

By compactness we can now cover the interval $[a, b]$ with a finite number of such neighborhoods $U_{\lambda_1}, \dots, U_{\lambda_n} \subseteq \mathbb{R}$. If

$$(3.1) \quad V := N(T(\lambda_1)^*) + N(T(\lambda_2)^*) + \dots + N(T(\lambda_n)^*) \subset L^\infty(\mathbb{R}),$$

then $\dim V < \infty$ and $R(T(\lambda)) + V = L^\infty(\mathbb{R})$ for all $\lambda \in [a, b]$.

(IV) This step is inspired by Step 3 in the proof of [41, Thm. 5.3]. Keeping $\tau > 0$ fixed, consider the family of operators

$$S(\lambda) : D(S(\lambda)) \rightarrow L^\infty[-\tau, \tau], \quad [S(\lambda)y](t) := \dot{y}(t) - A(t, \lambda)y(t)$$

for a.a. $t \in [-\tau, \tau]$, which due to Lemma 2.1(a) is well-defined on the domain

$$D(S(\lambda)) := \{u \in W^{1,\infty}[-\tau, \tau] \mid u(-\tau) \in N(P_\lambda^-(-\tau)), u(\tau) \in R(P_\lambda^+(\tau))\}.$$

(IV.1) Claim: $\dim N(S(\lambda)) = \dim N(T(\lambda)) < \infty$.

Consider the commutative diagram

$$(3.2) \quad \begin{array}{ccc} W^{1,\infty}(\mathbb{R}) & \xrightarrow{T(\lambda)} & L^\infty(\mathbb{R}) \\ \uparrow i_\lambda & & \downarrow p \\ D(S(\lambda)) & \xrightarrow{S(\lambda)} & L^\infty[-\tau, \tau], \end{array}$$

where p is the restriction of functions in $L^\infty(\mathbb{R})$ to $L^\infty[-\tau, \tau]$ given by $p(u) := u|_{[-\tau, \tau]}$ and a canonical map $i_\lambda : D(S(\lambda)) \rightarrow W^{1,\infty}(\mathbb{R})$ defined by extending a given function $u \in D(S(\lambda))$ to the intervals $(-\infty, -\tau)$ and (τ, ∞) as solution of (V_λ) . Observe that i_λ is injective and $i_\lambda(N(S(\lambda))) = N(T(\lambda))$ holds, where $i_\lambda(N(S(\lambda))) \subseteq N(T(\lambda))$ results directly due to the construction of i_λ , while $i_\lambda(N(S(\lambda))) \supseteq N(T(\lambda))$ as converse inclusion follows from the fact that provided $u \in N(T(\lambda))$, then $u(\tau) \in R(P_\lambda^+(\tau))$ and $u(-\tau) \in N(P_\lambda^-(-\tau))$ for any $\tau > 0$ (recall (2.3)). Finally, this yields the assertion.

(IV.2) We decompose $[-\tau, \tau] = [-\tau, 0] \cup [0, \tau]$ and define the spaces

$$\begin{aligned} X_+ &:= \{u \in W^{1,\infty}[0, \tau] \mid u(\tau) \in R(P_\lambda^+(\tau))\}, & Y_+ &:= L^\infty[0, \tau], \\ X_- &:= \{u \in W^{1,\infty}[-\tau, 0] \mid u(-\tau) \in N(P_\lambda^-(-\tau))\}, & Y_- &:= L^\infty[-\tau, 0]. \end{aligned}$$

Consider the following linear operators $S^\pm(\lambda) : X_\pm \rightarrow Y_\pm$ pointwise defined as

$$[S^\pm(\lambda)y](t) = \dot{y}(t) - D_2 f(t, \phi_\lambda(t), \lambda)y(t) \quad \text{for a.e. } t \in \mathbb{R}_\pm.$$

Next consider the following commutative diagram

$$(3.3) \quad \begin{array}{ccc} X_- \oplus X_+ & \xrightarrow{S^-(\lambda) \oplus S^+(\lambda)} & Y_- \oplus Y_+ \\ \uparrow J_\lambda & & \uparrow J \\ D(S(\lambda)) & \xrightarrow{S(\lambda)} & L^\infty[-\tau, \tau], \end{array}$$

where $J : L^\infty[-\tau, \tau] \rightarrow Y_- \oplus Y_+$ and $J_\lambda : D(S(\lambda)) \rightarrow X_- \oplus X_+$ are defined by

$$Ju := (u_-, u_+) \text{ and } J_\lambda v := (v_-, v_+)$$

with u_+ and v_+ (resp. u_- and v_-) being the corresponding restrictions to $[0, \tau]$ (resp. to the interval $[-\tau, 0]$). It is clear that J is an isomorphism, while J_λ is injective with range $R(J_\lambda) = \{(v_-, v_+) \mid v_-(0) = v_+(0)\}$. Defining the mapping $\Sigma : X_- \oplus X_+ \rightarrow \mathbb{R}^d$ by