

with  $n$  trials and success probability  $p$ . Recall that  $\mathbb{P} = \mathbb{P}_n, \mathbb{Q} = \mathbb{Q}_n$  are two probability measures on a pair of random graphs on  $[n] = \{1, \dots, n\}$ . Denote  $\mathfrak{S}_n$  the set of permutations in  $[n]$  and  $\mu = \mu_n$  the uniform distribution on  $\mathfrak{S}_n$ . In addition, denote by  $\nu = \nu_n$  the uniform distribution on  $\{-1, +1\}^n$ . We will use the following notation conventions for graphs.

- *Labeled graphs.* Denote by  $\mathcal{K}_n$  the complete graph with vertex set  $[n]$  and edge set  $\mathcal{U}_n$ . For any graph  $H$ , let  $V(H)$  denote the vertex set of  $H$  and let  $E(H)$  denote the edge set of  $H$ . We say  $H$  is a subgraph of  $G$ , denoted by  $H \subset G$ , if  $V(H) \subset V(G)$  and  $E(H) \subset E(G)$ . We say  $\varphi : V(H) \rightarrow V(S)$  is an injection, if for all  $(i, j) \in E(H)$  we have  $(\varphi(i), \varphi(j)) \in E(S)$ . For  $H, S \subset \mathcal{K}_n$ , denote by  $H \cap S$  the graph with vertex set given by  $V(H) \cap V(S)$  and edge set given by  $E(H) \cap E(S)$ , and denote by  $S \cup H$  the graph with vertex set given by  $V(H) \cup V(S)$  and edge set given by  $E(H) \cup E(S)$ . For any graph  $H$ , denote the excess of  $H$  by  $\tau(H) = |E(H)| - |V(H)|$ . Given  $u \in V(H)$ , define  $\text{Nei}_H(u)$  to be the set of neighbors of  $u$  in  $H$ . For two vertices  $u, v \in V(H)$ , we define  $\text{Dist}_H(u, v)$  to be their graph distance. Denote the diameter of a connected graph  $H$  by  $\text{Diam}(H) = \max_{u, v \in V(H)} \text{Dist}_H(u, v)$ .
- *Graph isomorphisms and unlabeled graphs.* Two graphs  $H$  and  $H'$  are isomorphic, denoted by  $H \cong H'$ , if there exists a bijection  $\pi : V(H) \rightarrow V(H')$  such that  $(\pi(u), \pi(v)) \in E(H')$  if and only if  $(u, v) \in E(H)$ . Denote by  $\mathcal{H}$  the isomorphism class of graphs; it is customary to refer to these isomorphic classes as unlabeled graphs. Let  $\text{Aut}(H)$  be the number of automorphisms of  $H$  (graph isomorphisms to itself). For any graph  $H$ , define  $\text{Fix}(H) = \{u \in V(H) : \varphi(u) = u, \forall \varphi \in \text{Aut}(H)\}$ .
- *Induced subgraphs.* For a graph  $H = (V, E)$  and a subset  $A \subset V$ , define  $H_A = (A, E_A)$  to be the induced subgraph of  $H$  in  $A$ , where  $E_A = \{(u, v) \in E : u, v \in A\}$ . Also, define  $H_{\setminus A} = (V, E_{\setminus A})$  to be the subgraph of  $H$  obtained by deleting all edges with both endpoints in  $A$ . Note that  $E_A \cup E_{\setminus A} = E$ .
- *Isolated vertices.* For  $u \in V(H)$ , we say  $u$  is an isolated vertex of  $H$  if there is no edge in  $E(H)$  incident to  $u$ . Denote  $\mathcal{I}(H)$  as the set of isolated vertices of  $H$ .
- *Paths, self-avoiding paths and non-backtracking paths.* We say a subgraph  $H \subset \mathcal{K}_n$  is a path with endpoints  $u, v$  (possibly with  $u = v$ ), if there exist  $w_1, \dots, w_m \in [n] \neq u, v$  such that  $V(H) = \{u, v, w_1, \dots, w_m\}$  and  $E(H) = \{(u, w_1), (w_1, w_2), \dots, (w_m, v)\}$  (we allow the occurrence of multiple vertices or edges). We say  $H$  is a self-avoiding path if  $w_0, w_1, \dots, w_m, w_{m+1}$  are distinct (where we denote  $w_0 = u$  and  $w_{m+1} = v$ ), and we say  $H$  is a non-backtracking path if  $w_{i+1} \neq w_{i-1}$  for  $1 \leq i \leq m$ . Denote  $\text{EndP}(P)$  as the set of endpoints of a path  $P$ .
- *Cycles and independent cycles.* We say a subgraph  $H$  is an  $m$ -cycle if  $V(H) = \{v_1, \dots, v_m\}$  and  $E(H) = \{(v_1, v_2), \dots, (v_{m-1}, v_m), (v_m, v_1)\}$ . For a subgraph  $K \subset H$ , we say  $K$  is an independent  $m$ -cycle of  $H$ , if  $K$  is an  $m$ -cycle and no edge in  $E(H) \setminus E(K)$  is incident to  $V(K)$ . Denote  $\mathcal{C}_m(H)$  as the set of independent  $m$ -cycles of  $H$ . For  $H \subset S$ , we define  $\mathfrak{C}_m(S, H)$  to be the set of independent  $m$ -cycles in  $S$  whose vertex set is disjoint from  $V(H)$ .
- *Leaves.* A vertex  $u \in V(H)$  is called a leaf of  $H$ , if the degree of  $u$  in  $H$  is 1; denote  $\mathcal{L}(H)$  as the set of leaves of  $H$ .
- *Trees and rooted trees.* We say a graph  $T = (V(T), E(T))$  is a tree, if  $T$  is connected and has no cycles. We say a pair  $(T, \mathfrak{R}(T))$  is a rooted tree with root  $\mathfrak{R}(T)$ , if  $T$  is a tree and  $\mathfrak{R}(T) \in V(T)$ . For a rooted tree  $T$  and  $u \in V(T)$ , we define  $\text{Dep}_T(u) = \text{Dist}_T(\mathfrak{R}(T), u)$  to be