

and then observing that the canonical function of  $Y$  writes as  $k(x) = \lambda e^{Ax-B|x|}$ , implying  $\tau_+^{-A} = \tau_-^A = \lambda \delta_B$ . The Dirac integration to which (5.25) reduces, produces

$$k_\rho(x) = \lambda e^{(A^2-B^2)x/2} = \lambda e^{-\theta_+\theta_-x/2}. \quad (5.27)$$

Therefore, the bilateral gamma process can be represented in law as  $Y_t = W_{T_t} + (\theta_- - \theta_+)t/2$  where  $\mathcal{L}(T_1) \sim \Gamma(\lambda, \theta_+\theta_-/2)$ . This has been known and used at least since Madan et al. (1998).

**Example 10.** *Subordinated representation of  $\text{CTS}_\alpha$  processes.* Madan and Yor (2008) through a series of propositions covering a substantial part of the article, explain how to represent a  $\text{CTS}_\alpha(\mathbb{R})$  process as a subordinated Brownian motion with drift. Based on the discussion so far, the main Proposition 2 therein trivialize. First of all notice that the assumptions in Theorem 6 are met also in the case of  $\text{CTS}_\alpha(\mathbb{R})$  processes with symmetric spherical part. Namely, assuming  $\lambda_+ = \lambda_-$  and letting again  $A = (\theta_- - \theta_+)/2$ ,  $B = (\theta_- + \theta_+)/2$ , leads to  $k(x) = x^{-\alpha} e^{Ax-B|x|}$ , so that using the Thorin measure found in Subsection 4.3 we deduce the required density relationship  $\tau_+^{-A}(y) = \tau_-^A(y) = \lambda(y-B)^{\alpha-1}(\Gamma(\alpha))^{-1} \mathbb{1}_{\{y \geq B\}}$ . It then follows from Corollary 6, and with the substitution  $w = (s-B)\sqrt{x}$ , that

$$\begin{aligned} k_\rho(x) &= \lambda \frac{e^{xA^2/2}}{\Gamma(\alpha)} \int_B^\infty e^{-\frac{s^2x}{2}} (s-B)^{\alpha-1} ds = \lambda \frac{e^{(A^2-B^2)x/2}}{\Gamma(\alpha)x^{\alpha/2}} \int_0^\infty e^{-\frac{w^2}{2}-wB\sqrt{x}} w^{\alpha-1} dw \\ &= \lambda \frac{e^{-\theta_+\theta_-x/2}}{x^{\alpha/2}} H_{-\alpha} \left( \frac{\theta_+ + \theta_-}{2} \sqrt{x} \right) \end{aligned} \quad (5.28)$$

where  $H_a$ ,  $a < 0$ , is the Hermite function, given by

$$H_a(z) = \frac{1}{\Gamma(-a)} \int_0^\infty e^{-x^2/2-xz} x^{-a-1} dx, \quad z > 0. \quad (5.29)$$

The corresponding Lévy density coincides with that determined in Madan and Yor (2008), Proposition 2.

**Example 11.** *Subordinated representation of generalized- $z$  processes.* Let  $\mu \in \text{GZD}_G(\mathbb{R})$  with Lévy measure of the form (4.20), i.e.  $c(\pm 1) = c_\pm > 0$ ,  $\sigma(\pm) = \sigma > 0$  and  $\lambda(du) = \lambda(\delta_1(du) + \delta_{-1}(du))$ ,  $\lambda > 0$ . We can rewrite again  $k(x) = e^{Ax/\sigma-B|x|/\sigma}/(1-e^{-|x|/\sigma})$ ,  $A = (c_- - c_+)/2$ ,  $B = (c_- + c_+)/2$  so we are under the assumption of Theorem 6. Based on (4.22) and (5.25), integrating term by term the series results in

$$\begin{aligned} k_\rho(x) &= \lambda e^{\frac{A^2x}{2\sigma^2}} \sum_{k=0}^\infty \exp \left( -x \frac{(k+B)^2}{2\sigma^2} \right) = \lambda \sum_{k=0}^\infty \exp \left( -x \frac{k^2 + 2Bk + B^2 - A^2}{2\sigma^2} \right) \\ &= \lambda \sum_{k=0}^\infty \exp \left( -x \frac{(k+c_+)(k+c_-)}{2\sigma^2} \right). \end{aligned} \quad (5.30)$$

The subordinator is then of the form of an infinite GGC convolution of gamma processes  $G^k = (G_t^k)_{t \geq 0}$ ,  $k \geq 0$ ,  $\mathcal{L}(G_1^k) \sim \Gamma \left( \lambda, \frac{(k+c_+)(k+c_-)}{2\sigma^2} \right)$  whose limit in law is not otherwise known.