

ON THE STABILITY OF THE PENALTY FUNCTION FOR THE \mathbb{Z}^2 -HARD SQUARE SHIFT

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ABSTRACT. We investigate the stability of maximizing measures for a penalty function of a two-dimensional subshift of finite type, building on the work of Gonschorowski et al. [GQS21]. In the one-dimensional case, such measures remain stable under Lipschitz perturbations for any subshift of finite type. However, instability arises for a penalty function of the Robinson tiling, which is a two-dimensional subshift of finite type with no periodic point and zero entropy. This raises the question of whether stability persists in two-dimensional subshifts of finite type with positive topological entropy. In this paper, we address this question by studying the \mathbb{Z}^2 -hard square shift, a well-known example of a two-dimensional subshift with positive entropy. Our main theorem establishes that, in contrast to previous results, a penalty function of the hard square shift remains stable under Lipschitz perturbations.

1. INTRODUCTION

Ergodic optimization is the study of maximizing measures. In its most basic form, let $T : X \rightarrow X$ be a continuous map on a compact metric space X and for a continuous function $\varphi : X \rightarrow \mathbb{R}$ we consider the *maximum ergodic average*

$$\beta(\varphi) = \sup_{\mu \in \mathcal{M}_T(X)} \int \varphi d\mu$$

where $\mathcal{M}_T(X)$ is the space of T -invariant Borel probability measures on X endowed with the weak*-topology. An invariant measure which attains the maximum is called a *maximizing measure* for φ and denote by $\mathcal{M}_{\max}(\varphi)$ the set of maximizing measures for φ .

The stability of maximizing measures for a penalty function of a subshift of finite type was established by Gonschorowski et al. [GQS21]. A penalty function is defined on the forbidden set of a subshift of finite type, assigning a value of 0 to admissible local configurations near the origin and -1 otherwise (see §2 for more details). It is straightforward to see that every maximizing measure of a penalty function is supported on the given subshift of finite type. In the one-dimensional case, maximizing measures remain supported on the given subshift under Lipschitz perturbations for any subshift of finite type. However, in the two-dimensional case, there exists a subshift of finite type where this stability fails.

In [GQS21], the authors highlight the difference between one and two dimensions, demonstrating that instability arises in a penalty function of the Robinson tiling, which is a two-dimensional subshift of finite type with no