

Note that one can choose any $\vec{\Theta}$ satisfying

$$\max\{-1, -1 - v_p(x_1)\} \leq v_p(\Theta_0) = 1 - v_p(\Theta_1) \leq \min\{0, v_p(x_0)\}.$$

We will divide $R(\vec{r}; \vec{k}')$ into following four areas, and in each area we will determine the mod- p reduction:

- (i) $\vec{t} \in R_{int}(\vec{r}; \vec{k}')$: Note that we have either $v_p(\Theta_0) \notin \{0, v_p(x_0)\}$ or $v_p(\Theta_0) \notin \{-1, -1 - v_p(x_1)\}$, in this case. Consider two $\overline{S}_{\mathbf{F}}$ -submodules $\mathcal{M}' = \overline{S}_{\mathbf{F}}(\overline{E}_1^{(0)}, \overline{E}_2^{(1)})$ and $\mathcal{M}'' = \overline{S}_{\mathbf{F}}(\overline{E}_2^{(0)}, \overline{E}_1^{(1)})$ of \mathcal{M} . Then one can observe that
- if $v_p(\Theta_0) \notin \{0, v_p(x_0)\}$, then \mathcal{M}' is a Breuil submodule of \mathcal{M} such that $\mathcal{M}' \cong \mathcal{M}(1, 1; -\beta_0, \alpha_1)$ and $\mathcal{M}/\mathcal{M}' \cong \mathcal{M}(1, 1; \alpha_0, -\beta_1)$;
 - if $v_p(\Theta_0) \notin \{-1, -1 - v_p(x_1)\}$, then \mathcal{M}'' is a Breuil submodule of \mathcal{M} such that $\mathcal{M}'' \cong \mathcal{M}(1, 1; \alpha_0, -\beta_1)$ and $\mathcal{M}/\mathcal{M}'' \cong \mathcal{M}(1, 1; -\beta_0, \alpha_1)$.

In particular, if $v_p(\Theta_0) \notin \{-1, 0, v_p(x_0), -1 - v_p(x_1)\}$, then $\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}''$ as Breuil modules. Otherwise, It is not difficult to show that the short exact sequence determined by $\mathcal{M}' \hookrightarrow \mathcal{M}$ or $\mathcal{M}'' \hookrightarrow \mathcal{M}$ is non-split. So, $\overline{\rho}^{ss}|_{I_{\mathbf{Q}_{p^2}}} \cong \omega_2^{p+1} \oplus \omega_2^{p+1}$, by Lemma 2.3.6 (i). Furthermore, the monodromy type is given as follow.

- (a) If $v_p(\Theta_0) = 0$, then $t_0 \geq 0$, and the monodromy type is $(1, 0)$.
 - (b) If $v_p(\Theta_0) = -1$, then $t_1 \geq 0$, and the monodromy type is $(0, 1)$.
 - (c) Otherwise, the monodromy type is $(0, 0)$.
- (ii) $t_1 = -1, t_0 \geq 0$: We have $v_p(\Theta_0) = 0 = -1 - v_p(x_1)$, and so $\frac{1}{\Theta_0}, \frac{x_1}{\Theta_1} \in \mathbf{F}^\times$ and $\frac{1}{\Theta_1} = 0$ in \mathbf{F} . In particular, the monodromy type is $(1, 0)$. Put $\mathcal{M}' = \overline{S}_{\mathbf{F}}(\overline{E}_2^{(0)}, \overline{E}_1^{(1)})$. It is easy to see that $\mathcal{M}' \cong \mathcal{M}(1, 2; \alpha_0, -px_1\beta_1)$ and $\mathcal{M}/\mathcal{M}' \cong \mathcal{M}(1, 0; -\beta_0, \frac{\alpha_1}{px_1})$, and so we conclude that $\overline{\rho}^{ss}|_{I_{\mathbf{Q}_{p^2}}} \cong \omega_2^p \oplus \omega_2^{p+2}$, by Lemma 2.3.6 (i), and $\overline{\rho}$ is non-split as the monodromy type is nonzero.
- (iii) $t_0 + t_1 = -1, -1 < t_0 < 0$: We have $v_p(\Theta_0) = v_p(x_0) = -1 - v_p(x_1) \notin \{-1, 0\}$, and so $\frac{x_0}{\Theta_0}, \frac{x_1}{\Theta_1} \in \mathbf{F}^\times$ and $\frac{1}{\Theta_0} = \frac{1}{\Theta_1} = 0$ in \mathbf{F} . In particular, the monodromy type is $(0, 0)$. Put

$$A := \begin{bmatrix} -\frac{x_1}{\Theta_1}\alpha_1 & \alpha_1 \\ -\beta_1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{x_0}{\Theta_0}\alpha_0 & \alpha_0 \\ -\beta_0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{x_0x_1}{\Theta_0\Theta_1}\alpha_0\alpha_1 - \beta_0\alpha_1 & -\frac{x_1}{\Theta_1}\alpha_0\alpha_1 \\ \frac{x_0}{\Theta_0}\alpha_0\beta_1 & -\alpha_0\beta_1 \end{bmatrix}.$$

Under the base change

$$\underline{E}'^{(1)} = \underline{E}^{(1)} \begin{bmatrix} 0 & -\frac{1}{\beta_1} \\ \frac{1}{\alpha_1} & -\frac{x_1}{\beta_1\Theta_1} \end{bmatrix}, \underline{F}'^{(0)} = \underline{F}^{(0)} \begin{bmatrix} 1 & 0 \\ -\frac{x_0}{\Theta_0} & 1 \end{bmatrix}, \underline{F}'^{(1)} = \underline{F}^{(1)} \begin{bmatrix} 0 & -\frac{1}{\beta_1} \\ \frac{1}{\alpha_1} & 0 \end{bmatrix},$$

we have

- $\text{Mat}_{\underline{E}^{(0)}, \underline{F}'^{(0)}}(\text{Fil}^r \mathcal{M}^{(0)}) = \text{Mat}_{\underline{E}'^{(1)}, \underline{F}'^{(1)}}(\text{Fil}^r \mathcal{M}^{(1)}) = uI_2$;
- $\text{Mat}_{\underline{E}'^{(1)}, \underline{F}'^{(0)}}(\phi_2^{(0)}) = A$ & $\text{Mat}_{\underline{E}^{(0)}, \underline{F}'^{(1)}}(\phi_2^{(1)}) = I_2$.

Note that we have $\frac{\alpha_0}{\beta_0} = \frac{\alpha_1}{\beta_1} = p\Theta_0\Theta_1$ and $\alpha_0\alpha_1 = \lambda^2\Theta_0\Theta_1$, and so one can describe the mod- p reduction as follow:

- if $px_0x_1 \neq 4$ in \mathbf{F} , then A is diagonalizable. So, extending \mathbf{F} large enough, we have $\overline{\rho}|_{I_{\mathbf{Q}_{p^2}}} \cong \omega_2^{p+1} \oplus \omega_2^{p+1}$ by Lemma 2.3.6 (i).