

Before turning to the proof of Theorem 3.15, we gather a few facts relating hyperplanes and relative contact complexes.

**Proposition 3.17.** *Let  $X$  be a quasi-median graph and  $\mathbb{G}$  be a collection of gated subgraphs. For a finite collection of hyperplanes  $\mathcal{J}$  of  $X$ , the following statements are equivalent:*

- $\mathcal{J}$  is a simplex of  $\text{Cont}^\Delta(X, \mathbb{G})$ ;
- there exists  $Y \in \mathbb{G}$  and  $x \in Y \cap \bigcap_{J \in \mathcal{J}} N(J)$  such that  $Y$  contains the clique of  $J$  containing  $x$  for each  $J \in \mathcal{J}$ .

Moreover, if  $\mathbb{G}$  is a star-covering collection of gated subgraphs, the above statements are equivalent to:

- for every  $x \in \bigcap_{J \in \mathcal{J}} N(J)$ , there exists  $Y \in \mathbb{G}$  that contains the clique of  $J$  containing  $x$  for each  $J \in \mathcal{J}$ .

*Proof.* Suppose  $\mathcal{J} = \{J_1, \dots, J_n\}$  is a simplex of  $\text{Cont}^\Delta(X, \mathbb{G})$ . Then there exists  $Y \in \mathbb{G}$  such that  $\{N(J_1), \dots, N(J_n)\} \cup \{Y\}$  is a collection of pairwise intersecting gated subgraphs, and the Helly property implies that  $Y \cap \bigcap_{i=1}^n N(J_i)$  is non-empty. Let  $x \in Y \cap \bigcap_{i=1}^n N(J_i)$ . Since  $J_i$  crosses  $Y$ , the clique of  $J_i$  containing  $x$  is contained in  $Y$  by Corollary 2.6. That the second and third statements imply the first one follows directly from the definitions.

Let us prove that the first statement implies the third one assuming that  $\mathbb{G}$  is star-covering. Let  $\mathcal{J} = \{J_1, \dots, J_n\}$  be a simplex of  $\text{Cont}^\Delta(X, \mathbb{G})$  and  $x \in \bigcap_{J \in \mathcal{J}} N(J)$ . The hyperplanes in  $\mathcal{J}$  are pairwise in contact, and there exists  $Z \in \mathbb{G}$  such that each  $J \in \mathcal{J}$  crosses  $Z$ . If  $x \in Z \cap \bigcap_{i=1}^n N(J_i)$ , then Corollary 2.6 implies that the clique of  $J$  containing  $x$  is contained in  $Z$  for each  $J \in \mathcal{J}$ ; in this case, it suffices to set  $Y := Z$ . Suppose that  $x \notin Z \cap \bigcap_{i=1}^n N(J_i)$  and let  $J$  be a hyperplane separating  $x$  from  $Z$ , which we can choose to be tangent to  $x$ . Let  $C$  be the clique of  $J$  containing  $x$ . Since  $\mathbb{G}$  is star-covering, there exists  $Y \in \mathbb{G}$  containing all the prisms that contain  $C$ . For every  $1 \leq i \leq n$ , let  $C_i$  be the clique of  $J_i$  containing  $x$ . Since  $J$  is necessarily transverse to  $J_i$ , Proposition 2.10 implies that  $C$  and  $C_i$  span a prism containing  $x$ , and hence  $C_i$  is contained in  $Y$ .  $\square$

The rest of the section is dedicated to the proof of Theorem 3.15. We fix a quasi-median graph  $X$  and a collection  $\mathbb{G}$  of gated subgraphs. The equivalence of the two statements of Theorem 3.15 will be proven in Lemma 3.20 by showing that for each vertex  $x \in X$ ,  $sL_{\mathbb{G}}(x)$  is homotopy equivalent to  $L_{\mathbb{G}}(x)$ .

Let  $X^\odot$  be the *perforation* of  $X$ , i.e. the space obtained from the prism-completion  $X^\square$  of  $X$  by removing a small open ball around each vertex of a fixed radius  $\epsilon < 1/2$ , if we endow  $X^\square$  with a length metric that extends the Euclidean metrics on its prisms. Given a vertex  $x \in X$ , the sphere  $S(x, \epsilon)$  can be identified with the link of  $x$  in the prism-completion  $X^\square$ . In other words, we can think of  $S(x, \epsilon)$  as the simplicial complex whose vertices are the edges of  $X$  containing  $x$  and whose simplices are given by collections of edges contained in a common prism of  $X$ .

When  $\mathbb{G}$  is prism-covering, the complex  $L_{\mathbb{G}}(x)$  naturally contains  $S(x, \epsilon)$  as a subcomplex, which allows us to define

$$X^{\mathbb{G}} := X^\odot \cup \bigcup_{x \in X} L_{\mathbb{G}}(x)$$

where each  $L_{\mathbb{G}}(x)$  is glued to  $X^\odot$  over  $S(x, \epsilon)$ .