ITÔ INTEGRALS FOR FRACTIONAL BROWNIAN MOTION AND APPLICATIONS TO OPTION PRICING

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ABSTRACT. In this paper, we develop an Itô type integration theory for fractional Brownian motions with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2})$ via rough path theory. The Itô type integrals are path-wise defined and have zero expectations. We establish the fundamental tools associated with this Itô type integration such as Itô formula, chain rule and etc. As an application we apply this path-wise Itô integration theory to the study of a fractional Black-Scholes model with Hurst parameter less than half, and prove that the corresponding fractional Black-Scholes market has no arbitrage under a class of trading strategies.

1. Introduction

Since Black and Scholes proposed a financial model for pricing options in their seminal paper [8] in 1973 (see also Merton [27] for a continuous model), most of papers published in the area of quantitative finance are written in the setting of Black-Scholes' model and its generalizations, where Brownian motion, Lévy processes and semi-martingales are used to model sources of noise in financial markets. The Markovian property of Black-Scholes' model is essential for a complete financial market, and is required by the market efficiency assumption. For financial markets, however, statistics of financial data show that value and price processes exhibit correlations in time. According to the theory of behavioral finance, people in general recognize that past information affects reactions of market participants. In order to model the memory effects of markets, it is natural to develop financial dynamic models with memory over time. A simple and good candidate of simulating market noise with certain memory is a special family of Gaussian processes called fractional Brownian motions (fBM for abbreviation), introduced by Mandelbrot and van Ness [26] in 1968. FBMs are extensions of the standard Brownian motion, which are different from Brownian motion in that fBMs capture dependencies with decay rate of polynomial order in time.

Many researchers have investigated integration theories with respect to fBM (see e.g. [12, 13, 5, 15, 21, 28, 2, 3] and the literature therein). Among them, Duncan, Hu and Pasik-Duncan [14] who are the first to propose Wick product approach to define fractional stochastic integrals against fBM, called Wick-Itô integrals. Hu and Oksendal [21], Elliott and van der Hoek [15] extended the idea of the Wick product approach, developed a fractional white noise calculus and applied their theory to financial models. The approach in [14] is limited to the persistent case, i.e. fBM with Hurst parameter $H > \frac{1}{2}$, while the approach in [15] works for the anti-persistent case, i.e. fBM with Hurst parameter $H < \frac{1}{2}$. In these papers, ordinary multiplications are replaced by Wick products to define stochastic integrals. The

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idea of Wick products is also applied to the study of portfolios and to the study of self-financing in fractional Black-Scholes' markets introduced in these papers, see below for a more precise definition. By a conceptual innovation, they showed the non-arbitrage property and completeness of the market where Black-Scholes model is replaced by the geometric fBM with Hurst parameter $H > \frac{1}{2}$. These papers also initiated an intense debate because of conceptual difference in the definition of non-arbitrage from the standard Black-Scholes' model. Severe critiques arose concerning the economic meaning of Wick products applying to the fundamental economic concepts, such as values of portfolio and self-financing (see, e.g. [6]). Wick products however are only defined as multiplications of two random variables, and cannot be computed in a path-wise sense, hence, if one only knows realizations $X(\omega)$, $Y(\omega)$ of two random variables X, Y, one is unable to work out the Wick product of $X(\omega)$ and $Y(\omega)$.

There are other approaches to define stochastic integrals with respect to fBM. The first is a path-wise approach by using the Hölder continuity of sample paths of fBM, developed in Ciesielski, Kerkyacharian and Roynette [11] and Zähle [32]. Ciesielski et al [11] proposed an integration theory based on the Besov-Orlicz spaces using wavelet expansions, and Zähle [32] developed a theory using fractional calculus and a generalization of the integration by parts formula. These integration theories are suitable however only for fBM with Hurst parameter $H > \frac{1}{2}$. The second approach, developed by Decreusefond and Üstünel [13], relies on the Malliavin calculus for fBM. Alos and Nualart [1], Carmona, Coutin and Montseny [9], and Cheridito and Nualart [10] investigated the theory proposed in [13] further. The third approach is an integration theory via the rough path theory, which was worked out by Coutin and Qian [12]. The rough path theory works for fBM with Hurst parameter $H > \frac{1}{4}$, which can be considered as an integration theory in Stratonovich's sense.

When applying Stratonovich's fractional integration to financial models such as option pricing, it has an obvious shortcoming, namely the expectations of Stratonovich's integrals are not zero in general, which makes it possible to construct explicitly an arbitrage trading strategy (see e.g. [31, 4]) in fractional Black-Scholes' markets. Since fBM with Hurst parameter $H \neq \frac{1}{2}$ is not a semi-martingale, there is no hope to apply Itô's theory of semi-martingales to fBM. The best we can hope is to define stochastic integrals in such way so that integrals against fBM have zero mean. This feature is at least more promising than the situation in the Stratonovich integration theory when considering arbitrage problems, and it implies that there occurs no systemic bias in price processes over all possible paths.

What we are going to do in this paper is to construct an Itô type fractional path-wise integration theory with respect to fBM with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2}]$, by using the rough path approach. It is therefore path-wise defined and has zero expectation property, and thus meets our initial expectations. It seems that the natural language to deal with fBM with Hurst parameter $H < \frac{1}{2}$ is based on the rough path theory of T. Lyons (see e.g. [22, 24, 23, 17, 16]). Under the setting of the analysis of rough paths, we establish the fundamental tools associated with Itô type integration such as relations with Stratonovich fractional integrals, fractional Itô formula, chain rule and etc. In a general setting, rough path integration theory for non-geometric rough paths have been also studied in [19, 20, 25] and etc. Here we however develop the theory only for fBM and for the propose of applications in Black-Scholes models. As an application we apply this path-wise Itô integration theory to fractional Black-Scholes model, and prove that the corresponding fractional Black-Scholes' market has no arbitrage under a class of allowed trading strategies which is more restrictive

than those allowed in a complete market. The study of arbitrage problems is just an example. There are many other problems which can be studied with this theory. We hope the present work can arise a new interest in applying geometric fBM to the study of quantitative finance.

Now we give a summary of our work. In section 2, we will recall the most important insights concerning rough path theory, and in section 3 we will present the fundamental facts about fBM and define the Itô factional Brownian rough path with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2}]$ by the theory introduced in section 2. The material in section 2 and section 3 can be found in standard literature such as [12, 24, 22, 16, 17]. Section 4 forms a core of this paper, we will give a definition of integration with respect to our Itô fractional Brownian rough path, establish the relation between Stratonovich fractional integrals and Itô's integrals, and work out the fractional Itô formula in our integration framework. Besides, we will also address the differential equation driven by rough paths and give several important examples of differential equations driven by fractional Brownian rough path, especially the fractional Black-Scholes model. Then we build the chain rule for fractional Black-Scholes' model, and finally show the zero expectation property of our Itô integration in this section. In section 5, we will apply the theory established above to financial market as an application, and show that there is no arbitrage in Itô fractional Black-Scholes' market under a restriction on the class of trading strategies.

As a remark, the path-wise Itô integration we construct is just for a one form $F(B_t^H)$ of fBM B_t^H . If the integrator is not a one form, the integral can still have a meaning under this natural rough path lift for fBM but no longer have zero expectation. For example, the fractional Ornstein-Uhlenbeck process $X_t = \int_0^t e^{-(t-s)} dB_s^H$ can not be a function of B_t^H , that is, not one form of fBM. If we want to define a path-wise Itô integral for fractional Ornstein-Uhlenbeck process with respect to fBM, i.e. make the integral $\int_0^t X_s dB_s^H$ have zero expectation, we should find another rough path lift for fBM as Itô lift. As a matter of fact, for constructing general path-wise Itô integration with respect to fBM, the Itô rough path lift of fBM, in particular its Lévy area part, has to be made to depend on the integrator, which is a striking difference from Itô's theory for Brownian motion. Itô's integrals with respect to fBM can be constructed, as we will demonstrate in the present paper, by using a single rough path lift of fBM for all one forms. When the integrator is a solution of a rough differential equation driven by fBM, however, the Itô rough path lift of fBM depends on the solution too.

2. Preliminaries on rough paths

In this section, we recall some basic notions concerning rough paths to establish several notations which will be used throughout the paper. Our exposition follows closely those presented in rough path literature (see e.g. [12, 16, 17, 24]). In particular, we mention the fundamental framework needed to ensure that fBM with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2}]$, which will be introduced in the next section, has a natural rough path lift.

For $N \in \mathbb{N}$, $T^{(N)}(\mathbb{R}^d)$ denotes the truncated tensor algebra defined by

$$T^{(N)}(\mathbb{R}^d) := \bigoplus_{n=0}^N (\mathbb{R}^d)^{\otimes n},$$

with the convention that $(\mathbb{R}^d)^{\otimes 0} = \mathbb{R}$. The space $T^{(N)}(\mathbb{R}^d)$ is equipped with a vector space structure and a multiplication \otimes defined by

$$(X \otimes Y)^k = \sum_{i=0}^k X^{k-i} Y^i, \quad k = 0, 1, \dots, N,$$

where $X = (1, X^1, \dots, X^N), Y = (1, Y^1, \dots, Y^N) \in T^{(N)}(\mathbb{R}^d).$

We will consider continuous \mathbb{R}^d -valued paths X on [0,T] with bounded variations, and their canonical lifts $X_{s,t} = (1, X_{s,t}^1, \cdots, X_{s,t}^N)$ in the space $T^{(N)}(\mathbb{R}^d)$, where

$$X_{s,t}^1 = X_t - X_s,$$

$$X_{s,t}^2 = \int_{s < t_1 < t_2 < t} dX_{t_1} \otimes dX_{t_2},$$

:

and

$$X_{s,t}^N = \int_{s < t_1 < \dots < t_N < t} dX_{t_1} \otimes \dots \otimes dX_{t_N}.$$

The lifted path satisfies 'Chen's identity':

$$(2.1) X_{s,t} = X_{s,u} \otimes X_{u,t}, \ \forall (s,u), \ (u,t) \in \Delta,$$

where Δ denotes the simplex $\{(s,t) : 0 \le s < t \le T\}$.

By definition, a continuous map X from the simplex Δ into a truncated tensor algebra $T^{(N)}(\mathbb{R}^d)$ is called a *rough path* (of roughness $p \geq 1$, where N = [p]), if it satisfies (2.1) and has finite p-variations, that is,

$$\sum_{i=1}^{N} \sup_{D} \sum_{\ell} |X_{t_{\ell-1},t_{\ell}}^{i}|^{p/i} < \infty,$$

where the sup runs over all finite partitions $D = \{0 = t_0 < t_1 < \cdots < t_n = T\}$. The p-variation distance is defined to be

$$d_p(X,Y) = \sum_{i=1}^{N} \left(\sup_{D} \sum_{\ell} |X_{t_{\ell-1},t_{\ell}}^i - Y_{t_{\ell-1},t_{\ell}}^i|^{p/i} \right)^{i/p}.$$

Equivalently, $X:\Delta\to T^{(N)}(\mathbb{R}^d)$ has finite *p*-variations if

$$|X_{s,t}^i| \le \omega(s,t)^{i/p}, \ \forall i = 1, \cdots, N, \ \forall (s,t) \in \Delta$$

for some function ω , where ω is a non-negative, continuous, super-additive function on Δ and satisfies $\omega(t,t)=0$. Such function ω is called a *control of the rough path X*.

The space of all p-rough paths is denoted by $\Omega_p(\mathbb{R}^d)$. A rough path X is called a geometric rough path if there is a sequence of X(n), where X(n) are the canonical lifts of their first level $X(n)^1$ which are continuous with finite variations, such that X is the limit of X(n) under p-variation distance d_p . The space of geometric rough paths is denoted by $G\Omega_p(\mathbb{R}^d)$.

As our interest lies in fBMs with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2}]$, which will be introduced later, we consider only rough paths valued in $T^{(2)}(\mathbb{R}^d)$. Thus, in what follows, we will

assume that $3 > p \ge 2$ so that [p] = 2. A rough path of roughness p can be written as $X_{s,t} = (1, X_{s,t}^1, X_{s,t}^2)$ for s < t, and the algebraic relation (Chen's identity) now becomes

$$(2.2) X_{s,t}^1 = X_t - X_s,$$

and

$$(2.3) X_{s,t}^2 - X_{s,u}^2 - X_{u,t}^2 = X_{s,u}^1 \otimes X_{u,t}^1,$$

for all (s, u), $(u, t) \in \Delta$, where X^2 should be (if it makes sense) considered as an iterated integral

(2.4)
$$\int_{s}^{t} X_{s,u}^{1} dX_{u}^{1} := X_{s,t}^{2}$$

which is of course not defined a priori.

