

We are now ready to present the proof of the Bianchi-Egnell type stability for the fractional Hardy-Sobolev inequality (1.1).

Proof of Theorem 1.3. Clearly the sharpness of the result follows from the Lemma 3.2 above.

Suppose that the theorem is not true. Then we can find a sequence $(u_k)_{k \geq 1}$ in $\dot{H}^s(\mathbb{R}^N)$ such that

$$\lim_{k \rightarrow \infty} \frac{\|u_k\|_{\dot{H}^s}^2 - \mu_{s,t} \| |x|^{-\frac{t}{2_s^*(t)}} u_k \|_{2_s^*(t)}^2}{d(u_k, \mathcal{M})^2} = 0 \quad (3.8)$$

Since the ratio in (3.8) is homogeneous of degree 2 in u_k , we may also suppose that $\|u_k\|_{\dot{H}^s}^2 = \mu_{s,t}$ for all $k \geq 1$. Moreover, $0 \in \mathcal{M}$ implies that $d(u_k, \mathcal{M}) \leq d(u_k, 0) = \|u_k\|_{\dot{H}^s} = \sqrt{\mu_{s,t}} < \infty$. Thus, up to a subsequence still denoted by u_k , we see that $d(u_k, \mathcal{M}) \rightarrow L \in [0, \sqrt{\mu_{s,t}}]$.

We now consider two cases, specifically $L = 0$ and $L > 0$.

When $L = 0 < \sqrt{\mu_{s,t}}$; we can assume (if required up to a further subsequence still denoted by u_k) that for all $k \geq 1$ one has $d(u_k, \mathcal{M}) < \sqrt{\mu_{s,t}} = \|u_k\|_{\dot{H}^s}$. Lemma 3.1 now gives that the above ratio is greater than or equal to $\alpha + o(1) \rightarrow \alpha > 0$ as $k \rightarrow \infty$ and it's a contradiction to (3.8).

Thus $L > 0$ and then we must have that the numerator of the above ratio goes to 0 as $k \rightarrow \infty$, this in turn satisfies the hypothesis of Lemma 3.3. Consequently, we obtain

$$L = \lim_{k \rightarrow \infty} d(u_k, \mathcal{M}) \leq \lim_{k \rightarrow \infty} \|u_k - U_{s,t}^{\lambda_k}\|_{\dot{H}^s} = 0$$

and this gives us the desired contradiction. \square

4. PALAIS-SMALE DECOMPOSITION

Although the energy functional associated with our problem (1.3) is given as in (3.7), for purely technical reasons, we consider the following normalized functional

$$I_{s,t}(u) := \frac{1}{2} \|u\|_{\dot{H}^s}^2 - \frac{1}{2_s^*(t)} \left\| |x|^{-\frac{t}{2_s^*(t)}} u \right\|_{2_s^*(t)}^{2_s^*(t)}. \quad (4.1)$$

There is a one-to-one correspondence between the critical points of (4.1) and the critical points of (3.7). Specifically, one has that $u \in \dot{H}^s(\mathbb{R}^N)$ is a weak solution to (1.3) (i.e., a critical point of (3.7)) if and only if $\mu_{s,t}^{\frac{1}{2_s^*(t)-2}} u$ is a weak solution to the normalized Euler-Lagrange equation

$$\begin{cases} (-\Delta)^s u = \frac{|u|^{2_s^*(t)-2} u}{|x|^t} & ; u \in \dot{H}^s(\mathbb{R}^N) \\ u > 0 \text{ in } \mathbb{R}^N \end{cases} \quad (4.2)$$

(i.e., a critical point of (4.1)). Due to this one-one correspondence, all the crucial properties of the energy functional are unchanged, therefore we only analyze the normalized functional moving onward.

Definition 4.1. We say that the sequence $(u_n)_{n \geq 1} \subset \dot{H}^s(\mathbb{R}^N)$ is a Palais-Smale sequence for $I_{s,t}$ at level β ($(PS)_\beta$ condition in-short), if $I_{s,t}(u_n) \rightarrow \beta$ and $(I_{s,t})'(u_n) \rightarrow 0$ in $(\dot{H}^s(\mathbb{R}^N))'$ as $n \rightarrow \infty$. The functional $I_{s,t}$ is said to satisfy $(PS)_\beta$ condition if every $(PS)_\beta$ sequence has a convergent subsequence in $\dot{H}^s(\mathbb{R}^N)$.

It is easy to see that the weak limit of a (PS) sequence solves (4.2) except for the positivity. However, the main difficulty is that the (PS) sequence may not converge strongly and the weak limit can be zero even if $\beta > 0$. The content of this section is the classification of (PS) sequences of the functional $I_{s,t}$ which is given in the next proposition and the proof follows by arguments analogous to [4, Theorem 2.1].

Proposition 4.2. Let $(u_n)_{n \geq 1} \subset \dot{H}^s(\mathbb{R}^N)$ be a Palais-Smale sequence for $I_{s,t}$ at level β . Then up to a subsequence (still denoted by u_n) the following properties hold: