

$\hat{D}_{23} = \bar{Y}_{.2.} - \bar{Y}_{.3.}$

$$\bar{Y}_{.2.} - \bar{Y}_{.3.} \pm B\sqrt{\frac{2MSE}{an}}$$

Code

```
## [1] -0.52308477 -0.05907816
```



For this family of confidence intervals, the following conclusions may be drawn with family confidence coefficient of 90 percent:

- The average days hospitalized for short and long duration do not differ significantly;
- The average days hospitalized for patients with mild weight gain is shorter than that for patients with moderate weight gain and substantial gain, respectively;
- The average days hospitalized for patients with moderate weight gain is shorter than that for patients with substantial weight gain.

Example

It is known from past experience that 30 percent of patients have mild weight gains, 40 percent have moderate weight gains, and 30 percent have severe weight gains, and that these proportions are the same for the two duration groups. Assume 50 percent of patient in each weight gain group receive short duration treatment, and the other 50 percent receive long duration treatment. Estimate the mean number of days hospitalized (in transformed units) in the entire population with a 95 percent confidence interval.

The linear combination of factor B levels is

$$L = .3\mu_{.1} + .4\mu_{.2} + .3\mu_{.3}$$

The 95 percent confidence interval: $\hat{L} \pm t(0.975{,}54)s(\hat{L}) = [0.617{,}0.784]$

Convert your confidence limits to the original units. Does it appear that the mean number of days is less than 7 ?

$$[10^{0.617} - 1{,}10^{0.784} - 1] = [3.14{,}5.08]$$

Since the confidence interval falls below 7, the mean number of days is less than 7.

Two-Factor Studies with Equal Sample Sizes

- Two-Way ANOVA Model
- Analysis of Variance
- Tests for Interaction effects and main effects
- Analysis of Factor A and B Main Effects (When Factors Do Not Interact)
- Analysis of Treatment Means when Interactions Are Present

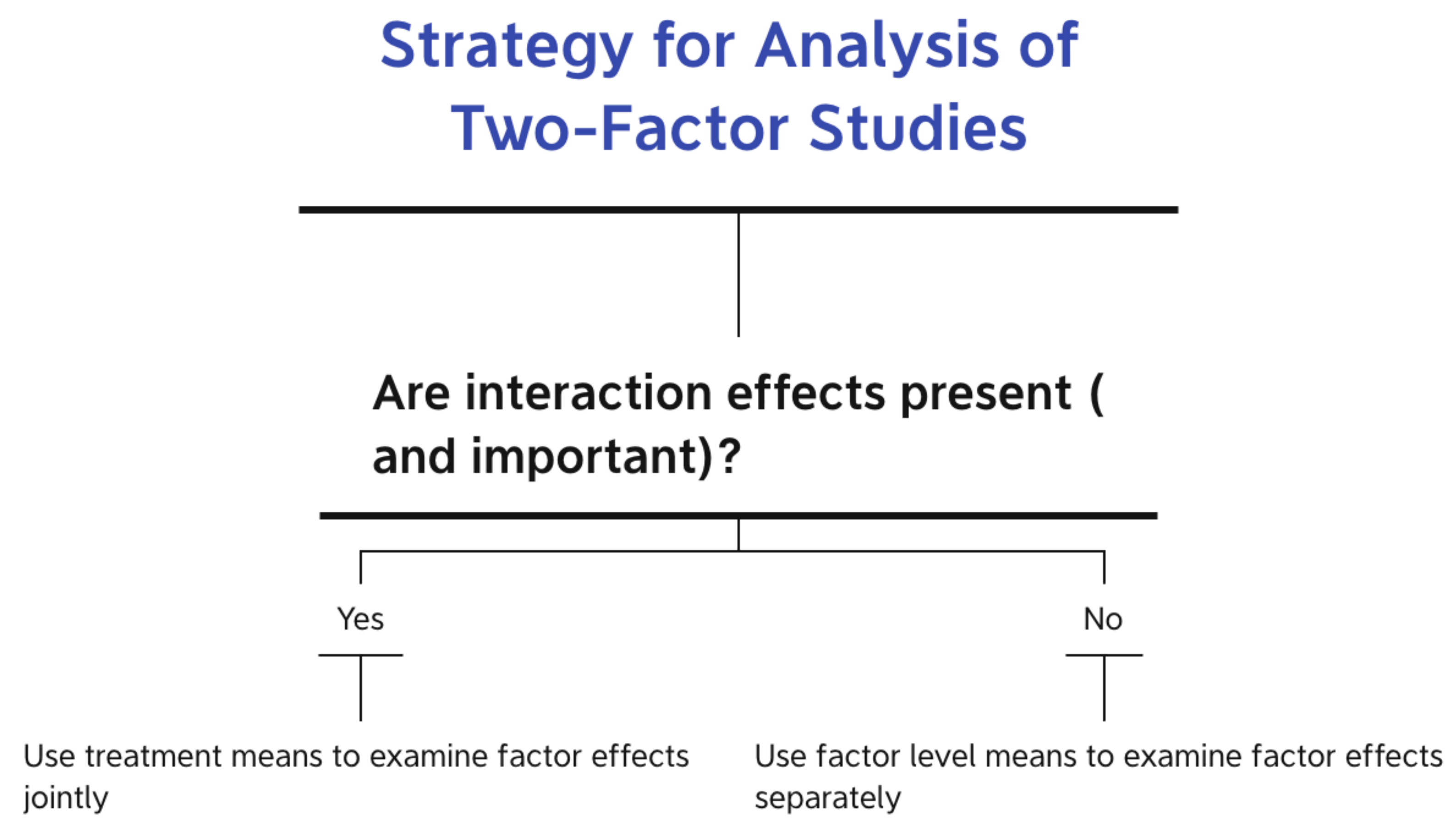
Strategy of Analysis of Two-Factor Studies

Scientific inquiry is guided by the principle:

simple, parsimonious explanations of observed phenomena tend to be the most effective

Additive factor effects : much simpler explanation of factor effects

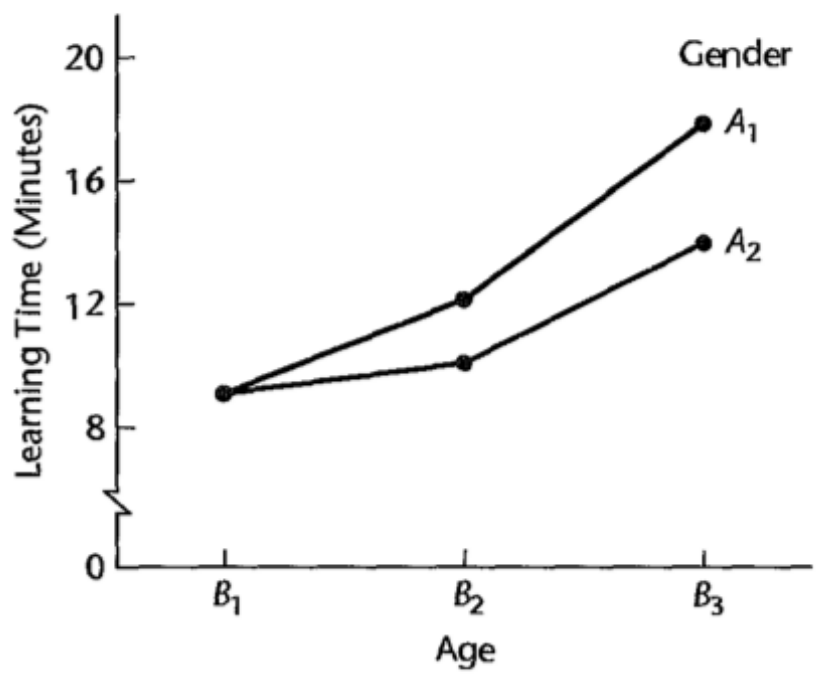
Interaction effects complicates the explanation



Analysis of Treatment Means when Interactions Are Present

When both main effects and interaction effects are present, only comparison of treatment means μ_{ij} makes sense, as treatment A has differential effect depending on the level of treatment B.

Often, insights can be gained by looking at how treatment means differ for levels of one factor, while fixing levels of another factor.



Compare effect of gender at different age levels?

Analysis of Treatment Means when Interactions Are Present

Multiple Comparison Procedure: Bonferroni

Suppose we're interested in making infernece about multiple quantities, that are linear combinations of treatment means,

$$\mathscr{L} = \{L_1 = \sum_i \sum_j c_{ij}^1 \mu_{ij}, \ldots, L_g = \sum_i \sum_j c_{ij}^g \mu_{ij}\}$$

$$\hat{L} = \sum_i \sum_j c_{ij} \bar{Y}_{ij.} \quad s^2(\hat{L}) = \frac{MSE}{n} \sum_i c_{ij}^2$$

Bonferroni's idea:
One very easy and conservative way to control family-wise error rate at α is to control individual test's significance level at $\alpha_0 = \frac{\alpha}{g}$

This procedure includes any inference about a single quantity as special case, just take g=1.

Analysis of Treatment Means when Interactions Are Present

$(1-\alpha)100\,\%$ confidence interval for individual quantity in this family:

$$\hat{L}_i \pm Bs\left(\hat{L}_i\right) \text{ for } i = 1 \ldots g$$

$$B = t\left(1 - \frac{\alpha}{2g}; ab(n-1)\right)$$

Guarantee:
family-wise confidence coefficient is at least $(1-\alpha)100\,\%$

Meaning:
in at least $(1-\alpha)100\,\%$ of repetition of experiments, all the intervals in the family cover the true corresponding $L_i's$ $\alpha\,\%$ of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

Hypothesis testing (t-test) for individual quantity in this family:

$$H_0^i : L_i = 0 \;\; H_a^i : L_i \neq 0$$

$$t^* = \frac{\hat{L}_i}{s(\hat{L}_i)} \sim t_{n_T - r} \text{if } H_0 \text{ is true}$$

If $|t^*| \leq B$, conclude H_0

If $|t^*| > B$, conclude H_a

Guarantee:
family-wise Type I error is at most α

Meaning:
in at most $\alpha\,\%$ of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

Analysis of Treatment Means when Interactions Are Present

Suppose we're interested in making inference about <u>all possible contracts of factor A level means</u>
i.e., a family containing all possible contracts of factor A level means

$$\mathscr{L} = \{L = \sum_i \sum_j c_{ij}\mu_{ij} \text{ where } \sum_i \sum_j c_{ij} = 0\}$$

Infinitely many claims or quantities

$$\hat{L} = \sum_i \sum_j c_{ij}\bar{Y}_{ij\cdot} \qquad s^2(\hat{L}) = \frac{MSE}{n}\sum_i \sum_j c_{ij}^2$$

Analysis of Treatment Means when Interactions Are Present

$(1-\alpha)100\,\%$ confidence interval for individual quantity in this family:

$$\hat{L}_i \pm Ss\left(\hat{L}_i\right)$$

$$S = \sqrt{(ab-1)F(1-\alpha;ab-1,ab(n-1))}$$

Guarantee:
family-wise confidence coefficient is at least $(1-\alpha)100\,\%$

Meaning:
in at least $(1-\alpha)100\,\%$ of repetition of experiments, all the intervals in the family cover the true corresponding $L_i's$ $\alpha\,\%$ of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

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If $|t^*| \leq S$, conclude H_0

If $|t^*| > S$, conclude H_a

Guarantee:
family-wise Type I error is at most α

Meaning:
in at most $\alpha\,\%$ of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

Analysis of Treatment Means when Interactions Are Present

Suppose we're interested in making inference about all pairwise comparisons of factor level means
i.e., a family containing <u>all pairwise comparisons of factor level means</u>

$$\mathscr{L} = \{D = \mu_{ij} - \mu_{i'j'} \text{ for } (i, j) \neq (i', j')\}$$

$\frac{ab(ab-1)}{2}$ Pairwise comparisons

$$\hat{D} = \bar{Y}_{ij\cdot} - \bar{Y}_{i'j'\cdot} \qquad s^2(\hat{D}) = MSE\frac{2}{n}$$

Analysis of Treatment Means when Interactions Are Present

$(1-\alpha)100\,\%$ confidence interval for individual quantity in this family:

$$\hat{D}_{ii'} \pm Ts\left(\hat{D}_{ii'}\right)$$

$$T = \frac{1}{\sqrt{2}} q(1-\alpha; ab, ab(n-1))$$

Guarantee:
family-wise confidence coefficient is at least $(1-\alpha)100\,\%$

Meaning:
in at least $(1-\alpha)100\,\%$ of repetition of experiments, all the intervals in the family cover the true corresponding $L_i's$ $\alpha\ \%$ of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

Hypothesis testing (t-test) for individual quantity in this family:

$$H_0^i : D_{ii'} = 0 \ H_a^i : D_{ii'} = \neq 0$$

$$q^* = \frac{\hat{D}_{ii'}}{s(\hat{D}_{ii'})}$$

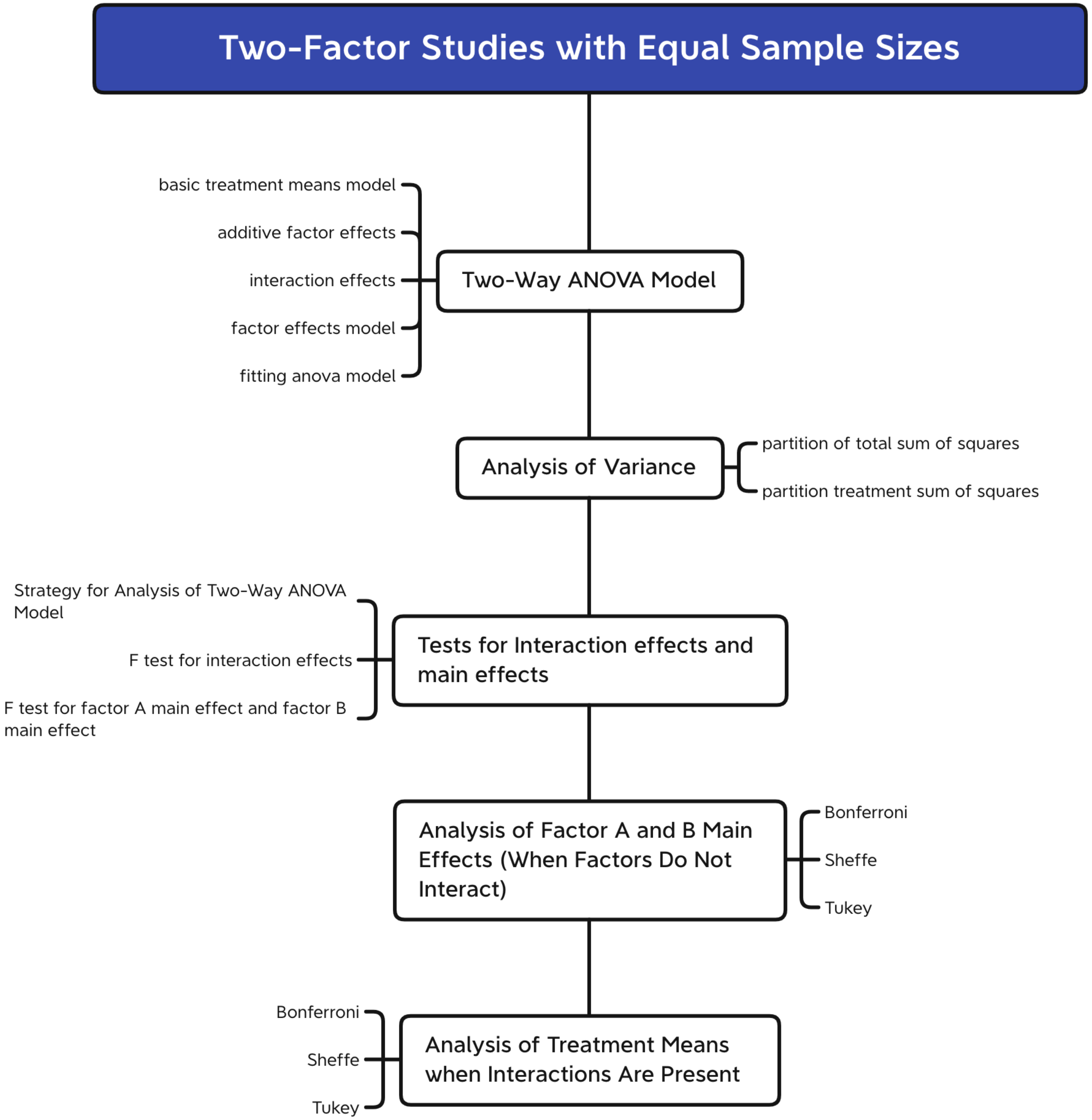
If $|q^*| \leq T$, conclude H_0

If $|q^*| > T$, conclude H_a

Guarantee:
family-wise Type I error is at most α

Meaning:
in at most $\alpha\ \%$ of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

Summary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basic treatment means model
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interaction effects
factor effects model
fitting anova model
Analysis of Variance
partition of total sum of squares
partition treatment sum of squares
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Strategy for Analysis of Two-Way ANOVA Model
F test for interaction effects
F test for factor A main effect and factor B main effect
Analysis of Factor A and B Main Effects (When Factors Do Not Interact)
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Sheffe
Tukey
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