The most convenient tool to construct rough paths is through almost rough paths. A function $Y = (1, Y^1, Y^2)$ from Δ to $T^{(2)}(\mathbb{R}^d)$ is called an almost rough path if it has finite p-variation, and for some control ω and constant $\theta > 1$,

$$|(Y_{s,t} \otimes Y_{t,u})^i - Y_{s,u}^i| \le \omega(s,u)^{\theta}, \ i = 1, 2,$$

for all $(s,t),(t,u) \in \Delta$. According to Theorem 3.2.1, [24], given an almost rough path $Y=(1,Y^1,Y^2)$, there exists a unique rough path $X=(1,X^1,X^2)$ such that

$$|X_{s,t}^i - Y_{s,t}^i| \le \omega(s,t)^{\theta}, \ i = 1, 2, \ \theta > 1$$

for some control ω , and all $(s,t) \in \Delta$.

3. Fractional Brownian motion

Fractional Brownian motion (fBM) is a continuous-time Gaussian process $B^H(t)$ (where $t \geq 0$), with mean zero for all $t \geq 0$, and the co-variance function given by

(3.1)
$$E[B^{H}(t)B^{H}(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}),$$

where H is a real number in (0,1), called the Hurst parameter.

If H>1/2, then the increments of the fBM are positively correlated, and in the case that H<1/2, then the increments of the fBM are negatively correlated. Therefore, for $H<\frac{1}{2}$, fBM has the property of counter persistence: that is, the persistence is increasing with respect to the past, it is more likely to decrease in the future, and vice versa. In contrast, for $H>\frac{1}{2}$, the fBM is persistent, it is more likely to keep trend than to break it. Therefore, fBM is a popular model for both short-range dependent and long-range dependent phenomena in various fields, including physics, biology, hydrology, network research, financial mathematics etc.

The fBM with Hurst parameter H has an integral representation in terms of Brownian motion

(3.2)
$$B^{H}(t) = \int_{0}^{t} K_{H}(t, s) dW(s),$$

where W(t) is a standard Brownian motion and

$$K_H(t,s) = C_H \left[\frac{2}{2H-1} \left(\frac{t(t-s)}{s} \right)^{H-\frac{1}{2}} - \int_s^t \left(\frac{u(u-s)}{s} \right)^{H-\frac{1}{2}} \frac{du}{u} \right] 1_{(0,t)}(s)$$

which is a singular kernel, and C_H is a normalised constant.

A theory of integration with respect to fBM with Hurst parameter $H > \frac{1}{2}$ may be established by using Young's integration theory or functional integration theories (see e.g.[11, 32]), while stochastic calculus with respect to fBM with Hurst parameter $H < \frac{1}{2}$ is better to be studied in the framework of rough path analysis. In [12], a construction of a canonical level-3 rough path B^H with Hurst parameter H > 1/4 is given, where iterated integrals of several dimensional fBM to level-3 are defined canonically in order to apply the integration theory of rough paths to fBM.

The second and third level processes are defined in terms of iterated Riemann-Stieltjes integrals along the dyadic piece-wise linear approximations, and geometric rough paths of fBM are the limits in p-variation distance. Here as our main concern is the fBM with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2}]$, so we only need the level-2 results about fBM. The method of [12] to construct the fractional Brownian rough $B^H = (1, (B^H)^1, (B^H)^2)$ implies that

(3.3)
$$(B^H)_{s,t}^2 := \lim_{m \to 0} \int_s^t B_{s,u}^{H,(m)} \otimes dB_u^{H,(m)}, \quad a.s.,$$

exists in p-variation distance as long as pH > 1, where the dyadic piece-wise linear approximations $B_t^{H,(m)}$ on interval [s,t] is defined by

$$B_r^{H,(m)} := B_{t_{\ell-1}^m}^H + 2^m \frac{r - t_{\ell-1}^m}{t_{\ell}^m - t_{\ell-1}^m} \Delta_{\ell}^m B^H,$$

with
$$\Delta_{\ell}^m B^H = B_{t_{\ell-1}}^H - B_{t_{\ell-1}}^H$$
, $t_{\ell}^m := s + \frac{\ell}{2^m} (t-s)$ for $\ell = 1, 2, \cdots, 2^m$.

Proposition 3.1. ([12], Theorem 2; [16], Theorem 10.4) Let $B^H = (B^{H,1}, \dots, B^{H,d})$ be a d-dimensional fBM with the Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2}]$. Then B^H , restricted to an finite interval [0,T], lifts via (3.3) to a geometric rough path $B^H = (1,(B^H)^1,(B^H)^2) \in G\Omega_p([0,T],\mathbb{R}^d)$, for all $p \in (\frac{1}{H},3)$.

The details of the proof of this proposition may be found from [12, 16]. We mention that the random rough path $B^H = (1, (B^H)^1, (B^H)^2)$ is called the canonical lift. It can be viewed as the Stratonovich lift of fBM. Since we will introduce a new natural lift of fBM in Itô's sense in this paper, we rewrite the Stratonovich fractional Brownian rough path $B^H = (1, (B^H)^1, (B^H)^2)$ as $B^S = (1, (B^S)^1, (B^S)^2)$. The fBM B^H is denoted by B for

The Itô rough path associated with fBM B^H is defined by

(3.4)
$$B_{s,t}^{I} := (1, (B^{I})_{s,t}^{1}, (B^{I})_{s,t}^{2}) = \left(1, B_{t} - B_{s}, (B^{S})_{s,t}^{2} - \frac{1}{2}(t^{2H} - s^{2H})\right),$$

where the Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2}]$. We may verify that $B_{s,t}^I = (1, (B^I)_{s,t}^1, (B^I)_{s,t}^2)$ is still a random rough path but not a geometric one. We call this non-geometric rough path as Itô fractional Brownian rough path as we can see from below that the rough path and its integration theory for B^{I} is an extension of standard Brownian motion and Itô stochastic integration. For simplicity, we denote it by $B_{s,t} = (1, B_{s,t}^1, B_{s,t}^2)$ if no confusion may arise.

4. Itô Integration against fractional Brownian motion

4.1. Itô integrals of one forms against fBM. The goal of this section is to give a meaning for Itô integrals of one forms with respect to fBM such as $\int_s^t F(B)dB$, where $B = (B^{(1)}, \dots, B^{(d)})$ is a d-dimensional fBM with a Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2}]$.

In order to define $\int_s^t F(B)dB$, where $F: \mathbb{R}^d \to L(\mathbb{R}^d, \mathbb{R}^e)$ satisfies some smooth conditions, according to the rough path theory of T. Lyons, we should take B as a rough path. Actually the symbol $\int_s^t F(B)dB$, to some extent, is a misleading statement. Recall that the rough integral $\int F(X)dX$ against a rough path $X = (1, X^1, X^2) \in \Omega_p(\mathbb{R}^d)$ with $2 \le p < 3$ is defined to be again a rough path. The rough integral is defined uniquely by its associated almost rough path (see [24], definition 5.2.1):

(4.1)
$$\widehat{Y}_{s,t}^{1} = F(X_s)X_{s,t}^{1} + DF(X_s)X_{s,t}^{2},$$

$$\widehat{Y}_{s,t}^2 = F(X_s) \otimes F(X_s) X_{s,t}^2,$$

(one can find a proof in [24], Theorem 5.2.1), and the integral $\int F(X)dX$ is defined to be the rough path Y uniquely associated with the almost rough path \widehat{Y} . The integral can be written in compensated Riemann sum form, that is,

(4.3)
$$Y_{s,t}^{1} = \int_{s}^{t} F(X)dX^{1}$$

$$= \lim_{|D| \to 0} \sum_{\ell} \left(F(X_{t_{\ell-1}}) X_{t_{\ell-1},t_{\ell}}^{1} + DF(X_{t_{\ell-1}}) X_{t_{\ell-1},t_{\ell}}^{2} \right)$$

and the second level

$$(4.4) Y_{s,t}^2 = \int_s^t F(X)dX^2$$

$$= \lim_{|D| \to 0} \sum_{\ell} \left(Y_{s,t_{\ell-1}}^1 \otimes Y_{t_{\ell-1},t_{\ell}}^1 + F(X_{t_{\ell-1}}) \otimes F(X_{t_{\ell-1}}) X_{t_{\ell-1},t_{\ell}}^2 \right).$$

These limits exist in the p-variation distance.

Apply the general definition above, we get the integral against the Stratonovich fractional Brownian rough path B^S , $X^S := \int F(B^S) dB^S$. For simplicity we will use $X^S = \int F(B) \circ dB$ to denote this Stratonovich integral, which is by definition the integral against rough path B^S .

Respectively, we also define the Itô integral $\int F(B^I)dB^I$ against rough path B^I (denoted by B in what follows), and denote it as $X^I := \int F(B)dB$.

4.1.1. Relation between Stratonovich and Itô rough integrals (I). Now we want to establish a relation between Stratonovich and Itô integrals.

Theorem 4.1. The relation between Stratonovich and Itô integral is given as the following. (i) For the first level,

(4.5)
$$(X^S)_{s,t}^1 - (X^I)_{s,t}^1 = \frac{1}{2} \int_0^t DF(B_u) du^{2H},$$

(ii) For the second level,

$$(X^{S})_{s,t}^{2} - (X^{I})_{s,t}^{2} = \frac{1}{2} \int_{s}^{t} F(B_{u}) \otimes F(B_{u}) du^{2H}$$

$$+ \frac{1}{2} \int_{s}^{t} \left(\int_{s}^{u} DF(B_{r}) dr^{2H} \right) \otimes d(X^{S})_{0,u}^{1}$$

$$+ \frac{1}{2} \int_{s}^{t} (X^{S})_{s,u}^{1} \otimes DF(B_{u}) du^{2H}$$

$$- \frac{1}{4} \int_{s}^{t} \left(\int_{s}^{u} DF(B_{r}) dr^{2H} \right) DF(B_{u}) du^{2H},$$

where the last four integrals are Young integrals.

This theorem is a corollary of Theorem 4.2 below.

4.1.2. Relation between Stratonovich and Itô rough integrals (II). Let us introduce the spacetime Stratonovich/Itô path $\widetilde{B} = (B, t)$, where the first level is given by

$$\widetilde{B}_{s,t}^{1} = (B_{s,t}^{1}, t - s),$$

and the second level is given by

$$\widetilde{B}_{s,t}^2 = \left(B_{s,t}^2, \int_s^t B_{s,u}^1 du, \int_s^t (u-s)dB_u, \frac{1}{2}(t-s)^2\right),$$

where the cross integrals are Young integrals, and B^2 is the Stratonovich or Itô lift of fBM. Naturally

(4.7)
$$\int F(B,t)dB := \int f(\widetilde{B})d\widetilde{B},$$

with $f(x,t)(\xi,\tau)=F(x,t)\xi$ and the right hand side is well defined as an integral for rough paths. We use the symbol $\int F(B,t)dB=:X^I$ as Itô integral and the symbol $\int F(B,t)\circ dB=:X^S$ as Stratonovich integral. We can establish the relationship between the Stratonovich and Itô integrals, too.

Theorem 4.2. The relation between Stratonovich and It integral is given as the following. (i) For the first level,

(4.8)
$$(X^S)_{s,t}^1 - (X^I)_{s,t}^1 = \frac{1}{2} \int_s^t D_x F(B_u, u) du^{2H},$$

(ii) For the second level,

$$(X^{S})_{s,t}^{2} - (X^{I})_{s,t}^{2} = \frac{1}{2} \int_{s}^{t} F(B_{u}, u) \otimes F(B_{u}, u) du^{2H}$$

$$+ \frac{1}{2} \int_{s}^{t} \left(\int_{s}^{u} D_{x} F(B_{r}) dr^{2H} \right) \otimes d(X^{S})_{0,u}^{1}$$

$$+ \frac{1}{2} \int_{s}^{t} (X^{S})_{s,u}^{1} \otimes D_{x} F(B_{u}) du^{2H}$$

$$- \frac{1}{4} \int_{s}^{t} \left(\int_{s}^{u} D_{x} F(B_{r}, r) dr^{2H} \right) D_{x} F(B_{u}, u) du^{2H},$$

where the last four integrals are Young integrals.

