

3.3. Global existence.

Theorem 3.10. *Assuming that (3.2) and (3.3) are satisfied, then the conclusions in Theorem 3.4 hold for any $T > 0$.*

Proof. By Theorem 3.4, we may assume that (3.1) has a unique solution (u, g, h) defined on some maximal time interval $(0, T_m)$ with $T_m \in (0, \infty]$, and

$$h, g \in C^{1+\frac{\beta}{2}}((0, T_m)), \quad u \in C^{1+\frac{\beta}{2}, 2+\beta}(\Omega_{T_m}),$$

where $\Omega_{T_m} := \{(t, x) : t \in (0, T_m), x \in [g(t), h(t)]\}$. To complete the proof, we must demonstrate that $T_m = \infty$. Suppose $T_m < \infty$. Then, by the proof of Theorem 3.4, along with Lemmas 3.8 and 3.9, there exist positive constants $C_1, C_2 = C_2(T_m)$, such that for $t \in [0, T_m)$ and $x \in [g(t), h(t)]$,

$$0 \leq u(x, t) \leq C_1, \quad |h'(t)| + |g'(t)| \leq C_2, \quad |h(t)|, |g(t)| \leq C_2 t + h_0.$$

For any small constant $\varepsilon > 0$, it follows from the proof of Theorem 3.4 and Lemmas 3.8 and 3.9 that $u, v \in C^{1+\beta, \frac{1+\beta}{2}}(\Omega_{T_m-\varepsilon})$. Thus, as in Step 4 of the proof of Theorem 3.4, applying Schauder's estimates, for any fixed $0 < T_0 < T_m - \varepsilon$, we obtain $\|u\|_{C^{2+\beta, 1+\frac{\beta}{2}}(\Omega_{T_m-\varepsilon} \setminus \Omega_{T_0})} \leq Q^*$, where Q^* depends on T_0, T_m , and C_i for $i = 1, 2$, but is independent of ε . Since $\varepsilon > 0$ can be made arbitrarily small, it follows that for any $t \in [T_0, T_m)$,

$$\|u(t, \cdot)\|_{C^{2+\beta}([g(t), h(t)])} \leq Q^*.$$

By repeating the arguments used in the proof of Theorem 3.4, we can conclude that there exists $T > 0$ small, depending on Q^* and C_i ($i = 1, 2$), such that the solution to (3.1) with initial time $T_m - \frac{T}{2}$ can be extended uniquely to $t = T_m - \frac{T}{2} + T > T_m$, a contradiction to the definition that T_m is the maximal time interval for the solution. Thus, we must have $T_m = \infty$. \square

3.4. Proof of Theorem 3.1. Let us first note that $f(t, x, 1) \equiv 0$ and $f(t, x, u) \leq \bar{f}(u) < 0$ for $u > 1$ imply $\bar{f}(1) = 0$. Let $M_0 = \max\{\|u_0\|_\infty, 1\}$ and $v(t)$ be the solution of

$$v' = \bar{f}(v), \quad v(0) = M_0.$$

Since $\bar{f}(v) < 0$ for $v > 1$, we clearly have $1 \leq v(t) \leq M_0$ and $v(t) \rightarrow 1$ as $t \rightarrow \infty$. Since $f(t, x, u) \leq \bar{f}(u)$ for $u \geq 1$, we obtain

$$v_t - dv_{xx} = \bar{f}(v) \geq f(t, x, v) \text{ for } t > 0, x \in [g(t), h(t)].$$

We also have $v \geq 1 > \delta = u$ for $x \in \{g(t), h(t)\}$ and $v(0) = M_0 \geq u(0, x)$ for $x \in [-h_0, h_0]$. Therefore the standard comparison principle over the region $\{(t, x) : t > 0, x \in [g(t), h(t)]\}$ infers $u(t, x) \leq v(t)$ in this region. It follows that

$$(3.31) \quad \limsup_{t \rightarrow \infty} u(t, x) \leq \lim_{t \rightarrow \infty} v(t) = 1 \text{ uniformly for } x \in [g(t), h(t)].$$

To bound $u(t, x)$ from below, we first make use of (3.4) to show that there exists $T_0 > 0$ such that

$$(3.32) \quad u(t, x) \geq \delta \text{ for } t \geq T_0 \text{ and } x \in [g(t), h(t)].$$

Similar to the proof of Theorem 1.1, since \underline{f} satisfies (\mathbf{f}_A) , we are able to choose a function $\hat{f} \in C^1$ sufficiently close to \underline{f} in L^∞ such that $\hat{f}(s) \leq f(s)$ for $s \geq 0$, and \hat{f} satisfies (\mathbf{F}_b) with $(P, Q) = (\hat{\theta}, \delta)$ for some $\hat{\theta} \in [\theta, \delta) \cap (0, \delta)$. Then, by Lemma 2.1, the traveling wave problem (2.4) has a solution pair $(c, q) = (c_0, q_0)$ with $c_0 > 0$ and $q_0(\cdot)$ strictly increasing.

The same reasoning as in the proof of Theorem 1.1 shows that for some $L > 0$ sufficiently large,

$$\underline{u}(t, x) := \max\{q_0(ct - x - L), q_0(ct + x - L)\}$$

satisfies (in the weak sense)

$$\underline{u}_t \leq d\underline{u}_{xx} + \underline{f}(\underline{u}) \leq f(t, x, \underline{u}) \quad \text{for } t > 0, x \in \mathbb{R}.$$

Additionally,

$$0 \leq \underline{u}(t, x) \leq \delta \leq u(t, x) \text{ for } t > 0, x \in \{g(t), h(t)\},$$

and

$$\underline{u}(0, x) \leq u_0(x) \quad \text{for } x \in [-h_0, h_0].$$

Therefore we can apply the standard comparison principle over $\{(t, x) : t > 0, x \in [g(t), h(t)]\}$ to deduce $u(t, x) \geq \underline{u}(t, x)$ in this region.

Moreover, using

$$\lim_{t \rightarrow \infty} \|\underline{u}(t, \cdot) - \delta\|_{L^\infty(\mathbb{R})} = 0,$$