

Proof of Theorem 3.7. First, note that f_A has a unique measure of maximal entropy η and its push forward measure $\mu = \pi_*\eta$ is a measure of maximal entropy of g_A , because π is bounded to one which guarantees $h_\mu(g_A) = h_\eta(f_A)$ (e.g. see [39]) and, hence, $h(g_A) = h(f_A) = h_\eta(f_A) = h_\mu(g_A)$ (see [25] for the definition of metric entropy).

Next let us prove the uniqueness of the measure of maximal entropy of g_A . To do so, let us fix any measure of maximal entropy $\hat{\mu}$ for g_A . Theorem 2.12 implies that g_A has the periodic specification property, which ensures the existence of a sequence $\hat{\mu}_n$ of measures supported on periodic points p_n such that $\hat{\mu}$ is the weak* limit of $\hat{\mu}_n$ (see [25, Proposition 21.8]). Let q_n be any periodic point $q_n \in \pi^{-1}(p_n)$ and let $\hat{\eta}_n$ be the measure supported on q_n . Then $\hat{\mu}_n = \pi_*(\hat{\eta}_n)$. Without loss of generality we may assume that $(\hat{\eta}_n)_{n \in \mathbb{N}}$ converge in the weak* topology to a measure $\hat{\eta}$. But then, by continuity of the push-forward operator π_* we have $\pi_*\hat{\eta} = \hat{\mu}$ and as a consequence

$$h(g_A) = h_{\hat{\mu}}(g_A) \leq h_{\hat{\eta}}(f_A) \leq h(f_A) = h(g_A).$$

This proves that $\hat{\eta} = \eta$ and consequently that $\hat{\mu} = \mu$, that is, μ is the unique measure of maximal entropy of g_A .

It remains to prove that μ is the weak* limit of μ_n . A difficulty that arise is how to lift precisely the sequence $(\mu_n)_{n \in \mathbb{N}}$ to the Torus since the sequence $(\eta_n)_{n \in \mathbb{N}}$ defined by

$$\eta_n = \frac{1}{\text{Per}_n(f_A)} \sum_{p \in P_n(f_A)} \delta_p,$$

which converge to η by Bowen's proof, does not project to $(\mu_n)_{n \in \mathbb{N}}$. Indeed, the antipodal periodic points are also in the pre-image of periodic points of g_A . An attempt would be to include the antipodal periodic points in the definition of these measures and consider the sequence $(\hat{\eta}_n)_{n \in \mathbb{N}}$ defined by

$$\hat{\eta}_n = \frac{1}{\text{Per}_n(f_A) + \text{Per}_n^-(f_A)} \sum_{p \in P_n(f_A) \cup P_n^-(f_A)} \delta_p.$$

But $\hat{\eta}_n$ still does not project to μ_n and the problem relies on the existence of the spines (points with a single pre-image). Thus, we rule out these points as follows: for each $n \in \mathbb{N}$, let

$$P_n^*(g_A) = \{p \in P_n(g_A) : \#\pi^{-1}(p) = 2\}, \quad \text{Per}_n^*(g_A) = \#P_n^*(g_A),$$

$$\text{and } \hat{\mu}_n = \frac{1}{\text{Per}_n^*(g_A)} \sum_{p \in P_n^*(g_A)} \delta_p.$$

Since there are only four points with a single pre-image, we have

$$|\text{Per}_n(g_A) - \text{Per}_n^*(g_A)| \leq 4.$$

Thus, $(\mu_n)_{n \in \mathbb{N}}$ and $(\hat{\mu}_n)_{n \in \mathbb{N}}$ converge weakly* to exactly the same measure, provided the limit exists. For each $n \in \mathbb{N}$, let

$$P_n^*(f_A) = \pi^{-1}(P_n^*(g_A)) \quad \text{and} \quad \text{Per}_n^*(f_A) = \#P_n^*(f_A) = 2 \text{Per}_n^*(g_A),$$

and note that for n sufficiently large we have

$$\frac{\text{Per}_n(f_A)}{2} \leq \text{Per}_n^*(f_A) \leq 3 \text{Per}_n(f_A).$$