Proof. (i) By definition of our integral, we have

$$\int_{s}^{t} F(B, u)dB^{1} = \int_{s}^{t} f(\widetilde{B})d\widetilde{B}^{1}$$

$$= \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} f(\widetilde{B}_{u})\widetilde{B}_{u,v}^{1} + Df(\widetilde{B}_{u})\widetilde{B}_{u,v}^{2}$$

$$= \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} F(B_{u}, u)B_{u,v}^{1} + D_{x}F(B_{u}, u)B_{u,v}^{2}$$

$$+ D_{u}F(B_{u}, u) \int_{u}^{v} B_{u,r}^{1} dr.$$

Since

$$\left| \int_{u}^{v} B_r dr - B_u(v - u) \right| = o(|v - u|) = o(|\mathcal{P}|),$$

so that

$$\int_{s}^{t} F(B, u) dB^{1} = \lim_{|\mathcal{P}| \to 0} \sum_{[u, v] \in \mathcal{P}} F(B_{u}, u) B_{u, v}^{1} + D_{x} F(B_{u}, u) B_{u, v}^{2}.$$

Therefore we may conclude that

$$\int_{s}^{t} F(B, u)dB^{1}$$

$$= \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} F(B_{u}, u)B_{u,v}^{1} + D_{x}F(B_{u}, u)((B^{S})_{u,v}^{2} - \frac{1}{2}I(v^{2H} - u^{2H}))$$

$$= \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} \left(F(B_{u}, u)B_{u,v}^{1} + D_{x}F(B_{u}, u)(B^{S})_{u,v}^{2} \right)$$

$$- \frac{1}{2} \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} D_{x}F(B_{u}, u)(v^{2H} - u^{2H})$$

$$= \int_{s}^{t} F(B, u) \circ dB^{1} - \frac{1}{2} \int_{s}^{t} D_{x}F(B_{u}, u)du^{2H}.$$

(ii) Now we show the second relation (4.9). It follows in the similar way as (i).

$$\int_{s}^{t} F(B, u) dB^{2} = \int_{s}^{t} f(\widetilde{B}) d\widetilde{B}^{2}$$

$$= \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} (X^{I})_{s,u}^{1} \otimes (X^{I})_{u,v}^{1} + f(\widetilde{B}_{u}) \otimes f(\widetilde{B}_{u}) \widetilde{B}_{u,v}^{2}.$$

By eqn (4.8), and $f(\widetilde{B}_u) \otimes f(\widetilde{B}_u)\widetilde{B}_{u,v}^2 = F(B_u, u) \otimes F(B_u, u)B_{u,v}^2$, we obtain

$$\begin{split} &\int_{s}^{t} F(B,u)dB^{2} \\ &= \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} \left((X^{S})_{s,u}^{1} - \frac{1}{2} \int_{s}^{u} D_{x} F(B_{r},r) dr^{2H} \right) \\ &\otimes \left((X^{S})_{u,v}^{1} - \frac{1}{2} \int_{u}^{v} D_{x} F(B_{r},r) dr^{2H} \right) \\ &+ F(B_{u},u) \otimes F(B_{u},u) \left((B^{S})_{u,v}^{2} - \frac{1}{2} (v^{2H} - u^{2H}) \right) \\ &= \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} (X^{S})_{s,u}^{1} \otimes (X^{S})_{u,v}^{1} + F(B_{u},u) \otimes F(B_{u},u) (B^{S})_{u,v}^{2} \\ &- \frac{1}{2} F(B_{u},u) \otimes F(B_{u},u) \left(v^{2H} - u^{2H} \right) \\ &- \frac{1}{2} \int_{s}^{u} D_{x} F(B_{r},r) dr^{2H} \otimes (X^{S})_{u,v}^{1} \\ &- (X^{S})_{s,u}^{1} \otimes \frac{1}{2} \int_{u}^{v} D_{x} F(B_{r},r) dr^{2H} \\ &+ \frac{1}{4} \int_{s}^{u} D_{x} F(B_{r},r) dr^{2H} \int_{u}^{v} D_{x} F(B_{r},r) dr^{2H}. \end{split}$$

Since B_t has finite p-variation with p > 1/H, therefore B^2 is of finite p/2-variation, $F(B_t, t)$, $D_x F(B_t, t)$ have finite p-variations, $\int_0^r D_x F(B_r, r) dr^{2H}$ has finite 1/2H-variation and $(X^S)^1$ has finite p-variation. Since $\frac{1}{H} and <math>\frac{1}{3} < H < \frac{1}{2}$, so that $\frac{1}{p} + 2H > 1$, and the last four sums on the right hand side converge to the Young integral. Eqn (4.9) therefore follows immediately.

4.2. The Itô formula.

4.2.1. Homogeneous Itô type formula. Let $X = (1, X^1, X^2) \in \Omega_p(\mathbb{R}^d)$, $2 \leq p < 3$ be a p-rough path, denote $X_t := X_{0,t}^1$, and $F : \mathbb{R}^d \to L(\mathbb{R}^d, \mathbb{R}^e)$ be a Lip (γ) function for some $\gamma > p$. Since we often do composition with rough paths, we want to lift the function $F(X^1)$ to a rough path $F_{\mathfrak{R}}(X) = (1, F_{\mathfrak{R}}(X)^1, F_{\mathfrak{R}}(X)^2)$. In terms of rough path integrals, we use the formula

(4.10)
$$F_{\mathfrak{R}}(X) = \int DF(X)dX$$

as a definition of image $F_{\mathfrak{R}}$ of function F. Actually $F_{\mathfrak{R}}$ sends a rough path to another rough path, so that $F_{\mathfrak{R}}: \Omega_p(\mathbb{R}^d) \to \Omega_p(\mathbb{R}^e)$, $2 \le p < 3$. By the definition of rough path integrals,

(4.11)
$$F_{\mathfrak{R}}(X)_{s,t}^{1} = \int_{s}^{t} DF(X)dX^{1}$$

$$= \lim_{|D| \to 0} \sum_{\ell} \left(DF(X_{t_{\ell-1}}) X_{t_{\ell-1},t_{\ell}}^{1} + D^{2}F(X_{t_{\ell-1}}) X_{t_{\ell-1},t_{\ell}}^{2} \right),$$

(4.12)
$$F_{\Re}(X)_{s,t}^{2} = \int_{s}^{t} DF(X)dX^{2}$$

$$= \lim_{|D| \to 0} \sum_{\ell} \left(F_{\Re}(X)_{s,t_{\ell-1}}^{1} \otimes F_{\Re}(X)_{t_{\ell-1},t_{\ell}}^{1} + DF(X_{t_{\ell-1}}) \otimes DF(X_{t_{\ell-1}}) X_{t_{\ell-1},t_{\ell}}^{2} \right).$$

Let $X(\phi)_{s,t} = (1, X(\phi)_{s,t}^1, X(\phi)_{s,t}^2) := (1, X_{s,t}^1, X_{s,t}^2 - \phi_{s,t})$ be a perturbation of the rough path $X = (1, X^1, X^2)$, where $\phi_{s,t} = \phi_t - \phi_s$ (additive) is finite q-variation with $q \leq \frac{p}{2}$. Assume that X is a geometric rough path, then $X(\phi)$ is no longer a geometric rough path in general. Define the composition $F_{\mathfrak{R}}(X(\phi)) = (1, F_{\mathfrak{R}}(X(\phi))^1, F_{\mathfrak{R}}(X(\phi))^2)$ as

(4.13)
$$F_{\mathfrak{R}}(X(\phi)) = \int DF(X(\phi))dX(\phi).$$

We have the following Itô type formula.

Theorem 4.3. (Itô type formula) Assume that $X \in G\Omega_p(\mathbb{R}^d)$ with $2 \leq p < 3$, $X(\phi)$ is a perturbation of the rough path X as above, and $F : \mathbb{R}^d \to L(\mathbb{R}^d, \mathbb{R}^e)$ is a $Lip(\gamma)$ function for some $\gamma > p$, then

- (i) $F_{\mathfrak{R}}(X)^{1}_{s,t} = F(X_t) F(X_s).$
- (ii) For the first level,

$$F(X_t) - F(X_s) = \int_s^t DF(X(\phi)) dX(\phi)^1 + \int_s^t D^2 F(X_u) d\phi_u.$$

(iii) For the second level,

$$F_{\mathfrak{R}}(X)_{s,t}^{2} = \int_{s}^{t} DF(X(\phi))dX(\phi)^{2} + \int_{s}^{t} DF(X_{u}) \otimes DF(X_{u})d\phi_{u}$$

$$+ \int_{s}^{t} \left(\int_{s}^{u} D^{2}F(X_{r})d\phi_{r}\right) \otimes dF(X_{u})$$

$$+ \int_{s}^{t} (F(X_{u}) - F(X_{s})) \otimes D^{2}F(X_{u})d\phi_{u}$$

$$+ \int_{s}^{t} \left(\int_{s}^{u} D^{2}F(X_{r})d\phi_{r}\right) D^{2}F(X_{u})d\phi_{u},$$

where the last four integrals are Young integrals, and $F_{\mathfrak{R}}(X)_{s,t}^2$ can be viewed as a kind of geometric increments of the second level process.

Proof. (i) The equality holds for any continuous path X with finite variation and its canonical lift as rough paths. Then by definition of geometric rough paths, it can be approximated by a sequence of path with finite variation in p-variation. By continuity of both sides, we know the equality still holds. (ii) and (iii) can be proved by the same arguments as in Theorem 4.2.

Remark 4.4. In integration theory for rough paths, if $X \in \Omega_p$, $2 \le p < 3$, $X_t = X_{0,t}^1$, one cannot just write a symbol $dF(X_t)$ as the ordinary case. There is no meaning for this symbol unless under the sense of Young integrals, if it is well defined. We can see an example below which says that the $dF(X_t)$ is undefined in general. Actually if we want to make sense the

differential symbol, we should lift $F(X_t)$ to a rough path $F_{\mathfrak{R}}(X)$ as above and then understand the differential symbol as

$$dF_{\mathfrak{R}}(X) = d(F_{\mathfrak{R}}(X)^{1}, F_{\mathfrak{R}}(X)^{2}).$$

In order to clarify the remark above, first we give a lemma below.

Lemma 4.5. Let $Y = (1, Y^1, Y^2)$ be a rough path. Then the integral

(4.14)
$$\int_{s}^{t} dY = (1, Y_{s,t}^{1}, Y_{s,t}^{2})$$

as expected.

Example 1. Take Y as $F_{\mathfrak{R}}(X)$, $F_{\mathfrak{R}}(X(\phi))$. Then by lemma 4.5, we have

$$\int_{s}^{t} dF_{\Re}(X) = (1, F_{\Re}(X)_{s,t}^{1}, F_{\Re}(X)_{s,t}^{2}),$$

$$\int_{s}^{t} dF_{\Re}(X(\phi)) = (1, F_{\Re}(X(\phi))_{s,t}^{1}, F_{\Re}(X(\phi))_{s,t}^{2}).$$

Actually $F(X(\phi)_t) = F(X_t)$, but $dF_{\mathfrak{R}}(X) \neq dF_{\mathfrak{R}}(X(\phi))$, even for the first level as we can see that $F_{\mathfrak{R}}(X)_{s,t}^1 \neq F_{\mathfrak{R}}(X(\phi))_{s,t}^1$ by It formula above. Therefore the symbol $dF(X_t)$ or $dF(X(\phi)_t)$ for rough paths can lead to confusion.

