

which implies

$$\sum_{k=0}^N \alpha_k \left( |\Theta(x^k)| + \frac{\|t^k\|^2}{2} \right) \leq \frac{1}{\beta} (\Phi_j(x^0) - \hat{y}_j), \quad \because \theta(x^k) < 0.$$

Since the right-hand side of the above inequality is finite and inequality holds for any positive integer  $N$ , then we get

$$\sum_{k=0}^{\infty} \alpha_k \left( |\Theta(x^k)| + \frac{\|t^k\|^2}{2} \right) < \infty.$$

The above inequality implies the result of this lemma.  $\square$

**Theorem 4.1.** *Suppose that  $\Phi$  is convex in component-wise sense (i.e.,  $\Phi$  is  $\mathbb{R}^m$ -convex) and the Assumption 1 holds. Then any sequence produced by Algorithm 3.1 converges to a WPOS  $x^* \in \mathbb{R}^n$ .*

*Proof.* Since by Algorithm 3.1,  $\{\Phi(x^k)\}$  is a component-wise decreasing sequence, then by assumption, there exists  $\tilde{x} \in \mathbb{R}^n$  such that

$$\Phi(\tilde{x}) \leq \Phi(x^k) \text{ for all } k = 0, 1, 2, \dots \quad (4.18)$$

It is observed that  $0 < \alpha_k \leq 1$  for all  $k$ , so

$$\begin{aligned} \|x^{k+1} - x^k\|^2 &\leq \frac{1}{\alpha_k} \|x^{k+1} - x^k\|^2 \text{ for all } k = 0, 1, 2, \dots \\ &\leq \frac{1}{\alpha_k} \|\alpha_k t^k\|^2 = \alpha_k \|t^k\|^2 \text{ for all } k = 0, 1, 2, \dots \quad (\because x^{k+1} = x^k + \alpha_k t^k). \end{aligned}$$

Therefore, by above inequality, (4.18), and Lemma (4.5) we obtained

$$\sum_{k=0}^{\infty} \|x^{k+1} - x^k\|^2 \leq \sum_{k=0}^{\infty} \alpha_k \|t^k\|^2 < \infty.$$

Thus,

$$\sum_{k=0}^{\infty} \|x^{k+1} - x^k\|^2 < \infty. \quad (4.19)$$

Let us define  $\tilde{L} = \{x \in \mathbb{R}^n : \Phi(x) \leq \Phi(x^k), k = 0, 1, 2, \dots\}$ . By the component-wise convexity of  $\Phi$  and Lemma 4.3, for any  $x \in \tilde{L}$  we have

$$\|x - x^{k+1}\|^2 \leq \|x - x^k\|^2 + \|x^k - x^{k+1}\|^2 \text{ for all } k = 0, 1, 2, \dots$$

As  $\tilde{L}$  is non empty because  $\tilde{x} \in \tilde{L}$ , by (4.19) and the above inequality, it follows that  $\{x^k\}$  is quasi-Fejer convergent to the set  $\tilde{L}$ . Then by Theorem 2.4,  $\{x^k\}$  is bounded and hence  $\{x^k\}$  has an accumulation point. Let  $x^*$  be one of them. Then by Lemma 4.4,  $x^* \in \tilde{L}$ . Then by Theorem 2.4, we observe that  $\{x^k\}$  converges to  $x^*$ . Therefore, by Theorem 3.3,  $x^*$  is a critical point, and hence  $\mathbb{R}^m$ -convexity implies that  $x^*$  is a weak Pareto optimal solution for  $\Phi$ .  $\square$