

But the emptiness of the regions  $\mathcal{T}(3, 2, k_3)$  for  $k_3 \geq 3$  follows from lemma 3(3c). Therefore,  $\mathcal{A}_3^\circ(3, 1, 2)$  consists of the triples (23).

Further, if  $(k_1, k_2, k_3) \in \mathcal{A}_3^*(3, 1, 2)$ , then we necessarily have  $(k_1, k_2) \in \mathcal{A}_2^*(3, 1, 2)$ . As it was stated before, we either have  $k_1 \equiv 0 \pmod{3}, k_1 \geq 3, k_2 = 1$ , or  $k_1 = 3, k_2 = 2$ .

For  $k_2 = 1$ , the condition (19) corresponding to  $i = r = 3$ , reduces to

$$k_1(k_3 - 1) \not\equiv 0 \pmod{3}.$$

Since  $k_1 \equiv 0 \pmod{3}$  in the first case, this condition can not be satisfied for any  $k_3$ . In the second case, the same condition takes the form

$$7k_3 \not\equiv 0 \pmod{3}.$$

For  $k_3 \geq 3$ , the emptiness of  $\mathcal{T}(3, 2, k_3)$  follows from lemma 3 (3c). Therefore, non-empty regions correspond to the values  $k_3 = 1, k_3 = 2$  only. Hence, the set  $\mathcal{A}_3^*(3, 1, 2)$  consists of the triples

$$(3, 2, 1), \quad (3, 2, 2). \quad (24)$$

It turns out that for any  $r \geq 4$  the set  $\mathcal{A}_r^*(3, 1, 2)$  consists of two sets:  $(3, 2^{r-2}, 1)$  and  $(3, 2^{r-1})$ . Let us prove it by induction with respect to  $r$ .

For  $r = 3$ , it was proved earlier. Suppose now that this assertion holds for any  $r$ ,  $3 \leq r \leq m - 1$ , and check it for  $r = m$ .

Indeed, let  $(k_1, \dots, k_m) \in \mathcal{A}_m^*(3, 1, 2)$ . Then  $(k_1, \dots, k_{m-1}) \in \mathcal{A}_{m-1}^*(3, 1, 2)$ , so, by the induction,  $(k_1, \dots, k_{m-1})$  is either  $(3, 2^{m-3}, 1)$  or  $(3, 2^{m-2})$ . Hence,  $(k_1, \dots, k_m)$  has the form  $(3, 2^{m-3}, 1, k)$  or  $(3, 2^{m-2}, k)$ , where  $k \geq 1$  is some integer.

In the first case, the condition (19) corresponding to  $i = m$ , has the form

$$\mathbb{K}_m(3, 2^{m-3}, 1, k) + \mathbb{K}_{m-1}(2^{m-3}, 1, k) \not\equiv 2 \pmod{3}. \quad (25)$$

By explicit formulas for the continuants, we get

$$\mathbb{K}_m(3, 2^{m-3}, 1, k) = 2(k - m) + 3, \quad \mathbb{K}_{m-1}(2^{m-3}, 1, k) = k - m + 2,$$

so (25) has the form  $3(k - m) + 5 \not\equiv 2 \pmod{3}$ . Therefore, it is not satisfied for any  $k$ .

In the second case, we have the condition

$$\mathbb{K}_m(3, 2^{m-2}, k) + \mathbb{K}_{m-1}(2^{m-2}, k) \not\equiv 2 \pmod{3}.$$

Using the formulas

$$\mathbb{K}_m(3, 2^{m-2}, k) = 2km - k - 2m + 3, \quad \mathbb{K}_m(2^{m-2}, k) = km - k - m + 2,$$

we rewrite it as follows

$$3m(k - 1) - 2k + 5 \not\equiv 2 \pmod{3}, \quad \text{that is, } k \not\equiv 0 \pmod{3}.$$

Now let us find all the tuples that contain the set  $\mathcal{A}_r^\circ(3, 1, 2)$  for  $r \geq 4$ .