Remark 4.6. If X is a geometric rough path, by Theorem 4.3, we have

$$F_{\mathfrak{R}}(X)_{s,t}^1 = F(X_t^1) - F(X_s^1),$$

i.e. $F_{\mathfrak{R}}(X)_{s,t}^1 = F(X^1)_{s,t}$. However, for the non-geometric rough path we do not have the equality alike. In fact, in general,

$$F_{\Re}(X(\phi))_{s,t}^1 \neq F(X(\phi)_t^1) - F(X(\phi)_s^1) (= F(X_t^1) - F(X_s^1)).$$

We next want to establish an Itô type formula for integrals against Itô fractional Brownian rough path. As a corollary, we have

Corollary 4.7. (Itô formula for fBM) Let B^S be fractional Brownian rough path with Hurst parameter $\frac{1}{3} < H \leq \frac{1}{2}$ enhanced under Stratonovich sense, B be the Itô fractional Brownian rough path, and $F: \mathbb{R}^d \to L(\mathbb{R}^d, \mathbb{R}^e)$ be a $\operatorname{Lip}(\gamma)$ function for some $H\gamma > 1$. Then

(i) $F(B_t) - F(B_s) = F_{\mathfrak{R}}(B^S)_{s,t}^1 = \int_s^t DF(B) \circ dB^1$

(ii) For the first level,

$$F(B_t) - F(B_s) = \int_s^t DF(B)dB^1 + \frac{1}{2} \int_s^t D^2F(B_u)du^{2H}.$$

(iii) For the second level,

$$F_{\Re}(B^{S})_{s,t}^{2} = \int_{s}^{t} DF(B)dB^{2} + \frac{1}{2} \int_{s}^{t} DF(B_{u}) \otimes DF(B_{u})du^{2H}$$

$$+ \frac{1}{2} \int_{s}^{t} \left(\int_{s}^{u} D^{2}F(B_{r})dr^{2H} \right) \otimes dF(B_{u})$$

$$+ \frac{1}{2} \int_{s}^{t} (F(B_{u}) - F(B_{s})) \otimes D^{2}F(B_{u})du^{2H}$$

$$- \frac{1}{4} \int_{s}^{t} \left(\int_{s}^{u} D^{2}F(B_{r})dr^{2H} \right) \otimes D^{2}F(B_{u})du^{2H},$$

where the last four integrals are Young integrals.

4.2.2. Inhomogeneous Itô formula. In the following, we want to make sense for $F_{\mathfrak{R}}((X,t))$ when the inhomogeneous function F(x,t) applied to a rough path $X=(1,X^1,X^2)\in\Omega_p$ and establish Itô formula for it. First, recall the space-time rough path $\widetilde{X}=(X,t)$, where the first level is given by

$$\widetilde{X}_{s,t}^{1} = (X_{s,t}^{1}, t - s)$$

and the second level is given by

$$\widetilde{X}_{s,t}^2 = \left(X_{s,t}^2, \int_s^t X_{s,u}^1 du, \int_s^t (u-s)dX_u^1, \frac{1}{2}(t-s)^2\right),$$

where the cross integrals are Young integrals. Define the rough path $F_{\mathfrak{R}}((X,t))$ by

(4.15)
$$F_{\mathfrak{R}}((X,t)) := F_{\mathfrak{R}}(\widetilde{X}) = \int DF(\widetilde{X})d\widetilde{X},$$

where $DF(x,t)(\xi,\tau) = D_x F(x,t)\xi + D_t F(x,t)\tau$.

Note that if $X(\phi)$ is a perturbation of the rough path X, and $\widetilde{X}(\phi)$ is its associated space-time rough path, then

$$\widetilde{X}(\phi)_{s,t}^{1} = \widetilde{X}_{s,t}^{1},$$

$$\widetilde{X}(\phi)_{s,t}^{2} = \left(X_{s,t}^{2} - \phi_{s,t}, \int_{s}^{t} X_{s,u}^{1} du, \int_{s}^{t} (u - s) dX_{u}^{1}, \frac{1}{2} (t - s)^{2}\right).$$

Note that only the first $d \times d$ dimensional components of the second level of \widetilde{X} are changed.

Theorem 4.8. (Itô formula) Assume $X \in G\Omega_p(\mathbb{R}^d)$ with $2 \leq p < 3$, $X(\phi)$ is a perturbation of the rough path X, and \widetilde{X} , $\widetilde{X}(\phi)$ are their associated space-time rough path respectively, $F: \mathbb{R}^{d+1} \to L(\mathbb{R}^{d+1}, \mathbb{R}^e)$ be a Lip (γ) function for some $\gamma > p$.

(i) We have the basic calculus formula:

(4.16)
$$F(X_t, t) - F(X_s, s) = \int_s^t DF(\widetilde{X}) d\widetilde{X}^1.$$

(ii) For the first level,

$$(4.17) F(X_t,t) - F(X_s,s) = \int_0^t DF(\widetilde{X}(\phi))d\widetilde{X}(\phi)^1 + \int_0^t D_x^2 F(X_u,u)d\phi_u.$$

(iii) For the second level,

$$F_{\mathfrak{R}}((X,t))_{s,t}^{2} = \int_{s}^{t} DF(\widetilde{X}(\phi))d\widetilde{X}(\phi)^{2} + \int_{s}^{t} D_{x}F(X_{u},u) \otimes D_{x}F(X_{u},u)d\phi_{u}$$

$$+ \int_{s}^{t} \left(\int_{s}^{u} D_{x}^{2}F(X_{r},r)d\phi_{r}\right) \otimes dF(X_{u},u)$$

$$+ \int_{s}^{t} (F(X_{u},u) - F(X_{s},s)) \otimes D_{x}^{2}F(X_{u},u)d\phi_{u}$$

$$+ \int_{s}^{t} \left(\int_{s}^{u} D_{x}^{2}F(X_{r},r)d\phi_{r}\right) \otimes D_{x}^{2}F(X_{u},u)d\phi_{u},$$

where the last four integrals are Young integrals.

Proof. (i) The proof is same as (i) of Theorem 4.3, first for p = 1 it holds, then the result for any geometric rough path follows from the continuity. For (ii) and (iii), note that

$$\left| \int_{s}^{t} X_{s,u}^{1} du \right| = o(|t - s|)$$

and

$$\left| \int_{s}^{t} (u-s)dX_{u} \right| = o(|t-s|),$$

the rest of proof is same as Theorem 4.2.

Note that if $t \to X_t := X_{0,t}^1$ is a continuous path with finite variation, then eqn (4.16) reads as

(4.18)
$$F(X_t, t) - F(X_s, s) = \int_s^t D_x F(X_u, u) dX_u + \int_s^t D_u F(X_u, u) du,$$

and eqn (4.17) becomes

(4.19)
$$F(X_t, t) - F(X_s, s) = \int_s^t D_x F(X_u, u) dX_u + \int_s^t D_u F(X_u, u) du + \int_s^t D_x^2 F(X_u, u) d\phi_u.$$

These equations are just like Itô formulas in terms of the Stratonovich and Itô integrals in stochastic calculus.

Now set \widetilde{B}^S , \widetilde{B} are the associated space-time rough paths of Stratonovich fractional Brownian rough path B^S and Itô rough path B, respectively.

Corollary 4.9. (Itô formula for fractional Brownian rough path) Let B^S be fractional Brownian rough path with Hurst parameter $\frac{1}{3} < H \le \frac{1}{2}$ enhanced under Stratonovich sense, B^I be the Itô fractional Brownian rough path (say B for short), and $F : \mathbb{R}^{d+1} \to L(\mathbb{R}^{d+1}, \mathbb{R}^e)$ be a Lip (γ) function for some $H\gamma > 1$, then

(i)
$$F(B_t, t) - F(B_s, s) = F_{\mathfrak{R}}((B^S, t))_{s,t}^1 = \int_s^t DF(\widetilde{B}) \circ d\widetilde{B}^1$$

(ii) For the first level,

(4.20)
$$F(B_t, t) - F(B_s, s) = \int_s^t DF(\widetilde{B}) d\widetilde{B}^1 + \frac{1}{2} \int_s^t D_x^2 F(B_u, u) du^{2H}.$$

(iii) For the second level,

$$F_{\Re}((B^{S},t))_{s,t}^{2} = \int_{s}^{t} DF(\widetilde{B})d\widetilde{B}^{2} + \frac{1}{2} \int_{s}^{t} D_{x}F(B_{u},u) \otimes D_{x}F(B_{u},u)du^{2H}$$

$$+ \frac{1}{2} \int_{s}^{t} \left(\int_{s}^{u} D_{x}^{2}F(B_{r},r)dr^{2H} \right) \otimes dF(B_{u},u)$$

$$+ \frac{1}{2} \int_{s}^{t} (F(B_{u},u) - F(B_{s},s)) \otimes D_{x}^{2}F(B_{u},u)du^{2H}$$

$$- \frac{1}{4} \int_{s}^{t} \left(\int_{s}^{u} D_{x}^{2}F(B_{r},r)dr^{2H} \right) \otimes D_{x}^{2}F(B_{u},u)du^{2H},$$

where the last four integrals are Young integrals.

4.3. Differential equations driven by rough paths.

4.3.1. Basics of differential equations. In this subsection, we readdress some basic facts about differential equations driven by rough paths, following the description of [24]. We recall a definition of differential equations driven by rough paths below.

Definition 4.10. Let $f: W \to L(V, W)$ be a vector field on W. Let $y_0 \in W$ and let $X \in \Omega_p(V)$, for $2 \le p < 3$. Then we say that a rough path $Y \in \Omega_p(W)$ is a solution to the following initial value problem:

$$(4.22) dY = f(Y)dX, Y_0 = y_0$$

if there is a rough path $Z \in \Omega_p(V \oplus W)$ such that $\pi_V(Z) = X$, $\pi_W(Z) = Y$ and

$$(4.23) Z = \int \widehat{f}(Z)dZ,$$

where $\widehat{f}: V \oplus W \to L(V \oplus W, V \oplus W)$ is defined by

$$\widehat{f}(x,y)(\xi,\eta) = (\xi, f(y_0+y)\xi), \ \forall (x,y), (\xi,\eta) \in V \oplus W.$$

If the vector field $f \in C^3(W, L(V, W))$ satisfies the linear growth and Lipschitz continuous conditions, then the existence and uniqueness of a solution are ensured (see [24], Theorem 6.2.1, Corollary 6.2.2). We will use $\Phi(y_0, X)$ to denote the unique solution Y, call the map $X \to \Phi(y_0, X)$ Itô map defined by differential equation (4.22), whose Lipschitz continuity in p-variation topology can be proved under the same conditions above (see [24], Theorem 6.2.2). This is an important result of Itô maps in the framework of differential equations driven by rough paths.

4.3.2. Relation between differential equations driven by different rough paths. In this subsection, our main goal is to show the relationship of differential equations driven by Stratonovich fractional Brownian rough path and Itô fractional Brownian rough path, respectively. Define $\Phi(x, B^S)$ as the Itô map of the differential equation

(4.24)
$$dX = f(X)dB^{S}, \ X_{0} = x,$$

where B^S is the Stratonovich fractional Brownian rough path, and we use dB^S to denote the equation driven by Stratonovich rough path, which sometime we use $\circ dB$ instead. Respectively, let I(x,B) denote the Itô map of the differential equations driven by the Itô fractional rough path

(4.25)
$$dX = f(X)dB, \ X_0 = x.$$

Now we want to ask what is the relationship between I(x, B) and $\Phi(x, B^S)$ or if the Itô differential equation has a representation in terms of a Stratonovich differential equation. First, we introduce a geometric rough path $B^{S,\varphi}$ defined by

$$(B^{S,\varphi})_{s,t}^1 := (B_{s,t}^1, t^{2H} - s^{2H}),$$

and

$$(B^{S,\varphi})_{s,t}^2 := \left((B^S)_{s,t}^2, \int_s^t B_{s,u}^1 du^{2H}, \right.$$
$$\left. \int_s^t (u^{2H} - s^{2H}) dB_u, \frac{1}{2} (t^{2H} - s^{2H})^2 \right),$$

where the cross integrals are Young integrals. Let $\Phi_{\tilde{f}}(x, B^{S,\varphi})$ be the Itô map defined by the differential equation

(4.26)
$$dX = \widetilde{f}(X)dB^{S,\varphi}, \ X_0 = x,$$

where $\widetilde{f}: \mathbb{R}^e \to L(\mathbb{R}^d \oplus \mathbb{R}, \mathbb{R}^e)$,

$$\widetilde{f}(x)(\xi,\eta) := f(x)\xi - \frac{1}{2}Df(x)f(x)(\eta),$$

for all $x \in \mathbb{R}^e$, $(\xi, \eta) \in \mathbb{R}^d \oplus \mathbb{R}$. Namely, $\Phi_{\widetilde{f}}(\cdot, B^{S,\varphi})$ is the Itô map of the rough differential equation

(4.27)
$$dX = f(X)dB^{S} - \frac{1}{2}Df(X_{t})f(X_{t})dt^{2H}.$$

Theorem 4.11. Let $f: \mathbb{R}^e \to L(\mathbb{R}^d, \mathbb{R}^e)$ be a C^4 vector field, and I(x, B), $\Phi_{\widetilde{f}}(x, B^{S,\varphi})$ be the Itô map defined by (4.25), (4.26), respectively. Then

$$(4.28) I(x,B)_{s,t}^{1} = \Phi_{\tilde{f}}(x,B^{S,\varphi})_{s,t}^{1}$$

and

(4.29)
$$I(x,B)_{s,t}^{2} = \Phi_{\widetilde{f}}(x,B^{S,\varphi})_{s,t}^{2} - \frac{1}{2} \int_{s}^{t} f(X_{u}) \otimes f(X_{u}) du^{2H},$$

where $X_t := \Phi_{\widetilde{f}}(x, B^{S,\varphi})_{0,t}^1$.

