

therefore implies that  $1 = \sum_{v \in \Lambda^0} \sum_{i=0}^{n-1} S_{v,w_i}$ . Consequently,

$$\begin{aligned} U_j U_j^* &= \left( \sum_{v \in \Lambda^0} \sum_{i=0}^{n-1} S_{(v,f_{i,j})}^* \right) \left( \sum_{u \in \Lambda^0} \sum_{m=0}^{n-1} S_{(u,f_{m,j})} \right) \\ &= \sum_{v \in \Lambda} \sum_{i=0}^{n-1} S_{(v,f_{i,j})}^* S_{(v,f_{i,j})} = \sum_{v \in \Lambda} \sum_{i=0}^{n-1} S_{v,w_i} \\ &= 1 = \left( \sum_{v \in \Lambda^0} \sum_{i=0}^{n-1} S_{(v,f_{i,j})} \right) \left( \sum_{u \in \Lambda^0} \sum_{m=0}^{n-1} S_{(u,f_{m,j})}^* \right) \\ &= U_j^* U_j. \end{aligned}$$

We conclude that  $\{U_j\}$  is a unitary representation of  $\mathbb{Z}^{k_2}$  contained in  $C^*(\Omega)$ .

To see that conjugation by  $U_j$  will induce the action of  $\rho_j$ , fix  $\lambda \in \Omega_1^{w_m} (\cong \Lambda)$ . By Lemma 2.9,

$$(\lambda, w_m)(s(\lambda), f_{m-1,j}) = (r(\lambda), f_{m-1,j})(\widehat{f_{m-1,j}}_{\triangleleft}(\lambda), w_{m-1})$$

since  $C_{n,k_2}$  is a stable quasi-factor. (CK2) now implies that

$$\begin{aligned} U_j S_{(\lambda, w_m)} U_j^* &= \left( \sum_{i=0}^{n-1} \sum_{v \in \Lambda^0} S_{(v,f_{i,j})}^* \right) S_{(\lambda, w_m)} \left( \sum_{\ell=0}^{n-1} \sum_{u \in \Lambda^0} S_{(u,f_{\ell,j})} \right) \\ &= S_{(r(\lambda), f_{m-1,j})}^* S_{(\lambda, w_m)} \left( \sum_{\ell=0}^{n-1} \sum_{u \in \Lambda^0} S_{(u,f_{\ell,j})} \right) \\ &= S_{(r(\lambda), f_{m-1,j})}^* S_{(\lambda, w_m)} S_{(s(\lambda), f_{m-1,j})} \\ &= S_{(r(\lambda), f_{m-1,j})}^* S_{(r(\lambda), f_{m-1,j})} S_{((\widehat{f_{m-1,j}}_{\triangleleft}(\lambda), w_{m-1}))} \\ &= S_{((\widehat{f_{m-1,j}}_{\triangleleft}(\lambda), w_{m-1}))} = \rho_{m-1,j}(S_{(\lambda, w_m)}). \end{aligned}$$

As each  $\rho_{i,j}$  is a  $*$ -homomorphism, and  $C^*(\Omega_1^{w_m})$  is generated by  $\{S_{(\lambda, w_m)}\}_\lambda$ , we conclude that the automorphism  $\rho_j$  of  $\bigoplus_{i=0}^{n-1} C^*(\Omega_1^{w_i})$  is indeed given by conjugation by  $U_j$ .

This demonstrates that  $\{U_j\}_{j=1}^{k_2}$  yields a unitary representation of  $\mathbb{Z}^{k_2}$  which induces  $\rho$  on  $C^*(\Omega_1) \cong \bigoplus_{i=0}^{n-1} C^*(\Omega_1^{w_i})$  via conjugation. The universal property of the crossed product now gives a  $*$ -homomorphism

$$\Phi : \left( \bigoplus_{i=0}^{n-1} C^*(\Omega_1^{w_i}) \right) \rtimes_{\rho} \mathbb{Z}^{k_2} \rightarrow C^*(\Omega)$$

with image  $\langle C^*(\Omega_1), U_j \rangle$ . It remains to show that  $\Phi$  is bijective.

We demonstrate surjectivity by observing first that each vertex projection  $S_{(v, w_i)}$  lies in  $C^*(\Omega_1)$  and therefore  $S_{(v, w_{i+1})} U_j^* = S_{(v, f_{i,j})} \in \text{Im } \Phi$  for all  $v, i, j$ . In particular, every generator of  $C^*(\Omega_2)$  lies in  $\text{Im } \Phi$ , making the image all of  $C^*(\Omega)$ .

To demonstrate injectivity, we construct a Cuntz–Krieger  $\Omega$ -family inside of  $(\bigoplus_i C^*(\Omega_1^{w_i})) \rtimes_{\rho} \mathbb{Z}^{k_2}$ , such that the associated  $*$ -homomorphism  $\Psi : C^*(\Omega) \rightarrow (\bigoplus_i C^*(\Omega_1^{w_i})) \rtimes_{\rho} \mathbb{Z}^{k_2}$  satisfies  $\Psi \Phi = \text{id}$ . To that end, write  $V_j$  for the canonical generators of  $\mathbb{Z}^{k_2}$  in the crossed product, and for each  $e \in G(\Lambda)^1$ ,  $f_{i,j} \in C_{n,k_2}^1$  and each vertex  $v \in \Lambda^0$ ,  $w_i \in C_{n,k_2}^0$ , define  $T_{(e, w_i)}, T_{(v, w_i)}, T_{(v, f_{i,j})} \in (\bigoplus_i C^*(\Omega_1^{w_i})) \rtimes_{\rho} \mathbb{Z}^{k_2}$