

- 1) $\Omega(t, t) = \text{id}$, $\Omega(t, t_0) = \Omega(t, s)\Omega(s, t_0)$ for all $t, s, t_0 \in [\tau, \infty)$;
 2) and for almost all $t, s \in [\tau, \infty)$ it holds that

$$\begin{aligned} \frac{\partial}{\partial t} \Omega(t, s) &= A(t)\Omega(t, s), \\ \frac{\partial}{\partial s} \Omega(t, s) &= -A(s)\Omega(t, s). \end{aligned} \quad (3)$$

A non-empty, closed, convex subset $K \subset \mathcal{H}$ is called a cone, if $\mathbb{R}_+ K \subset K$ and $K \cap (-K) = \{0\}$ hold. A cone K is called generating if any $x \in \mathcal{H}$ can be written as $x = x^+ - x^-$ for some $x^\pm \in K$. In a Hilbert space any generating cone is nonflat, that is there exists a constant $a_K > 0$, independent on x , such that in the representation $x = x^+ - x^-$ the vectors x^\pm can be chosen so that

$$\|x^\pm\| \leq a_K \|x\|. \quad (4)$$

The set $K^* = \{f \in L(\mathcal{H}; \mathbb{R}) : f(K) \subset \mathbb{R}_+\} \subset \mathcal{H}^* = \mathcal{H}$ is called adjoint cone. If K is generating, then K^* is normal, i.e., there is a constant $b_K > 0$ such that $x \in K$, $y - x \in K$ imply $\|x\| \leq b_K \|y\|$. A cone K is called selfadjoint if $K^* = K$. In a Hilbert space any selfadjoint cone is normal [13].

Definition 1: Let K be a generating and selfadjoint cone in \mathcal{H} . System (1) is called Wazewski [14] with respect to K (also called monotone [15]) if its evolution operator Ω satisfies $\Omega(t, s)K \subset K$ for all $t \geq s \geq \tau$.

Definition 2: We say that (1) is stable in K if for any $\varepsilon > 0$ and any $t_0 \geq \tau$ there is a $\delta = \delta(\varepsilon, t_0) > 0$ such that $x_0 \in B_\delta \cap K \implies \|x(t; t_0, x_0)\| < \varepsilon$, $t \geq t_0$; if, in addition, the δ can be chosen independent on t_0 , then (1) is called uniformly stable in K ;

asymptotically stable in K if it is stable in K and there is $\eta = \eta(t_0) > 0$ such that $x_0 \in B_\eta \cap K \implies \lim_{t \rightarrow \infty} \|x(t; t_0, x_0)\| = 0$; if, in addition, η can be chosen independent on t_0 , then (1) is called uniformly asymptotically stable in K .

If any of the above properties holds with \mathcal{H} on the place of K , then we drop "in K " in their definitions.

Let system (1) be decomposed in two subsystems as follows

$$\dot{x}_i = A_{ii}(t)x_i + A_{ij}(t)x_j, \quad i \neq j, \quad i, j = 1, 2, \quad (5)$$

with $x_i \in \mathcal{H}_i$, $A_{ij} \in L(H_j, H_i)$ for $i, j = 1, 2$. Let K_i be a solid selfadjoint cone in \mathcal{H}_i , then $K = K_1 \oplus K_2$ is a solid and selfadjoint cone in \mathcal{H} .

Let $\Omega_i \in C([\tau, \infty) \times [\tau, \infty); L(\mathcal{H}_i))$ denote the evolution operator of the decoupled subsystem

$$\dot{y}_i = A_{ii}(t)y_i, \quad y_i \in \mathcal{H}_i, \quad i = 1, 2 \quad (6)$$

It can be verified by definition that (1), written as (5), is a Wazewski system for the cone K , if and only if

$$\Omega_i(t, s)K_i \subset K_i \quad \text{and} \quad A_{ij}(t)K_j \subset K_i, \quad t \geq s \geq \tau. \quad (7)$$

Definition 3: An operator $O \in L(\mathcal{H}_j; \mathcal{H}_i)$ is called positive, if it satisfies $OK_j \subset K_i$ and it is denoted by $O \geq 0$.

