Rough Path Renormalization from Stratonovich to Itô for Fractional Brownian Motion

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Abstract   This paper develops an Itô-type fractional pathwise integration theory for fractional Brownian motion with Hurst parameters $H \in (\frac{1}{3}, \frac{1}{2}]$, using the Lyons' rough path framework. This approach is designed to fill gaps in conventional stochastic calculus models that fail to account for temporal persistence prevalent in dynamic systems such as those found in economics, finance, and engineering. The pathwise-defined method not only meets the zero expectation criterion but also addresses the challenges of integrating non-semimartingale processes, which traditional Itô calculus cannot handle. We apply this theory to fractional Black–Scholes models and high-dimensional fractional Ornstein–Uhlenbeck processes, illustrating the advantages of this approach. Additionally, the paper discusses the generalization of Itô integrals to rough differential equations (RDE) driven by fBM, emphasizing the necessity of integrand-specific adaptations in the Itô rough path lift for stochastic modeling.

Keywords   Rough paths, Itô integration, fractional Brownian motions, fractional Black–Scholes model, fractional Ornstein–Uhlenbeck process, renormalization

MR(2020) Subject Classification   60H05, 60H10, 91B70, 91G80

1   Introduction

Stochastic calculus, underpinned by Brownian motion and semi-martingales, plays a pivotal role in modeling dynamic stochastic environments across fields such as economics, finance, and engineering. A cornerstone of Itô calculus is the exploitation of the martingale property, with the Itô integral defined in relation to the complete law of the driving process, rather than specific realizations. However, empirical analyses of various data often reveal dynamic processes characterized by temporal persistence not accounted for by semi-martingale-driven dynamics. This discrepancy is exemplified by rough volatility models and more broadly, rough path models. A prominent approach to modeling memory effects over time involves the use of

fractional Brownian motions (fBMs), a class of Gaussian processes introduced by Mandelbrot and van Ness in 1968 ([27]). These fBMs extend standard Brownian motion by capturing dependencies that decay polynomially over time.

The exploration of integration theories with respect to fractional Brownian motion has garnered significant interest among researchers, as evidenced by a wide range of studies (see, for example, [3, 4, 6, 13, 14, 16, 22, 29] and references therein). Notably, Duncan, Hu, and Pasik-Duncan [15] pioneered the use of the Wick product to define fractional stochastic integrals against fBM, termed Wick–Itô integrals. Subsequently, Hu and Øksendal [22] and Elliott and van der Hoek [16] expanded this methodology, developing a fractional white noise calculus applicable to financial modeling. The method proposed by [15] is applicable specifically to the persistent case of fBM, where the Hurst parameter $H > \frac{1}{2}$, while the model in [16] addresses the anti-persistent case with $H < \frac{1}{2}$. These developments employ Wick products instead of ordinary multiplication to define stochastic integrals. Additionally, the concept of Wick products has been adapted for analyzing portfolios and self-financing strategies within fractional Black–Scholes markets. Through innovative concepts, the authors demonstrated the non-arbitrage properties and market completeness when replacing the traditional Black–Scholes model [9] with a geometric fBM characterized by a Hurst parameter $H > \frac{1}{2}$. However, the use of Wick products has sparked significant debate due to conceptual divergences from the conventional Black–Scholes model, particularly regarding the economic interpretation of fundamental concepts like portfolio values and self-financing (see, e.g., [7]). It is crucial to understand that Wick products, defined merely as the multiplication of two random variables, lack a pathwise computational basis. Consequently, from the realizations $X(\omega)$ and $Y(\omega)$ of two random variables X and Y, it is not possible to compute the Wick product $X(\omega)$ and $Y(\omega)$ directly. Perhaps it should be possible, in particular for the fBM with Hurst parameter greater than a half, to demonstrate the integral theory via Wick product, or in general in terms of Malliavin calculus, can be reformulated in terms of the rough path theory, which is however beyond the scope of the present paper. While, due to its generality, the Itô's theory offered in the current paper does not coincide with the integration theory via the Malliavin calculus.

Multiple methodologies have been advanced to define stochastic integrals with respect to fBM. One notable pathwise approach, developed by Ciesielski, Kerkyacharian, and Roynette [12], and furthered by Zähle [34], leverages the Hölder continuity of fBM sample paths. Specifically, Ciesielski et al. [12] introduced an integration framework using wavelet expansions within Besov–Orlicz spaces, and Zähle [34] extended this by employing fractional calculus alongside a generalization of the integration by parts formula. These theories, however, are applicable predominantly when the Hurst parameter $H > \frac{1}{2}$. A second method, pioneered by Decreusefond and Üstünel [14], utilizes Malliavin calculus tailored to fBM. This approach has been extensively explored in subsequent research by Alos and Nualart [2], Carmona, Coutin, and Montseny [10], and Cheridito and Nualart [11].

A third method, integrating via rough path theory, was introduced by Coutin and Qian [13]. This theory, applicable for fBM with a Hurst parameter $H > \frac{1}{4}$, aligns with Stratonovich's concept of integration. Despite its theoretical robustness, applying Stratonovich's fractional integration to financial models, such as option pricing, reveals a significant drawback: the

expected values of Stratonovich integrals are generally non-zero. This characteristic potentially allows for the construction of arbitrage strategies within fractional Black–Scholes markets (see, e.g., [5, 33]). Since fBM does not qualify as a semi-martingale when $H \neq \frac{1}{2}$, it is infeasible to apply Itô's semi-martingale theory directly to fBM. A more viable goal is to develop a stochastic integral where integrations against fBM yield zero-mean outcomes. Such an approach would be more advantageous than Stratonovich's, as it minimizes systemic bias in pricing processes across all possible paths, thereby mitigating arbitrage opportunities.

This paper endeavors to develop an Itô-type fractional pathwise integration theory tailored to fBM with Hurst parameters in the range $H \in (\frac{1}{3}, \frac{1}{2}]$, employing a rough path approach. Our proposed theory, which is defined pathwise, achieves zero expectation, aligning with our initial objectives. Rough path theory, as articulated by Lyons, provides a robust framework for handling fBMs with $H < \frac{1}{2}$ (see, e.g., [17, 18, 23–25]). Within this framework, we develop essential tools characteristic of Itô-type integration, such as connections to Stratonovich fractional integrals, fractional Itô's formula, chain rule applications, and more. While previous studies have examined rough path integration theory for non-geometric paths (see [20, 21, 26]), our focus is on specifically advancing this theory for fBM, aiming to enhance its application in areas like fractional Black–Scholes (fBS) models, fractional Ornstein–Uhlenbeck (fOU) processes, and more broadly, rough differential equations (RDEs) driven by fBMs.

As a practical application, we employ our pathwise Itô integration theory on fractional Black–Scholes models and demonstrate that the corresponding fractional Black–Scholes market remains arbitrage-free under a more restrictive set of trading strategies than those typically permissible in a complete market. While our focus on arbitrage issues illustrates one potential use of this theory, there are numerous other areas where it could be applied. Additionally, we extend the application of our theory to high-dimensional fractional Ornstein–Uhlenbeck processes. This allows us to address estimation challenges associated with fOUs, which are instrumental in modeling and forecasting realized volatility (see also [30]).

The structure of the paper is as follows: Section 2 offers a concise review of essential insights related to rough path theory and fractional Brownian motion. Section 3 forms the crux of our discussion, where we define integration with respect to the Itô-fractional Brownian rough path, clarify the relationship between Stratonovich fractional integrals and Itô integrals, and derive the fractional Itô formula within our framework. We also examine differential equations driven by rough paths, with a focus on the fractional Black–Scholes model, establishing the chain rule for this model and demonstrating the zero expectation property of our Itô integration. In Section 4, we broaden our discussion to more general Itô integrals beyond the simple one-form associated with fBM, particularly for fractional Ornstein–Uhlenbeck processes. We also consider the integration for solutions to rough differential equations, discussing how renormalization is contingent upon the nature of the integrand. Section 5 applies our developed theories to financial markets, illustrating the absence of arbitrage in the Itô fractional Black–Scholes market under specific trading strategy restrictions. Finally, Section 6 focuses on the application of our integration theory to the estimation problems in fOU processes.

As a note, the development of a general pathwise Itô integration theory tailored to fractional Brownian motion necessitates a specialized approach to the Itô rough path lift of fBM,

specifically its Levy area, which uniquely depends on the integrand. This requirement starkly contrasts with the classical Itô's theory for Brownian motion. In this paper, we will demonstrate that Itô integrals with respect to fBM can be constructed using a universal rough path lift of fBM, applicable across all one-forms. However, when the integrand itself is a solution to a rough differential equation driven by fBM, the specific Itô rough path lift must also adapt to accommodate the solution's dynamics. This approach bears resemblance to the methodology in regularity structures, where the models are intricately dependent on the underlying dynamics.

2 Preliminaries on Rough Paths and Fractional Brownian Motion

2.1 Rough Paths

This section aims to establish several notations that will be used throughout the paper by recalling some basic notions concerning rough paths. Our exposition closely follows the rough path literature (e.g., [1, 13, 17, 18, 25]). We will focus on the fundamental framework required to ensure that fBM with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2}]$, which we introduce in the next subsection, has a natural rough path lift.

For $N \in \mathbb{N}$, $T^{(N)}(\mathbb{R}^d)$ denotes the truncated tensor algebra defined by

$$T^{(N)}(\mathbb{R}^d) := \bigoplus_{n=0}^{N} (\mathbb{R}^d)^{\otimes n},$$

with the convention that $(\mathbb{R}^d)^{\otimes 0} = \mathbb{R}$. The space $T^{(N)}(\mathbb{R}^d)$ is equipped with a vector space structure and a multiplication $\otimes$ defined by

$$(\boldsymbol{X} \otimes \boldsymbol{Y})^k = \sum_{i=0}^{k} \boldsymbol{X}^{k-i} \boldsymbol{Y}^i, \quad k = 0, 1, \ldots, N,$$

where $\boldsymbol{X} = (1, \boldsymbol{X}^1, \ldots, \boldsymbol{X}^N)$, $\boldsymbol{Y} = (1, \boldsymbol{Y}^1, \ldots, \boldsymbol{Y}^N) \in T^{(N)}(\mathbb{R}^d)$.

We will consider continuous $\mathbb{R}^d$-valued paths X on $[0, T]$ with bounded variations, and their canonical lifts $\boldsymbol{X}_{s,t} = (1, \boldsymbol{X}_{s,t}^1, \ldots, \boldsymbol{X}_{s,t}^N)$ in the space $T^{(N)}(\mathbb{R}^d)$, where

$$\boldsymbol{X}_{s,t}^1 = X_t - X_s,$$

$$\boldsymbol{X}_{s,t}^2 = \int_{s < t_1 < t_2 < t} dX_{t_1} \otimes dX_{t_2},$$

$$\vdots$$

and

$$\boldsymbol{X}_{s,t}^N = \int_{s < t_1 < \cdots < t_N < t} dX_{t_1} \otimes \cdots \otimes dX_{t_N}.$$

The lifted path satisfies *"Chen's identity"* as following:

$$\boldsymbol{X}_{s,t} = \boldsymbol{X}_{s,u} \otimes \boldsymbol{X}_{u,t}, \quad \forall (s,u), (u,t) \in \Delta, \tag{2.1}$$

where Δ denotes the simplex $\{(s,t) : 0 \leq s < t \leq T\}$.

By definition, a continuous map $\boldsymbol{X}$ from the simplex Δ into a truncated tensor algebra $T^{(N)}(\mathbb{R}^d)$ is called a *rough path* (of roughness $p \geq 1$, where $N = [p]$), if it satisfies (2.1) and has finite p-variations, that is,

$$\sum_{i=1}^{N} \sup_{D} \sum_{\ell} |\boldsymbol{X}_{t_{\ell-1}, t_\ell}^i|^{\frac{p}{i}} < \infty,$$

where the sup runs over all finite partitions $D = \{0 = t_0 < t_1 < \cdots < t_n = T\}$. The p-variation distance is defined to be

$$d_p(\boldsymbol{X}, \boldsymbol{Y}) = \sum_{i=1}^{N} \left(\sup_D \sum_\ell |\boldsymbol{X}_{t_{\ell-1},t_\ell}^i - \boldsymbol{Y}_{t_{\ell-1},t_\ell}^i|^{\frac{p}{i}} \right)^{\frac{i}{p}}.$$

Equivalently, $\boldsymbol{X} : \Delta \to T^{(N)}(\mathbb{R}^d)$ has finite p-variations if

$$|\boldsymbol{X}_{s,t}^i| \le \omega(s,t)^{\frac{i}{p}}, \quad \forall i = 1, \ldots, N, \quad \forall (s,t) \in \Delta$$

for some function ω, where ω is a non-negative, continuous, super-additive function on Δ and satisfies $\omega(t,t) = 0$. Such function ω is called a *control of the rough path* $\boldsymbol{X}$.

The space of all p-rough paths is denoted by $\Omega_p(\mathbb{R}^d)$. A rough path $\boldsymbol{X}$ is called a geometric rough path if there is a sequence of $\boldsymbol{X}(n)$, where $\boldsymbol{X}(n)$ are the canonical lifts of their first level $\boldsymbol{X}(n)^1$ which are continuous with finite variations, such that $\boldsymbol{X}$ is the limit of $\boldsymbol{X}(n)$ under p-variation distance d_p. The space of geometric rough paths is denoted by $G\Omega_p(\mathbb{R}^d)$.

As our interest lies in fBMs with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2}]$, which will be introduced later, we consider only rough paths valued in $T^{(2)}(\mathbb{R}^d)$. Thus, in what follows, we will assume that $3 > p \ge 2$ so that $[p] = 2$. A rough path of roughness p can be written as $\boldsymbol{X}_{s,t} = (1, \boldsymbol{X}_{s,t}^1, \boldsymbol{X}_{s,t}^2)$ for $s < t$, and the algebraic relation (Chen's identity) now becomes

$$\boldsymbol{X}_{s,t}^1 = X_t - X_s, \tag{2.2}$$

and

$$\boldsymbol{X}_{s,t}^2 - \boldsymbol{X}_{s,u}^2 - \boldsymbol{X}_{u,t}^2 = \boldsymbol{X}_{s,u}^1 \otimes \boldsymbol{X}_{u,t}^1, \tag{2.3}$$

for all $(s,u), (u,t) \in \Delta$, where $\boldsymbol{X}^2$ should be (if it makes sense) considered as an iterated integral

$$\int_s^t \boldsymbol{X}_{s,u}^1 d\boldsymbol{X}_u^1 := \boldsymbol{X}_{s,t}^2 \tag{2.4}$$

which is of course not defined a priori.

The most convenient tool to construct rough paths is through *almost rough path*s. A function $\boldsymbol{Y} = (1, \boldsymbol{Y}^1, \boldsymbol{Y}^2)$ from Δ to $T^{(2)}(\mathbb{R}^d)$ is called an *almost rough path* if it has finite p-variation, and for some control ω and constant $\theta > 1$,

$$|(\boldsymbol{Y}_{s,t} \otimes \boldsymbol{Y}_{t,u})^i - \boldsymbol{Y}_{s,u}^i| \le \omega(s,u)^\theta, \quad i = 1, 2, \tag{2.5}$$

for all $(s,t), (t,u) \in \Delta$. According to Theorem 3.2.1 in Lyons and Qian [25], given an almost rough path $\boldsymbol{Y} = (1, \boldsymbol{Y}^1, \boldsymbol{Y}^2)$, there exists a unique rough path $\boldsymbol{X} = (1, \boldsymbol{X}^1, \boldsymbol{X}^2)$ such that

$$|\boldsymbol{X}_{s,t}^i - \boldsymbol{Y}_{s,t}^i| \le \omega(s,t)^\theta, \quad i = 1, 2, \ \theta > 1 \tag{2.6}$$

for some control ω, and all $(s,t) \in \Delta$.

As a clarification, unless otherwise indicated, we may utilize the symbol $\boldsymbol{X} = (1, X, \mathbb{X})$ as an alternative representation of $\boldsymbol{X} = (1, \boldsymbol{X}^1, \boldsymbol{X}^2)$, provided this does not introduce any ambiguity.

2.2 Fractional Brownian Motion

Fractional Brownian motion (fBM) $B^H(t)$ is a continuous-time Gaussian process, where $t \ge 0$. It has a mean of zero for all $t \ge 0$, and the covariance function is defined by the following

equation:

$$E[B^H(t)B^H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \tag{2.7}$$

where $H \in (0,1)$ is the Hurst parameter.

When $H > \frac{1}{2}$, the increments of fractional Brownian motion (fBM) exhibit positive correlation, indicating persistence: a tendency for future increments to continue in the same direction as past increments. Conversely, for $H < \frac{1}{2}$, the increments are negatively correlated, demonstrating counter-persistence: a tendency for future increments to reverse the direction of past increments. Thus, when $H < \frac{1}{2}$, fBM is likely to show a decrease following an increase and vice versa. This dual characteristic makes fBM a versatile model for phenomena exhibiting both short-range and long-range dependencies across various disciplines, including physics, biology, hydrology, network research, and financial mathematics.

