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|---|---|---|---|
| 1 | 4 | 3 | |
| 0 | 2 | 1 | |
| 0 | 3 | 0 | 1 |

Figure 6: An \mathbb{N} -labeling of $\nu = (4, 3, 3)$. The descents occur at cells $(2, 3), (3, 1), (3, 2), (3, 3)$ yielding $\text{maj}(\sigma) = 2 + 1 + 1 + 1 = 5$. The inversion triples are formed by $((3, 2), (3, 1), (3, 3)), ((1, 2), (1, 3)),$ and $((1, 2), (1, 4))$ yielding $\text{inv}(\sigma) = 3$.

The *reading order* of a diagram ν is the order on cells that goes left-to-right along rows starting from the top row and working down. The *standardization* of a labeling $\sigma : \nu \rightarrow \mathbb{N}$ with content $\alpha = (\alpha_0, \alpha_1, \dots)$, denoted by $\text{st}(\sigma)$, is an injective labeling $\sigma : \nu \rightarrow \{1, \dots, |\nu|\}$ that labels the 0's appearing in ν $1, 2, \dots, \alpha_0$ in the order that they appear in the reading order. The 1's appearing in ν are labeled $\alpha_0 + 1, \dots, \alpha_0 + \alpha_1$ in reading order and so on. A triple consists of cells with row-column coordinates of the form $u = (r, c), v = (r - 1, c)$ and $w = (r, c + k)$ for $r, c, k \geq 1$, where u, w must be cells of ν , but v is allowed to be a cell immediately below ν . Extend $\text{st}(\sigma)$ to cells immediately below ν by labeling them $-\infty$. A triple is an **inversion triple** of σ if the labels $\text{st}(\sigma)(u), \text{st}(\sigma)(w), \text{st}(\sigma)(v)$ are decreasing clockwise. The statistic $\text{inv}(\sigma)$ counts the number of inversion triples of σ .

A *descent* of σ is a cell $u = (r + 1, c)$ in ν , such that $v = (r, c)$ is also in ν and $\sigma(u) > \sigma(v)$. Letting $\text{Des}(\sigma)$ denote that set of descents we define the **major index** statistic

$$\text{maj}(\sigma) := \sum_{u \in \text{Des}(\sigma)} (\text{leg}(u) + 1),$$

where $\text{leg}(u)$ is the number of cells of ν in the same column as u and above u . See Fig 6 for an example.

Lemma 5.2.1. *The monomial coefficient $\langle [q^i t^j] \tilde{H}_{\mu[n]}, h_{\eta[n]} \rangle$ stabilizes once $n \geq \max(|\mu| + \mu_1 + |\eta| + i, |\eta| + \eta_1)$.*

Proof. Fix partitions μ, ν and i, j and let $M^{(n)}$ denote the set of being enumerated in (26). Then

$$\langle [q^i t^j] \tilde{H}_{\mu[n]}, h_{\eta[n]} \rangle = |M^{(n)}|$$

and we wish to show $|M^{(n)}|$ stabilizes.

Suppose $n \geq |\mu| + \mu_1 + |\eta| + i$ and $n \geq |\eta| + \eta_1$ so that $\mu[n]$ and $\eta[n]$ are well-defined. Define the map $\gamma_n : M^{(n)} \rightarrow M^{(n+1)}$ as follows. For $\sigma \in M^{(n)}$, shift the bottom row of the corresponding diagram to the right by one cell and insert a new cell labelled 0 in the bottom left corner. So for instance:

$$\gamma_n : \begin{array}{|c|c|c|} \hline 1 & 0 & 2 \\ \hline 0 & 0 & 4 \\ \hline \end{array} \begin{array}{|c|c|c|c|c|c|} \hline & & & 3 & 1 & 0 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline 1 & 0 & 2 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \begin{array}{|c|c|c|c|c|c|} \hline & & & 4 & 3 & 1 & 0 \\ \hline \end{array}$$

We have to check that the map is well-defined and surjective. We claim that the first $\mu_1 + 1$ entries of the bottom row of any $\sigma' \in M^{(n+1)}$ are 0. Suppose for contradiction that there was a non-zero value a among these first $\mu_1 + 1$ cells. Then there are at least $n + 1 - |\mu| - (\mu_1 + 1) = n - |\mu| - \mu_1$ entries right of a . Of these, at most $|\eta| - 1$ are allowed to be non-zero (since a is one of the $|\eta|$ non-zero values in the labeling). Hence a would contribute at least $n - |\mu| - \mu_1 - (|\eta| - 1) > i$ inversion triples formed by taking a , the phantom cell below it, and a 0 entry to the right of it. This contradicts that $\text{inv}(\sigma') = i$ for $\sigma' \in M^{(n+1)}$.

This implies that if we let σ be obtained from a $\sigma' \in M^{(n+1)}$ by deleting the bottom left cell and shifting the bottom row to the left by one, then $\sigma \in M^{(n)}$ and $\gamma_n(\sigma) = \sigma'$, which implies that γ_n is well-defined and bijective. Indeed, the first μ_1 cells of the bottom row of σ will be labeled 0 and so the status of descents and inversion triples doesn't change when going from σ' to σ . Thus, σ' and σ have the same inv and maj values. \square