

Let  $U = (P, S, M)$ , we rewrite the system of equations (9) into the following form:

$$\begin{cases} \frac{\partial U}{\partial t}(t, x) + \mathcal{A}(U)(t, x) = G(U)(t, x), & (t, x) \in [0, T] \times \Omega_R \\ U(t, x) = U_R, & (t, x) \in [0, T] \times \partial\Omega_R \\ U(0, x) = U_0(x). & x \in \Omega_R \end{cases} \quad (10)$$

with

$$\begin{aligned} \mathcal{A}(U)(t, x) &= \left[ \operatorname{div} \left( \vec{\alpha}_p(t, x) \left( \int_{\Omega_R} \gamma_p(x-y) P(t, y) dy \right) P(t, x) \right), \operatorname{div} \left( \vec{\alpha}_s(t, x) \left( \int_{\Omega_R} \gamma_s(x-y) S(t, y) dy \right) S(t, x) \right), -D\Delta M \right], \\ G(U) &= \begin{pmatrix} g_1(U) \\ g_2(U) \\ g_3(U) \end{pmatrix} = \begin{pmatrix} mP(H(M) - a_1\lambda M) \\ -ma_2\lambda MS \\ m_s S(1-M) - \eta MP \end{pmatrix}, \quad U_R = (0, S_R, M_R) \quad \text{and} \quad U_0 = (P_0, S_0, M_0). \end{aligned}$$

## 2. Existence and uniqueness of solution

Our aim in this section is to prove the existence and uniqueness of solutions for the system outlined in equation (9). We begin by examining the local dynamics through the isolation of the convolution term and leveraging established principles from the theory of semilinear evolution equations. Following this, we integrate insights from the theory of nonlocal balance equations, as elaborated in [21] and [22], to affirm the existence and uniqueness of the solutions for the system mentioned in equation (2).

Before proceeding with the existence and uniqueness proofs, we need to define the following spaces.

Let  $L^\infty(\Omega_R)$  be the space of essentially bounded measurable functions on  $\Omega_R$ , equipped with the norm:

$$\|f\|_{L^\infty(\Omega_R)} = \operatorname{ess\,sup}_{x \in \Omega_R} |f(x)|.$$

Let  $C(\Omega_R)$  be the space of continuous functions on  $\Omega_R$ , with the uniform norm:

$$\|f\|_{C(\Omega_R)} = \sup_{x \in \Omega_R} |f(x)|.$$

Let  $C_b^1(\Omega_R)$  be the space of continuously differentiable functions on  $\Omega_R$  with bounded derivatives, normed by:

$$\|f\|_{C_b^1(\Omega_R)} = \|f\|_{C(\Omega_R)} + \|\nabla f\|_{C(\Omega_R)}.$$

For a Banach space  $(X, \|\cdot\|_X)$ , let  $C([0, T]; X)$  denote the space of continuous functions from  $[0, T]$  to  $X$ , with the norm:

$$\|f\|_{C([0, T]; X)} = \sup_{t \in [0, T]} \|f(t)\|_X.$$

Let  $L^1([0, T]; X)$  be the space of Bochner integrable functions from  $[0, T]$  to  $X$ , with the norm:

$$\|f\|_{L^1([0, T]; X)} = \int_0^T \|f(t)\|_X dt.$$

Let  $W^{2,1}(\Omega_R)$  be the Sobolev space defined as:

$$W^{2,1}(\Omega_R) = \{u \in L^1(\Omega_R) : D^\alpha u \in L^1(\Omega_R) \quad \text{for all} \quad |\alpha| \leq 2\}$$

### 2.1. Existence and uniqueness of the solution of the local system

Let  $w_p$  and  $w_s$  be two fixed functions belonging to  $C([0, T], C_b^1(\mathbb{R}^d))$ , and let  $w := (w_p, w_s)$ . We consider the associated local system to (10) and written as follows:

$$\begin{cases} \frac{\partial U_w}{\partial t}(t, x) + \mathcal{A}_w(U_w)(t, x) = G(U_w)(t, x), & (t, x) \in [0, T] \times \Omega_R \\ U_w(t, x) = U_R, & (t, x) \in [0, T] \times \partial\Omega_R \\ U_w(0, x) = U_0(x). & x \in \Omega_R \end{cases} \quad (11)$$