The fBM with Hurst parameter H has an integral representation in terms of Brownian motion

$$B^H(t) = \int_0^t K_H(t,s)dW(s), \tag{2.8}$$

where $W(t)$ is a standard Brownian motion and

$$K_H(t,s) = C_H \left[\frac{2}{2H-1} \left(\frac{t(t-s)}{s} \right)^{H-\frac{1}{2}} - \int_s^t \left(\frac{u(u-s)}{s} \right)^{H-\frac{1}{2}} \frac{du}{u} \right] 1_{(0,t)}(s)$$

which is a singular kernel, and C_H is a normalised constant.

Integration theories for fractional Brownian motion with a Hurst parameter $H > \frac{1}{2}$ can be developed using Young's integration theory or functional integration approaches, as outlined in works such as [12, 34]. On the other hand, stochastic calculus for fBM with $H < \frac{1}{2}$ is more suitably addressed within the framework of rough path analysis. In their seminal work, Coutin and Qian [13] describe the construction of a canonical level-3 rough path B^H for fBM with $H > \frac{1}{4}$. This construction includes defining iterated integrals of multi-dimensional fBM up to level-3, enabling the application of rough path integration theory to effectively handle fBM.

The second and third level processes are defined in terms of iterated Riemann-Stieltjes integrals along the dyadic piece-wise linear approximations, and geometric rough paths of fBM are the limits in p-variation distance. Here as our main concern is the fBM with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2}]$, so we only need the level-2 results about fBM. The method of [13] to construct the fractional Brownian rough path $\boldsymbol{B}^H = (1, B^H, \mathbb{B}^H)$ implies that

$$\mathbb{B}_{s,t}^H := \lim_{m \to 0} \int_s^t B_{s,u}^{H,(m)} \otimes dB_u^{H,(m)}, \quad \text{a.s.}, \tag{2.9}$$

exists in p-variation distance as long as $pH > 1$, where the dyadic piece-wise linear approximations $B_t^{H,(m)}$ on interval $[s,t]$ is defined by

$$B_r^{H,(m)} := B_{t_{\ell-1}^m}^H + \frac{r - t_{\ell-1}^m}{t_\ell^m - t_{\ell-1}^m} \Delta_\ell^m B^H, \quad \text{for } r \in [t_{\ell-1}^m, t_\ell^m]$$

with $\Delta_\ell^m B^H = B_{t_\ell^m}^H - B_{t_{\ell-1}^m}^H$, $t_\ell^m := s + \frac{\ell}{2^m}(t-s)$ for $\ell = 1, 2, \ldots, 2^m$.

Proposition 2.1 ([13, Theorem 2], [17, Theorem 10.4]) *Let $B^H = (B^{H,1}, \ldots, B^{H,d})$ be a d-dimensional fBM with the Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2}]$. Then B^H, restricted to an finite*

interval $[0, T]$, *lifts via* (2.9) *to a geometric rough path* $\boldsymbol{B}^H = (1, B^H, \mathbb{B}^H) \in G\Omega_p([0, T], \mathbb{R}^d)$, *for all* $p \in (\frac{1}{H}, 3)$.

The proof details for this proposition are documented in [13, 17]. Here, the random rough path $\boldsymbol{B}^H = (1, B^H, \mathbb{B}^H)$ is referred to as the canonical lift, which parallels the Stratonovich lift of fBM. In this paper, however, we introduce a novel natural lift of fBM in the Itô sense, prompting us to designate the Stratonovich fractional Brownian rough path $\boldsymbol{B}^H = (1, B^H, \mathbb{B}^H)$ as $\boldsymbol{B}^{H,\mathrm{Str}} = (1, B^H, \mathbb{B}^{H,\mathrm{Str}})$. For brevity, we simply use B to denote B^H when there is no risk of confusion.

We define the Itô rough path associated with fBM B^H as follows:

$$\boldsymbol{B}_{s,t}^{H,\mathrm{It\hat{o}}} = (1, B_{s,t}^H, \mathbb{B}_{s,t}^{H,\mathrm{It\hat{o}}}) = (1, B_t^H - B_s^H, \mathbb{B}_{s,t}^{H,\mathrm{Str}} - \varphi_{s,t}),$$

where the Hurst parameter H is within $(\frac{1}{3}, \frac{1}{2}]$, and $\varphi_{s,t} = \varphi(t) - \varphi(s)$ with $\varphi(t)$ being a continuous path of finite $\frac{p}{2}$-variation. It is established that $\boldsymbol{B}^{H,\mathrm{It\hat{o}}} = (1, B^H, \mathbb{B}^{H,\mathrm{It\hat{o}}})$ constitutes a random rough path, albeit a non-geometric one. We term this non-geometric rough path the Itô fractional Brownian rough path, as the rough path and its integration theory extend the traditional frameworks of standard Brownian motion and Itô stochastic integration. To ensure that the expectation of the first-level integral of an integrand against the Itô fractional Brownian rough path is zero, the function $\varphi(t)$ depends on the integrand. This dependency will be determined in later sections. To simplify the notation, we denote it by $\boldsymbol{B}_{s,t} = (1, B_{s,t}, \mathbb{B}_{s,t})$ where appropriate to avoid confusion.

3 Itô Integration Against Fractional Brownian Motion

3.1 Itô Integrals of one Forms Against fBM

The objective of this section is to define Itô integrals of one-forms with respect to fractional Brownian motion, exemplified by expressions such as $\int_s^t F(B)\,dB$, where $B = (B^{(1)}, \ldots, B^{(d)})$ represents a d-dimensional fBM characterized by a Hurst parameter H in the range $(\frac{1}{3}, \frac{1}{2}]$.

In order to define $\int_s^t F(B)dB$, where $F : \mathbb{R}^d \to L(\mathbb{R}^d, \mathbb{R}^e)$ satisfies some smoothness conditions, according to the rough path theory of Lyons, we should take B as a rough path. Actually the symbol $\int_s^t F(B)dB$, to some extent, is a misleading statement. Recall that the rough integral $\int F(X)d\boldsymbol{X}$ against a rough path $\boldsymbol{X} = (1, X, \mathbb{X}) \in \Omega_p(\mathbb{R}^d)$ with $2 \leq p < 3$ is defined to be again a rough path. The rough integral is defined uniquely by its associated almost rough path $\widehat{\boldsymbol{Y}} = (1, \widehat{Y}, \widehat{\mathbb{Y}})$ as following (see e.g. Definition 5.2.1 in Lyons and Qian [25]):

$$\widehat{Y}_{s,t} = F(X_s)X_{s,t} + DF(X_s)\mathbb{X}_{s,t}, \tag{3.1}$$

$$\widehat{\mathbb{Y}}_{s,t} = F(X_s) \otimes F(X_s)\mathbb{X}_{s,t}, \tag{3.2}$$

and the integral $\int F(X)d\boldsymbol{X}$ is defined to be the rough path $\boldsymbol{Y} = (1, Y, \mathbb{Y})$ uniquely associated with the almost rough path $\widehat{\boldsymbol{Y}} = (1, \widehat{Y}, \widehat{\mathbb{Y}})$ (see e.g. Theorem 5.2.1 in [25]). The integral can be written in compensated Riemann sum form, that is,

$$Y_{s,t} = \int_s^t F(X)d_{\mathfrak{R}_1}\boldsymbol{X} := \lim_{|D| \to 0} \sum_\ell \left(F(X_{t_{\ell-1}})X_{t_{\ell-1},t_\ell} + DF(X_{t_{\ell-1}})\mathbb{X}_{t_{\ell-1},t_\ell} \right) \tag{3.3}$$

and the second level

$$\mathbb{Y}_{s,t} = \int_s^t F(X) d_{\mathfrak{R}_2} \boldsymbol{X} := \lim_{|D| \to 0} \sum_\ell \left(Y_{s,t_{\ell-1}} \otimes Y_{t_{\ell-1},t_\ell} + F(X_{t_{\ell-1}}) \otimes F(X_{t_{\ell-1}}) \mathbb{X}_{t_{\ell-1},t_\ell} \right). \quad (3.4)$$

These limits exist in the p-variation distance.

Using the general definition provided earlier, we calculate the integral against the Stratonovich fractional Brownian rough path $\boldsymbol{B}^{H,\mathrm{Str}}$, denoted as $\boldsymbol{X}^{\mathrm{Str}} := \int F(B^H) \circ d\boldsymbol{B}^{H,\mathrm{Str}}$. For clarity and simplicity, we use $\boldsymbol{X}^{\mathrm{Str}} = \int F(B) \circ d\boldsymbol{B}$ to represent this Stratonovich integral, which is defined against the rough path $\boldsymbol{B}^{H,\mathrm{Str}}$.

Similarly, we define the Itô integral $\int F(B^H) d\boldsymbol{B}^{H,\mathrm{It\hat{o}}}$ against the rough path $\boldsymbol{B}^{H,\mathrm{It\hat{o}}}$, where $\varphi(t) = \frac{1}{2} t^{2H}$. This integral is denoted as $\boldsymbol{X}^{\mathrm{It\hat{o}}} := \int F(B) d\boldsymbol{B}$, simplifying the notation for practical purposes.

3.1.1   Relation Between Stratonovich and Itô Rough Integrals (I)

Now we establish a relation between Stratonovich and Itô integrals.

Theorem 3.1   *The relation between Stratonovich and Itô integral is given as the following.*

(i) *For the first level,*

$$X_{s,t}^{\mathrm{Str}} - X_{s,t}^{\mathrm{It\hat{o}}} = \frac{1}{2} \int_s^t DF(B_u) du^{2H}, \quad (3.5)$$

(ii) *For the second level,*

$$\mathbb{X}_{s,t}^{\mathrm{Str}} - \mathbb{X}_{s,t}^{\mathrm{It\hat{o}}} = \frac{1}{2} \int_s^t F(B_u) \otimes F(B_u) du^{2H} + \frac{1}{2} \int_s^t \left(\int_s^u DF(B_r) dr^{2H} \right) \otimes dX_{0,u}^{\mathrm{Str}}$$

$$+ \frac{1}{2} \int_s^t X_{s,u}^{\mathrm{Str}} \otimes DF(B_u) du^{2H} - \frac{1}{4} \int_s^t \left(\int_s^u DF(B_r) dr^{2H} \right) DF(B_u) du^{2H}, \quad (3.6)$$

where the last four integrals are Young integrals.

See Appendix 6.2 for the proof. The Stratonovich–Itô correction follows from more general results on the relation between geometric and non-geometric lifts, see e.g. Theorem 5.4.2 in Lyons and Qian [25] with $\varphi_t = \frac{1}{2} I t^{2H}$ there.

3.1.2   Relation Between Stratonovich and Itô Rough Integrals (II)

Let us introduce the space-time Stratonovich/Itô path $\widetilde{B} = (B, t)$, where the first level is given by

$$\widetilde{B}_{s,t} = (B_{s,t}, t - s),$$

and the second level is given by

$$\widetilde{\mathbb{B}}_{s,t} = \left(\mathbb{B}_{s,t}, \int_s^t B_{s,u} du, \int_s^t (u - s) dB_u, \frac{1}{2} (t - s)^2 \right),$$

where the cross integrals are Young integrals, and $\mathbb{B}$ is the second level Stratonovich or Itô lift of fBM. Naturally

$$\int F(B, t) d\boldsymbol{B} := \int f(\widetilde{B}) d\widetilde{\boldsymbol{B}}, \quad (3.7)$$

with $f(x, t)(\xi, \tau) = F(x, t)\xi$ and the right hand side is well defined as an integral for rough paths. We use the symbol $\int F(B, t) d\boldsymbol{B} =: \boldsymbol{X}^{\mathrm{It\hat{o}}}$ as Itô integral and the symbol $\int F(B, t) \circ d\boldsymbol{B} =: \boldsymbol{X}^{\mathrm{Str}}$ as Stratonovich integral. We can also establish the relationship between the Stratonovich and Itô integrals.

Theorem 3.2 *The relation between Stratonovich and Itô integral is given as the following.*

(i) *For the first level,*

$$X_{s,t}^{\mathrm{Str}} - X_{s,t}^{\mathrm{Itô}} = \frac{1}{2}\int_s^t D_x F(B_u, u) du^{2H}, \tag{3.8}$$

(ii) *For the second level,*

$$\mathbb{X}_{s,t}^{\mathrm{Str}} - \mathbb{X}_{s,t}^{\mathrm{Itô}} = \frac{1}{2}\int_s^t F(B_u, u) \otimes F(B_u, u) du^{2H} + \frac{1}{2}\int_s^t \left(\int_s^u D_x F(B_r) dr^{2H}\right) \otimes dX_{0,u}^{\mathrm{Str}}$$

$$+ \frac{1}{2}\int_s^t X_{s,u}^{\mathrm{Str}} \otimes D_x F(B_u) du^{2H}$$

$$- \frac{1}{4}\int_s^t \left(\int_s^u D_x F(B_r, r) dr^{2H}\right) D_x F(B_u, u) du^{2H}, \tag{3.9}$$

where the last four integrals are Young integrals.

The proof can be found in Appendix 6.2. The inhomogeneous case follows with the space-time lift. See also, for example, Theorem 5.4.2 in Lyon and Qian [25].

3.2 The Itô Formula

3.2.1 Homogeneous Itô Type Formula

Let $\boldsymbol{X} = (1, X, \mathbb{X}) \in \Omega_p(\mathbb{R}^d)$, $2 \leq p < 3$ be a p-rough path, and $F : \mathbb{R}^d \to L(\mathbb{R}^d, \mathbb{R}^e)$ be a $\mathrm{Lip}(\gamma)$ function for some $\gamma > p$ (see Definition 5.1.1 in [25]). Since often composes with rough paths, we want to make the function $F(X)$ into a rough path $F_{\mathfrak{R}}(\boldsymbol{X}) = (1, F_{\mathfrak{R}}(\boldsymbol{X})^1, F_{\mathfrak{R}}(\boldsymbol{X})^2)$. In terms of rough path integrals, we use the formula

$$F_{\mathfrak{R}}(\boldsymbol{X}) = \int DF(X) d\boldsymbol{X} \tag{3.10}$$

as a definition of image $F_{\mathfrak{R}}$ of function F. Actually $F_{\mathfrak{R}}$ sends a rough path to another rough path, so that $F_{\mathfrak{R}} : \Omega_p(\mathbb{R}^d) \to \Omega_p(\mathbb{R}^e)$, $2 \leq p < 3$. By the definition of rough path integrals,

$$F_{\mathfrak{R}}(\boldsymbol{X})_{s,t}^1 = \int_s^t DF(X) d_{\mathfrak{R}_1}\boldsymbol{X} = \lim_{|D| \to 0}\sum_\ell (DF(X_{t_{\ell-1}})X_{t_{\ell-1}, t_\ell} + D^2 F(X_{t_{\ell-1}})\mathbb{X}_{t_{\ell-1}, t_\ell}), \tag{3.11}$$

$$F_{\mathfrak{R}}(\boldsymbol{X})_{s,t}^2 = \int_s^t DF(X) d_{\mathfrak{R}_2}\boldsymbol{X}$$

$$= \lim_{|D| \to 0}\sum_\ell (F_{\mathfrak{R}}(\boldsymbol{X})_{s,t_{\ell-1}}^1 \otimes F_{\mathfrak{R}}(\boldsymbol{X})_{t_{\ell-1}, t_\ell}^1$$

$$+ DF(X_{t_{\ell-1}}) \otimes DF(X_{t_{\ell-1}})\mathbb{X}_{t_{\ell-1}, t_\ell}). \tag{3.12}$$

Let $\boldsymbol{X}(\varphi)_{s,t} = (1, \boldsymbol{X}(\varphi)_{s,t}^1, \boldsymbol{X}(\varphi)_{s,t}^2) := (1, X_{s,t}, \mathbb{X}_{s,t} + \varphi_{s,t})$ be a perturbation of the rough path $\boldsymbol{X} = (1, X, \mathbb{X})$, where $\varphi_{s,t} = \varphi(t) - \varphi(s)$ (additive) is a continuous finite q-variation with $q \leq \frac{p}{2}$. Assume that $\boldsymbol{X}$ is a geometric rough path, then $\boldsymbol{X}(\varphi)$ is no longer a geometric rough path in general. Define the composition $F_{\mathfrak{R}}(\boldsymbol{X}(\varphi)) = (1, F_{\mathfrak{R}}(\boldsymbol{X}(\varphi))^1, F_{\mathfrak{R}}(\boldsymbol{X}(\varphi))^2)$ as

$$F_{\mathfrak{R}}(\boldsymbol{X}(\varphi)) = \int DF(X(\varphi)) d\boldsymbol{X}(\varphi). \tag{3.13}$$

We have the following Itô type formula.

Theorem 3.3 (Itô type formula)   *Assume that $\boldsymbol{X} \in \Omega_p(\mathbb{R}^d)$ with $2 \le p < 3$, $\boldsymbol{X}(\varphi)$ is a perturbation of the rough path $\boldsymbol{X}$ as above, and $F : \mathbb{R}^d \to L(\mathbb{R}^d, \mathbb{R}^e)$ is a $\mathrm{Lip}(\gamma)$ function for some $\gamma > p$, then*

(i) $F_{\mathfrak{R}}(\boldsymbol{X}(\varphi))^1_{s,t} = F(X_t) - F(X_s)$.

