Lecture 4: Two-Factor Studies with Equal Sample Sizes

STA 106: Analysis of Variance

Suggested reading: ALSM Chapter 19

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Two-Factor Studies with Equal Sample Sizes

- Two-Way ANOVA Model
- Analysis of Variance
- Tests for Interaction effects and main effects
- Analysis of Factor A and B Main Effects (When Factors Do Not Interact)
- Analysis of Treatment Means when Interactions Are Present

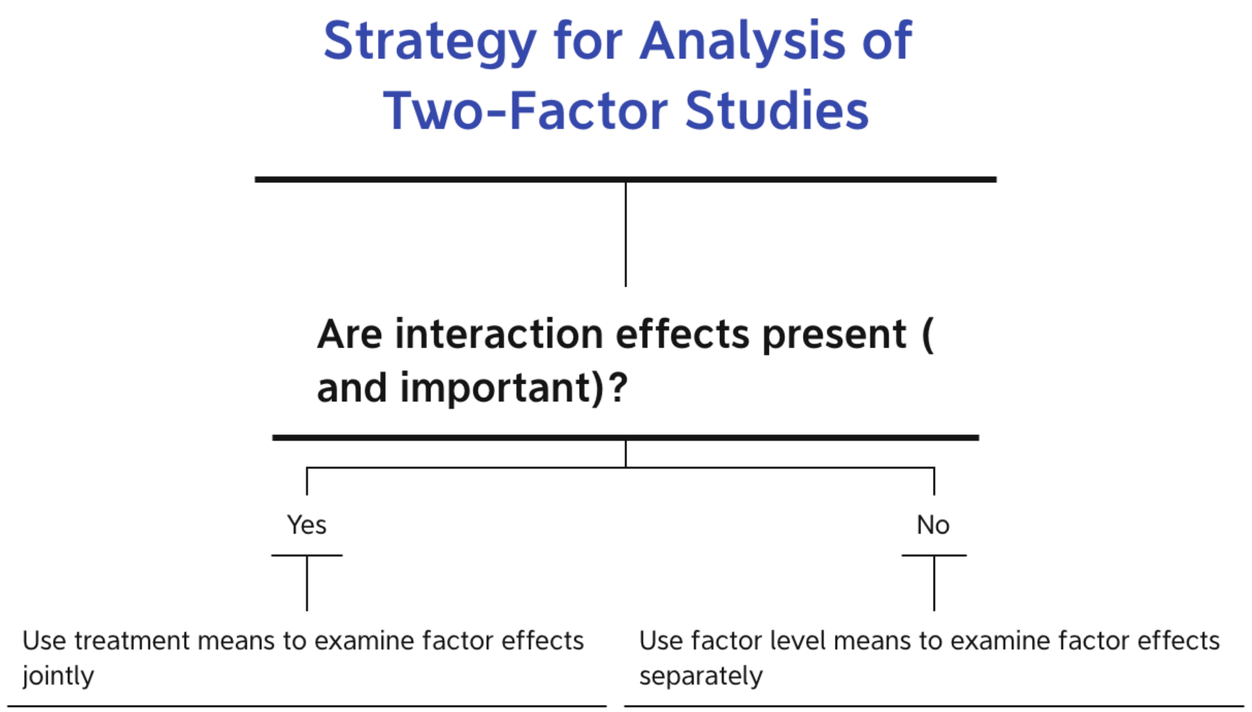
Strategy of Analysis of Two-Factor Studies

Scientific inquiry is guided by the principle:

simple, parsimonious explanations of observed phenomena tend to be the most effective

Additive factor effects : much simpler explanation of factor effects

Interaction effects complicates the explanation



All the analytic techniques stay the same as in One-factor Studies!

Analysis of Factor A (and B) Main Effects (When Factors Do Not Interact)

Multiple Comparison Procedure: Bonferroni

Suppose we're interested in making infernece about multiple quantities,

that are linear combinations of factor A level means,

i.e., a family containing g linear combinations of factor level means

$$\mathscr{L} = \{L_1 = \sum_{i=1}^{r} c_{1i}\mu_{i\cdot}, \ldots, L_g = \sum_{i=1}^{r} c_{gi}\mu_{i\cdot}\}$$

$$\hat{L} = \sum_i c_i \bar{Y}_{i\cdot\cdot} \qquad s^2(\hat{L}) = \frac{MSE}{bn}\sum_i c_i^2$$

Bonferroni's idea:

One very easy and conservative way to control family-wise error rate at α is to control individual test's significance level at $\alpha_0 = \dfrac{\alpha}{g}$

This procedure includes any inference about a single quantity as special case, just take g=1.

