Topic 3: Random Variables and Important Probability Distributions

Optional Reading: Chapter 5

Xiner Zhou

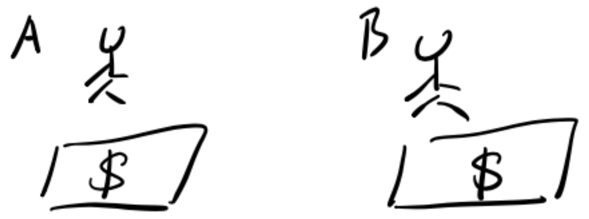
Department of Statistics

University of California, Davis

- What is Random Variable and its Distribution?
- Discrete random variable
- Important discrete distribution
- Continuous random variable
- Important continuous distribution

We have been using “verbal description” about experiment & event, when calculating probabilities.

But such tedious verbal description soon becomes convoluted and obscure how the quantities of interest are related.



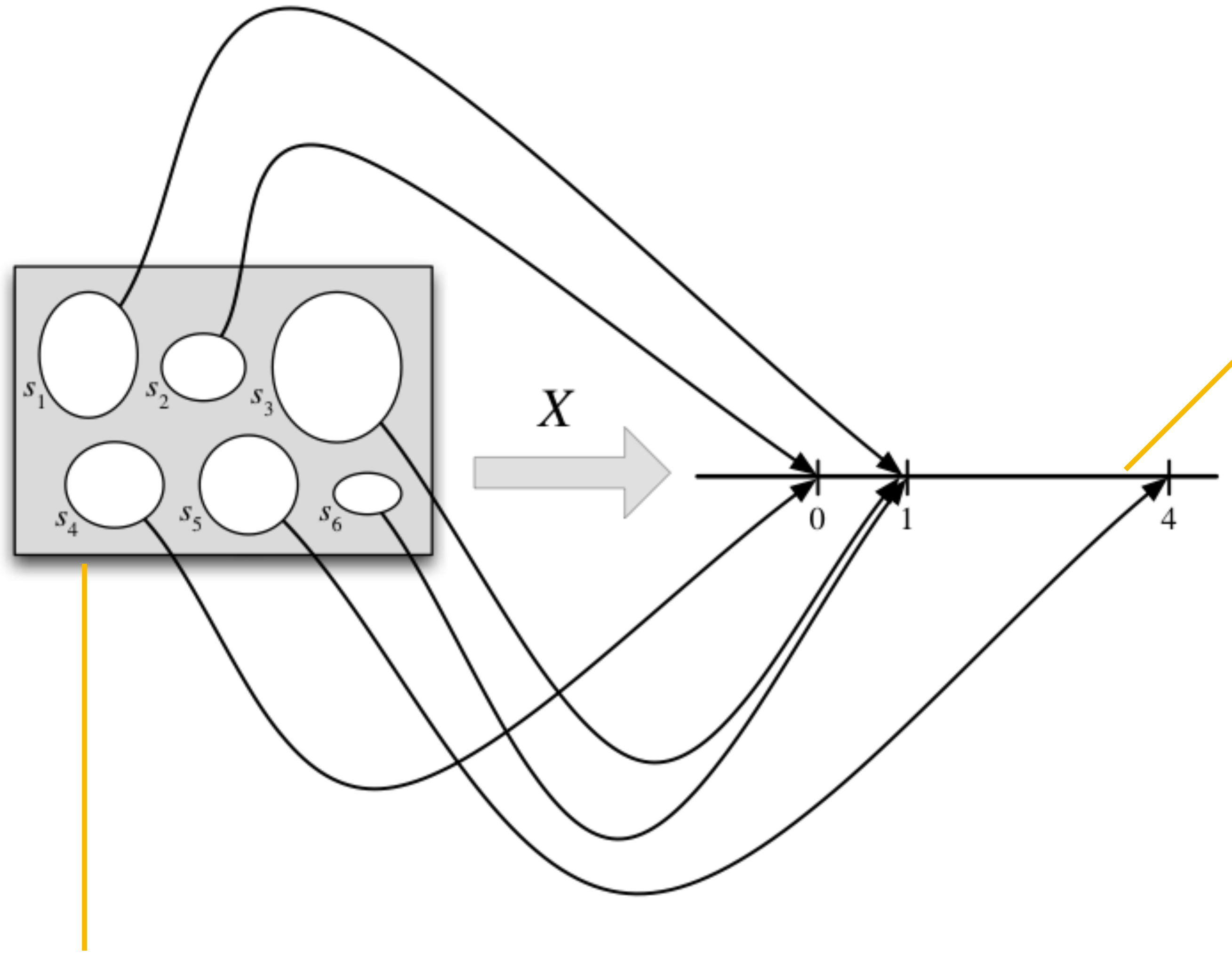
e.g. gambler A’s from Europe, gambler B is from US, what’s the probability that gambler A’s wealth is more than the gambler B’s wealth?

=> P(gambler A’s wealth * exchange rate < gambler B’s wealth)

But, wouldn’t be nice if we could just say something like this? $P(X_A < X_B)$

The notion of random variable will allow us to do exactly this!

Given an experiment with sample space S, a random variable (r.v.) is a function from the sample space S to the real numbers.



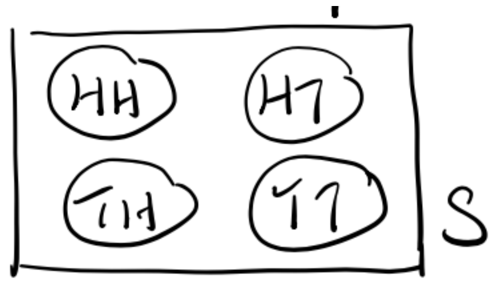
r.v. X records the assigned value

Sample space has 6 elements/simple events, each has an assigned value from {0,1,4} coming from some characteristics that we are interested in

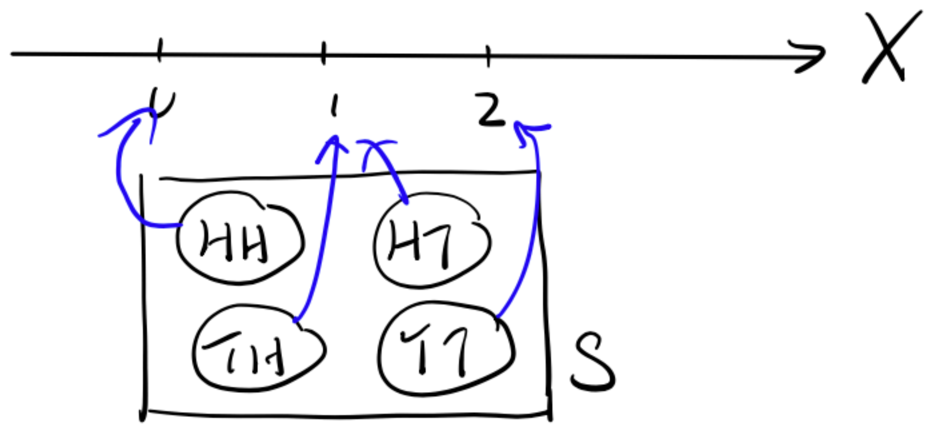
- The randomness in random variable come from the fact that we have a random experiment
- This definition is abstract but fundamental

Each r.v. is a numerical summary of some aspect of experiment that we are interested to study

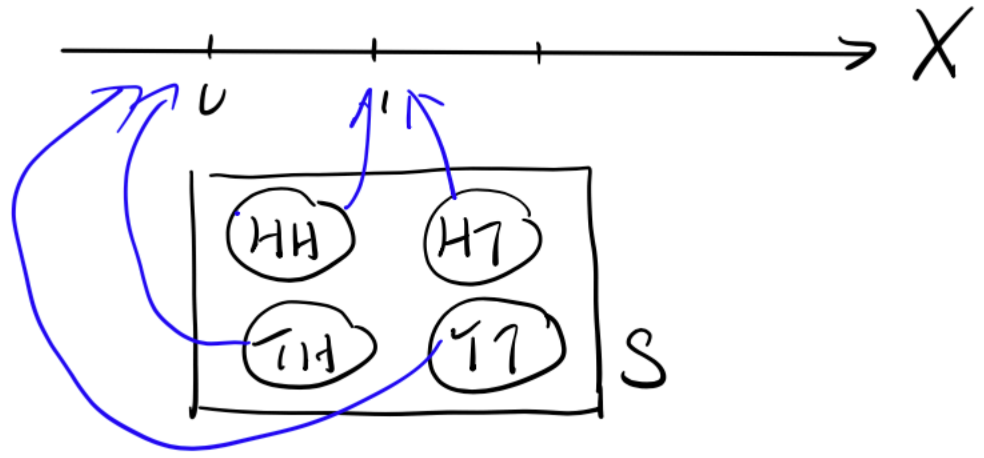
- (Two coins) Consider an experiment where we toss a fair coin twice.



