

Therefore, assuming that the first child of the root is $j \in \{1, \dots, d\}$ (equivalently, that the minimal label of $\sigma^{(j)} \sqcup \bigsqcup_{l \neq j} \sigma^{(l)}$ belongs to the j th component), and using this counting principle, one get the recursive formula

$$d(\emptyset, \sigma) = \left(\sum_{j=1}^d \frac{|\sigma^{(j)}|}{|\sigma| - 1} w_{\emptyset, j} \right) \binom{|\sigma| - 1}{|\sigma^{(1)}|, \dots, |\sigma^{(d)}|} \prod_{k=1}^d d_k(\emptyset, \sigma^{(k)}), \quad (4.17)$$

where d_k denotes the combinatorial dimension in which the weights $w_{i,ij}$ are replaced by $w_{ki,kij}$, and $\binom{n}{k_1, \dots, k_d}$ denotes the usual multinomial coefficient.

Thereafter, the proof of the lemma follows by induction. \square

By using Lemma 4.1, we obtain results similar to those in Theorem 3.1. The proof follows from simple calculations and is omitted.

Corollary 4.3. *The normalized extremal NNHF of this WBD take the form*

$$\varphi(\tau) = \prod_{i \in \tau} \alpha_i \prod_{i \in \tau^\circ} \frac{\alpha_i}{\sum_{j=1}^d \alpha_{ij} w_{i,ij}}, \quad (4.18)$$

with $\alpha_\emptyset = 1$ and $\alpha_i = \alpha_{i1} + \dots + \alpha_{id} \in [0, 1]$ for any $i \in \mathcal{S}_d$. Again, the α_i represents the asymptotic proportion of descendants of i in a regular path.

Besides, the corresponding saturated ergodic central Markov kernel is then given by

$$p(\tau, \tau \sqcup \{ij\}) = \begin{cases} \frac{\alpha_{ij} w_{i,ij}}{\sum_{k=1}^d \alpha_{ik} w_{i,ik}} \alpha_i, & \text{if } i \in \partial\tau, \\ \alpha_{ij}, & \text{if } i \notin \partial\tau \text{ and } ij \notin \tau, \end{cases} \quad (4.19)$$

for any $i \in \tau$ and $j \in \{1, \dots, d\}$.

As a consequence, we obtain the following generalization of Han's hook length formula (4.3).

Corollary 4.4. *Let $\mathcal{B}_d(n)$ be the set of all d -ary trees with n vertices. Then*

$$\sum_{\tau \in \mathcal{B}_d(n)} \frac{\prod_{v \in \tau^\circ} \frac{\sum_{j=1}^d |\tau^{(vj)}| w_j}{|\tau^{(v)}| - 1}}{\prod_{v \in \tau} |\tau^{(v)}| d^{|\tau^{(v)}| - 1}} \left(\frac{w_1 + \dots + w_d}{d} \right)^{|\partial\tau|} = \frac{1}{n!} \left(\frac{w_1 + \dots + w_d}{d} \right)^n. \quad (4.20)$$

Proof. Consider the uniform fragmentation measure case as in Section 4.1, that is $\alpha_i = d^{-|i|}$ and $w_{i,ij} \equiv w_j$ for all $i \in \mathcal{S}_d$ and $1 \leq j \leq d$.

By using (4.2) and by noting that

$$\prod_{i \in \tau^\circ} \frac{\alpha_i}{\sum_{j=1}^d \alpha_{ij} w_{i,ij}} = \left(\frac{d}{\sum_{j=1}^d w_j} \right)^{|\tau| - |\partial\tau|}, \quad (4.21)$$

we deduce the result from (4.18) and the general combinatorial identity (7.5). \square

As an example, when $d = 2$ and $n = 3$, we obtain

$$\frac{1}{6} \left(\frac{w_1 + w_2}{2} \right)^3 = \frac{1}{12} \left(\frac{w_1 + w_2}{2} \right)^3 + \frac{1}{48} \sum_{1 \leq i, j \leq 2} w_i w_j \left(\frac{w_1 + w_2}{2} \right).$$

It is much more difficult to describe the MERWs in full generality. However, drawing on the previous situation, one can construct them using appropriate IDLA-like random walks under some additional assumptions.