

# Periodic Inscription of Isosceles Trapezoids

Ali Naseri Sadr

## Abstract

We prove that a pair of continuous disjoint periodic curves in  $\mathbb{C}$  inscribes an isosceles trapezoid with any similarity type. The case of smooth curves can be identified with a Lagrangian intersection problem for a pair of Lagrangian cylinders in  $\mathbb{R} \times S^1 \times \mathbb{C}$ , and the continuous case follows from the smooth one by a standard convergence argument.

## 1 Introduction

Let  $\gamma_1, \gamma_2: \mathbb{R} \rightarrow \mathbb{C}$  be two continuous embeddings of the real line into  $\mathbb{C}$  that satisfy the periodicity condition

$$\gamma_i(t+1) = \gamma_i(t) + \sqrt{-1}$$

for every  $t$  and  $i = 1, 2$ . Furthermore, assume the images of  $\gamma_1$  and  $\gamma_2$  are disjoint. Tao conjectured in [10] that there exist four points in  $\gamma_1(\mathbb{R}) \cup \gamma_2(\mathbb{R})$  which are vertices of a square; this is a variation of the Toeplitz square peg problem for periodic curves, and Hugelmeyer proved it in [5].

For any given isosceles trapezoid  $Q$ , we show there are four points in  $\gamma_1(\mathbb{R}) \cup \gamma_2(\mathbb{R})$  that are vertices of a quadrilateral similar to  $Q$ . The approach of [5] does not directly generalize even to the case of rectangles. By contrast, in this article, we use a different approach to prove not only that every pair of periodic curves inscribes every similarity type of rectangles, but also every similarity type of isosceles trapezoids.

**Definition 1.1.** *Assume  $Q$  is an isosceles trapezoid. We say that the pair  $(\gamma_1, \gamma_2)$  admits a balanced inscription of  $Q$  if there exist  $p_1, p_2 \in \gamma_1(\mathbb{R})$  and  $p_3, p_4 \in \gamma_2(\mathbb{R})$  such that the quadrilateral formed by  $p_1, p_2, p_3, p_4$  is similar to  $Q$ , the line segments  $\overline{p_1 p_2}$  and  $\overline{p_3 p_4}$  are parallel, and  $|\overline{p_1 p_2}| \leq |\overline{p_3 p_4}|$ .*

Note that our definition depends on the order of the pair  $(\gamma_1, \gamma_2)$  unless  $Q$  is a rectangle.

**Theorem 1.2.** *Suppose  $\gamma_1$  and  $\gamma_2$  are two continuous disjoint periodic embeddings of the real line into the plane, and suppose  $Q$  is an isosceles trapezoid. Then  $(\gamma_1, \gamma_2)$  admits a balanced inscription of  $Q$ . Furthermore, there is a generic subset of smooth disjoint periodic pairs such that each pair in this set admits at least two balanced inscriptions of  $Q$  that are not related under translation by  $\sqrt{-1}$ .*

**Corollary 1.3.** *Let  $\theta \in (0, \frac{\pi}{2}]$ ; then every pair of continuous disjoint periodic curves in the plane inscribes a rectangle with angle  $\theta$  between its two diagonals.*

We conjecture that Theorem 1.2 is optimal, in the following sense.

**Conjecture 1.4.** *Let  $Q$  be a quadrilateral that admits an inscription in any pair of disjoint periodic curves in  $\mathbb{C}$ . Then  $Q$  is an isosceles trapezoid.*