· X = number of tails



· $X = \begin{cases} 1, & \text{if first toss is head} \\ 0, & \text{otherwise} \end{cases}$



- Examples:

- Number of defects on a randomly selected piece of furniture

- SAT score for a randomly selected college applicant

- Number of telephone calls received by a crisis intervention hotline during a randomly selected time period

We assign each possible outcome of an experiment a number, this quantification process results in random variables.

There is one quantity associated with each value of a random variable most important:

- the "probability" of the random variable taking that value.

This information is contained in the probability distribution of the random variable.

For a random variable, X, its probability distribution lists:

- What values x of X can occur

- The chance (probability or likelihood) of each value x that can occur.

Once we know the probability distribution of the random variable, probability of any events can be calculated quite straightforward!

There are different ways to describe probability distribution, depending on the types of random variables.

The possible set of values that a random variable could take could be either discrete or continuous.

- A r.v. X is said to be discrete if:

  X can take finite number, or infinite by countably many (1,2,3…) of values

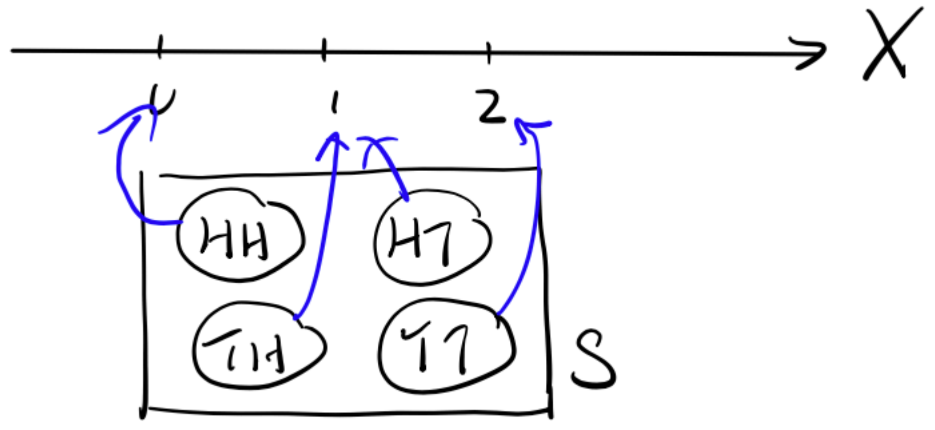
- Probability Mass Function (PMF) is the probability distribution for

  discrete random variables.

  $P_X(x) = P(X = x)$ for each possible value of r.v. X

(Two coins) Consider an experiment where we toss a fair coin twice.

