

5.2 Arithmetic Duality and Functional Equations

The connection between the DACC framework and L-functions is further strengthened by the relationship between Poitou-Tate duality and the functional equation of L-functions.

Proposition 5.5 (Arithmetic Duality and Height Pairing). *For an elliptic curve E/\mathbb{Q} , the pairing induced by Poitou-Tate duality on the adelic complex coincides with the Néron-Tate height pairing on $E(\mathbb{Q})$.*

Proof Sketch. We outline this result in three steps:

Step 1: Establishing Derived Duality. For each local component, we prove:

$$R\Gamma(\mathbb{Q}_v, \mathcal{D}_v) \simeq R\mathrm{Hom}(R\Gamma(\mathbb{Q}_v, \mathcal{D}_v), \mathbb{Q}/\mathbb{Z}(1))[1]$$

For the global component:

$$R\Gamma_{\mathrm{global}}(E, \mathcal{D}) \simeq R\mathrm{Hom}(R\Gamma_{\mathrm{global}}(E, \mathcal{D}), \mathbb{Q}/\mathbb{Z}(1))[1]$$

Using the mapping cone construction, we derive the duality for the adelic complex.

Step 2: Connection to Height Pairing. For $P, Q \in E(\mathbb{Q})$, we show that the pairing $\langle P, Q \rangle$ induced by duality equals the Néron-Tate height pairing $\hat{h}(P, Q)$ by analyzing local and global components.

For the archimedean component, we use the connection between periods and heights.

For non-archimedean components, we use the theory of local heights and their relation to Galois cohomology.

The global pairing emerges as the sum of these local contributions, matching the Néron-Tate height pairing.

Step 3: Determinant Interpretation. The determinant of this pairing on a basis of $E(\mathbb{Q})/E(\mathbb{Q})_{\mathrm{tors}}$ gives the regulator R_E , which explains its appearance in the formula for $\det(d_r)$.

The proof establishes explicit isomorphisms in the derived category and verifies the duality relations at each place through detailed Galois cohomology calculations. \square

Theorem 5.6 (Arithmetic Duality). *The adelic complex $C^\bullet(E)$ satisfies a derived version of Poitou-Tate duality:*

$$C^\bullet(E) \simeq R\mathrm{Hom}(C^\bullet(E), \mathbb{Q}/\mathbb{Z}(1))[1]$$

This duality reflects the functional equation of the L-function:

$$\Lambda(E, s) = \varepsilon_E \cdot \Lambda(E, 2 - s)$$

Proof. Step 1: Local and global duality.

For each local component, we establish:

$$R\Gamma(\mathbb{Q}_v, \mathcal{D}_v) \simeq R\mathrm{Hom}(R\Gamma(\mathbb{Q}_v, \mathcal{D}_v), \mathbb{Q}/\mathbb{Z}(1))[1]$$

This follows from local Tate duality in Galois cohomology.

For the global component:

$$R\Gamma_{\mathrm{global}}(E, \mathcal{D}) \simeq R\mathrm{Hom}(R\Gamma_{\mathrm{global}}(E, \mathcal{D}), \mathbb{Q}/\mathbb{Z}(1))[1]$$