

admits a finite index subgroup with positive first Betti number. Since the quotient of the unit ball in \mathbf{C}^n by a discrete cocompact subgroup has nonzero Euler characteristic, the proposition 3.1 can be applied.

Remark 3.5. — Llosa Isenrich and Py showed in fact the existence of infinitely many, pairwise not commensurable, word hyperbolic groups admitting subgroups of type \mathcal{F}_n and not of type \mathcal{F}_{n+1} .

4. Construction from right-angled polytopes

Hereafter the article Italiano, Martelli, and Migliorini (2023) will be mentioned as **IMM5** and the article Italiano, Martelli, and Migliorini (2022) will be mentioned as **IMM8**.

The approach developed by **IMM5** for the proof of theorem 1.2 is more combinatorial in nature. It starts by constructing a finite volume hyperbolic 5-manifold from a right-angled polytope in the hyperbolic space \mathbb{H}_5 , then a fibration $f: M \rightarrow S^1$ is constructed using a natural cubulation of the manifold M . In order to produce a compact object (and hence a word hyperbolic group) one needs to cap the boundary components of M to obtained a metric space M^\vee ; this can be done maintaining the negatively curved metric on M^\vee and an extension of the fibration exists.

4.1. The polytope and the manifold

The chosen model for hyperbolic space \mathbb{H}_5 is the Klein model: the unit ball in \mathbf{R}^5 with geodesic given by Euclidean segments.

The polytope P_5 in \mathbb{H}_5 is described as the intersections of the half-spaces

$$\underline{\varepsilon} \cdot \underline{x} = \sum_{i=1}^5 \varepsilon_i x_i \leq 1, \quad \underline{x} \in \mathbb{H}_5$$

where $\underline{\varepsilon}$ varies in the subgroup of $\{\pm 1\}^5$ defined by $\prod \varepsilon_i = 1$, i.e. an even number of the ε_i are equal to -1 . We refer to **IMM5** (section 1.1) for a complete description, and to **IMM8** for further details on that polytope as well as a related series of right-angled polytopes in dimensions $3, \dots, 8$. These polytopes were previously studied by Potyagailo and Vinberg (2005) who explained them starting from certain hyperbolic simplices. They are related (by duality) to a series of semiregular polytopes discovered by Gosset (1899).

The polytope P_5 has finite volume, is right-angled, and has 16 facets given by the hyperplanes where equality is achieved in the equation above. It has a big group of symmetries: the permutation of coordinates as well as the coordinate-wise pluttification by $\underline{\varepsilon}$ (cf. Lindgren, 1945, for this classical operation); this produces a group of symmetries of type D_4 and of order $2^4 \times 5! = 1920$. The hyperbolic reflections through the 16 hyperplanes bounding P_5 generate a discrete subgroup Γ of $\text{Isom}(\mathbb{H}_5)$ that is known to be isomorphic to the congruence two subgroup of the group of integral matrices in