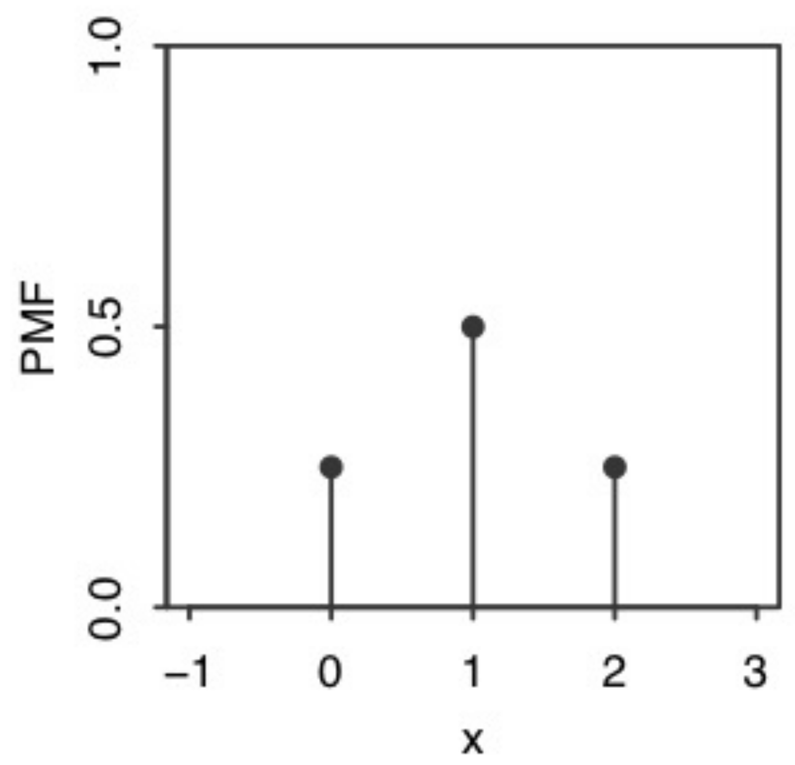
- X = number of tails



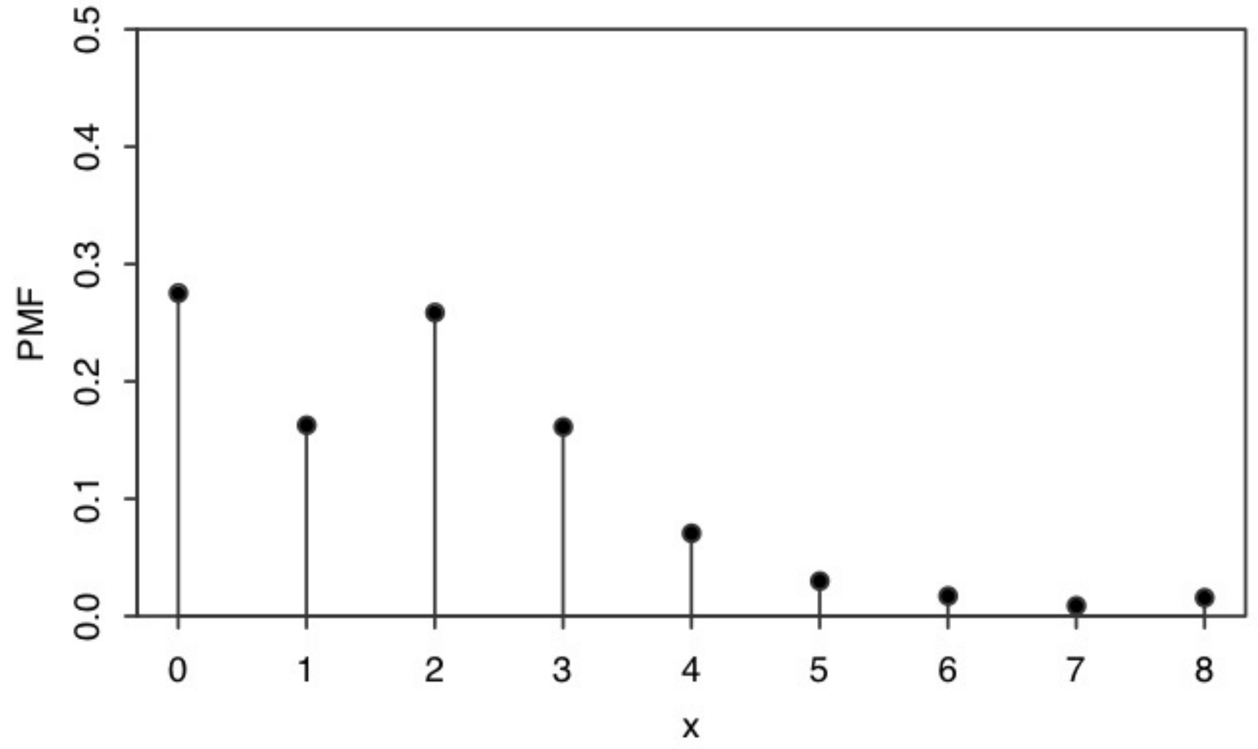
$P_X(0) = P(X = 0) = 1/4$

$P_X(1) = P(X = 1) = 1/2$

$P_X(2) = P(X = 2) = 1/4$



# of children in US households

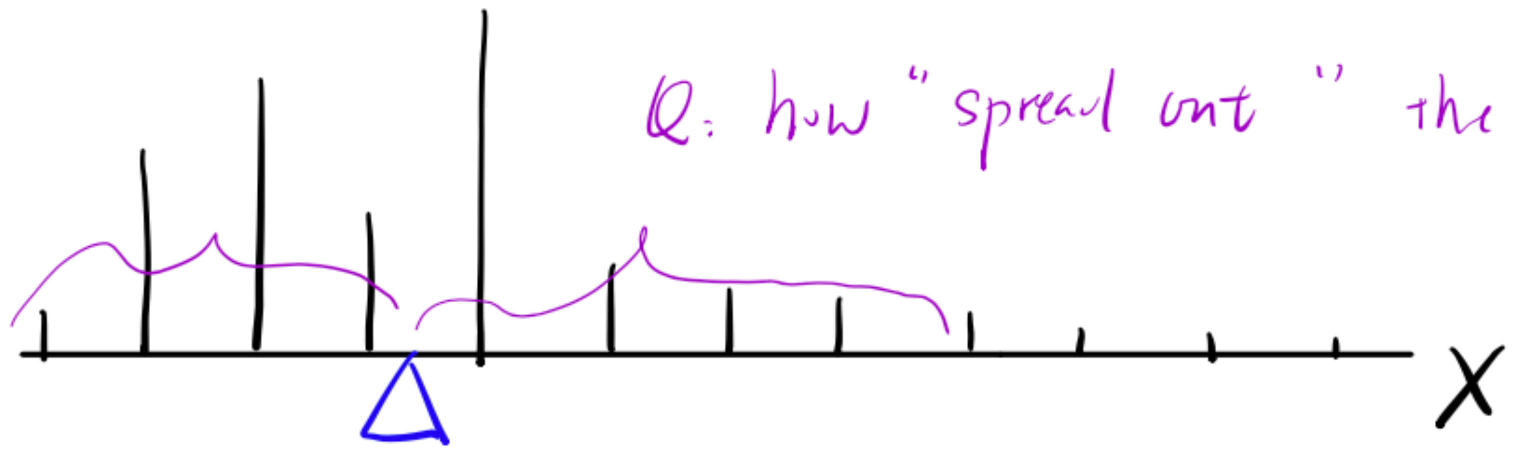


X= # children in a household in US

The distribution of a r.v. gives us full information about probabilities of any events associated with this r.v.

But often, to manage/look at the whole distribution is too much and unnecessary.

We want a few number that summarize key aspects of the distribution.

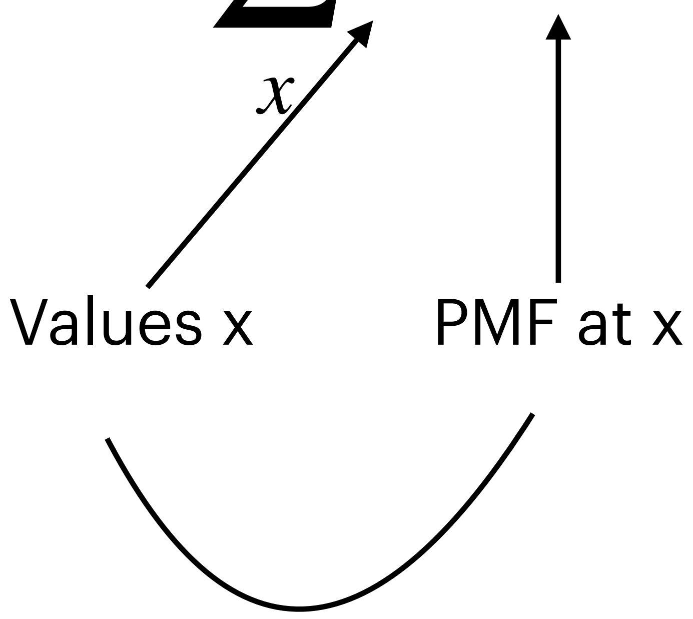


Q: how "spread out" the distribution is? ⟹ variation, standard deviation

Q: where is the "center" / "average"? ⟹ expected value or mean

Expectation / Expected Value/ Mean of a discrete r.v. which takes values in $\{x_1, x_2, \ldots\}$ is:

$$\mu = E(X) = \sum_x xP(X = x)$$

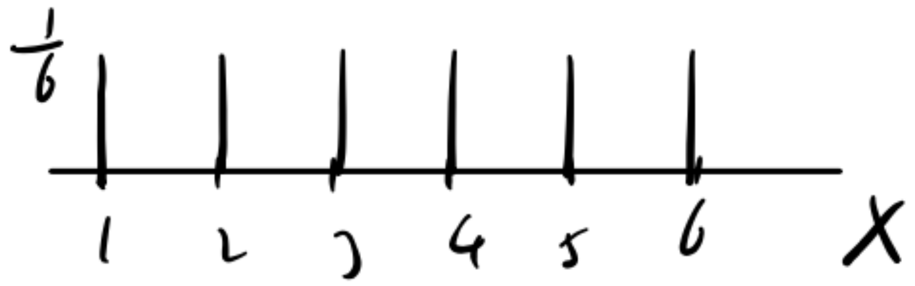


Values x

PMF at x

Expectation of X is a weighted sum of possible values that X can take, weight being the probability that r.v. X =x

X: result of rolling a fair 6-sided die



$$E(x) = \frac{1}{6} \times 1 + \frac{1}{6} \times 2 + \cdots + \frac{1}{6} \times 6 = 3.5$$

- Variance of a discrete r.v. X is:

$$\underbrace{\sigma^2}_{} = Var(X) = E(X - EX)^2 = \underbrace{\sum_x (x - EX)^2 P(X = x)}_{}$$

Variability/dispersion: How far away X is from its center, on average

Weighted sum of squared distance between each possible value x to the center mean E(X),
Weight being the probability of X=x

- The square root of variance is called standard deviation (SD),

  which measures variability in the original unit.

$$SD(X) = \sqrt{\mathrm{Var}(X)}$$

Why study famous distributions that have designate names?

The most famous distributions in Statistics all have stories,

Which explain why they are so often used as models for data.

They serve as models for a large number of applications.

Thinking about these named distributions in terms of their stories, not by their distribution formula!

What's important is understanding the stories, it helps <u>pattern recognition</u>.

See two problems are essentially identical in structure, just in difference cloths!

The Binomial r.v. and its distribution

Story: A coin-tossing experiment is a simple example of binomial random variable.

- Toss a coin for n times
- The tosses are independent
- Each toss, I either get a Head or a Tail
- The probability of getting a Head is p, which is the same for all tosses; the probability of getting a Tail is q=1-p
- We are interested in the total number of Heads I get in the fixed n tosses

Denote X= total number of heads

=> X is a binomial random variable whose distribution is called "Binomial Distribution”

We usually denote X ~ Bin(n,p), n is fixed quantity, p is the only parameter of the distribution.

Many practical problems are essentially the same structure as coin-toss.

- Political polls used to predict voter preferences in elections
    - Each voter = a coin
    - Each voter either votes for Rep or Dem = Head or Tail
    - The proportion of voters who favor one party = probability of getting a head

- A sociology is interested in the proportion of elementary school teachers who are men
- A soft drink marketer is interested in the proportion of cola drinkers who prefer her brand
- A geneticist is interested in the proportion of the population who possess a gene linked to Alzheimer's disease

A Binomial experiment has 5 characteristics:

1. The experiment consists of n identical trials

2. The trails are independent

3. Each trial results in one of two outcomes.

We usually call the one outcome that we care about "a success", and the other "a failure"

4. The probability of success on a single trail is equal to p, which is the same for all trials.

The probability of failure is then q=1-p

5. We are interested in the number of successes observed during a fixed n trials,

denote the number of successes as X, which is the binomial random variable.

We denote $X \sim \mathrm{Bin}(n, p), X = 0,1,2,\ldots n$

What is good about using binomial as the model to describe the problems?

