

Proof. Let $B_t = \sum_{i=1}^k a_i C_{i,t}$. Then B_t is big and nef. If B_t is not ample, then there exists an irreducible curve D_t on Y_t such that $B_t \cdot D_t = 0$. Note that $D_t \neq C_{j,t}$ for any $1 \leq j \leq k$. Thus, $D_t \cdot C_{j,t} = 0$ for all j . Let E_t be the exceptional divisor of f_t and D be the image of D_t on X . By adjunction formula,

$$-2 \leq (K_{Y_t} + D_t) \cdot D_t = \left(-\sum_i C_{i,t} - E_t + D_t \right) \cdot D_t = D_t^2 - E_t \cdot D_t < -E_t \cdot D_t,$$

so $0 \leq D_t \cdot E_t < 2$. Note that $D_t \cdot E_t > 0$ because $f_t^*(\sum_i C_i) \cdot D_t = -K_X \cdot D > 0$. Furthermore, D_t and E_t intersect at smooth points of Y_t because every singular point of Y_t lies on the support of $\sum_i C_{i,t}$. Thus, $D_t \cdot E_t = 1$. This implies that

$$-K_X \cdot D = f_t^* \left(\sum_i C_i \right) \cdot D_t = \left((p+q)E_t + \sum_i C_{i,t} \right) \cdot D_t = p+q$$

and

$$f_t^* D = D_t + pqE_t$$

since $E_t^2 = -\frac{1}{pq}$. Thus, $D_t^2 = D^2 - pq$ is an integer. Since $-2 \leq D_t^2 - E_t \cdot D_t$ (by adjunction) and $D_t^2 < 0$, we have $D_t^2 = -1$ and $D^2 = pq - 1$. The equality

$$(K_{Y_t} + D_t) \cdot D_t = -2$$

also implies that D_t is a smooth rational curve on Y_t . In particular, D is smooth or has a unique singular point at x .

Next, we show that D is a (p, q) -unicspidal curve well-formed with respect to (C, x) . Let x_1, x_2 be a local (analytic) coordinate near x such that the local equation of C is $x_1 x_2 = 0$. Since D is irreducible, we can write the local equation of D as

$$c_1 x_1^\alpha + c_2 x_2^\beta + g(x_1, x_2)$$

for some $c_1, c_2 \neq 0$ and a power series g , such that the coefficients of x_1^i and x_2^j in g vanish whenever $i \leq \alpha$ and $j \leq \beta$. Then the local intersection product at x satisfies

$$(D \cdot C)_x = \alpha + \beta.$$

Let $v = \text{ord}_{E_t} \in \text{QM}(X, C)$. Then

$$v(D) = v(f_t^* D) = pq.$$

Since $v(D) \leq p\alpha, q\beta$ by the definition of a quasi-monomial valuation, we have $\alpha \geq p$ and $\beta \geq q$. However,

$$p+q = D \cdot C \geq (D \cdot C)_x = \alpha + \beta \geq p+q,$$

so $\alpha = q$ and $\beta = p$. This implies that the Newton polygon of D in x_1, x_2 coordinate is precisely the region above the line connecting $(q, 0)$ and $(0, p)$, so D is a (p, q) -unicspidal curve which is well-formed with respect to (C, x) . \square

We need the following property about the set of unicspidal curves on X well-formed with respect to (C, x) .

Lemma 3.9. *Let T denote the set of $t = \frac{q}{p}$ such that*

- *There exists a (p, q) -unicspidal curve D on X which is well-formed with respect to (C, x) ,*
- *$-K_X \cdot D = p+q$,*
- *$D^2 = pq - 1$, and*
- *E_t is a special divisor over X .*