As a remark, we can see that the differential equations (4.25) and (4.26) (or (4.27)) are not equivalent. As far as the first level concerned, they define the same solution. This agrees with classical stochastic differential equations (SDE) driven by standard Brownian motion, i.e. we translate SDE (4.25) into SDE (4.27) when H = 1/2. However, in terms of the second level, two differential equations are different. The reason is that $I(x, B)^2$ can be viewed as the iterated integral of the first level in Itô's sense, while $\Phi_{\tilde{f}}(x, B^{S,\varphi})^2$ is the iterated integral of the first level in Stratonovich sense.

Besides, the relationship between $\Phi(x,B^S)$ and $\Phi_{\widetilde{f}}(x,B^{S,\varphi})$ is obvious. They are all understood in the Stratonovich sense, the difference is their drift terms. By the continuity of Itô's maps, they can all be approximated in variation topology by the solution of differential equations driven by piece-wise linear approximations of fBM and its iterated path integrals. In summary, we can establish the relationship between I(x,B), $\Phi(x,B^S)$ and $\Phi_{\widetilde{f}}(x,B^{S,\varphi})$.

Proof. Suppose $\{\overline{B}_t, t \geq 0\}$ is a piece-wise linear/smooth approximation with finite variation of fractional Brownian motion $\{B_t, t \geq 0\}$. Set $\overline{B}_{s,t}^1 = \overline{B}_t - \overline{B}_s$ and let $\overline{B}_{s,t}^2$ be the difference of the iterated integral over [s,t] of \overline{B}^1 and $\frac{1}{2}(t^{2H} - s^{2H})$. Consider the rough differential equation

$$(4.30) dX = f(X)d\overline{B}.$$

that is, the integral equation

(4.31)
$$Z = \int \widehat{f}(Z)dZ, \ \pi_d(Z) = \overline{B},$$

where $\widehat{f}(x,y)(\xi,\eta) := (\xi, f(y)\xi)$, and π_d is projection operator to \mathbb{R}^d , which is solved by the Picard iteration, that is

$$Z(n+1) = \int \widehat{f}(Z(n))dZ(n), \quad Z(0) = (\overline{B}, 0).$$

More precisely, define almost rough paths

(4.32)
$$\widehat{Z}(n+1)_{s,t}^1 := \widehat{f}(Z(n)_s)Z(n)_{s,t}^1 + D\widehat{f}(Z(n)_s)Z(n)_{s,t}^2,$$

(4.33)
$$\widehat{Z}(n+1)_{s,t}^{2} := \widehat{f}(Z(n)_{s}) \otimes \widehat{f}(Z(n)_{s}) Z(n)_{s,t}^{2},$$

and define the corresponding rough paths

(4.34)
$$Z(n+1)_{s,t}^{1} = \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} \widehat{Z}(n+1)_{u,v}^{1},$$

(4.35)
$$Z(n+1)_{s,t}^2 = \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} Z(n)_{s,u}^1 \otimes Z(n)_{u,v}^1 + \widehat{Z}(n+1)_{u,v}^2,$$

where \mathcal{P} is a partition of the interval [s,t]. Then

$$(4.36) |Z(n)_{s,t}^i - \widehat{Z}(n)_{s,t}^i| \le \omega(s,t)^{\theta}, \ i = 1, 2, \ \theta > 1,$$

for some control ω .

(i) Now we prove (4.28). Set $\widehat{Z}(n)_{s,t}^1 = (\overline{B}_{s,t}^1, X(n)_{s,t}^1)$, by definition of \widehat{f} and (4.32), we have

$$\widehat{Z}(n+1)_{s,t}^{1} = (\overline{B}_{s,t}^{1}, f(X(n)_{s})\overline{B}_{s,t}^{1}) + D\widehat{f}(Z(n)_{s})Z(n)_{s,t}^{2}
\simeq (\overline{B}_{s,t}^{1}, f(X(n)_{s})\overline{B}_{s,t}^{1}) + D\widehat{f}(Z(n)_{s})\widehat{Z}(n)_{s,t}^{2}
\simeq (\overline{B}_{s,t}^{1}, f(X(n)_{s})\overline{B}_{s,t}^{1}) + (0, Df(X(n)_{s})f(X(n)_{s})\overline{B}_{s,t}^{2}
\simeq (\overline{B}_{s,t}^{1}, f(X(n)_{s})\overline{B}_{s,t}^{1}) - \frac{1}{2}(0, Df(X(n)_{s})f(X(n)_{s})(t^{2H} - s^{2H}),$$

where \simeq means the error can be controlled by $\omega(s,t)^{\theta}$ with $\theta > 1$. Hence,

$$X(n+1)_{s,t}^1 \simeq f(X(n)_s)\overline{B}_{s,t}^1 - \frac{1}{2}Df(X(n)_s)f(X(n)_s)(t^{2H} - s^{2H}).$$

Since \overline{B}^1 has finite variation and $\sum_{[u,v]\in\mathcal{P}} X(n)^1_{u,v} = X(n)^1_{s,t}$, the formula above implies that

$$X(n+1)_{s,t}^{1} = \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} f(X(n)_u) \overline{B}_{u,v}^{1} - \frac{1}{2} Df(X(n)_u) f(X(n)_u) (v^{2H} - u^{2H}),$$

as n goes to infinity, the limit above is identified as

$$X_t = X_s + \int_s^t f(X_u)d\overline{B}_u - \frac{1}{2} \int_s^t Df(X_u)f(X_u)du^{2H}.$$

On the other hand,

$$\lim_{n \to \infty} X(n)_{s,t}^1 = \Phi(\cdot, \overline{B})_{s,t}^1,$$

so we get (4.28) when the system is driven by \overline{B} . By the continuity of Itô's maps, we conclude that (4.28) holds in real fractional Brownian rough path case.

- (ii) Similarly, by (4.35) and the continuity of Itô's maps again, we conclude that (4.29) holds. $\hfill\Box$
- 4.4. **Examples and applications.** In this subsection, we will give some interesting examples as applications of fractional Brownian rough paths.

Example 2. Consider the differential equation in dimension d=1,

$$dX = \sigma X \circ dB$$
.

Then $X = (1, X^1, X^2)$ is the solution of this differential equation, where $X_t = \exp(\sigma B_t)$,

$$X_{s,t}^1 = X_t - X_s = \exp(\sigma B_t) - \exp(\sigma B_s),$$

$$X_{s,t}^2 = \frac{1}{2}(X_{s,t}^1)^2 = \frac{1}{2}(\exp(\sigma B_t) - \exp(\sigma B_s))^2.$$

Example 3. Now we consider the Itô rough differential equation

$$dX = \sigma X dB$$
.

(i) Set $X_t = \exp(\sigma B_t - \frac{1}{2}\sigma^2 t^{2H}) =: F(B_t, t)$ and $X_{s,t}^1 := X_t - X_s$. Then

$$X_{s,t}^1 = \int_s^t \sigma X dB^1,$$

and

$$X_{s,t}^{2} := \int_{s}^{t} \sigma F(B, u) \circ dB^{2} - \frac{\sigma^{2}}{2} \int_{s}^{t} (F(B_{u}, u))^{2} du^{2H}$$

$$- \frac{\sigma^{2}}{2} \int_{s}^{t} \left(\int_{s}^{u} F(B_{r}, r) dr^{2H} \right) dY_{u} - \frac{\sigma^{2}}{2} \int_{s}^{t} Y_{s,u}^{1} F(B_{u}, u) du^{2H}$$

$$+ \frac{\sigma^{4}}{4} \int_{s}^{t} \left(\int_{s}^{u} F(B_{r}, r) dr^{2H} \right) F(B_{u}, u) du^{2H},$$

where $Y_{s,t}^1 = \int_s^t \sigma F(B,u) \circ dB^1$, $Y_t := Y_{0,t}^1$ and the last four integrals are Young integrals. **Example 4.** Geometric fBM (or fractional Black-Scholes model). First consider the fractional Black-Scholes model in Stratonovich sense

$$(4.37) dX = \mu X_t dt + \sigma X \circ dB,$$

where μ, σ are constants. The solution X can be constructed as above. Set

$$(4.38) X_t = \exp\left(\sigma B_t + \mu t\right) =: F(B_t, t),$$

and

$$X_{s,t}^1 = X_t - X_s,$$

$$X_{s,t}^2 = \lim_{n \to \infty} \int_s^t F(B_u^n, u) dF(B_u^n, u),$$

where B_u^n is the linear approximation of B, i.e.

$$B_u^n = B_{t_{\ell-1}} + \frac{B_{t_{\ell}} - B_{t_{\ell-1}}}{t_{\ell} - t_{\ell-1}} (u - t_{\ell-1})$$

on interval $[t_{\ell-1}, t_{\ell}]$ and $\{t_{\ell}^n, \ell = 0, 1, \dots, n\}$ is any partition of [s, t]. We can verify $X = (1, X^1, X^2)$ is the solution.

In this situation, the corresponding fractional Black-Scholes market has arbitrage. We change the Stratonovich integral into an Itô integral, i.e. we consider the fractional differential equation in Itô's sense

$$(4.39) dX = \mu X_t dt + \sigma X dB.$$

We demonstrate that the corresponding Itô fractional Black-Scholes market is arbitrage free in a restricted sense.

(i) Let

(4.40)
$$X_t = \exp\left(\sigma B_t + \mu t - \frac{1}{2}\sigma^2 t^{2H}\right) =: F(B_t, t).$$

Then $X_{s,t}^1 := X_t - X_s$ and

$$X_{s,t}^1 = \int_s^t \sigma X dB^1 + \int_s^t \mu X_u du.$$

By the relation between Stratonovich integrals and Itô integrals in time dependent case, we have

$$RHS = \int_{s}^{t} \sigma F(B, u) dB^{1} + \int_{s}^{t} \mu F(B_{u}, u) du$$

$$= \int_{s}^{t} \sigma F(B, u) \circ dB^{1} - \frac{\sigma^{2}}{2} \int_{s}^{t} F(B_{u}, u) du^{2H} + \int_{s}^{t} \mu F(B_{u}, u) du$$

$$= \int_{s}^{t} D_{x} F(B, u) \circ dB^{1} + \int_{s}^{t} D_{u} F(B_{u}, u) du$$

$$= \int_{s}^{t} DF(\widetilde{B}) \circ d\widetilde{B}^{1} = F(\widetilde{B}_{t}) - F(\widetilde{B}_{s})$$

$$= F(B_{t}, t) - F(B_{s}, s) = X_{t} - X_{s} = LHS.$$

(ii) Now set

$$X_{s,t}^{2} := Z_{s,t}^{2} + \int_{s}^{t} \mu Z_{s,u}^{1} F(B_{u}, u) du + \int_{s}^{t} \left(\int_{s}^{u} \mu F(B_{r}, r) dr \right) dZ_{u}$$
$$+ \int_{s}^{t} \left(\int_{s}^{u} \mu F(B_{r}, r) dr \right) \mu F(B_{u}, u) du,$$

where

$$Z_{s,t}^{1} = X_{s,t}^{1} - \int_{s}^{t} \mu F(B_{u}, u) du, \ Z_{t} = Z_{0,t}^{1}$$

$$Z_{s,t}^{2} = \int_{s}^{t} \sigma F(B, u) \circ dB^{2} - \frac{\sigma^{2}}{2} \int_{s}^{t} (F(B_{u}, u))^{2} du^{2H}$$

$$- \frac{\sigma^{2}}{2} \int_{s}^{t} \left(\int_{s}^{u} F(B_{r}, r) dr^{2H} \right) dY_{u} - \frac{\sigma^{2}}{2} \int_{s}^{t} Y_{s,u}^{1} F(B_{u}, u) du^{2H}$$

$$+ \frac{\sigma^{4}}{4} \int_{s}^{t} \left(\int_{s}^{u} F(B_{r}, r) dr^{2H} \right) F(B_{u}, u) du^{2H},$$

and $Y_{s,t}^1 = \int_s^t \sigma F(B,u) \circ dB^1$, $Y_t = Y_{0,t}^1$, and all the integrals except ones involving $\circ dB$ are Young integrals, and the integral against $\circ dB$ is Stratonovich rough integral which can be

computed by linear approximations. Actually, by the relation between Stratonovich rough integrals and It rough integrals, we have

$$Z_{s,t}^1 = \int_s^t \sigma F(B, u) dB^1$$
, and $Z_{s,t}^2 = \int_s^t \sigma F(B, u) dB^2$.

