

Notice that $n_i = n_i^1 \wedge n_i^2$. Thus,

$$\bigwedge_{i=1}^2 n_i = \bigwedge_{i=1}^2 (n_i^1 \wedge n_i^2).$$

Since $(a_1 \wedge a_2) \wedge n_i \neq 0$ for each i , we get

$$0 \neq (a_1 \wedge a_2) \wedge n_i = (a_1 \wedge a_2) \wedge (n_i^1 \wedge n_i^2) = (a_1 \wedge n_i^1) \wedge (a_2 \wedge n_i^2).$$

This implies that $a_1 \wedge n_i^1 \neq 0$ and $a_2 \wedge n_i^2 \neq 0$. Since $a_1 \sim_\mu b_1^\downarrow$ and $a_2 \sim_\mu b_2^\downarrow$, we have

$$a_1 \wedge \left(\bigwedge_{i=1}^2 n_i^1 \right) \neq 0 \quad \text{and} \quad a_2 \wedge \left(\bigwedge_{i=1}^2 n_i^2 \right) \neq 0.$$

Thus,

$$(a_1 \wedge a_2) \wedge \left(\bigwedge_{i=1}^2 n_i \right) = (a_1 \wedge a_2) \wedge \left(\bigwedge_{i=1}^2 (n_i^1 \wedge n_i^2) \right) \neq 0.$$

Therefore, $(a_1 \wedge a_2) \sim_\mu (b_1 \wedge b_2)^\downarrow$.

(3) Let x_1 and x_2 be nonzero elements in b^\downarrow such that $x_1 \wedge x_2 \neq 0$ and $(a \wedge b) \wedge x_k \neq 0$ for each k . This implies $a \wedge (b \wedge x_k) \neq 0$ for each k . Also, observe that $\bigwedge_{k=1}^2 (b \wedge x_k) = b \wedge (x_1 \wedge x_2) = x_1 \wedge x_2 \neq 0$. Since a is a μ -element in L , this implies that

$$(a \wedge b) \wedge (x_1 \wedge x_2) = a \wedge \left(\bigwedge_{k=1}^2 (b \wedge x_k) \right) \neq 0,$$

proving the desired claim.

(4) We can assume $a \neq 0$ and $a \neq 1$, because otherwise $b = 1$ or $b = 0$ respectively, and the thesis holds. We can also assume that $c \not\leq b$, because otherwise $b \vee c = b \sim_\mu b^\uparrow$. Let $x_1 > b$ and $x_2 > b$ such that $(x_1 \wedge x_2) > b$ and $((b \vee c) \wedge x_k) > b$, for each k . The latter condition is equivalent to $b \vee (c \wedge x_k) > b$, which is equivalent to $c \wedge x_k \not\leq b$. We need to show that

$$(b \vee c) \wedge x_1 \wedge x_2 > b,$$

which is equivalent to $c \wedge x_1 \wedge x_2 \not\leq b$. Since b is a pseudo-complement of a in L , $c \wedge x_1 \wedge x_2 \not\leq b$ holds if and only if

$$a \wedge c \wedge x_1 \wedge x_2 \neq 0.$$

Now, consider the elements $a, x_1, x_2 \in L$. $a \wedge x_1 \wedge x_2 \neq 0$ because otherwise we would have $x_1 \wedge x_2 \leq b$, as b is a pseudo-complement of a . Moreover $c \wedge x_k \neq 0$ for each k , because otherwise we would have $c \wedge x_k \leq b$. Finally $c \wedge a \neq 0$, because otherwise $c \leq b$, contradicting our assumption. As c is a μ -element in L , we conclude that $c \wedge a \wedge x_1 \wedge x_2 \neq 0$, which proves the claim. \square

Remark 3.16. — An analogue of part (2) of Proposition 3.15 does not hold for upsets. Consider the frame of power set $\mathcal{P}(X)$ of the set $X = \{1, 2, 3, 4, 5\}$. We have $\{1, 2\} \sim_\mu \{1\}^\uparrow$ and $\{3, 4\} \sim_\mu \{3\}^\uparrow$, by Remark 3.12. However, it is easy to see that $\{1, 2\} \vee \{3, 4\} = \{1, 2, 3, 4\}$ is not a μ -element in $\{1, 3\}^\uparrow = \{1\} \vee \{3\}^\uparrow$. Moreover, it is also not a μ -element in $\mathcal{P}(X)$.

Proposition 3.17. — Let L be a frame, $a \in L$, and b a pseudo-complement of a in L . If $c \in L$ is maximal relative to the properties $a \leq c$ and $b \wedge c = 0$, then $a \sim_\mu c^\downarrow$.