

dynamical delocalization result. However, assumption (1.3) is stronger than simply assuming the spectrum is AC, and this stronger assumption seems to be essentially necessary for ergodicity, see [7, Prp. 1.5] for a related result.

Some basic examples were given in [2, §3.2]: if  $\nu = 1$ , i.e. the fundamental cell is simply one vertex, then the limiting measure is uniform. This covers the continuous quantum walk  $e^{itA}$  on  $\mathbb{Z}^d$  and the triangular lattice for example. The same property is true for the hexagonal lattice and the infinite ladder, both of which have  $\nu = 2$ . It was also shown that in the cases where  $\Gamma$  is a  $1d$  strip of width 3, or a cylinder  $\mathbb{Z} \square C_4$ , then the limiting distribution is not uniform.

**1.2. Main results.** In this note we analyze further families of graphs which satisfy our assumptions, and compute the limiting average  $\langle a \rangle_p$  explicitly.

We first give the following result, extracted from [7].

**Proposition 1.2** (Case of Cartesian and Tensor Products). *Suppose  $\Gamma_0$  is a periodic graph with  $\nu = 1$  (for example  $\Gamma_0 = \mathbb{Z}^d$  or the triangular lattice), and let  $G_F$  be any finite graph with  $\nu_F = |G_F|$  vertices. Let  $\Gamma_1 = \Gamma_0 \square G_F$  be the Cartesian product,  $\Gamma_2 = \Gamma_0 \times G_F$  be the tensor product and  $\Gamma_3 = \Gamma_0 \boxtimes G_F$  be the strong product of  $\Gamma_0$  and  $G_F$ . Let  $E_{\Gamma_0}(\theta)$  be the band function of  $\Gamma_0$ ,  $(w_j)$  an orthonormal eigenbasis of  $A_{G_F}$  and  $(\mu_j)$  the corresponding eigenvalues,  $j \leq \nu_F$ . Then*

- (1) *The band functions of  $\mathcal{A}_{\Gamma_1}$  are given by  $E_j(\theta) = E_{\Gamma_0}(\theta) + \mu_j$ .*
- (2) *The band functions of  $\mathcal{A}_{\Gamma_2}$  are given by  $E_j(\theta) = \mu_j E_{\Gamma_0}(\theta)$ .*
- (3) *The band functions of  $\mathcal{A}_{\Gamma_3}$  are given by  $E_j(\theta) = (1 + \mu_j)E_{\Gamma_0}(\theta) + \mu_j$ .*
- (4) *Assumption (1.3) is satisfied for  $\mathcal{A}_{\Gamma_1}$  but not necessarily for  $\mathcal{A}_{\Gamma_2}$  or  $\mathcal{A}_{\Gamma_3}$ .*
- (5) *For each of  $\mathcal{A}_{\Gamma_1}$ ,  $\mathcal{A}_{\Gamma_2}$  and  $\mathcal{A}_{\Gamma_3}$ , we have*

$$\langle a \rangle_p = \sum_{q=1}^{\nu_F} \langle a(\cdot + v_q) \rangle \sum_{s=1}^{\nu'_F} |P_{\mu_s}(v_p, v_q)|^2,$$

where  $P_{\mu_s}(v_p, v_q) = \sum_{j: \mu_j = \mu_s} w_j(v_p) \overline{w_j(v_q)}$  is the (kernel) of the orthogonal projection for the distinct eigenvalues of the finite graph.

For example, if  $\Gamma_0 = \mathbb{Z}^d$ , then  $E_{\Gamma_0}(\theta) = 2 \sum_{i=1}^d \cos 2\pi\theta_i$  and if  $\Gamma_0$  is the triangular lattice, then  $E_{\Gamma_0}(\theta) = 2 \cos 2\pi\theta_1 + 2 \cos 2\pi\theta_2 + 2 \cos 2\pi(\theta_1 + \theta_2)$ , for  $\theta_i \in [0, 1)$ . Here  $\Gamma_{1,2,3}$  are viewed as periodic graphs with fundamental domain containing  $\nu_F$  vertices, cf. [7, Lemma 3.1, §3.4], with  $(v_i)$  the vertices of  $G_F$ .

Arguing as before, we get for these more special graphs that

$$(1.6) \quad \mu_{T, v_p + \mathbf{n}_a}^N(\mathbf{k}_a + v_q) \approx \frac{1}{N^d} \sum_{s=1}^{\nu'} |P_{\mu_s}(v_p, v_q)|^2 =: \frac{1}{N^d} d(p, q).$$

whenever (1.3) is satisfied. This gives a more satisfactory concept of a quantum limiting distribution than in [7] where quantum ergodicity was assessed by the behaviour of eigenvector bases, and it was shown in [7, §4.5] that such a limiting distribution depends on the eigenvector basis. In contrast, here the RHS of (1.6) depends only on the graph.

Our main target now is to compute the weights  $d(p, q)$  for specific finite graphs  $G_F$ . Because of point (4) above, the theorems are illustrated only for the Cartesian product, but some hold more generally. For definiteness, the reader can assume  $\Gamma_0 = \mathbb{Z}$  in all these results, which is already interesting. However, nothing changes for any  $\Gamma_0$  having a single vertex in its fundamental domain, such as  $\mathbb{Z}^d$  and the triangular lattice.

A nice simplification in the family of Cartesian products is that the limiting weight  $d(p, q)$  in (1.6) depends only on the finite graph  $G_F$ , compared to the general case (1.5), where the weight depends on the full Floquet matrix and computations become more daunting. Still, as we will see, Cartesian products already offer interesting contrasting