Acta Mathematica Scientia, 2024, **44B**(5): 1–31
https://doi.org/10.1007/s10473-024-

Acta Mathematica Scientia 数学物理学报
http://actams.apm.ac.cn

LÉVY AREA ANALYSIS AND PARAMETER ESTIMATION FOR FOU PROCESSES VIA NON-GEOMETRIC ROUGH PATH THEORY

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Abstract This paper addresses the estimation problem of an unknown drift parameter matrix for a fractional Ornstein-Uhlenbeck process in a multi-dimensional setting. To tackle this problem, we propose a novel approach based on rough path theory that allows us to construct pathwise rough path estimators from both continuous and discrete observations of a single path. Our approach is particularly suitable for high-frequency data. To formulate the parameter estimators, we introduce a theory of pathwise Itô integrals with respect to fractional Brownian motion. By establishing the regularity of fractional Ornstein-Uhlenbeck processes and analyzing the long-term behavior of the associated Lévy area processes, we demonstrate that our estimators are strong consistent and pathwise stable. Our findings offer a new perspective on estimating the drift parameter matrix for fractional Ornstein-Uhlenbeck processes in multi-dimensional settings and may have practical implications for various fields, including finance, economics, and engineering.

Key words Itô integration, Lévy area, Non-geometric rough path, FOU processes, Pathwise stability, Long time asymptotic, High-frequency data.

2020 MR Subject Classification 60H05, 62F12, 62M09, 91G30

1 Introduction

The field of statistical analysis of time series and random processes involves parameter and non-parameter estimations, and statistical inferences as well. The majority of research in this area has focused on models described in terms of diffusion processes and semi-martingales. Standard references, such as [27, 30, 42, 43], are among those that have concentrated on these models. However, applications that require the consideration of long-time memory effects have

Received March 04, 2023; revised April 29, 2024.
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prompted some attention to models that are not semi-martingales, such as those discussed in [24, 28, 45].

This article focuses on multi-dimensional Ornstein-Uhlenbeck (OU) processes driven by fractional Brownian motions (fBM). These processes are commonly referred to as fractional Ornstein-Uhlenbeck (fOU) processes and are defined as the solution to the stochastic differential equation (SDE):

$$dX_t = -\Gamma X_t dt + \Sigma dB_t^H,\ X_0 = x_0. \tag{1.1}$$

Here, B^H is a d-dimensional fBM with a Hurst parameter $H \in (0,1)$, $\Gamma \in \mathbb{R}^{d\times d}$ is the drift matrix which is symmetric and positive-definite, and $\Sigma \in \mathbb{R}^{d\times d}$ is the non-degenerate volatility matrix. The SDE must be interpreted as the stochastic integral equation:

$$X_t = x_0 - \int_0^t \Gamma X_s ds + \Sigma B_t^H,$$

which has a unique solution given by:

$$X_t = e^{-\Gamma t} x_0 + \int_0^t e^{-\Gamma(t-s)} \Sigma dB_s^H. \tag{1.2}$$

The integral on the right-hand side is understood as a Young's integral. Therefore, like ordinary OU processes, (X_t) is a Gaussian process.

The multi-dimensional fOU processes can be used to describe systems with linear interactions perturbed by Gaussian noise. For instance, inter-banking lending is a real-world example that can be modeled using Equation (1.1), as demonstrated in [8, 19]. A crucial question in such applications is to estimate the interaction structure Γ from a single path observation of the process, assuming that Σ is known and the single path $X(\omega)$ can be continuously or discretely observed.

In the one-dimensional case, maximum likelihood estimators (MLE) and least square estimators (LSE) have been studied extensively, and their properties have been documented in literature [24, 25, 28, 45]. The MLE based on continuous observation has been studied by Kleptsyna and Le Breton [28] and Tudor and Viens [45], who obtained the strong consistency of the MLE as T goes to infinity. Hu and Nualart [24] investigated the LSE for the case where the Hurst parameter $H > \frac{1}{2}$ and proved its strong consistency as $T \to \infty$ for $H > \frac{1}{2}$. They also established a central limit theorem if $\frac{1}{2} < H < \frac{3}{4}$. The results were extended by Hu, Nualart and Zhou [25] for all $H \in (0,1)$.

However, few research has been conducted on parameter estimation for multi-dimensional fOU processes. This paper aims to fill this gap. Firstly, we present an estimator based on the rough path theory for continuous observation of a single path. To formulate the parameter estimator, we define an Itô type integration theory for multi-dimensional fBM.

Coutin and Qian [16] developed a theory of Stratonovich integration for multi-dimensional fBM with Hurst parameter $H > \frac{1}{4}$ using rough path analysis. However, the problem of constructing a rough path theory for fBM with $H \le \frac{1}{4}$ remains unsolved. Here, we focus on fBM with H such that $\frac{1}{3} < H \le \frac{1}{2}$, where both fBM and fOU processes have finite p-variation with $2 \le p < 3$. We canonically enhance these processes to geometric rough paths, allowing us to define Itô-type integrals with respect to fBM and fOU processes by correcting their enhanced

Lévy area processes. We then apply this theory to investigate the parameter estimation problem for fOU processes based on continuous observation. To establish the strong consistency of the parameter estimator, we also explore the regularity of fOU processes and the long-term asymptotic behavior of their Lévy area processes, which we believe are interesting in their own right.

We also address the parameter estimation problem for fOU processes based on discrete observation. In practice, observations are often discrete rather than continuous, even though the sampling frequency can be increased, as in the case of high-frequency financial data. To tackle this statistical inference problem, we recommend [1–5, 15, 35] and related references. In this paper, we construct a parameter estimator based on high-frequency discrete observation using rough path theory and establish its strong consistency. It is worth noting that Diehl, Friz, and Mai [17] used rough path analysis to study maximum likelihood estimators for diffusion processes and initiated research on estimators for the fractional case, but only for small ε when $H = \frac{1}{2} - \varepsilon$.

The methodology proposed in this paper offers several advantages over existing methods in the literature. Firstly, our estimators are applicable to multi-dimensional fOU processes, which reveal the non-trivial role played by Lévy area processes, and are fundamentally different from the one-dimensional case. Secondly, the parameter estimators are pathwise defined and can be computed based on observations of a single path. Thirdly, our parameter estimators exhibit pathwise stability and robustness, in the sense that if two observations are close in the so-called p-variation distance (as defined in the main text below), then their corresponding estimators are also close. Fourthly, our estimators can be constructed using both continuous and discrete observation data, particularly useful for high-frequency financial data.

We note that our approach can be extended to the Ornstein-Uhlenbeck process X_t driven by a general Gaussian noise G_t satisfying certain technical conditions, where

$$X_t = e^{-\Gamma t}x_0 + \int_0^t e^{-\Gamma(t-s)}\Sigma dG_s. \tag{1.3}$$

The integral on the right-hand side is well-defined as long as $t \to G_t$ is α-Hölder continuous for some $\alpha > 0$. These singular OU processes may have practical applications.

The structure of this paper is organized as follows. In Section 2, we begin by introducing some preliminary concepts of rough path theory and present a framework for pathwise Itô integrals for both fBM and fOU processes. In Section 3, we explore the regularity of fOU processes and examine the long time behavior of their associated Lévy area processes. Then, in Section 4, we construct a continuous rough path estimator and present a complete proof for its almost sure convergence and pathwise stability. In Section 5, we present the discrete rough path estimator that is based on high-frequency data.

2 Rough paths and Itô integration

In this section, we present the notations used in the rough path theory, following established references such as [20–22, 33, 34]. We provide a precise definition of Itô integrals for both fBM and fOU processes.

2.1 Preliminary of rough paths

We define the truncated tensor algebra $T^{(2)}(\mathbb{R}^d)$ as follows: $T^{(2)}(\mathbb{R}^d) := \oplus_{n=0}^{2}(\mathbb{R}^d)^{\otimes n}$, where we adopt the convention that $(\mathbb{R}^d)^{\otimes 0} = \mathbb{R}$. In this context, we use Δ to represent the simplex given by $\{(s,t) : 0 \le s < t \le T\}$. Let X_t be a continuous path with finite p-variation, where $2 < p < 3$, defined on the interval $[0,T]$. We denote $\mathbf{X}_{s,t} = (1, X_{s,t}, \mathbb{X}_{s,t})$ as an element of the space $T^{(2)}(\mathbb{R}^d)$, where $X_{s,t} = X_t - X_s \in \mathbb{R}^d$ and $\mathbb{X}_{s,t} \in \mathbb{R}^d \otimes \mathbb{R}^d$. To further illustrate the concept, if the process X_t is of finite variation, then $\mathbb{X}_{s,t} = \int_{s<t_1<t_2<t} dX_{t_1} \otimes dX_{t_2}$, where the integral represents the tensor product of the two differentials. We refer to $\mathbf{X}_{s,t}$ as a lift of the process X to the space $T^{(2)}(\mathbb{R}^d)$ if it satisfies both finite p-variation and Chen's identity.

The initial motivation behind this concept is to define integrals with respect to X by increasing the information on X. We recall Chen's identity, which relates to algebraic information, and the definition of finite p-variation, which relates to analysis information.

We call that $\mathbf{X}_{s,t} = (1, X_{s,t}, \mathbb{X}_{s,t})$ satisfies *Chen's identity* if

$$X_{s,t} = X_t - X_s, \tag{2.1}$$

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = X_{s,u} \otimes X_{u,t}, \tag{2.2}$$

for all $(s,u),\ (u,t) \in \Delta$.

$\mathbf{X} = (1, X_{s,t}, \mathbb{X}_{s,t})$ has *finite p-variations* if

$$\sup_{\mathcal{P}} \sum_{[s,t]\in\mathcal{P}} |X_{s,t}|^p < \infty,\ \sup_{\mathcal{P}} \sum_{[s,t]\in\mathcal{P}} |\mathbb{X}_{s,t}|^{p/2} < \infty,$$

where $\mathcal{P}$ is a partition of $[0,T]$. It is equivalent to that there exists a control $\omega(s,t)$ such that

$$|X_{s,t}| \le \omega(s,t)^{1/p},\ |\mathbb{X}_{s,t}| \le \omega(s,t)^{2/p},\ \forall (s,t) \in \Delta.$$

A control ω is a non-negative, continuous, super-additive function on Δ and satisfies that $\omega(t,t) = 0$.

Let $2 < p < 3$ be a constant. A function $\mathbf{X} = (1, X, \mathbb{X})$ from Δ to $T^{(2)}(\mathbb{R}^d)$ is called a *p-rough path* if it has finite p-variation, and satisfies Chen's identity. Denote the space of p-rough paths as $\Omega_p(\mathbb{R}^d)$.

According to Lyons and Qian [34], the integration operator is defined as a linear map from $\Omega_p(\mathbb{R}^d)$ to $\Omega_p(\mathbb{R}^e)$, i.e. $\int F : \Omega_p(\mathbb{R}^d) \to \Omega_p(\mathbb{R}^e)$, and denote the integral by $Y = \int F(X) d_{\mathfrak{R}} \mathbf{X}$, where

$$Y^1_{u,v} \equiv \int_u^v F(X) d_{\mathfrak{R}_1} \mathbf{X} := \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} F(X_s) X_{s,t} + DF(X_s)\mathbb{X}_{s,t}, \tag{2.3}$$

and the second level Y^2 by

$$Y^2_{u,v} \equiv \int_u^v F(X) d_{\mathfrak{R}_2} \mathbf{X} := \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} Y^1_{u,s} \otimes Y^1_{s,t} + F(X_s) \otimes F(X_s)\mathbb{X}_{s,t}, \tag{2.4}$$

where the limit takes over all finite partitions $\mathcal{P}$ of interval $[u,v]$.

2.2 FBM as rough paths

Almost all sample paths of a d-dimensional fractional Brownian motion (fBM) with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2}]$ possess finite p-variation with $2 < \frac{1}{H} < p < 3$, and can be canonically enhanced to geometric rough paths. Coutin and Qian [16] constructed the canonical rough path enhancement $\mathbf{B}^{H,\mathrm{Str}} = (1, B^H, \mathbb{B}^{H,\mathrm{Str}})$ in the Stratonovich sense using dyadic approximations of fBM and their iterated integrals. However, for the parameter estimation problem discussed in this paper, understanding the stochastic integral in the estimator (see Section 4) in the Stratonovich sense would almost surely result in convergence to 0, rendering the estimator unreasonable and useless. Therefore, we require a theory of Itô-type integration (non-geometric rough path) for both fBM and fOU processes. In [41], Qian and Xu constructed a non-geometric rough path enhancement $\tilde{\mathbf{B}}^H = (1, B^H, \tilde{\mathbb{B}}^H)$ associated with an fBM by setting

$$\tilde{\mathbb{B}}^H_{s,t} = \mathbb{B}^{H,\mathrm{Str}}_{s,t} - \frac{1}{2} I(t^{2H} - s^{2H}),$$

where I denotes the $d \times d$ identity matrix. This construction of $\tilde{\mathbb{B}}^H_{s,t}$ allows for the definition of pathwise integrals with respect to the enhanced rough path $\tilde{\mathbf{B}}^H$. For the first level of this integral,

$$\int F(B^H) d_{\mathfrak{R}_1} \tilde{\mathbf{B}}^H(\omega) = \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} F(B^H_s(\omega)) B^H_{s,t}(\omega) + DF(B^H_s(\omega)) \tilde{\mathbb{B}}^H_{s,t}(\omega),$$

for every $\omega \in N^c$, where N is a null set. The second level is defined similarly as (2.4).

