

② Exactly one vertex in T has outgoing neighbor in $V(BS_n \setminus (BS_{n-1}^1 \oplus BS_{n-1}^2 \oplus BS_{n-1}^3))$.

Refer to Figure 4 Subcase 3.2 ②, w.l.o.g, assume that b has an outgoing neighbor b' in $V(BS_n \setminus (BS_{n-1}^1 \oplus BS_{n-1}^2 \oplus BS_{n-1}^3))$, $\{c^+\} \subseteq V(BS_{n-1}^1)$, $\{a^-, c^-\} \subseteq V(BS_{n-1}^2)$, $\{a^+\} \subseteq V(BS_{n-1}^3)$. Similarly to the proof in Subcase 3.1 in Part 1, we can find $M_i \subseteq H_i$ such that $|M_i| = 2k - 1$ ($i = 1, 3$) and $M_2 \subseteq H_2$ such that $|M_2| = 2k$.

Since $\kappa(BS_{n-1}) = 4k + 1$, we can construct one $4k$ -fan from a to $M_1 \cup M_2 \cup \{c^+\}$, one $4k$ -fan from b to $M_1^+ \cup M_3 \cup \{a^-, c^-\}$, one $4k$ -fan from c to $M_2^+ \cup M_3^+ \cup \{a^+\}$. Thus, we obtain the desired paths structure. Let $b_1 = b'$. Additionally, there must exist $b_2 \in N_{BS_{n-1}^2}(b)$ such that b_2 does not appear on any of $(b, M_1^+ \cup M_3 \cup \{a^-, c^-\})$ -paths, since $\kappa(BS_{n-1}^2 - \{b_2\}) = (2n - 5) - 1 = 4k$ and $|M_1^+ \cup M_3 \cup \{a^-, c^-\}| = 4k$.

③ Exactly two vertices in T have outgoing neighbor in $V(BS_n \setminus (BS_{n-1}^1 \oplus BS_{n-1}^2 \oplus BS_{n-1}^3))$.

Refer to Figure 4 Subcase 3.2 ③, w.l.o.g, assume that $\{a^-\} \subseteq V(BS_{n-1}^2)$, $\{a^+\} \subseteq V(BS_{n-1}^3)$ and b, c have outgoing neighbors b', c' in $V(BS_n \setminus (BS_{n-1}^1 \oplus BS_{n-1}^2 \oplus BS_{n-1}^3))$, respectively. Similarly to the proof of Subcase 3.1 in Part 1, we can find $M_i \subseteq H_i$ such that $|M_i| = 2k - 1$ ($i = 1, 3$) and $M_2 \subseteq H_2$ such that $|M_i| = 2k + 1$. In addition, there must exist one (b', c') -path in $BS_n \setminus (BS_{n-1}^1 \oplus BS_{n-1}^2 \oplus BS_{n-1}^3)$.

Since $\kappa(BS_{n-1}) = 4k + 1$, we can construct one $4k$ -fan from a to $M_1 \cup M_2$, one $(4k - 1)$ -fan from b to $M_1^+ \cup M_3 \cup \{a^-\}$, one $(4k + 1)$ -fan from c to $M_2^+ \cup M_3^+ \cup \{a^+\}$. Thus, we obtain the desired paths structure. Moreover, there must exist $b_1, b_2 \in N_{BS_{n-1}^2}(b)$ such that b_1, b_2 do not appear on any of $(b, M_1^+ \cup M_3 \cup \{a^+\})$ -paths, since $\kappa(BS_{n-1}^2 - \{b_1, b_2\}) = (2n - 5) - 2 = 4k - 1$ and $|M_1^+ \cup M_3 \cup \{a^+\}| = 4k - 1$.

④ Each vertex in T has at least one outgoing neighbor in $V(BS_n \setminus (BS_{n-1}^1 \oplus BS_{n-1}^2 \oplus BS_{n-1}^3))$.

Refer to Figure 4 Subcase 3.2 ④, w.l.o.g, assume that a, b and c have outgoing neighbors a', b' and c' in $V(BS_n \setminus (BS_{n-1}^1 \oplus BS_{n-1}^2 \oplus BS_{n-1}^3))$, respectively. Since a', b' and c' cannot all be equal, suppose that $b' \neq a'$ and $b' \neq c'$. Similarly to the proof in Subcase 3.1 in Part 1, we can find $M_i \subseteq H_i$ such that $|M_i| = 2k$ ($i = 1, 3$) and $M_2 \subseteq H_2$ such that $|M_2| = 2k + 1$. In addition, there must exist one (a', c') -path in $BS_n \setminus (BS_{n-1}^1 \oplus BS_{n-1}^2 \oplus BS_{n-1}^3)$.

Since $\kappa(BS_{n-1}) = 4k + 1$, we can construct one $(4k + 1)$ -fan from a to $M_1 \cup M_2$, one $4k$ -fan from b to $M_1^+ \cup M_3$, one $(4k + 1)$ -fan from c to $M_2^+ \cup M_3^+$. Thus, we obtain the desired paths structure. Let $b_1 = b'$. Additionally, there must exist $b_2 \in N_{BS_{n-1}^2}(b)$ such that b_2 does not occur on any of $(b, M_1^+ \cup M_3)$ -paths, since $\kappa(BS_{n-1}^2 - \{b_2\}) = (2n - 5) - 1 = 4k$ and $|M_1^+ \cup M_3| = 4k$.

Combining the results of Part 1 and Part 2, the proof is completed. \square