

with

$$\rho(y, x, \eta') = \varphi(y, \eta') - \varphi(x, \eta') - (y - x)\varphi_x(x, \eta'),$$

$\rho$  vanishes at order  $\geq 2$  in  $y = x$ . For checking (3.10), we see

$$(3.11) \quad Q(ae^{i\frac{\varphi}{h}}) = \int \int e^{i\frac{1}{h}((x'-y')\theta' + \varphi(y, \eta'))} \frac{q(x, \theta')}{(2\pi h)^{n-1}} a(y, \eta') dy' d\theta'.$$

a having compact support and  $\varphi$  being a symbol, one has  $|\varphi'_y| \leq C'$  on  $\text{supp}(qa)$ , so for  $|\theta'| \geq C > 0$ , large, the phase of (3.11)  $H = (x' - y')\theta' + \varphi(y, \eta')$  satisfies

$$|H'_{y'}| \geq c(1 + |\theta'|), \quad \text{for } |\theta'| \geq C > 0, \text{ large}$$

and if we split the integrand of (3.11) in  $qa = \chi(\theta')qa + (1 - \chi(\theta'))qa$ , we integrate the second term by parts and obtain

$$(3.12) \quad Q(ae^{i\frac{\varphi}{h}}) = Q'_\varphi(a)e^{i\frac{\varphi}{h}} + R(a),$$

where  $Q'_\varphi$  is a  $\mathcal{G}^s$  symbol of order  $\tilde{S}_s^{m,k+1}$  having the expansion (3.10) by the stationary phase lemma as  $R(a)$  is an  $\mathcal{O}_s(h^\infty)$  remainder. It is easy to see in view of these arguments that  $a_1 \in \tilde{S}_s^{m-1,k}$ . Moreover, it is to be observed that the above expansion is only a formal Gevrey  $2s - 1$  symbol.

The microlocal invertibility of FIO reduces to the PDO case. We refer to [5] for a proof of the Gevrey elliptic result in classes  $S_s^m$ .

For proving Theorem 2, we rewrite (3.8) in the form

$$(hD_{x_1} + Q(x, hD_x; h))Fu = F(hD_{x_1} + Q')u,$$

close to  $(x_0, \xi_0; x_0, \xi_0)$  for some PDO  $Q'(x, hD_x; h)$  of bi-order  $(-1, 0)$  in using a left microlocal inverse of  $F$  close to  $(x_0, \xi'_0)$ . Indeed, we compute  $FF^*$  and  $F^*F$ , and one has writing  $y = (x_1, y')$ .

One has, following Eskin [8],

$$FF^*u(x, h) = \frac{1}{(2\pi h)^{n-1}} \iint e^{i\frac{1}{h}(\varphi(x, \xi') - \varphi(y, \xi'))} a(x, \xi') \overline{a(y, \xi')} u(x_1, y') dy' d\xi',$$

$\varphi(x, \xi')$  having been obtained in (3.6). We split the integral above into two terms. The first is a  $h$ -PDO, the second is a smoothing operator. First, we note that the map:

$$(x, y', \xi') \rightarrow (x, y', \Sigma(x, y', \xi')),$$

with

$$(3.13) \quad \Sigma(x, y', \xi') = \int_0^1 \varphi'_{x'}(x_1, y' + t(x' - y'), \xi') dt$$

is a  $\mathcal{G}^s$ -diffeo in a neighbourhood of  $(x_0, y'_0, \eta'_0)$  with  $|x_1| \leq \delta$ ,  $|x' - y'| \leq \delta$ ,  $0 < \delta$  small, close to the identity.

Let  $(x, y', \eta') \rightarrow (x, y', \Sigma^{-1}(x, y', \eta'))$  be an inverse map.

One has obviously  $\varphi(x, \xi') - \varphi(y, \xi') = \Sigma(x, y', \xi')(x' - y')$ , and

$$(3.14) \quad FF^*u(x, h) = K_1u(x, h) + K_2u(x, h),$$