(ii) *For the first level,*

$$F(X_t) - F(X_s) = \int_s^t DF(X) d_{\mathfrak{R}_1} \boldsymbol{X} + \int_s^t D^2 F(X_u) d\varphi_u.$$

(iii) *For the second level,*

$$F_{\mathfrak{R}}(\boldsymbol{X}(\varphi))^2_{s,t} = \int_s^t DF(X) d_{\mathfrak{R}_2} \boldsymbol{X} + \int_s^t DF(X_u) \otimes DF(X_u) d\varphi_u$$

$$+ \int_s^t \left(\int_s^u D^2 F(X_r) d\varphi_r \right) \otimes dF(X_u) + \int_s^t (F(X_u) - F(X_s)) \otimes D^2 F(X_u) d\varphi_u$$

$$- \int_s^t \left(\int_s^u D^2 F(X_r) d\varphi_r \right) D^2 F(X_u) d\varphi_u,$$

where the last four integrals are Young integrals, and $F_{\mathfrak{R}}(\boldsymbol{X}(\varphi))^2_{s,t}$ can be viewed as a kind of geometric increments of the second level process.

Proof   (i) The equality holds for any continuous path X with finite variation and its canonical lift as rough paths. Then by definition of geometric rough paths, it can be approximated by a sequence of path with finite variation in p-variation. By continuity of both sides, we know the equality still holds. (ii) and (iii) can be proved by the same arguments as in Theorem 3.2. A proof can also be found in Friz and Hairer [17] as the proof of Equation (5.5) there.   □

Remark 3.4   In integration theory for rough paths, if $\boldsymbol{X} \in \Omega_p$, $2 \le p < 3$, one cannot just write a symbol $dF(X_t)$ as the ordinary case. There is no meaning for this symbol unless under the sense of Young integrals, if it is well defined. We can see an example below which says that the $dF(X_t)$ is undefined in general. Actually if we want to make sense the differential symbol, we should lift $F(X_t)$ to a rough path $F_{\mathfrak{R}}(\boldsymbol{X})$ as above and then understand the differential symbol as

$$dF_{\mathfrak{R}}(\boldsymbol{X}) = d(F_{\mathfrak{R}}(\boldsymbol{X})^1, F_{\mathfrak{R}}(\boldsymbol{X})^2).$$

To clarify the above remark, we first give a lemma below.

Lemma 3.5   *Let $\boldsymbol{Y} = (1, Y, \mathbb{Y})$ be a rough path. Then the integral*

$$\int_s^t d\boldsymbol{Y} = (1, Y_{s,t}, \mathbb{Y}_{s,t}) \tag{3.14}$$

as expected.

Example 3.6   Take $\boldsymbol{Y}$ as $F_{\mathfrak{R}}(\boldsymbol{X})$, $F_{\mathfrak{R}}(\boldsymbol{X}(\varphi))$. Then by Lemma 3.5, we have

$$\int_s^t dF_{\mathfrak{R}}(\boldsymbol{X}) = (1, F_{\mathfrak{R}}(\boldsymbol{X})^1_{s,t}, F_{\mathfrak{R}}(\boldsymbol{X})^2_{s,t}),$$

$$\int_s^t dF_{\mathfrak{R}}(\boldsymbol{X}(\varphi)) = (1, F_{\mathfrak{R}}(\boldsymbol{X}(\varphi))^1_{s,t}, F_{\mathfrak{R}}(\boldsymbol{X}(\varphi))^2_{s,t}).$$

Actually $F(X(\varphi)_t) = F(X_t)$, but $dF_{\mathfrak{R}}(\boldsymbol{X}) \ne dF_{\mathfrak{R}}(\boldsymbol{X}(\varphi))$, even for the first level as we can see that $F_{\mathfrak{R}}(\boldsymbol{X})^1_{s,t} \ne F_{\mathfrak{R}}(\boldsymbol{X}(\varphi))^1_{s,t}$ by Itô formula above. Therefore the symbol $dF(X_t)$ or $dF(X(\varphi)_t)$ for rough paths can lead to confusion.

Remark 3.7 If X is a geometric rough path, by Theorem 3.3, we have

$$F_{\mathfrak{R}}(\boldsymbol{X})^1_{s,t} = F(X^1_t) - F(X^1_s),$$

i.e. $F_{\mathfrak{R}}(\boldsymbol{X})^1_{s,t} = F(X^1)_{s,t}$. However, for the non-geometric rough path we do not have the equality alike. In fact, in general,

$$F_{\mathfrak{R}}(\boldsymbol{X}(\varphi))^1_{s,t} \neq F(X(\varphi)^1_t) - F(X(\varphi)^1_s)(= F(X^1_t) - F(X^1_s)).$$

We next want to establish an Itô type formula for integrals against Itô fractional Brownian rough path. Set $\boldsymbol{X}(\varphi) = \boldsymbol{B}^{\mathrm{Str}}$, $\boldsymbol{X} = \boldsymbol{B}^{\mathrm{Itô}}$. As a corollary, we have

Corollary 3.8 (Itô formula for fBM) *Let $\boldsymbol{B}^{\mathrm{Str}}$ be fractional Brownian rough path with Hurst parameter $\frac{1}{3} < H \leq \frac{1}{2}$ enhanced under Stratonovich sense, $\boldsymbol{B}^{\mathrm{Itô}}$ be the Itô fractional Brownian rough path, and $F : \mathbb{R}^d \to L(\mathbb{R}^d, \mathbb{R}^e)$ be a $\mathrm{Lip}(\gamma)$ function for some $H\gamma > 1$. Then*

(i) $F(B_t) - F(B_s) = F_{\mathfrak{R}}(\boldsymbol{B}^{\mathrm{Str}})^1_{s,t} = \int_s^t DF(B) \circ d_{\mathfrak{R}_1} \boldsymbol{B}^{\mathrm{Str}}$.

(ii) *For the first level,*

$$F(B_t) - F(B_s) = \int_s^t DF(B) d_{\mathfrak{R}_1} \boldsymbol{B}^{\mathrm{Itô}} + \frac{1}{2} \int_s^t D^2 F(B_u) du^{2H}.$$

(iii) *For the second level,*

$$\begin{aligned}
F_{\mathfrak{R}}(\boldsymbol{B}^{\mathrm{Str}})^2_{s,t} &= \int_s^t DF(B) d_{\mathfrak{R}_2} \boldsymbol{B}^{\mathrm{Itô}} + \frac{1}{2} \int_s^t DF(B_u) \otimes DF(B_u) du^{2H} \\
&\quad + \frac{1}{2} \int_s^t \left(\int_s^u D^2 F(B_r) dr^{2H} \right) \otimes dF(B_u) \\
&\quad + \frac{1}{2} \int_s^t (F(B_u) - F(B_s)) \otimes D^2 F(B_u) du^{2H} \\
&\quad - \frac{1}{4} \int_s^t \left(\int_s^u D^2 F(B_r) dr^{2H} \right) \otimes D^2 F(B_u) du^{2H},
\end{aligned}$$

where the last four integrals are Young integrals.

3.2.2 Inhomogeneous Itô Formula

In the following, we want to make sense of $F_{\mathfrak{R}}((\boldsymbol{X}, t))$ when the inhomogeneous function $F(x, t)$ is applied to a rough path $\boldsymbol{X} = (1, X, \mathbb{X}) \in \Omega_p$ and establish Itô formula for it. First, recall the space-time rough path $\widetilde{\boldsymbol{X}} = (\boldsymbol{X}, t)$, where the first level is given by

$$\widetilde{X}_{s,t} = (X_{s,t}, t - s),$$

and the second level is given by

$$\widetilde{\mathbb{X}}_{s,t} = \left(\mathbb{X}_{s,t}, \int_s^t X_{s,u} du, \int_s^t (u - s) dX_u, \frac{1}{2}(t - s)^2 \right),$$

where the cross integrals are Young integrals. Define the rough path $F_{\mathfrak{R}}((\boldsymbol{X}, t))$ by

$$F_{\mathfrak{R}}((\boldsymbol{X}, t)) := F_{\mathfrak{R}}(\widetilde{\boldsymbol{X}}) = \int DF(\widetilde{X}) d\widetilde{\boldsymbol{X}}, \tag{3.15}$$

where $DF(x, t)(\xi, \tau) = D_x F(x, t)\xi + D_t F(x, t)\tau$.

Note that if $\boldsymbol{X}(\varphi)$ is a perturbation of the rough path $\boldsymbol{X}$, and $\widetilde{\boldsymbol{X}}(\varphi)$ is its associated space-time rough path, then

$$\widetilde{X}(\varphi)_{s,t} = \widetilde{X}_{s,t},$$

$$\widetilde{\mathbb{X}}(\varphi)_{s,t} = \left(\mathbb{X}_{s,t} - \varphi_{s,t}, \int_s^t X_{s,u} du, \int_s^t (u-s) dX_u, \frac{1}{2}(t-s)^2 \right).$$

Note that only the first $d \times d$ dimensional components of the second level of $\widetilde{\boldsymbol{X}}$ are changed.

Theorem 3.9 (Itô formula)   *Assume* $\boldsymbol{X} \in \Omega_p(\mathbb{R}^d)$ *with* $2 \le p < 3$, $\boldsymbol{X}(\varphi)$ *is a perturbation of the rough path* $\boldsymbol{X}$, *and* $\widetilde{\boldsymbol{X}}$, $\widetilde{\boldsymbol{X}}(\varphi)$ *are their associated space-time rough path respectively,* $F : \mathbb{R}^{d+1} \to L(\mathbb{R}^{d+1}, \mathbb{R}^e)$ *be a* $\mathrm{Lip}(\gamma)$ *function for some* $\gamma > p$.

(i) *We have the basic calculus formula*:

$$F(X_t, t) - F(X_s, s) = \int_s^t DF(\widetilde{X}(\varphi)) d_{\mathfrak{R}_1} \widetilde{\boldsymbol{X}}(\varphi). \tag{3.16}$$

(ii) *For the first level,*

$$F(X_t, t) - F(X_s, s) = \int_s^t DF(\widetilde{X}) d_{\mathfrak{R}_1} \widetilde{\boldsymbol{X}} + \int_s^t D_x^2 F(X_u, u) d\varphi_u. \tag{3.17}$$

(iii) *For the second level,*

$$\begin{aligned}
F_{\mathfrak{R}}((\boldsymbol{X}(\varphi), t))_{s,t}^2 &= \int_s^t DF(\widetilde{X}) d_{\mathfrak{R}_2} \widetilde{\boldsymbol{X}} + \int_s^t D_x F(X_u, u) \otimes D_x F(X_u, u) d\varphi_u \\
&\quad + \int_s^t \left(\int_s^u D_x^2 F(X_r, r) d\varphi_r \right) \otimes dF(X_u, u) \\
&\quad + \int_s^t (F(X_u, u) - F(X_s, s)) \otimes D_x^2 F(X_u, u) d\varphi_u \\
&\quad - \int_s^t \left(\int_s^u D_x^2 F(X_r, r) d\varphi_r \right) \otimes D_x^2 F(X_u, u) d\varphi_u,
\end{aligned}$$

where the last four integrals are Young integrals.

Proof   (i) The proof is same as (i) of Theorem 3.3, first for $p = 1$ it holds, then the result for any geometric rough path follows from the continuity. For (ii) and (iii), note that

$$\left| \int_s^t X_{s,u} du \right| = o(|t - s|)$$

and

$$\left| \int_s^t (u - s) d_{\mathfrak{R}_1} \boldsymbol{X}_u \right| = o(|t - s|),$$

the rest of proof is same as Theorem 3.2. $\square$

Note that if X_t is a continuous path with finite variation, then Equation (3.16) reads as

$$F(X_t, t) - F(X_s, s) = \int_s^t D_x F(X_u, u) dX_u + \int_s^t D_u F(X_u, u) du, \tag{3.18}$$

and Equation (3.17) becomes

$$F(X_t, t) - F(X_s, s) = \int_s^t D_x F(X_u, u) dX_u + \int_s^t D_u F(X_u, u) du + \int_s^t D_x^2 F(X_u, u) d\varphi_u. \tag{3.19}$$

These equations are just like Itô formulas in terms of the Stratonovich and Itô integrals in stochastic calculus.

Now set $\widetilde{\boldsymbol{X}}(\varphi) = \widetilde{\boldsymbol{B}}^{\mathrm{Str}}$, $\widetilde{\boldsymbol{X}} = \widetilde{\boldsymbol{B}}^{\mathrm{Itô}}$ are the associated space-time rough paths of Stratonovich fractional Brownian rough path $\boldsymbol{B}^{\mathrm{Str}}$ and Itô rough path $\boldsymbol{B}^{\mathrm{Itô}}$, respectively.

Corollary 3.10 (Itô formula for fractional Brownian rough path) *Let $\boldsymbol{B}^{\text{Str}}$ be fractional Brownian rough path with Hurst parameter $\frac{1}{3} < H \leq \frac{1}{2}$ enhanced under Stratonovich sense, $\boldsymbol{B}^{\text{Itô}}$ be the Itô fractional Brownian rough path, and $F : \mathbb{R}^{d+1} \to L(\mathbb{R}^{d+1}, \mathbb{R}^e)$ be a $\text{Lip}(\gamma)$ function for some $H\gamma > 1$, then*

(i) $F(B_t, t) - F(B_s, s) = F_{\mathfrak{R}}((\boldsymbol{B}^{\text{Str}}, t))^1_{s,t} = \int_s^t DF(\widetilde{B}) \circ d_{\mathfrak{R}_1} \widetilde{\boldsymbol{B}}^{\text{Str}}.$

(ii) *For the first level,*

$$F(B_t, t) - F(B_s, s) = \int_s^t DF(\widetilde{B}) d_{\mathfrak{R}_1} \widetilde{\boldsymbol{B}}^{\text{Itô}} + \frac{1}{2} \int_s^t D_x^2 F(B_u, u) du^{2H}. \tag{3.20}$$

(iii) *For the second level,*

$$
\begin{aligned}
F_{\mathfrak{R}}((\boldsymbol{B}^{\text{Str}}, t))^2_{s,t} = {} & \int_s^t DF(\widetilde{B}) d_{\mathfrak{R}_2} \widetilde{B}^{\text{Itô}} + \frac{1}{2} \int_s^t D_x F(B_u, u) \otimes D_x F(B_u, u) du^{2H} \\
& + \frac{1}{2} \int_s^t \left(\int_s^u D_x^2 F(B_r, r) dr^{2H} \right) \otimes dF(B_u, u) \\
& + \frac{1}{2} \int_s^t (F(B_u, u) - F(B_s, s)) \otimes D_x^2 F(B_u, u) du^{2H} \\
& - \frac{1}{4} \int_s^t \left(\int_s^u D_x^2 F(B_r, r) dr^{2H} \right) \otimes D_x^2 F(B_u, u) du^{2H},
\end{aligned}
\tag{3.21}
$$

where the last four integrals are Young integrals.

3.3 Differential Equations Driven by Rough Paths

3.3.1 Basics of Differential Equations

In this subsection, we summarize key concepts regarding differential equations driven by rough paths, referencing the framework outlined in [25]. We will revisit the definition of these differential equations to provide an understanding for the discussions that follow.

Definition 3.11 *Let $f : W \to L(V, W)$ be a vector field on W. Let $y_0 \in W$ and let $\boldsymbol{X} \in \Omega_p(V)$, for $2 \leq p < 3$. Then we say that a rough path $\boldsymbol{Y} \in \Omega_p(W)$ is a solution to the following initial value problem:*

$$d\boldsymbol{Y} = f(Y) d\boldsymbol{X}, \quad Y_0 = y_0 \tag{3.22}$$

if there is a rough path $\boldsymbol{Z} \in \Omega_p(V \oplus W)$ such that $\pi_V(\boldsymbol{Z}) = \boldsymbol{X}$, $\pi_W(\boldsymbol{Z}) = \boldsymbol{Y}$ and

$$\boldsymbol{Z} = \int \widehat{f}(Z) d\boldsymbol{Z}, \tag{3.23}$$

where $\widehat{f} : V \oplus W \to L(V \oplus W, V \oplus W)$ is defined by

$$\widehat{f}(x, y)(\xi, \eta) = (\xi, f(y_0 + y)\xi), \quad \forall (x, y), (\xi, \eta) \in V \oplus W.$$

If the vector field $f \in C^3(W, L(V, W))$ satisfies the linear growth and Lipschitz continuous conditions, then the existence and uniqueness of a solution are ensured (see Theorem 6.2.1 and Corollary 6.2.2 in [25]). We will use $\Phi(y_0, \boldsymbol{X})$ to denote the unique solution $\boldsymbol{Y}$, call the map $\boldsymbol{X} \to \Phi(y_0, \boldsymbol{X})$ Itô map defined by differential equation (3.22), whose Lipschitz continuity in p-variation topology can be proved under the same conditions above (see Theorem 6.2.2 in [25]). This is an important result of Itô maps in the framework of differential equations driven by rough paths.

