

Proof. These operators are limit point at both end points $r = 0$, and $r = \infty$. Indeed, any solution of $Lf = 0$ or $L_j f = 0$ is asymptotic to a linear combination of $r^{\frac{3}{2}}, r^{-\frac{1}{2}}$ as $r \rightarrow 0+$, so no boundary condition is needed at $r = 0$. The other endpoint $r = \infty$ is standard, see [53, Theorem X.8]. The claim about essential self-adjointness is [53, Theorem X.7]. See [30, Section 4] for the Bessel operator L_0 and its domain. The Kato-Rellich theorem applies to L, L_1, L_2 , which are relatively bounded perturbations of L_0 , see [17, Section 1.4] (in fact, these operators are perturbations of L_0 by bounded operators). The Weyl criterium, see [54, Theorem XIII.14], implies that $\text{spec}_{\text{ess}}(L_0) = \text{spec}_{\text{ess}}(L_1) = [0, \infty)$. If $\lambda \in \text{spec}(L_1)$ for some $\lambda < 0$, then there exists a ground state of negative energy, i.e., $L_1 \phi = \lambda_0 \phi$ for some $\lambda_0 < 0$ and $\phi \in \mathcal{D}$ with $\phi > 0$ (in fact, ϕ is smooth and $\phi(r) \sim cr^{\frac{3}{2}}$ as $r \rightarrow 0+$ for some constant c). Let $\chi(r) = 1$ for $0 \leq r \leq 1$, $\chi \in C^\infty([0, \infty))$ with compact support, and set $\chi_b(r) = \chi(r/b)$ with $b \geq 1$. Then

$$\langle L_1 \phi, \chi_b(r) r^{\frac{1}{2}} \rho_1 \rangle = \lambda_0 \langle \phi, \chi_b(r) r^{\frac{1}{2}} \rho_1 \rangle$$

Integrating by parts on the left-hand side and sending $b \rightarrow \infty$ now leads to a contradiction because of the vanishing $L_1(r^{\frac{1}{2}} \rho_1) = 0$. Hence $L_1 > 0$ as stated (note that L_1 cannot have a zero energy eigenfunction because the unique 0-energy solution is not in X_0). Pure a.c. spectrum is a consequence of the construction of the Weyl, Titchmarsh m -function for these operators, see [30, 40].

The essential spectrum of L_2 follows from the Weyl criterium as before. Since $L_2 = L_1 + 2\rho_1^2 > 0$, we conclude that the discrete spectrum of L_2 – if it exists – is strictly positive. \square

Remark 2.3. *The exact value of $c_0 > 0$ is not known, but the approximate value $c_0 \approx 1.3326$ is obtained in [51] via a numerically assisted argument. Moreover, [51] shows that L_2 has infinitely many eigenvalues in $(c_0, 2)$ and that 2 is a resonance.*

Next, we determine the spectrum of $i\mathcal{L}$ and define the evolution $e^{t\mathcal{L}}$ using the Hille-Yosida theorem. As we do not have a selfadjoint reference operator available as required for the standard version of Weyl's theorem as in [54, Theorem XIII.14], we need to proceed differently. To this end we first obtain a proper understanding of the resolvent $(i\mathcal{L}_0 - z)^{-1}$ of the *free* operator

$$\mathcal{L}_0 := \begin{bmatrix} 0 & -\partial_r^2 + \frac{3}{4r^2} \\ -(-\partial_r^2 + \frac{3}{4r^2} + 2) & 0 \end{bmatrix}.$$

Let $\tilde{p}_+(\zeta)$ denote the modified Hankel function

$$\tilde{p}_+(\zeta) := \sqrt{\zeta} H_1^{(1)}(\zeta) = \sqrt{\zeta} (J_1(\zeta) + iY_1(\zeta)),$$

and let $\tilde{q}_+(\zeta)$ denote the modified Bessel function

$$\tilde{q}_+(\zeta) := \sqrt{\zeta} J_1(\zeta).$$

Here $H_1^{(1)}$, J_1 , and Y_1 denote the order one Hankel function, Bessel function of the first kind, and Bessel function of the second kind respectively. \tilde{p}_+ and \tilde{q}_+ satisfy the ODE

$$-\frac{d^2}{d\zeta^2} \tilde{p}_+(\zeta) + \frac{3}{4\zeta^2} \tilde{p}_+(\zeta) = \tilde{p}_+(\zeta), \quad -\frac{d^2}{d\zeta^2} \tilde{q}_+(\zeta) + \frac{3}{4\zeta^2} \tilde{q}_+(\zeta) = \tilde{q}_+(\zeta).$$

Then by direct inspection, the vectors

$$\begin{aligned} \tilde{\Psi}_1(r, z) &:= \begin{bmatrix} \frac{ik_1(z)^2}{z} \tilde{p}_+(k_1(z)r) \\ \tilde{p}_+(k_1(z)r) \end{bmatrix}, & \tilde{\Psi}_2(r, z) &:= \begin{bmatrix} \frac{ik_2(z)^2}{z} \tilde{p}_+(k_2(z)r) \\ \tilde{p}_+(k_2(z)r) \end{bmatrix}, \\ \tilde{\Psi}_3(r, z) &:= \begin{bmatrix} \frac{ik_1(z)^2}{z} \tilde{q}_+(k_1(z)r) \\ \tilde{q}_+(k_1(z)r) \end{bmatrix}, & \tilde{\Psi}_4(r, z) &:= \begin{bmatrix} \frac{ik_2(z)^2}{z} \tilde{q}_+(k_2(z)r) \\ \tilde{q}_+(k_2(z)r) \end{bmatrix}, \end{aligned}$$