

Remark 4.2. For any $\epsilon > 0$, a computation yields

$$\tilde{s}_{C,\epsilon}(p, q, r) = \begin{cases} -\frac{2}{q} + \epsilon, & \text{for } q \geq 3p' \text{ and } \frac{1}{p} \leq \frac{1}{4}; \\ \frac{2}{p} - \frac{2}{q} - \frac{1}{2} + \epsilon, & \text{for } q \geq 3p' \text{ and } \frac{1}{4} < \frac{1}{p} < \frac{1}{2} - \frac{1}{q}; \\ \frac{1}{p} - \frac{3}{q} + \epsilon, & \text{for } q \geq 3p' \text{ and } \frac{1}{p} \geq \frac{1}{2} - \frac{1}{q}; \\ \frac{3}{2p} - \frac{3}{2q} - \frac{1}{2} + \epsilon, & \text{for } p' < q < 3p'; \\ \frac{2}{p} - \frac{1}{q} - 1 + \epsilon, & \text{for } q \leq p'. \end{cases}$$

If $\tilde{s}_{C,0}(p, q, r) < 0$, then (4.8) holds with a decay in k , where $(\frac{1}{p}, \frac{1}{q})$ belongs to the region with vertices $(0, 0)$, $(\frac{1}{4}, 0)$, $(\frac{3}{8}, \frac{1}{8})$, $(\frac{1}{2}, \frac{1}{6})$, $(\frac{2}{3}, \frac{1}{3})$, and $(1, 1)$. Furthermore, the point $(\frac{3}{8}, \frac{1}{8})$ can be removed by interpolation between $(\frac{1}{4}, 0)$ and $(\frac{1}{2}, \frac{1}{6})$. Thus, (4.8) holds with some $s < 0$ if $(\frac{1}{p}, \frac{1}{q})$ satisfies

$$\begin{cases} \frac{1}{q} > \frac{2}{3p} - \frac{1}{6}, & \text{for } q \geq 3p'; \\ \frac{1}{q} > \frac{1}{p} - \frac{1}{3}, & \text{for } p' < q < 3p'; \\ \frac{1}{q} > \frac{2}{p} - 1, & \text{for } q \leq p'. \end{cases}$$

Combining $1 \leq r < p \leq q \leq \infty$ with the necessary condition $1 + (1 + \omega)\left(\frac{1}{q} - \frac{1}{p}\right) > 0$, the region of $(\frac{1}{p}, \frac{1}{q})$ in Theorem 1.4 for this case is illustrated in Figure 5. Here, we assume $1 \leq r \leq \frac{3}{2}$, and the figures for other ranges of r can be constructed similarly.

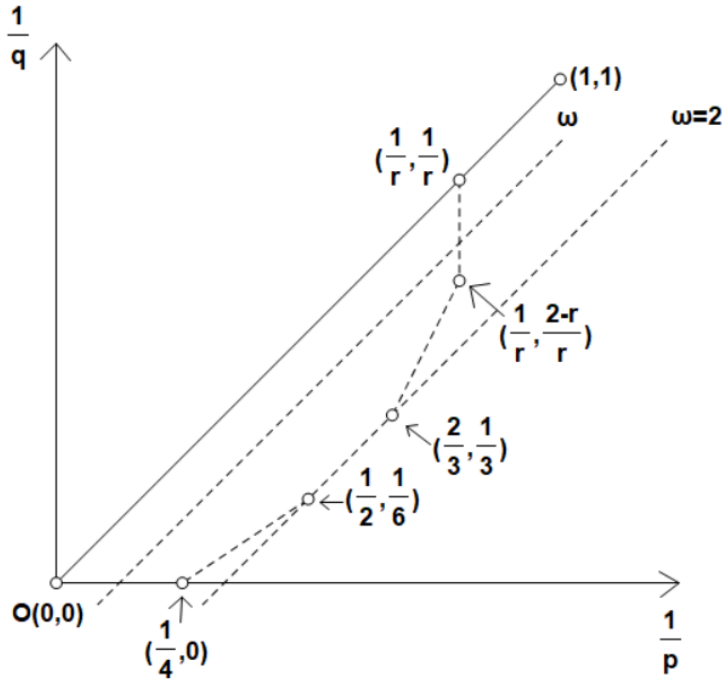


Figure 5: The region of boundedness in (1.4) for the case $1 \leq r < p \leq q \leq \infty$ and $1 \leq r \leq \frac{3}{2}$.