

and this last expression is equal to the expectation in (3.4), from the definition of the process $\Pi(\cdot)$ and the fact that τ is \mathcal{F}_τ -measurable.

Remark 3.2. The expectation in (3.4) is still well-defined for the infinite time horizon $T = \infty$. Indeed, on the event $\{\tau < \infty\}$ this is obvious, while on the event $\{\tau = \infty\}$ the expression is well-defined since the functions $c_i(\cdot)$, $i = 0, 1$ are defined on $[0, T]$, while the process $\Pi(\cdot)$ of (3.2) has a limit at infinity (by the Martingale Convergence Theorem).

The identity (3.4) and the fact that the processes $X(\cdot)$ and $\Pi(\cdot)$ generate the same filtrations enable us to reinterpret the optimal stopping problem (2.9) as maximizing the expression (3.4) over all stopping times τ of the filtration generated by the process $\Pi(\cdot)$. To solve this problem, we embed it into a general Markovian framework allowing for arbitrary initial data (t, π) .

3.1.1. General Markovian Framework

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, supporting a standard Brownian motion $B = \{B(t), 0 \leq t < \infty\}$. For each $\pi \in [0, 1]$, we consider a stochastic process $\Pi^\pi = \{\Pi^\pi(t), 0 \leq t < \infty\}$ with state space $[0, 1]$, which satisfies

$$\Pi^\pi(t) = \pi + \int_0^t a \Pi^\pi(s)(1 - \Pi^\pi(s)) dB(s), \quad 0 \leq t < \infty. \quad (3.5)$$

The processes $\Pi^\pi(\cdot)$, $\pi \in [0, 1]$ mimic the conditional process (3.2) with arbitrary starting positions π . For any $\pi \in [0, 1]$, the Lipschitz property of the coefficients on $[0, 1]$ guarantees the existence and uniqueness of the strong solution to the equation (3.5) (see, e.g., Chapter 5.2.B in Karatzas & Shreve [30]). Moreover, the processes $B(\cdot)$ and $\Pi^\pi(\cdot)$, $\pi \in (0, 1)$, generate the same filtration, as in the proof of Lemma 3.1. Hence, regardless of π , we let $\mathbb{F} := \{\mathcal{F}(t)\}_{t \geq 0}$ be the augmentation of the filtration generated by the processes $\Pi^\pi(\cdot)$ $\pi \in (0, 1)$, i.e., we set $\mathcal{F}(t) := \bar{\sigma}(B(s), 0 \leq s \leq t)$. In addition, we denote by \mathcal{T}_T the collection of all \mathbb{F} -stopping times τ such that $\mathbb{P}(\tau \leq T) = 1$.

For fixed $\pi \in [0, 1]$, time horizon T , and initial time t , we now consider the corresponding optimal stopping problem of maximizing the expected reward

$$J_T(t, \pi, \tau) := \mathbb{E} \left[\left(c_1(t + \tau) \Pi^\pi(\tau) - c_0(t + \tau) (1 - \Pi^\pi(\tau)) \right)^+ \right] \quad (3.6)$$

over all stopping times $\tau \in \mathcal{T}_{T-t}$, where the functions $c_i : [0, T] \rightarrow [0, 1]$, $i = 0, 1$, satisfy the assumptions (A1)–(A5). We denote the gain function of this problem by

$$G(t, \pi) := (c_1(t)\pi - c_0(t)(1 - \pi))^+, \quad (3.7)$$

and its value function by

$$V_T(t, \pi) := \sup_{\tau \in \mathcal{T}_{T-t}} J_T(t, \pi, \tau) = \sup_{\tau \in \mathcal{T}_{T-t}} \mathbb{E}[G(t + \tau, \Pi^\pi(\tau))]. \quad (3.8)$$

Clearly, the original optimal stopping problem (2.9) can be embedded into the one above by setting $\pi = p$, $t = 0$. Therefore, if we find the value function $V_T(\cdot, \cdot)$ of (3.8) and the corresponding optimal stopping time, we will automatically solve the original optimal stopping problem as well. For some future arguments, it will be convenient to treat the processes $\Pi^\pi(\cdot)$, $\pi \in [0, 1]$ of the new