

The maximum number of cliques in disjoint copies of graphs

Zhipeng Gao^a, Ping Li^b, Changhong Lu^c, Rui Sun^{c,*}, Long-Tu Yuan^c

^a*School of Mathematics and Statistics, Xidian University, Xi'an, 710071, China.*

^b*School of Mathematics and Statistics, Shaanxi Normal University, Xi'an, Shaanxi, China.*

^c*School of Mathematical Sciences, Shanghai Key Laboratory of PMMP, East China Normal University, Shanghai 200241, China.*

Abstract

The problem of determining the maximum number of copies of T in an H -free graph, for any graphs T and H , was considered by Alon and Shikhelman. This is a variant of Turán's classical extremal problem. We show lower and upper bounds for the maximum number of s -cliques in a graph with no disjoint copies of arbitrary graph. We also determine the maximum number of s -cliques in an n -vertex graph that does not contain a disjoint union of k paths of length two when $k = 2, 3$, or $s \geq k + 2$, or n is sufficiently large, this partly confirms a conjecture posed by Chen, Yang, Yuan, and Zhang [1].

1. Introduction

A graph G is said to be H -free if it does not contain any subgraph isomorphic to H . The classical extremal function $\text{ex}(n, H)$ denoted as $\text{ex}(n, H)$, is formally defined as the maximum number of edges on an n -vertex H -free graph. This function extends naturally to a scenario where the objective is not merely to maximize the number of edges, but rather to maximize the number of copies of a specified graph T within an n -vertex graph that is H -free. In accordance with the notation introduced by Alon and Shikhelman [2], the more general function is denoted as $\text{ex}(n, T, H)$, which is commonly referred to the generalized Turán number of H . Let P_k be the path with k vertices. Obviously, $\text{ex}(n, P_2, H) = \text{ex}(n, H)$. We denote the cycle with ℓ vertices by C_ℓ and the complete graph with s vertices by K_s .

The exploration of such problems dates back to the foundational work of Erdős [3], who established the values of $\text{ex}(n, K_s, K_t)$ for any two cliques. After these seminal discoveries, a series of related findings were made, with one of the most notable being the resolution of $\text{ex}(n, C_5, C_3)$ by Hatami et al. [4], and Grzesik [5], each working independently. A wealth of additional research has further expanded our understanding of generalized extremal numbers, as evidenced by various studies referenced in the literature [6, 7, 8, 9].

Scholars have studied the generalized Turán numbers for the vertex-disjoint union of graphs. Let kH represent the collection of k disjoint copies of the graph H . Gerbner, Methuku, and Vizer [10] delved into the function $\text{ex}(n, T, kH)$, where H

*Corresponding author.

Email addresses: gaozhipeng@xidian.edu.cn (Zhipeng Gao), lp-math@snnu.edu.cn (Ping Li), chlu@math.ecnu.edu.cn (Changhong Lu), sunruicaicaicai@163.com (Rui Sun), ltyuan@math.ecnu.edu.cn (Long-Tu Yuan)