

Here B_n denotes the ball of radius n centred at the origin. The following property is readily deduced from parabolic estimates:

$$\lim_{n \rightarrow +\infty} u_n(t, x) = \bar{p}(x) \quad \text{locally uniformly in } (t, x) \in [0, +\infty) \times \mathbb{R}^n. \quad (4.10)$$

We claim that the function u_n fulfils (4.9) for n sufficiently large, depending on ε , hence by the previous step it satisfies (4.8). Assume by contradiction that this is not the case. Then, for any $n \in \mathbb{N}$, it holds that

$$t_n := \inf\{t \geq 0 : \exists x \in t\widetilde{W}, u_n(t, x) \leq \underline{u}(x)\} < +\infty.$$

We know from (4.10) that $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$. In particular, $t_n > 0$ for n sufficiently large and it follows from the definition of t_n that

$$\forall t \in [0, t_n), \forall x \in t\widetilde{W}, \quad u_n(t, x) > \underline{u}(x), \quad (4.11)$$

and, moreover, being $t_n\widetilde{W}$ compact, that there exists $x_n \in t_n\widetilde{W}$ such that

$$u_n(t_n, x_n) = \underline{u}(x_n).$$

Recall that \underline{u} is a strict subsolution, hence the parabolic strong maximum principle necessarily implies that $x_n \in \partial(t_n\widetilde{W})$, that is, $x_n/t_n \in \partial\widetilde{W}$.

Consider now $h_n \in \mathbb{Z}^N$ such that $\xi_n := x_n - h_n \in [0, 1)^N$. We define

$$\tilde{u}_n(t, x) := u_n(t_n + t, h_n + x).$$

Up to extraction of a subsequence, the following limits exist:

$$\xi_n \rightarrow \xi_\infty \in [0, 1]^N, \quad x_n/t_n \rightarrow \zeta \in \partial\widetilde{W}.$$

Also, always up to subsequences, by standard parabolic estimates and spatial periodicity of the equation, the functions \tilde{u}_n converge to \tilde{u}_∞ , an entire solution of (1.1) which fulfils by construction (and by periodicity of \underline{u})

$$\tilde{u}_\infty(0, \xi_\infty) = \underline{u}(\xi_\infty). \quad (4.12)$$

Moreover, (4.11) rewrites for the \tilde{u}_n as

$$\forall t \in [-t_n, 0), \forall x \in (t_n + t)\widetilde{W} - \{x_n\}, \quad \tilde{u}_n(t, x + \xi_n) > \underline{u}(x + \xi_n). \quad (4.13)$$

We assert that this entails

$$\forall t \leq 0, \forall x \cdot \nu \leq \left(1 - \frac{\varepsilon}{3}\right) c_1(\nu) t, \quad \tilde{u}_\infty(t, x + \xi_\infty) \geq \underline{u}(x + \xi_\infty), \quad (4.14)$$

where ν is the outward unit normal vector to \widetilde{W} at the point ζ and, we recall, $c_1(\nu)$ is the speed of the uppermost front of the terrace \mathcal{T}^ν in the direction ν .

The first crucial observation to derive (4.14) is that the t -dependent sets $(t_n + t)\widetilde{W}$ expand at a given boundary point $(t_n + t)\tilde{w}(e)e$ with the (positive) constant normal speed $\tilde{w}(e)e \cdot \tilde{\nu}$, where $\tilde{\nu}$ is the outward normal at that point, hence $e \cdot \tilde{\nu} > 0$. The second observation is that $0 \in \partial((t_n + t)\widetilde{W} - \{x_n\})$, for any $n \in \mathbb{N}$, and that the normal at that point converges to ν . The last one is that, because \widetilde{W} is compact and smooth, it satisfies uniform interior and exterior sphere conditions of some radius $\rho > 0$ on the boundary, whence its dilation $(t_n + t)\widetilde{W}$ fulfils these conditions with radius $(t_n + t)\rho$, which for any t tends to $+\infty$ as $n \rightarrow +\infty$. This means that $(t_n + t)\widetilde{W}$ “flattens” to a half-space around each of its boundary points as $n \rightarrow +\infty$. These geometric observations are made rigorous in