

From now on we will drop the subscript from the notation $C^{\leq k, w}$.

For a sequence s denote by $\sigma(s)$ and $d(s)$ respectively its Schnirelmann density and asymptotic density. Namely,

$$\sigma(s) = \inf_n \frac{A(n)}{n},$$

and

$$d(s) = \lim_{n \rightarrow \infty} \frac{A(n)}{n} \quad \text{if it exists,}$$

where $A(n) = |\{i : s_i \leq n\}|$.

Recall Mann's theorem, relating sumsets to Schnirelmann densities:

Theorem 3.4 (Mann's Theorem [Man42]). *For subsets A and B of \mathbb{N} , if the sumset $A + B \neq \mathbb{N}$, then*

$$\sigma(A + B) \geq \sigma(A) + \sigma(B) \quad \text{holds.}$$

We now use Mann's theorem together with the results above to establish Theorem 1.4.

Proof of Theorem 1.4. By Lemma 3.3 it is enough to prove $P(C^{\leq k, w}) = 1$. By Lemma 3.1 $C^{k, w}$ is a tail event, and by Lemma 3.3 we have $P(C^{\leq k, w}) = P(C^{k, w})$, so it is enough to prove that $P(C^{\leq k, w}) > 0$. From $\mathbb{E}X < k$ it follows that $d(\{W_n\}_{n \in \mathbb{N}}) > \frac{1}{k}$. Introduce the random variable N by

$$N = \min \left\{ n : \forall m \geq n, \quad \frac{A(m)}{m} > \frac{1}{k} \right\}.$$

Since $\sum_{n=1}^{\infty} P(N = n) = 1$, it follows that $\exists n_0$ such that $P(N = n_0) > 0$.

We now distinguish two cases.

Case 1: $1 \in \text{Supp}X$.

We have

$$P(N = n_0 \text{ for the sequence generated by } \{X_i\}_{i > n_0}) > 0,$$

and since $1 \in \text{Supp}X$, we have

$$P(X_1 = X_2 = \dots = X_{n_0} = 1) > 0.$$

Since these two events deal with distinct parts of the i.i.d. sequence $\{X_i\}_{i > n_0}$, with positive probability both events occur simultaneously. Now, let us evaluate the Schnirelmann density $\sigma(\{W_i\}_{i \geq 1})$ for such sequences of gaps. We have

$$\frac{A(n)}{n} \geq \begin{cases} 1 & \text{if } n \leq n_0 \\ \frac{1}{2} & \text{if } n_0 < n \leq 2n_0, \\ \frac{1}{k} & \text{if } n > 2n_0 \end{cases}$$

where the first two inequalities follow from the choice of the first n_0 gaps and the last one from the condition $N = n_0$.

Thus we get $P(\sigma(W) \geq \frac{1}{k}) > 0$ and Mann's theorem implies asymptotic $\leq k$ -completeness with the same positive probability.

Case 2: $1 \notin \text{Supp}X$.