

Moreover, the indicator  $\mathbb{1}_{\{X_{(k-1)/n} < \rho_0 \leq \rho_0 + \theta < X_{k/n}\}}$  is larger, the smaller  $\theta$ . Thus,

$$\log \left( \frac{P_3^{\rho_0 + \theta}(X_{(k-1)/n}, X_{k/n}; 1/n)}{P_3^{\rho_0}(X_{(k-1)/n}, X_{k/n}; 1/n)} \right) \mathbb{1}_{I_{3,k}^\theta} \leq \exp \left( -\frac{(X_{(k-1)/n} - \rho_0)^2}{2\alpha^2/n} \right) \mathbb{1}_{I_{3,k}^{K\sqrt{n}}}$$

and

$$\begin{aligned} & \sup_{\theta \in \Theta_1 \cup \Theta_2} \sum_{k=1}^n \log \left( \frac{P_3^{\rho_0 + \theta}(X_{(k-1)/n}, X_{k/n}; 1/n)}{P_3^{\rho_0}(X_{(k-1)/n}, X_{k/n}; 1/n)} \right) \mathbb{1}_{I_{3,k}^\theta} \\ & \leq \sum_{k=1}^n \exp \left( -\frac{(X_{(k-1)/n} - \rho_0)^2}{2\alpha^2/n} \right) \mathbb{1}_{I_{3,k}^{K\sqrt{n}}} =: \Xi_n^{(3)}(K) \end{aligned}$$

Again, with Corollary B.2 we find  $\sup_{K \geq 0} \mathbb{E}_{\rho_0} [|\Xi_n^{(3)}(K)|/\sqrt{n}] \leq C_{\alpha, \beta}$ .

$\mathcal{I}_4(\theta)$ : Follows by the same reasoning as  $\mathcal{I}_2(\theta)$  by interchanging the roles of  $\alpha$  and  $\beta$  and gives an upper bound  $\Xi_n^{(4)}$  with  $\mathbb{E}_{\rho_0} [|\Xi_n^{(4)}|/\sqrt{n}] \leq C_{\alpha, \beta}$ .

$\mathcal{I}_5(\theta)$ : First, we split

$$\begin{aligned} & \frac{P_1^{\rho_0 + \theta}(X_{(k-1)/n}, X_{k/n}; 1/n)}{P_2^{\rho_0}(X_{(k-1)/n}, X_{k/n}; 1/n)} \\ & = \frac{\beta \exp \left( -\frac{(X_{k/n} - X_{(k-1)/n})^2}{2\alpha^2/n} \right) 1 - \frac{\alpha - \beta}{\alpha + \beta} \exp \left( -\frac{2}{\alpha^2/n} (X_{k/n} - \rho_0 - \theta)(X_{(k-1)/n} - \rho_0 - \theta) \right)}{\alpha \exp \left( -\frac{(X_{k/n} - X_{(k-1)/n})^2}{2\beta^2/n} \right) 1 + \frac{\alpha - \beta}{\alpha + \beta} \exp \left( -\frac{2}{\beta^2/n} (X_{k/n} - \rho_0)(X_{(k-1)/n} - \rho_0) \right)}. \end{aligned}$$

Taking the logarithm on both sides then yields

$$\log \left( \frac{P_1^{\rho_0 + \theta}(X_{(k-1)/n}, X_{k/n}; 1/n)}{P_2^{\rho_0}(X_{(k-1)/n}, X_{k/n}; 1/n)} \right) = \log \left( \frac{\beta}{\alpha} \right) - \frac{(X_{k/n} - X_{(k-1)/n})^2}{2/n} \left( \frac{1}{\alpha^2} - \frac{1}{\beta^2} \right) + R_{k, \theta}$$

with

$$R_{k, \theta} = \log \left( \frac{1 - \frac{\alpha - \beta}{\alpha + \beta} \exp \left( -\frac{2}{\alpha^2/n} (X_{k/n} - \rho_0 - \theta)(X_{(k-1)/n} - \rho_0 - \theta) \right)}{1 + \frac{\alpha - \beta}{\alpha + \beta} \exp \left( -\frac{2}{\beta^2/n} (X_{k/n} - \rho_0)(X_{(k-1)/n} - \rho_0) \right)} \right).$$

Denoting

$$g_k := \log \left( \frac{\beta}{\alpha} \right) - \frac{n}{2} \left( \frac{1}{\alpha^2} - \frac{1}{\beta^2} \right) (X_{k/n} - X_{(k-1)/n})^2$$

(which is independent of  $\theta$ ) and decomposing

$$\begin{aligned} \mathbb{1}_{I_{5,k}^\theta} &= \mathbb{1}_{\{\rho_0 \leq X_{(k-1)/n} < \rho_0 + \theta\}} \mathbb{1}_{I_{5,k}^\theta} \\ &= \mathbb{1}_{\{\rho_0 + L/\sqrt{n} \leq X_{(k-1)/n} < \rho_0 + \theta\}} - \mathbb{1}_{\{\rho_0 + L/\sqrt{n} \leq X_{(k-1)/n} < \rho_0 + \theta\}} (1 - \mathbb{1}_{\{\rho_0 < X_{k/n} \leq \rho_0 + \theta\}}) \\ &\quad + \mathbb{1}_{\{\rho_0 \leq X_{(k-1)/n} < \rho_0 + L/\sqrt{n}\}} \mathbb{1}_{\{\rho_0 < X_{k/n} \leq \rho_0 + \theta\}} \end{aligned}$$

then gives the decomposition

$$(F.1) \quad \log \left( \frac{P_1^{\rho_0 + \theta}(X_{(k-1)/n}, X_{k/n}; 1/n)}{P_2^{\rho_0}(X_{(k-1)/n}, X_{k/n}; 1/n)} \right) \mathbb{1}_{I_{5,k}^\theta} = S_{1, \theta}(k) + S_{2, \theta}(k) + S_{3, \theta}(k) + S_{4, \theta}(k),$$