

3. SKEW SHAPED POSITROID VARIETIES

3.1. Positroid varieties. First, we recall equivalent ways of describing positroid varieties in the Grassmannian: Grassmann necklaces and bounded affine permutations.

Definition 3.1.1. A (k, n) -*source Grassmann necklace* $\mathcal{I} = (I_1, I_2, \dots, I_n)$ is an n -tuple of k -element subsets $I_i \subseteq [n]$ where

- if $i \in I_i$ then there exists $j \in [n]$ such that $I_{i-1} = (I_i \setminus \{i\}) \cup \{j\}$, and
- if $i \notin I_i$ then $I_i = I_{i-1}$.

We denote by $\text{GN}(k, n)$ the set of (k, n) -Grassmann necklaces.

Remark 3.1.2. Note that if $i \in I_i$ then it may be that $I_{i-1} = I_i = (I_i \setminus \{i\}) \cup \{i\}$. Finally, note that $I_{i-1} \setminus I_i$ is either empty or a singleton.

Remark 3.1.3. Grassmann necklaces were first defined in [24]. Our definition is slightly different, as we use I_{i-1} instead of I_{i+1} , this difference is indicated by the word **source** in the name.

Definition 3.1.4 ([19]). A bijection $f : \mathbb{Z} \rightarrow \mathbb{Z}$ is called a (k, n) -**bounded affine permutation** if it satisfies the following conditions:

- (1) $f(i + n) = f(i) + n$ for every $i \in \mathbb{Z}$.
- (2) For every $i \in \mathbb{Z}$, $i \leq f(i) \leq f(i) + n$.
- (3) We have:

$$\sum_{i=1}^n (f(i) - i) = nk.$$

Note that, by (1), f is uniquely determined by the values $f(1), \dots, f(n)$, which are pairwise distinct modulo n . We often simply denote $f = [f(1), \dots, f(n)]$. Note also that, upon the presence of (1) and (2), (3) is equivalent to $|\{i \in [n] : f(i) > n\}| = k$. We denote the set of (k, n) -bounded affine permutations by $\text{BA}(k, n)$. We say that f is a **bounded n -affine permutation** if there exists k such that f is a (k, n) -bounded affine permutation.

Lemma 3.1.5. *The sets $\text{GN}(k, n)$ and $\text{BA}(k, n)$ are in natural bijection.*

In the proof, we will need the following definition.

Definition 3.1.6. We define the cyclic order \leq_i on the set $[n]$: $i <_i i + 1 <_i i + 2 <_i \dots <_i n <_i 1 <_i \dots <_i i - 1$.

Proof of Lemma 3.1.5. This is essentially a combination of [19, Corollary 3.13] and [23, Remark 2.4]. For the reader's convenience and to fix notation, we provide the bijection $\varphi : \text{GN}(k, n) \rightarrow \text{BA}(k, n)$. Let us first describe $\bar{f}_{\mathcal{I}} := \varphi(\mathcal{I}) \pmod{n}$ through its inverse $\bar{f}_{\mathcal{I}}^{-1}$: we have $\bar{f}_{\mathcal{I}}^{-1}(i) = i$ if $I_{i-1} = I_i$, and $\bar{f}_{\mathcal{I}}^{-1}(i) = j$ if $I_{i-1} \setminus I_i = \{j\}$. To lift this to a bounded affine permutation, we set $f_{\mathcal{I}}(i) = i + n$ if $I_{i-1} = I_i$ and $i \in I_i$, and $f_{\mathcal{I}}(i) = i$ if $i \notin I_i$. On the other hand, if $f : \mathbb{Z} \rightarrow \mathbb{Z}$ is a k -bounded affine permutation, let $I = \{i \in [n] \mid f(i) = i + n\}$, and $\bar{f} = f \pmod{n}$. Then I_i consists of I together with the elements $\{a \in [n] \mid a <_i \bar{f}(a)\}$. The collection $\mathcal{I}_f = (I_1, \dots, I_n)$ is a Grassmann necklace, and $f \mapsto \mathcal{I}_f$, $\mathcal{I} \mapsto f_{\mathcal{I}}$ are inverse bijections. \square

Now, we associate a Grassmann necklace and a bounded affine permutation to an element V in the Grassmannian $\text{Gr}(k, n)$ following [19, 24]. We represent V by a $k \times n$ matrix of rank k , up to row operations, and denote by $v_1, \dots, v_n \in \mathbb{C}^k$ the columns of V . Furthermore, we define v_i for all $i \in \mathbb{Z}$ by setting $v_{i+n} := (-1)^{k-1} v_i$. Given an (ordered) k -element subset