3.3.2 Relation Between Differential Equations Driven by Different Rough Paths

In this subsection, our main goal is to show the relationship of differential equations driven by Stratonovich fractional Brownian rough path and Itô fractional Brownian rough path, respectively. Define $\Phi(x, \boldsymbol{B}^{\mathrm{Str}})$ as the Itô map of the differential equation

$$d\boldsymbol{X} = f(X)d\boldsymbol{B}^{\mathrm{Str}}, \quad X_0 = x, \tag{3.24}$$

where $\boldsymbol{B}^{\mathrm{Str}}$ is the Stratonovich fractional Brownian rough path, and we use $d\boldsymbol{B}^{\mathrm{Str}}$ to denote the equation driven by Stratonovich rough path, which sometime we use $\circ d\boldsymbol{B}^{\mathrm{Str}}$ instead. Respectively, let $I(x, \boldsymbol{B}^{\mathrm{Itô}})$ denote the Itô map of the differential equations driven by the Itô fractional rough path

$$d\boldsymbol{X} = f(X)d\boldsymbol{B}^{\mathrm{Itô}}, \quad X_0 = x. \tag{3.25}$$

Now we want to ask what is the relationship between $I(x, \boldsymbol{B}^{\mathrm{Itô}})$ and $\Phi(x, \boldsymbol{B}^{\mathrm{Str}})$ or if the Itô differential equation has a representation in terms of a Stratonovich differential equation. First, we introduce a geometric rough path $\boldsymbol{B}^{\mathrm{Str},\varphi}$ defined by

$$B_{s,t}^{\mathrm{Str},\varphi} := (B_{s,t}, t^{2H} - s^{2H}),$$

and

$$\mathbb{B}_{s,t}^{\mathrm{Str},\varphi} := \left(\mathbb{B}_{s,t}^{\mathrm{Str}}, \int_s^t B_{s,u} du^{2H}, \int_s^t (u^{2H} - s^{2H}) dB_u, \frac{1}{2}(t^{2H} - s^{2H})^2 \right),$$

where the cross integrals are Young integrals. Let $\Phi_{\widetilde{f}}(x, \boldsymbol{B}^{\mathrm{Str},\varphi})$ be the Itô map defined by the differential equation

$$d\boldsymbol{X} = \widetilde{f}(X)d\boldsymbol{B}^{\mathrm{Str},\varphi}, \quad X_0 = x, \tag{3.26}$$

where $\widetilde{f} : \mathbb{R}^e \to L(\mathbb{R}^d \oplus \mathbb{R}, \mathbb{R}^e)$,

$$\widetilde{f}(x)(\xi, \eta) := f(x)\xi - \frac{1}{2}\eta Df(x)f(x),$$

for all $x \in \mathbb{R}^e$, $(\xi, \eta) \in \mathbb{R}^d \oplus \mathbb{R}$. Namely, $\Phi_{\widetilde{f}}(\cdot, \boldsymbol{B}^{\mathrm{Str},\varphi})$ is the Itô map of the rough differential equation

$$d\boldsymbol{X} = f(X)d\boldsymbol{B}^{\mathrm{Str}} - \frac{1}{2}Df(X_t)f(X_t)dt^{2H}. \tag{3.27}$$

Theorem 3.12 *Let $f : \mathbb{R}^e \to L(\mathbb{R}^d, \mathbb{R}^e)$ be a C^4 vector field, and $I(x, \boldsymbol{B}^{\mathrm{Itô}}), \Phi_{\widetilde{f}}(x, \boldsymbol{B}^{\mathrm{Str},\varphi})$ be the Itô map defined by* $(3.25), (3.26)$, *respectively. Then*

$$I(x, \boldsymbol{B}^{\mathrm{Itô}})_{s,t}^1 = \Phi_{\widetilde{f}}(x, \boldsymbol{B}^{\mathrm{Str},\varphi})_{s,t}^1 \tag{3.28}$$

and

$$I(x, \boldsymbol{B}^{\mathrm{Itô}})_{s,t}^2 = \Phi_{\widetilde{f}}(x, \boldsymbol{B}^{\mathrm{Str},\varphi})_{s,t}^2 - \frac{1}{2}\int_s^t f(X_u) \otimes f(X_u)du^{2H}, \tag{3.29}$$

where $X_t := \Phi_{\widetilde{f}}(x, \boldsymbol{B}^{\mathrm{Str},\varphi})_{0,t}^1$.

The proof can be found in Appendix 6.2. As a remark, we can see that the differential equations (3.25) and (3.26) (or (3.27)) are not equivalent. As far as the first level concerned, they define the same solution. This agrees with classical stochastic differential equations (SDE) driven by standard Brownian motion, i.e. we translate SDE (3.25) into SDE (3.27) when $H = \frac{1}{2}$. However, in terms of the second level, the two differential equations are different. The reason

is that $I(x, \boldsymbol{B}^{\text{Itô}})^2$ can be viewed as the iterated integral of the first level in Itô's sense, while $\Phi_{\widetilde{f}}(x, \boldsymbol{B}^{\text{Str},\varphi})^2$ is the iterated integral of the first level in Stratonovich sense.

Besides, the relationship between $\Phi(x, \boldsymbol{B}^{\text{Str}})$ and $\Phi_{\widetilde{f}}(x, \boldsymbol{B}^{\text{Str},\varphi})$ is obvious. They are all understood in the Stratonovich sense, the difference is their drift terms. By the continuity of Itô's maps, they can all be approximated in variation topology by the solution of differential equations driven by piece-wise linear approximations of fBM and its iterated path integrals. In summary, we can establish the relationship between $I(x, \boldsymbol{B}^{\text{Itô}})$, $\Phi(x, \boldsymbol{B}^{\text{Str}})$ and $\Phi_{\widetilde{f}}(x, \boldsymbol{B}^{\text{Str},\varphi})$.

3.4 Examples and Applications

In this subsection, we will give some interesting examples as applications of fractional Brownian rough paths.

Example 3.13 Consider the differential equation in dimension $d = 1$,

$$d\boldsymbol{X} = \sigma X \circ d\boldsymbol{B}^{\text{Str}}.$$

Then $\boldsymbol{X} = (1, X, \mathbb{X})$ is the solution of this differential equation, where $X_t = \exp(\sigma B_t)$,

$$X_{s,t} = X_t - X_s = \exp(\sigma B_t) - \exp(\sigma B_s),$$

$$\mathbb{X}_{s,t} = \frac{1}{2}(X_{s,t})^2 = \frac{1}{2}(\exp(\sigma B_t) - \exp(\sigma B_s))^2.$$

Example 3.14 Now we consider the Itô rough differential equation

$$d\boldsymbol{X} = \sigma X d\boldsymbol{B}^{\text{Itô}}.$$

(i) Set $X_t = \exp(\sigma B_t - \frac{1}{2}\sigma^2 t^{2H}) =: F(B_t, t)$ and $X_{s,t} := X_t - X_s$. Then

$$X_{s,t} = \int_s^t \sigma X d_{\mathfrak{R}_1} \boldsymbol{B}^{\text{Itô}},$$

and

$$\mathbb{X}_{s,t} := \int_s^t \sigma F(B, u) \circ d_{\mathfrak{R}_2} \boldsymbol{B}^{\text{Str}} - \frac{\sigma^2}{2} \int_s^t (F(B_u, u))^2 du^{2H}$$

$$- \frac{\sigma^2}{2} \int_s^t \left(\int_s^u F(B_r, r) dr^{2H} \right) dY_u - \frac{\sigma^2}{2} \int_s^t Y_{s,u} F(B_u, u) du^{2H}$$

$$+ \frac{\sigma^4}{4} \int_s^t \left(\int_s^u F(B_r, r) dr^{2H} \right) F(B_u, u) du^{2H},$$

where $Y_{s,t} = \int_s^t \sigma F(B, u) \circ d_{\mathfrak{R}_1} \boldsymbol{B}^{\text{Str}}$, and the last four integrals are Young integrals.

Example 3.15 (Geometric fBM or fractional Black–Scholes model) First consider the fractional Black–Scholes model in Stratonovich sense

$$d\boldsymbol{X} = \mu X_t dt + \sigma X \circ d\boldsymbol{B}^{\text{Str}}, \tag{3.30}$$

where μ, σ are constants. The solution $\boldsymbol{X}$ can be constructed as above. Set

$$X_t = \exp(\sigma B_t + \mu t) =: F(B_t, t), \tag{3.31}$$

and

$$X_{s,t} = X_t - X_s,$$

$$\mathbb{X}_{s,t} = \lim_{n\to\infty} \int_s^t F(B_u^n, u) dF(B_u^n, u),$$

where B_u^n is the linear approximation of B, i.e.

$$B_u^n = B_{t_{\ell-1}} + \frac{B_{t_\ell} - B_{t_{\ell-1}}}{t_\ell - t_{\ell-1}}(u - t_{\ell-1})$$

on interval $[t_{\ell-1}, t_\ell]$ and $\{t_\ell^n, \ell = 0, 1, \ldots, n\}$ is any partition of $[s,t]$. We can verify $\boldsymbol{X} = (1, X, \mathbb{X})$ is the solution.

In this situation, the corresponding fractional Black–Scholes market has arbitrage. We change the Stratonovich integral into an Itô integral, i.e. we consider the fractional differential equation in Itô's sense

$$d\boldsymbol{X} = \mu X_t dt + \sigma X d\boldsymbol{B}^{\text{Itô}}. \tag{3.32}$$

We demonstrate later that the corresponding Itô fractional Black–Scholes market is arbitrage free in a restricted sense.

(i) Let

$$X_t = \exp\left(\sigma B_t + \mu t - \frac{1}{2}\sigma^2 t^{2H}\right) =: F(B_t, t). \tag{3.33}$$

Then $X_{s,t} := X_t - X_s$ and

$$X_{s,t} = \int_s^t \sigma X d_{\mathfrak{R}_1} \boldsymbol{B}^{\text{Itô}} + \int_s^t \mu X_u du.$$

By the relation between Stratonovich integrals and Itô integrals in time dependent case, we have

$$\begin{aligned}
RHS &= \int_s^t \sigma F(B, u) d_{\mathfrak{R}_1} \boldsymbol{B}^{\text{Itô}} + \int_s^t \mu F(B_u, u) du \\
&= \int_s^t \sigma F(B, u) \circ d_{\mathfrak{R}_1} \boldsymbol{B}^{\text{Str}} - \frac{\sigma^2}{2} \int_s^t F(B_u, u) du^{2H} + \int_s^t \mu F(B_u, u) du \\
&= \int_s^t D_x F(B, u) \circ d_{\mathfrak{R}_1} \boldsymbol{B}^{\text{Str}} + \int_s^t D_u F(B_u, u) du \\
&= \int_s^t DF(\widetilde{B}) \circ d_{\mathfrak{R}_1} \widetilde{\boldsymbol{B}}^{\text{Str}} \\
&= F(\widetilde{B}_t) - F(\widetilde{B}_s) \\
&= F(B_t, t) - F(B_s, s) \\
&= X_t - X_s \\
&= LHS.
\end{aligned}$$

(ii) Now set

$$\begin{aligned}
\mathbb{X}_{s,t} &:= \mathbb{Z}_{s,t} + \int_s^t \mu Z_{s,u} F(B_u, u) du + \int_s^t \left(\int_s^u \mu F(B_r, r) dr\right) dZ_u \\
&\quad + \int_s^t \left(\int_s^u \mu F(B_r, r) dr\right) \mu F(B_u, u) du,
\end{aligned}$$

where

$$Z_{s,t} = X_{s,t} - \int_s^t \mu F(B_u, u) du,$$

$$\mathbb{Z}_{s,t} = \int_s^t \sigma F(B,u) \circ d_{\mathfrak{R}_2} \boldsymbol{B}^{\mathrm{Str}} - \frac{\sigma^2}{2} \int_s^t (F(B_u,u))^2 du^{2H}$$

$$- \frac{\sigma^2}{2} \int_s^t \left(\int_s^u F(B_r,r) dr^{2H} \right) dY_u - \frac{\sigma^2}{2} \int_s^t Y_{s,u} F(B_u,u) du^{2H}$$

$$+ \frac{\sigma^4}{4} \int_s^t \left(\int_s^u F(B_r,r) dr^{2H} \right) F(B_u,u) du^{2H},$$

and $Y_{s,t} = \int_s^t \sigma F(B,u) \circ d_{\mathfrak{R}_1} \boldsymbol{B}^{\mathrm{Str}}$, and all the integrals except ones involving $\circ d\boldsymbol{B}^{\mathrm{Str}}$ are Young integrals, and the integral against $\circ d\boldsymbol{B}^{\mathrm{Str}}$ is Stratonovich rough integral which can be computed by linear approximations. Actually, by the relation between Stratonovich rough integrals and Itô rough integrals, we have

$$Z_{s,t} = \int_s^t \sigma F(B,u) d_{\mathfrak{R}_1} \boldsymbol{B}^{\mathrm{It\hat{o}}} \quad \text{and} \quad \mathbb{Z}_{s,t} = \int_s^t \sigma F(B,u) d_{\mathfrak{R}_2} \boldsymbol{B}^{\mathrm{It\hat{o}}}.$$

If we define $f(x,t)(\xi,\tau) := \sigma F(x,t)\xi + \mu F(x,t)\tau$, $\widetilde{B} = (\boldsymbol{B},t)$ the space-time rough path of $\boldsymbol{B}$, then $\mathbb{X}_{s,t} = \int_s^t f(\widetilde{B}) d_{\mathfrak{R}_2} \widetilde{\boldsymbol{B}}^{\mathrm{It\hat{o}}}$. Combining (i) and (ii), we have verified that $\boldsymbol{X}_{s,t} = \int_s^t f(\widetilde{B}) d\widetilde{\boldsymbol{B}}^{\mathrm{It\hat{o}}}$. The right hand side indeed coincides with the right hand side of the differential equation (3.32). So we have constructed the solution of the Itô fractional Black–Scholes equation (3.32).

Remark 3.16 As a remark, we have two ways to understand the integrals on the right hand side of above differential equation with drift (3.32). On the one hand, we can define $f(x,t)(\xi,\tau) := \sigma F(x,t)\xi + \mu F(x,t)\tau$, then $\boldsymbol{X}_{s,t} = \int_s^t f(\widetilde{B}) d\widetilde{\boldsymbol{B}}^{\mathrm{It\hat{o}}}$ is well defined. On the other hand, we can define $g(x,t)(\xi,\tau) := \sigma F(x,t)\xi$, $h_t := \int_0^t \mu F(B_u,u) du$ and see $\int_s^t \sigma F(B,u) d\boldsymbol{B}^{\mathrm{It\hat{o}}}$ as $\int_s^t g(\widetilde{B}) d\widetilde{\boldsymbol{B}}^{\mathrm{It\hat{o}}}$ (This integral is well defined). Then view the right hand side of differential equation (3.32) as a perturbation of $\int g(\widetilde{B}) d\widetilde{\boldsymbol{B}}^{\mathrm{It\hat{o}}}$ by h. We want to say that the two ways are consistent, and they give the same results, i.e.,

$$\int_s^t f(\widetilde{B}) d_{\mathfrak{R}_1} \widetilde{\boldsymbol{B}}^{\mathrm{It\hat{o}}} = \int_s^t g(\widetilde{B}) d_{\mathfrak{R}_1} \widetilde{\boldsymbol{B}}^{\mathrm{It\hat{o}}} + \int_s^t \mu F(B_u,u) du, \tag{3.34}$$

$$\int_s^t f(\widetilde{B}) d_{\mathfrak{R}_2} \widetilde{\boldsymbol{B}}^{\mathrm{It\hat{o}}} = \int_s^t g(\widetilde{B}) d_{\mathfrak{R}_2} \widetilde{\boldsymbol{B}}^{\mathrm{It\hat{o}}} + \int_s^t \mu Z_{s,u}^1 F(B_u,u) du + \int_s^t \left(\int_s^u \mu F(B_r,r) dr \right) dZ_u$$

$$+ \int_s^t \left(\int_s^u \mu F(B_r,r) dr \right) \mu F(B_u,u) du, \tag{3.35}$$

where $Z_{s,t} := \int_s^t g(\widetilde{B}) d_{\mathfrak{R}_1} \widetilde{\boldsymbol{B}}^{\mathrm{It\hat{o}}}$, $Z_t := Z_0 + Z_{0,t}$, and the last three integrals are Young Integral. A proof of Equation (3.34) and (3.35) can be found in Appendix 6.2.