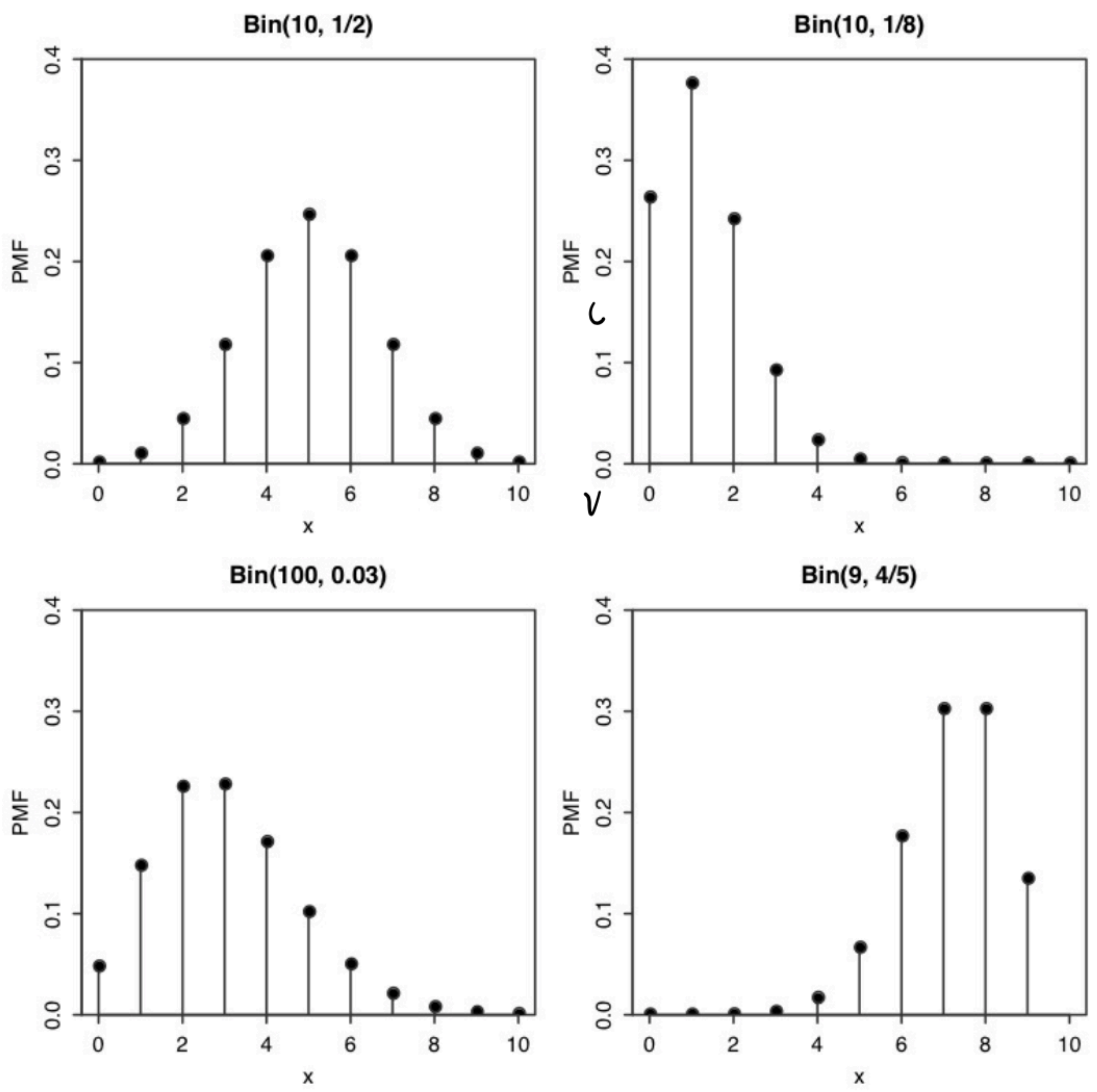
- Only one parameter describe all possible question of interested

Probability Mass Function to describe its Probability Distribution:

$$p(x) = C_k^n p^k q^{n-k} = \frac{n!}{k!(n-k)!} p^k q^{n-k}$$

Where:

- p= Probability of a success on a single trial
- q=1-p
- n= Number of trails
- x= number of successes in n trials



Mean: $\mu = np$

Variance: $\sigma^2 = npq$

Standard deviation: $\sigma = \sqrt{npq}$

Find $P(X = 2)$ for a binomial random variable X with $n = 10$ and $p = 0.1$.

A market research firm hires operators to conduct telephone surveys. The computer randomly dials a telephone number, and the operator asks the respondent whether or not he has time to answer some questions. Let X be the number of telephone calls made until the first respondent is willing to answer the operators questions. Is this a binomial experiment? Explain.

A home security system is designed to have a 99% reliability rate.

Suppose that nine homes equipped with this system experience an attempted burglary.

Find the probabilities of these events:

a. At least one of the alarms is triggered.

b. More than seven of the alarms are triggered.

c. Eight or fewer alarms are triggered.

Over a long period of time, it has been observed that a professional basketball player can make a free throw on a given trial with probability equal to .8. Suppose he shoots four free throws.

1. What is the probability that he will make exactly two free throws?

2. What is the probability that he will make at least one free throw?

Would you rather take a multiple-choice or a full recall test? If you have absolutely no knowledge of the material, you will score zero on a full recall test. However, if you are given five choices for each question, you have at least one chance in five of guessing correctly! If a multiple-choice exam contains 100 questions, each with five possible answers, what is the expected score for a student who is guessing on each question? Within what limits will the no-knowledge scores fall?

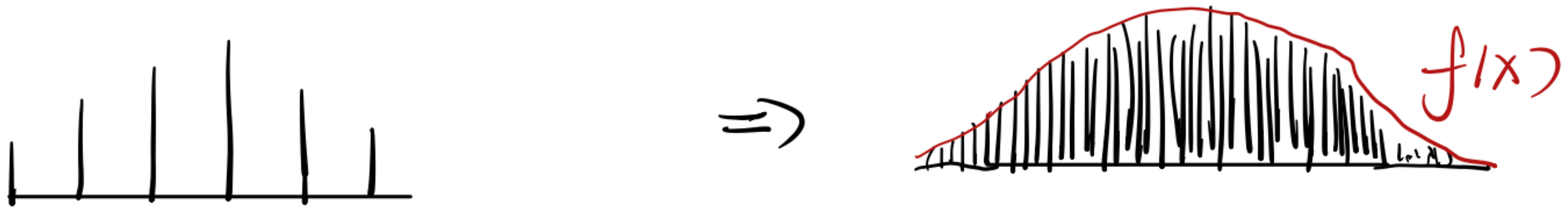
A r.v. X is said to be continuous if:

X can take on any value in an interval in the real line.

Probability Density Function (PDF) is the probability distribution for Continuous random variables

f(x) with domain (a,b) that satisfies two properties:

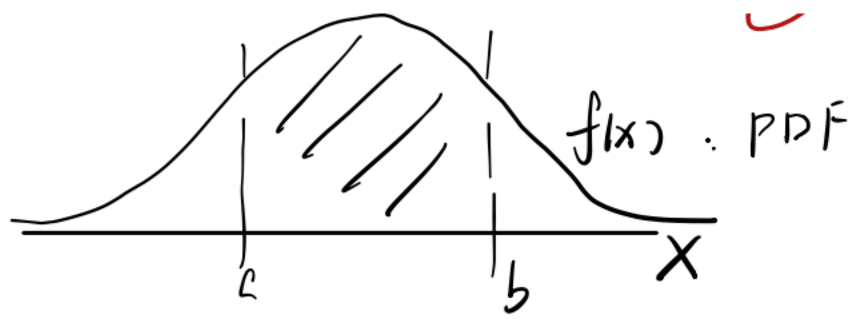
- $f(x) \geq 0$
- $\int_a^b f(x) = 1$



Discrete r.v.: PMF tells us the probability at each possible value

Think continuous r.r. as we increase possible values that X can take. PDF tells us relative frequency at all possible values.

