

turns into

$$\mu_{u^\perp} + \mu_{v^\perp} = 0, \quad (4.27)$$

where the index  $\perp$  refers to the projection to the subspace orthogonal to  $w, w^*$ , i.e.  $\mu_u^\perp = \sum_{i=2}^n (u_i \otimes u_i^\dagger - u_i^* \otimes u_i^t)$ . Clearly, the latter equation means that  $\text{Span}\left(\left\{u_i\right\}_{i=2}^n\right) = \text{Span}\left(\left\{v_i^*\right\}_{i=2}^n\right)$ .

It is also clear from above that the map  $(w, u^\perp) \mapsto (u, v, w)$  is the one described in the proposition.  $\square$

## 5. General embeddings $\mu = 0$

To start with, we prove the following lemma (we assume  $\mu^{-1}(0)$  connected):

**Lemma 1.** *An orbit  $L \subset \mu^{-1}(0)$  is Lagrangian only if  $L \simeq \mu^{-1}(0)$ .*

*Proof.* Start with the definition of the moment map:  $d\mu[v] = \omega(\bullet, v)$ , where  $v$  is a vector field generating the Lie group action. Take a point  $p \in L$ : contracting the above equality with an arbitrary vector field tangent to  $\mu^{-1}(0)$  we find that

$$0 = \omega(w, v), \quad \text{where } v \in T_p L, \quad w \in T_p \mu^{-1}(0) \quad (5.1)$$

Now, suppose  $v_1 \dots v_\ell$  is a basis of vectors tangent to the orbit  $L \subset \mu^{-1}(0)$  but not a basis in  $T_p \mu^{-1}(0)$ . In this case there is a non-zero vector  $w \in T_p \mu^{-1}(0)/T_p L$ . By (5.1) the restriction of  $\omega$  to  $\text{Span}(T_p L, w)$  is zero. As a result,  $L$  is not Lagrangian unless  $T_p \mu^{-1}(0) \simeq T_p L$  at every point implying that  $\mu^{-1}(0) \simeq L$ .  $\square$

Thus, if one wishes to find Lagrangian submanifolds within  $\mu^{-1}(0)$ , one should search for cases where  $\mu^{-1}(0)$  is a single orbit. Let us recall how this works in the case of  $\text{SU}(n)$  flag manifolds.

**5.1.  $\text{SU}(n)$  flag manifolds.** We will now use the method employed in the proof of Proposition 4 to show that the flag manifold featuring in Proposition 2 is  $\mu^{-1}(0)$ . Indeed, for any matrix  $\mu_i := g_i \Lambda_i g_i^\dagger$  in the orbit of  $\Lambda_i$  given by (3.4) one has the