

**Proposition 5.1.** *Let  $(M, \omega)$  be a closed symplectic manifold. Assume that Shelukhin's quasimorphism  $\mathfrak{S}_M$  on  $\widetilde{\text{Ham}}^c(M, \omega)$  is not extendable to  $\widetilde{\text{Symp}}_0^c(M, \omega)$ . Then  $R_0 \in H^2(\text{Symp}_0^c(M, \omega))$  is non-zero. In particular, the Reznikov class  $R \in H^2(\text{Symp}^c(M, \omega))$  is non-zero.*

Before proving the proposition, we discuss the vanishing of the Reznikov class in a more general setting. Let  $M$  be a closed symplectic manifold. Let  $\tilde{G}$  be a subgroup of  $\widetilde{\text{Symp}}_0^c(M, \omega)$  which contains  $\widetilde{\text{Ham}}^c(M, \omega)$  and set  $G = p(\tilde{G})$ , where  $p: \widetilde{\text{Symp}}_0^c(M, \omega) \rightarrow \text{Symp}_0^c(M, \omega)$  is the universal covering map. Consider the following commutative diagram:

$$\begin{array}{ccccc} \widetilde{\text{Ham}}^c(M, \omega) & \xrightarrow{\tilde{i}_0} & \tilde{G} & \xrightarrow{\tilde{i}_1} & \widetilde{\text{Symp}}_0^c(M, \omega) \\ \downarrow p & & \downarrow p & & \downarrow p \\ \text{Ham}^c(M, \omega) & \xrightarrow{i_0} & G & \xrightarrow{i_1} & \text{Symp}_0^c(M, \omega). \end{array}$$

Here  $\tilde{i}_0, \tilde{i}_1, i_0$  and  $i_1$  are the inclusions.

**Lemma 5.2.** *The following are equivalent.*

- (1)  $i_1^* R_0 \in H^2(G)$  is zero.
- (2) There exists  $\phi \in Q(G)$  such that  $\tilde{i}_0^* p^* \phi = \mathfrak{S}_M$ .

*Proof.* To prove that (1) implies (2), we assume that  $i_1^* R_0 = 0$ . Recall that  $b_J$  is a cocycle representing the Reznikov class. Then there exists  $u \in C^1(G)$  such that  $i_1^* b_J = \delta u$ . Since  $b_J$  is a bounded cocycle,  $u$  is a quasimorphism. Recall from (2.4) that  $-\delta \nu_J = p^* i_0^* i_1^* b_J$ . Hence we have

$$-\delta \nu_J = p^* i_0^* \delta u = \delta(p^* i_0^* u) = \delta(\tilde{i}_0^* p^* u).$$

This implies that  $\nu_J + \tilde{i}_0^* p^* u: \widetilde{\text{Ham}}^c(M, \omega) \rightarrow \mathbb{R}$  is a homomorphism. Because  $\widetilde{\text{Ham}}^c(M, \omega)$  is perfect ([Ban78]), we have  $\nu_J = -\tilde{i}_0^* p^* u$ . Let  $\phi$  be the homogenization of  $-u$ . Then we have  $\mathfrak{S}_M = \tilde{i}_0^* p^* \phi$ .

To prove that (2) implies (1), we assume (2) and take  $\phi \in Q(G)$  satisfying  $\tilde{i}_0^* p^* \phi = \mathfrak{S}_M$ . Since  $\mathfrak{S}_M$  is the homogenization of  $\nu_J$ , there exists a bounded function  $v: \widetilde{\text{Ham}}^c(M, \omega) \rightarrow \mathbb{R}$  such that  $\mathfrak{S}_M = \nu_J + v$ . Then we have

$$-\delta \nu_J = -\delta(\mathfrak{S}_M - v) = \delta v - \delta \tilde{i}_0^* p^* \phi = \delta v - p^* i_0^* \delta \phi.$$

Together with  $-\delta \nu_J = p^* i_0^* i_1^* b_J$ , we have

$$p^* i_0^* (i_1^* b_J + \delta \phi) = \delta v.$$

Note that  $i_1^* b_J + \delta \phi$  is a bounded cocycle on  $G$ . Since  $v$  is a bounded function, the second bounded cohomology class  $p^* i_0^* [i_1^* b_J + \delta \phi]$  of  $\widetilde{\text{Ham}}^c(M, \omega)$  is zero. Since  $G/\text{Ham}^c(M, \omega)$  is abelian, the bounded cohomology  $H_b^2(G/\text{Ham}^c(M, \omega))$  is zero. In particular, the map  $i_0^*: H_b^2(G) \rightarrow H_b^2(\text{Ham}^c(M, \omega))$  is injective by (2.2). Moreover, since  $p: \widetilde{\text{Ham}}^c(M, \omega) \rightarrow \text{Ham}^c(M, \omega)$  is surjective, the map