Calculating probability of events using PDF



$$P(a < X < b) = \int_a^b f(x)dx$$

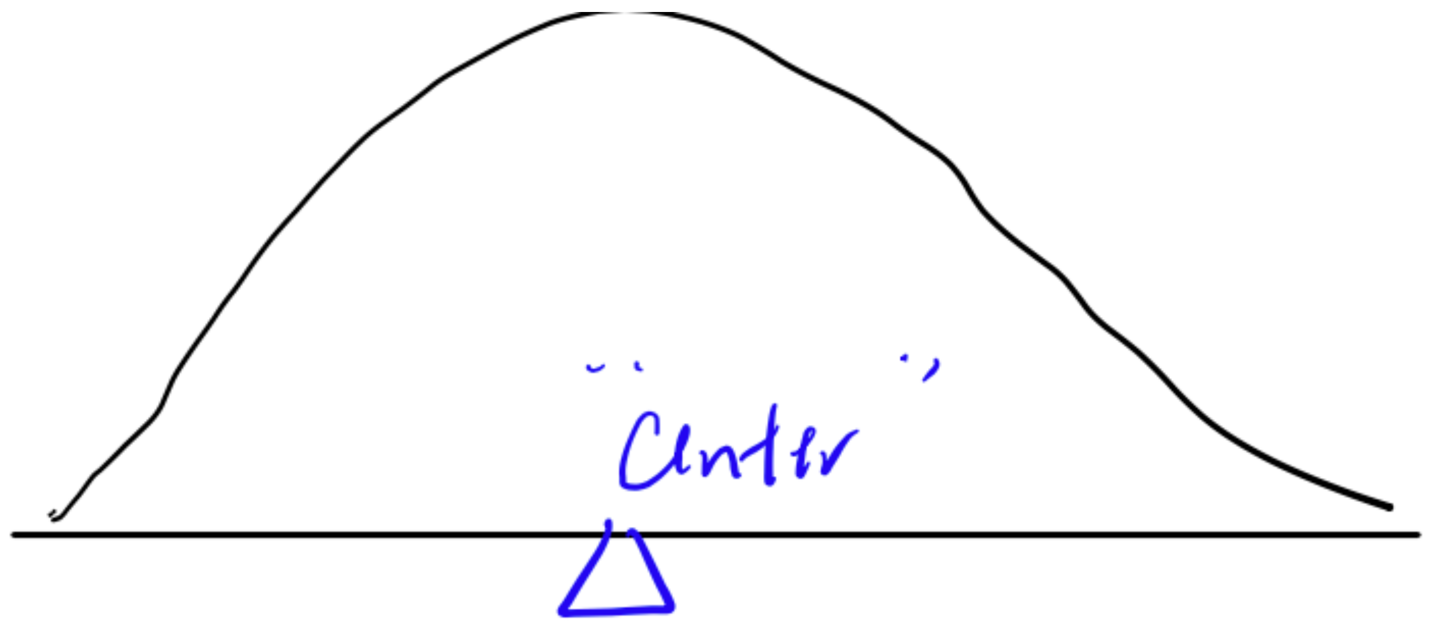
Probability is the area under the curve f(x) for x lies between a and b

Expectation / Expected Value/ Mean of a continuous r.v.:

$$\mu = E(X) = \int_{-\infty}^{\infty} \underset{\text{Values x}}{\uparrow x}\,\underset{\text{PDF at x}}{\uparrow f(x)}\,dx$$

Values x PDF at x

Expectation of X is a weighted sum of possible values that X can take, weight being the probability that r.v. X =x



Variance of a continuous r.v. X is:

$$\sigma^2 = Var(X) = E(X - EX)^2 = \int_{-\infty}^{\infty} (x - EX)^2 f(x)dx$$

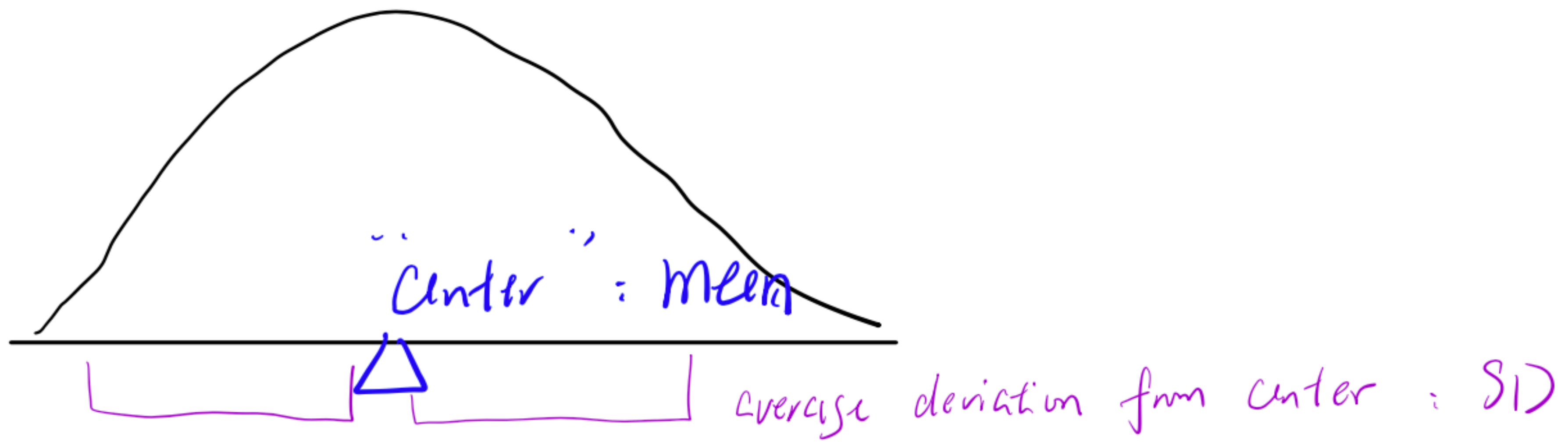
Variability/dispersion: How far away X is from its center, on average

Weighted sum of squared distance between each possible value x to the center mean E(X), Weight being the probability of X=x

The square root of variance is called standard deviation (SD),

which measures variability in the original unit.

$$SD(X) = \sqrt{\text{Var}(X)}$$



Among continuous distributions, Normal distribution is the most famous one.

It’s extremely widely used because:

1. Many variables are nearly normal, because most measurement has large number of influencing factors, when those factors acting together, their additive effects often make the measurement we are interested in close to a Normal curve
2. One of the most important theorem — Central Limit Theorem — says that, the average of a large number of independent and identically distributed r.v.s has an approximately Normal distribution, regardless of the actual distribution.

Meaning: averages are basically following Normal distribution.

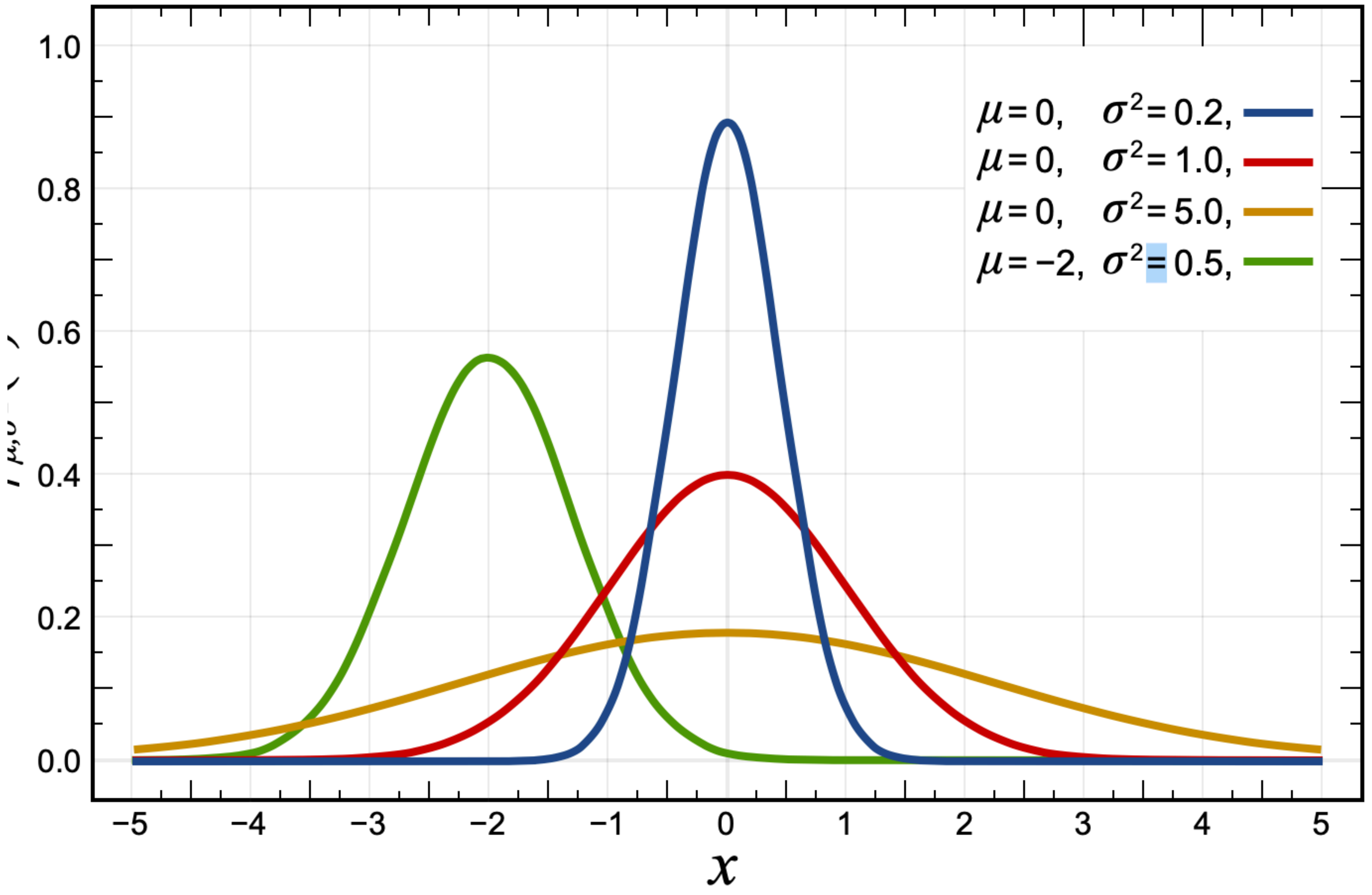
Averages are what we care about most of times!