3.5 Chain Rule of fBM

This subsection is the continuation of Example 3.15 which plays an important role in the remainder of the paper. Let $\boldsymbol{X}$ be the solution of the fractional Black–Scholes equation (3.32) and let G be a good function, then the integral $\int G(X,t) d\boldsymbol{X}$ is well defined. Since X_t has the explicit representation (3.33),

$$\int \sigma G(X,t) X d\boldsymbol{B}^{\mathrm{It\hat{o}}} + \int \mu G(X_t,t) X_t dt$$

is defined in terms of the rough path $\boldsymbol{B}^{\mathrm{It\hat{o}}}$, which can be defined as $\int f(\widetilde{B}) d\widetilde{\boldsymbol{B}}^{\mathrm{It\hat{o}}}$, where

$$f(x,t)(\xi,\eta) := f^1(x,t)\xi + f^2(x,t)\eta,$$

$$f^1(x,t) = \sigma G(F(x,t),t)F(x,t),$$
$$f^2(x,t) = \mu G(F(x,t),t)F(x,t),$$

and

$$F(x,t) = \exp(\sigma x - \sigma^2 t^{2H}/2 + \mu t).$$

Heuristically, the two integrals should equal. Our aim in this subsection is to show that they are indeed the same in this case. This kind of formula is usually called *Chain Rule*. Namely, we want to show the theorem below.

Theorem 3.17 (Chain rule) *Let $\boldsymbol{B}^{\text{Itô}}$ be the Itô fractional Brownian rough path with $H \in (\frac{1}{3}, \frac{1}{2}]$, and $\boldsymbol{X}$ be the geometric fBM with parameters σ and μ, G be a function smooth enough. Then*

$$\int G(X,t)d\boldsymbol{X} = \int \sigma G(X,t)X d\boldsymbol{B}^{\text{Itô}} + \int \mu G(X_t,t)X_t dt, \tag{3.36}$$

where the integrals are understood as above.

The proof can be found in Appendix 6.2. We mention that with the Stratonovich rough paths, the chain rule still holds by the same argument as Itô rough paths above.

3.6   Zero Mean Property of Itô Integrals

Now, we must verify whether the first level of our It integrals, as defined above, exhibit zero mean. Specifically, we need to confirm that:

$$\mathbb{E}\left[\int_s^t f(B)\, d\boldsymbol{B}^{\text{Itô}} \right] = 0.$$

3.6.1   One Dimensional Case

We prove it in the case that dimension $d = 1$. First, we suppose that f has first and second continuous derivatives, by Itô formula proved above, we can show that

$$\mathbb{E}\left[\int_s^t f'(B) \circ d_{\mathfrak{R}_1}\boldsymbol{B}^{\text{Str}} \right] = \mathbb{E}\left[\frac{1}{2} \int_s^t f''(B_r)dr^{2H} \right]. \tag{3.37}$$

The computation is routine, so we omit the details.

For the general case, set $F(x) = \int_{-\infty}^x f(y)dy$, $F(-\infty) = 0$, so that $F'(x) = f(x)$. By (3.37) we get

$$\mathbb{E}\left[\int_s^t F'(B)d_{\mathfrak{R}_1}\boldsymbol{B}^{\text{Itô}} \right] = \mathbb{E}\left[\frac{1}{2} \int_s^t F''(B_r)dr^{2H} \right]. \tag{3.38}$$

Thus the expectation of the first level of the Itô integral

$$\mathbb{E}\left[\int_s^t f(B)d_{\mathfrak{R}_1}\boldsymbol{B}^{\text{Itô}} \right] = 0. \tag{3.39}$$

3.6.2   High Dimensional Case

Now we can prove that eqn (3.37) still holds when dimension $d \geq 2$ and for any function $F : \mathbb{R}^d \to L(\mathbb{R}^d, \mathbb{R}^e)$. Let $\boldsymbol{X}_{s,t} := \int_s^t F(B)d\boldsymbol{B}^{\text{Itô}}$. The i-th component of first level of this Itô integral is

$$\boldsymbol{X}_{s,t}^{1;i} = \left[\sum_{j=1}^d \int_s^t F^{ij}(B)d_{\mathfrak{R}_1}\boldsymbol{B}^{\text{Itô},(j)} \right].$$

As a remark here, the integral $\int_s^t F^{ij}(B)d\boldsymbol{B}^{\text{Itô},(j)}$ is well-defined, which can be understood as $\int_s^t \widetilde{F}^{ij}(B)d\boldsymbol{B}^{\text{Itô}}$, where $\widetilde{F}^{ij}(x_1, \ldots, x_d)(\xi_1, \ldots, \xi_d) = F^{ij}(x_1, \ldots, x_d)\xi_j$. Therefore, we have the equation $\mathbb{E}[\int_s^t F^{ij}(B)d_{\mathfrak{R}_1}\boldsymbol{B}^{\text{Itô},(j)}] = 0$, which yields that

$$\mathbb{E}\left[\int_s^t F(B)d_{\mathfrak{R}_1}\boldsymbol{B}^{\text{Itô}}\right] = 0. \tag{3.40}$$

3.6.3 Zero Mean Property of Time-Dependent Functions

In this subsection, we will show that for the time-dependent function $F(x,t)$ we can still have the mean zero property, i.e.

$$\mathbb{E}\left[\int_s^t F(B,u)d_{\mathfrak{R}_1}\boldsymbol{B}^{\text{Itô}}\right] = 0. \tag{3.41}$$

As the time independent case, we first show the one dimensional case, then by conditional expectation technique we conclude the high dimensional cases. By the Itô formula (3.20) and Remark 3.16, we know that it is equivalent to

$$F(B_t, t) - F(B_s, s) = \int_s^t D_x F(B, u)d_{\mathfrak{R}_1}\boldsymbol{B}^{\text{Itô}} + \int_s^t D_u F(B_u, u)du$$

$$+ \frac{1}{2}\int_s^t D_x^2 F(B_u, u)du^{2H}. \tag{3.42}$$

Then in order to prove the zero mean property, we should verify that

$$\mathbb{E}(F(B_t, t) - F(B_s, s)) = \int_s^t \mathbb{E}[D_u F(B_u, u)]du + \frac{1}{2}\int_s^t \mathbb{E}[D_x^2 F(B_u, u)]du^{2H}.$$

For the one dimension case, the left-hand side above can be computed as the following

$$\mathbb{E}(F(B_t, t) - F(B_s, s)) = \int_{\mathbb{R}} (F(t^H x, t) - F(s^H x, s))\varphi(x)dx, \tag{3.43}$$

where φ is the standard normal probability density function.

On the other hand,

$$\int_s^t \mathbb{E}[D_u F(B_u, u)]du = \int_s^t \left[\int_{\mathbb{R}} \partial_2 F(u^H x, u)\varphi(x)dx\right]du$$

$$= \int_{\mathbb{R}} \left[(F(t^H x, t) - F(s^H x, s)) - \int_s^t x\partial_1 F(u^H x, u)du^H\right]\varphi(x)dx \tag{3.44}$$

and

$$\frac{1}{2}\int_s^t \mathbb{E}[D_x^2 F(B_u, u)]du^{2H} = \frac{1}{2}\int_s^t \left[\int_{\mathbb{R}} \partial_1^2 F(u^H x, u)\varphi(x)dx\right]du^{2H}$$

$$= \int_s^t \left[\int_{\mathbb{R}} \partial_1 F(u^H x, u)x\varphi(x)dx\right]du^H$$

$$= \int_{\mathbb{R}} \left[\int_s^t \partial_1 F(u^H x, u)du^H\right]x\varphi(x)dx. \tag{3.45}$$

Combining (3.43), (3.44) and (3.45), we thus obtain that

$$\mathbb{E}\left[\int_s^t D_x F(B,u)d_{\mathfrak{R}_1}\boldsymbol{B}^{\text{Itô}}\right] = 0. \tag{3.46}$$

By using $\widetilde{F}(x,t) = \int_{-\infty}^x F(y,t)dy$, we finally get (3.41) as for one dimensional case of one form above.

Now we turn to the high dimensional case. Let

$$\boldsymbol{X}_{s,t}^{1;i} := \left[\sum_{j=1}^{d} \int_{s}^{t} F^{ij}(B,u) d_{\mathfrak{R}_1} \boldsymbol{B}^{\mathrm{It\hat{o}},(j)} \right], \quad i = 1, \ldots, d.$$

We need to prove that $\mathbb{E}[\int_{s}^{t} F^{ij}(B,u) d_{\mathfrak{R}_1} \boldsymbol{B}^{\mathrm{It\hat{o}},(j)}] = 0$. Let $F^{ij} =: f$ for simplicity. Then

$$\mathbb{E}\left[\int_{s}^{t} f(B,u) d_{\mathfrak{R}_1} \boldsymbol{B}^{\mathrm{It\hat{o}},(j)} \right] = \mathbb{E}\left[\mathbb{E}\left[\left(\int_{s}^{t} f(x_1, \ldots, B^{(j)}, \ldots, x_d, u) d_{\mathfrak{R}_1} \boldsymbol{B}^{\mathrm{It\hat{o}},(j)} \right) \right]_{x_i = B^{(i)}, i \neq j} \right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[\left(\int_{s}^{t} \widetilde{f}(B^{(j)}, u) d_{\mathfrak{R}_1} \boldsymbol{B}^{\mathrm{It\hat{o}},(j)} \right) \right]_{x_i = B^{(i)}, i \neq j} \right]$$

$$= 0.$$

Hence $\mathbb{E}[\boldsymbol{X}_{s,t}^{1;i}] = 0$ for $i = 1, \ldots, d$. Thus we have proved that the expectation of time-dependent function is also zero, i.e. Equation (3.41) holds.

4   General Itô Integration for fBM

In the previous section, we introduced the Itô integral for one-forms of fBM. For integrands that are not one-forms, we can similarly extend the concept of Itô integration to accommodate them by adopting a renormalization strategy that transitions from the Stratonovich integral to the Itô interpretation.

4.1   Itô Integration for fOU

In the work by Qian and Xu (2024), [30], the authors introduced the Itô integral for fractional Ornstein–Uhlenbeck (fOU) processes and applied this integral to the problem of parameter estimation for the fOU process.

The multi-dimensional Ornstein–Uhlenbeck (OU) processes, driven by fractional Brownian motions (fBM), are commonly known as fractional Ornstein–Uhlenbeck (fOU) processes. These processes are defined by the solution to the stochastic differential equation (SDE):

$$dX_t = -\Gamma X_t dt + \Sigma dB_t^H, \quad X_0 = x_0.$$

In this equation, B^H represents a d-dimensional fBM with a Hurst parameter H within the interval $(0, 1)$. The matrix Γ in $\mathbb{R}^{d \times d}$ is the symmetric and positive-definite drift matrix, while Σ in $\mathbb{R}^{d \times d}$ is the non-degenerate volatility matrix. This SDE is interpreted as a stochastic integral equation:

$$X_t = x_0 - \int_0^t \Gamma X_s ds + \Sigma B_t^H,$$

which admits a unique solution given by

$$X_t = \mathrm{e}^{-\Gamma t} x_0 + \int_0^t \mathrm{e}^{-\Gamma(t-s)} \Sigma dB_s^H.$$

The integral on the right-hand side is interpreted using Young's integral approach. As a result, like the standard OU processes, the (X_t) series forms a Gaussian process.

To establish a non-geometric Itô rough path enhancement compatible with fBM for analyzing fOU processes where $\frac{1}{3} < H \leq \frac{1}{2}$, we define $\varphi(t) := \frac{1}{2} I t^{2H} - U(t)$, where

$$U(t) = H\Gamma \int_0^t \int_0^s \mathrm{e}^{-\Gamma(s-u)} (s^{2H-1} - (s-u)^{2H-1}) du ds.$$

This function demonstrates finite q-variation, with $q = \frac{1}{2H}$, facilitating the definition of a non-geometric Itô-type fractional Brownian rough path lift for B^H as

$$\boldsymbol{B}_{s,t}^{H,\text{Itô}} = (1, B_{s,t}^H, \mathbb{B}_{s,t}^{H,\text{Itô}}) := (1, B_{s,t}^H, \mathbb{B}_{s,t}^{H,\text{Str}} - \varphi_{s,t}),$$

where $\varphi_{s,t} = \varphi(t) - \varphi(s)$.

Based on the theory of differential equations driven by rough paths and the previously defined integrals, and assuming the coefficient matrices Γ and Σ commute for simplicity, we establish the following relationship:

$$\int_0^t X_s d_{\mathfrak{R}_1} \boldsymbol{B}^{H,\text{Itô}} = \int_0^t X_s \circ d_{\mathfrak{R}_1} \boldsymbol{B}^{H,\text{Str}} - \Sigma \varphi(t).$$

Given that

$$\mathbb{E}\left[\int_0^t X_s \circ d_{\mathfrak{R}_1} \boldsymbol{B}^{H,\text{Str}} \right] = \Sigma \varphi(t),$$

we observe the zero expectation property for fOU processes, meaning:

$$\mathbb{E}\left[\int_0^t X_s d_{\mathfrak{R}_1} \boldsymbol{B}^{H,\text{Itô}} \right] = 0.$$

This formulation reflects the expectation neutrality in the integrals of fOU processes under the specified rough path dynamics. For a more detailed exposition, we refer the reader to the work of Qian and Xu (2024), [30].

4.2 Itô Integration for RDE

In a more general context, consider a stochastic process $\boldsymbol{X}$ that solves the rough differential equation (RDE) driven by a Stratonovich fractional Brownian rough path, formulated as

$$d\boldsymbol{X} = f(X) \circ d\boldsymbol{B}^{H,\text{Str}}, \quad X_0 = x,$$

where $\boldsymbol{B}^{H,\text{Str}}$ is the Stratonovich fractional Brownian rough path with $\frac{1}{3} < H \leq \frac{1}{2}$. To establish a corresponding Itô integral for this process X, we need to convert the Stratonovich integral into an Itô integral, considering the nature of the driving fractional Brownian motion.

The transformation from a Stratonovich integral to an Itô integral, in the context of fractional Brownian motion, is non-trivial. Similar to the above, we approach this by defining a non-geometric Itô-type fractional Brownian rough path lift for B^H as follows:

$$\boldsymbol{B}_{s,t}^{H,\text{Itô}} = (1, B_{s,t}^H, \mathbb{B}_{s,t}^{H,\text{Itô}}) := (1, B_{s,t}^H, \mathbb{B}_{s,t}^{H,\text{Str}} - \varphi_{s,t}),$$

where $\varphi_{s,t} = \varphi(t) - \varphi(s)$.

Consequently, the integral can be related to its Stratonovich counterpart:

$$\int_s^t X_u d_{\mathfrak{R}_1} \boldsymbol{B}^{H,\text{Itô}} = \int_s^t X_u \circ d_{\mathfrak{R}_1} \boldsymbol{B}^{H,\text{Str}} - \int_s^t f(X_u) d\varphi(u).$$

To ensure the expectation neutrality of the Itô integral $\int_s^t X_u d_{\mathfrak{R}_1} \boldsymbol{B}^{H,\text{Itô}}$, the function $\varphi(t)$ must satisfy the following equation:

$$\mathbb{E}\left[\int_s^t X_u \circ d_{\mathfrak{R}_1} \boldsymbol{B}^{H,\text{Str}} \right] = \mathbb{E}\left[\int_s^t f(X_u) d\varphi(u) \right].$$

Here, $\varphi(t) = \varphi(t; H, f)$ depends on the vector field f and Hurst parameter H, $\varphi(0) = 0$, and some regularity assumption needed.

From a computational perspective, the function $\varphi(t)$ can be approximated using a deep neural network $\varphi_\theta(t; H, f)$ with parameters θ that are trainable. To implement this, simulate a batch of N path samples $\{X^i\}_{i=1}^N$ under the Stratonovich framework. The objective is to minimize the loss function defined as

$$L(\theta) = \sum_{[s,t] \subset [0,T]} \left\| \frac{1}{N} \sum_{i=1}^N \left(\int_s^t X_u^i \circ d_{\mathfrak{R}_1} \boldsymbol{B}^{H,\mathrm{Str}} - \int_s^t f(X_u^i) d\varphi_\theta(u) \right) \right\|^2,$$

where both integrals are approximated by their respective discrete sums, i.e.

$$\int_s^t X_u \circ d_{\mathfrak{R}_1} \boldsymbol{B}^{H,\mathrm{Str}} \approx \sum_\ell (X_{t_{\ell-1}} B_{t_{\ell-1},t_\ell}^H + f(X_{t_{\ell-1}}) \mathbb{B}_{t_{\ell-1},t_\ell}^{H,Str}),$$

and

$$\int_s^t f(X_u) d\varphi_\theta(u) \approx \sum_\ell f(X_{t_{\ell-1}})(\varphi_\theta(t_\ell) - \varphi_\theta(t_{\ell-1})).$$

This leaves opportunities for further research in the future.

5 Application in the Fractional Black–Scholes Model

In this section, we extend the study of Example 3.15 presented in Section 3.4 by considering the Itô fractional Black–Scholes model, denoted by fBS, and its associated market. Our goal is to demonstrate that the market is arbitrage-free when the class of trading strategies is restricted. To achieve this, we first provide an arbitrage strategy under the Stratonovich fractional Black–Scholes market.

5.1 Arbitrage Strategy in Stratonovich fBS Market

Since the Stratonovich integral does not have zero mean property, therefore fractional Black–Scholes market based on Stratonovich integral suggests the possibility of existence of arbitrage. Shiryayev gave an arbitrage trading strategy in [33] under Stratonovich fBS market but driven by fBM with Hurst parameter $H > \frac{1}{2}$. We adapt this strategy in our case when $H \in (\frac{1}{3}, \frac{1}{2}]$.

