

Lemma 4. *The function $\det A_{n;3}$ has the same sign as δ_n in some neighborhood of the triangular boundary of the region $0 < x_1 < x_2 < 1$. Hence, it is negative in this neighborhood if $n \leq 472$ and positive si $n \geq 473$.*

The proof follows by an analysis of the dominant terms in different pieces of the boundary.

Lemma 5. *In the Newtonian case if $456 \leq n \leq 472$, the function $A_n(x, \sqrt{x})$ vanishes at least once on each of the intervals $(0, 0.098)$ and $(0.098, 1)$. If $473 \leq n \leq 874$, the function $A_n(x, \sqrt{x})$ vanishes at least once on each of the intervals $(0, 0.97)$ and $(0.97, 1)$.*

Proof of Lemma 5 We must study the roots in the interval $(0, 1)$ of the equation

$$\det A_{n;3}(x, \sqrt{x}) = \delta_n^3 + p_n(x, \sqrt{x})\delta_n + q_n(x, \sqrt{x}) = 0,$$

where

$$\begin{aligned} p_n(x, \sqrt{x}) &= -k_n(x)k_n(1/x) - 2k_n(\sqrt{x})k_n(1/\sqrt{x}) \\ q_n(x, \sqrt{x}) &= k_n(x)k_n(1/\sqrt{x})^2 + k_n(1/x)k_n(\sqrt{x})^2. \end{aligned}$$

From the analysis of $p_n(x, \sqrt{x})$ and $q_n(x, \sqrt{x})$ for x close to 0 and x close to 1 it follows that if $n \leq 472$ (resp. $n \geq 473$) the determinant $A_{n;3}(x, \sqrt{x})$ is negative (resp. positive) in a neighborhood of $x = 0$ and $x = 1$.

Hence, to prove the Lemma, it is enough to check numerically that

- for each integer $456 \leq n \leq 472$, $A_{n;3}(0.098, \sqrt{0.098})$ is positive.
- for each integer $473 \leq n \leq 874$, $A_{n;3}(0.97, \sqrt{0.97})$ is negative.

This amounts to a finite number of checks.

The question of the positivity of the masses is more involved. In fact, not all the (x_1, x_2) for which $\det A_{n;3}(x_1, x_2) = 0$ have corresponding masses such that all of them are positive.

The figure below displays curves in (x_1, x_2) for which $\det A_{n;3}(x_1, x_2) = 0$ for the admissible range of n . The curve for $n = 456$ looks like a small ellipse around $P = (0.098, \sqrt{0.098})$, while the one for $n = 874$ looks like a small ellipse around $Q = (0.97, \sqrt{0.97})$. The last curve of index 1 around P has $n = 472$, while the first curve of index 1 around Q has $n = 473$.