

Proof. Let $g \in \Lambda$ have a non-trivial projection on $\mathbb{R}_{\mathbf{H}}$, and let $z \in \Lambda \cap Z$. We have $g^k z g^{-k} = a^k(z)$, where $a = r(g)$. Since the actions of $\mathbb{R}_{\mathbf{L}}$ and K on the center of \mathbf{Heis} are trivial, and the action of $\mathbb{R}_{\mathbf{H}}$ is contracting (up to forward or backward iteration), it follows that the sequence $(a^k(z))_{k \in \mathbb{Z}}$ has an accumulation point at 1. As Λ is discrete, the set $\{a^k(z), k \in \mathbb{N}\} \subset \Lambda$ must be finite, which forces $z = 1$. \square

Lemma 3.7. ([17, Lemma 5.3]) *Let Q be the semi-direct product $Q = \mathbb{R} \ltimes C$ where C is a compact connected Lie group. Then Q is isomorphic to the product $\mathbb{R} \times C$.*

The last lemma we will need in this section is Lemma 3.8 below. It is similar to [16, Proposition 5.4], but we present it here in a form that allows for systematic application. The proof however is the same, so to avoid unnecessary lengthening of the paper, we will not rewrite the proof.

Let $Q \ltimes C \subset \text{Aut}(\mathbf{Heis})$ preserving the decomposition $\mathfrak{heis} = \mathfrak{a} \oplus \mathfrak{z}$, and such that C is compact. Define

$$G := (Q \ltimes C) \ltimes N.$$

Let Γ be a discrete subgroup of G . Under some conditions, the proposition below ensures the existence of a nilpotent syndetic hull of Γ , i.e. a connected nilpotent subgroup of G containing Γ as a lattice.

Lemma 3.8. *Let Γ be a discrete subgroup of G , whose projection on Q is dense. Then, up to finite index, Γ is a lattice in a connected closed nilpotent subgroup of G .*

3.2. Proof of Theorem 3.1. We will examine all possible closures of the projection of Γ to $\mathbb{R}_{\mathbf{H}} \times \mathbb{R}_{\mathbf{L}} \simeq \mathbb{R}^2$.

Proposition 3.9. *If $\overline{p(\Gamma)} = \mathbb{R}_{\mathbf{H}} \times \mathbb{R}_{\mathbf{L}}$, then Γ does not act properly discontinuously and cocompactly on X .*

Proof. By Lemma 3.8, Γ is a cocompact lattice in a connected nilpotent subgroup N of G . We will show that $N \cap \mathbf{Heis} \neq \{1\}$. Our claim is that $\dim N \geq n + 2$. Indeed, Γ acts properly and cocompactly on the $K(\pi, 1)$ space $N/C \simeq \mathbb{R}^k$, where C is the maximal compact subgroup of N . So its cohomological dimension is equal to $\dim(N/C)$. On the other hand, since Γ also acts properly and cocompactly on $X = \mathbb{R}^{n+2}$, its cohomological dimension must be $n + 2$. This implies that $\dim N/C = n + 2$. Now, decompose N as $N = \mathbb{R} \ltimes N_1$, where the \mathbb{R} -factor is generated by a one-parameter subgroup of G with a nontrivial projection on $\mathbb{R}_{\mathbf{L}}$, and where

$$N_1 := N \cap (\mathbb{R}_{\mathbf{H}} \times K) \ltimes \mathbf{Heis},$$

satisfies $\dim N_1 \geq n + 1$. We have $p_{\mathbf{H}}(N_1) = \mathbb{R}$, so we further decompose N_1 as $N_1 = \mathbb{R} \ltimes N_0$, where the \mathbb{R} -factor is a one-parameter subgroup $s(t)$ of $(\mathbb{R}_{\mathbf{H}} \times K) \ltimes \mathbf{Heis}$ that has a nontrivial projection on $\mathbb{R}_{\mathbf{H}}$, and

$$N_0 := N_1 \cap (K \ltimes \mathbf{Heis}),$$

with $\dim N_0 \geq n$. Since N_0 is nilpotent, its projection to K is an abelian subgroup of $O(n)$, whose maximal possible dimension is $\lfloor \frac{n}{2} \rfloor$. Consequently, N_0 must intersect \mathbf{Heis} non-trivially. Moreover, N_1 contains the subgroup $s(t) \ltimes (N_0 \cap \mathbf{Heis})$, which is therefore nilpotent. However, since the action of $s(t)$ on $N_0 \cap \mathbf{Heis}$ is semisimple (and unipotent), it must be trivial. This is impossible since $s(t)$ projects non-trivially on $\mathbb{R}_{\mathbf{H}}$, so its real eigenvalues are all different from 1 (the real eigenvalues of its projection to K are all equal to 1). \square

Proposition 3.10. *If $\overline{p(\Gamma)} = \mathbb{R}$, the action of Γ is properly discontinuous and cocompact if and only if Γ is contained in the isometry group.*

Proof. We have a short exact sequence $1 \rightarrow K \ltimes \mathbf{Heis} \rightarrow G \rightarrow \mathbb{R}_{\mathbf{L}} \times \mathbb{R}_{\mathbf{H}} \rightarrow 1$. Let $\mathbb{R}_{\mathbf{T}} = \{e^{tT}, t \in \mathbb{R}\}$ be a one-parameter subgroup of G such that $p(\mathbb{R}_{\mathbf{T}}) = \overline{p(\Gamma)}$. Then, Γ is contained in $G_1 := \mathbb{R}_{\mathbf{T}} \ltimes (K \ltimes \mathbf{Heis})$.

We will show that this group is, in fact, isomorphic to a group of the form $(\mathbb{R} \times K) \ltimes \mathbf{Heis}$. Write $\mathbf{T} = \lambda + \Phi + \omega$, with $\lambda \in \mathbb{R}H \oplus \mathbb{R}L$, $\Phi \in \mathfrak{k}$, and $\omega \in \mathfrak{heis}$. For any $\Psi \in \mathfrak{k}$, we have

$$[\mathbf{T}, \Psi] = [\Phi, \Psi] + [\omega, \Psi] \in \mathfrak{k} \ltimes \mathfrak{heis}. \quad (2)$$