

Therefore σ conjugates φ and Q . In particular it follows that $\text{Fix}(Q) \subset \mathbb{R}^m$ is sub vector space of dimension r . From (6) we know that Q is an involution. Thus from statements (i) and (ii) we have that there is a linear change of variables such that in these new variables Q is given by,

$$Q(x_1, \dots, x_m) = (x_1, \dots, x_r, -x_{r+1}, \dots, -x_m).$$

The result now follows from the fact that φ is C^k -conjugated to Q by σ . \square

APPENDIX B. POINCARÉ COMPACTIFICATION

Let X be a planar *polynomial* vector field of degree n as our polynomial differential systems of Theorem 1. The *Poincaré compactified vector field* $p(X)$ is an analytic vector field on \mathbb{S}^2 constructed as follow (for more details see [17, Chapter 5]).

First we identify \mathbb{R}^2 with the plane $(x_1, x_2, 1)$ in \mathbb{R}^3 and define the *Poincaré sphere* as $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$. We define the *northern hemisphere*, the *southern hemisphere* and the *equator* respectively by $H_+ = \{y \in \mathbb{S}^2 : y_3 > 0\}$, $H_- = \{y \in \mathbb{S}^2 : y_3 < 0\}$ and $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$.

Consider now the projections $f_{\pm} : \mathbb{R}^2 \rightarrow H_{\pm}$ given by

$$f_{\pm}(x_1, x_2) = \pm \Delta(x_1, x_2)(x_1, x_2, 1),$$

where $\Delta(x_1, x_2) = (x_1^2 + x_2^2 + 1)^{-\frac{1}{2}}$. These two maps define two copies of X , one copy X^+ in H_+ and one copy X^- in H_- . Consider the vector field $X' = X^+ \cup X^-$ defined in $\mathbb{S}^2 \setminus \mathbb{S}^1$. Note that the *infinity* of \mathbb{R}^2 is identified with the equator \mathbb{S}^1 . The Poincaré compactified vector field $p(X)$ is the analytic extension of X' from $\mathbb{S}^2 \setminus \mathbb{S}^1$ to \mathbb{S}^2 given by $y_3^{n-1} X'$. The *Poincaré disk* \mathbb{D} is the projection of the closed northern hemisphere to $y_3 = 0$ under $(y_1, y_2, y_3) \mapsto (y_1, y_2)$ (the vector field given by this projection is also denoted by $p(X)$). Note that to know the behavior $p(X)$ near \mathbb{S}^1 is the same than to know the behavior of X near the infinity. We define the local charts of \mathbb{S}^2 by $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$ and $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$ for $i \in \{1, 2, 3\}$. In these charts we define $\phi_i : U_i \rightarrow \mathbb{R}^2$ and $\psi_i : V_i \rightarrow \mathbb{R}^2$ by

$$\phi_i(y_1, y_2, y_3) = -\psi_i(y_1, y_2, y_3) = \left(\frac{y_m}{y_i}, \frac{y_n}{y_i} \right),$$

where $m \neq i$, $n \neq i$ and $m < n$. Denoting by (u, v) the image of ϕ_i and ψ_i in every chart (therefore (u, v) play different roles in each chart) one can see the following expressions for $p(X)$:

$$\begin{aligned} & v^n m(u, v) \left(Q \left(\frac{1}{v}, \frac{u}{v} \right) - uP \left(\frac{1}{v}, \frac{u}{v} \right), -vP \left(\frac{1}{v}, \frac{u}{v} \right) \right) \text{ in } U_1, \\ & v^n m(u, v) \left(P \left(\frac{u}{v}, \frac{1}{v} \right) - uQ \left(\frac{u}{v}, \frac{1}{v} \right), -vQ \left(\frac{u}{v}, \frac{1}{v} \right) \right) \text{ in } U_2, \\ & m(u, v)(P(u, v), Q(u, v)) \text{ in } U_3, \end{aligned}$$

where $m(u, v) = (u^2 + v^2 + 1)^{-\frac{1}{2}(n-1)}$. We can omit the term $m(u, v)$ by a time rescaling of $p(X)$. Therefore, we obtain a polynomial expression of $p(X)$ in each U_i . The expressions of $p(X)$ in each V_i is the same as that for each U_i , except by a multiplicative factor of $(-1)^{n-1}$. In these coordinates for $i \in \{1, 2\}$, $v = 0$ always represents the points of \mathbb{S}^1 and thus the infinity of \mathbb{R}^2 . Note that \mathbb{S}^1 is invariant under the flow of $p(X)$.