

Lemma 3.1. Assume that $T \in \mathcal{T}_h$ and $S \in \mathcal{S}_h$ satisfy $\emptyset \neq T \cap \Gamma_h \subset S$. Then we have

$$\|\nabla^m(v_h - \mathring{I}_h v_h)\|_{L^p(T)} \leq Ch_T^{1/p-m} \|v_h\|_{L^p(S)} \quad \forall v_h \in V_h,$$

where $m = 0, 1, \dots$ and $1 \leq p \leq \infty$.

We recall from [3, Theorem 5] standard interpolation error estimates for I_h :

Lemma 3.2. Let $l, m \in \mathbb{N}$ satisfy $0 \leq l \leq m \leq k+1$ and $p \in [1, \infty]$. Assume the embedding $W^{m,p} \hookrightarrow C^0$ holds for subsets in \mathbb{R}^N . Then we have

$$\|v - I_h v\|_{W^{l,p}(T)} \leq Ch^{m-l} \|v\|_{W^{m,p}(T)} \quad (T \in \mathcal{T}_h, v \in W^{m,p}(T)).$$

Estimates for \mathring{I}_h are, however, more involved because of domain perturbation ($u = 0$ on Γ does not necessarily imply $\tilde{u} = 0$ on Γ_h). We state it in the following form, whose proof is similar to that of Proposition 5.1 below (we only have to consider global Ω_h and set $v_2 = 0$ there) and thus omitted here.

Proposition 3.1. Under the assumptions of Lemma 3.2, let $m \geq 2$ and $v \in W^{m,p}(\tilde{\Omega})$ satisfy $v = 0$ on Γ . Then we have

$$\left(\sum_{T \in \mathcal{T}_h} \|\nabla^l(v - \mathring{I}_h v)\|_{L^p(T)}^p \right)^{1/p} \leq Ch^{m-l} \|v\|_{W^{m,p}(\Omega_h)} + Ch^{k+1-l} \|\nabla^2 v\|_{L^p(\Gamma(\delta))},$$

with the obvious modification for $p = \infty$.

4. REDUCTION TO $W^{1,1}$ -ANALYSIS OF A REGULARIZED GREEN FUNCTION

Fixing arbitrary $K \in \mathcal{T}_h$ and $z \in K$, we try to bound the pointwise error $\tilde{u}(z) - u_h(z)$. We construct a regularized delta function; the proof is given in the appendix.

Proposition 4.1. For $K \in \mathcal{T}_h$ and $z \in K$, there exists $\eta = \eta_{K,z} \in C_0^\infty(K)$ such that $\text{dist}(\text{supp } \eta, \partial K) \geq Ch_K$, $\|\nabla^m \eta\|_{L^\infty(K)} \leq Ch_K^{-N-m}$ ($m = 0, 1$), and

$$(v_h, \eta)_K = v_h(z) \quad \text{for } v_h = \hat{v}_h \circ \mathbf{F}_K^{-1} \text{ with arbitrary } \hat{v}_h \in \mathbb{P}_k(\hat{T}),$$

where the constant C is independent of K , z , and h_K .

Next we introduce a “dyadic decomposition” of Ω_h . We set a sequence of scales:

$$d_0 = Lh, \quad d_j = 2^j d_0 \quad \text{for } j = 1, \dots, J := \left\lceil \frac{\log(\text{diam } \Omega_h / d_0)}{\log 2} \right\rceil,$$

where L means the ratio of the “initial stride” d_0 to the “minimum scale” h . As we see later, L will be taken sufficiently large (but independently of h). Then we define a subset $\Omega_{h,j}$ of Ω_h —which has the scale d_j in terms of the distance from K —by

$$\begin{aligned} \Omega_{h0} &= \bigcup \{T \in \mathcal{T}_h \mid d(T, K) \leq d_0\}, \\ \Omega_{h,j} &= \bigcup \{T \in \mathcal{T}_h \mid d_{j-1} < d(T, K) \leq d_j\} \quad (j = 1, \dots, J), \end{aligned}$$

where $d(T, T') = \min\{|\mathbf{x} - \mathbf{x}'| \mid \mathbf{x} \in T, \mathbf{x}' \in T'\}$ denotes a distance function between two elements $T, T' \in \mathcal{T}_h$. They are compatible with a standard ball $B(\mathbf{z}; r) = \{\mathbf{x} \mid |\mathbf{x} - \mathbf{z}| \leq r\}$ and annulus $A(\mathbf{z}; r, R) = \{\mathbf{x} \mid r \leq |\mathbf{x} - \mathbf{z}| \leq R\}$. In fact, by triangle inequalities, combined with $d_J \geq \text{diam } \Omega_h$ and $\text{diam } T \leq Ch$ for $T \in \mathcal{T}_h$, we obtain

$$\begin{aligned} \Omega_h &= \bigcup_{j=0}^J \Omega_{h,j} \quad (\text{disjoint union}), \quad \Omega_{h0} \subset \Omega_h \cap B(\mathbf{z}; 2d_0), \\ \Omega_{h,j} &\subset \Omega_h \cap A_j^{(s)} \subset \Omega_{h,j-1} \cup \Omega_{h,j} \cup \Omega_{h,j+1} =: \Omega'_{h,j} \quad (j \geq 1), \end{aligned}$$

where $A_j^{(s)} := A(\mathbf{z}; (1 - \frac{s}{2})d_{j-1}, (1 + s)d_j)$ for all $s \in (0, 1)$, provided that L is sufficiently large. We also remark that $\sum_{j=\ell_1}^{\ell_2} d_j^\alpha$ is bounded by $Cd_{\ell_1}^\alpha$ if $\alpha < 0$, by $C|\log d_0|$ if $\alpha = 0$, and by $Cd_{\ell_2}^\alpha$ if $\alpha > 0$, for $0 \leq \ell_1 \leq \ell_2 \leq J$.

Now let us start the first part of the proof of Theorem 1.1. For any $v_h \in \mathring{V}_h$ we use the regularized delta function η constructed in Proposition 4.1 to get

$$(\tilde{u} - u_h)(z) = (\tilde{u} - v_h)(z) + (v_h - \tilde{u}, \eta)_{\Omega_h} + (\tilde{u} - u_h, \eta)_{\Omega_h}.$$

The first two terms on the right-hand side are bounded by $C\|\tilde{u} - v_h\|_{L^\infty(K)}$. To address the last term we define a regularized Green function $g \in W^{3,\infty}(\Omega)$ by solving

$$-\Delta g = \eta \quad \text{in } \Omega, \quad g = 0 \quad \text{on } \Gamma,$$

where η is extended by zero outside $\text{supp } \eta \subset \Omega$ (this inclusion holds if h is small). We also utilize its finite element approximation $g_h \in \mathring{V}_h$ obtained by solving

$$a_h(v_h, g_h) = (\nabla v_h, \nabla g_h)_{\Omega_h} = (v_h, \eta)_{\Omega_h} \quad \forall v_h \in \mathring{V}_h.$$