Lecture 4: Two-Factor Studies with Equal Sample Sizes

STA 106: Analysis of Variance

Suggested reading: ALSM Chapter 19

Xiner Zhou
Department of Statistics, University of California, Davis

Two-Factor Studies with Equal Sample Sizes

1. Two-Way ANOVA Model
2. Analysis of Variance
3. Tests for Interaction effects and main effects
4. Analysis of Factor A and B Main Effects (When Factors Do Not Interact)
5. Analysis of Treatment Means when Interactions Are Present

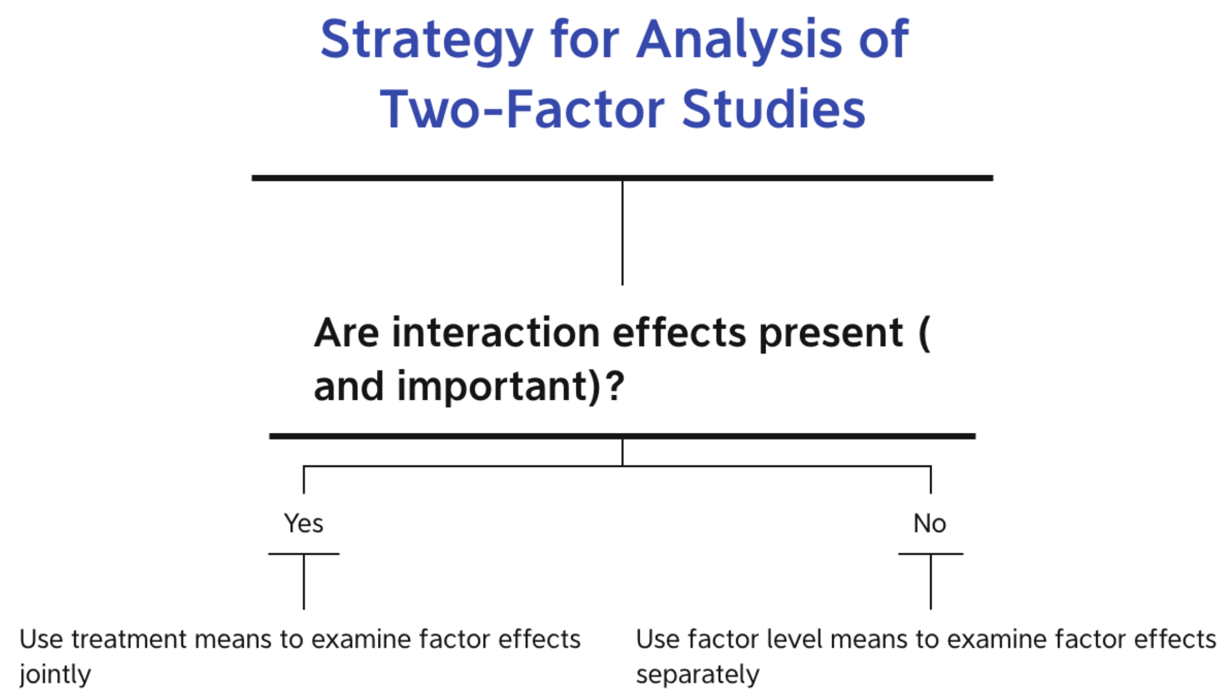
Strategy of Analysis of Two-Factor Studies

Scientific inquiry is guided by the principle:

simple, parsimonious explanations of observed phenomena tend to be the most effective

Additive factor effects : much simpler explanation of factor effects

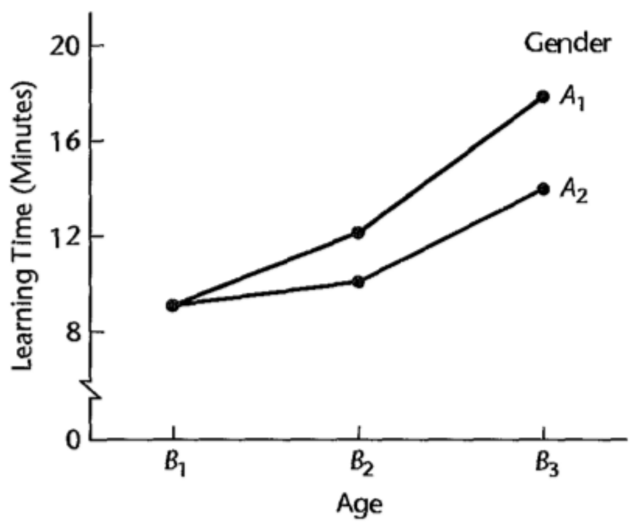
Interaction effects complicates the explanation



Analysis of Treatment Means when Interactions Are Present

When both main effects and interaction effects are present, only comparison of treatment means μ_{ij} makes sense, as treatment A has differential effect depending on the level of treatment B.

Often, insights can be gained by looking at how treatment means differ for levels of one factor, while fixing levels of another factor.