The market has a stock (the risky asset) $\boldsymbol{X}$ whose price process is $X_t := \boldsymbol{X}_{0,t}^1$ at time t. We assume that $\boldsymbol{X}$ satisfies the differential equation driven by Stratonovich fractional Brownian rough path with $H \in (\frac{1}{3}, \frac{1}{2}]$ as the first part of Example 3.15, i.e.

$$d\boldsymbol{X} = \mu X_t dt + \sigma X \circ d\boldsymbol{B}^{\mathrm{Str}}, \quad X_0 = x, \quad t \in [0, T]. \tag{5.1}$$

The solution $\boldsymbol{X}$ of this equation has been constructed in Example 3.15 and $X_t := X_{0,t}^1 = x e^{\sigma B_t + \mu t}$. It is assumed that there is a money market (the risk-less asset) M, that is, an asset whose price at time t is not subject to uncertainty. Namely, the price process M_t satisfies the following equation

$$dM_t = r M_t dt, \quad M_0 = 1, \quad t \in [0, T], \tag{5.2}$$

where $r > 0$ is a constant, i.e. $M_t = e^{rt}$.

A portfolio (γ_t, ζ_t) gives the number of units γ_t, ζ_t held at time t in the money market and stock market, respectively. The value process $V_t \in \mathbb{R}$ of the portfolio is given by

$$V_t = \gamma_t M_t + \zeta_t X_t. \tag{5.3}$$

The portfolio is called *self-financing* if

$$V_t = V_0 + \int_0^t \gamma_s dM_s + \int_0^t \zeta \circ d_{\mathfrak{R}_1} \boldsymbol{X}. \tag{5.4}$$

Note that the second integral on the right hand side is the first level of the Stratonovich integral against rough path X defined in (5.1).

Now consider the following portfolio

$$\gamma_t = 1 - e^{2\sigma B_t + 2(\mu-r)t}, \tag{5.5}$$

$$\zeta_t = 2x^{-1}(e^{\sigma B_t + (\mu-r)t} - 1), \tag{5.6}$$

we will show that this trading strategy is an arbitrage one. First, by (5.5) and (5.6), we get the value process of the portfolio

$$V_t = (1 - e^{2\sigma B_t + 2(\mu-r)t})e^{rt} + 2(e^{\sigma B_t + (\mu-r)t} - 1)e^{\sigma B_t + \mu t} = e^{rt}(e^{\sigma B_t + (\mu-r)t} - 1)^2 \geq 0.$$

By applying the basic principle/Itô formula for Stratonovich integral to $V_t =: f(B_t, t)$, we have

$$\begin{aligned}
V_t = V_0 &+ \int_0^t re^{rs}(e^{\sigma B_s + (\mu-r)s} - 1)^2 ds \\
&+ \int_0^t 2(\mu - r)e^{\sigma B_s + (\mu-r)s} e^{rs}(e^{\sigma B_s + (\mu-r)s} - 1) ds \\
&+ \int_0^t 2\sigma e^{rs}(e^{\sigma B_s + (\mu-r)s} - 1)e^{\sigma B_s + (\mu-r)s} \circ d_{\mathfrak{R}_1} \boldsymbol{B}_s^{\text{Str}} \\
&= \int_0^t r\gamma_s e^{rs} ds + \int_0^t \mu\zeta_s X_s ds + \int_0^t \sigma\zeta_s X_s \circ d_{\mathfrak{R}_1} \boldsymbol{B}_s^{\text{Str}} \\
&= \int_0^t \gamma_s dM_s + \int_0^t \zeta_s \circ d_{\mathfrak{R}_1} \boldsymbol{X}_s.
\end{aligned}$$

The last equality is by the chain rule of Stratonovich integral in section 3.5.

Hence, the portfolio (5.5), (5.6) is self-financing in this financial market. Note that the initial payment at $t = 0$ is $V_0 = 0$, but after that the value of this portfolio is positive almost surely. This means one gets free lunch with no risk.

5.2 Arbitrage Free Under a Class of Trading Strategies

Now we consider the Itô fractional Black–Scholes market. As for the Stratonovich fBS market, we suppose that the market has a stock $\boldsymbol{X}$ (the risky asset) whose price process is $X_t := \boldsymbol{X}_{0,t}^1$ but now it satisfies the differential equation driven by Itô fractional Brownian rough path.

$$d\boldsymbol{X} = \mu X_t dt + \sigma X d\boldsymbol{B}^{\text{Itô}}, \quad X_0 = x, \quad t \in [0, T]. \tag{5.7}$$

The solution is also constructed in Example 3.15. The risk-less asset money market M satisfies Equation (5.2), i.e. $M_t = e^{rt}$.

Suppose a portfolio (γ_t, ζ_t) gives the value process $V_t \in \mathbb{R}$ by

$$V_t = \gamma_t M_t + \zeta_t X_t. \tag{5.8}$$

In this Itô fBS market, we restrict the class of trading strategies. We call a portfolio is *admissible* if $\gamma_t = \gamma(X_t, t)$, and $\zeta_t = \zeta(X_t, t)$. Besides, a portfolio is called *self-financing* if

$$V_t = V_0 + \int_0^t \gamma_s dM_s + \int_0^t \zeta d_{\mathfrak{R}_1} \boldsymbol{X},$$

where the second integral on the right hand side is the first level of the Itô integral against rough path $\boldsymbol{X}$ defined in (5.7).

Then by the chain rule of Itô fractional Brownian rough path, we have

$$
\begin{aligned}
V_t &= V_0 + \int_0^t \gamma_s dM_s + \int_0^t \zeta d_{\mathfrak{R}_1} \boldsymbol{X} \\
&= V_0 + \int_0^t r e^{rs} \gamma_s ds + \int_0^t \mu \zeta_s X_s ds + \int_0^t \sigma \zeta_s X_s d_{\mathfrak{R}_1} \boldsymbol{B}^{\mathrm{It\hat{o}}}.
\end{aligned}
$$

By (5.8) we also have

$$
\gamma_t = \mathrm{e}^{-rt}(V_t - \zeta_t X_t), \tag{5.9}
$$

plugging it into last equality, we get

$$
V_t = V_0 + \int_0^t r V_s ds + \int_0^t \sigma \zeta_s X_s \left(\frac{\mu - r}{\sigma} ds + d_{\mathfrak{R}_1} \boldsymbol{B}^{\mathrm{It\hat{o}}} \right). \tag{5.10}
$$

In order to prove there is no arbitrage in this case, we first introduce the Girsanov theorem for fBM with Hurst parameter $H \leq \frac{1}{2}$.

5.3 Girsanov's Theorem

The following version of Girsanov theorem for the fBM has been obtained in [14, Theorem 4.9], and we also suggest reader to see [8, Theorem 4.1] and proof therein. In our case, we would like to show that there is a new probability measure $\widehat{\mathbb{P}}$ such that

$$
\widehat{B}_t = B_t + \frac{\mu - r}{\sigma} t, \tag{5.11}
$$

which is still an fBM under this measure $\widehat{\mathbb{P}}$. This is what Girsanov theorem says in usual. Now let $K_H(t, s)$ be a square integrable kernel given by

$$
K_H(t, s) = C_H \left[\frac{2}{2H - 1} \left(\frac{t(t-s)}{s} \right)^{H - \frac{1}{2}} - \int_s^t \left(\frac{u(u-s)}{s} \right)^{H - \frac{1}{2}} \frac{du}{u} \right] 1_{(0,t)}(s). \tag{5.12}
$$

Define the operator K_H on $L^2([0,T])$ associated with the kernel $K_H(t, s)$ as

$$
(K_H f)(s) = \int_0^T f(t) K_H(t, s) dt. \tag{5.13}
$$

Given an adapted and integrable process $u = \{u_t, \ t \in [0, T]\}$, consider the transformation

$$
\widehat{B}_t = B_t + \int_0^t u_s ds, \tag{5.14}
$$

since fBM B can be represented by the integral along standard Brownian motion W, we can write (5.14) into

$$
\widehat{B}_t = B_t + \int_0^t u_s ds = \int_0^t K_H(t, s) dW_s + \int_0^t u_s ds = \int_0^t K_H(t, s) d\widetilde{W}_s,
$$

where W_t is a standard Brownian motion and

$$
\widetilde{W}_t = W_t + \int_0^t K_H^{-1} \left(\int_0^{\cdot} u_r dr \right)(s) ds. \tag{5.15}
$$

By the standard Girsanov theorem for Brownian motion applied to (5.15), as a consequence, we have the following version of the Girsanov theorem for the fBM with Hurst parameter $H \leq \frac{1}{2}$, which has obtained in [14], [29] and [8].

Theorem 5.1 (Girsanov theorem for fBM with $H \leq \frac{1}{2}$) ([14, Theorem 4.9]; [29, Theorem 2]; [8, Theorem 4.1]) *Let B be a fBM with Hurst parameter $H \in (0, \frac{1}{2}]$, and*

$$v(s) := K_H^{-1}\left(\int_0^{\cdot} u_r dr\right)(s).$$

Consider the shifted process (5.14). Assume that

(i) $\int_0^T u_t^2 dt < \infty$, *almost surely.*

(ii) $\mathbb{E}(Z_T) = 1$, *where*

$$Z_T = \exp\left(-\int_0^T v(s)dW_s - \frac{1}{2}\int_0^T (v(s))^2 ds\right),$$

Then the shifted process $\widehat{B}$ is an $\mathcal{F}_t^B$-fBM with Hurst parameter H under the new probability measure $\widehat{\mathbb{P}}$ defined by $\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} = Z_T$.

Remark 5.2 Here when u satisfies the condition (i) in Theorem 5.1 with $H \leq \frac{1}{2}$, then $v = K_H^{-1}\left(\int_0^{\cdot} u_r dr\right)$ is well-defined, and $K_H^{-1}\left(\int_0^{\cdot} u_r dr\right) \in L^2([0,T])$, where K_H^{-1} is the inverse of the operator K_H.

In our case,

$$\widehat{B}_t = B_t + \frac{\mu - r}{\sigma}t,$$

so we can apply the Girsanov Theorem 5.1 to it. Let $\mathbb{P}$ be the distribution of fBM B, and $\widehat{\mathbb{P}}$ be the distribution constructed from $\mathbb{P}$ by Girsanov theorem. In terms of $\widehat{B}_t$, we can write (5.10), under $\widehat{\mathbb{P}}$, as

$$V_t = V_0 + \int_0^t rV_s ds + \int_0^t \sigma\zeta_s X d_{\mathfrak{R}_1}\widehat{\boldsymbol{B}}^{\text{Itô}}. \tag{5.16}$$

Transforming this equation, we have

$$\mathrm{e}^{-rt}V_t = V_0 + \sigma\int_0^t \mathrm{e}^{-rs}\zeta_s X d_{\mathfrak{R}_1}\widehat{\boldsymbol{B}}^{\text{Itô}}, \quad t \in [0, T]. \tag{5.17}$$

Taking expectation under the measure $\widehat{\mathbb{P}}$, we have

$$\mathrm{e}^{-rT}\mathbb{E}_{\widehat{\mathbb{P}}}[V_T] = V_0 + \mathbb{E}_{\widehat{\mathbb{P}}}\left[\sigma\int_0^T \mathrm{e}^{-rs}\zeta_s X d_{\mathfrak{R}_1}\widehat{\boldsymbol{B}}^{\text{Itô}}\right]. \tag{5.18}$$

By Girsanov's theorem and our zero mean property of integrals, we get that

$$\mathbb{E}_{\widehat{\mathbb{P}}}\left[\int_0^T \mathrm{e}^{-rs}\zeta_s X d_{\mathfrak{R}_1}\widehat{\boldsymbol{B}}^{\text{Itô}}\right] = 0. \tag{5.19}$$

Thus we may conclude that

$$\mathrm{e}^{-rT}\mathbb{E}_{\widehat{\mathbb{P}}}[V_T] = V_0. \tag{5.20}$$

Hence the probability measure $\widehat{\mathbb{P}}$ defined in Theorem 5.1 is a risk neutral measure. Then this Itô fBS market has no arbitrage under the class of admissible trading strategy.

5.4 Option Pricing Formula

Furthermore, we aim to derive a pricing formula for the financial derivative F at time $t = 0$ under the risk-neutral measure $\widehat{\mathbb{P}}$. The arbitrage-free property of our Itô fBS market under $\widehat{\mathbb{P}}$ is established in Theorem 5.1, which allows us to compute the price using the formula

$$V_0 = \mathrm{e}^{-rT}\mathbb{E}_{\widehat{\mathbb{P}}}[V_T]. \tag{5.21}$$

Once we have determined the price, we can state the following theorem.

Theorem 5.3 (Fractional Black–Scholes pricing formula) *The price of claim $F(X_T)$ under fractional Black–Scholes model and risk-neutral measure $\widehat{\mathbb{P}}$ in Theorem 5.1 is*

$$V_0 = e^{-rT} \int_{\mathbb{R}} F(X_0 e^{\sigma T^H y + rT - \frac{1}{2}\sigma^2 T^{2H}})\varphi(y)dy, \tag{5.22}$$

where $\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ is the standard normal density function.

The proof of Theorem 5.3 can be found in Appendix 6.2. For the *European call option*, $F(X_T) = (X_T - K)^+$, where $K > 0$ is the striking price, we have the following conclusion.

Corollary 5.4 (European call) *The pricing formula of the European call option under fractional Black–Scholes model and risk-neutral measure $\widehat{\mathbb{P}}$ in Theorem 5.1 is*

$$V_0 = X_0(1 - \Phi(c_-)) - Ke^{-rT}(1 - \Phi(c_+)), \tag{5.23}$$

where

$$c_- = \frac{1}{\sigma T^H} \log\left(\frac{K}{X_0}\right) - \frac{r}{\sigma}T^{1-H} - \frac{1}{2}\sigma T^H,$$

$$c_+ = \frac{1}{\sigma T^H} \log\left(\frac{K}{X_0}\right) - \frac{r}{\sigma}T^{1-H} + \frac{1}{2}\sigma T^H,$$

and $\Phi(x) = \int_{-\infty}^{x} \varphi(y)dy$ is standard normal distribution function.
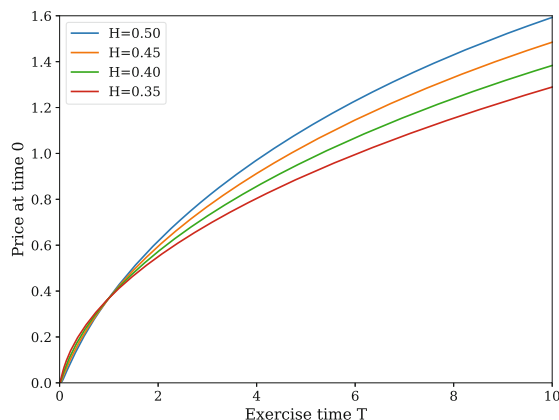

Figure 1   Price of European call at time 0 for different exercise time T and varying Hurst parameter $H = 0.50, 0.45, 0.40, 0.35$, where we take $\sigma = 0.50$, $K = 3$, $X_0 = 2.5$, $r = 0.05$ as an example

Let us comment on the pricing formula itself, which is similar to that of the classical Black–Scholes model. The values are exactly the same if the maturity time $T = 1$, which seems strange at the first glance. The Hurst parameter H comes in to play a role only through the exercise time T, and appears as a power in the exercise time T through $c_\pm$ and to modify the volatility $\sigma^2 T$ (for Black–Scholes' market) into $\sigma^2 T^{2H}$ (for the fractional BS market case), so that the intensity of the volatility is reduced as $H < \frac{1}{2}$ due to the long time memory. In fact the numerics $c_\pm$ can be rewritten as

$$c_\pm = \frac{1}{\sigma T^H} \log\left(\frac{K}{X_0}\right) - \frac{rT}{\sigma T^H} \pm \frac{1}{2}\sigma T^H.$$

Of course one has to understand the scale of the maturity T in time unit has no economic meaning, and its scale is in fact fixed by the interest rate through e^{rT}, and therefore it looks natural that H should appear in the power of T to have its effect on the option pricing. See Figure 5.4, we show prices of European call option at time 0 for different exercise time T and Hurst parameter H.

6 Application in the Fractional Ornstein–Uhlenbeck Process

In Qian and Xu (2024), [30], they introduced Itô integration for fractional Ornstein–Uhlenbeck (fOU) processes, and applied the Itô integration to the parameter estimation problem for the fOU process. In this section, we present the theoretical results regarding the parameter estimation problem for fOU processes. Furthermore, we showcase the efficacy of this rough path estimator via Monte Carlo simulations conducted on a 2-dimensional fOU process.

6.1 Theoretical Results

Let X be a multi-dimensional fOU process, i.e. the solution to the following stochastic differential equation driven by multi-dimensional fractional Brownian motion

$$dX_t = -\Gamma X_t dt + \sigma dB_t^H, \tag{6.1}$$

where Γ is a parametric matrix, which is symmetric and positive-definite. We can construct an estimator based on either continuous or discrete observation via rough path theory. We suppose that the rough path enhancement $(X_{0,t}(\omega), \mathbb{X}_{0,t}(\omega))$ of the fOU process $X_t(\omega)$ can be observed continuously or discretely in Itô sense. The main results are as follows. The proofs can be found in [30].