The theory of above rough path enhancement and associated Itô integration is limited to one forms, meaning it only works well for functions of B^H_t. Thus, this theory is not suitable for dealing with fOU processes, which depend on the whole path of B^H_s, $0 \le s \le t$. In this paper, we reveal that a different integration theory is needed, one with different rough paths associated with fBM.

To define a non-geometric Itô rough path enhancement associated with fBM suitable for the study of fOU processes, we take $\varphi(t) := \frac{1}{2} I t^{2H} - U(t)$, where

$$U(t) := H\Gamma \int_0^t \int_0^s e^{-\Gamma(s-u)} (s^{2H-1} - (s-u)^{2H-1}) du ds. \tag{2.5}$$

This function has finite q-variation with $q = \frac{1}{2H}$, allowing us to define the non-geometric Itô type fractional Brownian rough path lift for B^H as

$$\mathbf{B}^{H,\text{Itô}}_{s,t} = (1, B^H_{s,t}, \mathbb{B}^{H,\text{Itô}}_{s,t}) := (1, B^H_{s,t}, \mathbb{B}^{H,\mathrm{Str}}_{s,t} - \varphi_{s,t}), \tag{2.6}$$

where $\varphi_{s,t} = \varphi(t) - \varphi(s)$.

Remark 2.1 One can verify that, if $H = \frac{1}{2}$, this Itô rough path enhancement is consistent with Itô theory for the standard Brownian motion. When $\Gamma = 0$, this enhancement is the same with the one form case defined in Qian and Xu [41]. In the following, we will illustrate why we call it as Itô rough path/Itô rough integration.

2.3 FOU as rough paths

For the fOU process X_t defined by stochastic differential equation (1.1), it can also be enhanced as a rough path according to the theory of rough path, which is the essence of the theory of rough differential equations. Although for the existence and uniqueness of the solution to (1.1), for this simple case, the theory of rough path is not needed. However, when we ask if X_t can be enhanced to a rough path, or when we want to integrate $F(X)$ with respect to X, the rough path analysis is a natural tool to deal with these problems.

We emphasize that the meaning of the solution $\mathbf{X}$ to a rough differential equation enhanced by (1.1) depends on the rough paths $\mathbf{B}^H$ we use. Here $\mathbf{B}^H$ can be either $\mathbf{B}^{H,\mathrm{Str}}$ (in Stratonovich sense) or $\mathbf{B}^{H,\mathrm{Itô}}$ (in Itô sense).

Let $Z_t=(B_t^H,X_t,t)$ and $\mathbf{Z}=(\mathbf{B}^H,\mathbf{X},\mathbf{t})$ to be its associated rough path enhancement. Then equation (1.1) is enhanced to

$$d_{\mathfrak{R}}\mathbf{Z}=f(Z)d_{\mathfrak{R}}\mathbf{Z}, \tag{2.7}$$

where $f(x,y,t)(\xi,\eta,\tau):=(\xi,-\Gamma y\tau+\Sigma\xi,\tau)$. According to Theorem 6.2.1 and Corollary 6.2.2 in [34], a unique solution $\mathbf{Z}$, which is a rough path, exists. Formally, $\mathbf{Z}=(1,Z,\mathbb{Z})$ has the following expression:

$$Z_{s,t}=(B^H_{s,t},X_{s,t},t-s), \tag{2.8}$$

$$\mathbb{Z}_{s,t}=\begin{pmatrix}\mathbb{B}^H_{s,t} & \int_s^t B^H_{s,u}dX_u & \int_s^t B^H_{s,u}du\\ \int_s^t X_{s,u}dB^H_u & \mathbb{X}_{s,t} & \int_s^t X_{s,u}du\\ \int_s^t (u-s)dB^H_u & \int_s^t(u-s)dX_u & \frac{1}{2}(t-s)^2\end{pmatrix}. \tag{2.9}$$

Each component of the second level $\mathbb{Z}_{s,t}$ is well-defined as parts of the solution to (2.7). More exactly, we denote Stratonovich solution of RDE (2.7) as $\mathbf{Z}^{\mathrm{Str}}=(1,Z,\mathbb{Z}^{\mathrm{Str}})$, where $\mathbb{Z}^{\mathrm{Str}}=(\mathbb{Z}^{\mathrm{Str},ij})_{i,j=1,2,3}$, and we denote Itô solution of RDE (2.7) as $\mathbf{Z}^{\mathrm{Itô}}=(1,Z,\mathbb{Z}^{\mathrm{Itô}})$, where $\mathbb{Z}^{\mathrm{Itô}}=(\mathbb{Z}^{\mathrm{Itô},ij})_{i,j=1,2,3}$.

We therefore may define Stratonovich integral (first level) of fOU process with respect to fBM as

$$\int_0^t X_s\circ d_{\mathfrak{R}_1}\mathbf{B}^{H,\mathrm{Str}}=\mathbb{Z}^{\mathrm{Str},21}_{0,t}+X_0B^H_{0,t}, \tag{2.10}$$

and Itô integral (first level) of fOU process with respect to fBM as

$$\int_0^t X_s d_{\mathfrak{R}_1}\mathbf{B}^{H,\mathrm{Itô}}=\mathbb{Z}^{\mathrm{Itô},21}_{0,t}+X_0B^H_{0,t}. \tag{2.11}$$

Now we can define stochastic integrals with respect to fOU rough path enhancement $\mathbf{X}$ by equations (2.3), (2.4). Note that these integrals are pathwise defined and continuous with respect to the sample path $\mathbf{X}(\omega)$ in p-variation metric. In what follows, we denote Stratonovich rough integral as

$$\int_0^t F(X_s)\circ d_{\mathfrak{R}}\mathbf{X}=\left(1,\int_0^t F(X_s)\circ d_{\mathfrak{R}_1}\mathbf{X},\int_0^t F(X_s)\circ d_{\mathfrak{R}_2}\mathbf{X}\right),$$

and Itô rough integral as

$$\int_0^t F(X_s)d_{\mathfrak{R}}\mathbf{X}=\left(1,\int_0^t F(X_s)d_{\mathfrak{R}_1}\mathbf{X},\int_0^t F(X_s)d_{\mathfrak{R}_2}\mathbf{X}\right).$$

As an application we will use the Itô rough integrals to construct the estimator for parametric matrix Γ and prove the asymptotic properties and pathwise stability in the following sections.

2.4 Zero expectation

Let us illustrate the reason for naming Itô rough paths and Itô rough integrals. In stochastic analysis, Itô integrals can be defined in terms of the martingale property, which is suitable for semi-martingales. While for processes which are not semi-martingales such as fBM, attempts of making integrals with respect to fBM being martingales are of course hopeless. We instead demand that the expectations of integrals with respect to fBM are constant (e.g., to be zero). We call this kind of integrals as Itô type integrals, which is in fact an extension of classical Itô integration theory.

Now let us verify that expectation of the Itô integral of fOU process with respect to fBM $\int_0^t X_s d_{\mathfrak{R}_1}\mathbf{B}^{H,\text{Itô}}$ (or write as $\int_0^t X_s \otimes d_{\mathfrak{R}_1}\mathbf{B}^{H,\text{Itô}}$) vanishes.

According to the theory of differential equations driven by rough paths and the definition of integrals above, and assuming that coefficient matrices Γ and Σ are commutative for simplicity, we have

$$\int_0^t X_s d_{\mathfrak{R}_1}\mathbf{B}^{H,\text{Itô}} = \int_0^t X_s \circ d_{\mathfrak{R}_1}\mathbf{B}^{H,\text{Str}} - \Sigma\varphi(t). \tag{2.12}$$

Since

$$X_t = e^{-\Gamma t}X_0 + \int_0^t e^{-\Gamma(t-s)}\Sigma dB_s^H, \tag{2.13}$$

where X_0 is a constant vector and the integral on the right hand side is Young's integral and equals $\int_0^t e^{-\Gamma(t-s)}\Sigma \circ d_{\mathfrak{R}_1}\mathbf{B}_s^{H,\text{Str}}$. Therefore

$$\begin{aligned}\mathbb{E}\left(\int_0^t X_s \circ d_{\mathfrak{R}_1}\mathbf{B}^{H,\text{Str}}\right) &= \mathbb{E}\left(\int_0^t e^{-\Gamma s}X_0 \circ d_{\mathfrak{R}_1}\mathbf{B}^{H,\text{Str}}\right)\\ &\quad + \mathbb{E}\left(\int_0^t\int_0^s e^{-\Gamma(s-u)}\Sigma \circ d_{\mathfrak{R}_1}\mathbf{B}_u^{H,\text{Str}} \circ d_{\mathfrak{R}_1}\mathbf{B}_s^{H,\text{Str}}\right).\end{aligned}$$

The first term on the right hand side is zero, and the second term

$$\begin{aligned}\mathbb{E}\left(\int_0^t\int_0^s e^{-\Gamma(s-u)}\Sigma \circ d_{\mathfrak{R}_1}\mathbf{B}_u^{H,\text{Str}} \circ d_{\mathfrak{R}_1}\mathbf{B}_s^{H,\text{Str}}\right) &= \int_0^t\int_0^s e^{-\Gamma(s-u)}\Sigma dR_H(u,s)\\ = \Sigma\left(\frac{1}{2}It^{2H} - H\Gamma\int_0^t\int_0^s e^{-\Gamma(s-u)}(s^{2H-1}-(s-u)^{2H-1})duds\right) &= \Sigma\varphi(t),\end{aligned}$$

where $R_H(u,s) = \mathbb{E}(B_u^H B_s^H) = \frac{1}{2}(u^{2H}+s^{2H}-|u-s|^{2H})$ is the covariance function of fBM and the integral against $R_H(u,s)$ is defined as an Young's integral in 2D sense (see e.g. [21]). Thus, combining equations above, we have proved the zero expectation property, i.e.

$$\mathbb{E}\left(\int_0^t X_s d_{\mathfrak{R}_1}\mathbf{B}^{H,\text{Itô}}\right) = 0. \tag{2.14}$$

3 Long time asymptotic of Lévy area of fOU processes

In this section, we examine the properties of fOU processes. We establish the α-Hölder continuity of fOU processes and prove a long-term asymptotic property of the Lévy area of fOU processes.

3.1 Regularity of fOU processes

3.1.1 The covariance of fOU processes

The covariance function of a general fOU process can be worked out explicitly. For simplicity, we first study a stationary version of fOU process in this section. Consider

$$X_t = \sigma\int_{-\infty}^t e^{-\lambda(t-s)}dB_s^H,$$

which is stationary and ergodic (see e.g. [14]), and B^H is fBM with Hurst parameter $H < \frac{1}{2}$. It is well known that the covariance $R_H(\cdot,\cdot)$ of B^H is of finite $\frac{1}{2H}$-variation.

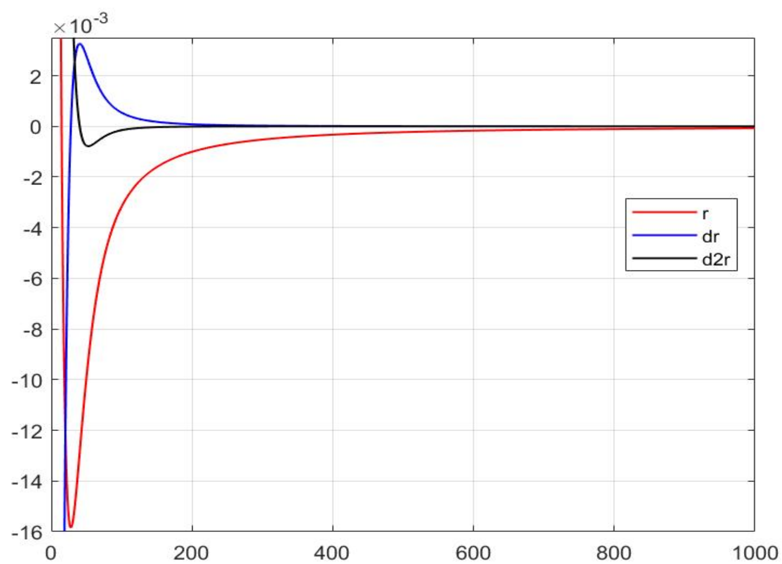
The covariance function of $\{X_t = \sigma \int_{-\infty}^{t} e^{-\lambda(t-s)} dB_s^H, t \geq 0\}$ is given by (see, e.g. [40])

$$
\begin{aligned}
r(t) &= \mathrm{Cov}(X_s, X_{s+t}) = \mathrm{Cov}(X_0, X_t) \\
&= \frac{\sigma^2}{\lambda^{2H}} \frac{G(2H+1)\sin(\pi H)}{\pi} \int_0^{\infty} \cos(\lambda t x) \frac{x^{1-2H}}{1+x^2} dx \\
&= \frac{\sigma^2}{2\lambda^{2H}} G(2H+1)\cosh(\lambda t) - \frac{\sigma^2}{2} t^{2H} {}_1F_2(1; H+\frac{1}{2}, H+1; \frac{1}{4}\lambda^2 t^2),
\end{aligned}
$$

where $G(\cdot)$ is the Gamma function, $\cosh(\cdot)$ the hyperbolic cosine function, ${}_1F_2(\cdot;\cdot,\cdot;\cdot)$ the generalized hypergeometric function, i.e.

$$
{}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!},
$$

and $(a)_0 = 1, (a)_n = a(a+1)\cdots(a+n-1)$, for $n \geq 1$. One can see the figure of this covariance function $r(\cdot)$ and its first and second derivatives below, where we take $H = 0.2$, $\sigma = \lambda = 1$ as an example.



[ht]

Figure 1 Graph of the covariance function $r(\cdot)$ of stationary fOU and its first two derivatives, e.g. $H = 0.2$, $\sigma = \lambda = 1$.

