

3.1 Zero Infimum Spacing

The next theorem gives a condition under which the infimum spacing is zero. Here our main result is Theorem 2, where we show that the existence of a nonzero repulsive fixed point of the spectral decimation function with multiplier, larger than the zero fixed point indicates zero infimum spacing in the spectrum.

Theorem 2. *Assume Δ admits spectral decimation with spectral decimation function R and suppose $0, \zeta > 0$ are fixed point of R with $|R'(\zeta)| > |R'(0)| > 1$. Then $\inf\{|\lambda - \lambda'| : \lambda \neq \lambda', \lambda, \lambda' \in \sigma(\Delta)\} = 0$.*

Proof. Let ϕ_ζ be the inverse branch of R with ζ in its range. Note that ζ is in the Julia set of R and is an attracting fixed point of ϕ_ζ and hence we may choose n , and $x_1, x_2 \in \sigma(\Delta_n)$ so that $\phi_\zeta^m(x_j) \rightarrow \zeta, j = 1, 2$ [HSTZ11]. Therefore, we have,

$$\begin{aligned} \left| \phi_\zeta^m(x_1) - \phi_\zeta^m(x_2) \right| &= \left| (\phi_\zeta^m(\gamma_m))' \right| |x_1 - x_2| \\ &= \left| \phi'_\zeta(\phi_\zeta^{m-1}(\gamma_m)) \right| \cdots \left| \phi'_\zeta(\gamma_m) \right| |x_1 - x_2| \end{aligned}$$

Note that ϕ_ζ is a continuous injective map and hence monotone. Therefore, it is no loss to assume, $\phi_\zeta^k(x_1) \leq \phi_\zeta^k(\gamma_m) \leq \phi_\zeta^k(x_2), \forall k$.

Thus, $\left| \phi_\zeta^k(\gamma_m) - \zeta \right| \leq \max_{j \in \{1, 2\}} \left| \phi_\zeta^k(x_j) - \zeta \right|$. Hence, by continuity of R' , given $\delta > 0$ there is N so that $m \geq k \geq N \implies |R'(\zeta)| - \delta \leq |R'(\phi_\zeta^k(\gamma_m))|$. Thus, $\frac{1}{|R'(\phi_\zeta^k(\gamma_m))|} \leq \frac{1}{(R'(\zeta) - \delta)}, \forall k \geq N$. So for $m > N$, we have,

$$\begin{aligned} \left| \phi_\zeta^m(x_1) - \phi_\zeta^m(x_2) \right| &\leq \frac{1}{|R'(\phi_\zeta^m(\gamma_m))|} \cdots \frac{1}{|R'(\phi_\zeta(\gamma_m))|} |x_1 - x_2| \\ &\leq \frac{1}{(R'(\zeta) - \delta)^{m-N}} \frac{1}{|R'(\phi_\zeta^N(\gamma_m))|} \cdots \frac{1}{|R'(\phi_\zeta(\gamma_m))|} |x_1 - x_2|. \end{aligned}$$

Now note that

$$\left| c_\Delta^{n+j+m} \phi_0^j \phi_\zeta^m(x_1) - c_\Delta^{n+j+m} \phi_0^j \phi_\zeta^m(x_2) \right| = c_\Delta^n \cdot c_\Delta^j \left| (\phi_0^j)'(\gamma_j) \right| \cdot c_\Delta^m \left| \phi_\zeta^m(x_1) - \phi_\zeta^m(x_2) \right|.$$

for some γ_j lying between $\phi_\zeta^m(x_1)$ and $\phi_\zeta^m(x_2)$. Choose δ so that $R'(0) < |R'(\zeta)| - \delta$ and observe that $c_\Delta^m \left| \phi_\zeta^m(x_1) - \phi_\zeta^m(x_2) \right| \rightarrow 0$ as $m \rightarrow \infty$.

Hence by [Shi96, Proposition 3.1], it remains to show that $c_\Delta^j \left| (\phi_0^j)'(\gamma_j) \right|$ converges to a finite number as $j \rightarrow \infty$.