Theorem 6.1 (1) *Suppose Γ is a parametric matrix and it is symmetric and positive-definite. Let $\widehat{\Gamma}_t$ be the rough path estimator of Γ:*

$$\widehat{\Gamma}_t^T \equiv -\left(\int_0^t X_s \otimes X_s ds\right)^{-1}\left(\int_0^t X_s \otimes d_{\mathfrak{R}_1}\boldsymbol{X}\right). \tag{6.2}$$

Then

$$\widehat{\Gamma}_t \to \Gamma, \quad a.s., \ as \ t \to \infty. \tag{6.3}$$

(2) *Suppose we take the stochastic integral $\int_0^t X_s \otimes d_{\mathfrak{R}_1}\boldsymbol{X}$ in the rough path estimator $\widehat{\Gamma}_t$ as Stratonovich rough integral rather than Itô rough integral as above, we have*

$$\widehat{\Gamma}_t \to 0, \quad a.s., \ as \ t \to \infty. \tag{6.4}$$

That is, we cannot use the Stratonovich rough integral to do this estimation problem.

Theorem 6.2 *Suppose the fOU process X with $H \in (\frac{1}{3}, \frac{1}{2}]$ can be observed at discrete time $\{t_\ell = \ell h, \ell = 0, 1, 2, \ldots, n\}$ and as sample size $n \to \infty$, n and h satisfy*

$$nh \to \infty, \quad h = h_n \to 0, \quad nh^p \to 0, \tag{6.5}$$

for some $p \in (1, \frac{1+H+\beta}{1+\beta})$, and $0 < \beta < 1$. Let

$$\widehat{\Gamma}_n^T \equiv -\left(\sum_{\ell=0}^n (X_{\ell h} \otimes X_{\ell h})h\right)^{-1}\left(\sum_{\ell=0}^{n-1} X_{\ell h}X_{\ell h,(\ell+1)h} + \mathbb{X}_{\ell h,(\ell+1)h}\right), \tag{6.6}$$

where $\widehat{\Gamma}^T$ denotes transpose of matrix $\widehat{\Gamma}$. Then

$$\widehat{\Gamma}_n \to \Gamma, \quad \text{a.s.} \tag{6.7}$$

as $n \to \infty$.

As a remark, the explicit dependence of the Lévy area of fOU processes on the drift parameter Γ can complicate the application of the aforementioned rough path estimator in real-world observations. Nevertheless, as indicated by Equation (6.2) or (6.6), we can approach this issue by regarding it as an equation to be solved iteratively for Γ. This involves initializing with an approximate guess for the parameters, followed by a Lévy area correction based on this preliminary estimate. Subsequently, we refine our parameter estimates in an iterative fashion.

6.2 Monte Carlo Exercise

In this subsection, we give a numerical example based on Monte Carlo simulation for a two dimensional fOU process, i.e. the dynamic

$$dX_t = -\Gamma X_t dt + dB_t^H, \quad X_0 = 0, \quad t \in [0, T], \tag{6.8}$$

with parameter matrix

$$\Gamma = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}.$$

That is,

$$dX_t^1 = -(X_t^1 + 2X_t^2)dt + dB_t^{H,1},$$
$$dX_t^2 = -(2X_t^1 + 5X_t^2)dt + dB_t^{H,2}.$$

We apply Euler scheme to draw n equidistant samples $\{X_h(\omega), X_{2h}(\omega), \ldots, X_{nh}(\omega)\}$ on time horizon T with frequency $h = \frac{T}{n}$. In order to estimate the parameter matrix, we should enhance sample paths to data in rough path sense. That is, to get $\{(X_{\ell h,(\ell+1)h}(\omega), \mathbb{X}_{\ell h,(\ell+1)h}(\omega))_{\ell=0,1,\ldots,n-1}\}$ as our observation data for estimation.

For dimension $d = 2$, the continuous rough path estimator $\widehat{\Gamma}_T(\omega) = (\widehat{\Gamma}_T^{ij}(\omega))_{i,j=1,2}$ is given by

$$\widehat{\Gamma}_T^{ij}(\omega) = -\frac{1}{V_T(X(\omega))} \left(\int_0^T (X_s^{3-j}(\omega))^2 ds \int_0^T X_s^j(\omega) d_{\mathfrak{R}_1} \boldsymbol{X}^i(\omega) \right.$$
$$\left. - \int_0^T X_s^i(\omega) X_s^{3-i}(\omega) ds \int_0^T X_s^{3-j}(\omega) d_{\mathfrak{R}_1} \boldsymbol{X}^i(\omega) \right), \tag{6.9}$$

where

$$V_T(X(\omega)) = \int_0^T (X_s^1(\omega))^2 ds \int_0^T (X_s^2(\omega))^2 ds - \left(\int_0^T X_s^1(\omega) X_s^2(\omega) ds \right)^2,$$

and all the rough integrals above are defined in our Itô sense. Discretizing every integral above, we obtain our high frequency rough path estimator. The attention we need pay to is the cross term of rough integral, i.e. Lévy area $\int X^i d_{\mathfrak{R}_1} \boldsymbol{X}^j$ or $\mathbb{X}^{ij}$. Since

$$\mathbb{X}_{s,t}^{ij} = \mathbb{X}_{s,t}^{\text{Str},ij} - \varphi_{s,t},$$

where $\varphi_{s,t}$ is defined in Section 4.1 and $\mathbb{X}_{s,t}^{\text{Str},ij}$ denotes the second level/Lévy area of fOU rough path enhancement in the Stratonovich sense. Now we can discretize the Stratonovich's Lévy

area $\mathbb{X}^{\text{Str},ij}$ by trapezoidal scheme. By this, we get $\{(X_{\ell h,(\ell+1)h}(\omega), \mathbb{X}_{\ell h,(\ell+1)h}(\omega))_{\ell=0,1,\dots,n-1}\}$ as our discrete observation data for estimation.

We illustrate our two dimensional simulation results in Table 1 below. In this case, we estimate the parameter matrix Γ using the simulated data $\{(X_{\ell h,(\ell+1)h}(\omega), \mathbb{X}_{\ell h,(\ell+1)h}(\omega))_{\ell=0,1,\dots,n-1}\}$. We draw 1000 sample paths by Monte Carlo simulation.

In Table 1, every component of "Mean" denotes average of the value of respective estimator based on 1000 Monte Carlo simulation. And the component of "Standard deviation (Std dev)" represents the fluctuation of estimation of parameter with corresponding index. One could see that, under proper time horizon T, sample size n and frequency h, the rough path estimator performs very well and the results are quite stable.

	$n = 2048$	$n = 4096$	$n = 8192$
$H = 0.50$	$T = 20,\ h = 0.0098$	$T = 30,\ h = 0.0073$	$T = 40,\ h = 0.0049$
Mean	$\begin{pmatrix} 1.0956 & 1.9720 \\ 1.9765 & 5.0381 \end{pmatrix}$	$\begin{pmatrix} 1.0548 & 1.9579 \\ 1.9918 & 5.0443 \end{pmatrix}$	$\begin{pmatrix} 1.0551 & 2.0091 \\ 1.9848 & 5.0146 \end{pmatrix}$
Std dev	$\begin{pmatrix} 0.3350 & 0.6824 \\ 0.3374 & 0.7399 \end{pmatrix}$	$\begin{pmatrix} 0.2755 & 0.5955 \\ 0.2719 & 0.5590 \end{pmatrix}$	$\begin{pmatrix} 0.2429 & 0.5142 \\ 0.2313 & 0.4984 \end{pmatrix}$
$H = 0.45$	$T = 13,\ h = 0.0063$	$T = 22,\ h = 0.0054$	$T = 40,\ h = 0.0049$
Mean	$\begin{pmatrix} 1.1163 & 1.9832 \\ 1.9889 & 5.1160 \end{pmatrix}$	$\begin{pmatrix} 1.0731 & 1.9943 \\ 1.9665 & 5.0106 \end{pmatrix}$	$\begin{pmatrix} 1.0449 & 2.0105 \\ 1.9744 & 4.9902 \end{pmatrix}$
Std dev	$\begin{pmatrix} 0.4135 & 0.9030 \\ 0.4429 & 0.8453 \end{pmatrix}$	$\begin{pmatrix} 0.3172 & 0.6914 \\ 0.3034 & 0.6128 \end{pmatrix}$	$\begin{pmatrix} 0.2169 & 0.4743 \\ 0.2166 & 0.4517 \end{pmatrix}$
$H = 0.40$	$T = 14,\ h = 0.0068$	$T = 20,\ h = 0.0049$	$T = 35,\ h = 0.0043$
Mean	$\begin{pmatrix} 1.1030 & 2.0077 \\ 1.9767 & 5.0416 \end{pmatrix}$	$\begin{pmatrix} 1.0585 & 1.9836 \\ 1.9895 & 5.0283 \end{pmatrix}$	$\begin{pmatrix} 1.0361 & 1.9946 \\ 1.9790 & 4.9894 \end{pmatrix}$
Std dev	$\begin{pmatrix} 0.3745 & 0.8473 \\ 0.3878 & 0.7160 \end{pmatrix}$	$\begin{pmatrix} 0.3200 & 0.7466 \\ 0.3242 & 0.6152 \end{pmatrix}$	$\begin{pmatrix} 0.2275 & 0.5441 \\ 0.2367 & 0.4517 \end{pmatrix}$
$H = 0.35$	$T = 14,\ h = 0.0068$	$T = 20,\ h = 0.0049$	$T = 30,\ h = 0.0037$
Mean	$\begin{pmatrix} 1.0724 & 1.9606 \\ 1.9588 & 4.9324 \end{pmatrix}$	$\begin{pmatrix} 1.0555 & 2.0065 \\ 1.9645 & 4.9839 \end{pmatrix}$	$\begin{pmatrix} 1.0298 & 1.9927 \\ 1.9796 & 4.9908 \end{pmatrix}$
Std dev	$\begin{pmatrix} 0.3612 & 0.8816 \\ 0.3796 & 0.6448 \end{pmatrix}$	$\begin{pmatrix} 0.3074 & 0.7608 \\ 0.3234 & 0.5628 \end{pmatrix}$	$\begin{pmatrix} 0.2448 & 0.6067 \\ 0.2488 & 0.4453 \end{pmatrix}$

Table 1   Mean and standard deviation of rough path estimators $\widehat{\Gamma}_n$ of the fOU process based on 1000 Monte Carlo simulation in dimension $d = 2$, n sample size, T time horizon, h sampling frequency, H Hurst parameter, and true parameter matrix $\Gamma = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 5 \end{smallmatrix}\right)$

As a remark, in dimension $d = 2$, the performance of discrete estimator becomes a little sensitive to sampling mode. One should adhere to the conditions about time horizon T, sample size n and frequency h in Theorem 6.2 in order to obtain better estimated values. In Table 1, we set frequency h becomes smaller as sample size n becomes larger and Hurst parameter H smaller.

Appendix Proofs

In this appendix, we provide the proofs for the results stated in the main text. For clarity, unless specified otherwise, we may use the notation $\boldsymbol{X} = (1, X, \mathbb{X})$ interchangeably with $\boldsymbol{X} = (1, \boldsymbol{X}^1, \boldsymbol{X}^2)$ for rough path $\boldsymbol{X}$, as long as this substitution does not lead to any confusion.

A.1 Proof of Theorems 3.1 and 3.2

Proof   (i) By definition of our integral, we have

$$
\int_s^t F(B, u) d_{\mathfrak{R}_1} \boldsymbol{B}^{\text{Itô}} = \int_s^t f(\widetilde{B}) d_{\mathfrak{R}_1} \widetilde{\boldsymbol{B}}^{\text{Itô}}
$$

$$
= \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} f(\widetilde{B}_u) \widetilde{B}_{u,v} + Df(\widetilde{B}_u) \widetilde{\mathbb{B}}_{u,v}^{\text{Itô}}
$$

$$
= \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} F(B_u, u) B_{u,v} + D_x F(B_u, u) \mathbb{B}_{u,v}^{\text{Itô}} + D_u F(B_u, u) \int_u^v B_{u,r} dr.
$$

Since

$$
\left| \int_u^v B_r dr - B_u(v - u) \right| = o(|v - u|) = o(|\mathcal{P}|),
$$

so that

$$
\int_s^t F(B, u) d_{\mathfrak{R}_1} \boldsymbol{B}^{\text{Itô}} = \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} F(B_u, u) B_{u,v} + D_x F(B_u, u) \mathbb{B}_{u,v}^{\text{Itô}}.
$$

Therefore we may conclude that

$$
\int_s^t F(B, u) d_{\mathfrak{R}_1} \boldsymbol{B}^{\text{Itô}} = \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} F(B_u, u) B_{u,v} + D_x F(B_u, u) \left(\mathbb{B}_{u,v}^{\text{Str}} - \frac{1}{2} I(v^{2H} - u^{2H}) \right)
$$

$$
= \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} (F(B_u, u) B_{u,v} + D_x F(B_u, u) \mathbb{B}_{u,v}^{\text{Str}})
$$

$$
- \frac{1}{2} \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} D_x F(B_u, u)(v^{2H} - u^{2H})
$$

$$
= \int_s^t F(B, u) \circ d_{\mathfrak{R}_1} \boldsymbol{B}^{\text{Str}} - \frac{1}{2} \int_s^t D_x F(B_u, u) du^{2H}.
$$

(ii) Now we show the second relation (3.9). It follows in the similar way as (i).

$$
\int_s^t F(B, u) d_{\mathfrak{R}_2} \boldsymbol{B}^{\text{Itô}} = \int_s^t f(\widetilde{B}) d_{\mathfrak{R}_2} \widetilde{\boldsymbol{B}}^{\text{Itô}}
$$

$$
= \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} X_{s,u}^{\text{Itô}} \otimes X_{u,v}^{\text{Itô}} + f(\widetilde{B}_u) \otimes f(\widetilde{B}_u) \widetilde{\mathbb{B}}_{u,v}^{\text{Itô}}.
$$

By Equation (3.8), and $f(\widetilde{B}_u) \otimes f(\widetilde{B}_u)\widetilde{\mathbb{B}}^{\text{Itô}}_{u,v} = F(B_u, u) \otimes F(B_u, u)\mathbb{B}^{\text{Itô}}_{u,v}$, we obtain

$$
\int_s^t F(B, u)d_{\mathfrak{R}_2}\boldsymbol{B}^{\text{Itô}} = \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} \left(X^{\text{Str}}_{s,u} - \frac{1}{2} \int_s^u D_x F(B_r, r)dr^{2H} \right)
$$

$$
\otimes \left(X^{\text{Str}}_{u,v} - \frac{1}{2} \int_u^v D_x F(B_r, r)dr^{2H} \right)
$$

$$
+ F(B_u, u) \otimes F(B_u, u)\left(\mathbb{B}^{\text{Str}}_{u,v} - \frac{1}{2}(v^{2H} - u^{2H}) \right)
$$

$$
= \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} X^{\text{Str}}_{s,u} \otimes X^{\text{Str}}_{u,v} + F(B_u, u) \otimes F(B_u, u)\mathbb{B}^{\text{Str}}_{u,v}
$$

$$
- \frac{1}{2}F(B_u, u) \otimes F(B_u, u)(v^{2H} - u^{2H})
$$

$$
- \frac{1}{2} \int_s^u D_x F(B_r, r)dr^{2H} \otimes X^{\text{Str}}_{u,v}
$$

$$
- X^{\text{Str}}_{s,u} \otimes \frac{1}{2} \int_u^v D_x F(B_r, r)dr^{2H}
$$

$$
+ \frac{1}{4} \int_s^u D_x F(B_r, r)dr^{2H} \int_u^v D_x F(B_r, r)dr^{2H}.
$$

Since B_t has finite p-variation with $p > \frac{1}{H}$, therefore $\mathbb{B}$ is of finite $\frac{p}{2}$-variation, $F(B_t, t)$, $D_x F(B_t, t)$ have finite p-variations, $\int_0^{\cdot} D_x F(B_r, r)dr^{2H}$ has finite $\frac{1}{2}H$-variation and X^{Str} has finite p-variation. Since $\frac{1}{H} < p < 3$ and $\frac{1}{3} < H < \frac{1}{2}$, so that $\frac{1}{p} + 2H > 1$, and the last four sums on the right hand side converge to the Young integral. Equation (3.9) therefore follows immediately. $\qquad\square$

A.2   Proof of Theorem 3.12

Proof   Suppose $\{\overline{B}_t,\ t \geq 0\}$ is a piece-wise linear/smooth approximation with finite variation of fractional Brownian motion $\{B_t,\ t \geq 0\}$. Set $\overline{B}_{s,t} = \overline{B}_t - \overline{B}_s$ and let $\overline{\mathbb{B}}_{s,t}$ be the difference of the iterated integral over $[s, t]$ of $\overline{B}$ and $\frac{1}{2}(t^{2H} - s^{2H})$. Consider the rough differential equation

$$
d\boldsymbol{X} = f(X)d\overline{\boldsymbol{B}}, \tag{A.1}
$$

that is, the integral equation

$$
\boldsymbol{Z} = \int \widehat{f}(Z)d\boldsymbol{Z}, \quad \pi_d(\boldsymbol{Z}) = \overline{\boldsymbol{B}}, \tag{A.2}
$$

where $\widehat{f}(x, y)(\xi, \eta) := (\xi, f(y)\xi)$, and π_d is projection operator to $\mathbb{R}^d$, which is solved by the Picard iteration, that is

$$
\boldsymbol{Z}(n+1) = \int \widehat{f}(Z(n))d\boldsymbol{Z}(n), \quad \boldsymbol{Z}(0) = (\overline{\boldsymbol{B}}, 0).
$$

More precisely, define almost rough paths

$$
\widehat{\boldsymbol{Z}}(n+1)^1_{s,t} := \widehat{f}(Z(n)_s)\boldsymbol{Z}(n)^1_{s,t} + D\widehat{f}(Z(n)_s)\boldsymbol{Z}(n)^2_{s,t}, \tag{A.3}
$$

$$
\widehat{\boldsymbol{Z}}(n+1)^2_{s,t} := \widehat{f}(Z(n)_s) \otimes \widehat{f}(Z(n)_s)\boldsymbol{Z}(n)^2_{s,t}, \tag{A.4}
$$

and define the corresponding rough paths

$$
\boldsymbol{Z}(n+1)^1_{s,t} = \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} \widehat{\boldsymbol{Z}}(n+1)^1_{u,v}, \tag{A.5}
$$

$$\boldsymbol{Z}(n+1)^2_{s,t} = \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} \boldsymbol{Z}(n)^1_{s,u} \otimes \boldsymbol{Z}(n)^1_{u,v} + \widehat{\boldsymbol{Z}}(n+1)^2_{u,v}, \tag{A.6}$$

where $\mathcal{P}$ is a partition of the interval $[s,t]$. Then

$$|\boldsymbol{Z}(n)^i_{s,t} - \widehat{\boldsymbol{Z}}(n)^i_{s,t}| \le \omega(s,t)^\theta, \quad i = 1,2, \ \theta > 1, \tag{A.7}$$

for some control ω.

