

distance at least K from each of u, v, w . We cannot have $d_G(w, Q(x, y)) < K$ since then we get $d_G(x, w) < K + K = 2K$, which is impossible as $x \in P(u, v)$. Similarly, we cannot have $d_G(u, Q(x, y)) < K$. Assume, by way of contradiction, $d_G(v, Q(x, y)) < K$. Then, there is a vertex $z \in Q(x, y)$ with $d_G(v, z) < K$. Assuming, without loss of generality, $d_G(x, z) \leq d_G(z, y)$, we get $d_G(v, x) \leq d_G(v, z) + d_G(z, x) \leq K - 1 + (K - 1)/2 = 3/2(K - 1) < \lfloor \frac{3}{2}K \rfloor$. The latter contradicts the choice of vertex v_u as being a vertex of $P(u, v) \cap D_G(v, \lfloor \frac{3}{2}K \rfloor)$ furthest (along the path $P(u, v)$) from v . This proves that the $K_{1,3}$ -minor constructed is K -fat.

Now, we can assume that all three paths $P(v_u, u_v)$, $P(v_w, w_v)$, $P(u_w, w_u)$ are pairwise at distance at least K . In this case, we can build a K -fat K_3 -minor in G . Set $H_v := G[D_G(v, \lfloor \frac{3}{2}K \rfloor)]$, $H_u := G[D_G(u, \lfloor \frac{3}{2}K \rfloor)]$, $H_w := G[D_G(w, \lfloor \frac{3}{2}K \rfloor)]$. It is easy to see that these connected subgraphs H_v, H_w, H_u and paths $P(v_u, u_v)$, $P(v_w, w_v)$, $P(u_w, w_u)$ form a K -fat K_3 -minor in G . Recall that $d_G(v, P(u, w)) \geq 4K$, $d_G(u, P(v, w)) \geq 4K$ and $d_G(w, P(u, v)) \geq 4K$. Hence, if $d_G(V(H_v), V(H_u)) < K$, then $d_G(v, u) < \lfloor \frac{3}{2}K \rfloor + K + \lfloor \frac{3}{2}K \rfloor \leq 4K$, which is impossible. If $d_G(V(H_v), P(u_w, w_u)) < K$, then $d_G(v, P(u, w)) < \lfloor \frac{3}{2}K \rfloor + K < 3K$, which is also impossible. By symmetries, the K_3 -minor constructed is K -fat. \square

Note that, in the proof of Lemma 12, we constructed very specific K -fat $(K_3, K_{1,3})$ -minors. In our K -fat K_3 -minor, the connected subgraphs H_1, H_2, H_3 are disks. In our K -fat $K_{1,3}$ -minor, the connected subgraphs H_1, H_2, H_3 are singletons and the paths $P_{i,0}$, $i = 1, 2, 3$, are shortest paths. Even more specific K -fat $(K_3, K_{1,3})$ -minors were obtained in [1]. It was shown that if a graph G contains no $(\geq K)$ -subdivision of K_3 as a geodesic subgraph and no $(\geq 3K)$ -subdivision of $K_{1,3}$ as a 3-quasi-geodesic subgraph, then $\text{pb}(G) \leq 18K + 2$ (see [1] for details and definitions).

From Lemma 12, we immediately get the following corollary.

Corollary 7. *If G has neither K -fat K_3 -minor nor K -fat $K_{1,3}$ -minor, then $\text{mci}(G) \leq 4K - 1$. In particular, $\text{mci}(G) \leq 4 \cdot \text{mfi}(G) + 3$.*

Proof. The first part of Corollary 7 follows from Lemma 12. For the second part, let $\text{mfi}(G) = K - 1$. Then, G has neither K -fat K_3 -minor nor K -fat $K_{1,3}$ -minor. By the first part, $\text{mci}(G) \leq 4K - 1 = 4(K - 1) + 3 = 4 \cdot \text{mfi}(G) + 3$. \square

Combining Lemma 11 and Corollary 7 with Theorem 4, we get.

Theorem 5. *For every graph G ,*

$$\begin{aligned} \text{mfi}(G) &\leq \text{mci}(G) \leq 4 \cdot \text{mfi}(G) + 3, \\ \text{mfi}(G) &\leq \text{pl}(G) \leq 16 \cdot \text{mfi}(G) + 10, \\ \frac{\text{mfi}(G)}{2} &\leq \text{pat}(G) \leq 8 \cdot \text{mfi}(G) + 5, \\ \frac{\text{mfi}(G) - 1}{2} &\leq \text{adc}(G) \leq 16 \cdot \text{mfi}(G) + 10, \\ \text{mfi}(G) &\leq \Delta(G) \leq 16 \cdot \text{mfi}(G) + 12, \\ \frac{\text{mfi}(G)}{4} &\leq \text{dsp}(G) \leq \text{dpr}(G) \leq 8 \cdot \text{mfi}(G) + 5, \\ \frac{\text{mfi}(G)}{2} &\leq \text{pcc}(G) \leq 16 \cdot \text{mfi}(G) + 10. \end{aligned}$$