For the evolution operator Ω_i of (6) let α_i , β_i , γ_i , $\delta_i \in C([\tau, \infty); \mathbb{R}_{>0})$ be such that

$$\begin{aligned} \|\Omega_i(t, s)\| &\leq \alpha_i(t)\beta_i(s), \quad t \geq s \geq \tau, \\ \|\Omega_i(t, s)\| &\leq (\gamma_i(t))^{-1}(\delta_i(s))^{-1}, \quad s \geq t \geq \tau, \end{aligned} \quad (8)$$

for example we can fix any $p \in (s, t)$ and take $\alpha_i(t) = \|\Omega_i(t, p)\|$, $\beta_i(s) = \|\Omega_i(p, s)\|$ due to $\Omega_i(t, s) = \Omega_i(t, p)\Omega_i(p, s)$.

Let $q_i \in C([\tau, \infty); \mathbb{R}_{>0})$ be suitable weight functions guaranteeing convergence of the next integrals for $t \in [\tau, \infty)$

$$\begin{aligned} \phi_i(t) &:= \gamma_i^2(t) \int_t^\infty q_i(s) \delta_i^2(s) ds, \\ g_i(t) &:= \beta_i^2(t) \int_t^\infty q_i(s) \alpha_i^2(s) ds. \end{aligned} \quad (9)$$

III. MAIN RESULTS

For simplicity and clearness in this work we consider the case of $r = 2$ interconnected subsystems in (5). An extension for $r \in \mathbb{N}$ will be developed elsewhere. We introduce the following notation for $1 \leq i \neq j \leq 2$: linear weighted integral gains for the interconnection (5) are defined as

$$\begin{aligned} \pi_{ii}(t_0) &= 2 \sup_{t \geq t_0} \omega_i(t) \beta_i^2(t) \\ &\times \int_t^\infty \alpha_i(s) \|A_{ji}(s)\| \beta_j(s) \int_s^\infty \frac{\alpha_i(p) \alpha_j(p) \|A_{ij}(p)\|}{\omega_i(p)} dp ds, \end{aligned} \quad (10)$$

$$\begin{aligned} \pi_{ji}(t_0) &= 2 \sup_{t \geq t_0} \omega_i(t) \beta_i^2(t) \\ &\times \int_t^\infty \alpha_i(s) \|A_{ji}(s)\| \beta_j(s) \int_s^\infty \frac{\alpha_i(p) \alpha_j(p) \|A_{ji}(p)\|}{\omega_j(p)} dp ds, \end{aligned} \quad (11)$$

where ω_i are suitable weight functions, which can help to enable convergence of the integrals. Also for $f \in C([t_0, T]; L(\mathcal{H}_i))$ we introduce the weighted norm

$$\|f\|_{\omega_i, T} := \max_{t \in [t_0, T]} \omega_i(t) \|f(t)\|. \quad (12)$$

Theorem 1: Let (1) be a Wazewski system with respect to a selfadjoint solid cone K and written as (5) with $r = 2$. Let $\alpha_i, \beta_i, \gamma_i, \delta_i$ be as in (8) and $\phi_i, g_i, q_i, \omega_i, \pi_{ij}$ as in (9), (10), (11). If the spectral radius of the matrix $\Pi(t_0) \in \mathbb{R}^{2 \times 2}$ defined by the weighted integral gains $\pi_{ij}(t_0)$ satisfies

$$r_\sigma(\Pi(t_0)) < 1 \quad (13)$$

then solutions to (1) satisfy the estimate

$$\|x(t; t_0, x_0)\| \leq 2a_K \sqrt{\frac{h(t_0, t_0)}{\phi(t)}} \exp\left(-\int_{t_0}^t \frac{q(s)}{2h(s, t_0)} ds\right) \|x_0\|, \quad (14)$$

where a_K is from (4), $q(t) := \min\{q_1(t), q_2(t)\}$, $\phi(t) := \min\{\phi_1(t), \phi_2(t)\}$ and $h(t, t_0) := \max\{h_{11}(t, t_0) +$