

polynomial may not achieve the maximum allowable number of roots given by the Fundamental Theorem of Algebra, but this bound is sharp, in the sense that given a natural number  $n$  there exist a polynomial of degree  $n$  which has exactly  $n$  zeros counting multiplicities.

**Theorem 2.4.** (Wilmschurst [14]) *Let  $h(z)$  and  $\overline{g(z)}$  be analytic polynomials of degree  $n$  and  $m$  respectively with  $n > m$ . Then  $f(z) = h(z) + \overline{g(z)}$ , has at most  $n^2$  zeros counting multiplicities. The case  $n = m$  could have infinite number of zeros.*

It was shown by Bshouty *et al.* [3] that there exists a complex-valued harmonic polynomial  $f = h + \overline{g}$ , such that  $h$  is an analytic polynomial of degree  $n$ ,  $g$  is an analytic polynomial of degree  $m \leq n$  and  $f$  has exactly  $n^2$  zeros counting with multiplicities in the field of complex numbers,  $\mathbb{C}$ .

## 3 Main Results

In this section we state and prove the main results of the paper. The first part of this section is results regarding products of complex-valued harmonic functions; the second part is results about the boundary values of the factors and products; and the third part is an auxiliary result concerning the the maximum number of zeros of the product, provided the product and factors are complex-valued harmonic polynomials.

### 3.1 Product of harmonic functions

We know that the square of real harmonic function  $u$  cannot be harmonic, unless  $u$  is a constant. Of course the square of analytic function is analytic and hence harmonic as any analytic function is complex-valued harmonic function. Similarly the square of anti-analytic function is anti-analytic and hence complex-valued harmonic function. But for the complex-valued harmonic function which is neither analytic nor anti-analytic we have the following result

**Theorem 3.1.** *Suppose  $f(z) = h(z) + \overline{g(z)}$ , is harmonic on  $\Omega$ . Then  $f^2$ , is harmonic on  $\Omega$  if and only if either  $h$  or  $g$  is a constant on  $\Omega$ .*

*Proof.* If either  $h$  or  $g$  is a constant on  $\Omega$ , then it easy to see that the square of analytic function or the square of co-analytic function is harmonic on  $\Omega$ . Assume that the square of a complex-valued harmonic function is harmonic on  $\Omega$ . Then

$$f^2 = (h + \overline{g})^2 = h^2 + \overline{g}^2 + 2h\overline{g}$$

is harmonic on  $\Omega$  implies that,

$$\frac{\partial^2 f^2}{\partial z \partial \overline{z}} = 0.$$

From which we obtain,

$$\frac{\partial^2 f^2}{\partial z \partial \overline{z}} = 2 \frac{\partial h}{\partial z} \frac{\partial \overline{g}}{\partial \overline{z}} = 0,$$

This on turn implies that either of the following two is satisfied on  $\Omega$ ,

$$\frac{\partial h}{\partial z} = 0, \quad \frac{\partial \overline{g}}{\partial \overline{z}} = 0.$$

Hence either  $h$  or  $\overline{g}$  is a constant  $\Omega$ . Which is the desired result. □