

Theorem 1.6 (Bars [5]). *The modular curve $X_0(N)$ has infinitely many points of degree 2 over \mathbb{Q} if and only if*

$$N \in \{1 - 33, 35 - 37, 39 - 41, 43, 46 - 50, 53, 59, 61, 65, 71, 79, 83, 89, 101, 131\}.$$

Theorem 1.7 (Jeon [18]). *The modular curve $X_0(N)$ has infinitely many points of degree 3 over \mathbb{Q} if and only if*

$$N \in \{1 - 29, 31, 32, 34, 36, 37, 43, 45, 49, 50, 54, 64, 81\}.$$

Theorem 1.8 (Hwang, Jeon; Derickx, Orlić [17, 11]). *The modular curve $X_0(N)$ has infinitely many points of degree 4 over \mathbb{Q} if and only if*

$$N \in \{1 - 75, 77 - 83, 85 - 89, 91, 92, 94 - 96, 98 - 101, 103, 104, 107, 111, \\ 118, 119, 121, 123, 125, 128, 131, 141 - 143, 145, 155, 159, 167, 191\}.$$

From these results, it is easy to determine the modular curves of the form $X_0(N)$ with density degree 1, 2, 3 and 4. For example, $X_0(N)$ has density degree 4 if and only if N occurs in the set of Theorem 1.8 but not in the sets of Theorems 1.6 and 1.7. Motivated by the recent work of Kadets and Vogt [23] on density degrees, we determine all curves $X_0(N)$ with density degree equal to 5. Our main result is the following theorem:

Theorem 1.9. *The modular curve $X_0(N)$ has density degree $\min(\delta(X_0(N)/\mathbb{Q}))$ equal to 5 if and only if $N = 109$.*

This main theorem is an important step in determining all curves $X_0(N)$ with infinitely many quintic points. In fact, if a curve has infinitely many quintic points, then it has to have density degree ≤ 5 . And our result completes the classification of all modular curves of density degree ≤ 5 . So, in order to determine all curves $X_0(N)$ with infinitely many quintic points, it suffices to determine for the explicit list $X_0(N)$ of density degree ≤ 4 whether they have infinitely many quintic points. With some extra work, we are able to prove the following:

Theorem 1.10. *The modular curve $X_0(N)$ has infinitely many points of degree 5 for*

$$N \in \{1 - 45, 47 - 58, 61, 63, 64, 65, 67, 68, 72, 73, 75, 80, 81, 91, 109, 121, 125\}.$$

The modular curve $X_0(N)$ has only finitely many points of degree 5 for $N \geq 192$ and

$$N \in \{46, 59, 60, 62, 66, 69, 70, 71, 76, 78, 84, 87, 90, 93, 94, 95, 97, 102, 104, 105, \\ 106, 108, 110, 112 - 117, 119, 120, 122, 124, 126, 127, 129, 130, 132 - 140, \\ 144, 146 - 154, 156, 157, 158, 160 - 166, 168 - 190\}.$$

Remark 1.11. With the above theorem, there are 30 levels N for which we do not know yet if there are infinitely many quintic points. These levels are

$$\{74, 77, 79, 82, 83, 85, 86, 88, 89, 92, 96, 98, 99, 100, 101, 103, 107, \\ 111, 118, 123, 128, 131, 141, 142, 143, 145, 155, 159, 167, 191\}.$$

The outline of the paper is as follows:

- In Section 2 we present the previously known results that will be used in later sections of the paper.