Compare effect of gender at different age levels?

Analysis of Treatment Means when Interactions Are Present

Multiple Comparison Procedure: Bonferroni

Suppose we're interested in making infernece about multiple quantities,
that are linear combinations of treatment means,

$$\mathscr{L} = \{L_1 = \sum_i \sum_j c_{ij}^1 \mu_{ij}, \ldots, L_g = \sum_i \sum_j c_{ij}^g \mu_{ij}\}$$

$$\hat{L} = \sum_i \sum_j c_{ij} \bar{Y}_{ij\cdot} \quad s^2(\hat{L}) = \frac{MSE}{n} \sum_i c_{ij}^2$$

Bonferroni's idea:
One very easy and conservative way to control family-wise error rate at α is to control individual test's significance level at $\alpha_0 = \dfrac{\alpha}{g}$

This procedure includes any inference about a single quantity as special case, just take g=1.

Analysis of Treatment Means when Interactions Are Present

$(1-\alpha)100\,\%$ confidence interval for individual quantity in this family:

$$\hat{L}_i \pm Bs\left(\hat{L}_i\right) \text{ for } i = 1\ldots g$$

$$B = t\left(1 - \frac{\alpha}{2g}; ab(n-1)\right)$$

Guarantee:
family-wise confidence coefficient is at least $(1-\alpha)100\,\%$

Meaning:
in at least $(1-\alpha)100\,\%$ of repetition of experiments, all the intervals in the family cover the true corresponding $L_i's$ $\alpha\,\%$ of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

Hypothesis testing (t-test) for individual quantity in this family:

$$H_0^i : L_i = 0 \;\; H_a^i : L_i \neq 0$$

$$t^* = \frac{\hat{L}_i}{s(\hat{L}_i)} \sim t_{n_T - r} \text{if } H_0 \text{ is true}$$

If $|t^*| \leq B$, conclude H_0

If $|t^*| > B$, conclude H_a

Guarantee:
family-wise Type I error is at most α

Meaning:
in at most $\alpha\,\%$ of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

Analysis of Treatment Means when Interactions Are Present

Suppose we're interested in making inference about <u>all possible contracts of factor A level means</u>

i.e., a family containing all possible contracts of factor A level means

$$\mathscr{L} = \{L = \sum_i \sum_j c_{ij}\mu_{ij} \text{ where } \sum_i \sum_j c_{ij} = 0\}$$

Infinitely many claims or quantities

$$\hat{L} = \sum_i \sum_j c_{ij}\bar{Y}_{ij\cdot} \qquad s^2(\hat{L}) = \frac{MSE}{n}\sum_i \sum_j c_{ij}^2$$

Analysis of Treatment Means when Interactions Are Present

$(1-\alpha)100\ \%$ confidence interval for individual quantity in this family:

$$\hat{L}_i \pm Ss\left(\hat{L}_i\right)$$

$$S = \sqrt{(ab-1)F(1-\alpha; ab-1, ab(n-1))}$$

Guarantee:
family-wise confidence coefficient is at least $(1-\alpha)100\ \%$

Meaning:
in at least $(1-\alpha)100\ \%$ of repetition of experiments, all the intervals in the family cover the true corresponding $L_i's$ $\alpha\ \%$ of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

Hypothesis testing (t-test) for individual quantity in this family:

$$H_0^i : L_i = 0 \ H_a^i : L_i \neq 0$$

$$t^* = \frac{\hat{L}_i}{s(\hat{L}_i)}$$

If $|t^*| \leq S$, conclude H_0

If $|t^*| > S$, conclude H_a

Guarantee:
family-wise Type I error is at most α

Meaning:
in at most $\alpha\ \%$ of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

Analysis of Treatment Means when Interactions Are Present

Suppose we're interested in making inference about all pairwise comparisons of factor level means i.e., a family containing <u>all pairwise comparisons of factor level means</u>

$$\mathscr{L} = \{D = \mu_{ij} - \mu_{i'j'} \text{ for } (i, j) \neq (i', j')\}$$

$\frac{ab(ab-1)}{2}$ Pairwise comparisons

$$\hat{D} = \bar{Y}_{ij\cdot} - \bar{Y}_{i'j'\cdot} \qquad s^2(\hat{D}) = MSE\frac{2}{n}$$

Analysis of Treatment Means when Interactions Are Present

$(1-\alpha)100\,\%$ confidence interval for individual quantity in this family:

$$\hat{D}_{ii'} \pm Ts\left(\hat{D}_{ii'}\right)$$

$$T = \frac{1}{\sqrt{2}} q(1-\alpha; ab, ab(n-1))$$

Guarantee:
family-wise confidence coefficient is at least $(1-\alpha)100\,\%$

Meaning:
in at least $(1-\alpha)100\,\%$ of repetition of experiments, all the intervals in the family cover the true corresponding $L_i's$ $\alpha\,\%$ of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

