

**Theorem 1.9.** *Given integers  $\Delta \geq 1$  and  $n \geq 2\Delta$ . There exist  $A, B \subseteq \mathbb{Z}$ ,  $|A| = |B| = n$  and a relation  $\mathcal{R} \subseteq A \times B$  with bounded degree  $\Delta$  from  $B$  such that*

$$|A +_{\mathcal{R}} B| = |A| + |B| - 1 - \left\lfloor \frac{5\Delta}{2} \right\rfloor.$$

We believe the above theorem is tight, which suggests Theorem 1.5(i) could potentially be strengthened to  $|A +_{\mathcal{R}} B| \geq |A| + |B| - 1 - \left\lfloor \frac{5\Delta}{2} \right\rfloor$ . It is worth noting that any improvement of the  $-3\Delta$  term in Theorem 1.5(i) would directly lead to a strengthening of Theorem 1.6(i), with the same parameters  $c_\varepsilon, p_0$  unchanged. The bottleneck of the current method for potential improvement of Theorem 1.6(i) lies in the case  $A, B \subseteq \mathbb{Z}$  after applying the rectifiability argument. The remaining part of the proof would remain valid without any modification.

**Conjecture 1.10.** *Suppose  $\Delta \geq 1$  is an integer,  $A, B \subseteq \mathbb{Z}$  satisfies  $|B| \leq |A|$ , and  $\mathcal{R} \subseteq A \times B$  is a binary relation between  $A$  and  $B$ . If the maximum degree of  $\mathcal{R}$  on  $B$  is at most  $\Delta$ , we have*

$$|A +_{\mathcal{R}} B| \geq |A| + |B| - 1 - \left\lfloor \frac{5\Delta}{2} \right\rfloor.$$

The second part of this paper presents two examples. The first explains why the additional requirement  $|B| = O_\varepsilon(p)$  is necessary for Corollary 1.8(i) to hold. The second demonstrates why a stronger assumption  $|A| + |B| \geq (1 + \delta)p$  is required for Conjecture 1.2 to hold in order to prove  $|A +_{\mathcal{R}} B| \geq p - 2$ , even in the case where  $|B| \leq \varepsilon p$ .

Theorem 1.11(i) restates an example originally given by Lev [Lev00b]. We reformulated it here for consistency with the notation and style used throughout this paper. Theorem 1.11(ii) was inspired from the same paper.

**Theorem 1.11.** *Suppose  $p$  is a prime number.*

(i) *For any integer  $k \geq 1$  and  $1 \leq \ell \leq \lfloor \frac{p-k-1}{2k-1} \rfloor$ , there exist subsets  $A, B \subseteq \mathbb{F}_p$  and a function  $R: B \rightarrow A$  such that  $|A| = p - (k-1)\ell - k + 1$ ,  $|B| = k\ell + 2$  and  $|A +_{\mathcal{R}} B| = p - k$ .*

(ii) *For any prime number  $p$ , there exists a subset  $A \subseteq \mathbb{F}_p$  and a symmetric relation  $\mathcal{R} \subseteq A \times A$  with maximum degree 1, such that  $|A| = 6\lfloor \frac{p}{11} \rfloor - 3$  and  $|A +_{\mathcal{R}} A| = p - 3$ .*

(iii) *For any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for any sufficiently large prime number  $p$ , there exist  $A, B \subseteq \mathbb{F}_p$  with  $|A| + |B| > (1 + \delta)p - O(1)$ ,  $|B| \leq \varepsilon p$ , and a relation  $\mathcal{R} \subseteq A \times B$  with maximum degree 1, such that  $|A +_{\mathcal{R}} B| = p - 3$ .*

Plugging  $\ell = \lfloor \frac{p-k-1}{2k-1} \rfloor$  and  $\ell = 1$  into Theorem 1.11(i), we have the following.

**Corollary 1.12.** *Suppose  $p$  is a prime number.*

(i) *For any integer  $k \geq 1$ , there exist subsets  $A, B \subseteq \mathbb{F}_p$  and a function  $R: B \rightarrow A$  such that  $|A| = p - (k-1)\lfloor \frac{p-k-1}{2k-1} \rfloor - k + 1$ ,  $|B| = k\lfloor \frac{p-k-1}{2k-1} \rfloor + 2$  and  $|A +_{\mathcal{R}} B| = p - k$ .*

(ii) *For any  $\varepsilon > 0$ , there exist  $A, B \subseteq \mathbb{F}_p$  and a function  $R: B \rightarrow A$  such that  $|A| = (1 - 2\varepsilon)p + O(1)$ ,  $|B| = \varepsilon p + O(1)$  and  $|A +_{\mathcal{R}} B| = |A| + |B| - 4$ , where the value of the  $O(1)$  term within the expressions of  $|A|$  and  $|B|$  are smaller than 3.*

Thus, for potential extensions of Corollary 1.8(i), in the regime where  $|A| + |B| \geq p$ , it is necessary to assume at least  $|A| + |B| \geq p - k + \lfloor \frac{p-k-1}{2k-1} \rfloor + 4 > \frac{2k}{2k-1}p - k + 2$  in order to establish  $|A +_{\mathcal{R}} B| \geq p - k + 1$ . In the regime  $|A| + |B| \leq (1 - \varepsilon)p$ , an additional assumption  $|B| \leq \varepsilon p$  is required.

Combining Corollary 1.8(i) and the examples above, we propose the following conjecture. The main obstacle in proving this conjecture lies in the fact that, in Theorem 1.6, the parameter  $c_\varepsilon$  is not linear in  $\varepsilon$ . So it remains unclear how to prove  $|A +_{\mathcal{R}} B| \geq |A| + |B| - 3$  under assumptions such as  $|A| + c|B| < p$  for any constant  $c$ .

**Conjecture 1.13.** *Suppose  $p$  is a prime number,  $A, B \subseteq \mathbb{F}_p$  with  $|B| \leq |A|$ . Let  $\mathcal{R}: B \rightarrow A$  be an arbitrary function from  $B$  to  $A$ . If  $|A| + 2|B| \leq p$ , then  $|A +_{\mathcal{R}} B| \geq |A| + |B| - 3$ .*