

Proof. The result follows from the observation that

$$\mathrm{rk}(\mathbb{1}_A^{C_0 \oplus C_1}) = \mathrm{rk}(\mathbb{1}_A^{C_0} \oplus \mathbb{1}_A^{C_1}) \geq \mathrm{rk}(\mathbb{1}_A^{C_0}),$$

for any measurable subset A of X . \square

Observe that $\mathcal{A}_\kappa(\mathcal{X})$ does not necessarily contains a faithful \mathcal{X} -module. For instance, if \mathcal{X} has \aleph_1 coarsely connected components, no coarse \mathcal{X} -module of rank \aleph_0 can be faithful. However, it is possible to fully characterise the cardinals for which the approximable category contains a faithful module.

Lemma 2.10. Let \mathcal{X} be a LFCM space with an infinite discrete partition. For any two discrete partitions $\{A_i\}_{i \in I}$ and $\{B_j\}_{j \in J}$, the cardinalities of I and J are equal. Moreover, for an infinite cardinal κ , the following statements are equivalent:

- (1) κ is greater than or equal to the cardinality of a discrete partition of \mathcal{X} ;
- (2) $\mathcal{A}_\kappa(\mathcal{X})$ contains a faithful \mathcal{X} -module;
- (3) $\mathcal{A}_\kappa(\mathcal{X})$ contains an ample \mathcal{X} -module.

Proof. Since $\{A_i\}_{i \in I}$ is a discrete partition of \mathcal{X} , for each $j \in J$, there exist only finitely many $i \in I$ such that $B_j \cap A_i \neq \emptyset$. Consequently, $|J| \leq |I| \times \aleph_0$. Similarly, one has $|I| \leq |J| \times \aleph_0$. Note that if \mathcal{X} admits an infinite discrete partition, then any discrete partition of \mathcal{X} must also be infinite. It follows that

$$|I| = |I| \times \aleph_0 = |J| \times \aleph_0 = |J|.$$

For the second part of the statement, note that every ample \mathcal{X} -module is necessarily faithful. Conversely, given a faithful \mathcal{X} -module C , let H be a separable, infinite-dimensional Hilbert space. Define the \mathcal{X} -module $C \otimes H$, where the underlying Hilbert space is $H_C \otimes H$ and the representation $\mathbb{1}_{\bullet}^{C \otimes H}$ is given by $\mathbb{1}_A^{C \otimes H} = \mathbb{1}_A^C \otimes \mathrm{id}_H$ for all subsets A . It is straightforward to verify that $C \otimes H$ is an ample \mathcal{X} -module with the same rank as C .

It remains to establish the equivalence of (1) and (2). If κ exceeds the cardinality of the discrete partition, then the module constructed in Example 2.1 is a faithful \mathcal{X} -module of rank $|I|$. Conversely, suppose C is a faithful \mathcal{X} -module of rank κ . Define $I_C = \{i \in I \mid A_i \cap \mathrm{dom}_1(C) \neq \emptyset\} \subseteq I$. By discretising \mathcal{X} and $\mathrm{dom}_1(C)$, there exists a gauge $E \in \mathcal{E}^I$ such that

$$I = E[I_C] = \bigcup_{i \in I_C} E[\{i\}].$$

It follows that the cardinality of I is given by $|I_C| \times \sup_{i \in I_C} |E[\{i\}]|$. Since $|E[\{i\}]|$ is finite for all $i \in I$, we deduce $|I| = |I_C|$. \square

We shall focus on the case of countably generated LFCM spaces. Accordingly, we aim to determine the cardinals κ for which the approximable category $\mathcal{A}_\kappa(\mathcal{X})$ contains a faithful \mathcal{X} -module. The following lemma addresses this question in the connected case.

Lemma 2.11. If \mathcal{X} is a countably generated LFCM space with a countable number of connected components, then $\mathcal{A}_{\aleph_0}(\mathcal{X})$ contains a faithful \mathcal{X} -module.

Proof. Let $\{A_i\}_{i \in I}$ be a discrete partition of \mathcal{X} . By considering the discretisation $\mathcal{I} = (I, \mathcal{E}^I, \mathcal{P}(I))$ of \mathcal{X} , as in Proposition 1.9, we obtain a locally finite, countably generated coarse space. We may select a generating set $\{E_n\}_{n \in \mathbb{N}}$ for \mathcal{I} satisfying the following conditions:

- (1) For every $n \in \mathbb{N}$, the inclusion $E_n \subset E_{n+1}$ holds;
- (2) For every $n \in \mathbb{N}$, the entourage E_n is a gauge;
- (3) For every entourage $F \in \mathcal{E}^I$, there exists $n \in \mathbb{N}$ such that $F \subset E_n$.

Since \mathcal{I} has a countable amount of connected components, there exists a sequence $\{i_k\}_{k \in \mathbb{N}} \subset I$ such that for every $j \in J$ there are $n, k \in \mathbb{N}$ such that $(i_k, j) \in E_n$. As \mathcal{I} is locally finite, the sets $E_n[i_k]$ are finite for all $n, k \in \mathbb{N}$. Hence,

$$I = \bigcup_{k \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} E_n[i_k]$$

is at most countable. Consequently, \mathcal{X} admits a countable discrete partition. By Lemma 2.10, the approximable category $\mathcal{A}_{\aleph_0}(\mathcal{X})$ contains a faithful \mathcal{X} -module. \square

In the case where \mathcal{X} has an uncountable number of connected components, the theorem above does not hold. For instance, consider an uncountable disjoint union of copies of \mathbb{N} , equipped with an extended metric

$$d(n, m) = \begin{cases} |n - m|, & \text{if } n, m \text{ belong to the same copy of } \mathbb{N}; \\ \infty, & \text{otherwise.} \end{cases}$$