

Definition 2.9. Given $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, define $\bar{z}^h := (z_1, \dots, z_{h-1}, \bar{z}_h, z_{h+1}, \dots, z_n)$, for any $h \in \{1, \dots, n\}$. A set $D \subset \mathbb{C}^n$ is called symmetric if it is invariant with respect to complex conjugation in any variable, i.e. if $z \in D \iff \bar{z}^h \in D$, for every $h = 1, \dots, n$.

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of \mathbb{R}^n and denote with $\{e_K\}_{K \in \mathcal{P}(n)}$ a basis of \mathbb{R}^{2^n} .

Definition 2.10. Let $D \subset \mathbb{C}^n$ be an open symmetric set and consider a function $F : D \subset \mathbb{C}^n \rightarrow \mathbb{R}_m \otimes \mathbb{R}^{2^n}$, $F(z) = \sum_{K \in \mathcal{P}(n)} e_K F_K(z)$ with $F_K : D \rightarrow \mathbb{R}_m$. We call F a stem function if $F_K(\bar{z}^h) = (-1)^{|K \cap \{h\}|} F_K(z)$ or equivalently

$$F_K(\bar{z}^h) = \begin{cases} F_K(z) & \text{if } h \notin K \\ -F_K(z) & \text{if } h \in K, \end{cases} \quad (8)$$

for every $z \in D$, every $K \in \mathcal{P}(n)$ and any $h \in \{1, \dots, n\}$. Again, we use the symbol $Stem(D)$ to denote the set of stem functions $F : D \rightarrow \mathbb{R}_m \otimes \mathbb{R}^{2^n}$.

Equip \mathbb{R}^{2^n} with the family of commutative complex structures $\mathcal{J} = \{\mathcal{J}_h : \mathbb{R}^{2^n} \rightarrow \mathbb{R}^{2^n}\}_{h=1}^n$, where each \mathcal{J}_h is defined over any basis element e_K of \mathbb{R}^{2^n} as

$$\mathcal{J}_h(e_K) := (-1)^{|K \cap \{h\}|} e_{K \Delta \{h\}} = \begin{cases} e_{K \cup \{h\}} & \text{if } h \notin K \\ -e_{K \setminus \{h\}} & \text{if } h \in K, \end{cases}$$

where $K \Delta H = (K \cup H) \setminus (K \cap H)$ and extend it by linearity to all \mathbb{R}^{2^n} . \mathcal{J} induces a family of commutative complex structure on $\mathbb{R}_m \otimes \mathbb{R}^{2^n}$ (by abuse of notation, we use the same symbol) $\mathcal{J} = \{\mathcal{J}_h : \mathbb{R}_m \otimes \mathbb{R}^{2^n} \rightarrow \mathbb{R}_m \otimes \mathbb{R}^{2^n}\}_{h=1}^n$ according to the formula

$$\mathcal{J}_h(x \otimes a) := x \otimes \mathcal{J}_h(a) \quad \forall x \in \mathbb{R}_m, \quad \forall a \in \mathbb{R}^{2^n}.$$

We can associate two Cauchy-Riemann operators to each complex structure \mathcal{J}_h .

Definition 2.11. Given a stem function $F \in Stem(D) \cap \mathcal{C}^1(D)$, we define

$$\partial_h F := \frac{1}{2} \left(\frac{\partial F}{\partial \alpha_h} - \mathcal{J}_h \left(\frac{\partial F}{\partial \beta_h} \right) \right), \quad \bar{\partial}_h F := \frac{1}{2} \left(\frac{\partial F}{\partial \alpha_h} + \mathcal{J}_h \left(\frac{\partial F}{\partial \beta_h} \right) \right).$$

We call $F = \sum_{K \in \mathcal{P}(n)} e_K F_K$ h -holomorphic (with respect to \mathcal{J}) if $F \in \ker \bar{\partial}_h$ and it is called holomorphic if it is h -holomorphic for every $h = 1, \dots, n$.

We can give the definition of holomorphic stem function through a system of Cauchy-Riemann equations.

Proposition 2.3 ([18], Lemma 3.12). *Let F be a stem function. Then F is h -holomorphic if and only if*

$$\frac{\partial F_K}{\partial \alpha_h} = \frac{\partial F_{K \cup \{h\}}}{\partial \beta_h}, \quad \frac{\partial F_K}{\partial \beta_h} = -\frac{\partial F_{K \cup \{h\}}}{\partial \alpha_h}, \quad \forall K \in \mathcal{P}(n), h \notin K. \quad (9)$$

For any $J_1, \dots, J_n \in \mathbb{S}$, define

$$\phi_{J_1} \times \dots \times \phi_{J_n} : \mathbb{C}^n \ni (z_1, \dots, z_n) \mapsto (\phi_{J_1}(z_1), \dots, \phi_{J_n}(z_n)) \in (\mathbb{R}^{m+1})^n,$$

where ϕ_J is defined in (3).