

is a subcomplex since

$$\begin{aligned}\overline{D}(z_i) &= m_i - e\bar{z}_i \in B_{d+1} \otimes \Lambda(V_1, \overline{V}_1), \overline{D}), \\ \overline{D}(\bar{z}_i) &= -\Theta(m_i) = 0,\end{aligned}$$

by Corollary 2.15, where  $z_i$  is a basis element of  $V_1$  with  $D(z_i) = m_i$  and the second equality holds as  $\Theta$  is  $B_{d+1}$ -linear. Denote this subcomplex by  $C_1$  and the cokernel by  $C_2$  and consider the short exact sequences

$$(2.21) \quad 0 \rightarrow C_1 \rightarrow (B_{d+1} \otimes \mathcal{C}_{CE}(H^*(\mathbb{S}^d) \otimes L_d), \overline{D}) \rightarrow C_2 \rightarrow 0.$$

The subcomplex  $C_1$  is a sub-cdga which is a pure Sullivan algebra [FHT01, Ch. 32], i.e. the differential is only non-trivial for odd degree generators and lies in the algebra of even degree generators. Hence, it has an additional homological grading  $F_k C_1 := B_{d+1} \otimes \Lambda \overline{V}_1 \otimes \Lambda^k V_1$  and

$$H_0(C_1) = \mathbb{Q}[e, p_1, \dots, p_n, \{\bar{z}_i\}] / (\{m_i - e\bar{z}_i\}).$$

The map  $B_{d+1} \rightarrow (B_{d+1} \otimes \mathcal{C}_{CE}(H^*(\mathbb{S}^d) \otimes L_d), \overline{D})$  factors through  $F_0 C_1$  and hence we need to understand the image of the connecting homomorphism of the above short exact sequence.

For  $d = 3$  the vector space  $V_1$  is 1-dimensional with generator  $z_1$  and  $D(z_1) = p_1^2$  by Corollary 2.19. Then the sequence (2.21) actually splits as a sequence of cochain complexes so that

$$H(C_1) \hookrightarrow H^*((B_{d+1} \otimes \mathcal{C}_{CE}(H^*(\mathbb{S}^d) \otimes L_d), \overline{D})).$$

This is because

$$\tilde{C}_2 := B_{d+1} \otimes \Lambda(V_1 \oplus \overline{V}_1) \otimes \Lambda^+(V_2 \oplus \overline{V}_2)$$

is a subcomplex isomorphic to  $C_2$  as the only way  $\overline{D}|_{\tilde{C}_2}$  can have image in  $C_1$  is if there is  $w \in V_2$  so that  $D(w) = f z_1 + \chi$  for  $f \in B_{d+1}$  and  $\chi \in B_{d+1} \otimes \Lambda V_1 \otimes \Lambda^+ V_2$ . Suppose  $w$  has the minimal degree where this happens, then

$$0 = D^2(w) = f p_1^2 + D(\chi)$$

and since  $D(\chi)$  cannot be contained in  $B_{d+1} \otimes 1$  by construction it follows that  $f = 0$ . Hence,  $D^1|_{V_2}$  has image in  $B_{d+1} \otimes V_2$  and consequently (2.21) splits as as cochain complexes. This proves the second part the theorem as

$$H(C_1) \cong \mathbb{Q}[e, p_1, \bar{z}_1] / (p_1^2 - e\bar{z}_1)$$

injects into  $H^*((B_{d+1} \otimes \mathcal{C}_{CE}(H^*(\mathbb{S}^d) \otimes L_d), \overline{D})) \cong H^*(\Gamma_{\mathbb{S}^3} // \text{SO}(4); \mathbb{Q})$ . The first part then follows as the map  $B_4 \rightarrow H(C_1)$  is injective.  $\square$

*Remark 2.22.* The formulation of Corollary 2.19 for all dimensions  $d$  is due to the author's attempt to prove a general version of Theorem 2.20, namely that  $H^*(\text{BSO}(d+1)) \rightarrow H^*(\Gamma_{\mathbb{S}^d} // \text{SO}(d+1))$  is injective for all odd  $d$ . However, for  $d \geq 5$  the free resolution of the kernel of  $H^*(\text{BSO}(d)) \rightarrow H^*(F_d // \text{SO}(d))$ , i.e. the ideal generated by monomials in the Pontrjagin classes of degree  $> 2d$ , has higher syzygies and the proof for  $d = 3$  does not generalize to show that  $D^1(V_2) \subset B_{d+1} \otimes V_2$ . In fact, one would need to show a stronger version of Corollary 2.19 to get a splitting of (2.21) for odd  $d > 3$ . Nonetheless, it follows from Nariman's results that the monomials in the Euler and Pontrjagin classes do not vanish and I expect that one can improve the statement of Corollary 2.19 or control the image of the connecting homomorphism of (2.21) in order to find a purely algebraic proof of Nariman's result. For this reason, I have kept a general version of Corollary 2.19 in this article in the hope that it can serve as a starting point to prove a more general version of Theorem 2.20 – it is hard to imagine that the statement doesn't generalize to all odd  $d$ .