

Finally, we proceed to the main result of this section. We show that if  $G$  is a nontrivial  $(k, g, s)$ -multipole with  $s \leq (k - 2)g$ , then it cannot happen that  $|G| \leq b_1(k, g, s)$ . Thus, we obtain a nontrivial lower bound on the size of  $G$ .

**Theorem 9.** Let  $G$  be a nontrivial  $(k, g, s)$ -multipole for some  $k \geq 3$ ,  $g \geq 5$  and  $s \leq (k - 2)g$ . Then

$$|G| \geq b_2(k, g, s) \geq \frac{1}{2}M(k, g).$$

*Proof.* Note that  $b_2(k, g, s) \geq \frac{1}{2}M(k, g)$ . It is easy to check that the statement is true for  $k = 3$  and  $g \leq 8$  (see Table 1), for other values we can use Lemma 7 and it is sufficient to prove  $|G| \geq b_2(k, g, (k - 2)g)$  which is equivalent to  $|G| > b_1(k, g, (k - 2)g)$ . This is clearly true for  $s = 0$ . We will proceed by induction as follows. Let  $G$  be a nontrivial  $(k, g, s)$ -multipole of order  $n$  with  $0 < s \leq (k - 2)g$  and assume that Theorem 9 holds for all nontrivial  $(k', g', s')$ -multipoles of order  $n'$  with  $s' \leq (k' - 2)g'$  where  $(k', g', s', n')$  is lexicographically smaller than  $(k, g, s, n)$ . By Lemma 3 we also assume that  $G$  contains no vertex with inner degree 1.

**Case 1:  $G$  contains a vertex  $v$  with inner degree 2**

If  $k \geq 4$ , then let  $G' = G - v$ . Clearly,  $G'$  is a  $(k, g, s - (k - 4))$ -multipole of order  $n - 1$ . Graph  $G'$  cannot be trivial, otherwise either  $G$  would be trivial, or it would have girth less than  $g$  ( $\geq 5$ ). But then  $G'$  fulfils the theorem condition and as  $(k, g, s - (k - 4), n - 1)$  is lexicographically smaller than  $(k, g, s, n)$  we have  $|G'| \geq b_2(k, g, s - (k - 4))$ . By Lemma 7,  $|G'| \geq b_2(k, g, (k - 2)g)$  and then  $|G| \geq b_2(k, g, (k - 2)g) + 1$ . Thus,  $k = 3$ .

For  $k = 3$  we already resolved the cases when  $g \leq 8$ . Thus, we may assume  $g \geq 9$ . We remove the vertex  $v$  from  $G$  and join the two links formerly incident with  $v$  to produce one new link. The constructed multipole  $G'$  is clearly a nontrivial  $(3, g - 1, s - 1)$ -multipole with  $s - 1 \leq (k - 2)(g - 1)$ . By the induction hypothesis and Lemma 7 we have  $|G| \geq b_2(3, g - 1, s - 1) \geq b_2(3, g - 1, g - 1)$ . So it is sufficient to show that

$$b_2(3, g - 1, g - 1) > b_1(3, g, g)$$

We manually checked that this is true for all  $g < 14$ . For  $g \geq 14$ , by Proposition 8 it is sufficient to show

$$M(3, g - 1) - \frac{(g - 1)^2}{2} > \frac{g^2}{2}, \quad \text{or equivalently,} \quad M(3, g - 1) > g^2 - g + \frac{1}{2}. \quad (8)$$

For  $g = 2d + 1$ , Inequality (8) is equivalent to

$$M(k, 2d) = 2 \cdot 2^d - 2 > 4d^2 + 2d + \frac{1}{2}, \quad (9)$$

which is true for  $g \geq 7$ .

For  $g = 2d + 2$ , Inequality (8) is equivalent to

$$M(3, 2d + 1) = 3 \cdot 2^d - 2 > 4d^2 + 6d + \frac{11}{2}, \quad (10)$$

which holds for  $d \geq 6$ , thus  $g \geq 14$ .