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## 2. GROUPOIDS

A *groupoid* is a set  $G$  with a partially defined composition. This consists of a map  $\mu : S \rightarrow G$ , where  $S$  is a subset of  $G \times G$ . If  $a, b \in G$  we say that the product  $a \star b$  is *defined* if  $(a, b) \in S$ , and then we write  $a \star b = \mu(a, b)$ . The groupoid is also required to have an “inverse map”  $x \mapsto x'$  from  $G \rightarrow G$ . The inverse map is more commonly denoted as  $x \mapsto x^{-1}$ , but we will be concerned with a groupoid whose elements are matrices, and we will reserve the notation  $x^{-1}$  for the matrix inverse. The following axioms are required.

**Axiom 1** (Associative Law). If  $a \star b$  and  $b \star c$  are defined then  $(a \star b) \star c$  and  $a \star (b \star c)$  are defined, and they are equal.

We say that  $a \star b \star c$  is *defined* if  $a \star b$  and  $b \star c$  are defined, and then we denote  $(a \star b) \star c = a \star (b \star c)$  as  $a \star b \star c$ .

**Axiom 2** (Inverse). The compositions  $a \star a'$  and  $a' \star a$  are always defined. Thus if  $a \star b$  is defined, then  $a \star b \star b'$  is defined, and this is required to equal  $a$ . Similarly  $a' \star a \star b$  is defined, and this is required to equal  $b$ .

**Example 2.1.** A category  $\mathcal{C}$  is *small* if its class of objects is a set. A small category is a *groupoid category* if every morphism is an isomorphism. Assuming this, the disjoint union

$$G = \bigsqcup_{A, B \in \mathcal{C}} \text{Hom}(A, B)$$

is a groupoid, with the  $\star$  operation being composition: thus if  $a \in \text{Hom}(A, B)$  and  $b \in \text{Hom}(C, D)$ , then  $a \star b$  is defined if and only if  $B = C$ . The groupoid axioms are clear.

**Lemma 2.2.** *In a groupoid, we have  $(a')' = a$ . Moreover if  $a \star b$  is defined then so is  $b' \star a'$  and  $(a \star b)' = b' \star a'$ .*

*Proof.* Since  $(a')' \star a'$  and  $a' \star a$  are both defined, by the Associative Law the product  $(a')' \star a' \star a$  is defined, and using the Inverse Axiom, this equals both  $(a')'$  and  $a$ . For the second assertion, assume  $a \star b$  is defined. It follows from the axioms that

$$(a \star b)' = (a \star b)' \star a \star b \star b' \star a' = b' \star a'. \quad \square$$

Given a groupoid  $G$ , let us say an element  $A$  is *idempotent* if  $A \star A$  is defined and  $A \star A = A$ .

**Lemma 2.3.** *An element  $A \in G$  is an idempotent if and only if  $A = g \star g'$  for some  $g \in G$ . If  $A$  is idempotent then  $A = A'$ .*

*Proof.* It is easy to check that  $g \star g'$  is idempotent. Conversely if  $A$  is idempotent, then  $A = A \star A'$  since  $A = A \star A = A \star A \star A' = A \star A'$ , and so  $A$  can be written  $g \star g'$  with  $g = A$ . Now if  $A = g \star g'$  then  $A = A'$  as a consequence of Lemma 2.2.  $\square$

**Lemma 2.4.** *If  $g \in G$  then there are unique idempotents  $A$  and  $B$  such that  $g = g \star A$  and  $g = B \star g$ .*

*Proof.* We can take  $A = g' \star g$ , and this is an idempotent such that  $g \star A = g$ . Conversely if  $A'$  is any other element such that  $g \star A' = g$ , then  $g^{-1} \star g = g^{-1} \star g \star A' = A'$ , so  $A' = A$ . The statements about  $B$  are proved similarly.  $\square$