Analysis of Factor A (and B) Main Effects (When Factors Do Not Interact)
Multiple Comparison Procedure: Bonferroni

$(1-\alpha)100\,\%$ confidence interval for individual quantity in this family:

$$\hat{L}_i \pm Bs\left(\hat{L}_i\right) \text{ for } i = 1\ldots g$$

$$B = t\left(1 - \frac{\alpha}{2g}; ab(n-1)\right)$$

Guarantee:
family-wise confidence coefficient is at least $(1-\alpha)100\,\%$

Meaning:
in at least $(1-\alpha)100\,\%$ of repetition of experiments, all the intervals in the family cover the true corresponding $L_i's$ $\alpha\,\%$ of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

Hypothesis testing (t-test) for individual quantity in this family:

$$H_0^i : L_i = 0 \ H_a^i : L_i \neq 0$$

$$t^* = \frac{\hat{L}_i}{s(\hat{L}_i)} \sim t_{n_T - r} \text{if } H_0 \text{ is true}$$

If $|t^*| \leq B$, conclude H_0

If $|t^*| > B$, conclude H_a

Guarantee:
family-wise Type I error is at most α

Meaning:
in at most $\alpha\,\%$ of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

Analysis of Factor A (and B) Main Effects (When Factors Do Not Interact)

Multiple Comparison Procedure: Sheffe

Suppose we're interested in making inference about <u>all possible contracts of factor A level means</u>
i.e., a family containing all possible contracts of factor A level means

$$\mathscr{L} = \{L = \sum_{i=1}^{r} c_i \mu_{i.} \text{ where } \sum_{i=1}^{r} c_i = 0\}$$

Infinitely many claims or quantities

$$\hat{L} = \sum_i c_i \bar{Y}_{i..} \qquad s^2(\hat{L}) = \frac{MSE}{bn} \sum_i c_i^2$$

Analysis of Factor A (and B) Main Effects (When Factors Do Not Interact)

Multiple Comparison Procedure: Sheffe

$(1-\alpha)100\,\%$ confidence interval for individual quantity in this family:

$$\hat{L}_i \pm Ss\left(\hat{L}_i\right)$$

$$S = \sqrt{(a-1)F(1-\alpha; a-1, ab(n-1))}$$

Guarantee:
family-wise confidence coefficient is at least $(1-\alpha)100\,\%$

Meaning:
in at least $(1-\alpha)100\,\%$ of repetition of experiments, all the intervals in the family cover the true corresponding $L_i's$ $\alpha\ \%$ of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

Hypothesis testing (t-test) for individual quantity in this family:

$$H_0^i : L_i = 0 \ H_a^i : L_i \neq 0$$

$$t^* = \frac{\hat{L}_i}{s(\hat{L}_i)}$$

If $|t^*| \leq S$, conclude H_0

If $|t^*| > S$, conclude H_a

Guarantee:
family-wise Type I error is at most α

Meaning:
in at most $\alpha\ \%$ of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

Analysis of Factor A (and B) Main Effects (When Factors Do Not Interact)

Multiple Comparison Procedure: Tukey

Suppose we're interested in making inference about all pairwise comparisons of factor level means
i.e., a family containing <u>all pairwise comparisons of factor level means</u>

$$\mathscr{L} = \{D_{ii'} = \mu_{i\cdot} - \mu_{i'\cdot} \text{ for } i \neq i'\}$$

$\frac{a(a-1)}{2}$ Pairwise comparisons

$$\hat{D}_{ii'} = \bar{Y}_{i\cdot\cdot} - \bar{Y}_{i'\cdot\cdot} \qquad s^2(\hat{D}_{ii'}) = MSE\,\frac{2}{bn}$$

Analysis of Factor A (and B) Main Effects (When Factors Do Not Interact)

Multiple Comparison Procedure: Tukey

$(1-\alpha)100\,\%$ confidence interval for individual quantity in this family:

$$\hat{D}_{ii'} \pm Ts\left(\hat{D}_{ii'}\right)$$

$$T = \frac{1}{\sqrt{2}}q(1-\alpha; a, ab(n-1))$$

Guarantee:
family-wise confidence coefficient is at least $(1-\alpha)100\,\%$

Meaning:
in at least $(1-\alpha)100\,\%$ of repetition of experiments, all the intervals in the family cover the true corresponding $L_i's$ α % of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

Hypothesis testing (t-test) for individual quantity in this family:

$$H_0^i : D_{ii'} = 0 \;\; H_a^i : D_{ii'} = \neq 0$$

$$q^* = \frac{\hat{D}_{ii'}}{s(\hat{D}_{ii'})}$$

If $|q^*| \leq T$, conclude H_0

If $|q^*| > T$, conclude H_a

Guarantee:
family-wise Type I error is at most α

Meaning:
in at most α % of repetition of experiments, some tests in the family made false discovery when the null hypothesis was true.

Analysis of Factor A (and B) Main Effects (When Factors Do Not Interact)

Combined Factor A and Factor B Family ?

When we are interested in composite statement involving both factor A and factor B mean effects

Bonferroni method

Sheffe method:

If contrasts among factor A main effects and among factor B main effects are interested:

contrasts for factor A using Sheffe method with a family confidence coefficient of .975,

contrasts for factor B using Sheffe method with a family confidence coefficient of .975,

Then, by Bonferroni, combine the two families together, we get a family confidence coefficient of .95 for the “bigger family”

Tukey method:

If pairwise comparisons among factor A main effects and among factor B main effects are interested:

Pairwise comparisons for factor A using Tukey method with a family confidence coefficient of .975,

Pairwise comparisons for factor B using Tukey method with a family confidence coefficient of .975,

Then, by Bonferroni, combine the two families together, we get a family confidence coefficient of .95 for the “bigger family”

Compare: Bonferroni vs Scheffe vs Tukey

Compare 3 Simultaneous Inference Procedure

- **Bonferroni**
  - most general: quantities of interest can be mixture of any type: factor level mean, difference between factor level mean, contrast, or linear combination
  - can't handle data snooping, since need to specify the number of "data snooping" beforehand
- **Sheffe**
  - all possible contrasts
  - allow for data snooping as long as they are all contrasts
  - since applications never involve all contrasts, the actual family confidence coefficient is larger than stated, and the actual significance is smaller than stated
  - if only contrasts are of interest, Sheffe should be preferred, especially when the number of contrasts are large. However, when the number of contrasts is not that large, Bonferron might be better at times. (narrower C.I.s and smaller Type I error)
- **Tukey**
  - all possible pairwise differences
  - allow for data snooping as long as they are all differences
  - since applications never involve all differences, the actual family confidence coefficient is larger than stated, and the actual significance is smaller than stated
  - if only pairwise differences are of interest, Tukey should be preferred, especially when the number of differences are large. However, when the number of differences is not that large, Bonferroni might be better at times. (narrower C.I.s and smaller Type I error)

Compare: Bonferroni vs Scheffe vs Tukey

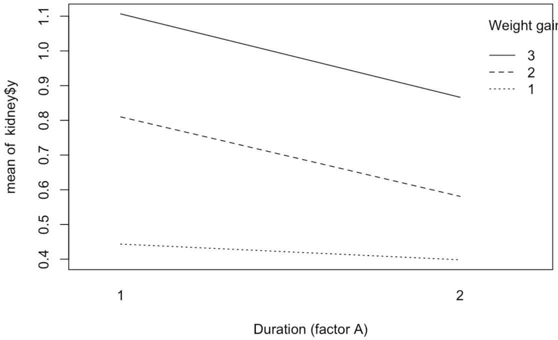
All three procedures are of the form "estimator $\pm$ multiplier $\times$SE."

The only difference among the three procedures is the multiplier.

In any given problem, one may compute the Bonferroni multiple as well as the Scheffé multiple and, when appropriate, the Tukey multiple, and select the one that is smallest.

Example

The researcher wishes to study the main effects of each of the two factors by making all pairwise comparisons of factor level means with a 90 percent family confidence coefficient for the entire set of comparisons. Which multiple comparison procedure is most efficient here?



There are 1 pairwise comparison for factor A and 3 pairwise comparisons for factor B, 4 in total.

Bonferroni method: $B = t(1 - \alpha/8, (n-1)ab) = 2.3$

Tukey method:

Pairwise comparisons for factor A using Tukey method with a family confidence coefficient of .95, with Tukey multiple:

$$T = \frac{1}{\sqrt{2}} q(1 - \alpha/2; a, ab(n-1)) = 2.83$$

Pairwise comparisons for factor B using Tukey method with a family confidence coefficient of .95, with Tukey multiple:

$$T = \frac{1}{\sqrt{2}} q(1 - \alpha/2; b, ab(n-1)) = 3.4$$

Then, combine the two families together, we get a family confidence coefficient of .90 for all pairwise comparisons of factor level means.

Sheffe method:

Pairwise comparisons for factor A using Sheffe method with a family confidence coefficient of .95, with Tukey multiple:

$$S = \sqrt{(a-1)F(1 - \alpha/2; a-1, ab(n-1))} = 2$$

Pairwise comparisons for factor B using Sheffe method with a family confidence coefficient of .95, with Tukey multiple:

$$S = \sqrt{(a-1)F(1 - \alpha/2; b-1, ab(n-1))} = 2.5$$

Then, combine the two families together, we get a family confidence coefficient of .90 for all pairwise comparisons of factor level means.

The Bonferroni procedure is the most efficient overall.

Example

Using the most efficient procedure, make all pairwise comparisons. State your findings.

For factor A, $\hat{D}_{12} = \bar{Y}_{1..} - \bar{Y}_{2..}$

$$\bar{Y}_{1..} - \bar{Y}_{2..} \pm B\sqrt{\frac{2MSE}{bn}}$$

Code

```
## [1] -0.01790859  0.36095122
```

For factor B, $\hat{D}_{12} = \bar{Y}_{.1.} - \bar{Y}_{.2.}$

$$\bar{Y}_{.1.} - \bar{Y}_{.2.} \pm B\sqrt{\frac{2MSE}{an}}$$

Code

```
## [1] -0.5066143 -0.0426077
```

$\hat{D}_{13} = \bar{Y}_{.1.} - \bar{Y}_{.3.}$

$$\bar{Y}_{.1.} - \bar{Y}_{.3.} \pm B\sqrt{\frac{2MSE}{an}}$$

Code

```
## [1] -0.7976958 -0.3336892
```

$\hat{D}_{23} = \bar{Y}_{.2.} - \bar{Y}_{.3.}$

$$\bar{Y}_{.2.} - \bar{Y}_{.3.} \pm B\sqrt{\frac{2MSE}{an}}$$

Code

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## [1] -0.52308477 -0.05907816
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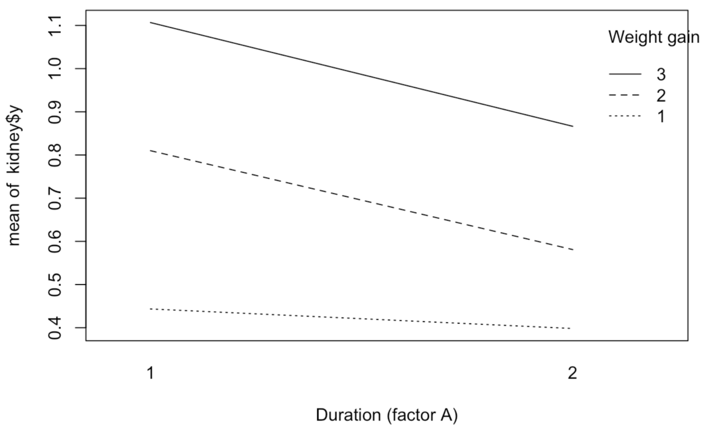

For this family of confidence intervals, the following conclusions may be drawn with family confidence coefficient of 90 percent:

- The average days hospitalized for short and long duration do not differ significantly;
- The average days hospitalized for patients with mild weight gain is shorter than that for patients with moderate weight gain and substantial gain, respectively;
- The average days hospitalized for patients with moderate weight gain is shorter than that for patients with substantial weight gain.

Example

It is known from past experience that 30 percent of patients have mild weight gains, 40 percent have moderate weight gains, and 30 percent have severe weight gains, and that these proportions are the same for the two duration groups. Assume 50 percent of patient in each weight gain group receive short duration treatment, and the other 50 percent receive long duration treatment. Estimate the mean number of days hospitalized (in transformed units) in the entire population with a 95 percent confidence interval.

The linear combination of factor B levels is

$$L = .3\mu_{.1} + .4\mu_{.2} + .3\mu_{.3}$$

The 95 percent confidence interval: $\hat{L} \pm t(0.975,54)s(\hat{L}) = [0.617,0.784]$

Convert your confidence limits to the original units. Does it appear that the mean number of days is less than 7 ?

$$[10^{0.617} - 1, 10^{0.784} - 1] = [3.14,5.08]$$

Since the confidence interval falls below 7, the mean number of days is less than 7.