

where  $F_h(x)$  enumerates independent sets fixed by non-identity elements  $h \in H_p$ . This proves part (ii)(c) of Theorem 3.1.

*Proof.* Each term  $b_j$  in the polynomial modulo  $p$  counts exactly those independent sets with nontrivial stabilizers. By definition, these are precisely sets that are fixed by some non-identity element  $h \in H_p$ . Thus, summing over non-identity elements gives the generating function for fixed independent sets, proving the equivalence stated in (ii)(c) of Theorem 3.1.  $\square$

The orbit structure established in Lemma 3.4 implies the modular collapse behavior described in Theorem 3.1.

### 3.3 Algebraic Completion via Frobenius

Building upon the results established in Lemmas 3.3 and 3.1, we now give an algebraic characterization involving polynomial congruences modulo prime divisors of  $n$ . This analysis directly ties into the orbit-stabilizer theorem and the modular collapse phenomenon, demonstrating how independent sets are constrained under the action of  $H_p$ .

For prime  $n$ , we have established that  $I(C_n^{\boxtimes d}, x) \equiv 1 \pmod{n}$ . For composite  $n$  with prime divisor  $p$ , the subgroup  $H_p$  partitions independent sets into orbits of size exactly  $p$ , ensuring that the number of such sets satisfies congruences modulo  $p$ . This leads to specific conditions under which  $I(C_n^{\boxtimes d}, x)$  exhibits structured congruences.

**Frobenius Automorphism and Its Role** The **Frobenius automorphism**, denoted  $\text{Frob}_p$ , is a key tool in understanding the structure of field extensions in characteristic  $p$ . It is defined as the mapping:

$$\text{Frob}_p : a \mapsto a^p$$

for any element  $a$  in a field of characteristic  $p$ . This automorphism preserves the algebraic structure of the field while raising each element to the power of  $p$ , which naturally extends to polynomials and their roots. Applying this iteratively, we obtain:

$$(x + 1)^{p^d} \equiv x^{p^d} + 1 \pmod{p}.$$

**Polynomial Congruences and Frobenius Completion** Let us examine when the congruence  $I(C_n^{\boxtimes d}, x) \equiv (x + 1)^m \pmod{p}$  holds for some  $m \leq p^d$ :

**Proposition 3.5.** *Let  $n$  be composite with prime divisor  $p$ . Then  $I(C_n^{\boxtimes d}, x) \equiv c \cdot (x + 1)^m \pmod{p}$  for some constant  $c \in \mathbb{F}_p^\times$  and  $m \leq p^d$  if and only if the only  $H_p$ -invariant independent sets are those consisting entirely of vertices fixed point wise by the action of  $H_p$ .*

*Proof.* Let  $\mathcal{F}_p$  be the set of all independent sets that are fixed by at least one non-identity element of  $H_p$ . The generating function for these sets is:

$$F(x) = \sum_{S \in \mathcal{F}_p} x^{|S|}$$

By the orbit-stabilizer theorem, each  $H_p$ -invariant independent set belongs to an orbit whose size divides  $p$ , leading to constraints on  $I(C_n^{\boxtimes d}, x)$ . When  $\mathcal{F}_p$  contains only independent sets consisting of vertices fixed point wise by  $H_p$ ,  $F(x)$  takes the form  $(x + 1)^m - 1$  for some  $m \leq p^d$ , representing all possible subsets of the fixed vertex set minus the empty set.

Under these conditions, we have:

$$I(C_n^{\boxtimes d}, x) \equiv 1 + F(x) \equiv (x + 1)^m \pmod{p}$$

Conversely, if  $I(C_n^{\boxtimes d}, x) \equiv c \cdot (x + 1)^m \pmod{p}$ , then  $\mathcal{F}_p$  must correspond to the structure  $(x + 1)^m - 1$ , implying that the only invariant sets are those built from fixed vertices.

Applying the Frobenius automorphism, we recall: