

As the proof will show, the value r_0 decreases as the value of L increases.

Proof. Let L , r , and p_0 be as in the statement of the lemma. We may assume that $p_0 = p_-$ is the south pole. Let ψ be the stereographic projection based at the north pole p_+ and let ϱ be its inverse. Set $r_1 := \operatorname{arccot}(L)$ and let κ denote the biLipschitz constant of the restriction of ψ to $S^2 \setminus B(p_+, r_1)$. Finally, set $r_0 := 2^{-1} \cdot \min\{r_1, L^{-1}, \kappa^{-2}\}$.

Let $\delta_r: \mathbb{C} \rightarrow \mathbb{C}$ be the scaling map given by $\delta_r(z) = r \cdot z$ and denote by $\eta: S^2 \rightarrow S^2$ the conformal diffeomorphism agreeing with $\varrho \circ \delta_r \circ \psi$ on $S^2 \setminus \{p_+\}$. It follows from (2.1) and (2.2) that

$$\eta(B(p_+, 2r_1)) = S^2 \setminus B(p_-, h(Lr)) \subset S^2 \setminus B(p_-, Lr)$$

since $h(Lr) \geq Lr$.

Now, let $p \in S^2$. We distinguish two cases. If $B(p, r_0)$ intersects $B(p_+, r_1)$ non-trivially then $B(p, r_0)$ is contained in $B(p_+, 2r_1)$ and hence, by the above, we have $\eta(B(p, r_0)) \subset S^2 \setminus B(p_-, Lr)$. If $B(p, r_0)$ does not intersect $B(p_+, r_1)$ then we have

$$\eta(B(p, r_0)) \subset B(\eta(p), r\kappa^2 r_0) \subset B(\eta(p), r)$$

since the restriction of ψ to $S^2 \setminus B(p_+, r_1)$ is κ -biLipschitz. \square

Proof of Proposition 9.2. Let $0 < \varepsilon < \varepsilon_{\mathbf{I}}$ and let $\varphi: S^2 \rightarrow X$ be an ε -indecomposable map. Define, as in the proof of Proposition 9.1,

$$\delta := \min \left\{ \frac{l_0^2}{2\pi}, \frac{\varepsilon}{10C_{\mathbf{I}}} \right\} \quad \text{and} \quad L := 2e^{\delta^{-1}k_{\mathbf{I}}(e_{\mathbf{I}}(\varphi)+10^{-1}\varepsilon)},$$

where $l_0 > 0$ is the scale up to which the isoperimetric inequality holds in X , $C_{\mathbf{I}}$ is as in (9.1), and $k_{\mathbf{I}}$ is as in (3.1). Let $0 < r_0 < L^{-1}$ be as in Lemma 9.3.

Let $u \in \Lambda(\varphi)$ be as in the statement of the proposition. For each $p \in S^2$ set

$$r(p) := \inf \left\{ r > 0 : E_{\mathbf{I}}(u|_{B(p,r)}) \geq 5^{-1}\varepsilon_{\mathbf{I}} \right\}$$

and let $\bar{r} > 0$ be the infimum of the $r(p)$ over all $p \in S^2$. We clearly have

$$E_{\mathbf{I}}(u|_{B(p,\bar{r})}) \leq \frac{\varepsilon_{\mathbf{I}}}{5}$$

for every $p \in S^2$ and there exists $\bar{p} \in S^2$ such that equality holds for $p = \bar{p}$. If $\bar{r} \geq r_0$ then the proposition holds with η being the identity mapping, so we may assume that $\bar{r} < r_0$. We claim that

$$(9.2) \quad E_{\mathbf{I}}(u|_{B(\bar{p}, L\bar{r})}) > E_{\mathbf{I}}(u) - \frac{\varepsilon_{\mathbf{I}}}{5}.$$

The proposition easily follows from this together with Lemma 9.3. Indeed, let $\eta: S^2 \rightarrow S^2$ be as in the lemma applied with $p_0 = \bar{p}$ and $r = \bar{r}$. Then for every $p \in S^2$ we have

$$\eta(B(p, r_0)) \subset B(\eta(p), \bar{r}) \quad \text{or} \quad \eta(B(p, r_0)) \cap B(\bar{p}, L\bar{r}) = \emptyset.$$

In the first case we obtain

$$E_{\mathbf{I}}(u \circ \eta|_{B(p,r_0)}) \leq E_{\mathbf{I}}(u|_{B(\eta(p), \bar{r})}) \leq \frac{\varepsilon_{\mathbf{I}}}{5}$$

and in the second case

$$E_{\mathbf{I}}(u \circ \eta|_{B(p,r_0)}) \leq E_{\mathbf{I}}(u|_{S^2 \setminus B(\bar{p}, L\bar{r})}) \leq \frac{\varepsilon_{\mathbf{I}}}{5}.$$

This establishes the proposition assuming (9.2).