Lemma 3.1 For the covariance function $r(\cdot)$ of stationary fOU process X with $H < \frac{1}{2}$, we have the following properties:

(i) $r(t)$ is $2H$-Hölder continuous on $\mathbb{R}_+$, that is,

$$
|r(t) - r(s)| \leq C_H |t-s|^{2H},
$$

for any $s, t \in \mathbb{R}_+$ and C_H depends on H, σ, λ only (we may ignore σ, λ).

(ii) There exist constants $0 < T_0 < T_1$ such that $r''(T_0) = 0$, and $r''(t) > 0$ on interval $(0, T_0)$, $r''(t) < 0$ on interval (T_1, ∞). That is, r is convex on $[0, T_0]$ and concave on (T_1, ∞).

Proof For covariance function $r(\cdot)$, near $t=0$,

$$r(t)=\sigma^2\lambda^{-2H}HG(2H)\left(1-\frac{\lambda^{2H}}{G(2H+1)}t^{2H}+o(t^{2H})\right),$$

and for t large enough (see Theorem 2.3, [14]),

$$r(t)=\frac{1}{2}\sigma^2\sum_{n=1}^{N}\lambda^{-2n}\left(\prod_{k=0}^{2n-1}(2H-k)\right)t^{2H-2n}+O(t^{2H-2N-2}).$$

Since $r(t)$ is continuous on $[0,\infty)$ and one can also see that $r(t)$ has polynomial decay to zero as t large from above equality.

For (i), we have $\max_{t\geq 0}|r(t)|=C<\infty$, for any $s,t\in\mathbb{R}_+$ and $|t-s|\geq 1$, then

$$|r(t)-r(s)|\leq 2C\leq 2C|t-s|^{2H}.$$

For any $s,t\in\mathbb{R}_+$ and $|t-s|<1$, we show the statement in three case: $s,t\in[0,1]$, $s,t\in[1,\infty)$ and $0\leq s<1<t(<2)$. For the first and third terms, we actually need to show that for any $s,t\in[0,2]$ and $|t-s|<1$, there exists a constant C such that $|r(t)-r(s)|\leq C|t-s|^{2H}$. Since $r(t)=-ct^{2H}+\varphi(t)$, where $\varphi(\cdot)$ is smooth on $\mathbb{R}_+$, then

$$|r(t)-r(s)|\leq c|t-s|^{2H}+\max_{0\leq u\leq 2}|\varphi'(u)||t-s|\leq C|t-s|^{2H}.$$

For the second case, i.e. for any $s,t\in[1,\infty)$ and $|t-s|<1$, we have

$$|r(t)-r(s)|\leq\max_{u\geq 1}|r'(u)||t-s|\leq C|t-s|^{2H}.$$

Thus we proved the statement (i).

For (ii), one can see that there exists a small number $\varepsilon>0$ such that $r''(\varepsilon)>0$ and a large number $T_1>0$ such that, for all $t\geq T_1$, $r''(t)<0$. By the continuity of r'' on $(0,\infty)$, there exists a $T_0\in(\varepsilon,T_1)$ satisfying $r''(T_0)=0$ and $r''(t)>0$ for any $t\in(0,T_0)$. □

Followings are important properties of fOU processes when $H<\frac{1}{2}$. It is well-known that, increments of fBM are negatively correlated when $H<\frac{1}{2}$, and positively correlated when $H>\frac{1}{2}$, while for $H=\frac{1}{2}$ increments over different time periods are independent. We found that for fOU process with $H<\frac{1}{2}$, the disjoint increments are *locally negative correlated*. If the distance of the intervals corresponding the disjoint increments is large, then they are positively correlated, we call it *long-range positive correlation*. See the theorem below. Heuristically, fOU process is locally like fBM so that it has the locally negative correlation property as fBM when $H<\frac{1}{2}$. For long distance the drift becomes the dominated force, so the fOU behaves positively correlated. In the case where $H=\frac{1}{2}$, the fOU is the standard OU process driven by standard Bownian motion. The properties of it are well known. Our main concern here is for the true fOU process case with $H<\frac{1}{2}$.

Theorem 3.2 Consider the stationary fOU process X with $H<\frac{1}{2}$. T_0,T_1 are given in the previous lemma.

(i) (Locally negative correlation) For any s_0 and $s_0\leq t_i<t_{i+1}\leq t_j<t_{j+1}\leq s_0+T_0$, then

$$\mathbb{E}(X_{t_{i+1}}-X_{t_i})(X_{t_{j+1}}-X_{t_j})\leq 0. \tag{3.1}$$

(ii) (Long-range positive correlation) For any $0\leq t_i<t_{i+1}<t_j<t_{j+1}$, and if $t_j-t_{i+1}>T_1$, then

$$\mathbb{E}(X_{t_{i+1}}-X_{t_i})(X_{t_{j+1}}-X_{t_j})\geq 0. \tag{3.2}$$

Proof (i) Since

$$\begin{aligned}
&\mathbb{E}(X_{t_{i+1}}-X_{t_i})(X_{t_{j+1}}-X_{t_j})\\
&=(r(t_{j+1}-t_{i+1})-r(t_{j+1}-t_i))-(r(t_j-t_{i+1})-r(t_j-t_i))\\
&=(r(x_3)-r(x_4))-(r(x_1)-r(x_2))\\
&=-\left[(r(x_4)-r(x_3))-(r(x_2)-r(x_1))\right],
\end{aligned}$$

where $x_1:=t_j-t_{i+1}$, $x_2:=t_j-t_i$, $x_3:=t_{j+1}-t_{i+1}$, $x_4:=t_{j+1}-t_i$, then we have $0\le x_1< x_2\le x_3<x_4\le T_0$ or $0\le x_1<x_3\le x_2<x_4\le T_0$, and

$$\frac{r(x_4)-r(x_3)}{x_4-x_3}\ge\frac{r(x_2)-r(x_1)}{x_2-x_1},$$

by convexity of r. This proves (3.1).
(ii) The proof of (3.2) is almost the same as (i). □

Recall that, the stationary fOU process X satisfies the inequality

$$\mathbb{E}|X_t-X_s|^2\le C_H|t-s|^{2H}, \tag{3.3}$$

for any $s,t\in\mathbb{R}_+$, and C_H depends on H,σ,λ (we ignore σ,λ in notation here). This result can be found in Lemma 5.7 of [29].

Proposition 3.3 Let X be the stationary fOU process with $H\in(0,\frac{1}{2})$. Then its covariance $R_X(s,t)=\mathbb{E}(X_sX_t)$ is of finite $\frac{1}{2H}$-variation on $[s_0,s_0+T_0]^2$ in $2D$ sense for any s_0. Moreover, there exist constants $C=C(H)$ and $T_0>0$ such that, for all $s<t$ in $[s_0,s_0+T_0]$,

$$|R_X|_{\frac{1}{2H}-var;[s,t]^2}\le C(H)|t-s|^{2H}, \tag{3.4}$$

where

$$|R_X|^{\rho}_{\rho-var;[s,t]^2}:=\sup\sum_{i,j}\left|\mathbb{E}\left[(X_{t_{i+1}}-X_{t_i})(X_{t'_{j+1}}-X_{t'_j})\right]\right|^{\rho}, \tag{3.5}$$

and $\mathcal{P}=\{t_i\}$, $\mathcal{P}'=\{t'_j\}$ are any two partitions of interval $[s,t]$.

Proof By Lemma 5.54 of [21], we just need to show the finite $\frac{1}{2H}$-variation by the same partition $\mathcal{P}=\{t_i\}$ of interval $[s,t]\subset[s_0,s_0+T_0]$. Let us consider

$$\sum_{i,j}\left|\mathbb{E}\left[(X_{t_{i+1}}-X_{t_i})(X_{t_{j+1}}-X_{t_j})\right]\right|^{\frac{1}{2H}}. \tag{3.6}$$

For a fixed i, and $i \neq j$, $\mathbb{E}\left[(X_{t_{i+1}} - X_{t_i})(X_{t_{j+1}} - X_{t_j})\right] \leq 0$ for $H < \frac{1}{2}$ by Theorem 3.2, hence,

$$\begin{aligned}
&\sum_j \left|\mathbb{E}\left[X_{t_i,t_{i+1}} X_{t_j,t_{j+1}}\right]\right|^{\frac{1}{2H}} \\
&= \sum_{j\neq i} \left|\mathbb{E}\left[X_{t_i,t_{i+1}} X_{t_j,t_{j+1}}\right]\right|^{\frac{1}{2H}} + \left(\mathbb{E}\left|X_{t_i,t_{i+1}}\right|^2\right)^{\frac{1}{2H}} \\
&\leq \left|\mathbb{E}\left(\sum_{j\neq i} X_{t_i,t_{i+1}} X_{t_j,t_{j+1}}\right)\right|^{\frac{1}{2H}} + \left(\mathbb{E}\left|X_{t_i,t_{i+1}}\right|^2\right)^{\frac{1}{2H}} \\
&\leq \left(2^{\frac{1}{2H}-1}\left|\mathbb{E}\left(\sum_j X_{t_i,t_{i+1}} X_{t_j,t_{j+1}}\right)\right|^{\frac{1}{2H}} + 2^{\frac{1}{2H}-1}\left(\mathbb{E}\left|X_{t_i,t_{i+1}}\right|^2\right)^{\frac{1}{2H}}\right) \\
&\quad + \left(\mathbb{E}\left|X_{t_i,t_{i+1}}\right|^2\right)^{\frac{1}{2H}} \\
&\leq C(H)\left|\mathbb{E}\left[X_{t_i,t_{i+1}} X_{s,t}\right]\right|^{\frac{1}{2H}} + C(H)\left(\mathbb{E}\left|X_{t_i,t_{i+1}}\right|^2\right)^{\frac{1}{2H}}.
\end{aligned}$$

Therefore, we have

$$\sum_{i,j} \left|\mathbb{E}\left[X_{t_i,t_{i+1}} X_{t_j,t_{j+1}}\right]\right|^{\frac{1}{2H}} \leq C(H)\sum_i \left|\mathbb{E}\left[X_{t_i,t_{i+1}} X_{s,t}\right]\right|^{\frac{1}{2H}} + C(H)\sum_i \left(\mathbb{E}\left|X_{t_i,t_{i+1}}\right|^2\right)^{\frac{1}{2H}}.$$

The second term on the right hand side is controlled by $C(H)|t-s|$. Now we show that

$$\sum_i \left|\mathbb{E}\left[X_{t_i,t_{i+1}} X_{s,t}\right]\right|^{\frac{1}{2H}} \leq C(H)|t-s|.$$

Since

$$\begin{aligned}
\left|\mathbb{E}\left[X_{t_i,t_{i+1}} X_{s,t}\right]\right| &= \left|\mathbb{E}\left(X_{t_{i+1}}X_t - X_{t_i}X_t + X_{t_i}X_s - X_{t_{i+1}}X_s\right)\right| \\
&= |r(t-t_{i+1}) - r(t-t_i) + r(t_i - s) - r(t_{i+1}-s)| \\
&\leq |r(t-t_{i+1}) - r(t-t_i)| + |r(t_i-s) - r(t_{i+1}-s)| \\
&\leq C_H\left|t_{i+1}-t_i\right|^{2H} + C_H\left|t_{i+1}-t_i\right|^{2H} \leq 2C_H\left|t_{i+1}-t_i\right|^{2H},
\end{aligned}$$

thus

$$\sum_i \left|\mathbb{E}\left[X_{t_i,t_{i+1}} X_{s,t}\right]\right|^{\frac{1}{2H}} \leq \sum_i C(H)\left|t_{i+1}-t_i\right| \leq C(H)|t-s|.$$

Now we have completed the proof. □

Corollary 3.4 Let X be the stationary fOU process with $H \in (0, \frac{1}{2})$. Then its covariance $R_X(s,t) = \mathbb{E}(X_sX_t)$ is of finite $\frac{1}{2H}$-variation on $[0,T]^2$ in $2D$ sense. Moreover, there exists a constant $C = C(H)$ such that, for all $s < t$ in $[0,T]$,

$$|R_X|^{\frac{1}{2H}}_{\frac{1}{2H}-var;[s,t]^2} \leq C(H)|t-s|. \tag{3.7}$$

Proof We divide the interval $[0,T]$ into $m+1 = \left[\frac{T}{T_0}\right]+1$ pieces, denote them as $[0,T_0]$, $[T_0, 2T_0]$, $\cdots$, $[(m-1)T_0, mT_0]$, $[mT_0, T]$. For any subinterval $[s,t] \subset [0,T]$, there exist $q_1, q_2 \in \mathbb{N}$ such that $s \in [(q_1-1)T_0, q_1T_0]$ and $t \in [q_2T_0, (q_2+1)T_0]$, by the subadditivity of $|R_X|^{\frac{1}{2H}}_{\frac{1}{2H}-var;[\cdot,\cdot]^2}$, then we have

$$
\begin{aligned}
|R_X|^{\frac{1}{2H}}_{\frac{1}{2H}-var;[s,t]^2} &\le |R_X|^{\frac{1}{2H}}_{\frac{1}{2H}-var;[s,q_1T_0]^2} + |R_X|^{\frac{1}{2H}}_{\frac{1}{2H}-var;[q_1T_0,(q_1+1)T_0]^2} + \cdots + |R_X|^{\frac{1}{2H}}_{\frac{1}{2H}-var;[q_2T_0,t]^2} \\
&\le C(H)(|q_1T_0 - s| + |2q_1T_0 - q_1T_0| + \cdots + |t - q_2T_0|) \\
&\le C(H)|t-s|.
\end{aligned}
$$

This completes the proof of the corollary. □

3.1.2 Regularity of fOU processes

In the following, we study the α-Hölder continuity of one dimensional, stationary fOU process $X_t = \sigma \int_{-\infty}^t e^{-\lambda(t-s)} dB_s^H$. Before showing the regularity, we recall the usual Garsia-Rodemich-Rumsey inequality (see e.g., page 60, Stroock and Varadhan [44]).

