

(resp. dual) closedness criterion. To this aim, let us recall the concept of closedness regarding a set [1].

Given two subsets  $U$  and  $V$  of a topological space  $Z$ , one says that  $U$  is closed regarding  $V$  if  $\overline{U} \cap V = U \cap V$ , where  $\overline{U}$  is the closure of  $U$  in  $Z$ . In particular, Given  $z \in Z$ ,  $U$  is closed regarding  $\{z\}$  if and only if either  $z \in U$  or  $z \notin \overline{U}$ ,  $U$  is closed regarding  $V$  if and only if  $U$  is closed regarding  $\{z\}$  for all  $z \in V$ , and  $U$  is closed if and only if it is closed regarding  $Z$ .

The term *characterization of Farkas' lemma* was introduced by V. Jeyakumar, S. Kum, and G. M. Lee [13] and that of *characterization of stable Farkas' lemma* by V. Jeyakumar and G. M. Lee [14], both papers published in 2008. Their characterization of stable Farkas' lemma consisted in the closedness of certain cone associated with the given conic system. After the introduction of the concept of closedness regarding a given set by Boř in 2010, successive characterizations of (stable) Farkas' lemma in terms of closedness of certain set regarding another set were obtained for different types of systems, e.g., systems involving convex, nonconvex composite functions [8], [9], [10], systems involving vector-valued functions ([4] and [6]), systems involving a family of finite subsets of the given index set [7], etc. The characterization of stable Farkas' lemma for nonconvex composite semi-infinite programming has been considered in [16] and [17].

Considering feasible sets of the form  $C \cap \mathbb{A}^{-1}(D)$ , with the convex set  $D$  not being necessarily a cone, is the main novelty of this paper, which is organized as follows. Section 2 characterizes the equivalence  $(\mathcal{A}) \iff (\mathcal{B})$  in terms of the closedness regarding  $\{(0_X, 0_Y, -1)\}$  and  $\{(0_Y, -1)\}$  of certain subsets of the primal spaces  $X \times Y \times \mathbb{R}$  and  $Y \times \mathbb{R}$ , respectively (see Theorems 1 and 2). Section 3, in turn, characterizes the equivalence  $(\mathcal{A}) \iff (\mathcal{B})$  in terms of the closedness regarding  $\{(0_{X'}, 0)\}$  of certain subset of the dual space  $X' \times \mathbb{R}$  (Theorem 11) as well as the non-emptiness of the feasible set  $C \cap \mathbb{A}^{-1}(D)$  (Proposition 9) and the stable Farkas' lemma (Proposition 16). Section 4 is devoted to linear infinite systems. Section 5 characterizes Farkas-type lemmas oriented to identify the minima of convex (resp. concave) functions on  $C \cap \mathbb{A}^{-1}(D)$ , reason why they are known in the literature as convex (resp. concave) Farkas' lemmas. More in detail, the convex Farkas' lemmas provided in Section 3 determine when a convex feasible set of the form  $C \cap \mathbb{A}^{-1}(D)$  is contained in the reverse-convex set  $\{x \in X : f(x) \geq 0\}$ , where  $f$  is convex. Similarly, Section 5 characterizes the containment of  $C \cap \mathbb{A}^{-1}(D)$  in the sublevel set of a convex function  $f$  (Theorem 23). Finally, Section 6 shows applications to constrained convex minimization problems (optimization, strong duality and stable strong duality theorems) and to functional approximation by polynomials (Farkas-type lemmas).

## 2 Primal closedness characterizations

Recall that  $f : X \longrightarrow \overline{\mathbb{R}}$  ( $f \in \overline{\mathbb{R}}^X$  in short) is a proper convex function. We consider the vector-valued mapping  $H : X \times Y \times \text{dom } f \longrightarrow X \times Y \times \mathbb{R}$  such that

$$H(x, y, v) = (x - v, \mathbb{A}x - y, f(v)),$$