

We finally show that the set A_C is contained within a neighborhood of size $N^{-l}e^{-M}$ around a hyperplane in $\mathbb{F}_q((X^{-1}))^n$. For any $x \in C$, define $y = e^{(n+1)lM}(x - x_0)$ in $\mathbb{Z}_{\mathcal{O}}^n$, so that

$$g_t u_x = u_y g_t u_{x_0}.$$

Consider a linear functional $\phi_{t,C}$ of norm 1 that is zero on the hyperplane $H_{t,C} = g_t u_{x_0} H_C$. Then, we have

$$d([e_1], g_t u_x H_C) = d([e_1], u_y H_{t,C}) \asymp d([u_{-y} e_1], H_{t,C}) \asymp \phi_{t,C}(u_{-y} e_1).$$

Thus, if $x \in A_C$, we obtain $\phi_{t,C}(u_{-y} e_1) \lesssim e^{-M}$, implying that y lies within a neighborhood of size $O(e^{-\frac{M}{n}})$ of the affine hyperplane in $\mathbb{F}_q((X^{-1}))^n$ defined by

$$\phi_{t,C}(e_1 - y_1 e_2 - \cdots - y_n e_{n+1}) = 0.$$

This means that x lies in a neighborhood of size $O(N^{-l}e^{-M})$ around an affine hyperplane in $\mathbb{F}_q((X^{-1}))^n$. The number of subcubes of C that intersect this neighborhood is bounded by

$$\lesssim_n N^n e^{-M},$$

and thus we obtain

$$\text{card } K_l(C) \geq N^n(1 - O_n(e^{-M})) = N^n - O_n(N^{n-\frac{1}{n+1}}).$$

□

Now, we proceed to the proof of Theorem 1.1.

Proof of Theorem 1.1. To obtain a lower bound for the Hausdorff dimension of K_∞ , we apply the Mass Distribution Principle. This argument follows identically from the work of [BdS23], but we include the proof here for the sake of completeness.

For $l \geq 1$, define

$$b_l = \begin{cases} \lfloor N^n(1 - R_3 e^{-\frac{M}{n}}) \rfloor & \text{if } l_{k-1}^+ < l \leq l_k^- \\ 1 & \text{if } l_k^- < l \leq l_k^+. \end{cases}$$

We also replace K_∞ by a Cantor subset F_∞ that is more regular in nature. Removing some cubes in K_l at each step, we obtain the subset $F_\infty \subset K_\infty$ given as

$$F_\infty = \bigcap_{l \geq 1} F_l,$$

where each cube C in F_{l-1} contains exactly b_l subcubes in F_l . As stated in Proposition 1.7 of [Fal90], there is a probability measure μ that is supported on F_∞ , and for each cube C at level l (with $C \subset F_l$), we have the relationship

$$\mu(C) = \frac{1}{b_1 b_2 \dots b_l}.$$

We propose that if α is smaller than the limit inferior

$$\liminf_{l \rightarrow \infty} \frac{\log(b_1 b_2 \dots b_l)}{l \log N},$$

then there exists a constant $C = C_{n,N,\alpha}$ such that for every $x \in \mathbb{F}_q((X^{-1}))$ and for all radii $r > 0$,

$$\mu(B(x, r)) \leq C r^\alpha.$$

To justify this, choose l such that $N^{-l} < r \leq N^{-l+1}$. The ball $B(x, r)$ can intersect no more than $(3N)^n$ cubes from F_l . Hence, we have

$$\mu(B(x, r)) \leq (3N)^n \cdot (b_1 b_2 \dots b_l)^{-1} \leq (3N)^n N^{-l\alpha} \leq (3N)^n r^\alpha$$