

verified that  $g$  is an eub for  $\vec{f} \restriction \alpha_\xi$ . But then  $g$  witnesses that  $\alpha_\xi$  is good for  $\vec{f}$ . Moreover, by construction,  $\alpha_\xi \in C$ . Since  $C$  was arbitrary, we have shown that there are stationarily many elements of  $S_\lambda^{\lambda^{+3}}$  that are good for  $\vec{f}$ .

By Theorem 12, it follows that there is an eub  $h$  for  $\vec{f}$  such that  $\text{cf}(h(i)) > \lambda$  for all  $i < \omega$ .

**Claim 14.**  $\text{cf}(h(i)) \geq \lambda^{+3}$  for all but finitely many  $i < \omega$ .

*Proof.* If not, then there exist  $k \in \{1, 2\}$  and an unbounded  $A \subseteq \omega$  such that, for all  $i \in A$ , we have  $\text{cf}(h(i)) = \lambda^{+k}$ . For each  $i \in A$ , let  $\{\delta_\eta^i : \eta < \lambda^{+k}\}$  enumerate, in increasing fashion, a set of ordinals cofinal in  $h(i)$ . For each  $\eta < \lambda^{+k}$ , define a function  $h_\eta$  from  $\omega$  to the ordinals by letting  $h_\eta(i) = \delta_\eta^i$  if  $i \in A$  and  $h_\eta(i) = 0$  otherwise. For each  $\eta < \lambda^{+k}$ , we have  $h_\eta <^* h$ , so, since  $h$  is an eub for  $\vec{f}$ , there is  $\beta_\eta < \lambda^{+3}$  such that  $h_\eta <^* f_{\beta_\eta}$ . Let  $\gamma = \sup\{\beta_\eta : \eta < \lambda^{+k}\}$ . Since  $k < 3$ , we have  $\gamma < \lambda^{+3}$ . Therefore, for all  $\eta < \lambda^{+k}$ , we have  $h_\eta <^* f_\gamma$ . Fix an unbounded  $B \subseteq \lambda^{+k}$  and an  $n < \omega$  such that, for all  $\eta \in B$ , we have  $h_\eta <_n f_\gamma$ . But then, for all  $i \in A \setminus n$ , we must have  $f_\gamma(i) \geq \sup\{\delta_\eta^i : \eta \in B\} = h(i)$ , contradicting the fact that  $h$  is an upper bound for  $\vec{f}$ .  $\square$

But this claim immediately contradicts the fact that  $\vec{f}$  is a sequence of functions from  $\omega$  to  $\epsilon$  and  $\epsilon < \lambda^{+3}$ . This is because, by the claim, we must have  $h(i) > \epsilon$  for all but finitely many  $i < \omega$ . But then the constant function, taking value  $\epsilon$ , witnesses that  $h$  fails to be an eub.  $\square$

The results in this section lead to the following corollary.

**Corollary 15.** *Suppose that  $3 \leq n < \omega$ .*

1. *If  $\eta < \omega_{n+1}$ , then there is no strongly increasing sequence  $\langle f_\alpha : \alpha < \omega_{n+1} \rangle$  of functions from  $\omega$  to  $\eta$ .*
2.  $(\aleph_{\omega+1}, \aleph_\omega) \not\rightarrow (\aleph_{n+1}, \aleph_n)$ .
3. *There are no inner models  $V \subseteq W$  of ZFC such that  $(\aleph_{\omega+1})^V = (\aleph_{n+1})^W$ .*

It also follows that the only regular cardinals that can possibly be lengths of strongly increasing sequences from  $\omega^\omega$  are  $\aleph_n$  for  $0 \leq n \leq 3$ . We have seen that there are always such sequences of length  $\aleph_0$  and  $\aleph_1$ . We will prove, in Section 3, the consistency of the existence of a strongly increasing sequence of length  $\aleph_2$ . The question about the consistency of the existence of a strongly increasing sequence of length  $\aleph_3$  remains open.

### 3 Consistency via a $\mathbb{P}_{\max}$ variation

In this section we use a natural variation of Woodin's partial order  $\mathbb{P}_{\max}$  to produce a very strongly increasing sequence in  $\omega^\omega$  of length  $\omega_2$ .

We refer the reader to [14] for background on  $\mathbb{P}_{\max}$ , especially Chapter 4 and Section 9.2. The article [8] may also be helpful. Conditions in our partial order  $\mathbb{P}$  are triples  $(M, F, a)$  such that