

## 2 The Mandelstam Region

A symmetric  $n \times n$  matrix  $S = [s_{ij}]$  of rank  $r$  is said to be a *Mandelstam matrix* if

- the diagonal entries are non-negative,  $s_{ii} \geq 0$  for  $i = 1, \dots, n$ ; and
- it has precisely one positive eigenvalue and  $r - 1$  negative eigenvalues.

We denote the set of all Mandelstam matrices of rank  $r$  by  $\mathcal{M}_{n,r}$ . This is a semialgebraic set in  $\mathbb{R}^{\binom{n+1}{2}}$ , the space of all symmetric matrices. The following is the Mandelstam analogue of the familiar characterization of positive semidefinite matrices in terms of principal minors.

**Lemma 2.1.** *A symmetric  $n \times n$  matrix  $S$  is Mandelstam if and only if*

$$(-1)^{|I|-1} \det(S_I) \geq 0 \quad \text{for all } I \subseteq [n], \quad (4)$$

where  $\det(S_I)$  are the principal minors of  $S$ .

*Proof.* This follows from the general results in [6]. We refer to Baker's exposition in [3]. The key step is Cauchy's interlacing theorem [12]. This states that the eigenvalues of  $S_I$  interlace the eigenvalues of  $S_J$  whenever  $I \subset J$ . Hence, if  $S_I$  has at most one positive eigenvalue then so does  $S_J$ . But  $S_J$  cannot have all negative eigenvalues because its trace is non-negative.  $\square$

The name of our matrices refers to the physicist Stanley Mandelstam (1928–2016) who is credited for introducing the variables  $s_{ij}$  in the context of scattering amplitudes. In [14] the role of  $\mathcal{M}_{n,r}$  as a kinematic space is recognized. A term more familiar to mathematicians might be “Lorentzian matrices.” These encode Lorentzian quadratic forms [5, 6]. We here use the term *Lorentzian matrix* for a Mandelstam matrix whose entries  $s_{ij}$  are all non-negative.

Mandelstam matrices arise as Gram matrices of momentum vectors in  $\mathbb{R}^{1+d}$  with the Lorentzian inner product. A non-zero momentum vector is any vector  $p \in \mathbb{R}^{1+d}$  of the form

$$p = \lambda(1, x_1, \dots, x_d), \quad (5)$$

for some scalar  $\lambda \neq 0$ , and  $x = (x_1, \dots, x_d)$  in the closed unit ball  $\mathbb{B}^d = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ . Given  $n$  momentum vectors,  $p^{(i)}$ , their Gram matrix  $S = [s_{ij}]$  has entries  $s_{ij} = p^{(i)} \cdot p^{(j)}$ . This is the matrix in (1). The entries of  $S$  may now be written as

$$s_{ij} = \lambda_i \lambda_j (1 - \langle x^{(i)}, x^{(j)} \rangle) \quad (6)$$

Here  $\cdot$  is the Lorentz inner product on  $\mathbb{R}^{1+d}$  and  $\langle \cdot, \cdot \rangle$  is the Euclidean inner product on  $\mathbb{R}^d$ .

**Lemma 2.2.** *A symmetric  $n \times n$  matrix  $S$  is Mandelstam, i.e.  $S$  lies in the region  $\mathcal{M}_{n, \leq 1+d}$ , if and only if it is the Gram matrix of  $n$  momentum vectors in  $(1+d)$ -dimensional spacetime.*

*Proof.* Assume that  $S$  has no zero rows or columns. For the only-if direction, take a Mandelstam matrix  $S$ . By Lemma 2.1 and diagonalization of symmetric matrices, it can be factorized as in (1). Namely, we write  $S = MDM^T$ , where  $D = \text{diag}(1, -1, -1, \dots, -1)$ . Let the