A continuous r.v. has a bell-shaped probability distribution or bell curve, is known as a normal random variable and its probability distribution is called a normal distribution.

Normal distribution has two parameters:

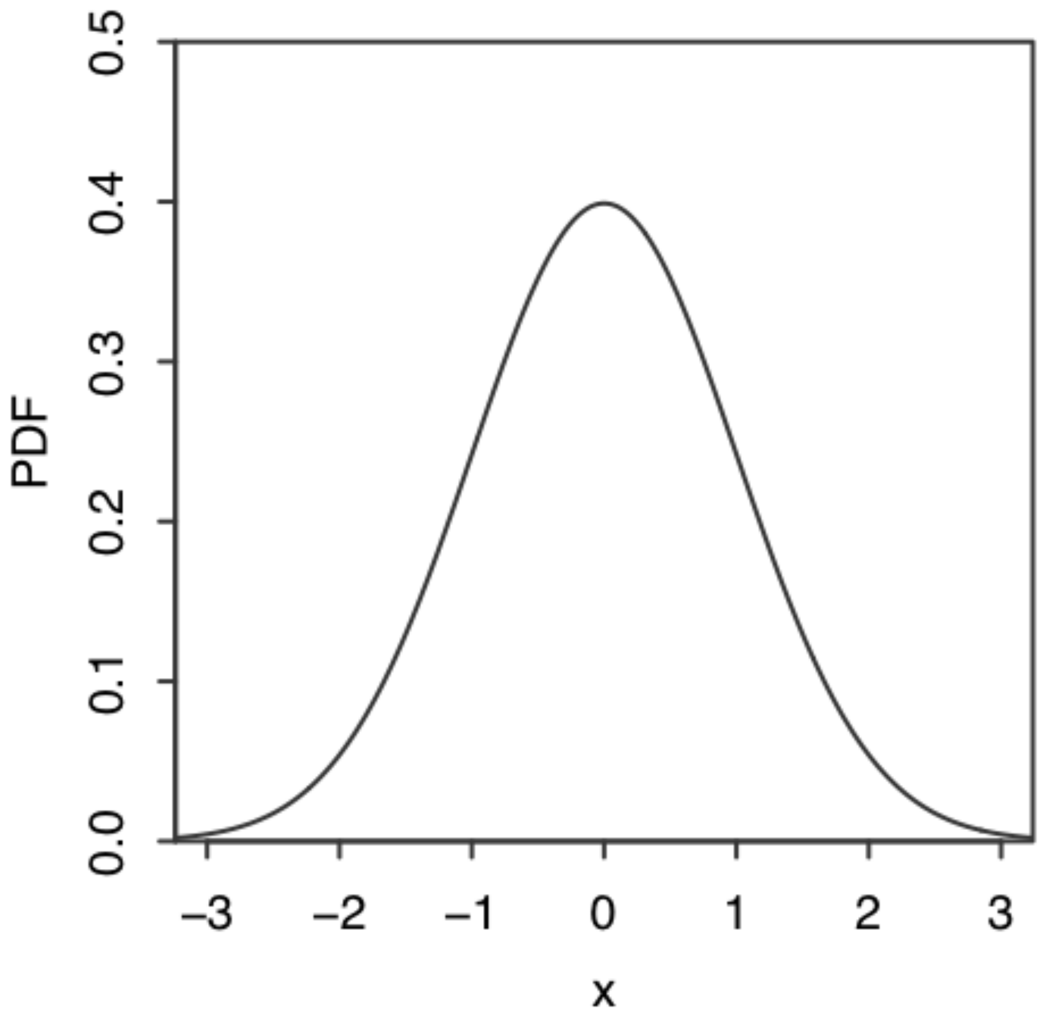
- Mean μ
- Standard deviation σ

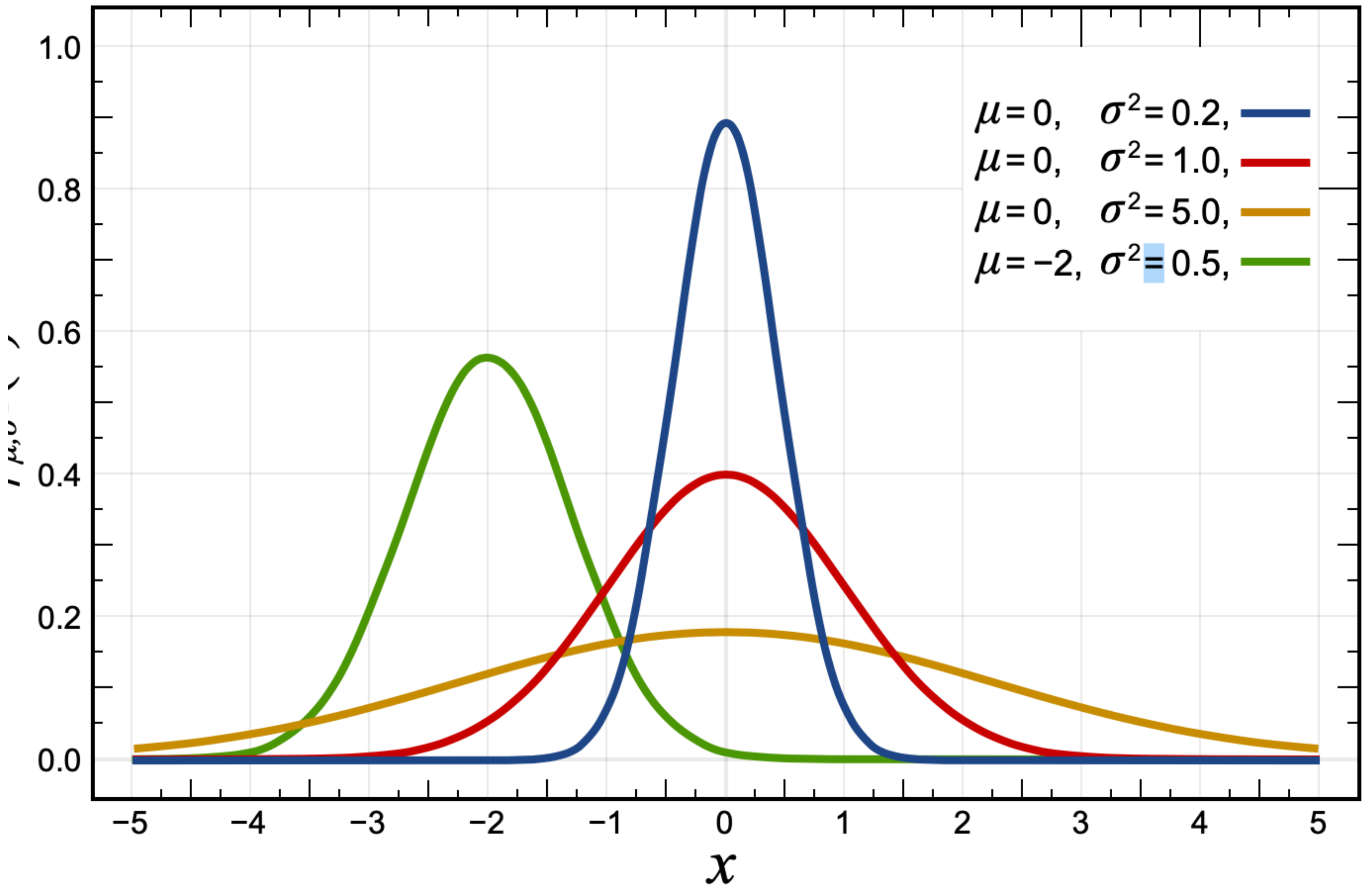
We denote $X \sim N(\mu, \sigma^2)$



The standard normal distribution is a normal distribution with $\mu = 0$ and $\sigma = 1$