(i) Now we prove (3.28). Set $\widehat{\boldsymbol{Z}}(n)^1_{s,t} = (\overline{\boldsymbol{B}}^1_{s,t}, \boldsymbol{X}(n)^1_{s,t})$. By definition of $\widehat{f}$ and (A.3), we have

$$\begin{aligned}
\widehat{\boldsymbol{Z}}(n+1)^1_{s,t} &= (\overline{\boldsymbol{B}}^1_{s,t}, f(X(n)_s)\overline{\boldsymbol{B}}^1_{s,t}) + D\widehat{f}(Z(n)_s)\boldsymbol{Z}(n)^2_{s,t} \\
&\simeq (\overline{\boldsymbol{B}}^1_{s,t}, f(X(n)_s)\overline{\boldsymbol{B}}^1_{s,t}) + D\widehat{f}(Z(n)_s)\widehat{\boldsymbol{Z}}(n)^2_{s,t} \\
&\simeq (\overline{\boldsymbol{B}}^1_{s,t}, f(X(n)_s)\overline{\boldsymbol{B}}^1_{s,t}) + (0, Df(X(n)_s)f(X(n)_s)\overline{\boldsymbol{B}}^2_{s,t} \\
&\simeq (\overline{\boldsymbol{B}}^1_{s,t}, f(X(n)_s)\overline{\boldsymbol{B}}^1_{s,t}) - \frac{1}{2}(0, Df(X(n)_s)f(X(n)_s)(t^{2H} - s^{2H}),
\end{aligned}$$

where $\simeq$ means the error can be controlled by $\omega(s,t)^\theta$ with $\theta > 1$. Hence,

$$\boldsymbol{X}(n+1)^1_{s,t} \simeq f(X(n)_s)\overline{\boldsymbol{B}}^1_{s,t} - \frac{1}{2}Df(X(n)_s)f(X(n)_s)(t^{2H} - s^{2H}).$$

Since $\overline{\boldsymbol{B}}^1$ has finite variation and $\sum_{[u,v] \in \mathcal{P}} \boldsymbol{X}(n)^1_{u,v} = \boldsymbol{X}(n)^1_{s,t}$, the formula above implies that

$$\boldsymbol{X}(n+1)^1_{s,t} = \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} f(X(n)_u)\overline{\boldsymbol{B}}^1_{u,v} - \frac{1}{2}Df(X(n)_u)f(X(n)_u)(v^{2H} - u^{2H}),$$

as n goes to infinity, the limit above is identified as

$$X_t = X_s + \int_s^t f(X_u)d_{\mathfrak{R}_1}\overline{\boldsymbol{B}}_u - \frac{1}{2}\int_s^t Df(X_u)f(X_u)du^{2H}.$$

On the other hand,

$$\lim_{n \to \infty} \boldsymbol{X}(n)^1_{s,t} = \Phi(\cdot, \overline{\boldsymbol{B}})^1_{s,t},$$

so we get (3.28) when the system is driven by $\overline{\boldsymbol{B}}$. By the continuity of Itô's maps, we conclude that (3.28) holds in real fractional Brownian rough path case.

(ii) Similarly, by (A.6) and the continuity of Itô's maps again, we conclude that (3.29) holds. $\qquad\square$

A.3 Proof of Theorem 3.17

Proof   Let $\boldsymbol{Y} = \int G(X,t)d\boldsymbol{X}$, $\boldsymbol{H} = \int f(\widetilde{B})d\widetilde{\boldsymbol{B}}$. Then their associated almost rough path are $\widehat{\boldsymbol{Y}}$, $\widehat{\boldsymbol{H}}$ respectively, where

$$\begin{aligned}
\widehat{\boldsymbol{Y}}^1_{s,t} &= G(X_s,s)\boldsymbol{X}^1_{s,t} + D_x G(X_s,s)\boldsymbol{X}^2_{s,t}, \\
\widehat{\boldsymbol{Y}}^2_{s,t} &= G(X_s,s) \otimes G(X_s,s)\boldsymbol{X}^2_{s,t},
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\boldsymbol{H}}^1_{s,t} &= f^1(B_s,s)\boldsymbol{B}^1_{s,t} + f^2(B_s,s)(t-s) + D_x f^1(B_s,s)\boldsymbol{B}^2_{s,t}, \\
\widehat{\boldsymbol{H}}^2_{s,t} &= f^1(B_s,s) \otimes f^1(B_s,s)\boldsymbol{B}^2_{s,t}.
\end{aligned}$$

We should prove that

$$|\widehat{\boldsymbol{Y}}_{s,t}^i - \widehat{\boldsymbol{H}}_{s,t}^i| \le \omega(s,t)^\theta, \quad i = 1, 2, \ \forall s < t, \ \exists\, \theta > 1, \tag{A.8}$$

for some control ω. If the difference of two quantities is controlled by ω like (A.8), we use the symbol $\simeq$ to represent it. So we should show $\widehat{\boldsymbol{Y}}_{s,t}^i \simeq \widehat{\boldsymbol{H}}_{s,t}^i$.

Denote $\boldsymbol{Z} = \int \sigma F(B, t) d\boldsymbol{B}$, $h_{s,t} = \int_s^t \mu F(B_t, t) dt$, and $\widehat{\boldsymbol{Z}}$, $\widehat{h}$ denote their respective almost rough path, that is,

$$\widehat{\boldsymbol{Z}}_{s,t}^1 = \sigma F(B_s, s)\boldsymbol{B}_{s,t}^1 + \sigma^2 F(B_s, s)\boldsymbol{B}_{s,t}^2,$$
$$\widehat{\boldsymbol{Z}}_{s,t}^2 = \sigma^2 F(B_s, s) \otimes F(B_s, s)\boldsymbol{B}_{s,t}^2,$$
$$\widehat{h}_{s,t} = \mu F(B_s, s)(t - s).$$

It is easy to verify that the almost rough path associated with $\boldsymbol{X}$ is given by

$$(1, \boldsymbol{Z}_{s,t}^1 + h_{s,t}, \boldsymbol{Z}_{s,t}^2) \simeq (1, \widehat{\boldsymbol{Z}}_{s,t}^1 + \widehat{h}_{s,t}, \boldsymbol{Z}_{s,t}^2). \tag{A.9}$$

Two quantities on both sides in (A.9) are all almost rough paths. So we have the following relations:

$$\boldsymbol{X}_{s,t}^1 = \boldsymbol{Z}_{s,t}^1 + h_{s,t} \simeq \widehat{\boldsymbol{Z}}_{s,t}^1 + \widehat{h}_{s,t}, \tag{A.10}$$
$$\boldsymbol{X}_{s,t}^2 \simeq \boldsymbol{Z}_{s,t}^2 \simeq \widehat{\boldsymbol{Z}}_{s,t}^2. \tag{A.11}$$

(i) For the first level,

$$\begin{aligned}
\widehat{\boldsymbol{Y}}_{s,t}^1 &= G(X_s, s)\boldsymbol{X}_{s,t}^1 + D_x G(X_s, s)\boldsymbol{X}_{s,t}^2 \\
&= G(F(B_s, s), s)\boldsymbol{X}_{s,t}^1 + \partial_1 G(F(B_s, s), s)\boldsymbol{X}_{s,t}^2 \\
&\simeq G(F(B_s, s), s)(\widehat{\boldsymbol{Z}}_{s,t}^1 + \widehat{h}_{s,t}) + \partial_1 G(F(B_s, s), s)\widehat{\boldsymbol{Z}}_{s,t}^2 \\
&= G(F(B_s, s), s)(\sigma F(B_s, s)\boldsymbol{B}_{s,t}^1 + \sigma^2 F(B_s, s)\boldsymbol{B}_{s,t}^2 + \mu F(B_s, s)(t - s)) \\
&\quad + \partial_1 G(F(B_s, s), s)(\sigma^2 F(B_s, s) \otimes F(B_s, s)\boldsymbol{B}_{s,t}^2).
\end{aligned}$$

Since

$$D_x f^1(x, s) = \sigma^2 \partial_1 G(F(x, s), s)F(x, s) \otimes F(x, s) + \sigma^2 G(F(x, s), s)F(x, s),$$

so that

$$\widehat{\boldsymbol{Y}}_{s,t}^1 \simeq f^1(B_s, s)\boldsymbol{B}_{s,t}^1 + f^2(B_s, s)(t - s) + D_x f^1(B_s, s)\boldsymbol{B}_{s,t}^2 = \widehat{\boldsymbol{H}}_{s,t}^1.$$

Thus we have proved the first part of the claim.

(ii) For the second level paths, we have

$$\begin{aligned}
\widehat{\boldsymbol{Y}}_{s,t}^2 &= G(X_s, s) \otimes G(X_s, s)\boldsymbol{X}_{s,t}^2 \\
&= G(F(B_s, s), s) \otimes G(F(B_s, s), s)\boldsymbol{X}_{s,t}^2 \\
&\simeq G(F(B_s, s), s) \otimes G(F(B_s, s), s)\widehat{\boldsymbol{Z}}_{s,t}^2 \\
&= G(F(B_s, s), s) \otimes G(F(B_s, s), s)(\sigma^2 F(B_s, s) \otimes F(B_s, s)\boldsymbol{B}_{s,t}^2) \\
&= \sigma^2 G(F(B_s, s), s)F(B_s, s) \otimes G(F(B_s, s), s)F(B_s, s)\boldsymbol{B}_{s,t}^2 \\
&= f^1(B_s, s) \otimes f^1(B_s, s)\boldsymbol{B}_{s,t}^2 \\
&= \widehat{\boldsymbol{H}}_{s,t}^2,
\end{aligned}$$

which thus completes our proof. $\qquad\square$

A.4　Proof of Remark 3.16

Proof　(i) For the first level, we have

$$\int_s^t f(\widetilde{B})d_{\mathfrak{R}_1}\widetilde{\boldsymbol{B}}^{\text{Itô}} = \lim_{|\mathcal{P}|\to 0}\sum_{[u,v]\in\mathcal{P}} f(\widetilde{B}_u)\widetilde{B}_{u,v} + Df(\widetilde{B}_u)\widetilde{\mathbb{B}}^{\text{Itô}}_{u,v}$$

$$= \lim_{|\mathcal{P}|\to 0}\sum_{[u,v]\in\mathcal{P}} \sigma F(B_u,u)B_{u,v} + \mu F(B_u,u)(v-u)$$

$$+ \sigma D_x F(B_u,u)\mathbb{B}^{\text{Itô}}_{u,v} + \sigma D_u F(B_u,u)\int_u^v (B_r - B_u)dr$$

$$+ \mu D_x F(B_u,u)\int_u^v (r-u)dB_r + \frac{\mu}{2}D_u F(B_u,u)(v-u)^2,$$

Since

$$\left|\int_u^v B_r dr - B_u(v-u)\right| = o(|v-u|) = o(|\mathcal{P}|)$$

and

$$\left|\int_u^v r dB_r - u(B_v - B_u)\right| = o(|v-u|) = o(|\mathcal{P}|),$$

we therefore have

$$\int_s^t f(\widetilde{B})d_{\mathfrak{R}_1}\widetilde{\boldsymbol{B}}^{\text{Itô}} = \lim_{|\mathcal{P}|\to 0}\sum_{[u,v]\in\mathcal{P}} \sigma F(B_u,u)B_{u,v} + \mu F(B_u,u)(v-u) + \sigma D_x F(B_u,u)\mathbb{B}^{\text{Itô}}_{u,v}$$

$$= \lim_{|\mathcal{P}|\to 0}\sum_{[u,v]\in\mathcal{P}} g(\widetilde{B}_u)\widetilde{B}_{u,v} + Dg(\widetilde{B}_u)\widetilde{\mathbb{B}}^{\text{Itô}}_{u,v} + \mu F(B_u,u)(v-u)$$

$$= \int_s^t g(\widetilde{B})d_{\mathfrak{R}_1}\widetilde{\boldsymbol{B}}^{\text{Itô}} + \int_s^t \mu F(B_u,u)du,$$

which completes the proof of eqn (3.34).

(ii) For the second level,

$$\int_s^t f(\widetilde{B})d_{\mathfrak{R}_2}\widetilde{\boldsymbol{B}}^{\text{Itô}} = \lim_{|\mathcal{P}|\to 0}\sum_{[u,v]\in\mathcal{P}} X^1_{s,u}\otimes X^1_{u,v} + f(\widetilde{B}_u)\otimes f(\widetilde{B}_u)\widetilde{\mathbb{B}}^{\text{Itô}}_{u,v}$$

$$= \lim_{|\mathcal{P}|\to 0}\sum_{[u,v]\in\mathcal{P}} \left(Z_{s,u} + \int_s^u \mu F(B_r,r)dr\right)\left(Z_{u,v} + \int_u^v \mu F(B_r,r)dr\right)$$

$$+ \sigma^2(F(B_u,u))^2\mathbb{B}^{\text{Itô}}_{u,v}\quad\text{(Some terms go to zero here as above.)}$$

$$= \int_s^t g(\widetilde{B})d_{\mathfrak{R}_2}\widetilde{\boldsymbol{B}}^{\text{Itô}} + \int_s^t \mu Z^1_{s,u}F(B_u,u)du + \int_s^t\left(\int_s^u \mu F(B_r,r)dr\right)dZ_u$$

$$+ \int_s^t\left(\int_s^u \mu F(B_r,r)dr\right)\mu F(B_u,u)du,$$

which yields eqn (3.35). □

A.5　Proof of Theorem 5.3

Proof　Since the price is

$$V_0 = \text{e}^{-rT}\mathbb{E}_{\widehat{\mathbb{P}}}\left[V_T\right] = \text{e}^{-rT}\mathbb{E}_{\widehat{\mathbb{P}}}\left[F(X_T)\right],$$

by the Girsanov theorem for fBM,

$$\mathbb{E}_{\widehat{\mathbb{P}}}\left[F\left(X_T\right)\right] = \mathbb{E}_{\widehat{\mathbb{P}}}\left[F\left(X_0 \exp\left(\sigma B_T^H + \mu T - \frac{1}{2}\sigma^2 T^{2H}\right)\right)\right]$$

$$= \mathbb{E}_{\widehat{\mathbb{P}}}\left[F\left(X_0 \exp\left(\sigma \widehat{B}_T^H + rT - \frac{1}{2}\sigma^2 T^{2H}\right)\right)\right]$$

$$= \mathbb{E}_{\mathbb{P}}\left[F\left(X_0 \exp\left(\sigma B_T^H + rT - \frac{1}{2}\sigma^2 T^{2H}\right)\right)\right]$$

$$= \int_{\mathbb{R}} F(X_0 \mathrm{e}^{\sigma T^H y + rT - \frac{1}{2}\sigma^2 T^{2H}})\varphi(y)dy,$$

which completes our proof. $\qquad\square$

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