

**Theorem 1.1** (equivalent to [Ale19, Theorem 22]). *Let  $\alpha = \alpha_1 \dots \alpha_n$  be a composition with  $\alpha_i = \alpha_{i+1}$  for some  $i \in [n-1]$ . Let  $\sigma \in \mathcal{S}_n$ , such that  $\sigma_{i+1} = \sigma_i \pm 1$ . Then,*

$$E_\alpha^\sigma(\mathbf{x}; q, t) = E_\alpha^{\sigma s_i}(\mathbf{x}; q, t).$$

Our main result is stated in the following theorem, which extends Theorem 1.1 by removing the assumption that  $\sigma_i$  and  $\sigma_{i+1}$  are consecutive.

**Theorem 1.2.** *Let  $\alpha = \alpha_1 \dots \alpha_n$  be a composition with  $\alpha_i = \alpha_{i+1}$  for some  $i \in [n-1]$ , and let  $\sigma \in \mathcal{S}_n$ . Then,*

$$E_\alpha^\sigma(\mathbf{x}; q, t) = E_\alpha^{\sigma s_i}(\mathbf{x}; q, t).$$

Theorem 1.2 is proved algebraically from Theorem 1.1 in Section 2.3. However, in this article, our focus is on providing a fully combinatorial proof of Theorems 1.1 and 1.2 by constructing a probabilistic bijection on non-attacking fillings.

The combinatorial interpretation of Theorem 1.2 is that swapping the basement entries in two adjacent columns of the same height does not change the generating function for non-attacking fillings. Our approach for proving this builds upon a (classical) bijection introduced in [CHMMW22] for unrestricted fillings of partition shape, in which two adjacent entries of the filling in the bottom row of columns of the same height are swapped while preserving the overall weight of the filling. In [Man24], the second author extended this technique by introducing a probabilistic bijection for partition-shaped non-attacking fillings. In this article, we generalize this probabilistic bijection to composition-shaped non-attacking fillings, thereby giving a bijective proof of Theorem 1.2.

As implications of Theorem 1.2, certain assumptions can be removed from theorems in [Ale19] and [CMW22], which we summarize in Section 4.

This article is organized as follows. Section 2 contains preliminaries on permuted-basement Macdonald polynomials and their tableaux formula in terms of non-attacking fillings. Section 3 defines a probabilistic bijection on non-attacking fillings, leading to a combinatorial proof of Theorem 1.2 in Section 3.1. Finally, Section 4 discusses some implications of Theorem 1.2.

**Acknowledgements.** We would like to thank Per Alexandersson, Sarah Mason, and Arun Ram for helpful discussions. OM was supported by NSERC grant RGPIN-2021-02568.

## 2. PRELIMINARIES

**2.1. Permutations and compositions.** Throughout the paper, let  $n$  be a nonnegative integer. The *symmetric group*  $\mathcal{S}_n$  is the group of all permutations of the set  $[n] = \{1, 2, \dots, n\}$ . We write permutations in one-line notation  $\sigma = [\sigma_1, \sigma_2, \dots, \sigma_n]$ , with brackets to distinguish them from compositions. Given two permutations  $\sigma, \pi \in \mathcal{S}_n$ , their product is the permutation  $\sigma\pi \in \mathcal{S}_n$  defined by  $(\sigma\pi)_i = \sigma_{\pi_i}$ . The *reverse*  $\text{rev}(\sigma)$  of a permutation  $\sigma \in \mathcal{S}_n$  is the permutation  $\sigma w_0 = [\sigma_n, \sigma_{n-1}, \dots, \sigma_1]$ , where  $w_0 = [n, n-1, \dots, 1]$ . The symmetric group  $\mathcal{S}_n$  is generated by the *simple transpositions*  $s_1, s_2, \dots, s_{n-1}$ , where  $s_i$  swaps  $i$  and  $i+1$ . The *length*  $\ell(\sigma)$  of a permutation  $\sigma \in \mathcal{S}_n$  is the smallest number of simple transpositions whose product equals  $\sigma$ . A *reduced expression* of a permutation  $\sigma$  is a sequence  $s_{i_1}, s_{i_2}, \dots, s_{i_{\ell(\sigma)}}$  of simple transpositions whose product  $s_{i_1} s_{i_2} \dots s_{i_{\ell(\sigma)}}$  equals  $\sigma$ .

A *composition of length  $n$*  is a sequence  $\alpha = \alpha_1 \alpha_2 \dots \alpha_n$  of nonnegative integers. A *partition* is a weakly decreasing composition, and an *antipartition* is a weakly increasing composition. Given a composition  $\alpha$ , let  $\text{dec}(\alpha)$  and  $\text{inc}(\alpha)$  denote the partition and antipartition obtained