We write $Z \sim N(0,1)$



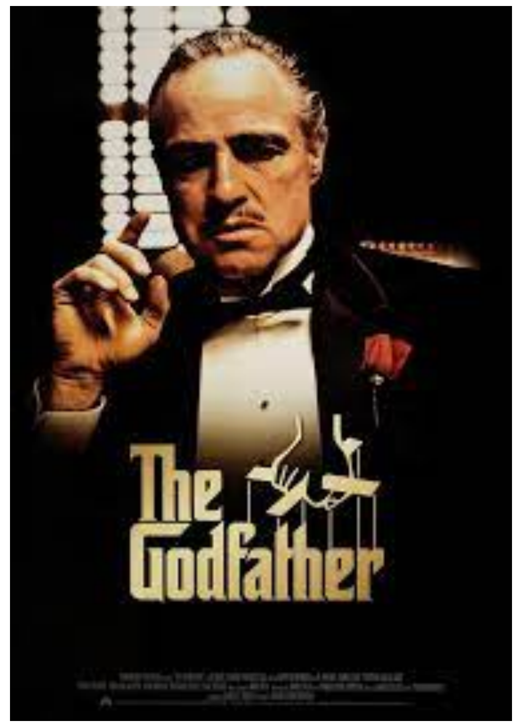


Symmetric around 0, bell-shaped curve

The Godfather:

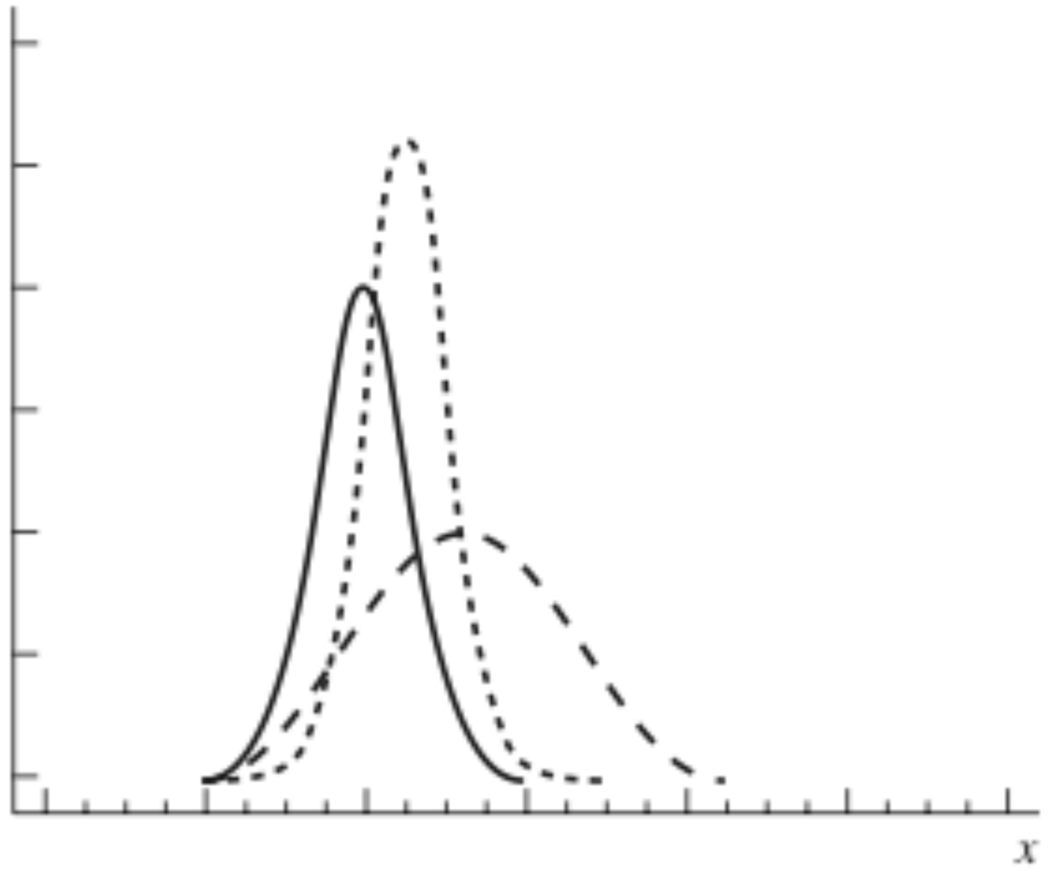
Standard normal distribution is just one member of the normal distribution family (Corleone family).

“There is nothing can’t be solved if you went to the Godfather for help.”



The general normal distribution is governed by:

- Mean μ — center
- Standard deviation σ — spread



Think of general normal distributions as a shift + a stretch upon standard normal.

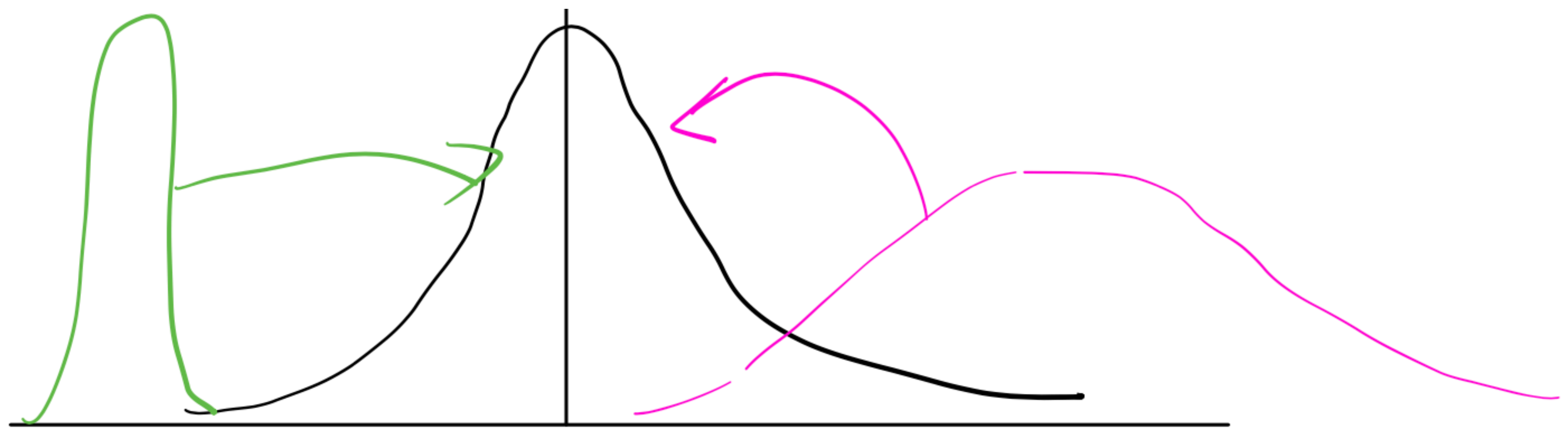
Z-score: describe how far X is away from its mean, in unit of its standard deviation

- If $X \sim N(\mu, \sigma^2)$, then $Z = \dfrac{X - \mu}{\sigma}$

  You can turn any normal distribution into a standard one!

- If $Z \sim N(0,1)$, then $X = \mu + \sigma Z \sim N(\mu, \sigma^2)$

  You can turn a standard one into any normal distribution!



Why do this? We know everything about the Godfather = Z!

To find normal probabilities, say P(X<a), one usually coverts X to its z-score, and uses standard normal table to find corresponding probabilities.

Steps for Finding a Probability Corresponding to a Normal Random Variable:

1. Sketch the normal distribution and indicate the mean of the random variable X. Then shade the area corresponding to the probability you want to find.

2. Convert the boundaries of the shaded area from X values to standard normal random variable z-score values by using the formula,

$$z = \frac{x - \mu}{\sigma}$$

Show the z values under the corresponding X values on your sketch.

