

with  $C'' := C_{\text{Nash},d}^{-2} |||E|||^{-2-\frac{4}{d}}$ , and finally

$$\frac{1}{2} \frac{d}{dt} \|\Gamma_\Omega(t, \cdot, y)\|_{L^2(\Omega)}^2 \leq -C'' \|\Gamma_\Omega(t, \cdot, y)\|_{L^2(\Omega)}^{2+\frac{4}{d}} + \|\Gamma_\Omega(t, \cdot, y)\|_{L^2(\Omega)}^2.$$

Denoting  $q(t) := \left( \|\Gamma_\Omega(t, \cdot, y)\|_{L^2(\Omega)}^2 \right)^{-\frac{2}{d}}$ , we see therefore that

$$q'(t) \geq \frac{4}{d} (C'' - q(t)),$$

which implies

$$q(t) \geq C'' \frac{4}{d} t e^{-\frac{4}{d} t},$$

so that in the end

$$\|\Gamma_\Omega(t, \cdot, y)\|_{L^2(\Omega)}^2 \leq \left( C'' \frac{4}{d} \right)^{-\frac{d}{2}} e^{2T} t^{-\frac{d}{2}}.$$

This yields the claimed  $L^2$  bound.

For the  $L^\infty$  bound, note first that the Laplace operator is self-adjoint, so that  $\Gamma_\Omega(t, x, y) = \Gamma_\Omega(t, y, x)$ . By the semigroup property, we therefore find that

$$\begin{aligned} \Gamma_\Omega(t, x, y) &= \int_{\Omega} \Gamma_\Omega(t/2, x, z) \Gamma_\Omega(t/2, z, y) dz \\ &\leq \|\Gamma_\Omega(t/2, x, \cdot)\|_{L^2(\Omega)} \|\Gamma_\Omega(t/2, \cdot, y)\|_{L^2(\Omega)} \\ &\leq \|\Gamma_\Omega(t/2, \cdot, x)\|_{L^2(\Omega)} \|\Gamma_\Omega(t/2, \cdot, y)\|_{L^2(\Omega)}, \end{aligned}$$

and we use the already established  $L^2$  bound to conclude.

A direct interpolation between the  $L^1$  and  $L^\infty$  estimates yields the  $L^p$  estimate stated in the Lemma. Finally, the self-adjointness argument leads to the second group of estimates.  $\square$

In the context of the domains appearing in the proof of Proposition 5 (but keeping the notations of Lemma 11), we write down for the related geometry the following statement:

**Lemma 12.** *Fix  $T > 0$ . Then there exists a constant  $C_{T,d}^*$  such that for any domain*

$$\Omega := B_d(0, R) \cap \{(x', x_d) \in \mathbb{R}^d : x_d > \phi(x')\}$$

for  $R > 0$  and  $\phi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  with  $\|\nabla \phi\|_\infty \leq \frac{1}{11d}$  with  $\partial\Omega_n = B_d(0, R) \cap \{(x', x_d) \in \mathbb{R}^d : x_d = \phi(x')\}$  and  $\partial\Omega_d = \partial\Omega \setminus \partial\Omega_n$  the constant  $C_{\Omega,T,d}$  appearing in Lemma 11 is bounded by  $C_{T,d}^*$ .

*Proof.* For the domain considered in the statement, we need an extension operator  $E$  for functions  $f \in H^1(\Omega)$  such that  $f|_{\partial\Omega_d} = 0$ , for the norms  $H^1$  and  $L^1$ .

We can first extend  $f$  to  $\Omega_+ := \{(x', x_d) \in \mathbb{R}^d : x_d > \phi(x')\}$  by setting  $E_1 f = 0$  on  $\Omega_+ \setminus \Omega$ , and  $E_1 f = f$  on  $\Omega$ . Recalling that  $f|_{\partial\Omega_d} = 0$ , we see that  $\|E_1\|_{H^1 \rightarrow H^1} = 1$  and  $\|E_1\|_{L^1 \rightarrow L^1} = 1$ . We then use Lemma 10 (with  $\Omega$  in this lemma corresponding to  $\Omega_+$  here) to build the operator  $E_2$  from  $H^1(\Omega_+)$  to  $H^1(\mathbb{R}^d)$ , so that still thanks to Lemma 10,  $|||E_2 E_1||| = |||E_2||| \leq 2\sqrt{1 + \frac{1}{121d^2}}$ , and we conclude the Lemma by setting  $E := E_2 E_1$ .  $\square$

We now write down an estimate which is a direct consequence of Lemma 11, and which will be used several times in the sequel.