

of system (29) we can see that e_n is a real and positive root of the polynomial

$$Q(s) := \alpha s^{m+1} + \alpha s - 1, \quad (30)$$

and, in view of Descartes' rule of signs, there is a unique such e_n . Then

$$e_j = \left(\prod_{k \geq j+1} \alpha_k \right) e_n, \text{ for all } j \in [n-1], \quad (31)$$

and hence $e \in \text{int}(\mathcal{B}_G)$ is unique.

Several studies (see e.g. Sanchez (2009b) and the references therein) derived conditions guaranteeing that the equilibrium $e \in \text{int}(\mathcal{B}_G)$ is globally asymptotically stable. Tyson (1975) analyzed the special case of (29) with $n = 3$. He noted that if e is locally asymptotically stable, then one may expect that all solutions converge to e , and proved that system (29) admits a periodic solution whenever e is unstable. For $n = 3$, the model can also be studied using the theory of competitive dynamical systems (Smith, 1995). The case $n = 3$ has also been analyzed using the theory of Hopf bifurcations (Woller et al., 2014). For a general n , the analysis using Hopf bifurcations becomes highly non-trivial and results exist only for special cases, e. g. under the additional assumption that all the α_i 's are equal, see Invernizzi and Treu (1991). Hastings et al. (1977) studied the general n -dimensional case and proved that, if the Jacobian of the vector field at the equilibrium has no repeated eigenvalues and at least one eigenvalue with a positive real part, then the system admits a non-trivial periodic orbit; the proof relies on the Brouwer fixed point theorem.

Our Theorem 2 allows us to prove the following result.

Corollary 3 *Consider the n -dimensional Goodwin model (29) with $n \geq 3$, and let e denote the unique equilibrium in $\text{int}(\mathcal{B}_G)$. Let $J : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}^{n \times n}$ denote the Jacobian of the vector field of the Goodwin model. Suppose that $J(e)$ has at least one eigenvalue with a positive real part. Then, for any initial condition $a \in \mathbb{R}_{\geq 0}^n \setminus \{e\}$ such that $s^-(a - e) \leq 1$, the solution $x(t, a)$ of (29) converges to a (non-trivial) periodic orbit as $t \rightarrow \infty$.*

PROOF.

The Jacobian of (29)

$$J(x) = \begin{bmatrix} -\alpha_1 & 0 & 0 & \dots & 0 & -\frac{mx_n^{m-1}}{(1+x_n^m)^2} \\ 1 & -\alpha_2 & 0 & \dots & 0 & 0 \\ 0 & 1 & -\alpha_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\alpha_{n-1} & 0 \\ 0 & 0 & 0 & \dots & 1 & -\alpha_n \end{bmatrix}$$

has the sign pattern \bar{A}_2 in (3) for all $x \in \mathbb{R}_{\geq 0}^n$, hence the system is 2-cooperative on $\mathbb{R}_{\geq 0}^n$. We