3. Use the standard normal table to find the areas corresponding to the z values. If necessary, use the symmetry of the normal distribution to find areas corresponding to negative z values and the fact that the total area on each side of the mean equals 0.5. to convert the areas from the table to the nrobabilities of the event you have selected

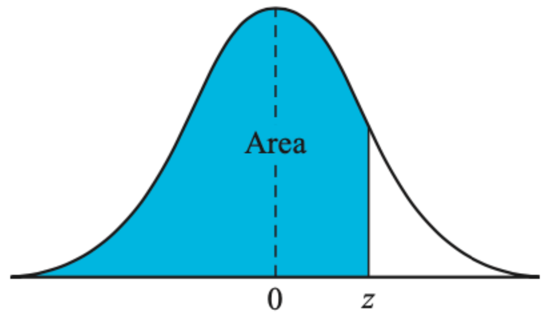


TABLE 3 **Areas under the Normal Curve**

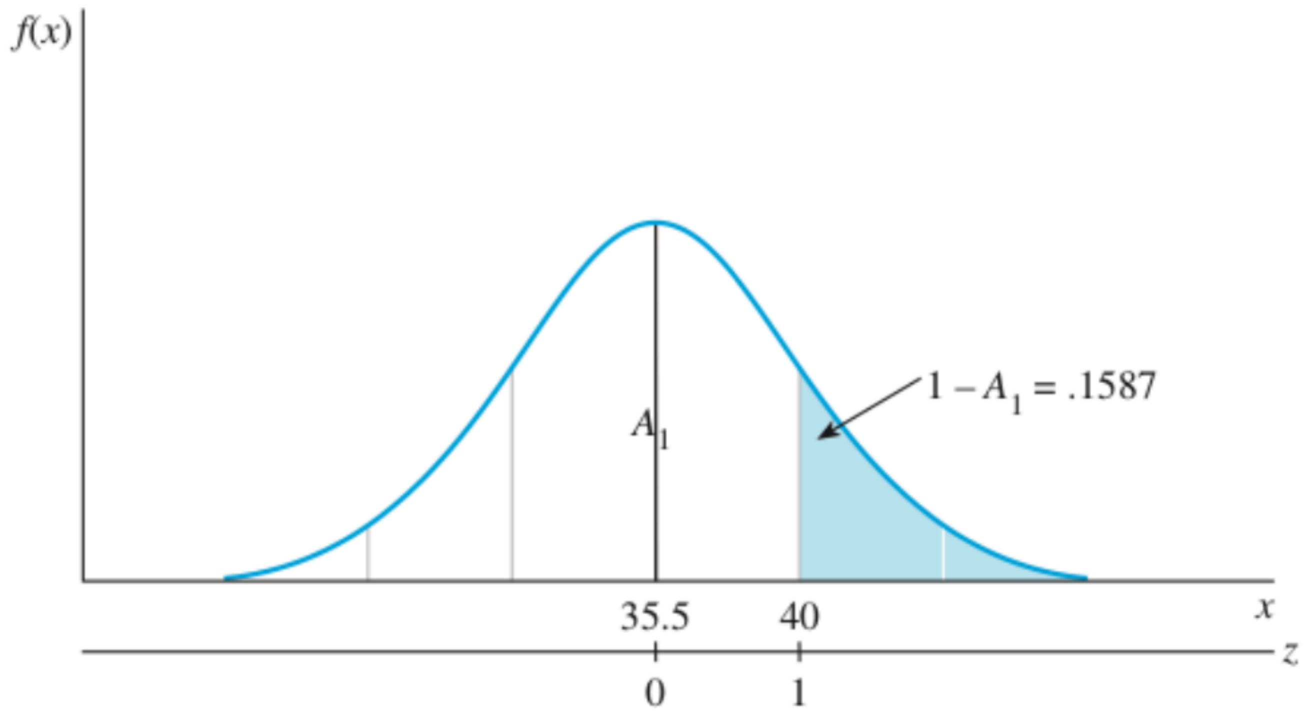
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0017	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0722	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

TABLE 3 *(continued)*

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5 871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

- $p(z > 0)$, $p(z \leq -1.5)$, $p(z \geq 1.5)$, $p(-1.5 < z < 1.5)$, $p(-2.4 < z \leq 3)$, $p(z > 3.5)$

Studies show that gasoline use for compact cars sold in the United States is normally distributed, with a mean of 35.5 miles per gallon (mpg) and a standard deviation of $4.5 mpg. What percentage of compacts get $40 mpg or more?



In times of scarce energy resources, a competitive advantage is given to an automobile manufacturer who can produce a car that has substantially better fuel economy than the competitors cars. If a manufacturer wishes to develop a compact car that outperforms 95% of the current compacts in fuel economy, what must the gasoline use rate for the new car be?

A normal random variable X has mean 50 and standard deviation 15.

Would it be unusual to observe the value $x=0$? Explain your answer.

For a car traveling 30 miles per hour (mph), the distance required to brake to a stop is normally distributed with a mean of 50 feet and a standard deviation of 8 feet. Suppose you are traveling 30 mph in a residential area and a car moves abruptly into your path at a distance of 60 feet.

a. If you apply your brakes, what is the probability that you will brake to a stop within 40 feet or less? Within 50 feet or less?

b. If the only way to avoid a collision is to brake to a stop, what is the probability that you will avoid the collision?

Once nice thing about Normal distribution is: easy to calculate probabilities of practical relevance based on the “68-95-99.7%” rule.

68-95-99.7% rule:

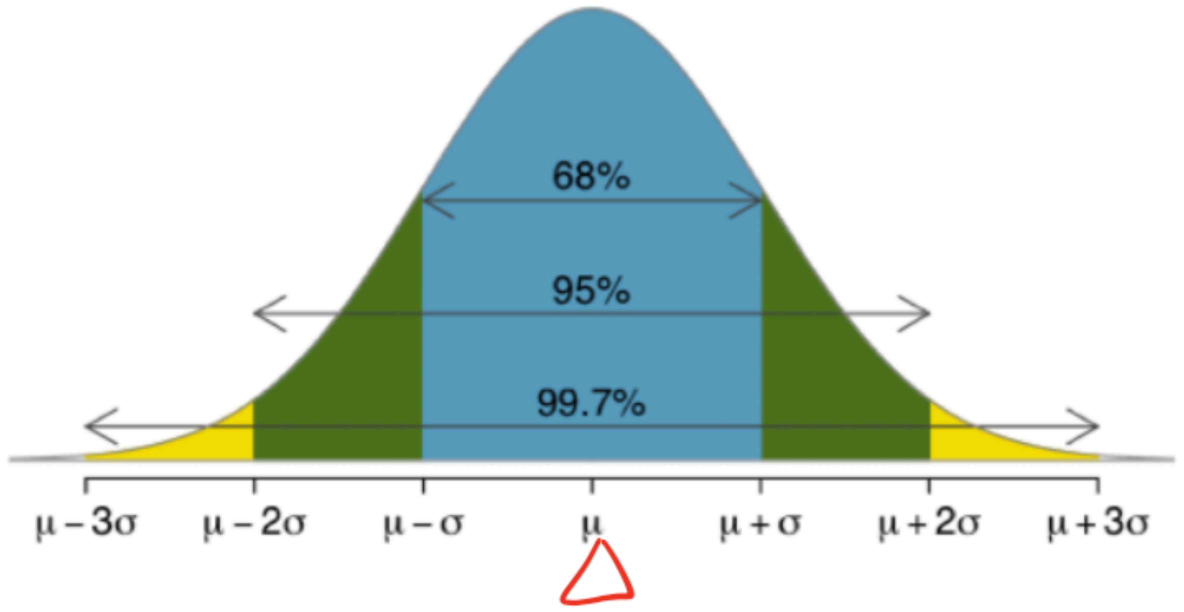
1. If $X \sim N\left(\mu, \sigma^2\right)$, then

- $P(|x-\mu| < \sigma) \approx 0.68$
- $P(|x-\mu| < 2\sigma) \approx 0.95$
- $P(|x-\mu| < 3\sigma) \approx 0.997$

About 95% of the time, a normal data will fall within + or - 2 standard deviation away from its center

2. If $Z \sim N(0,1)$, then

- $P(|Z| < 1) \approx 0.68$
- $P(|Z| < 2\ ) \approx 0.95$
- $P(|Z| < 3) \approx 0.997$



$\underbrace{\mu\text{: center}}$

σ: average distance from center

Let $X \sim N(-1,4)$, what is $P(|X|<3)$ approximately?