

We may refer to the triple (C, Δ, ε) just as C if the comultiplication and counit are understood. We shall occasionally use the Sweedler notation and write $\Delta(c) = \sum_i c_{(1)i} \otimes c_{(2)i}$ simply as:

$$\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)} \in C \otimes C.$$

We say C is *cocommutative* if $\tau \circ \Delta = \Delta$, where $\tau: C \otimes C \rightarrow C \otimes C$ swap the terms, i.e., $\tau(c \otimes c') = c' \otimes c$. In other words, C is cocommutative if for all $c \in C$:

$$\sum_{(c)} c_{(1)} \otimes c_{(2)} = \sum_{(c)} c_{(2)} \otimes c_{(1)}.$$

A *homomorphism of coalgebras* $(C, \Delta_C, \varepsilon_C) \rightarrow (D, \Delta_D, \varepsilon_D)$ consists of a k -linear homomorphism $f: C \rightarrow D$ such that $\Delta_D \circ f = (f \otimes f) \circ \Delta_C$ and $\varepsilon_D \circ f = \varepsilon_C$. We denote by coAlg_k the induced category of k -coalgebras with coalgebra homomorphisms. Notice that coAlg_k is equivalent to $\text{Alg}(\text{Vect}_k^{\text{op}})$, the category of algebra objects in $\text{Vect}_k^{\text{op}}$.

Definition 2.2. A *k -bialgebra* H is a k -vector space H together with a k -coalgebra structure (H, Δ, ε) and a k -algebra structure (H, μ, η) such that $\Delta: H \rightarrow H \otimes H$ and $\varepsilon: H \rightarrow k$ are algebra homomorphisms (or equivalently, $\mu: H \otimes H \rightarrow H$ and $\eta: k \rightarrow H$ are coalgebra homomorphisms). We say H is *commutative* if it is commutative as an algebra, and we say H is *cocommutative* if it is cocommutative as a coalgebra. We say H is a *Hopf algebra* if there exists a k -linear function $S: H \rightarrow H$ (necessarily unique) such that $\mu \circ (\text{id} \otimes S) \circ \Delta = \eta \circ \varepsilon = \mu \circ (S \otimes \text{id}) \circ \Delta$.

Definition 2.3. Given a k -coalgebra C , a *right C -comodule* (M, ρ) consists of a k -vector space M together with a k -linear homomorphism $\rho: M \rightarrow M \otimes C$ that is coassociative and counital: $(\text{id}_M \otimes \Delta) \circ \rho = (\rho \otimes \text{id}_C) \circ \rho$ and $(\text{id}_M \otimes \varepsilon) \circ \rho = \text{id}_M$. A (right) *C -colinear homomorphism* $f: (M, \rho) \rightarrow (M', \rho')$ is a k -linear homomorphism $f: M \rightarrow M'$ such that $\rho' \circ f = (f \otimes \text{id}_C) \circ \rho$. Let coMod_C denote the category of right C -comodules with colinear homomorphisms. *Left C -comodules* are defined completely analogously. If C is cocommutative, then left and right C -comodules are equivalent and we will simply refer to them as C -comodules. We shall occasionally use the Sweedler notation for the coaction and write simply $\sum_{(m)} m_{(0)} \otimes m_{(1)}$ for $\rho(m) = \sum_i m_{(0)i} \otimes m_{(1)i} \in M \otimes C$ for any $m \in M$.

Definition 2.4. A right C -comodule M is *finitely cogenerated* if there exists a C -colinear monomorphism $M \hookrightarrow C^{\oplus n} := k^{\oplus n} \otimes C$.

Definition 2.5. A right C -comodule M is *injective* if for every C -colinear monomorphism $\iota: X \hookrightarrow Y$ and any C -colinear homomorphism $f: X \rightarrow M$, there exists a C -colinear homomorphism $g: Y \rightarrow M$ such that $g \circ \iota = f$.

In particular, if M is a finitely cogenerated and injective right C -comodule, there exists another finitely cogenerated and injective right C -comodule N such that $M \oplus N \cong C^{\oplus n}$ as comodules, for some $n \geq 0$. Let $\text{Inj}_{\text{fc}}(C)$ denote the category of finitely cogenerated and injective right C -comodules. Then, we have the following result.

Proposition 2.6. *The category $\text{Inj}_{\text{fc}}(C)$ is an exact category.*

Proof. The category coMod_C is an abelian category as finite limits and colimits in coMod_C are created under the forgetful functor $\text{coMod}_C \rightarrow \text{Vect}_k$. Consider a short exact sequence of right C -comodules:

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0.$$

If M and P are finitely cogenerated and injective, then so is $M \oplus P$, thus as $N \cong M \oplus P$ we can conclude. \square

Definition 2.7 ([KP25]). Given a k -coalgebra C , define its *coalgebraic K -theory* $K^c(C)$ to be the algebraic K -theory spectrum $K(\text{Inj}_{\text{fc}}(C))$ of the exact category $\text{Inj}_{\text{fc}}(C)$ of finitely cogenerated and injective right C -comodules.

The class of finitely cogenerated and injective comodules forms the class of dualizable objects in comodules with respect to a monoidal structure we now make precise. Recall that given a right C -comodule (M, ρ) and a left C -comodule (N, λ) , the relative cotensor product $M \square_C N$ is defined as the equalizer in Vect_k :

$$M \square_C N \longrightarrow M \otimes N \xrightarrow[1 \otimes \lambda]{\rho \otimes 1} M \otimes C \otimes N.$$