

Repeatedly applying the same trick from right to left we obtain that

$$Y_1 \vee (\cdots (Y_{n-1} \vee Y_n)) = q_2^n(x, y_1[1], \dots, y_n[1]).$$

Proceeding similarly for every component of (\star) , we get that

$$\begin{aligned} q_m^n(\varphi(x), \varphi(y_1), \dots, \varphi(y_n)) &= (q_2^n(x, y_1[1], \dots, y_n[1]), \dots, q_2^n(x, y_1[m], \dots, y_n[m])) \\ &= (q_m^n(x, y_1, \dots, y_n)[1], \dots, q_m^n(x, y_1, \dots, y_n)[m]) \\ &= \varphi(q_m^n(x, y_1, \dots, y_n)) \end{aligned}$$

as desired. \square

As a consequence the following representation theorem holds.

Corollary 3.12. *Every partition clone \mathbf{T} is isomorphic to the partition-of-unity clone of the Boolean algebra $T(2)$.*

4. HYPERVARIETIES

We recall the following definitions for an algebra \mathbf{A} :

- \mathbf{A} is *directly indecomposable* if it cannot be expressed as an isomorphic product of two nontrivial algebras;
- \mathbf{A} is *subdirectly irreducible* if its lattice of congruences possesses a unique nontrivial minimal element;
- \mathbf{A} is *simple* if its lattice of congruences is isomorphic of the two-element chain $\{\Delta < \nabla\}$.

Stone famously proved that $2 = \{0, 1\}$ is the unique nontrivial directly indecomposable Boolean algebra. Consequently, 2 is also the only subdirectly irreducible Boolean algebra (and, incidentally, the only simple Boolean algebra). By Theorem 2.1, this implies that every Boolean algebra is isomorphic to a subdirect power of 2 . As a consequence, the variety of Boolean algebras is generated by 2 .

The first purpose of this section is to lift these considerations to the theory of partition clones. Here the role of the Boolean algebra 2 is played by the clone of projections, which corresponds to 2 in the equivalence of Theorem 3.11.

Lemma 4.1. *Let \mathbf{T} be a partition clone. The following are equivalent:*

- (1) \mathbf{T} is simple;
- (2) \mathbf{T} is subdirectly irreducible;
- (3) \mathbf{T} is directly indecomposable;
- (4) $\mathbf{T} \simeq \mathbf{N}$.

Proof. The implications (1) \implies (2) and (2) \implies (3) are standard. (It is easy to see that the argument for single-sorted algebras lifts to the many-sorted case; at any rate, cf. [25, Proposition 4.8]). We now prove that $\neg(4) \implies \neg(3)$. Up to passing to coordinates, we can assume that the clone \mathbf{T} is the partition-of-unity clone of a Boolean algebra B . Let $a := (b_1, \dots, b_n) \in T(n)$, with a different from e_1^n, \dots, e_n^n . This implies that there exists i such that $b_i \notin \{0, 1\}$. Letting $c := b_i$, and consequently having $\neg c = \bigvee \{a_k : k \neq i\}$, it is a standard fact regarding Boolean algebras that $B \simeq B/\theta(c, 1) \times B/\theta(c, 0)$, where $\theta(c, 1)$ (resp. $\theta(c, 0)$) is the smallest congruence identifying c and 1 (resp. c and 0). To conclude the argument, observe that if $(b_1, \dots, b_n) \in T(n)$, then, $(b_1, \dots, b_n, 0, \dots, 0) \in T(m)$ for all $m > n$. Therefore, for each n we have a decomposition of $T(n)$ in (at least) two factors and \mathbf{T} is not directly indecomposable. To prove that (4) \implies (1), assume that $\mathbf{T} \simeq \mathbf{N}$, and let θ be a congruence of \mathbf{T} different than Δ . Then there is $n \geq 2$ and $1 \leq i \neq j \leq n$ such that $(e_i^n, e_j^n) \in \theta$. But then for all $m \in \omega$ and all $x, y \in T(m)$

$$(q_m^n(e_i^n, z_1, \dots, z_n), q_m^n(e_j^n, z_1, \dots, z_n)) \in \theta$$

where z_1, \dots, z_n are any elements of $T(m)$ such that $z_i = x$ and $z_j = y$, so that, in light of (C2), $(x, y) \in \theta$, i.e. $\theta = \nabla$. \square