

Remark 3.15. Another interesting situation is when we have two foliated manifolds (M, \mathcal{F}^M) and (N, \mathcal{F}^N) , and a local diffeomorphism — say, a covering map — $F : M \rightarrow N$ which sends leaves of \mathcal{F}^M into leaves of \mathcal{F}^N , and we want to move affine information between them. Note that F_* maps \mathcal{F}^M -vertical vectors into \mathcal{F}^N -vertical vectors, but not necessarily surjectively. It will become clear that it is necessary that only \mathcal{F}^M -vertical vectors are mapped into \mathcal{F}^N -vertical vectors. For example, consider $id : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, where the first \mathbb{R}^3 is foliated by $T\mathcal{F}^1 = \mathbb{R} \frac{\partial}{\partial y}$ and the second one is foliated by $T\mathcal{F}^2 = \mathbb{R} \frac{\partial}{\partial x} \oplus \mathbb{R} \frac{\partial}{\partial y}$. It is clear that id is a foliated diffeomorphism. The usual flat connection ∇^{flat} of \mathbb{R}^3 is a transverse affine connection to both foliations, albeit associated with different partner connections ω^1 and ω^2 (each of them is just ∇^{flat} restricted to the respective vertical distributions). However, $id^* \nabla^{\text{flat}} = \nabla^{\text{flat}}$ and $id^* \omega^2 = \omega^2$, which does not result in a transverse affine connection in the source foliation.

To achieve our goal, we must require furthermore that \mathcal{F}^M and \mathcal{F}^N are of the same dimension, so that vertical vectors on TM are in one-to-one correspondence with the vertical vectors in TN . Indeed, if (N, \mathcal{F}^N) is endowed with a transverse affine connection $\hat{\nabla}^N$, we can define the *pullback* $F^* \hat{\nabla}^N$ as follows: for $X, Y \in \mathfrak{X}(M)$, we set $(F^* \hat{\nabla}^N)_X Y = Z$, where Z is given by: For $p \in M$, let \mathcal{U} be any neighborhood of p for which $\Phi := F|_{\mathcal{U}}$ is a diffeomorphism onto its (open) image. Then, $Z(p) := d\Phi_p^{-1} \left(\hat{\nabla}_{\Phi_* X}^N \Phi_* Y \right)_{\Phi(p)}$. If ω^N is the partner connection associated with $\hat{\nabla}^N$, then we can check that, for any $V \in \mathfrak{X}(\mathcal{F}^M)$ and any $X \in \mathfrak{X}(M)$ we have

$$\begin{aligned} (F^* \hat{\nabla}^N)_X V &= (F^* \omega^N)_X V, \\ (F^* \hat{\nabla}^N)_V X &= [V, X] + (F^* \omega^N)_X V. \end{aligned}$$

As for the holonomy invariance condition, given $V \in \mathfrak{X}(\mathcal{F}^M)$, and $X, Y \in \mathfrak{X}(\mathcal{F}^M)$ we have

$$(\mathcal{L}_V(F^* \hat{\nabla}^N))(X, Y) = [V, (F^* \hat{\nabla}^N)_X Y] - (F^* \hat{\nabla}^N)_{[V, X]} Y - (F^* \hat{\nabla}^N)_X [V, Y].$$

Note that the two properties that we already established imply that the two last terms above are vertical. As for the first one, note that since F is locally a diffeomorphism, on a neighborhood of each point we can write $V = \Phi_*^{-1}(W)$ for some $W \in \mathfrak{X}(\mathcal{F}^N)$. Then we are left with

$$[V, (F^* \hat{\nabla}^N)_X Y] = \left[\Phi_*^{-1}(W), \Phi_*^{-1} \left(\hat{\nabla}_{\Phi_* X}^N \Phi_* Y \right) \right] = \Phi_*^{-1} \left[W, \hat{\nabla}_{\Phi_* X}^N \Phi_* Y \right],$$

which is vertical because $\hat{\nabla}^N$ is a transverse affine connection and Φ_* establishes a one-to-one correspondence between vertical vectors.

In the converse direction, if now (M, \mathcal{F}^M) is given a transverse affine connection $\hat{\nabla}^M$, the necessary and sufficient condition for the existence of the pushforward via F is that F is one-to-one, (that is, a full diffeomorphism). When this is the case, we can set the *pushforward* $(F_* \hat{\nabla}^M)_X Y = W$, for $X, Y \in \mathfrak{X}(N)$, where W is constructed as: given $q \in N$, let \mathcal{U} be any neighborhood of $p = F^{-1}(q)$ for which $\Phi := F|_{\mathcal{U}}$ is a diffeomorphism. Put $W(q) := d\Phi_p \left(\hat{\nabla}_{\Phi_* X}^M \Phi_* Y \right)_p$. As before, since vertical vectors in TM and in TN are in bijection, we obtain that $F_* \hat{\nabla}^M$ is a transverse affine connection with partner connection given by $F_* \omega^M$.

The next important consistency check arises by considering semi-Riemannian foliations. We show that any one such gives rise to a unique transverse affine structure — a result that works as an analogue of the fundamental theorem of semi-Riemannian geometry.

Theorem 3.16. *Let $g_{\mathcal{T}}$ be a transverse semi-Riemannian metric on (M, \mathcal{F}) . There exists a unique transverse affine structure $[\bar{\nabla}]$ on (M, \mathcal{F}) such that, for any $\hat{\nabla} \in [\bar{\nabla}]$*

- (i) *the torsion tensor $\text{Tor}(\hat{\nabla})$ takes values in $\Gamma(T\mathcal{F})$, and*
- (ii) *$X g_{\mathcal{T}}(Y, Z) = g_{\mathcal{T}}(\hat{\nabla}_X Y, Z) + g_{\mathcal{T}}(Y, \hat{\nabla}_X Z), \forall X, Y, Z \in \mathfrak{X}(M)$.*