Lemma 3.5 (Garsia-Rodemich-Rumsey inequality) Let $p(\cdot)$ and $\Psi(\cdot)$ be continuous, strictly increasing functions on $[0,\infty)$ such that

$$p(0) = \Psi(0) = 0, \quad \text{and} \quad \lim_{t\to\infty} \Psi(t) = \infty.$$

Given $T > 0$ and $\phi \in C([0,T], \mathbb{R}^d)$, if there is a constant B such that

$$\int_0^T \int_0^T \Psi\left(\frac{|\phi(t) - \phi(s)|}{p(|t-s|)}\right) dsdt \le B, \tag{3.8}$$

then for all $0 \le s \le t \le T$,

$$|\phi(t) - \phi(s)| \le 8 \int_0^{|t-s|} \Psi^{-1}\left(\frac{4B}{u^2}\right) p(du). \tag{3.9}$$

As an application of this lemma above, we have

Proposition 3.6 Let X be a one dimensional, stationary fOU process with $H \in (0, \frac{1}{2})$ on $[0,T]$. Then there exist a constant $0 < \beta < 1$ and an almost surely finite random variable C independent of T such that

$$|X_t - X_s| \le CT^\beta |t-s|^\alpha, \ a.s. \tag{3.10}$$

for any $\alpha \in (0,H)$, any $0 \le s, t \le T$.

Proof Recall that

$$\mathbb{E}|X_t - X_s|^2 \le C_H |t-s|^{2H}. \tag{3.11}$$

Since X_t is Gaussian process, all the norms are equivalent, we get

$$\mathbb{E}|X_t - X_s|^p \le C_p (\mathbb{E}|X_t - X_s|^2)^{\frac{p}{2}} \le C_{p,H} |t-s|^{pH}, \tag{3.12}$$

for any $p > 2$.

Next, we apply the Garsia-Rodemich-Rumsey inequality. Take $\Psi(x) = x^p$ and $p(x) = x^H$. Then inequality (3.12) implies that

$$\mathbb{E}\left(\int_0^T \int_0^T \Psi\left(\frac{|X_t - X_s|}{p(|t-s|)}\right) dsdt\right) \le C_{p,H} T^2.$$

Define

$$B_T := \int_0^T \int_0^T \Psi\left(\frac{|X_t - X_s|}{p(|t-s|)}\right) dsdt = \int_0^T \int_0^T \frac{|X_t - X_s|^p}{|t-s|^{pH}} dsdt.$$

Then for any $q > 3$, we get

$$\mathbb{E}\left(\sum_{n=1}^{\infty} \frac{B_n}{n^q}\right) = \sum_{n=1}^{\infty} \frac{\mathbb{E}(B_n)}{n^q} \leq \sum_{n=1}^{\infty} \frac{Cn^2}{n^q} < \infty.$$

Thus there exists an almost surely finite random variable R independent of n such that

$$\sum_{n=1}^{\infty} \frac{B_n}{n^q} \leq R, \ a.s.$$

So we have

$$B_n \leq Rn^q, \ a.s. \quad \forall n \geq 1, \ q > 3.$$

Take $n = [T]$, then

$$B_T \leq B_{n+1} \leq R(n+1)^q \leq CRT^q, \ a.s. \quad \forall T > 0, \ q > 3.$$

Then the Garsia-Rodemich-Rumsey inequality gives that

$$\begin{aligned}
|X_t - X_s| &\leq 8 \int_0^{|t-s|} \Psi^{-1}\left(\frac{4B_T}{u^2}\right) p(du) \\
&\leq C(4B_T)^{\frac{1}{p}} |t-s|^{H-2/p} \leq CR^{\frac{1}{p}} T^{\frac{q}{p}} |t-s|^{\alpha},
\end{aligned}$$

for any $\alpha \in (0, H)$, $p > 3 \vee \left[\frac{2}{H-\alpha}\right]$ and $3 < q < p$. This concludes the lemma. $\square$

Remark 3.7 When $X_t = \sigma \int_0^t e^{-\lambda(t-s)} dB_s^H$, it still satisfies inequality (3.10).

Additionally, we prove a proposition for a function of fOU processes, which will be applied in section 5. Here, we introduce the process $\overline{X}_t^i$ defined as follows:

$$\overline{X}_t^i = \sigma \int_{-\infty}^t e^{-\lambda_i(t-s)} dB_s^{H,i}, \ i = 1, 2, \cdots, d. \tag{3.13}$$

It is worth noting that the processes $\{\overline{X}_t^i, t \geq 0\}$ are stationary, ergodic, Gaussian processes, as discussed in [14].

Proposition 3.8 Let $B^H = (B^{H,1}, B^{H,2}, \cdots, B^{H,d})$ be a d-dimensional fBM with $H \in (0, \frac{1}{2})$, $X = (X^1, X^2, \cdots, X^d)$ a d-dimensional fOU process, where $X_t^i = \sigma \int_0^t e^{-\lambda_i(t-s)} dB_s^{H,i}$, $\lambda_i > 0$, $\sigma \in \mathbb{R}$. Define $F(X_t) := X_t \otimes X_t = (X_t^i X_t^j)_{i,j=1,2,\cdots,d}$, and the norm of matrix A as $\|A\| = \sum_{i,j=1}^d |a_{ij}|$. Then there exist a constant $0 < \beta < 1$, an almost surely finite random variable C (independent of T) and a random variable R_T (tends to zero almost surely as $T \to \infty$) such that

$$\sup_{s \neq t} \frac{\|F(X_t) - F(X_s)\|}{|t-s|^{\alpha}} \leq CR_T T^{\beta}, \tag{3.14}$$

for any $0 \leq s, t \leq T$, any $\alpha \in (0, H)$.

Proof First, we present a fact about supremum of one dimensional, stationary fOU process $(\overline{X}^i)_t^* := \sup_{0 \leq s \leq t} |\overline{X}_s^i|$ below. Since we know that $\overline{X}^i$ and $-\overline{X}^i$ have the same distribution and their covariance function is

$$r_i(t) = \mathrm{Cov}(\overline{X}_{s+t}^i, \overline{X}_s^i) = C\left(1 - \frac{\lambda_i^{2H}}{G(2H+1)} t^{2H} + o(t^{2H})\right), \tag{3.15}$$

for t small, where $C = \sigma^2 \lambda_i^{-2H} HG(2H)$ and $G(\cdot)$ is Gamma function. So by Theorem 3.1 of Pickands [39], we know that for t tending to infinity

$$\frac{1}{t^{\delta}} \sup_{0 \leq s \leq t} \overline{X}_s^i \to 0, \ a.s., \quad \frac{1}{t^{\delta}} \sup_{0 \leq s \leq t} (-\overline{X}_s^i) \to 0, \ a.s.,$$

for any $\delta > 0$. Since $(\overline{X}^i)^*_t = (\sup_{0\le s\le t} \overline{X}^i_s) \vee (\sup_{0\le s\le t}(-\overline{X}^i_s))$, then

$$\frac{(\overline{X}^i)^*_t}{t^\delta} \to 0,\ a.s. \tag{3.16}$$

Since $X^i_t = \overline{X}^i_t - e^{-\lambda t}\overline{X}^i_0$, so we also have $\frac{(X^i)^*_t}{t^\delta} \to 0,\ a.s$, where $(X^i)^*_t := \sup_{0\le s\le t} |X^i_s|$.

Now define $R_t = \sup_{i=1,\cdots,d} \frac{(X^i)^*_t}{t^\delta}$, then $R_t \to 0,\ a.s.$ as $t \to \infty$. For any $i, j = 1, 2, \cdots, d$, and $0 \le s, t \le T$,

$$\begin{aligned}
|X^i_tX^j_t - X^i_sX^j_s| &= |(X^i_t - X^i_s)X^j_t + X^i_s(X^j_t - X^j_s)| \\
&\le |X^j_t||X^i_t - X^i_s| + |X^i_s||X^j_t - X^j_s| \\
&\le (X^j)^*_T|X^i_t - X^i_s| + (X^i)^*_T|X^j_t - X^j_s| \\
&\le CR_TT^\delta T^\beta|t-s|^\alpha + CR_TT^\delta T^\beta|t-s|^\alpha \\
&\le CR_TT^{\delta+\beta}|t-s|^\alpha,
\end{aligned}$$

where the last second inequality is followed from Proposition 3.6. One can choose δ, β such that $0 < \delta + \beta =: \beta' < 1$. This completes the proof of the statement. □

3.1.3 Lévy area of multi-dimensional fOU processes

In this subsection, let $B^H = (B^{H,1}, B^{H,2}, \cdots, B^{H,d})$ be a d-dimensional fBM with $H \in (\frac{1}{3}, \frac{1}{2})$, $X = (X^1, X^2, \cdots, X^d)$ a d-dimensional fOU process, where $X^i_t = \sigma\int_{-\infty}^t e^{-\lambda_i(t-s)}dB^{H,i}_s$, $\lambda_i > 0$, $\sigma \in \mathbb{R}$. Then $X = (X^1, X^2, \cdots, X^d)$ is stationary (see [14]). Its covariance function is given by

$$R_X(s,t) = \mathrm{diag}(R_1(s,t), \cdots, R_d(s,t)),$$

where $R_i(s,t) = \mathbb{E}(X^i_sX^i_t)$.

In this subsection, we will show one estimate for off-diagonal elements of Lévy area $\int_0^t X^i_u \circ d_{\mathfrak{R}_1}\mathbf{X}^j$ of the multi-dimensional fOU process X. We denote Stratonovich's Lévy area of fOU process X as

$$A(t) := \int_0^t X_u \circ d_{\mathfrak{R}_1}\mathbf{X} = \left(\int_0^t X^i_u \circ d_{\mathfrak{R}_1}\mathbf{X}^j\right)_{i,j=1,2,\cdots,d},$$

and $A_{ij}(t)$ as its components.

Before showing the estimate of off-diagonal elements, we recall a lemma based on Wiener chaos. We denote $\mathcal{H}_n(\mathbb{P})$ as homogeneous Wiener chaos of order n and $\mathcal{C}^n(\mathbb{P}) := \oplus_{j=0}^n\mathcal{H}_j(\mathbb{P})$ the Wiener chaos (or non-homogeneous chaos) of order n. The lemma below gives the hypercontractivity of Wiener chaos.

Lemma 3.9 (Refer to, e.g., Lemma 15.21, [21]) Let $q \in \mathbb{N}$ and $Z \in \mathcal{C}^q(\mathbb{P})$. Then, for $p > 2$,

$$(\mathbb{E}|Z|^2)^{\frac{1}{2}} \le (\mathbb{E}|Z|^p)^{\frac{1}{p}} \le (q+1)(p-1)^{\frac{q}{2}}(\mathbb{E}|Z|^2)^{\frac{1}{2}}. \tag{3.17}$$

Now we illustrate one estimate for off-diagonal elements, i.e. when $i \ne j$, we have the following proposition.

Proposition 3.10 Let $X = (X^1, \cdots, X^d)$ be a d-dimensional, stationary fOU process with $H \in (\frac{1}{3}, \frac{1}{2})$, and $A_{ij}(t) = \int_0^t X^i_u \circ d_{\mathfrak{R}_1}\mathbf{X}^j, i \ne j$, be the off-diagonal elements of Stratonovich's Lévy area of X. Then there exist $0 < \beta < 1$ and an almost surely finite random variable $\widetilde{C}$ such that

$$|A_{ij}(t) - A_{ij}(s)| \le \widetilde{C}n^\beta,\ a.s. \tag{3.18}$$

for any $s, t \in [n-1, n]$ and any integer $n \geq 1$.