If we define $f(x,t)(\xi,\tau) := \sigma F(x,t)\xi + \mu F(x,t)\tau$, $\widetilde{B} = (B,t)$ the space-time rough path of B, then $X_{s,t}^2 = \int_s^t f(\widetilde{B})d\widetilde{B}^2$. Combining (i) and (ii), we have verified that $X_{s,t} = \int_s^t f(\widetilde{B})d\widetilde{B}$. The right hand side indeed coincides with the right hand side of the differential equation (4.39). So we have constructed the solution of the Itô fractional Black-Scholes equation (4.39).

Remark 4.12. As a remark, we have two ways to understand the integrals on the right hand side of above differential equation with drift (4.39). On the one hand, we can define $f(x,t)(\xi,\tau) := \sigma F(x,t)\xi + \mu F(x,t)\tau$, then $X_{s,t} = \int_s^t f(\widetilde{B})d\widetilde{B}$ is well defined. On the other hand, we can define $g(x,t)(\xi,\tau) := \sigma F(x,t)\xi$, $h_t := \int_0^t \mu F(B_u,u)du$ and see $\int_s^t \sigma F(B,u)dB$ as $\int_s^t g(\widetilde{B})d\widetilde{B}$ (This integral is well defined). Then view the right hand side of differential equation (4.39) as a perturbation of $\int g(\widetilde{B})d\widetilde{B}$ by h. We want to say that the two ways are consistent, they give the same results, i.e.

$$(4.41) \qquad \int_{s}^{t} f(\widetilde{B})d\widetilde{B}^{1} = \int_{s}^{t} g(\widetilde{B})d\widetilde{B}^{1} + \int_{s}^{t} \mu F(B_{u}, u)du,$$

$$\int_{s}^{t} f(\widetilde{B})d\widetilde{B}^{2} = \int_{s}^{t} g(\widetilde{B})d\widetilde{B}^{2} + \int_{s}^{t} \mu Z_{s,u}^{1} F(B_{u}, u)du$$

$$+ \int_{s}^{t} \left(\int_{s}^{u} \mu F(B_{r}, r)dr \right) dZ_{u}$$

$$+ \int_{s}^{t} \left(\int_{s}^{u} \mu F(B_{r}, r)dr \right) \mu F(B_{u}, u)du,$$

where $Z_{s,t}^1 := \int_s^t g(\widetilde{B}) d\widetilde{B}^1$, $Z_t := Z_{0,t}^1$, and the last three integrals are Young Integral. Now we give a proof of eqn (4.41), (4.42) below.

Proof. (i) For the first level, we have

$$\int_{s}^{t} f(\widetilde{B})d\widetilde{B}^{1} = \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} f(\widetilde{B}_{u})\widetilde{B}_{u,v}^{1} + Df(\widetilde{B}_{u})\widetilde{B}_{u,v}^{2}$$

$$= \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} \sigma F(B_{u},u)B_{u,v}^{1} + \mu F(B_{u},u)(v-u)$$

$$+ \sigma D_{x}F(B_{u},u)B_{u,v}^{2} + \sigma D_{u}F(B_{u},u) \int_{u}^{v} (B_{r} - B_{u})dr$$

$$+ \mu D_{x}F(B_{u},u) \int_{u}^{v} (r-u)dB_{r} + \frac{\mu}{2}D_{u}F(B_{u},u)(v-u)^{2},$$
Since
$$\left| \int_{u}^{v} B_{r}dr - B_{u}(v-u) \right| = o(|v-u|) = o(|\mathcal{P}|)$$
and
$$\left| \int_{u}^{v} rdB_{r} - u(B_{v} - B_{u}) \right| = o(|v-u|) = o(|\mathcal{P}|),$$

we therefore have

$$\int_{s}^{t} f(\widetilde{B})d\widetilde{B}^{1} = \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} \sigma F(B_{u}, u) B_{u,v}^{1} + \mu F(B_{u}, u)(v - u)
+ \sigma D_{x} F(B_{u}, u) B_{u,v}^{2}
= \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} g(\widetilde{B}_{u}) \widetilde{B}_{u,v}^{1} + Dg(\widetilde{B}_{u}) \widetilde{B}_{u,v}^{2} + \mu F(B_{u}, u)(v - u)
= \int_{s}^{t} g(\widetilde{B}) d\widetilde{B}^{1} + \int_{s}^{t} \mu F(B_{u}, u) du,$$

which completes the proof of eqn (4.41).

(ii) For the second level,

$$\int_{s}^{t} f(\widetilde{B})d\widetilde{B}^{2}$$

$$= \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} X_{s,u}^{1} X_{u,v}^{1} + f(\widetilde{B}_{u}) \otimes f(\widetilde{B}_{u}) \widetilde{B}_{u,v}^{2}$$

$$= \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} \left(Z_{s,u}^{1} + \int_{s}^{u} \mu F(B_{r},r) dr \right) \left(Z_{u,v}^{1} + \int_{u}^{v} \mu F(B_{r},r) dr \right)$$

$$+ \sigma^{2} (F(B_{u},u))^{2} B_{u,v}^{2} \text{ (Some terms go to zero here as above.)}$$

$$= \int_{s}^{t} g(\widetilde{B}) d\widetilde{B}^{2} + \int_{s}^{t} \mu Z_{s,u}^{1} F(B_{u},u) du + \int_{s}^{t} \left(\int_{s}^{u} \mu F(B_{r},r) dr \right) dZ_{u}$$

$$+ \int_{s}^{t} \left(\int_{s}^{u} \mu F(B_{r},r) dr \right) \mu F(B_{u},u) du,$$

which yields eqn (4.42).

4.5. Chain rule of fBM. This subsection is the continuity of Example 4 which plays an important role in the remainder of the paper. Let X be the solution of fractional Black-Scholes equation (4.39) and G is a good function, then the integral $\int G(X,t)dX$ is well defined. Since X_t has the explicit representation (4.40),

$$\int \sigma G(X,t)XdB + \int \mu G(X_t,t)X_tdt$$

is defined in terms of the rough path B, which can be defined as $\int f(\widetilde{B})d\widetilde{B}$, where

$$f(x,t)(\xi,\eta) := f^{1}(x,t)\xi + f^{2}(x,t)\eta,$$

$$f^{1}(x,t) = \sigma G(F(x,t),t)F(x,t),$$

$$f^{2}(x,t) = \mu G(F(x,t),t)F(x,t),$$

and

$$F(x,t) = \exp(\sigma x - \sigma^2 t^{2H}/2 + \mu t).$$

Heuristically, the two integrals should equal. Our aim in this subsection is to show that they are indeed the same in this case. This kind of formula is usually called *Chain Rule*. Namely, we want to show the theorem below.

Theorem 4.13. (Chain rule) Let B be the Itô fractional Brownian rough path with $H \in (\frac{1}{3}, \frac{1}{2}]$, and X be the geometric fBM with parameters σ and μ , G be a function smooth enough.

(4.43)
$$\int G(X,t)dX = \int \sigma G(X,t)XdB + \int \mu G(X_t,t)X_tdt,$$

where the integrals are understood as above.

Proof. Let $Y = \int G(X,t)dX$, $H = \int f(\widetilde{B})d\widetilde{B}$. Then their associated almost rough path are \widehat{Y} , \widehat{H} respectively, where

$$\widehat{Y}_{s,t}^{1} = G(X_{s}, s) X_{s,t}^{1} + D_{x} G(X_{s}, s) X_{s,t}^{2},$$

$$\widehat{Y}_{s,t}^{2} = G(X_{s}, s) \otimes G(X_{s}, s) X_{s,t}^{2},$$

and

$$\widehat{H}_{s,t}^{1} = f^{1}(B_{s}, s)B_{s,t}^{1} + f^{2}(B_{s}, s)(t - s) + D_{x}f^{1}(B_{s}, s)B_{s,t}^{2},$$

$$\widehat{H}_{s,t}^{2} = f^{1}(B_{s}, s) \otimes f^{1}(B_{s}, s)B_{s,t}^{2}.$$

We should prove that

$$|\widehat{Y}_{s,t}^{i} - \widehat{H}_{s,t}^{i}| \le \omega(s,t)^{\theta}, \ i = 1, 2, \ \forall \ s < t, \ \exists \ \theta > 1,$$

for some control ω . If the difference of two quantities is controlled by ω like (4.44), we use the symbol \simeq to represent it. So we should show $\widehat{Y}_{s,t}^i \simeq \widehat{H}_{s,t}^i$.

Denote $Z = \int \sigma F(B, t) dB$, $h_{s,t} = \int_s^t \mu F(B_t, t) dt$, and \widehat{Z} , \widehat{h} denote their respective almost rough path, that is,

$$\widehat{Z}_{s,t}^{1} = \sigma F(B_{s}, s) B_{s,t}^{1} + \sigma^{2} F(B_{s}, s) B_{s,t}^{2},$$

$$\widehat{Z}_{s,t}^{2} = \sigma^{2} F(B_{s}, s) \otimes F(B_{s}, s) B_{s,t}^{2},$$

$$\widehat{h}_{s,t} = \mu F(B_{s}, s) (t - s).$$

It is easy to verify that the almost rough path associated with X is given by

$$(4.45) (1, Z_{s,t}^1 + h_{s,t}, Z_{s,t}^2) \simeq (1, Z_{s,t}^1 + \widehat{h}_{s,t}, Z_{s,t}^2).$$

Two quantities on both sides in (4.45) are all almost rough paths. So we have the following relations:

$$(4.46) X_{s,t}^1 = Z_{s,t}^1 + h_{s,t} \simeq \widehat{Z}_{s,t}^1 + \widehat{h}_{s,t},$$

$$(4.47) X_{s,t}^2 \simeq Z_{s,t}^2 \simeq \widehat{Z}_{s,t}^2$$

(i) For the first level,

$$\widehat{Y}_{s,t}^{1} = G(X_{s}, s)X_{s,t}^{1} + D_{x}G(X_{s}, s)X_{s,t}^{2}$$

$$= G(F(B_{s}, s), s)X_{s,t}^{1} + \partial_{1}G(F(B_{s}, s), s)X_{s,t}^{2}$$

$$\simeq G(F(B_{s}, s), s)(\widehat{Z}_{s,t}^{1} + \widehat{h}_{s,t}) + \partial_{1}G(F(B_{s}, s), s)\widehat{Z}_{s,t}^{2}$$

$$= G(F(B_{s}, s), s)(\sigma F(B_{s}, s)B_{s,t}^{1}$$

$$+ \sigma^{2}F(B_{s}, s)B_{s,t}^{2} + \mu F(B_{s}, s)(t - s))$$

$$+ \partial_{1}G(F(B_{s}, s), s)(\sigma^{2}F(B_{s}, s) \otimes F(B_{s}, s)B_{s,t}^{2}).$$

Since

$$D_x f^1(x,s) = \sigma^2 \partial_1 G(F(x,s),s) F(x,s) \otimes F(x,s) + \sigma^2 G(F(x,s),s) F(x,s),$$

so that

$$\widehat{Y}_{s,t}^1 \simeq f^1(B_s, s)B_{s,t}^1 + f^2(B_s, s)(t-s) + D_x f^1(B_s, s)B_{s,t}^2 = \widehat{H}_{s,t}^1$$

Thus we have proved the first part of the claim.

