

We point out that there are no conditions on spaces and maps, except normality, making the theorem as broad as possible.

Proof. We keep the same data as in Lemma 34, so $\text{liftcat}_f^{\text{op}}(h) = \text{liftcat}_{j \circ f}^{\text{op}}(\mu)$. On the other hand, Proposition 12 shows that $\text{liftcat}_{j \circ f}(\mu) = \text{liftcat}_f(h)$. So we can work with μ and $f' = j \circ f$ rather than with h and f .

By definition, $\text{liftcat}_{f'}(\mu) \leq n$ if and only if there exists a map $\Phi: X \rightarrow T^n(\mu)$ making the following diagram homotopy commutative:

$$\begin{array}{ccccc} & & & T^n(\mu) & \\ & & \nearrow \Phi & \downarrow t_n & \\ X & \xrightarrow{f'} & Z_h & \xrightarrow{\Delta_{n+1}} & Z_h^{n+1}. \end{array}$$

From Remark 6, we regard the cofibrations μ and t_n as inclusions and consider that $T^n(\mu) = \bigcup_{i=0}^n p_i^{-1}(A) \subseteq Z_h^{n+1}$ where p_i is the i -th projection $p_i: Z_h^{n+1} \rightarrow Z_h$.

Assume we have the map Φ and the homotopy $H: X \times I \rightarrow Z_h^{n+1}$ between $\Delta_{n+1} \circ f'$ and $t_n \circ \Phi$.

Consider the open neighborhood \mathcal{N} of A in Z_h which is deformable in Z_h into A with some homotopy $G \text{ rel } A$. For $0 \leq i \leq n$, let $h_i = p_i \circ t_n \circ \Phi$ and $U_i = h_i^{-1}(\mathcal{N})$. As $(t_n \circ \Phi)(X) \subseteq T^n(\mu) = \bigcup_{i=0}^n p_i^{-1}(A)$, we have $X = \bigcup_{i=0}^n U_i$.

Now we can define homotopies $L_i: U_i \times I \rightarrow Z_h$ by:

$$L_i(u, t) = \begin{cases} p_i(H(u, 2t)) & \text{if } 0 \leq t \leq \frac{1}{2} \\ G(h_i(u), 2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Each L_i is well defined because $p_i(H(u, 1)) = p_i((t_n \circ \Phi)(u)) = h_i(u) = G(h_i(u), 0)$. We have $L_i(u, 0) = f'(u)$ and $L_i(u, 1) \in A$, so L_i is a homotopy between $f'|_{U_i}$ and $\mu \circ l_i$ where $l_i = L_i(-, 1)$.

Conversely assume we have an open cover $(U_i)_{0 \leq i \leq n}$ of X and homotopies $K_i: U_i \times I \rightarrow Z_h$ with $K_i(u, 0) = f'(u)$ and $K_i(u, 1) \in A$. Choose a closed subset $F_i \subset U_i$ (possibly the empty set) in each U_i . If X is normal, we have two more open covers $(V_i)_{0 \leq i \leq n}$ and $(W_i)_{0 \leq i \leq n}$ of X such that

$$F_i \subset W_i \subset \overline{W_i} \subset V_i \subset \overline{V_i} \subset U_i.$$

Moreover, as $\overline{W_i}$ and $\mathbb{C}V_i = X \setminus V_i$ are disjoint closed subspaces of the normal space X , we have an Urysohn map $\varphi_i: X \rightarrow I$ with $\varphi_i(\overline{W_i}) = 1$ and $\varphi_i(\mathbb{C}V_i) = 0$.

Now we define $H_i: X \times I \rightarrow Z_h$ by:

$$H_i(x, t) = \begin{cases} f'(x) & \text{if } x \in \mathbb{C}\overline{V_i} \\ K_i(x, t\varphi_i(x)) & \text{if } x \in U_i. \end{cases}$$

Each H_i is well defined because if $x \in \mathbb{C}\overline{V_i} \cap U_i$, then $K_i(x, t\varphi_i(x)) = K_i(x, 0) = f'(x)$.

Let $H = (H_0, \dots, H_n): X \times I \rightarrow Z_h^{n+1}$ and $\Phi = H(-, 1)$. We have $H(-, 0) = \Delta_{n+1} \circ f'$. On the other hand, for any $x \in X$, there is (at least) one i such that $x \in W_i$, so $\varphi_i(x) = 1$ and $H_i(x, 1) = K_i(x, 1) \in A$; this means that $\Phi(X) \subseteq T^n(\mu)$. Thus H is a homotopy between $\Delta_{n+1} \circ f'$ and $t_n \circ \Phi$. \square