Proof First, we rewrite $R_i(s,t)$ as

$$R_i\begin{pmatrix} s \\ t \end{pmatrix} = \mathbb{E}X_s^i X_t^i,$$

and denote

$$R_i\begin{pmatrix} s \\ u,v \end{pmatrix} = \mathbb{E}X_s^i X_{u,v}^i,\ R_i\begin{pmatrix} s,t \\ u \end{pmatrix} = \mathbb{E}X_{s,t}^i X_u^i,\ R_i\begin{pmatrix} s,t \\ u,v \end{pmatrix} = \mathbb{E}X_{s,t}^i X_{u,v}^i.$$

For the second moment of the Lévy area,

$$\begin{aligned}
\mathbb{E}\left(\left|\int_s^t X_u^i \circ d_{\mathfrak{R}_1}\mathbf{X}^j\right|^2\right) &= \mathbb{E}\left(\int_s^t\int_s^t X_u^i X_v^i \circ d_{\mathfrak{R}_1}\mathbf{X}^j \circ d_{\mathfrak{R}_1}\mathbf{X}^j\right) \\
&= \int_s^t\int_s^t \mathbb{E}(X_u^i X_v^i) d\mathbb{E}(X_u^j X_v^j) \\
&= \int_s^t\int_s^t R_i\begin{pmatrix} u \\ v \end{pmatrix} dR_j\begin{pmatrix} u \\ v \end{pmatrix},
\end{aligned}$$

where the integral which appears on the right hand side above can be viewed as a 2-dimensional (2D) Young's integral (see e.g. Section 6.4 of Friz and Victoir [21]). Then we have

$$\begin{aligned}
\int_s^t\int_s^t R_i\begin{pmatrix} u \\ v \end{pmatrix} dR_j\begin{pmatrix} u \\ v \end{pmatrix} &= \int_s^t\int_s^t R_i\begin{pmatrix} s,u \\ s,v \end{pmatrix} dR_j\begin{pmatrix} u \\ v \end{pmatrix} + \int_s^t\int_s^t R_i\begin{pmatrix} s,u \\ s \end{pmatrix} dR_j\begin{pmatrix} u \\ v \end{pmatrix} \\
&\quad + \int_s^t\int_s^t R_i\begin{pmatrix} s \\ s,v \end{pmatrix} dR_j\begin{pmatrix} u \\ v \end{pmatrix} + R_i\begin{pmatrix} s \\ s \end{pmatrix}\int_s^t\int_s^t dR_j\begin{pmatrix} u \\ v \end{pmatrix} \\
&=: I + II + III + IV.
\end{aligned}$$

For the first term I, by Young-Loève-Towghi inequality (see e.g. Theorem 6.18 of [21]), we have

$$\begin{aligned}
I &\leq C|R_i|_{\frac{1}{2H}-var;[s,t]^2}|R_j|_{\frac{1}{2H}-var;[s,t]^2} \\
&\leq C\max\{|R_i|^2_{\frac{1}{2H}-var;[s,t]^2}, |R_j|^2_{\frac{1}{2H}-var;[s,t]^2}\}.
\end{aligned}$$

Then by Corollary 3.4, we have that

$$I = \int_s^t\int_s^t R_i\begin{pmatrix} s,u \\ s,v \end{pmatrix} dR_j\begin{pmatrix} u \\ v \end{pmatrix} \leq C|t-s|^{4H}. \tag{3.19}$$

For the second term II, by Young 1D estimate (see e.g. Theorem 6.8 of [21]), we have

$$\begin{aligned}
II &= \int_s^t R_i\begin{pmatrix} s,u \\ s \end{pmatrix} dR_j\begin{pmatrix} u \\ s,t \end{pmatrix} \\
&\leq C\left|R_i\begin{pmatrix} \cdot \\ s \end{pmatrix}\right|_{\frac{1}{2H}-var;[s,t]}\left|R_j\begin{pmatrix} \cdot \\ s,t \end{pmatrix}\right|_{\frac{1}{2H}-var;[s,t]},
\end{aligned}$$

where

$$\begin{aligned}
\left|R_i\begin{pmatrix} \cdot \\ s \end{pmatrix}\right|^{\frac{1}{2H}}_{\frac{1}{2H}-var;[s,t]} &= \sup_{\mathcal{P}}\sum_\ell\left|R_i\begin{pmatrix} t_{\ell+1} \\ s \end{pmatrix} - R_i\begin{pmatrix} t_\ell \\ s \end{pmatrix}\right|^{\frac{1}{2H}} \\
&= \sup_{\mathcal{P}}\sum_\ell|r_i(t_{\ell+1}-s) - r_i(t_\ell - s)|^{\frac{1}{2H}} \\
&\leq \sup_{\mathcal{P}}\sum_\ell C_H\,|t_{\ell+1}-t_\ell| \leq C_H|t-s|,
\end{aligned}$$

and

$$\begin{aligned}\left|R_j\begin{pmatrix}\cdot\\ s,t\end{pmatrix}\right|_{\frac{1}{2H}-var;[s,t]}^{\frac{1}{2H}} &= \sup_{\mathcal{P}}\sum_{\ell}\left|R_j\begin{pmatrix}t_{\ell+1}\\ s,t\end{pmatrix}-R_j\begin{pmatrix}t_\ell\\ s,t\end{pmatrix}\right|^{\frac{1}{2H}}\\ &= \sup_{\mathcal{P}}\sum_{\ell}\left|\mathbb{E}(X^j_{t_\ell,t_{\ell+1}}X^j_{s,t})\right|^{\frac{1}{2H}} \le |R_j|^{\frac{1}{2H}}_{\frac{1}{2H}-var;[s,t]^2}.\end{aligned}$$

In above estimate, function r_i is the covariance $r_i(t)=\mathbb{E}(X^i_sX^i_{s+t})$. Thus, we have

$$II=\int_s^t\int_s^t R_i\begin{pmatrix}s,u\\ s\end{pmatrix}dR_j\begin{pmatrix}u\\ v\end{pmatrix}\le C|t-s|^{4H}. \tag{3.20}$$

For the third term III, it is the same with the second term II line by line. So

$$III=\int_s^t\int_s^t R_i\begin{pmatrix}s\\ s,v\end{pmatrix}dR_j\begin{pmatrix}u\\ v\end{pmatrix}\le C|t-s|^{4H}. \tag{3.21}$$

For the last term IV,

$$\begin{aligned}IV&=R_i\begin{pmatrix}s\\ s\end{pmatrix}\left(R_j\begin{pmatrix}t\\ t\end{pmatrix}-R_j\begin{pmatrix}s\\ t\end{pmatrix}-R_j\begin{pmatrix}t\\ s\end{pmatrix}+R_j\begin{pmatrix}s\\ s\end{pmatrix}\right)\\ &=r_i(0)(2r_j(0)-2r_j(t-s))\le C|t-s|^{2H}.\end{aligned} \tag{3.22}$$

Now combing inequalities (3.19),(3.20),(3.21) and (3.22), we get

$$\mathbb{E}\left(\left|\int_s^t X^i_u\circ d_{\mathfrak{R}_1}\mathbf{X}^j\right|^2\right)\le C|t-s|^{4H}+C|t-s|^{2H}. \tag{3.23}$$

Let $s<t$ and $s,t\in[n-1,n]$, so we have

$$\mathbb{E}\left(\left|\int_s^t X^i_u\circ d_{\mathfrak{R}_1}\mathbf{X}^j\right|^2\right)\le C|t-s|^{2H}.$$

Now we turn to prove the estimate, for arbitrary $p\ge 2$, by the hypercontractivity of Wiener chaos (see Lemma 3.9), we further have

$$\begin{aligned}\mathbb{E}[|A_{ij}(t)-A_{ij}(s)|^p]&=\mathbb{E}\left(\left|\int_s^t X^i_u\circ d_{\mathfrak{R}_1}\mathbf{X}^j\right|^p\right)\\ &\le 3^p(p-1)^p\left(\mathbb{E}\left|\int_s^t X^i_u\circ d_{\mathfrak{R}_1}\mathbf{X}^j\right|^2\right)^{\frac{p}{2}}\\ &\le C|t-s|^{pH}.\end{aligned}$$

Take $\Psi(x)=x^p$ and $p(x)=x^H$, the above inequality implies that

$$\mathbb{E}\left(\int_n^{n+1}\int_n^{n+1}\Psi\left(\frac{|A_{ij}(t)-A_{ij}(s)|}{p(|t-s|)}\right)dsdt\right)\le C.$$

Define

$$B_n:=\int_n^{n+1}\int_n^{n+1}\Psi\left(\frac{|A_{ij}(t)-A_{ij}(s)|}{p(|t-s|)}\right)dsdt=\int_n^{n+1}\int_n^{n+1}\frac{|A_{ij}(t)-A_{ij}(s)|^p}{|t-s|^{pH}}dsdt.$$

Then for any $q>1$, we get

$$\mathbb{E}\left(\sum_{n=1}^{\infty}\frac{B_n}{n^q}\right)=\sum_{n=1}^{\infty}\frac{\mathbb{E}(B_n)}{n^q}\le\sum_{n=1}^{\infty}\frac{C}{n^q}<\infty.$$

Thus there exists an almost surely finite random variable R independent of n such that

$$\sum_{n=1}^{\infty} \frac{B_n}{n^q} \leq R, \ a.s.$$

So we have

$$B_n \leq R n^q, \ a.s. \quad \forall n \geq 1, \ q > 1. \tag{3.24}$$

Apply the Garsia-Rodemich-Rumsey inequality, and for any $n-1 \leq s < t \leq n$, we get

$$\begin{aligned} |A_{ij}(t) - A_{ij}(s)| &\leq 8 \int_0^{|t-s|} \Psi^{-1}\left(\frac{4B_n}{u^2}\right) p(du) \leq 8H \int_0^1 \left(\frac{4B_n}{u^2}\right)^{\frac{1}{p}} u^{H-1} du \\ &= \frac{8H}{H - 2/p} (4B_n)^{\frac{1}{p}} \leq C R^{\frac{1}{p}} n^{\frac{q}{p}}, \quad a.s. \end{aligned}$$

for any $p > \frac{2}{H}$ and $1 < q < p$. Thus we complete this proof. □

3.2 Long time asymptotic of Lévy area

Now in this subsection, we consider the multi-dimensional fOU process which is the solution to stochastic differential equation

$$dX_t = -\Gamma X_t dt + \sigma dB_t^H, \ X_0 = 0, \tag{3.25}$$

where Γ is a symmetric, positive-definite matrix, σ is a constant, and $B^H = (B^{H,1}, B^{H,2}, \cdots, B^{H,d})$ is a d-dimensional fBM. Our aim in this section is to show a long time asymptotic property of Lévy area $A(t) = \int_0^t X_s \circ d_{\mathfrak{R}_1}\mathbf{X}$ of fOU processes X. That is to show

$$\frac{1}{t} A(t) = \frac{1}{t} \int_0^t X_s \circ d_{\mathfrak{R}_1}\mathbf{X} \to 0, \ a.s.$$

as t goes to infinity.

The components of solution process X are not independent since the interactions between each other. We first make an orthogonal transformation for this dynamical system. Since the drift matrix Γ is symmetric and positive-definite, there exists an orthogonal matrix $\overline{\Sigma}$ such that

$$\overline{\Sigma} \Gamma \overline{\Sigma}^T = \Lambda, \tag{3.26}$$

where $\Lambda = \mathrm{diag}\{\lambda_1, \cdots, \lambda_d\}$ and $0 < \lambda_1 \leq \cdots \leq \lambda_d$.

Define $\widetilde{X}_t := \overline{\Sigma} X_t$, and $\widetilde{B}_t^H := \overline{\Sigma} B_t^H$. Since $\overline{\Sigma}$ is an orthogonal matrix, $\widetilde{B}_t^H$ is still a d-dimensional fBM with Hurst parameter H. The stochastic differential equation (3.25) becomes

$$d\widetilde{X}_t = -\Lambda \widetilde{X}_t dt + \sigma d\widetilde{B}_t^H. \tag{3.27}$$

Now the fOU process $\widetilde{X}_t$ has independent components, so that

$$\int_0^t X_s \circ d_{\mathfrak{R}_1}\mathbf{X} = \overline{\Sigma}^T \left(\int_0^t \widetilde{X}_s \circ d_{\mathfrak{R}_1} \widetilde{\mathbf{X}} \right) \overline{\Sigma}.$$

What we should prove is therefore that

$$\frac{1}{t} \int_0^t \widetilde{X}_s \circ d_{\mathfrak{R}_1} \widetilde{\mathbf{X}} \to 0, \ a.s.$$

as t goes to infinity.

We may ignore the symbol tilde and use X, B^H to denote $\widetilde{X}$ and $\widetilde{B}^H$, respectively, for simplicity. Now the d-dimensional fOU process $X = (X^1, X^2, \cdots, X^d)$ has independent components and satisfies

$$X_t^i = \sigma \int_0^t e^{-\lambda_i(t-s)} dB_s^{H,i}, \ i = 1, 2, \cdots, d. \tag{3.28}$$

3.2.1 On-diagonal case

Lemma 3.11 For the on-diagonal components of Lévy area $A(t) = \int_0^t X_s \circ d_{\mathfrak{R}_1}\mathbf{X}$, we have

$$\frac{1}{t}A_{ii}(t) = \frac{1}{t}\int_0^t X_s^i \circ d_{\mathfrak{R}_1}\mathbf{X}^i \to 0, \ a.s., \ \forall\ i = 1, 2, \cdots, d. \tag{3.29}$$

as t tends to infinity.

Proof Recall that

$$\overline{X}_t^i = \sigma \int_{-\infty}^t e^{-\lambda_i(t-s)} dB_s^{H,i}, \ i = 1, 2, \cdots, d. \tag{3.30}$$

As we have shown, in Proposition 3.8, that for any $\alpha > 0$,

$$\lim_{t\to\infty} \frac{\overline{X}_t^i}{t^\alpha} = 0, \ a.s. \tag{3.31}$$

Since $X_t^i = \overline{X}_t^i - e^{-\lambda_i t}\overline{X}_0^i$ and

$$\int_0^t X_s^i \circ d_{\mathfrak{R}_1}\mathbf{X}^i = \frac{1}{2}(X_t^i)^2,$$

then from (3.31), it follows that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t X_s^i \circ d_{\mathfrak{R}_1}\mathbf{X}^i = 0, \ a.s.$$

Thus, we conclude this lemma for on-diagonal case. □

3.2.2 Off-diagonal case

Let $X = (X^1, X^2, \cdots, X^d)$ be the d-dimensional, stationary Gaussian process given by (3.30). Its covariance function is given by

$$R_X(s,t) = \mathrm{diag}(R_1(s,t), \cdots, R_d(s,t)),$$

where $R_i(s,t) := \mathbb{E}(X_s^i X_t^i)$.