(ii) For the second level paths, we have

$$\begin{split} \widehat{Y}_{s,t}^2 &= G(X_s,s) \otimes G(X_s,s) X_{s,t}^2 \\ &= G(F(B_s,s),s) \otimes G(F(B_s,s),s) X_{s,t}^2 \\ &\simeq G(F(B_s,s),s) \otimes G(F(B_s,s),s) \widehat{Z}_{s,t}^2 \\ &= G(F(B_s,s),s) \otimes G(F(B_s,s),s) (\sigma^2 F(B_s,s) \otimes F(B_s,s) B_{s,t}^2) \\ &= \sigma^2 G(F(B_s,s),s) F(B_s,s) \otimes G(F(B_s,s),s) F(B_s,s) B_{s,t}^2 \\ &= f^1(B_s,s) \otimes f^1(B_s,s) B_{s,t}^2 = \widehat{H}_{s,t}^2, \end{split}$$

which thus completes our proof.

We mention that with the Stratonovich rough paths, the chain rule still holds by the same argument as Itô rough paths above.

4.6. **Zero mean property of Itô Integrals.** Now we need check if the first level of our Itô integrals defined above have zero mean, that is, we need to verify that

(4.48)
$$\mathbb{E}\left[\int_{s}^{t} f(B)dB\right]^{1} = 0.$$

4.6.1. One dimension case. We prove it in the case that dimension d = 1. First, we suppose that f has first and second continuous derivatives, by It formula proved above, we can show that

(4.49)
$$\mathbb{E}\left[\int_{s}^{t} f'(B) \circ dB\right]^{1} = \mathbb{E}\left[\frac{1}{2} \int_{s}^{t} f''(B_{r}) dr^{2H}\right].$$

The computation is routine, so we omit the details.

For the general case, set $F(x) = \int_{-\infty}^{x} f(y) dy$, $F(-\infty) = 0$, so that F'(x) = f(x). By (4.49) we get

(4.50)
$$\mathbb{E}\left[\int_{s}^{t} F'(B)dB\right]^{1} = \mathbb{E}\left[\frac{1}{2}\int_{s}^{t} F''(B_{r})dr^{2H}\right].$$

Thus the expectation of the first level of the Itô integral

(4.51)
$$\mathbb{E}\left[\int_{s}^{t} f(B)dB\right]^{1} = 0.$$

4.6.2. High dimension case. Now we can prove that eqn (4.49) still holds when dimension $d \geq 2$ and for any function $F : \mathbb{R}^d \to L(\mathbb{R}^d, \mathbb{R}^e)$. Let $X_{s,t} := \int_s^t F(B)dB$. The *i*-th component of first level of this Itô integral is

$$X_{s,t}^{1,i} = \left[\sum_{j=1}^{d} \int_{s}^{t} F^{ij}(B) dB^{(j)}\right]^{1}.$$

As a remark here, the integral $\int_s^t F^{ij}(B)dB^{(j)}$ is well-defined, which can be understood as $\int_s^t \widetilde{F}^{ij}(B)dB$, where $\widetilde{F}^{ij}(x_1, \dots, x_d)(\xi_1, \dots, \xi_d) = F^{ij}(x_1, \dots, x_d)\xi_j$. Therefore $\mathbb{E}\left[\int_s^t F^{ij}(B)dB^{(j)}\right]^1 = 0$, which yields that

(4.52)
$$\mathbb{E}\left[\int_{s}^{t} F(B)dB\right]^{1} = 0.$$

4.6.3. Zero mean property of time-dependent functions. In this subsection, we will show that for the time-dependent function F(x,t) we can still have the mean zero property, i.e.

(4.53)
$$\mathbb{E}\left[\int_{s}^{t} F(B, u) dB^{1}\right] = 0.$$

As the time independent case, we first show the one dimensional case, then by conditional expectation technique we conclude the high dimensional cases. By the It formula (4.20) and Remark 4.12, we know that it is equivalent to

(4.54)
$$F(B_t, t) - F(B_s, s) = \int_s^t D_x F(B, u) dB^1 + \int_s^t D_u F(B_u, u) du + \frac{1}{2} \int_s^t D_x^2 F(B_u, u) du^{2H}.$$

Then in order to prove the zero mean property, we should verify that

$$\mathbb{E}(F(B_t, t) - F(B_s, s)) = \int_s^t \mathbb{E}[D_u F(B_u, u)] du + \frac{1}{2} \int_s^t \mathbb{E}[D_x^2 F(B_u, u)] du^{2H}.$$

For the one dimension case, the left-hand side above can be computed as the following

(4.55)
$$\mathbb{E}\left(F(B_t,t) - F(B_s,s)\right) = \int_{\mathbb{R}} \left(F(t^H x,t) - F(s^H x,s)\right) \varphi(x) dx,$$

where φ is the standard normal probability density function. On the other hand,

$$\int_{s}^{t} \mathbb{E}[D_{u}F(B_{u},u)]du$$

$$= \int_{s}^{t} \left[\int_{\mathbb{R}} \partial_{2}F(u^{H}x,u)\varphi(x)dx \right] du$$

$$= \int_{\mathbb{R}} \left[\left(F(t^{H}x,t) - F(s^{H}x,s) \right) - \int_{s}^{t} x \partial_{1}F(u^{H}x,u)du^{H} \right] \varphi(x)dx.$$

and

(4.57)
$$\frac{1}{2} \int_{s}^{t} \mathbb{E}[D_{x}^{2}F(B_{u}, u)]du^{2H}$$

$$= \frac{1}{2} \int_{s}^{t} \left[\int_{\mathbb{R}} \partial_{1}^{2}F(u^{H}x, u)\varphi(x)dx \right] du^{2H}$$

$$= \int_{s}^{t} \left[\int_{\mathbb{R}} \partial_{1}F(u^{H}x, u)x\varphi(x)dx \right] du^{H}$$

$$= \int_{\mathbb{R}} \left[\int_{s}^{t} \partial_{1}F(u^{H}x, u)du^{H} \right] x\varphi(x)dx.$$

Combining (4.55),(4.56) and (4.57), we thus obtain that

(4.58)
$$\mathbb{E}\left[\int_{s}^{t} D_{x} F(B, u) dB^{1}\right] = 0.$$

By using $\widetilde{F}(x,t) = \int_{-\infty}^{x} F(y,t)dy$, we finally get (4.53) as for one dimensional case of one form above.

Now we turn to the high dimension case. Let

$$X_{s,t}^{1;i} := \left[\sum_{j=1}^{d} \int_{s}^{t} F^{ij}(B, u) dB^{(j)}\right]^{1}, i = 1, \dots, d.$$

We need to prove that $\mathbb{E}\left[\int_s^t F^{ij}(B,u)dB^{(j)}\right]^1 = 0$. Let $F^{ij} =: f$ for simplicity. Then

$$\mathbb{E}\left[\int_{s}^{t} f(B, u)dB^{(j)}\right]^{1}$$

$$= \mathbb{E}\left[\mathbb{E}\left[\left(\int_{s}^{t} f(x_{1}, \cdots, B^{(j)}, \cdots, x_{d}, u)dB^{(j)}\right)^{1}\right]_{x_{i} = B^{(i)}, i \neq j}\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[\left(\int_{s}^{t} \widetilde{f}(B^{(j)}, u)dB^{(j)}\right)^{1}\right]_{x_{i} = B^{(i)}, i \neq j}\right]$$

$$= 0.$$

Hence $\mathbb{E}\left[X_{s,t}^{1,i}\right]=0$ for $i=1,\cdots,d$. Thus we have proved that the expectation of time-dependent function is also zero, i.e. eqn (4.53) holds.

5. Fractional Black-Scholes model

We continue the study of Example 4 in section 4.4. We want to study the Itô fractional Black-Scholes model, fBS for simplicity, and its corresponding market. We want to show the market is arbitrage free under a restriction of the class of trading strategies. To this end, we first give the arbitrage strategy under Stratonovich fractional Black-Scholes market.

5.1. Arbitrage strategy in Stratonovich fBS market. Since the Stratonovich integral does not have zero mean property, therefore fractional Black-Scholes market based on Stratonovich integral suggests the possibility of existence of arbitrage. Shiryayev gave an arbitrage trading strategy in [31] under Stratonovich fBS market but driven by fBM with Hurst parameter H > 1/2. We adapt this strategy in our case when $H \in (\frac{1}{3}, \frac{1}{2}]$.

The market has a stock (the risky asset) X whose price process is $X_t := X_{0,t}^1$ at time t. We assume that X satisfies the differential equation driven by Stratonovich fractional Brownian rough path with $H \in (\frac{1}{3}, \frac{1}{2}]$ as the first part of example 4, i.e.

(5.1)
$$dX = \mu X_t dt + \sigma X \circ dB, \ X_0 = x, \ t \in [0, T].$$

The solution X of this equation has been constructed in example 4 and $X_t := X_{0,t}^1 = xe^{\sigma B_t + \mu t}$. It is assumed that there is a money market (the risk-less asset) M, that is, an asset whose price at time t is not subject to uncertainty. Namely, the price process M_t satisfies the following equation

(5.2)
$$dM_t = rM_t dt, \ M_0 = 1, \ t \in [0, T],$$

where r > 0 is a constant, i.e. $M_t = e^{rt}$.

A portfolio (γ_t, ζ_t) gives the number of units γ_t, ζ_t held at time t in the money market and stock market, respectively. The value process $V_t \in \mathbb{R}$ of the portfolio is given by

$$(5.3) V_t = \gamma_t M_t + \zeta_t X_t.$$

The portfolio is called *self-financing* if

(5.4)
$$V_{t} = V_{0} + \int_{0}^{t} \gamma_{s} dM_{s} + \int_{0}^{t} \zeta \circ dX^{1}.$$

Note that the second integral on the right hand side is the first level of the Stratonovich integral against rough path X defined in (5.1).

Now consider the following portfolio

(5.5)
$$\gamma_t = 1 - e^{2\sigma B_t + 2(\mu - r)t},$$

(5.6)
$$\zeta_t = 2x^{-1}(e^{\sigma B_t + (\mu - r)t} - 1),$$

we will show that this trading strategy is an arbitrage one. First, by (5.5) and (5.6), we get the value process of the portfolio

$$V_t = (1 - e^{2\sigma B_t + 2(\mu - r)t})e^{rt} + 2(e^{\sigma B_t + (\mu - r)t} - 1)e^{\sigma B_t + \mu t}$$
$$= e^{rt} \left(e^{\sigma B_t + (\mu - r)t} - 1\right)^2 \ge 0.$$

By applying the basic principle/Itô formula for Stratonovich integral to $V_t =: f(B_t, t)$, we have

$$V_{t} = V_{0} + \int_{0}^{t} re^{rs} \left(e^{\sigma B_{s} + (\mu - r)s} - 1 \right)^{2} ds$$

$$+ \int_{0}^{t} 2(\mu - r) e^{\sigma B_{s} + (\mu - r)s} e^{rs} \left(e^{\sigma B_{s} + (\mu - r)s} - 1 \right) ds$$

$$+ \int_{0}^{t} 2\sigma e^{rs} \left(e^{\sigma B_{s} + (\mu - r)s} - 1 \right) e^{\sigma B_{s} + (\mu - r)s} \circ dB_{s}$$

$$= \int_{0}^{t} r\gamma_{s} e^{rs} ds + \int_{0}^{t} \mu \zeta_{s} X_{s} ds + \int_{0}^{t} \sigma \zeta_{s} X_{s} \circ dB_{s}$$

$$= \int_{0}^{t} \gamma_{s} dM_{s} + \int_{0}^{t} \zeta_{s} \circ dX_{s}.$$

The last equality is by the chain rule of Stratonovich integral in section 4.5.

Hence, the portfolio (5.5), (5.6) is self-financing in this financial market. Note that the initial payment at t = 0 is $V_0 = 0$, but after that the value of this portfolio is positive almost surely. This means one gets free lunch with no risk.

5.2. Arbitrage free under a class of trading strategies. Now we consider the Itô fractional Black-Scholes market. As for the Stratonovich fBS market, we suppose that the market has a stock X (the risky asset) whose price process is $X_t := X_{0,t}^1$ but now it satisfies the differential equation driven by Itô fractional Brownian rough path.

(5.7)
$$dX = \mu X_t dt + \sigma X dB, \ X_0 = x, \ t \in [0, T].$$

The solution is also constructed in example 4. The risk-less asset money market M satisfies the equation (5.2), i.e. $M_t = e^{rt}$.

