

particular with semisimple isometry group G . The holonomy group at a point p coincides with the isotropy K of the factor Y (see [2, Proposition 10.79]). On the other hand, the fixed space of the K -action at p is exactly X^L . Therefore, the stabilizer of p preserves the factor X^L . The rest follows similarly as above. \square

3. INJECTIVITY OF THE DEVELOPING MAP

In this section we assume that $X = X^L \times Y$ is a simply connected Lorentz symmetric space whose Lorentz factor $X^L = X_\kappa$ is the universal model of constant curvature κ . A compact manifold M locally isometric to X is an $(\text{Isom}(X), X)$ -manifold in the sense of (G, X) -structures (see [15, 38]). That is, we have a local diffeomorphism $D : \widetilde{M} \rightarrow X$ and a representation $\rho : \pi_1(M) \rightarrow \text{Isom}(X)$ satisfying the equivariance property $D\gamma = \rho(\gamma)D$ for all $\gamma \in \pi_1(M)$. The local diffeomorphism D is called the *developing map* and ρ is called the *holonomy representation*.

3.1. Natural foliations. Proposition 2.5 implies that a compact manifold M locally isometric to $X = X^L \times Y$ inherits two transverse foliations \mathcal{F}_1 and \mathcal{F}_2 where the leaves of \mathcal{F}_1 are Lorentz of constant curvature locally modeled on X_κ and the leaves of \mathcal{F}_2 are Riemannian modeled on Y . The same applies to \widetilde{M} , we have two foliations $\widetilde{\mathcal{F}}_1$ and $\widetilde{\mathcal{F}}_2$.

Lemma 3.1. *Let M be a compact manifold locally isometric to X . Each leaf of $\widetilde{\mathcal{F}}_2$ is mapped, under the developing map, bijectively and isometrically onto a vertical Riemannian fiber $\{x_0\} \times Y$.*

Proof. The compactness of the manifold M implies that each leaf of the foliation \mathcal{F}_2 is complete (with respect to its Riemannian structure locally modeled on Y). Hence each leaf $\widetilde{\mathcal{F}}_2(p)$ of the foliation $\widetilde{\mathcal{F}}_2$ is also complete. The developing map D , restricted to $\widetilde{\mathcal{F}}_2(p)$, is a local isometry between $\widetilde{\mathcal{F}}_2(p)$ and a corresponding fiber $\{x_0\} \times Y$. Since $\widetilde{\mathcal{F}}_2(p)$ is complete, then $D : \widetilde{\mathcal{F}}_2(p) \rightarrow \{x_0\} \times Y$ is a covering map. But Y being simply connected implies that $D : \widetilde{\mathcal{F}}_2(p) \rightarrow \{x_0\} \times Y$ is in fact an isometric diffeomorphism. \square

Corollary 3.2. *The universal cover \widetilde{M} is mapped, under the the developing map, onto $\Omega \times Y$ where $\Omega \subset X_\kappa$ is an open subset.*

3.2. A natural action on \widetilde{M} . Since each leaf of $\widetilde{\mathcal{F}}_2$ is identified canonically under the developing map to $Y = G/K$, we obtain a well-defined G -action on \widetilde{M} for which the developing map is G -equivariant. Since the developing map is isometric, we have

Corollary 3.3. *The G -action on \widetilde{M} is isometric, proper, and with orbits the $\widetilde{\mathcal{F}}_2$ -leaves.*

3.3. Product structure of the universal cover. We have seen in Corollary 3.2 that $D(\widetilde{M}) = \Omega \times Y$ where $\Omega \subset X_\kappa$ is an open subset. The subset $\widehat{\Omega} = D^{-1}(\Omega \times \{y\})$ is a global cross section of the $\widetilde{\mathcal{F}}_2$ -foliation. Indeed, consider the map $\sigma : \widetilde{M} \rightarrow Y$ where $\sigma = \pi \circ D$ and $\pi : X_\kappa \times Y \rightarrow Y$ is the natural projection. We have $\sigma^{-1}(y) = \widehat{\Omega}$ and, by construction, σ is G -equivariant. Since each leaf $\widetilde{\mathcal{F}}_2(p)$ is identified under the map σ with Y , then $\widetilde{\mathcal{F}}_2(p)$ intersects $\sigma^{-1}(y) = \widehat{\Omega}$ exactly in one point. In other words, $\widehat{\Omega}$ is a leaf of $\widetilde{\mathcal{F}}_1$ which is a global cross section of the foliation $\widetilde{\mathcal{F}}_2$.

Corollary 3.4. *The universal cover \widetilde{M} is globally isometric to $\widehat{\Omega} \times Y$.*

In particular, $\widehat{\Omega}$ is a connected and simply connected Lorentz manifold of constant curvature κ . The injectivity of the developing map $D : \widetilde{M} \rightarrow X_\kappa \times Y$ is then equivalent to the fact that $\widehat{\Omega}$ isometrically embeds in X_κ .

3.4. The injectivity. We have seen in Corollary 3.4 that $\widetilde{M} = \widehat{\Omega} \times Y$ and $\pi_1(M) = \Gamma \subset \text{Isom}(\widehat{\Omega}) \times G$ acts freely, properly, and cocompactly on \widetilde{M} where G is the isometry group of Y . Frances in [10, Sect. 6] considers, among many other things, a similar situation (in fact a more general setting of warped products). He observed [10, Proposition 6.9] that the injectivity result in the works of Carrière and Klingler [7, 24] can be adapted to the product setting. That is,