

Obviously  $W$  contains an open neighborhood of zero. Now let  $z = (r(f_1), r(f_2), \dots, r(f_n))$ . By assumption and the fact that  $\overline{W}$  is compact,  $z \notin \overline{W}$ . Since  $\overline{W}$  compact we can find a bounded linear functional that strictly separates  $\overline{W}$  and  $z_0$ . Namely we can find  $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{C}^n$  such that:

$$\sum_{i \leq n} \lambda_i r(f_i) > 1 \quad \text{and} \quad \forall (x_1, \dots, x_n) \in \overline{W}, \quad \sum_{i \leq n} \lambda_i x_i \leq 1$$

Therefore  $\|\sum_{i \leq n} \lambda_i f_i\| \leq 1$  and hence:

$$1 < r \left( \sum_{i \leq n} \lambda_i f_i \right) \leq \|r\| \left\| \sum_{i \leq n} \lambda_i f_i \right\| \leq 1$$

which is absurd. Hence for any  $\{f_i\}_{i \leq n} \subseteq X^*$  and  $\sigma \in (0, 1)$ :

$$(r(f_1), \dots, r(f_n)) \in \overline{\{(f_1(x), \dots, f_n(x)) \mid x \in X_{\leq 1}\}} \subseteq (1 + \sigma) \{(f_1(x), \dots, f_n(x)) \mid x \in X_{\leq 1}\}$$

or

$$(r(f_1), \dots, r(f_n)) \in \{(f_1(x), \dots, f_n(x)) \mid x \in X_{\leq 1+\sigma}\}$$

□

**Proposition 5.3.** *Any non-reflexive Banach space  $X$  will have a bounded bi-orthogonal system  $(e_i, f_j)_{i,j \in \mathbb{N}}$  such that  $\sup_n \|\sum_{i \leq n} e_i\| < \infty$  or  $\sup_n \|\sum_{i \leq n} f_i\| < \infty$*

*Proof.* Let  $r \in X^{**} \setminus Q(X)$  and suppose  $H$  is an arbitrary finite dimensional subspace of  $X$ . Define:

$$\rho = \sup \{ |r(y)| : y \in X_{=1}^* \cap H^\perp \}.$$

Suppose  $H = \text{Span}\{h_1, h_2, \dots, h_n\}$  and  $H^* = \text{Span}\{f_1, f_2, \dots, f_n\}$  where  $f_i(h_j) = 1$  iff  $i = j$ . If  $\rho = 0$ , then  $r \in (H^\perp)^\perp$ . Observe that  $Q(H)^\perp = Q'(H^\perp)$  and  $(H^\perp)^\perp \subseteq Q'(H^\perp)^\perp$ . Then  $r \in [Q(H)^\perp]^\perp = Q(H) \subseteq Q(X)$ , which is absurd. Hence  $\rho > 0$ . Below we will start constructing the desired bi-orthogonal system.

Suppose  $\|r\| = 1$ . Fix  $\sigma \in (0, 1)$ .  $y_1 \in X_{=1}^*$  so that  $\beta_1 = r(y_1) > \frac{1}{2}$ . By **Proposition 5.2** we can find  $b_1 \in X_{\leq 1+\sigma}$  so that  $y_1(b_1) = \beta_1$ . Define  $E_1 = \text{Span}\{b_1\}$  and:

$$\rho_1 = \sup \{ |r(y)| : y \in X_{=1}^* \cap E_1^\perp \}$$

By the previous remark, we have  $\rho_1 > 0$ . Next find  $y_2 \in X_{\leq 1}^* \cap E_1^\perp$  so that  $\beta_2 = r(y_2) > \frac{1}{2}\rho_1$ . Again by **Proposition 5.2**, find  $b_2 \in X_{\leq 1+\sigma}$  so that  $y_1(b_2) = \beta_1, y_2(b_2) = \beta_2$ . Define  $E_2 = \text{Span}\{b_1, b_2\}$  and:

$$\rho_2 = \sup \{ |r(y)| : y \in X_{=1}^* \cap E_2^\perp \}$$

and similarly  $\rho_2 > 0$ . By induction for each  $n \in \mathbb{N}$  we will have  $\{b_i\}_{i \leq n} \subseteq X_{\leq 1+\sigma}$ ,  $E_n = \text{Span}\{b_i\}_{i \leq n}$ ,  $y_{i+1} \in X_{=1}^* \cap E_i^\perp$ ,  $\{\beta_i\}_{i \leq n}$  and:

$$\rho_n = \sup \{ |r(y)| : y \in X_{=1}^* \cap E_n^\perp \}$$

such that  $\beta_1 = r(y_1) > \frac{1}{2}$ ,  $\beta_{n+1} = r(y_{n+1}) \in \left( \frac{1}{2}\rho_n, \rho_n \right]$  and for each  $i, j \in \{1, 2, \dots, n\}$ :

$$y_i(b_j) = \begin{cases} r(y_i), & 1 \leq i \leq j \leq n \\ 0, & 1 \leq j < i \leq n \end{cases}$$

Since  $H_n^\perp$  is decreasing as  $n$  increases, we have  $\{\rho_n\}$  is non-increasing sequence in  $(0, 1]$  and hence convergent. Assume that  $\inf_n \rho_n = \lim_n \rho_n = 0$ . For each  $\epsilon \in (0, 1)$  and then suppose  $\rho_n < \epsilon$  for all  $n \geq N$ . Fix  $f \in X_{\leq 1}^*$  and for each  $n \in \mathbb{N}$ , define  $\tau_n(f) = \max_{i \leq n} |f(b_i)|$  and:

$$z_n(f) = \frac{1}{\beta_1} f(b_1) y_1 + \sum_{1 < i \leq n} \frac{1}{\beta_i} [f(b_i) - f(b_{i-1})] y_i$$

Since for each  $n \in \mathbb{N}$ :

$$0 < \frac{1}{2}\rho_{n+1} < \beta_n \leq \rho_n$$