

with arbitrary λ and arbitrary coefficients c_i . They satisfy

$$u \in H^s(\Omega), \quad p \in H^{s-1}(\Omega) \quad \forall s < 1 + \lambda,$$

and the parameter λ can be chosen such that the test example has the desired regularity.

3.2 Boundary conditions

As in Subsection 2.2, the coefficients c_i and the parameter λ can be used to satisfy homogeneous boundary conditions. Two boundary conditions for both $\theta = 0$ and $\theta = \omega$ give a homogeneous linear system of 4 equations which has a non-trivial solution iff the determinant vanishes. This condition is used to find again a countable number of values of λ . Let us sketch this approach for the case of Dirichlet boundary conditions and $\lambda \neq 0$.

The condition $U(0) = 0$ leads to

$$\begin{pmatrix} U_r(0) \\ U_\theta(0) \end{pmatrix} = \begin{pmatrix} c_1 + (1 - \lambda)c_3 \\ c_2 + (1 + \lambda)c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{i. e.} \quad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -(1 - \lambda)c_3 \\ -(1 + \lambda)c_4 \end{pmatrix},$$

hence

$$U(\omega) = -(1 - \lambda)c_3 U^{(1)}(\omega) - (1 + \lambda)c_4 U^{(2)}(\omega) + c_3 U^{(3)}(\omega) + c_4 U^{(4)}(\omega).$$

The 2×2 linear system $U(\omega) = 0$ for the coefficients c_3 and c_4 has the determinant

$$4(\sin^2 \lambda\omega - \lambda^2 \sin^2 \omega) = 4(\sin \lambda\omega - \lambda \sin \omega)(\sin \lambda\omega + \lambda \sin \omega). \quad (3.2)$$

This means that for given angle ω one gets the corresponding exponents $\lambda \in \mathbb{C}$ by solving (separately) the two transcendental, scalar equations $\sin \lambda\omega = \pm \lambda \sin \omega$. All values $\text{Re} \lambda \in [\frac{1}{2}, 4]$ are given for $\omega_k = k\pi/10$, $k = 4, 5, \dots, 20$, in [Dau89].

3.3 Weak and very weak solutions

As in Subsection 2.3, the pair (u, p) is a weak solution for $\lambda > 0$ and a very weak solution for $-\min(1, \xi) < \lambda \leq 0$, where

$$\xi = \min\{\text{Re} \lambda > 0 : \lambda \text{ satisfies (3.2)}\}$$

in the case of Dirichlet boundary conditions. The weak solution $(u, p) \in H^1(\Omega) \times L_0^2(\Omega)$ is defined by

$$\begin{aligned} (\nabla u, \nabla v) - (\nabla \cdot v, p) &= 0 \quad \forall v \in H_0^1(\Omega), \\ (\nabla \cdot u, q) &= 0 \quad \forall q \in L_0^2(\Omega). \end{aligned}$$

The very weak solution $(y, p) \in L^2(\Omega) \times P'$ with $P = \{v \in H^1(\Omega) \cap L_0^2(\Omega) : r^{-1}v \in L^2(\Omega)\}$ is defined by

$$(u, -\Delta v + \nabla q) - (\nabla \cdot v, p) = \langle u, qn - \partial_n v \rangle_\Gamma \quad \forall (v, q) \in \mathcal{V}$$

where $\mathcal{V} := \{(v, q) \in H_0^1(\Omega) \times L_0^2(\Omega) : -\Delta v + \nabla q \in L^2(\Omega), \nabla \cdot v \in P\}$, see [ALP24].