

On contrary, assume that $0 \notin \text{Conv}\left\{\bigcup_{j \in \Lambda} \partial H_j(x^*)\right\}$. Since, $\text{Conv}\left\{\bigcup_{j \in \Lambda} \partial H_j(x^*)\right\}$ and $\{0\}$ are closed and convex sets then with the help of theorem of separation, there exists $v \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $v^T 0 \geq b$ and $v^T d < b \forall d \in \text{Conv}\left\{\bigcup_{j \in \Lambda} \partial H_j(x^*)\right\}$. Jointly both inequality contradicts (3.1). Hence, $0 \in \text{Conv}\left\{\bigcup_{j \in \Lambda} \partial H_j(x^*)\right\}$.

Conversely, it needs to be proven that if $0 \in \text{Conv}\left\{\bigcup_{j \in \Lambda} \partial H_j(x^*)\right\}$, then x^* is a Pareto critical point for H . For this purpose, define $\check{H}(x) = \max_{j \in \Lambda} H_j(x) - H_j(x^*)$. Then, by item (ii) of Theorem 2.2, $\partial \check{H}(x) = \text{Conv}\left\{\bigcup_{i \in \Lambda} \partial H_j(x)\right\}$. Hence, the assumption leads to $0 \in \text{Conv}\left\{\bigcup_{i \in \Lambda} \partial H_j(x)\right\}$, implying $x^* = \arg \min_{x \in D} \check{H}(x)$. On the contrary, if x^* is not a Pareto critical point, then according to Definition 2.3, there exists $s \in D$ such that $\nabla h_j(x^*, \xi_i)^T s < 0$, for all $i \in I_j(x^*)$, $j \in \Lambda$, i.e., $H'_j(x^*, s) < 0$ for all j . Then there exists some $\eta > 0$ sufficiently small such that $H_j(x^* + \eta s) < H_j(x^*)$ for all j which implies $\check{H}(x^* + \eta s) < 0 = \check{H}(x^*)$ holds for some $(x^* + \eta s) \in D$. This contradicts the fact that $x^* = \arg \min_{x \in D} \check{H}(x)$. As a consequence, the assumption that x^* is not a Pareto critical point is incorrect, and x^* is indeed a Pareto critical point for H . \square

Theorem 3.1. *If $h_j(x, \xi_i)$ is continuously differentiable and convex for each $j \in \Lambda$ and $\xi_i \in U$, then $x^* \in D$ is a weak efficient solution solution for $OWC_{P(U)}$ if and only if*

$$0 \in \text{conv}\left(\bigcup_{j=1}^m \partial H_j(x^*)\right).$$

Proof. Let x^* be a weak efficient solution solution for $OWC_{P(U)}$. It must be shown that $0 \in \text{Conv}\bigcup_{j \in \Lambda} \partial H_j(x^*)$. Since given function $h_j(x, \xi_i)$ is continuously differentiable and convex for each j and $\xi_i \in U$, then $h_j(x, \xi_i)$ will be locally Lipschitz continuous for all $i \in \bar{\Lambda}$. Then $0 \in \text{Conv}\{\bigcup_{j \in \Lambda} \partial H_j(x^*)\}$ (see Theorem 4.3 in [71]).

Conversely, by assumption $0 \in \text{Conv}\{\bigcup_{j \in \Lambda} \partial H_j(x^*)\}$ it is clear that x^* is Pareto critical point. Then for atleast one j^0 , it is established that $H'_{j^0}(x^*, d) \geq 0, \forall d \in D - \{x^*\}$. Now, by using the Definition 2.2, it follows that

$$\nabla h_{j^0}(x^*, \xi_i)^T d \geq 0, \forall d \in D, i \in I_{j^0}(x^*). \quad (3.2)$$

By convexity of H_j and $h_j(x, \xi_i)$, it is obtained that

$$h_{j^0}(x, \xi_i) \geq h_{j^0}(x^*, \xi_i) + \nabla h_{j^0}(x^*, \xi_i)^T (x - x^*), \forall i \in I_{j^0}(x^*) \text{ and } x, x^* \in D.$$

Since the last term of the latest inequality is positive by (3.2), it is established that

$$h_{j^0}(x, \xi_i) \geq h_{j^0}(x^*, \xi_i), \forall i \in I_{j^0}(x^*),$$

and therefore

$$H_{j^0}(x) \geq H_{j^0}(x^*), \forall x \in D,$$

i.e., x^* is weak efficient solution. \square