Suppose a portfolio (γ_t, ζ_t) gives the value process $V_t \in \mathbb{R}$ by

$$(5.8) V_t = \gamma_t M_t + \zeta_t X_t.$$

In this Itô fBS market, we restrict the class of trading strategies. We call a portfolio is admissible if $\gamma_t = \gamma(X_t, t)$, and $\zeta_t = \zeta(X_t, t)$. Besides, a portfolio is called *self-financing* if

$$V_t = V_0 + \int_0^t \gamma_s dM_s + \int_0^t \zeta dX^1,$$

where the second integral on the right hand side is the first level of the Itô integral against rough path X defined in (5.7).

Then by the chain rule of Itô fractional Brownian rough path, we have

$$V_t = V_0 + \int_0^t \gamma_s dM_s + \int_0^t \zeta dX^1$$

$$= V_0 + \int_0^t re^{rs} \gamma_s ds$$

$$+ \int_0^t \mu \zeta_s X_s ds + \int_0^t \sigma \zeta_s X_s dB^1$$

By (5.8) we also have

(5.9)
$$\gamma_t = e^{-rt}(V_t - \zeta_t X_t),$$

plugging it into last equality, we get

(5.10)
$$V_t = V_0 + \int_0^t r V_s ds + \int_0^t \sigma \zeta_s X_s \left(\frac{\mu - r}{\sigma} ds + dB^1 \right).$$

In order to prove there is no arbitrage in this case, we first introduce the Girsanov theorem for fBM with Hurst parameter $H \leq 1/2$.

5.3. **Girsanov's theorem.** The following version of Girsanov theorem for the fBM has been obtained in ([13], Theorem 4.9), and we also suggest reader to see [7], Theorem 4.1 and proof therein. In our case, we would like to show that there is a new probability measure $\widehat{\mathbb{P}}$ such that

$$\widehat{B}_t = B_t + \frac{\mu - r}{\sigma}t,$$

which is still an fBM under this measure $\widehat{\mathbb{P}}$. This is what Girsanov theorem says in usual. Now let $K_H(t,s)$ be a square integrable kernel given by

(5.12)
$$K_H(t,s) = C_H \left[\frac{2}{2H-1} \left(\frac{t(t-s)}{s} \right)^{H-\frac{1}{2}} - \int_s^t \left(\frac{u(u-s)}{s} \right)^{H-\frac{1}{2}} \frac{du}{u} \right] 1_{(0,t)}(s).$$

Define the operator K_H on $L^2([0,T])$ associated with the kernel $K_H(t,s)$ as

(5.13)
$$(K_H f)(s) = \int_0^T f(t) K_H(t, s) dt.$$

Given an adapted and integrable process $u = \{u_t, t \in [0, T]\}$, consider the transformation

$$\widehat{B}_t = B_t + \int_0^t u_s ds,$$

since fBM B can be represented by the integral along standard Brownian motion W, we can write (5.14) into

$$\widehat{B}_{t} = B_{t} + \int_{0}^{t} u_{s} ds = \int_{0}^{t} K_{H}(t, s) dW_{s} + \int_{0}^{t} u_{s} ds = \int_{0}^{t} K_{H}(t, s) d\widetilde{W}_{s},$$

where W_t is a standard Brownian motion and

$$\widetilde{W}_t = W_t + \int_0^t K_H^{-1} \left(\int_0^{\cdot} u_r dr \right) (s) ds.$$

By the standard Girsanov theorem for Brownian motion applied to (5.15), as a consequence, we have the following version of the Girsanov theorem for the fBM with Hurst parameter $H \leq \frac{1}{2}$, which has obtained in [13], [28] and [7].

Theorem 5.1. (Girsanov theorem for fBM with $H \leq \frac{1}{2}$)([13], Theorem 4.9; [28], Theorem 2; [7], Theorem 4.1) Let B be a fBM with Hurst parameter $H \in (0, \frac{1}{2}]$, and

$$v(s) := K_H^{-1} \left(\int_0^{\cdot} u_r dr \right) (s).$$

Consider the shifted process (5.14). Assume that

(i) $\int_0^T u_t^2 dt < \infty$, almost surely.

(ii) $\mathbb{E}(Z_T) = 1$, where

$$Z_T = \exp\left(-\int_0^T v(s)dW_s - \frac{1}{2}\int_0^T (v(s))^2 ds\right),$$

Then the shifted process \widehat{B} is an \mathcal{F}_t^B -fBM with Hurst parameter H under the new probability measure $\widehat{\mathbb{P}}$ defined by $\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} = Z_T$.

Remark 5.2. Here when u satisfies the condition (i) in Theorem 5.1 with $H \leq \frac{1}{2}$, then $v = K_H^{-1}\left(\int_0^{\cdot} u_r dr\right)$ is well-defined, and $K_H^{-1}\left(\int_0^{\cdot} u_r dr\right) \in L^2([0,T])$, where K_H^{-1} is the inverse of the operator K_H .

In our case,

$$\widehat{B}_t = B_t + \frac{\mu - r}{\sigma}t,$$

so we can apply the Girsanov Theorem 5.1 to it. Let \mathbb{P} be the distribution of fBM B, and $\widehat{\mathbb{P}}$ be the distribution constructed from \mathbb{P} by Girsanov theorem. In terms of \widehat{B}_t , we can write (5.10), under $\widehat{\mathbb{P}}$, as

$$(5.16) V_t = V_0 + \int_0^t r V_s ds + \int_0^t \sigma \zeta_s X d\widehat{B}^1.$$

Transforming this equation, we have

(5.17)
$$e^{-rt}V_t = V_0 + \sigma \int_0^t e^{-rs} \zeta_s X d\widehat{B}^1, \ t \in [0, T].$$

Taking expectation under the measure $\widehat{\mathbb{P}}$, we have

(5.18)
$$e^{-rT}\mathbb{E}_{\widehat{\mathbb{P}}}[V_T] = V_0 + \mathbb{E}_{\widehat{\mathbb{P}}}\left[\sigma \int_0^T e^{-rs} \zeta_s X d\widehat{B}^1\right].$$

By Girsanov's theorem and our zero mean property of integrals, we get that

(5.19)
$$\mathbb{E}_{\widehat{\mathbb{P}}}\left[\int_0^T e^{-rs} \zeta_s X d\widehat{B}^1\right] = 0.$$

Thus we may conclude that

$$(5.20) e^{-rT} \mathbb{E}_{\widehat{\mathbb{P}}} \left[V_T \right] = V_0.$$

Hence the probability measure $\widehat{\mathbb{P}}$ defined in Theorem 5.1 is a risk neutral measure. Then this Itô fBS market has no arbitrage under the class of admissible trading strategy.

5.4. **Option pricing formula.** Moreover, we want to give a pricing formula under risk-neutral measure $\widehat{\mathbb{P}}$ for the financial derivative F at time t=0. Our Itô fBS market is arbitrage free under this risk-neutral measure $\widehat{\mathbb{P}}$ in Theorem 5.1. Hence, the price under this risk-neutral measure $\widehat{\mathbb{P}}$ defined in section 5.3 is

$$(5.21) V_0 = e^{-rT} \mathbb{E}_{\widehat{\mathbb{p}}} \left[V_T \right].$$

So we can work out explicitly the price, then get the theorem below.

Theorem 5.3. (Fractional Black-Scholes pricing formula) The price of claim $F(X_T)$ under fractional Black-Scholes model and risk-neutral measure $\widehat{\mathbb{P}}$ in Theorem 5.1 is

(5.22)
$$V_0 = e^{-rT} \int_{\mathbb{R}} F(X_0 e^{\sigma T^H y + rT - \frac{1}{2}\sigma^2 T^{2H}}) \varphi(y) dy,$$

where $\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ is the standard normal density function.

Proof. Since the price is

$$V_0 = e^{-rT} \mathbb{E}_{\widehat{\mathbb{P}}} \left[V_T \right] = e^{-rT} \mathbb{E}_{\widehat{\mathbb{P}}} \left[F(X_T) \right],$$

by the Girsanov theorem for fBM,

$$\mathbb{E}_{\widehat{\mathbb{P}}}\left[F\left(X_{T}\right)\right]$$

$$= \mathbb{E}_{\widehat{\mathbb{P}}}\left[F\left(X_{0}\exp\left(\sigma B_{T}^{H} + \mu T - \frac{1}{2}\sigma^{2}T^{2H}\right)\right)\right]$$

$$= \mathbb{E}_{\widehat{\mathbb{P}}}\left[F\left(X_{0}\exp\left(\sigma \widehat{B}_{T}^{H} + rT - \frac{1}{2}\sigma^{2}T^{2H}\right)\right)\right]$$

$$= \mathbb{E}_{\mathbb{P}}\left[F\left(X_{0}\exp\left(\sigma B_{T}^{H} + rT - \frac{1}{2}\sigma^{2}T^{2H}\right)\right)\right]$$

$$= \int_{\mathbb{R}}F(X_{0}e^{\sigma T^{H}y + rT - \frac{1}{2}\sigma^{2}T^{2H}})\varphi(y)dy,$$

which completes our proof.

For the European call option, $F(X_T) = (X_T - K)^+$, where K > 0 is the striking price, we have the following conclusion.

Corollary 5.4. (European call) The pricing formula of the European call option under fractional Black-Scholes model and risk-neutral measure $\widehat{\mathbb{P}}$ in Theorem 5.1 is

(5.23)
$$V_0 = X_0 (1 - \Phi(c_-)) - Ke^{-rT} (1 - \Phi(c_+)),$$

where

$$c_{-} = \frac{1}{\sigma T^{H}} \log \left(\frac{K}{X_{0}}\right) - \frac{r}{\sigma} T^{1-H} - \frac{1}{2} \sigma T^{H},$$

$$c_{+} = \frac{1}{\sigma T^{H}} \log \left(\frac{K}{X_{0}}\right) - \frac{r}{\sigma} T^{1-H} + \frac{1}{2} \sigma T^{H},$$

and $\Phi(x) = \int_{-\infty}^{x} \varphi(y) dy$ is standard normal distribution function.

We should point out that the above option pricing formula is derived based on the noarbitrage pricing technique but with a serve restriction on the trading strategies where only Markovian type portfolios are allowed. Probably in practice this is the case – tradings are done based on current market prices. This looks not so reasonable in contrast with fractional BS markets which are not Markovian, and therefore is not so satisfactory theoretically. However, at the current technology, this is best we can do within the fractional BS markets due to lack of a flexible stochastic integration theory beyond the semi-martingale setting. Still the rough path theory provides the only way forward with a fractional BS market where the Hurst parameter is less than half.

Finally let us comment on the pricing formula itself, which is similar to that of the classical Black-Scholes model. The values are exactly the same if the maturity time T = 1, which

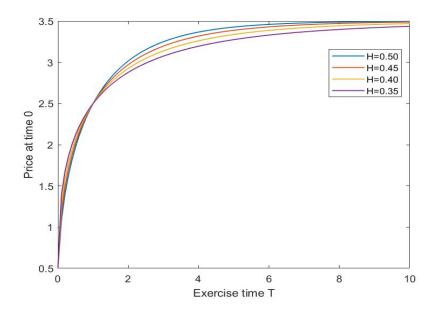


FIGURE 1. Price of European call at time 0 for different exercise time T and varying Hurst parameter H=0.50,0.45,0.40,0.35, where we take $\sigma=2,$ K=3, $X_0=3.5,$ r=0.05 as an example.

seems strange at the first glance. The Hurst parameter H comes in to play a role only through the exercise time T, and appears as a power in the exercise time T through c_{\pm} and to modify the volatility $\sigma^2 T$ (for Black-Scholes' market) into $\sigma^2 T^{2H}$ (for the fractional BS market case), so that the intensity of the volatility is reduced as $H < \frac{1}{2}$ due to the long time memory. In fact the numerics c_{\pm} can be rewritten as

$$c_{\pm} = \frac{1}{\sigma T^H} \log \left(\frac{K}{X_0} \right) - \frac{rT}{\sigma T^H} \pm \frac{1}{2} \sigma T^H.$$

Of course one has to understand the scale of the maturity T in time unit has no economic meaning, and its scale is in fact fixed by the interest rate through e^{rT} , and therefore it looks natural that H should appear in the power of T to have its effect on the option pricing. See Figure 1, we show prices of European call option at time 0 for different exercise time T and Hurst parameter H.

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