

Lorentz spaces $L^{p,q}(R, \mu)$ are an important generalization of Lebesgue spaces, where either $p \in (1, \infty)$ and $q \in [1, \infty]$ or $p = q = 1$ or $p = q = \infty$. The corresponding r.i. function norm $\|\cdot\|_{L^{p,q}(R, \mu)}$ is defined as

$$\|f\|_{L^{p,q}(R, \mu)} = \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L^q(0, \infty)}, \quad f \in \mathcal{M}^+(R, \mu).$$

However, one needs to be more careful here. The functional $\|\cdot\|_{L^{p,q}(R, \mu)}$ is not an r.i. function norm when $1 < p < q \leq \infty$, because it is not subadditive. When $1 < p < q \leq \infty$, the functional $\|\cdot\|_{L^{p,q}(R, \mu)}$ is merely equivalent to an r.i. function norm. More precisely, the functional

$$\|f\|_{L^{(p,q)}(R, \mu)} = \|f^{**}\|_{L^{p,q}(0, \infty)}, \quad f \in \mathcal{M}^+(R, \mu),$$

is an r.i. function norm, and there are positive constants C_1 and C_2 such that

$$C_1 \|f\|_{L^{(p,q)}(R, \mu)} \leq \|f\|_{L^{p,q}(R, \mu)} \leq C_2 \|f\|_{L^{(p,q)}(R, \mu)} \quad \text{for every } f \in \mathcal{M}^+(R, \mu),$$

provided that either $p \in (1, \infty)$ and $q \in [1, \infty]$ or $p = q = \infty$. The interested reader can find more information in [6, Chapter 4, Section 4] or [33]. In view of that, we will consider $L^{p,q}(R, \mu)$ an r.i. space even when $1 < p < q \leq \infty$. Note that

$$\|\cdot\|_{L^p(R, \mu)} = \|\cdot\|_{L^{p,p}(R, \mu)} \quad \text{for every } p \in [1, \infty].$$

Furthermore, when $p \in (1, \infty)$ and $1 \leq q_1 < q_2 \leq \infty$, we have

$$L^{p,q_1}(R, \mu) \subsetneq L^{p,q_2}(R, \mu),$$

regardless of whether $\mu(R) < \infty$ or not. *Orlicz spaces* $L^A(R, \mu)$ are another very important generalization of Lebesgue spaces. The corresponding r.i. function norm $\|\cdot\|_{L^A(R, \mu)}$ is defined as

$$\|f\|_{L^A(R, \mu)} = \inf \left\{ \lambda > 0 : \int_R A\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \leq 1 \right\}, \quad f \in \mathcal{M}^+(R, \mu),$$

where $A: [0, \infty] \rightarrow [0, \infty]$ is a Young function. A function $A: [0, \infty] \rightarrow [0, \infty]$ is called a Young function if it is convex, left-continuous, vanishing at 0, and not constant on the entire interval $(0, \infty)$. For example, when $p \in [1, \infty)$, we have $\|\cdot\|_{L^p(R, \mu)} = \|\cdot\|_{L^A(R, \mu)}$ with $A(t) = t^p$, $t \geq 0$. We also have $\|\cdot\|_{L^\infty(R, \mu)} = \|\cdot\|_{L^A(R, \mu)}$ with $A(t) = \infty \cdot \chi_{(1, \infty]}(t)$, $t \geq 0$. Besides the classical textbooks [6, 53], the interested reader can find more information on the contemporary theory of Orlicz spaces and in particular Orlicz–Sobolev spaces in [16, 46].

An analogue of Fatou's lemma is at our disposal in the framework of r.i. spaces. More precisely, if $\mathcal{M}(R, \mu) \ni f_k \rightarrow f$ pointwise μ -a.e., then

$$(2.1) \quad \|f\|_{X(R, \mu)} \leq \liminf_{k \rightarrow \infty} \|f_k\|_{X(R, \mu)}.$$

With any r.i. function norm $\|\cdot\|_{X(R, \mu)}$, there is associated another r.i. function norm, $\|\cdot\|_{X'(R, \mu)}$, defined for $g \in \mathcal{M}^+(R, \mu)$ as

$$(2.2) \quad \|g\|_{X'(R, \mu)} = \sup_{\|f\|_{X(R, \mu)} \leq 1} \int_R |f(x)|g(x) d\mu(x), \quad g \in \mathcal{M}^+(R, \mu).$$

The r.i. function norm $\|\cdot\|_{X'(R, \mu)}$ is called the *associate norm* of $\|\cdot\|_{X(R, \mu)}$. The resulting r.i. space $X'(R, \mu)$ is called the *associate space*. The definition of $\|\cdot\|_{X'(R, \mu)}$ immediately gives us that the Hölder inequality

$$(2.3) \quad \int_R |f|g d\mu \leq \|f\|_{X(R, \mu)} \|g\|_{X'(R, \mu)} \quad \text{for all } f, g \in \mathcal{M}(R, \mu)$$