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2. ISOMETRIES ON HYPERBOLIC PLANE

Let \mathbb{H} be the upper half-plane model in hyperbolic geometry. The group of isometries of \mathbb{H} is given by

$$\mathrm{PSL}_2^\pm(\mathbb{R}) = \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{R}, ad - bc = \pm 1 \right\}.$$

Each isometry $g \in \mathrm{PSL}_2^\pm(\mathbb{R})$ is classified according to the absolute value of its trace (for example, see [5]):

- Case 1: $\det g = 1$ (Orientation-preserving isometries)
 - **Elliptic** if $|\mathrm{Tr} g| < 2$. In this case, g has a unique fixed point in \mathbb{H} .
 - **Parabolic** if $|\mathrm{Tr} g| = 2$. In this case, g has a unique fixed point on the real axis.
 - **Hyperbolic** if $|\mathrm{Tr} g| > 2$. In this case, g has exactly two fixed points on the real axis.
- Case 2: $\det g = -1$ (Orientation-reversing isometries)
 - **Reflection** if $\mathrm{Tr} g = 0$. In this case, g is an involution with a geodesic of fixed points.
 - **Glide-reflection** if $\mathrm{Tr} g \neq 0$. In this case, g has two fixed points on the real axis.

For each $g \in \mathrm{PSL}_2^\pm(\mathbb{R})$, the translation length t_g is defined as:

$$t_g = \inf_{z \in \mathbb{H}} d(z, gz).$$

If t_g is positive, we call g a *positive translation isometry*. For a positive translation isometry g , the absolute value of the trace of g satisfies the following:

$$|\mathrm{Tr} g| = \begin{cases} 2 \cosh \frac{t_g}{2}, & \text{if } g \text{ is hyperbolic,} \\ 2 \sinh \frac{t_g}{2}, & \text{if } g \text{ is a glide-reflection.} \end{cases} \quad (1)$$

For a positive translation isometry $g \in \mathrm{PSL}_2^\pm(\mathbb{R})$, we define A_g as the axis of g , i.e., the geodesic joining the two fixed points of g , and let ρ_{A_g} be the reflection with respect to A_g .

We state the following result from [1, Theorem 7.38.6] because it will be used later.

Theorem 2.1. [1, Theorem 7.38.6] *Let g and h be hyperbolic transformations of the hyperbolic plane and suppose that A_g and A_h intersect at a point P . Denote by θ_P the angle at P between forward direction of A_g and A_h . Then the composition $g \circ h$ is hyperbolic and*

$$\frac{1}{2} |\mathrm{Tr} gh| = \cosh \left(\frac{t_g}{2} \right) \cosh \left(\frac{t_h}{2} \right) + \sinh \left(\frac{t_g}{2} \right) \sinh \left(\frac{t_h}{2} \right) \cos(\theta_P). \quad (2)$$

Now, we generalize the above theorem to positive translation isometries. The proof of [1, Theorem 7.38.6] uses the law of cosines for the triangle formed by the three axes A_g , A_h , and A_{gh} to compute the absolute value of the trace. In this paper we compute the absolute value of the trace by matrix computations as follows.

Theorem 2.2. *Let g and h be positive translation isometries and suppose that A_g and A_h intersect at a point P . Denote by θ_P the angle at P between forward direction of A_g and A_h . Then, the following hold.*

Case 1: *If g is a glide-reflection and h is a hyperbolic element,*

$$\frac{1}{2} |\mathrm{Tr} gh| = \left| \sinh \left(\frac{t_g}{2} \right) \cosh \left(\frac{t_h}{2} \right) + \cosh \left(\frac{t_g}{2} \right) \sinh \left(\frac{t_h}{2} \right) \cos \theta_P \right|. \quad (3)$$

Case 2: *If both g and h are glide-reflections,*

$$\frac{1}{2} |\mathrm{Tr} gh| = \left| \sinh \left(\frac{t_g}{2} \right) \sinh \left(\frac{t_h}{2} \right) + \cosh \left(\frac{t_g}{2} \right) \cosh \left(\frac{t_h}{2} \right) \cos \theta_P \right|. \quad (4)$$

Proof. Without loss of generality, we may assume the axis A_g equals the imaginary axis with the intersection point $P \in A_g \cap A_h$ located at $i \in \mathbb{H}$. See Figure 1.