

$\sigma(i') = 1$ for some $i' \in J$, then $\Xi_1(b_{i'}^{\sigma(i')}) = \Xi_1(\sim b_{i'}) = 0$, so $\Xi\left(g(\sigma) \wedge \bigwedge_{i \in J} b_i^{\sigma(i)}\right) = 0$. In conclusion, $\Xi(a) = \Xi_0(g(\sigma^0))$, where σ^0 is the constant map onto 0. $\square_{3.15}$

We wonder whether the results of this section are valid when restricting to fams with the uap. These are not in general. For example, let $a \subseteq \omega$ be the set of even numbers, $\mathcal{B} := \mathcal{P}(\omega)$, let \mathcal{B}_0 be the field of sets over ω generated by a and let $F \subseteq \mathcal{P}(\omega)$ be the filter of subsets of ω containing $\{0, 1\}$. Let Ξ_0 be the (unique) probability measure on \mathcal{B} such that $\Xi_0(a) = \frac{2}{3}$ (which is clearly uniformly supported). By [Theorem 3.15](#), there is a unique probability fam Ξ on $\langle \mathcal{B}_0 \cup F \rangle$ extending Ξ_0 such that $\Xi(b) = 1$ for all $b \in F$, in particular, $\Xi(\{0, 1\}) = 1$. Then, $\Xi(\{0\}) = \Xi(a \cap \{0, 1\}) = \Xi_0(a) = \frac{2}{3}$ and, naturally, $\Xi(\{1\}) = \frac{1}{3}$. However, Ξ cannot have the uap because no singleton can have measure $\frac{2}{3}$ under a probability fam with the uap (by [Theorem 2.18](#)). This also shows that [Theorem 3.8](#) is not valid in general when restricting to the uap: both Ξ_0 and Ξ_F are uniformly supported, but any common extension extends Ξ and thus cannot have the uap.

We can validate [Theorem 3.15](#) to fams with the uap under further conditions.

Theorem 3.16. *In [Theorem 3.15](#), assume that $\mathcal{B} = \mathcal{P}(X)$ and that \mathcal{B}_0 contains all the finite subsets of X . If Ξ_0 has the uap, then so does Ξ . Likewise for “uniformly supported”.*

Proof. If all finite sets have Ξ_0 -measure zero, then the same is true for Ξ . So assume that Ξ_0 is uniformly supported with support P . Since \mathcal{B}_0 contains all finite sets, all finite sets in P with positive measure must be singletons (by minimality). It is not hard to show that P is a support of Ξ , so Ξ is uniformly supported. $\square_{3.16}$

Similarly, we can validate the extension results when adding one set to the subalgebra.

Lemma 3.17. *In the context of [Lemma 3.11](#), assume that $\mathcal{B} = \mathcal{P}(X)$ and that \mathcal{C} contains all the singletons. Then, if Ξ has the uap, then Ξ' can be found with the uap. As a consequence, Ξ can be extended to a fam into K with the uap on the whole $\mathcal{P}(X)$.*

Here, “uap” can be replaced by “uniformly supported”, even more, extensions of Ξ can be found with the same support.

Proof. The result is trivial when all singletons have Ξ -measure zero. So assume that Ξ is uniformly supported. The argument in the proof of [Lemma 3.11](#), when replacing A by a support of Ξ , allows to find a uniformly supported Ξ' with the same support as Ξ . The rest follows by Zorn’s Lemma. $\square_{3.17}$

We ask whether (iii) \Rightarrow (i) of [Theorem 3.8](#) is valid when Ξ_0 and Ξ_1 have the uap for finding Ξ with the uap, under the condition that both \mathcal{B}_0 and \mathcal{B}_1 contain all singletons. If, say, Ξ_0 is free then Ξ_1 and Ξ must be free, but we do not know how to obtain an uniformly supported Ξ when Ξ_0 and Ξ_1 are uniformly supported.

Question 3.18. *Assume that \mathcal{B}_0 and \mathcal{B}_1 are fields of sets over X , containing all the singletons, and Ξ_0 and Ξ_1 are uniformly supported fams with the uap on \mathcal{B}_0 and \mathcal{B}_1 , respectively. Assume that Ξ_0 and Ξ_1 are compatible (in the sense of [Theorem 3.8 \(iii\)](#)). Is there a uniformly supported fam that extends both Ξ_0 and Ξ_1 ?*

A strategy to solve this problem may suggest a short proof of [Theorem 3.8](#).