

**Proposition 3.6** ([15, Corollary 1.4]). *Consider the assumption in Theorem 3.5. Then,  $\text{Sel}_p^\pm(E/K_\infty)$  has no proper  $\Lambda$ -submodules with finite index.*

By the above proposition and a property of the Fitting ideal (cf. [10, Lemma A.7]), we have

$$\text{Fitt}_\Lambda(\text{Sel}_p^\pm(E/K_\infty)^\vee) = \text{char}_\Lambda(\text{Sel}_p^\pm(E/K_\infty)^\vee). \quad (3.1)$$

Last, we prove the control theorem for  $\text{Sel}_p^\pm(E/K_\infty)$  in our situation by the similar arguments to [1]. We prepare the following lemma for the proof. Let  $w$  be a prime of  $K_\infty$  above  $p$ .

**Lemma 3.7.** *The natural map  $f_n^\pm : H_\pm^1(K_{n,p}, E[p^\infty]) \rightarrow H_\pm^1(K_{\infty,w}, E[p^\infty])[\omega_n^\pm]$  is injective, and the cokernel of  $f_n^\pm$  is a finite group for any positive integer  $n$ . Here, we define  $H_\pm^1(K_{\infty,w}, E[p^\infty]) := \varinjlim H_\pm^1(K_{n,p}, E[p^\infty])$ .*

*Proof.* Note that we have  $E[p^\infty]^{G_{K_\infty,p}} = 0$  by [15, Proposition 3.2]. By Inflation-Restriction exact sequence, we see that the canonical map

$$f_n : H^1(K_{n,p}, E[p^\infty]) \rightarrow H^1(K_{\infty,p}, E[p^\infty])^{\text{Gal}(K_{\infty,p}/F_{n,p})} = H^1(K_{\infty,p}, E[p^\infty])[\omega_n]$$

is injective. Since the map  $f_n^\pm$  is the restriction of  $f_n$  to  $H_\pm^1(K_{n,p}, E[p^\infty])$ ,  $f_n^\pm$  is also injective.

For the claim for Coker  $f_n^\pm$ , it suffices to show that both the  $\mathbb{Z}_p$ -coranks of  $H_\pm^1(K_{n,p}, E[p^\infty])$  and  $H_\pm^1(K_{\infty,w}, E[p^\infty])[\omega_n^\pm]$  are same. Since we have  $H_\pm^1(K_{\infty,w}, E[p^\infty])^\vee \simeq \Lambda^2$  as  $\Lambda$ -module by the Rubin conjecture, we see that

$$\begin{aligned} H_\pm^1(K_{\infty,w}, E[p^\infty])[\omega_n^\pm]^\vee &\simeq (H_\pm^1(K_{\infty,w}, E[p^\infty]))^\vee / \omega_n^\pm (H_\pm^1(K_{\infty,w}, E[p^\infty]))^\vee \\ &\simeq \Lambda^2 / \omega_n^\pm \Lambda^2. \end{aligned}$$

On the other hand, we have  $H_\pm^1(K_{n,p}, E[p^\infty]) = \widehat{E}(K_{n,p})^\pm \otimes (\mathbb{Q}_p / \mathbb{Z}_p)$  by the definition. Thus, we see that  $H_\pm^1(K_{n,p}, E[p^\infty])^\vee \simeq \Lambda_n^2 / \omega_n^\pm \Lambda_n^2$  by Proposition 3.4. Therefore, the  $\mathbb{Z}_p$ -rank of  $H_\pm^1(K_{n,p}, E[p^\infty])$  and  $H_\pm^1(K_{\infty,w}, E[p^\infty])[\omega_n^\pm]$  are the same.  $\square$

**Proposition 3.8.** *Consider the assumption in Theorem 3.5. Then, the canonical homomorphism*

$$\text{Sel}_p^\pm(E/K_\infty)[\omega_n^\pm] \rightarrow \text{Sel}_p^\pm(E/K_n)[\omega_n^\pm]$$

*is injective, and the order of the cokernel is finite for any  $n$ .*

*Proof.* We take the finite subset  $\Sigma = \{p\} \cup \{\text{bad primes of } E\}$ . Then, we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Sel}_p^\pm(E/K_n)[\omega_n^\pm] & \longrightarrow & H^1(K_\Sigma/K_n, E[p^\infty])[\omega_n^\pm] & \xrightarrow{a} & \prod_{\substack{v_n|v \\ v \in \Sigma}} \frac{H^1(K_{n,v_n}, E[p^\infty])[\omega_n^\pm]}{H_\pm^1(K_{n,v_n}, E[p^\infty])}, \\ & & \downarrow s_n^\pm & & \downarrow h_n^\pm & & \downarrow g_n^\pm = \prod g_{n,v_n}^\pm \\ 0 & \longrightarrow & \text{Sel}_p^\pm(E/K_\infty)[\omega_n^\pm] & \longrightarrow & H^1(K_\Sigma/K_\infty, E[p^\infty])[\omega_n^\pm] & \longrightarrow & \prod_{\substack{v_\infty|v \\ v \in \Sigma}} \frac{H^1(K_{\infty,v_\infty}, E[p^\infty])[\omega_n^\pm]}{H_\pm^1(K_{\infty,v_\infty}, E[p^\infty])[\omega_n^\pm]}. \end{array}$$

Since we have  $E[p^\infty]^{G_{K_\infty}} = 0$ , the map  $H^1(K_\Sigma/K_n, E[p^\infty]) \rightarrow H^1(K_\Sigma/K_\infty, E[p^\infty])[\omega_n]$  induced by the restriction map is isomorphism by the Inflation-Restriction exact sequence. Therefore,  $h_n^\pm$  is also isomorphism. By the snake lemma,  $s_n^\pm$  is injective, and it suffices to calculate the kernel of  $g_n^\pm$ .

We first consider the case  $v \in \Sigma \setminus \{p\}$ . Then,  $K_{\infty,v_\infty}/K_{n,v_n}$  is the trivial extension or the unramified  $\mathbb{Z}_p$ -extension. If the extension  $K_{\infty,v_\infty}/K_{n,v_n}$  is trivial, then it is clear that  $\text{Ker } g_{n,v_n}^\pm = 0$ . Assume that  $K_{\infty,v_\infty}/K_{n,v_n}$  is the unramified  $\mathbb{Z}_p$ -extension. Write  $B_{v_\infty} := E[p^\infty]^{G_{K_\infty,v_\infty}}$ . We consider the exact sequence

$$0 \longrightarrow H^1(K_{\infty,v_\infty}/K_{n,v_n}, B_{v_\infty}) \longrightarrow H^1(K_{n,v_n}, E[p^\infty]) \longrightarrow H^1(K_{\infty,v_\infty}, E[p^\infty])$$