

Assumption 1.5. *The growth function Θ is such that the associated generalized modulus of continuity $\varphi_\Theta : [0, +\infty) \rightarrow [0, +\infty)$ is continuous, where*

$$\varphi_\Theta(s) := \begin{cases} 0 & \text{if } s = 0, \\ s|\log s|\Theta(|\log s|) & \text{if } 0 < s < e^{-d-1}, \\ e^{-d-1}(d+1)\Theta(d+1) & \text{if } s \geq e^{-d-1}. \end{cases}$$

Indeed, φ_Θ is the modulus of continuity of the force field given that the macroscopic density is Yudovich, which is enforced by [Assumption 1.4](#) (see Crippa, Inversi, Saffirio and Stefani [\[4, Lemma 1.1 and Assumption 1.3\]](#)); as explained in [\[4\]](#), the value e^{-d-1} in the definition of φ_Θ is essentially irrelevant and included solely to make φ_Θ more appealing. Under [Assumption 1.4](#) and [Assumption 1.5](#), one can define weak solutions f to (VP) through

$$\int_0^T \int_{\mathcal{X} \times \mathbb{R}^d} [(\partial_t \phi + v \cdot \nabla_x \phi - \nabla U_f \cdot \nabla_v \phi) f](t; x, v) \, dx dv \, dt = - \int_{\mathcal{X} \times \mathbb{R}^d} \phi(0, x) f(0; x, v) \, dx dv$$

for all test functions $\phi \in C_c^\infty([0, T] \times (\mathcal{X} \times \mathbb{R}^d))$, since then the product of the solution with the force field is integrable; i.e., $\|f(t) \nabla U_f(t)\|_{L^1(\mathcal{X} \times \mathbb{R}^d)} \in L^1([0, T])$.

While [\[4, Theorem 1.6\]](#) assumed φ_Θ to be nondecreasing concave in some regime for the 1-Wasserstein stability of (VP), in the p -Wasserstein setting, we instead assume a p -modified version of φ_Θ to be nondecreasing concave in some regime as follows:

Assumption 1.6. *The growth function Θ is such that $\varphi_{p,\Theta} : [0, +\infty) \rightarrow [0, +\infty)$ is nondecreasing concave on $[0, c_{p,\Theta;d})$ for some positive constant $c_{p,\Theta;d} < 1/e$ that depends only on p , Θ , and d , where $\varphi_{p,\Theta}$ is given by*

$$\varphi_{p,\Theta}(s) := \begin{cases} 0 & \text{if } s = 0, \\ s|\log s|^p \Theta^p(|\log s|) & \text{if } 0 < s \leq c_{p,\Theta;d}, \\ \varphi_{p,\Theta}(c_{p,\Theta;d}) & \text{if } s \geq c_{p,\Theta;d}. \end{cases}$$

This encompasses for instance the bounded case with $\Theta(r) = 1$ ($c_{p,\Theta;d} = e^{-\max\{p,d+1\}}$), the exponential Orlicz space with $\Theta(r) = r^{1/\alpha}$ and $1 \leq \alpha < +\infty$ ($c_{p,\Theta;d} = e^{-\max\{p\beta,d+1\}}$, $\beta := 1 + 1/\alpha$), and also a countable family of iterated logarithms due to Yudovich [\[18, Section 3\]](#) for two-dimensional Euler's equations in vorticity form; $\Theta_n : [0, +\infty) \rightarrow [0, +\infty)$ ($c_{p,\Theta;d} = \min\{\exp_{n+1}^{-2p}(1), e^{-d-1}\}$) given by

$$\Theta_n(r) := \begin{cases} r|\log_1(r)|^2 |\log_2(r)|^2 \cdots |\log_n(r)|^2 & \text{if } r \geq \exp_n(1), \\ \Theta_n(\exp_n(1)) & \text{else,} \end{cases}$$

where $\exp_0(1) := 1$, $\exp_{n+1}(1) := e^{\exp_n(1)}$, and

$$\log_n(r) := \begin{cases} r & \text{if } n = 0, \\ \underbrace{\log \circ \log \circ \cdots \circ \log}_{(n-1)\text{times}} |\log r| & \text{otherwise.} \end{cases}$$

Moreover, each of these cases satisfies the following two assumptions: