

single symbol  $\omega \in \Omega_\vartheta$ , we write  $\mathbb{E}[X] = \int_{\Omega_\vartheta} X(\omega) d\mathbb{P}(\omega)$ . Recall that a measure  $\mu$  on  $\mathbb{T}$  is called a stationary measure if

$$\int_{\Sigma_\vartheta} \mu \left( (T_\omega)^{-1}(A) \right) d\mathbb{P}(\omega) = \mu(A),$$

for any (Borel) measurable set  $A \subset \mathbb{T}$ .

The skew product map  $\Theta : \Sigma_\vartheta \times \mathbb{T} \rightarrow \Sigma_\vartheta \times \mathbb{T}$  is defined by

$$\Theta(\omega, x) := (\sigma\omega, T_{\omega_0}(x)).$$

Here  $\sigma$  is the left shift operator  $\sigma\omega := (\omega_{i+1})_{i \in \mathbb{N}}$ . With a slight abuse of notation we write

$$T_\omega^n(x) := T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_0}(x)$$

for iterates.

We compare two different trajectories by studying the random dynamical system. For  $\omega \in \Sigma_\vartheta$ , the two-point map  $(x, y) \mapsto T_\omega^{(2)}(x, y)$  on  $\mathbb{T}^2$  is the product

$$(x, y) \mapsto (T_\omega(x), T_\omega(y)).$$

This yields the random dynamical system

$$(x_{n+1}, y_{n+1}) = T_{\omega_n}^{(2)}(x_n, y_n). \quad (2.3)$$

The two-point skew product map  $\Theta^{(2)} : \Sigma_\vartheta \times \mathbb{T}^2 \rightarrow \Sigma_\vartheta \times \mathbb{T}^2$  is denoted by

$$\Theta^{(2)}(\omega, x, y) = (\sigma\omega, T_\omega^{(2)}(x, y)).$$

A measure  $\mu^{(2)}$  on  $\mathbb{T}^2$  is a stationary measure of the random dynamical system  $T_\omega^{(2)}$  on  $\mathbb{T}^2$  if

$$\int_{\Sigma_\vartheta} \mu^{(2)} \left( \left( T_\omega^{(2)} \right)^{-1}(A) \right) d\mathbb{P}(\omega) = \mu^{(2)}(A),$$

for any (Borel) measurable set  $A \subset \mathbb{T}^2$ .

**2.1. Hypotheses.** We focus on random circle endomorphisms whose trajectories are not confined to subintervals of the circle but spread over the entire circle.

There is  $k > 0$  so that for any  $x, y \in \mathbb{T}$ , there is  $\omega \in \Sigma_\vartheta$  so that  $T_\omega^k(x) = y$ . **(H2)**

This hypothesis guarantees the existence of a unique absolutely continuous stationary measure of full support, but also has further applications that are used throughout the paper.

**Proposition 2.1.** *Suppose the random dynamical system described by (2.2) with  $\omega_n$  i.i.d. picked from a uniform distribution for  $[-\vartheta, \vartheta]$ , adheres to Hypotheses (H1), (H2).*

*Then the random dynamical system admits an absolutely continuous stationary measure  $\mu$  with full support and smooth density.*