

Related work. Visibility graphs of polygons (and other geometric creatures) have been studied extensively both from a combinatorial and a computational point of view, see, e.g., [8, 10, 9, 13, 25]. It is worth mentioning a recent result of Davies, Krawczyk, McCarty and Walczak [3] who proved that curve pseudo-visibility graphs are χ -bounded, that is, their chromatic number can be bounded from above in terms of their clique number.

If the maximum size of any $K_{t,t}$ -free n -vertex graph in a certain family of graphs is at most $c_t n$, where c_t depends only on t , then this family of graphs is called *degree-bounded*.⁴ By a remarkable result of Girão and Hunter [11], if a hereditary family of graphs is degree-bounded, then c_t can be bounded by a polynomial function. For a further discussion on degree-boundedness see the recent survey of Du and McCarty [4].

Organization. The bounds for curve pseudo-visibility graphs are proved in Section 3. In Section 4 we prove the bounds for polygon visibility graphs including the special cases of star-shaped and x -monotone polygons.

2 Terminology and tools

An *ordered graph* is a graph whose vertices are linearly ordered. In a *cyclically-ordered graph* $G = (V, E, <)$ the vertices are ordered cyclically. For any vertex v in G we denote by $G_v = (V, E, <_v)$ the ordered graph such that $<_v$ is the linear order induced by $<$ when setting v as the smallest element.

Let $G = (V, E, <)$ be an ordered graph and let $A, B \subseteq V$ be two vertex subsets. We write $A < B$ if $a < b$ for every $a \in A$ and $b \in B$. If G is cyclically-ordered, then $A < B$ means that $A <_x B$ for some $x \in V$ (which also implies that $B < A$). In both cases $v < A$ stands for $\{v\} < A$. Two edges $(a, b), (c, d)$ in an (cyclically-) ordered graph *cross* if $a < c < b < d$.⁵ The set of neighbors of a vertex v in G is denoted by $N_G(v)$.

Lemma 2.1. *Every ordered graph $G = (V, E, <)$ has an ordered bipartite subgraph $G' = (L \cup R, E', <)$ such that $v < N_{G'}(v)$ for every vertex $v \in L$; $v > N_{G'}(v)$ for every vertex $v \in R$ and $|E'| \geq |E|/4$.*

Proof. For every vertex v we toss a fair coin. If the result is ‘heads’, then we place v in L and delete all the edges (u, v) such that $u < v$. Otherwise, if the result is ‘tails’ we place v in R and delete all the edges (u, v) such that $u > v$. Clearly the resulting subgraph G' is bipartite, $v < N_{G'}(v)$ for every vertex $v \in L$ and $v > N_{G'}(v)$ for every vertex $v \in R$. Since every edge of G survives with probability $1/4$, the expected number of edges in G' is $|E|/4$. Therefore, there is a series of coin tosses that yields at least that many edges. \square

For points lying on a Jordan curve K we denote by $<_K$ their clockwise cyclic order along K . The clockwise sub-curve from a point $p \in K$ to a point $q \in K$ is denoted by $p \rightsquigarrow_K q$. Any curve pseudo-visibility graph $G_{\mathcal{L}}(K, V)$ can naturally be considered as a cyclically-ordered graph where $<_K$ is the cyclic order of its vertices. Furthermore, edges cross in the sense of cyclically-ordered graphs if and only if they cross in the geometric sense in these graphs. This fact is trivial for polygon and curve visibility graphs, and is not hard to prove for curve pseudo-visibility graphs, see [3, Lemma 2.3].

Let u and v be two vertices of an ordered graph G such that $u < v$. Following [3] we say that a sequence of edges e_1, e_2, \dots, e_k is a *crossing sequence from u to v* if u is

⁴Since every $K_{t,t}$ -free graph in the family has a vertex of a small degree – at most $2c_t$.

⁵When we write $v_1 < v_2 < \dots < v_k$ for some $k \geq 3$ vertices of a cyclically-ordered graph $G = (V, E, <)$, we mean that there is a vertex $x \in V$ such that $v_1 <_x v_2 <_x \dots <_x v_k$.