

with global minima at index i_0 there are exactly $\sum_{i=1}^{i_0-1} (k-1)k^{i-1}$ descents by Theorem 5.3. The number of inversions can be expressed through the values of the permutation before the minimum: $I(k, n, w) = \frac{k^n(k-1)}{4} \sum_{i=1}^n (k^{n-i} - k^{n-i-(c_w)_i}) = \frac{k^n(k-1)}{4} \sum_{i=1}^{i_0-1} (k^{n-i} - k^{n-i-w_i+1})$.

Now, we describe some bounds on the number of descents and inversion in the stable configuration depending on the permutation. We start with permutations with an increasing tail.

Proposition 6.1. *Given permutation w , if there exists an $i_0 \in \mathbb{N}$ such that for all indices $i \geq i_0$, we have $w_i < w_{i+1}$, then $\mathcal{C}_{k,n,w}$ has at most $\sum_{j=1}^{i_0-1} (k-1)k^{j-1}$ descents and at most $\binom{k}{2} \sum_{i=1}^{i_0-1} k^{i-1} \binom{k^{n-i}}{2}$ inversions.*

Proof. Because $w_i < w_{i+1}$ for all integers i such that $i \geq i_0$, we find that $(c_w)_i = 0$ for all $i \geq i_0$. Therefore, the support of c_w is a subset of $[i_0 - 1]$. Thus, by Theorem 5.3, we have that there are at most $\sum_{j=1}^{i_0-1} (k-1)k^{j-1}$ descents in $\mathcal{C}_{k,n,w}$.

Because $(c_w)_i = 0$ for all $i \geq i_0$, we have

$$I(k, n, w) = \binom{k}{2} \sum_{i=1}^{i_0-1} k^{i-1} \binom{k^{(c_w)_i}}{2} k^{2n-2i-2(c_w)_i} = \frac{k-1}{4} \sum_{i=1}^{i_0-1} (k^{2n-i} - k^{2n-i-(c_w)_i}).$$

Since for each i , we have $(c_w)_i \leq n - i$ by definition of Lehmer code, we obtain

$$I(k, n, w) \leq \frac{k^n(k-1)}{4} \sum_{i=1}^{i_0-1} (k^{n-i} - k^{n-i-(n-i)}) = \frac{k(k-1)}{4} \sum_{i=1}^{i_0-1} k^{n-1} (k^{n-i} - 1) = \binom{k}{2} \sum_{i=1}^{i_0-1} k^{i-1} \binom{k^{n-i}}{2}.$$

□

One can observe that the upper bounds on the number of inversions and descents in Proposition 6.1 are tight.

Example 24. Consider any positive integers i_0, n, k such that $i_0 < n$ and $k \geq 2$. Let w be a permutation in S_n defined by $w_i = n + 1 - i$ for $i \in [i_0 - 1]$ and $w_i = i - i_0 + 1$ for $i \in \{i_0, i_0 + 1, \dots, n\}$. This is a special case of a valley permutation, where the increasing part consists of smaller numbers than the decreasing part. We obtain that the Lehmer code of this permutation is $(c_w)_j = n - i$ for each $i \in [i_0 - 1]$ and $(c_w)_i = 0$ for $i \in \{i + 1, i + 2, \dots, n\}$. Therefore we obtain from Theorem 4.1 that the number of inversions in the stable configuration $\mathcal{C}_{k,n,w}$ resulting from firing strategy F_w is $\binom{k}{2} \sum_{i=1}^{i_0-1} k^{i-1} \binom{k^{n-i}}{2}$. This is exactly the upper bound on the number of inversions in $\mathcal{C}_{k,n,w}$ for w with increasing tail starting at i_0 .

Also observe that Theorem 5.3 and the fact that $\text{supp}(c_w) = [i_0 - 1]$ imply that $\mathcal{C}_{k,n,w}$ has exactly $\sum_{i=1}^{i_0-1} (k-1)k^{i-1}$ descents. This is equal to the upper bound on the number of descents in $\mathcal{C}_{k,n,w}$ from Proposition 6.1.

On a similar note, we calculate the lower bound for the number of inversions and descents in $\mathcal{C}_{k,n,w}$ in the case where w has a decreasing tail.