

quantized spectrum:

$$\begin{aligned}\hat{S}_0(\mathbf{k}) &= \tau^z & \hat{Q}_0 &= \sum_{\mathbf{k}} d_{\mathbf{k}}^\dagger \hat{S}_0(\mathbf{k}) d_{\mathbf{k}} \\ \hat{S}_1(\mathbf{k}) &= \cos k_z \tau^z + \sin k_z \tau^x & \hat{Q}_1 &= \sum_{\mathbf{k}} d_{\mathbf{k}}^\dagger \hat{S}_1(\mathbf{k}) d_{\mathbf{k}}.\end{aligned}\quad (10)$$

The first is the usual $U(1)$ symmetry, while the second is composed of Kitaev Majorana chains [16] along z -axis wires. These generators do not commute, instead they generate an infinite-dimensional Lie algebra known as the Onsager algebra, introduced in [17]. This algebra has recently appeared in the study of the 1+1D chiral anomaly on the lattice [18, 19], and offers another route to defining exact symmetries on the lattice giving anomalous symmetries in the IR.

We can actually write a Hamiltonian that has this symmetry and *two* Weyl nodes, which in the BdG formalism above (using the $d_{\mathbf{k}}^\dagger$ basis) is

$$\begin{aligned}h_{\text{double Weyl}}^{\text{BdG}}(\mathbf{k}) &= \mathbb{1}_\tau \otimes \frac{1}{2} \left[\sin k_x \sigma^x + \sin k_y \sigma^z \right. \\ &\quad \left. + [\cos k_z - \cos K + m(\mathbf{k})] \sigma^y \right],\end{aligned}\quad (11)$$

where the identity in the τ basis ensures it has both $U(1)$ symmetries, K is a parameter, and $m(\mathbf{k})$ is the same as in (3). This model is a magnetic Weyl semimetal model that has the two Weyl nodes at $\mathbf{k} = \pm \mathbf{K}$ where $\mathbf{K} = (0, 0, K)$.

Let's linearize around the Weyl nodes. We get

$$h_l^{\text{BdG}} = \mathbb{1}_\tau \otimes \frac{1}{2} (k_x \sigma^x + k_y \sigma^z - \sin K k_z \sigma^y). \quad (12)$$

This shows that for $K \neq 0, \pi$, the two Weyl nodes have an opposite handedness. To figure out the effect of $\hat{S}_1(\mathbf{k})$ at low energy, we can also linearize it, and obtain

$$\hat{S}_{1,K}(\mathbf{k}) = \cos K \tau^z + \sin K \tau^x. \quad (13)$$

The important feature for $K \neq 0, \pi$ is that the second term is non-zero.

Thus, together with $\hat{S}_0 = \tau^z$, these generate an $su(2)$ algebra acting on the low energy theory. Note that τ^x acts by exchanging particles at \mathbf{K} with holes at $-\mathbf{K}$. Thus, it is convenient to apply a charge conjugation the right-handed Weyl fermion, to give a low energy theory in terms of two left-handed Weyl fermions, now with *opposite* charge w.r.t. τ^z rotations. τ^x rotations meanwhile act by a flavor rotation exchanging the two Weyl fermions. Thus, our symmetry generators \hat{Q}_0, \hat{Q}_1 correspond to two $su(2)$ generators in the flavor symmetry of the low energy, at an angle of K . For $K \neq 0, \pi$, they thus generate the whole chiral symmetry.

We can demonstrate that this symmetry protects the gapless Weyl points. To do so, we must break translation symmetry, since otherwise z -axis translations also

act as a discrete axial symmetry and help to stabilize the Weyl nodes [12, 13]. To analyze translation-symmetry breaking, we consider an extended basis $e_{\mathbf{k}} \equiv (c_{\mathbf{k}-\mathbf{K}}, c_{-\mathbf{k}+\mathbf{K}}^\dagger, c_{\mathbf{k}+\mathbf{K}}, c_{-\mathbf{k}-\mathbf{K}}^\dagger)^T$ (we suppress the spin component). Hamiltonians in this basis may couple states at \mathbf{k} with $\mathbf{k} + 2\mathbf{K}$ but are automatically \hat{Q}_0 preserving. In this basis the symmetry action of \hat{Q}_1 becomes

$$\begin{aligned}U(1)_{\hat{Q}_1} : \delta e_{\mathbf{k}} &= i(\cos k_z \cos K \tau^z + \sin k_z \sin K \eta^z \tau^z \\ &\quad + \sin k_z \cos K \tau^x - \cos k_z \sin K \eta^z \tau^x) e_{\mathbf{k}},\end{aligned}\quad (14)$$

which prohibits all mass terms except $m_j(\mathbf{k}) \eta^z \sigma^j$. However, these terms always commute with at least one term in the original Hamiltonian, so the result is a shift in the gapless modes rather than a gap. At a large enough perturbation, we can move the modes until $K = 0$ or π , where the symmetry generators are aligned and no longer generate the whole $SU(2)$ symmetry. At these special points, we will be able to open a symmetric gap.

IV. TIME REVERSAL SYMMETRIC SINGLE WEYL FERMION IN 3+1D

So far, we have considered time-reversal breaking models. We can also construct time-reversal invariant models, at the cost of making the symmetry generator slightly more not-on-site. As long as $S(\mathbf{k})$ is a smooth function of the momentum, then the charge density in real space will be a sum of terms with faster-than-polynomial decay. Such ‘‘almost-local’’ operators share many properties with local operators, while being closed under Hamiltonian evolution generated by such terms [20, 21]. This will allow us to employ bump functions and partitions of unity in momentum space.

To construct a time-reversal invariant model with a single protected Weyl fermion, we begin with a model on a cubic lattice with eight Weyl nodes. We use the BdG formalism with the basis $d_{\mathbf{k}}^\dagger \equiv (c_{\mathbf{k}\uparrow}^\dagger, c_{\mathbf{k}\downarrow}^\dagger, c_{-\mathbf{k}\uparrow}, c_{-\mathbf{k}\downarrow})$ used above, giving the Hamiltonian

$$h_8^{\text{BdG}}(\mathbf{k}) = \frac{1}{2} [\sin k_x \sigma^x + \sin k_y \sigma^z + \sin k_z \tau^z \sigma^y]. \quad (15)$$

This model has Weyl nodes at all eight time-reversal-invariant-momentum (TRIM) points of the Brillouin zone, as well as a time-reversal symmetry $\Theta = i\sigma^y \mathcal{K}$, where \mathcal{K} is complex conjugation, satisfying $\Theta^2 = -1$.

We will now add a $U(1)$ symmetry-breaking term that will gap out all Weyl nodes except the one at $\mathbf{k} = 0$. In order to facilitate our discussion, let us first define a bump function $B(k)$ given by

$$B(\mathbf{k}, w) = \begin{cases} e^{-\frac{w^2}{|\mathbf{k}|^2 - w^2}} & \text{for } |\mathbf{k}| < w \\ 0 & \text{for } |\mathbf{k}| \geq w \end{cases} \quad (16)$$