

$\sigma(y_i, t + s)$ as functions of s . These solve $\partial_s \sigma(y_i, t + s) = F(\sigma(y_i, t + s), t + s)$ and $\sigma(y_1, t + 0) = \sigma(y_2, t + 0)$. Again by uniqueness, $\sigma(y_1, t + s) = \sigma(y_2, t + s)$ for all $|s| < \delta$, for some $\delta > 0$, that is, $(t - \delta, t + \delta) \subset J$ and so J is open.

2.3. Minimization flows and OT. Here we use the approach from [AHT03]. Let $T : (\Omega_0, \rho_0) \rightarrow (\Omega_1, \rho_1)$ be a measure preserving map, i.e.,

$$\int_{T^{-1}(E)} \rho_0(x) dx = \int_E \rho_1(x) dx$$

for each Borel set $E \subset \Omega_1$. Set $\sigma_t(\cdot) = \sigma(\cdot, t)$ where σ is as before, the flow corresponding to a vector field F satisfying (2.2). The family of maps $T \circ (\sigma_t)^{-1} : (\Omega_0, \rho_0) \rightarrow (\Omega_1, \rho_1)$ are measure preserving since

$$(T \circ (\sigma_t)^{-1})^{-1}(E) = \sigma_t \circ T^{-1}(E), \quad E \subset \Omega_1$$

and

$$\begin{aligned} \int_{(T \circ (\sigma_t)^{-1})^{-1}(E)} \rho_0(x) dx &= \int_{\sigma_t \circ T^{-1}(E)} \rho_0(x) dx \\ &= \int_{T^{-1}(E)} \rho_0(x) dx \quad \text{since } \sigma_t \text{ preserves the measure} \\ &= \int_E \rho_1(x) dx. \end{aligned}$$

Consider the function of t

$$G(t) = \int_{\Omega_0} c(x, T \circ (\sigma_t)^{-1}(x)) \rho_0(x) dx;$$

here $c(x, y)$ is a general cost. Making the change of variables $x = \sigma(z, t)$ yields

$$\begin{aligned} G(t) &= \int_{\Omega_0} c(\sigma(z, t), T(z)) \rho_0(\sigma(z, t)) J_\sigma(z) dz \\ &= \int_{\Omega_0} c(\sigma(z, t), T(z)) \rho_0(z) dz \end{aligned}$$

since $\rho_0(\sigma(z, t)) J_\sigma(z) = \rho_0(z)$ because $\int_{\sigma(E, t)} \rho_0(x) dx = \int_E \rho_0(x) dx$ for all t and all $E \subset \Omega_0$. If T is an optimal map with respect to the cost c , then $G'(t) = 0$ when