Hypothesis testing (t-test) for individual quantity in this family:

$$H_0^i : D_{ii'} = 0 \; H_a^i : D_{ii'} = \neq 0$$

$$q^* = \frac{\hat{D}_{ii'}}{s(\hat{D}_{ii'})}$$

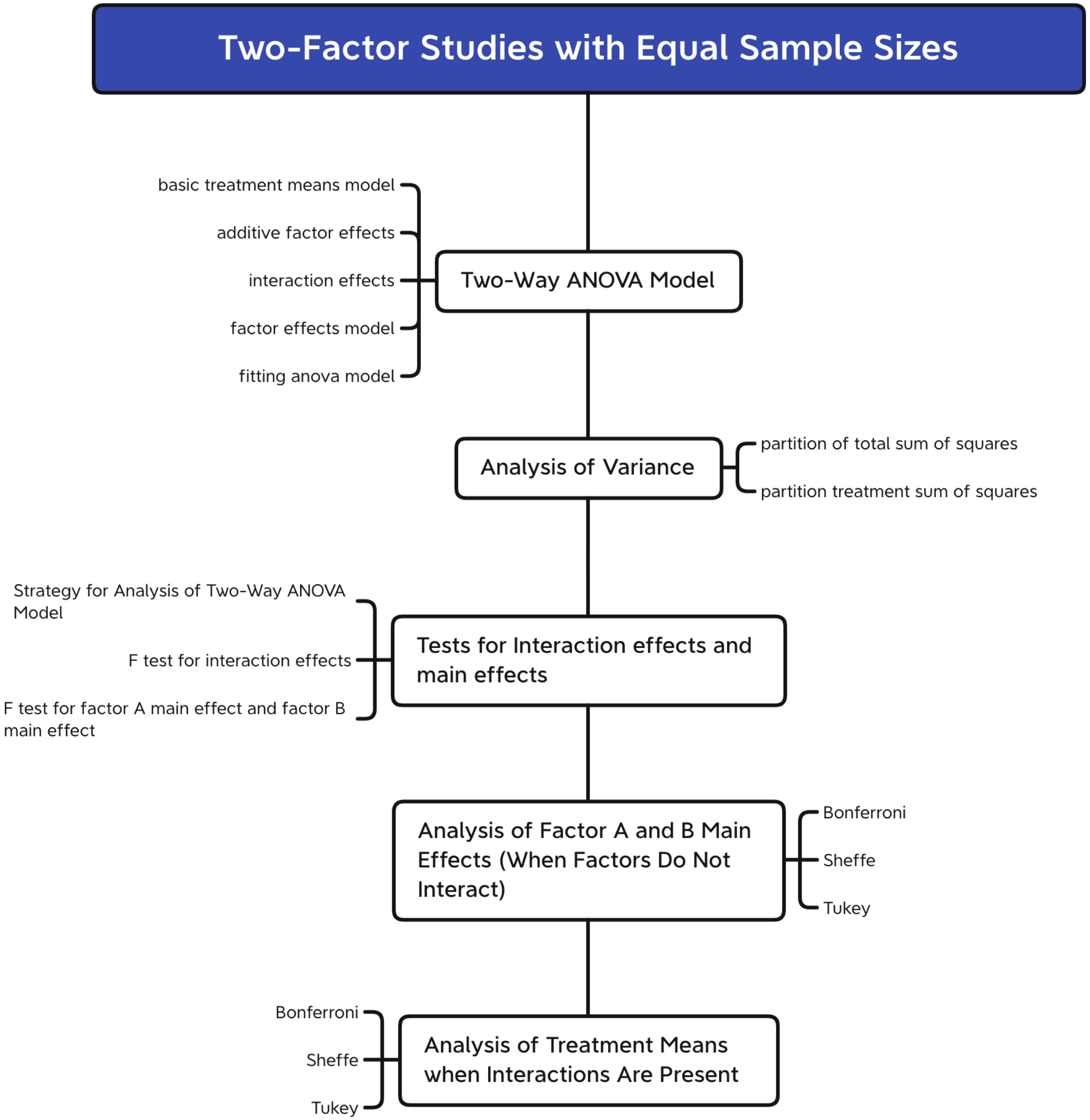
If $|q^*| \leq T$, conclude H_0

If $|q^*| > T$, conclude H_a

Guarantee:
family-wise Type I error is at most α

Meaning:
in at most $\alpha\,\%$ of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

Summary


Two-Factor Studies with Equal Sample Sizes
Two-Way ANOVA Model
basic treatment means model
additive factor effects
interaction effects
factor effects model
fitting anova model
Analysis of Variance
partition of total sum of squares
partition treatment sum of squares
Tests for Interaction effects and main effects
Strategy for Analysis of Two-Way ANOVA Model
F test for interaction effects
F test for factor A main effect and factor B main effect
Analysis of Factor A and B Main Effects (When Factors Do Not Interact)
Bonferroni
Sheffe
Tukey
Analysis of Treatment Means when Interactions Are Present
Bonferroni
Sheffe
Tukey