When $i \neq j$, we have (as proof of equation (3.23) in Proposition 3.10) that

$$\mathbb{E}\left(\left|\int_0^t X_s^i \circ d_{\mathfrak{R}_1}\mathbf{X}^j\right|^2\right) \leq Ct^{4H} + Ct^{2H}. \tag{3.32}$$

When $t \geq 1$, we have

$$\mathbb{E}\left(\left|\int_0^t X_s^i \circ d_{\mathfrak{R}_1}\mathbf{X}^j\right|^2\right) \leq Ct^{4H}. \tag{3.33}$$

Now we define $A_{ij}(t) = \int_0^t X_s^i \circ d_{\mathfrak{R}_1}\mathbf{X}^j$ as in subsection 3.1.3, and $Z_n^{ij} := n^{-2H}A_{ij}(n)$ we first show that when $t = n \in \mathbb{N}$ (discrete sequence), we have

$$\lim_{n\to\infty} \frac{1}{n}A_{ij}(n) = 0, \ a.s. \tag{3.34}$$

Proposition 3.12 For the discrete sequence $\{\frac{1}{n}A_{ij}(n), n \geq 1\}$ and $H \in (\frac{1}{3}, \frac{1}{2})$, we have

$$\frac{1}{n}A_{ij}(n) \to 0, \ a.s. \tag{3.35}$$

as n goes to infinity.

Proof By the inequality (3.33), we have

$$\mathbb{E}|A_{ij}(n)|^2 \leq Cn^{4H}.$$

Then

$$\sup_n \mathbb{E}|Z_n^{ij}|^2 \leq C.$$

According Proposition 15.20 of [21], we know that Z_n^{ij} belongs to the second Wiener chaos $\mathcal{C}^2(\mathbb{P})$. By Lemma 3.9, we have

$$\sup_n \mathbb{E}|Z_n^{ij}|^p \leq 3^p(p-1)^p \sup_n (\mathbb{E}|Z_n^{ij}|^2)^{\frac{p}{2}} < \infty.$$

For any $\epsilon > 0$, by Chebyshev inequality, we have

$$\mathbb{P}\left(|A_{ij}(n)| > n\epsilon\right) = \mathbb{P}\left(|Z_n^{ij}| > n^{1-2H}\epsilon\right) \leq \frac{1}{n^{p(1-2H)}\epsilon^p} \sup_n \mathbb{E}|Z_n^{ij}|^p$$

where $p > \frac{1}{1-2H}$.

Then,

$$\sum_n \mathbb{P}\left(|A_{ij}(n)| > n\epsilon\right) \leq \sum_n \frac{C}{n^{p(1-2H)}\epsilon^p} < \infty.$$

The almost sure convergence follows from the Borel-Cantelli lemma. □

Now we can conclude this subsection, that is, to show the limit for arbitrary t rather than at discrete time $\mathbb{N}_+$.

Theorem 3.13 Suppose stochastic process X_t is fOU process which is the solution to stochastic differential equation (3.25) and Γ is symmetric and positive-definite. Then

$$\frac{1}{t}A(t) = \frac{1}{t}\int_0^t X_s \otimes \circ d_{\mathfrak{R}_1}\mathbf{X} \to 0, \ a.s.,$$

as $t \to \infty$, where the above integral is in Stratonovich sense.

Proof First, assume that X is the stationary fOU process as in equation (3.30). The on-diagonal case is proved in Lemma 3.11. For the off-diagonal case, since

$$\frac{1}{t}|A_{ij}(t)| \leq \frac{1}{t}|A_{ij}(t) - A_{ij}(n)| + \frac{n}{t}\frac{1}{n}|A_{ij}(n)|, \tag{3.36}$$

and setting $n = [t]$, by Proposition 3.10, we have that the first term on the right hand side is controlled by $\widetilde{C}t^{-1}n^{\beta} \leq \widetilde{C}n^{\beta-1} \to 0, \ a.s.$ And the second term also tends to zero by Proposition 3.12. Thus we have completed the proof of Theorem 3.13 when fOU process X is stationary.

If X is not stationary but starts at point 0 at $t = 0$, we can also prove this asymptotic for their Stratonovich integrals. Now let $\overline{X} = (\overline{X}^1, \cdots, \overline{X}^d)$ be the stationary version as above. Then fOU process $X_t^i = \overline{X}_t^i - e^{-\lambda_i t}\overline{X}_0^i$, $i = 1, 2, \cdots, d$. So

$$\begin{aligned}\frac{1}{t}\int_0^t X_u^i \circ d_{\mathfrak{R}_1}\mathbf{X}^j = & \frac{1}{t}\int_0^t \overline{X}_u^i \circ d_{\mathfrak{R}_1}\overline{\mathbf{X}}^j + \frac{1}{t}\int_0^t \lambda_j e^{-\lambda_j u}\overline{X}_u^i du \overline{X}_0^j \\ & - \frac{1}{t}\int_0^t e^{-\lambda_i u} d\overline{X}_u^j \overline{X}_0^i - \frac{1}{t}\int_0^t \lambda_j e^{-(\lambda_i+\lambda_j)u} du \overline{X}_0^i \overline{X}_0^j,\end{aligned}$$

where the last three integrals are Young's integrals.

The first term on the right hand side tends to zero almost surely, which has been proved above. The last term also goes to zero almost surely, which can be proved easily. For the second and third terms, we can see that $\int_0^t \lambda_j e^{-\lambda_j u}\overline{X}_u^i du$ and $\int_0^t e^{-\lambda_i u} d\overline{X}_u^j$ are two Gaussian processes. By almost the same arguments as the proof of the limit $\frac{1}{t}\int_0^t \overline{X}_u^i \circ d_{\mathfrak{R}_1}\overline{\mathbf{X}}^j \to 0,\ a.s$, we can also prove that the second and third terms both converge to zero almost surely. Here we just give a sketch of proof for the second term.

Define $Z_t = \int_0^t \lambda_j e^{-\lambda_j u}\overline{X}_u^i du$, and $\xi = \overline{X}_0^j$. First, we show that $\frac{1}{n}(\xi Z_n) \to 0,\ a.s$ for integer subsequence. Since

$$\begin{aligned}\mathbb{E}|Z_n|^2 &= \mathbb{E}\left(\int_0^n \lambda_j e^{-\lambda_j u}\overline{X}_u^i du\right)^2 = \int_0^n\int_0^n r_i(u-v)e^{-\lambda_j(u+v)}dudv\\ &\le \max_{t\ge 0}|r_i(t)|\int_0^n\int_0^n e^{-\lambda_j(u+v)}dudv \le \frac{C}{\lambda_j^2}(e^{-\lambda_j n}-1)^2 \le \widetilde{C},\end{aligned}$$

where $C, \widetilde{C}$ independent of n. Then

$$\mathbb{P}\left(\frac{1}{n}|\xi Z_n| > \varepsilon\right) \le \frac{\mathbb{E}|\xi Z_n|^2}{n^2\varepsilon^2} \le \frac{(\mathbb{E}\xi^4)^{\frac{1}{2}} + (\mathbb{E}Z_n^4)^{\frac{1}{2}}}{n^2\varepsilon^2} \le \frac{C}{n^2\varepsilon^2},$$

by Borel-Cantelli lemma, we proved that $\frac{1}{n}(\xi Z_n) \to 0,\ a.s.$

Now we show for any $n \ge 1$ and any $s,t \in [n, n+1]$, there exist a constant $\beta \in (0,1)$ and an almost surely finite random variable R such that $|Z_t - Z_s| \le Rn^\beta,\ a.s.$. Since

$$\mathbb{E}|Z_t - Z_s|^2 = \mathbb{E}\left(\int_s^t \lambda_j e^{-\lambda_j u}\overline{X}_u^i du\right)^2 = \int_s^t\int_s^t r_i(u-v)e^{-\lambda_j(u+v)}dudv \le C|t-s|^2,$$

where C is a universal constant, applying Garsia-Rodemich-Rumsey inequality as Proposition 3.10, we get $|Z_t - Z_s| \le Rn^\beta,\ a.s.$ Then, choose $n = [t]$,

$$\frac{1}{t}|\xi Z_t| \le \frac{1}{t}|\xi||Z_t - Z_n| + \frac{n}{t}\frac{1}{n}|\xi Z_n| \le R|\xi|n^{\beta-1} + \frac{1}{n}|\xi Z_n| \to 0,\ a.s.$$

Thus we proved the limit of the second term. The third term follows likely as above. By taking an orthogonal transformation for X (independent components), we get the same limit for Stratonovich integral of solution to equation (3.25). Therefore, we conclude this theorem. □

4 Pathwise stable estimators

4.1 Continuous rough path estimator

In this section, let X be fOU process, i.e. the solution to the following stochastic differential equation

$$dX_t = -\Gamma X_t dt + \Sigma dB_t^H. \tag{4.1}$$

We construct an estimator based on continuous observation via rough path theory. We suppose that the rough path enhancement $(X_{0,t}(\omega), \mathbb{X}_{0,t}(\omega))$ of fOU process $X_t(\omega)$ could be continuously observed in Itô sense defined in section 2. It may leave the users with the question of how to understand data as a rough path in practice. For this direction, there are in fact works on how to inverse data to rough paths. We recommend those who may be interested in these questions to look at the literature on rough path analysis, in particular [6].

For the construction of estimator, we adapt the idea of the least square estimator of Hu and Nualart [24]. Hu and Nualart in the mentioned paper have derived the estimator in one dimensional case, which is formally taken as the minimizer

$$\widehat{\gamma}_T := \arg\min_{\gamma\in\Theta} \int_0^T |\dot{X}_t - (-\gamma X_t)|^2 dt, \tag{4.2}$$

where Θ is the parameter space. In multi-dimensional case, we consider (formally) the estimator as the minimizer, which is also discussed in [26],

$$\widehat{\Gamma}_t := \arg\min_{\Gamma\in\Theta} \int_0^t \|\Sigma^{-1}\dot{X}_s - (-\Gamma\Sigma^{-1}X_s)\|^2 ds,\ t\in[0,T], \tag{4.3}$$

which leads to the solution

$$\widehat{\Gamma}_t = -\mathcal{L}_t^{-1} S_t, \tag{4.4}$$

where

$$\mathcal{L}_t = \int_0^t (I\otimes X_s)^T Q^{-1}(I\otimes X_s) ds \in L(V,V^*), \tag{4.5}$$

$$S_t = \int_0^t (I\otimes X_s)^T Q^{-1} d_{\mathfrak{R}_1}\mathbf{X} \in V^*, \tag{4.6}$$

and space $V=\mathbb{R}^{d\times d}$, $\mathcal{L}_t^{-1}$ is the inverse of $\mathcal{L}_t$, $Q=\Sigma\Sigma^T$, $I\otimes X = (\delta^i_j X^k)_{i,j,k=1,\cdots,d}$, and M^T denotes transpose of matrix M. The integral S_t is taken as Itô rough integral of X defined in section 2. We call this estimator as *rough path estimator*.

When $\Sigma=\sigma I$ (I is identity matrix, σ is a constant), the estimator becomes

$$\widehat{\Gamma}_t^T = -\left(\int_0^t X_s\otimes X_s ds\right)^{-1}\left(\int_0^t X_s\otimes d_{\mathfrak{R}_1}\mathbf{X}\right). \tag{4.7}$$

Acctually, we can make a rotation to dynamical system (4.1), i.e. act Σ^{-1} to X_t, then we get the above diagonal case. So without loss of generality, we can suppose that $\Sigma=\sigma I$.

Now we give two examples for cases $d=1,2$. For one dimensional case, the rough path estimator is

$$\widehat{\gamma}_t = -\frac{\int_0^t X_s d_{\mathfrak{R}_1}\mathbf{X}}{\int_0^t X_s^2 ds} = -\frac{\mathbb{X}_{0,t} + X_0 X_{0,t}}{\int_0^t X_s^2 ds}. \tag{4.8}$$

For $d=2$, the transpose of the rough path estimator is

$$\begin{aligned}\widehat{\Gamma}_t^T = -\frac{1}{\det(\mathcal{L}_t(X))} &\begin{pmatrix} \int_0^t (X_s^2)^2 ds & -\int_0^t X_s^1 X_s^2 ds \\ -\int_0^t X_s^1 X_s^2 ds & \int_0^t (X_s^1)^2 ds \end{pmatrix} \\ &\times \begin{pmatrix} \int_0^t X_s^1 d_{\mathfrak{R}_1}\mathbf{X}^1 & \int_0^t X_s^1 d_{\mathfrak{R}_1}\mathbf{X}^2 \\ \int_0^t X_s^2 d_{\mathfrak{R}_1}\mathbf{X}^1 & \int_0^t X_s^2 d_{\mathfrak{R}_1}\mathbf{X}^2 \end{pmatrix},\end{aligned} \tag{4.9}$$

where

$$\det(\mathcal{L}_t(X)) = \int_0^t (X_s^1)^2 ds \int_0^t (X_s^2)^2 ds - \left(\int_0^t X_s^1 X_s^2 ds\right)^2, \tag{4.10}$$

$$\int_0^t X_s^i d_{\mathfrak{R}_1}\mathbf{X}^j = \mathbb{X}_{0,t}^{ij} + X_0^i X_{0,t}^j,\quad i,j=1,2. \tag{4.11}$$

As a remark, we mention that here in our paper $X(\omega),\mathbb{X}(\omega)$, and $\widehat{\Gamma}(\omega)$ are pathwise-defined almost surely.

4.2 Strong consistency

Now we consider the asymptotic behavior of this rough path estimator $\widehat{\Gamma}_t$. The solution X to (4.1) is given by

$$X_t = e^{-\Gamma t} X_0 + \int_0^t e^{-\Gamma(t-s)} \Sigma dB_s^H. \tag{4.12}$$

Without loss of generality, we suppose that $X_0 = 0$.

In the following, we will prove chain rules for our rough integrals, and then show the almost sure convergence of our rough path estimator.

4.2.1 Chain rules

First, we have the following lemma.

Lemma 4.1 For $H \in (\frac{1}{3}, \frac{1}{2}]$, we have

$$\int_0^t X_s \otimes d_{\mathfrak{R}_1} \mathbf{X} = -\left(\int_0^t X_s \otimes X_s ds\right) \Gamma^T + \sigma \int_0^t X_s \otimes d_{\mathfrak{R}_1} \mathbf{B}^H. \tag{4.13}$$

Here, the integrals can be either Stratonovich's or Itô's rough integrals.

Proof We use the relationship between almost rough paths and rough paths, see Theorem 3.2.1 in [34], to prove this lemma. To simplify notations, we show $d = 1$ case, i.e. to prove

$$\int_0^t X_s d_{\mathfrak{R}_1} \mathbf{X} = -\gamma \int_0^t (X_s)^2 ds + \sigma \int_0^t X_s d_{\mathfrak{R}_1} \mathbf{B}^H. \tag{4.14}$$

First, by the theory of rough differential equations and (2.7), we know that

$$Z_{s,t} \simeq f(Z_s) Z_{s,t} + Df(Z_s) \mathbb{Z}_{s,t}, \tag{4.15}$$

$$\mathbb{Z}_{s,t} \simeq f(Z_s) \otimes f(Z_s) \mathbb{Z}_{s,t}, \tag{4.16}$$

where the right hand sides are actually almost rough paths associated $\mathbf{Z}_{s,t}$, and $\simeq$ means the difference is controlled by $\omega(s,t)^\theta$ with $\theta > 1$, for all $(s,t) \in \Delta$. So by (4.15), we have

$$Z_{s,t} \simeq \left(B_{s,t}^H, -\gamma X_s (t-s) + \sigma B_{s,t}^H, t-s\right) + \left(0, -\gamma \int_s^t X_{s,u} du, 0\right).$$

Since

$$\left|\int_s^t X_{s,u} du\right| = \left|\int_s^t X_u du - X_s(t-s)\right| = o(|t-s|),$$

so we have

$$Z_{s,t} \simeq \left(B_{s,t}^H, -\gamma X_s (t-s) + \sigma B_{s,t}^H, t-s\right).$$

This implies

$$X_{s,t} \simeq -\gamma X_s (t-s) + \sigma B_{s,t}^H. \tag{4.17}$$

Actually, this above formula could be seen from stochastic differential equation (4.1) directly. Now by (4.16), we have

$$\mathbb{Z}_{s,t} \simeq \begin{pmatrix} \mathbb{B}_{s,t}^H & \int_s^t B_{s,u}^H dX_u & \int_s^t B_{s,u}^H du \\ M_1 & M_2 & M_3 \\ \int_s^t (u-s) dB_u^H & \int_s^t (u-s) dX_u & \frac{1}{2}(t-s)^2 \end{pmatrix},$$

where

$$\begin{aligned}
M_1 &= \sigma \mathbb{B}^H_{s,t} - \gamma X_s \int_s^t (u-s) dB^H_u, \\
M_2 &= \sigma \int_s^t B^H_{s,u} dX_u - \gamma X_s \int_s^t (u-s) dX_u, \\
M_3 &= \sigma \int_s^t B^H_{s,u} du - \frac{1}{2}\gamma X_s (t-s)^2.
\end{aligned}$$

Hence,

$$\int_s^t X_{s,u} dB^H_u \simeq \sigma \mathbb{B}^H_{s,t} - \gamma X_s \int_s^t (u-s) dB^H_u \simeq \sigma \mathbb{B}^H_{s,t}, \tag{4.18}$$

$$\mathbb{X}_{s,t} \simeq \sigma \int_s^t B^H_{s,u} dX_u - \gamma X_s \int_s^t (u-s) dX_u \simeq \sigma \int_s^t B^H_{s,u} dX_u, \tag{4.19}$$

$$\int_s^t X_{s,u} du \simeq \sigma \int_s^t B^H_{s,u} du - \frac{1}{2}\gamma X_s (t-s)^2 = o(|t-s|). \tag{4.20}$$

Combine (4.17) and (4.19), we further have $\mathbb{X}_{s,t} \simeq \sigma^2 \mathbb{B}^H_{s,t}$.

Now using the results above, we can show the equation (4.14) since we have

$$\begin{aligned}
LHS &\simeq X_s X_{s,t} + \mathbb{X}_{s,t} \\
&\simeq -\gamma (X_s)^2 (t-s) + \sigma X_s B^H_{s,t} + \sigma^2 \mathbb{B}^H_{s,t} \\
&\simeq RHS.
\end{aligned}$$

Thus we have completed the proof of this lemma. □

As a corollary, we have

Corollary 4.2 For $H \in (\frac{1}{3}, \frac{1}{2}]$, the rough path estimator $\widehat{\Gamma}_t$ has the following expression:

$$\widehat{\Gamma}^T_t = \Gamma^T - \left(\int_0^t X_s \otimes X_s ds\right)^{-1} \left(\int_0^t X_s \otimes d_{\mathfrak{R}_1} \mathbf{B}^{H,\text{Itô}}\right). \tag{4.21}$$

4.2.2 Almost sure convergence

In order to establish the strong consistency of the rough path estimator $\widehat{\Gamma}_t$, i.e.

$$\widehat{\Gamma}_t \to \Gamma, \ a.s. \text{ as } t \to \infty, \tag{4.22}$$

our aim now is to prove that

$$\frac{1}{t}\int_0^t X_s \otimes X_s ds \to C_1(H), \ a.s., \tag{4.23}$$

$$\frac{1}{t}\int_0^t X_s \otimes d_{\mathfrak{R}_1} \mathbf{B}^{H,\text{Itô}} \to 0, \ a.s.. \tag{4.24}$$

Then combining Slutsky Theorem and Corollary 4.2, we can get (4.22).

Proposition 4.3 Suppose stochastic process X_t is the fOU process to stochastic differential equation (4.1) and Γ is symmetric and positive-definite, then

$$\frac{1}{t}\int_0^t X_s \otimes X_s ds \to C_1(H), \ a.s.,$$

where the above integral on the left hand side is Lebesgue integral and the constant matrix $C_1(H) = \sigma^2 H \int_0^\infty x^{2H-1} e^{-\Gamma x} dx$.

Proof Define the process

$$\overline{X}_t := \sigma \int_{-\infty}^{t} e^{-\Gamma(t-s)} dB_s^H, \tag{4.25}$$

then the process $\overline{X}$ is stationary Gaussian process and it is ergodic (see subsection 3.2). By the ergodic theorem (see [14]), we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \overline{X}_s \otimes \overline{X}_s ds = \mathbb{E}(\overline{X}_0 \otimes \overline{X}_0),\ a.s. \tag{4.26}$$

Since

$$X_t = \overline{X}_t - e^{-\Gamma t}\overline{X}_0,$$

so that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t X_s \otimes X_s ds = \mathbb{E}(\overline{X}_0 \otimes \overline{X}_0),\ a.s. \tag{4.27}$$

For the right hand side, applying integration by parts, we have

$$\begin{aligned}
\mathbb{E}(\overline{X}_0 \otimes \overline{X}_0) &= \sigma^2 \mathbb{E}\left(\int_{-\infty}^0 e^{\Gamma s} dB_s^H\right) \otimes \left(\int_{-\infty}^0 e^{\Gamma s} dB_s^H\right) \\
&= \sigma^2\Gamma^2 \mathbb{E}\left(\int_{-\infty}^0 \int_{-\infty}^0 e^{\Gamma(s+u)} (B_s^H \otimes B_u^H) du ds\right) \\
&= \sigma^2\Gamma^2 \int_0^\infty \int_0^\infty e^{-\Gamma(s+u)} \frac{1}{2} I(s^{2H} + u^{2H} - |s-u|^{2H}) du ds \\
&= \sigma^2 \Gamma \int_0^\infty x^{2H} e^{-\Gamma x} dx,
\end{aligned}$$

and

$$\Gamma \int_0^\infty x^{2H} e^{-\Gamma x} dx = H \int_0^\infty x^{2H-1} e^{-\Gamma x} dx.$$

Thus we have proved this lemma. □

Proposition 4.4 Suppose stochastic process X_t is the fOU process to stochastic differential equation (4.1) and Γ is symmetric and positive-definite, then

$$\frac{1}{t} \int_0^t X_s \otimes d_{\mathfrak{R}_1} \mathbf{B}^{H,\text{Itô}} \to 0,\ a.s.,$$

where $\mathbf{B}^{H,\text{Itô}}$ is Itô type rough path enhancement of fBM B_t^H as in section 2 with $H \in (\frac{1}{3}, \frac{1}{2}]$.

Proof Applying integration by parts, we have

$$X_t = \sigma \int_0^t e^{-\Gamma(t-s)} dB_s^H = \sigma \left(B_t^H - \Gamma \int_0^t e^{-\Gamma(t-s)} B_s^H ds\right). \tag{4.28}$$

By definitions of Itô integration and Stratonovich integration with respect to fBM for fOU process (see rough differential equation (2.7)), we have

$$\int_0^t X_s \otimes d_{\mathfrak{R}_1} \mathbf{B}^{H,\text{Itô}} = \int_0^t X_s \otimes \circ d_{\mathfrak{R}_1} \mathbf{B}^{H,\text{Str}} - \sigma\varphi(t), \tag{4.29}$$

where

$$\varphi(t) = \frac{1}{2} I t^{2H} - U(t),$$

and

$$U(t) = H\Gamma \int_0^t \int_0^s e^{-\Gamma(s-u)} (s^{2H-1} - (s-u)^{2H-1}) du ds.$$

For the first term on the right hand side, it is defined as Stratonovich integral, and has the following expression (by Lemma 4.1)

$$\sigma \int_0^t X_s \otimes \circ d_{\mathfrak{R}_1} \mathbf{B}^{H,\mathrm{Str}} = \int_0^t X_s \otimes \circ d_{\mathfrak{R}_1} \mathbf{X} + \Gamma \int_0^t X_s \otimes X_s ds. \tag{4.30}$$

Now we represent $U(t)$ as

$$\begin{aligned} U(t) = \frac{1}{2} I t^{2H} - H \int_0^t e^{-\Gamma s} s^{2H-1} ds - H\Gamma t \int_0^t e^{-\Gamma s} s^{2H-1} ds \\ + H\Gamma \int_0^t e^{-\Gamma s} s^{2H} ds, \end{aligned}$$

and we have

$$\begin{aligned} \varphi(t) = H \int_0^t e^{-\Gamma s} s^{2H-1} ds + H\Gamma t \int_0^t e^{-\Gamma s} s^{2H-1} ds \\ - H\Gamma \int_0^t e^{-\Gamma s} s^{2H} ds, \end{aligned}$$

Since $\int_0^t e^{-\Gamma s} s^{\alpha-1} ds \uparrow \int_0^\infty e^{-\Gamma s} s^{\alpha-1} ds \leq C$ as $t \to \infty$, then

$$\lim_{t\to\infty} \frac{1}{t} \varphi(t) = H\Gamma \int_0^\infty e^{-\Gamma s} s^{2H-1} ds, \ a.s. \tag{4.31}$$

Then Combing equations (4.29), (4.30) and Theorem 3.13, and Proposition 4.3, for

$$\sigma \int_0^t X_s \otimes d_{\mathfrak{R}_1} \mathbf{B}^{H,\mathrm{Itô}} = \int_0^t X_s \otimes \circ d_{\mathfrak{R}_1} \mathbf{X} + \Gamma \int_0^t X_s \otimes X_s ds - \sigma^2 \varphi(t),$$

we have

$$\lim_{t\to\infty} \frac{\sigma}{t} \int_0^t X_s \otimes d_{\mathfrak{R}_1} \mathbf{B}^{H,\mathrm{Itô}} = \Gamma C_1(H) - \sigma^2 H\Gamma \int_0^\infty e^{-\Gamma s} s^{2H-1} ds = 0, \ a.s.$$

Thus we conclude this proposition. □

As a corollary of Theorem 3.13, now we have the following statement, in which the integral is in Itô sense.

Corollary 4.5 Suppose stochastic process X_t is the fOU process to stochastic differential equation (4.1) and Γ is symmetric and positive-definite, then

$$\frac{1}{t} \int_0^t X_s \otimes d_{\mathfrak{R}_1} \mathbf{X} \to C_2(H), \ a.s., \tag{4.32}$$

where the above integral is in Itô sense.

Proof As we can see from the definition of rough integral and Lemma 4.1 that

$$\begin{aligned} \int_s^t X \otimes d_{\mathfrak{R}_1} \mathbf{X} &\simeq X_s X_{s,t} + \mathbb{X}_{s,t} \\ &\simeq X_s X_{s,t} + \sigma^2 \mathbb{B}^{H,\mathrm{Itô}}_{s,t} \\ &\simeq X_s X_{s,t} + \sigma^2 (\mathbb{B}^{H,\mathrm{Str}}_{s,t} - \varphi_{s,t}) \\ &\simeq \int_s^t X \otimes \circ d_{\mathfrak{R}_1} \mathbf{X} - \sigma^2 \varphi_{s,t}, \end{aligned}$$

for any $(s,t) \in \Delta$. Thus we have

$$\int_0^t X_s \otimes d_{\mathfrak{R}_1} \mathbf{X} = \int_0^t X_s \otimes \circ d_{\mathfrak{R}_1} \mathbf{X} - \sigma^2 \varphi(t). \tag{4.33}$$

By Theorem 3.13 and the limit in equation (4.31), we get

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t X_s\otimes d_{\mathfrak{R}_1}\mathbf{X} = -\sigma^2 H\Gamma\int_0^\infty e^{-\Gamma s}s^{2H-1}ds =: C_2(H),\ a.s. \tag{4.34}$$

Besides, by the definition of $C_1(H)$, we also have the relation between $C_2(H)$ and $C_1(H)$ as $C_2(H) = -\Gamma C_1(H)$. □

Now we have strong consistency of the rough path estimator $\widehat{\Gamma}_t$ as t tends to infinity.

