

whose center vertex is x , by adding an edge vx and a pendent 4-cycle at vertex v . In that case, by Proposition 2.6, there is a hop dominating set S of H , such that $x \in S$ and $|S| \leq \frac{2(n-4)}{5}$, which implies that $S \cup \{v\}$ is a hop dominating set S of G , a contradiction. Thus such a path P exists. Note that $k \geq 2$.

Claim 3.10. $k \in \{2, 3\}$

Proof. Suppose that $k \geq 4$. Let $G' = (G - xx_1) + x_1x_4$. Note that $x_1x_2x_3x_4v_1$ is a pendent 4-cycle at x_4 in G' . Note that $|V(G')| = n$, $|E(G')| = |E(G)|$, and G' has more pendent 4-cycles than G .

Suppose that $\deg_G(x) \geq 3$. Then $\delta(G') \geq 2$ and G' has no connected component in \mathcal{B} . There is a minimum hop dominating set S' of G' such that $x_4, v, x \in S'$ and $N_G(x_4) \cap S' \neq \emptyset$ by Proposition 2.7. By the choice of G , $|S'| \leq \frac{2n}{5}$. Thus S' is a hop dominating set of G , which is a contradiction.

Suppose that $\deg_G(x) = 2$. Then let H be the connected component of G' containing x_4 . Note that $|V(H)| \leq n - 5$. There is a minimum hop dominating set S' of G' such that $x_4 \in S'$, $N_{G'}(x_4) \cap S' \neq \emptyset$ by Proposition 2.7. If S' contains x_1 , then we replace x_1 with x_3 so that S' does not have x_1 . Then $S' \cup \{v, x\}$ is a minimum hop dominating set of G , a contradiction. \square

Let $G' = G - \{x_1, \dots, x_{k-1}\}$. Suppose that $\deg_G(x) \geq 3$. Then $\delta(G') \geq 2$. If G' has no connected component in \mathcal{B} , then take a minimum hop dominating set S' of G' such that $v, x \in S'$ by Proposition 2.7, which implies that S' is a minimum hop dominating set of G , a contradiction. Thus, G' has two connected components D_1 and D_2 . We may assume that $D_1 \notin \mathcal{B}$ and $D_2 \in \mathcal{B}$. There is a hop dominating set S_1 of D_1 such that $v, x \in S_1$ and $|S_1| \leq \frac{2|V(D_1)|}{5}$ by Proposition 2.7. If $k = 3$ or $D_2 \neq C_8$, then by (3.1), there is a minimum hop dominating set S_2 of D_2 such that $|S_2| \leq \frac{2|V(D_2)| + 2(k-1)}{5}$. If $k = 2$ and $D_2 = C_8$, then we take a minimum hop dominating set S_2 of a path $D_2 - x_2$ such that $|S_2| = 3 \leq \frac{2|V(D_2)| + 2(k-1)}{5}$. Note that $S_1 \cup S_2$ is a hop dominating set of G . Then

$$|S_1 \cup S_2| \leq \frac{2|V(D_1)| + 2|V(D_2)| + 2(k-1)}{5} = \frac{2n}{5},$$

which is a contradiction.

Suppose that $\deg_G(x) = 2$. Let H be the connected component containing x_k . Then $|V(H)| \leq n - 6$. If $H \neq C_8$, then for a minimum hop dominating set S of H , we have $|S| \leq \frac{2(n-6)+2}{5}$ by (3.1), and so $S \cup \{v, x\}$ is a hop dominating set of G whose size is at most $\frac{2n}{5}$, a contradiction. Thus $H = C_8$. Then G is one of the last two graphs in Figure 6, which shows that $\gamma_h(G) \leq \frac{2n}{5}$. This completes the proof of our main theorem.

4 An open problem

It would be a natural extension to consider giving a sharp upper bound on the hop domination number for a graph with a large girth. So we propose the following open problem.