Theorem 4.6 Suppose Γ is a parametric matrix and it is symmetric and positive-definite. Let $\widehat{\Gamma}_t$ be the rough path estimator as (4.7) of Γ for the stochastic differential equation (4.1). Then

$$\widehat{\Gamma}_t \to \Gamma,\ a.s.,\ \text{as } t\to\infty. \tag{4.35}$$

Proof Applying Proposition 4.3 and Proposition 4.4, and by Slutsky Theorem, we have

$$\left(\frac{1}{t}\int_0^t X_s\otimes X_s ds\right)^{-1}\left(\frac{1}{t}\int_0^t X_s\otimes d_{\mathfrak{R}_1}\mathbf{B}^H\right)\to 0,\ a.s.$$

as t goes to infinity. Hence, by Corollary 4.2, the rough path estimator $\widehat{\Gamma}_t$ almost surely converges to Γ. □

Remark 4.7 Suppose we take the stochastic integral $\int_0^t X_s\otimes d_{\mathfrak{R}_1}\mathbf{X}$ in the rough path estimator $\widehat{\Gamma}_t$ (equation (4.7)) as Stratonovich rough integral rather than Itô rough integral as above, we can see that

$$\widehat{\Gamma}_t \to 0,\ a.s.,\ \text{as } t\to\infty, \tag{4.36}$$

by Theorem 3.13 and Proposition 4.3. That is to say, we cannot use Stratonovich rough integral to do this estimation problem.

Remark 4.8 The explicit dependence of the Lévy area of fOU processes on the drift parameter Γ can make it challenging to apply the estimator in practical observations. However, as suggested by equation (4.4), we can treat it as an equation for Γ and solve it iteratively.

4.3 Pathwise stability

In this subsection, we will show that our rough path estimator is pathwise stable and robust. Note that $\widehat{\Gamma}_T$ is a functional on the path space $C([0,T],\mathbb{R}^d)$, or exactly on the rough path space $\Omega_p([0,T],\mathbb{R}^d)$. For every observation sample path $X(\omega)$ or rough path enhancement $\mathbf{X}(\omega) = (X(\omega),\mathbb{X}(\omega))$, one has a corresponding estimator $\widehat{\Gamma}_T(X(\omega))$ or $\widehat{\Gamma}_T(\mathbf{X}(\omega)) = \widehat{\Gamma}_T((X(\omega),\mathbb{X}(\omega)))$. In the following, we will use the rough path notation rather than sample path, since our continuous rough path estimator depends on $\mathbf{X}(\omega) = (X(\omega),\mathbb{X}(\omega))$ rather than just the first level sample path $X(\omega)$.

A natural question about robustness of estimator arise: if two observations $\mathbf{X}$ and $\widetilde{\mathbf{X}}$ are very close in some sense, e.g. uniform distance or p-variation distance etc, does it give arise to close estimations $\widehat{\Gamma}_T(\mathbf{X}) \approx \widehat{\Gamma}_T(\widetilde{\mathbf{X}})$? In other words, is the estimator $\widehat{\Gamma}_T(\cdot)$ continuous in some distance?

Actually, the rough path idea gives us a good solution to this problem. As well-known, in rough path space, rough integration is continuous with respect to p-variation distance. Now we first recall the p-variation rough path distance d_p:

$$d_p(\mathbf{X}, \mathbf{Y}) = \max_{i=1,2} \sup_{\mathcal{P}} \left(\sum_{\ell} |\mathbf{X}^i_{t_{\ell-1},t_\ell} - \mathbf{Y}^i_{t_{\ell-1},t_\ell}|^{\frac{p}{i}} \right)^{\frac{i}{p}}, \tag{4.37}$$

where $\mathbf{X} = (\mathbf{X}^1, \mathbf{X}^2)$ and $\mathbf{Y} = (\mathbf{Y}^1, \mathbf{Y}^2)$ are two rough paths in rough path space $\Omega_p([0,T], \mathbb{R}^d)$, and $\mathcal{P}$ is any partition of interval $[0,T]$.

Now we give the continuity of estimator $\widehat{\Gamma}_T(\cdot)$ under p-variation distance d_p.

Theorem 4.9 Let X be a fOU process driven by fBM with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2}]$ and $(X, \mathbb{X})$ be the Itô rough path enhancement. Then rough path estimator $\widehat{\Gamma}_T : (X(\omega), \mathbb{X}(\omega)) \to \widehat{\Gamma}_T((X(\omega), \mathbb{X}(\omega)))$ is continuous with respect to p-variation distance d_p for $\frac{1}{H} < p < 3$.

Proof The statement is a corollary of Theorem 5.3.1 of Lyons and Qian [34]. □

5 Rough path estimator based on high-frequency data

In the preceding sections, the estimator we have examined assumes access to continuous observations. However, in practical scenarios, the process is typically observed only at discrete intervals. Consequently, it is imperative to devise an estimator that operates on discrete data. We refer to the work [46], in which the authors estimated the parameters of the fOU process using discrete-sampled observations in the one-dimensional case. In this section, based on our continuous rough path estimator, we construct a discrete rough path estimator and it still has favorable properties. We assume that the fOU process X can be enhanced to an Itô rough path $\mathbf{X} = (1, X, \mathbb{X})$ as section 2 and can be observed at discrete time $\{t_\ell = \ell h, \ell = 0, 1, 2, \cdots, n\}$, or equivalently we can get the discrete data $\{(X_{t_0,t_1}, \mathbb{X}_{t_0,t_1}), (X_{t_1,t_2}, \mathbb{X}_{t_1,t_2}), \cdots, (X_{t_{n-1},t_n}, \mathbb{X}_{t_{n-1},t_n})\}$ in Itô sense as in section 2. Here, n is sample size, $h = h_n$ is the observation frequency, and $t := nh$ is the time horizon. We further assume that as the sample size n tends to infinity, the observation frequency $h = h_n \to 0$ and time horizon $t = nh \to \infty$. In other words, the data is high-frequency. Besides, we also should give more assumptions to balance the rate of sample size n and the frequency h in order to get good estimator below. Now we give the theorem of almost sure convergence for our high-frequency rough path estimator.

Theorem 5.1 Suppose the fOU process X which is the solution to stochastic differential equation (3.25) with $H \in (\frac{1}{3}, \frac{1}{2}]$ can be observed at discrete time $\{t_\ell = \ell h, \ell = 0, 1, 2, \cdots, n\}$ and as sample size $n \to \infty$, n and h satisfy

$$nh \to \infty,\ h = h_n \to 0,\ nh^p \to 0, \tag{5.1}$$

for some $p \in (1, \frac{1+H+\beta}{1+\beta})$, and $0 < \beta < 1$. Let

$$\widetilde{\Gamma}_n^T = -\left(\sum_{\ell=0}^{n} (X_{\ell h} \otimes X_{\ell h}) h \right)^{-1} \left(\sum_{\ell=0}^{n-1} X_{\ell h} X_{\ell h,(\ell+1)h} + \mathbb{X}_{\ell h,(\ell+1)h} \right), \tag{5.2}$$

where $\widetilde{\Gamma}^T$ denotes transpose of matrix $\widetilde{\Gamma}$. Then

$$\widetilde{\Gamma}_n \to \Gamma,\ a.s. \tag{5.3}$$

as $n \to \infty$.

Proof Let

$$\mathcal{L}_{nh} = \int_0^{nh} X_u \otimes X_u du, \ A_{nh} = \int_0^{nh} X_u \otimes d_{\mathfrak{R}_1} \mathbf{X},$$

and

$$\widetilde{\mathcal{L}}_n = \sum_{\ell=0}^{n} (X_{\ell h} \otimes X_{\ell h}) h, \ \widetilde{A}_n = \sum_{\ell=0}^{n-1} X_{\ell h} X_{\ell h,(\ell+1)h} + \mathbb{X}_{\ell h,(\ell+1)h}.$$

By Proposition 4.3, we know that

$$\frac{1}{nh} \mathcal{L}_{nh} = \frac{1}{nh} \int_0^{nh} X_u \otimes X_u du \to C_1(H), \ a.s. \text{ as } n \to \infty. \tag{5.4}$$

From Corollary 4.5, we have

$$\frac{1}{nh} A_{nh} = \frac{1}{nh} \int_0^{nh} X_u \otimes d_{\mathfrak{R}_1} \mathbf{X} \to C_2(H), \ a.s. \text{ as } n \to \infty. \tag{5.5}$$

In the following, we show that

$$\frac{1}{nh} \left(\mathcal{L}_{nh} - \widetilde{\mathcal{L}}_n \right) \to 0, \ a.s., \tag{5.6}$$

and

$$\frac{1}{nh} \left(A_{nh} - \widetilde{A}_n \right) \to 0, \ a.s. \tag{5.7}$$

If so, combining (5.4), (5.5), (5.6) and (5.7), we can conclude this theorem, that is,

$$\begin{aligned} -\widetilde{\mathcal{L}}_n^{-1} \widetilde{A}_n &= -\left(\frac{1}{nh} \left(\widetilde{\mathcal{L}}_n - \mathcal{L}_{nh} \right) + \frac{1}{nh} \mathcal{L}_{nh} \right)^{-1} \\ &\quad \times \left(\frac{1}{nh} \left(\widetilde{A}_n - A_{nh} \right) + \frac{1}{nh} A_{nh} \right) \\ &\to -C_1(H)^{-1} C_2(H) = \Gamma, \ a.s. \end{aligned}$$

Now we first show the limit (5.7), we have

$$\begin{aligned} \frac{1}{nh} \left| A_{nh} - \widetilde{A}_n \right| &= \frac{1}{nh} \left| \int_0^{nh} X_u \otimes d_{\mathfrak{R}_1} \mathbf{X} - \sum_{\ell=0}^{n-1} (X_{\ell h} X_{\ell h,(\ell+1)h} + \mathbb{X}_{\ell h,(\ell+1)h}) \right| \\ &= \frac{1}{nh} \left| \sum_{\ell=0}^{n-1} \left(\int_{\ell h}^{(\ell+1)h} X_u \otimes d_{\mathfrak{R}_1} \mathbf{X} - (X_{\ell h} X_{\ell h,(\ell+1)h} + \mathbb{X}_{\ell h,(\ell+1)h}) \right) \right| \\ &\le \frac{1}{nh} \sum_{\ell=0}^{n-1} \left| \int_{\ell h}^{(\ell+1)h} X_u \otimes d_{\mathfrak{R}_1} \mathbf{X} - (X_{\ell h} X_{\ell h,(\ell+1)h} + \mathbb{X}_{\ell h,(\ell+1)h}) \right|. \end{aligned}$$

Since

$$\mathbb{X}_{\ell h,(\ell+1)h} = \int_{\ell h}^{(\ell+1)h} X_{\ell h,u} \otimes d_{\mathfrak{R}_1} \mathbf{X} = \int_{\ell h}^{(\ell+1)h} X_u \otimes d_{\mathfrak{R}_1} \mathbf{X} - X_{\ell h} X_{\ell h,(\ell+1)h},$$

so we have got

$$\frac{1}{nh} \left| A_{nh} - \widetilde{A}_n \right| = 0.$$

For the limit (5.6),

$$\begin{aligned}\frac{1}{nh}\left|\mathcal{L}_{nh}-\widetilde{\mathcal{L}}_n\right| &= \frac{1}{nh}\left|\int_0^{nh} X_u\otimes X_u du-\sum_{\ell=0}^{n}(X_{\ell h}\otimes X_{\ell h})h\right| \\ &= \frac{1}{nh}\left|\sum_{\ell=0}^{n-1}\left(\int_{\ell h}^{(\ell+1)h} X_u\otimes X_u du-(X_{\ell h}\otimes X_{\ell h})h\right)\right| \\ &\le \frac{1}{nh}\sum_{\ell=0}^{n-1}\left|\int_{\ell h}^{(\ell+1)h} X_u\otimes X_u du-(X_{\ell h}\otimes X_{\ell h})h\right|.\end{aligned}$$

Let $F(X_t)=X_t\otimes X_t$, and any $0\le s<t\le T$. Then by Proposition 3.8, we get

$$\left|\int_s^t F(X_u)du-F(X_s)(t-s)\right|\le CR_T T^{\beta}|t-s|^{1+\alpha},\ \forall\alpha\in(0,H).$$

Take $s=\ell h$, $t=(\ell+1)h$, and $T=nh$, we have

$$\left|\mathcal{L}_{nh}-\widetilde{\mathcal{L}}_n\right|\le\sum_{\ell=0}^{n-1}CR_{nh}(nh)^{\beta}h^{1+\alpha}=CR_{nh}n^{1+\beta}h^{1+\alpha+\beta}=CR_{nh}\left(nh^{\frac{1+\alpha+\beta}{1+\beta}}\right)^{1+\beta}.$$

By assumption, there exists a number $p\in(1,\frac{1+H+\beta}{1+\beta})$ such that $nh^p\to 0$ and $nh\to\infty$ as $n\to\infty$. And $R_{nh}\to 0$, *a.s.* So we get $\mathcal{L}_{nh}-\widetilde{\mathcal{L}}_n\to 0$, *a.s.* (We may assume that the components of fOU process X are independent, we should make an orthogonal transformation for X.) Thus, we have completed the proof of Theorem 5.1. □

Remark 5.2 The explicit dependence of the Lévy area of fOU processes on the drift parameter Γ can make it challenging to apply the estimator in practical observations. However, as suggested by equation (5.2), we can treat it as an equation for Γ and solve it iteratively.

Acknowledgements The authors thank the anonymous reviewers for their valuable comments and suggestions which helped improve and clarify this manuscript.

Conflict of Interest The authors declare no conflict of interest.

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