

Define $a_0 := \infty$, $a_n := \sum_{k \geq n} \mu_+^{-1}(\delta^k)/k$ for $n \in \mathbb{N}$, and the function ψ by

$$\psi(t) := \sum_{n \in \mathbb{N}_0} \delta^n \mathbb{1}_{(a_{n+1}, a_n]}(t), \quad t \geq 0.$$

Then

$$F(t) \leq \psi(4t) \quad \text{for all } t \leq t_0 := \frac{\delta}{4(2+\delta)} \mu_+^{-1}(1).$$

Proof. The assumptions imply that μ_+ is continuous and strictly increasing and hence is the cumulative mass function of a continuous measure on $[0, \infty)$. Moreover, μ_+ is a bijection of $[0, \infty)$, and its generalized inverse μ_+^{-1} is simply the inverse function. Let $\psi_0 = \psi$, and for $n \in \mathbb{N}$, define ψ_n recursively by

$$\psi_{n+1}(t) := 1 - \exp\left(-\psi_n(t) - \int_0^{t/4} \psi_n(t-4x) \mu_+(dx)\right), \quad t \geq 0.$$

Denote by \mathbf{T} denoting the smoothing transform associated with the Poisson point process with intensity measure $\mu = \delta_0 + \mu_+$ as given by (3.2) and its representation (5.1). By induction, we observe that

$$1 - \psi_n(4t) = \mathbf{T}^n[1 - \psi(4(\cdot))](t), \quad t \geq 0.$$

Thus, by Theorem 3.4(b), we infer that $\psi_n(4t) \rightarrow F(t)$ as $n \rightarrow \infty$ for all $t > 0$.

To complete the proof, it remains to show that $\psi_n(t) \leq \psi(t)$ holds for all $t \leq 4t_0$ and $n \in \mathbb{N}_0$. We will prove the slightly stronger statement $\psi_n(t) \leq \psi(t)$ for all $t \leq a_{k_0}$ via induction on n , where $k_0 \in \mathbb{N}_0$ is chosen such that

$$a_{k_0+1} < 4t_0 \leq a_{k_0}. \quad (5.3)$$

For $n = 0$, the claim is trivially true (base case). For the inductive step, assume $\psi_n(t) \leq \psi(t)$ for all $t \leq a_{k_0}$ and some $n \in \mathbb{N}_0$. We will show that $\psi_{n+1}(t) \leq \psi(t)$ for all $t \in (a_{k+1}, a_k]$ and $k \geq k_0$. If $k_0 = 0$, for $t > a_1$ we clearly have $\psi(t) = 1 \geq \psi_{n+1}(t)$. Thus we can assume $k \geq k_0 \vee 1$. Let $t \in (a_{k+1}, a_k]$ for $k \geq k_0 \vee 1$. Then we have

$$\begin{aligned} \psi_{n+1}(t) &\leq \psi_{n+1}(a_k) = 1 - \exp\left(-\psi_n(a_k) - \int_0^{a_k/4} \psi_n(a_k - 4x) \mu_+(dx)\right) \\ &\leq 1 - \exp\left(-\psi(a_k) - \int_0^{a_k/4} \psi(a_k - 4x) \mu_+(dx)\right) \\ &= 1 - \exp\left(-(\psi(t) + I_k)\right) \end{aligned}$$

where I_k represents the integral in the exponent. To estimate I_k , we first note that

$$\begin{aligned} I_k &= \int_0^{a_k/4} \psi(a_k - 4x) \mu_+(dx) = \sum_{j \geq k} \int_{[a_k - a_j, a_k - a_{j+1})/4} \psi(a_k - 4x) \mu_+(dx) \\ &= \sum_{j \geq k} \delta^j \left[\mu_+\left(\frac{a_k - a_{j+1}}{4}\right) - \mu_+\left(\frac{a_k - a_j}{4}\right) \right] \\ &= \delta^k \sum_{j=0}^{\infty} \delta^j \left[\mu_+\left(\frac{a_k - a_{k+j+1}}{4}\right) - \mu_+\left(\frac{a_k - a_{k+j}}{4}\right) \right] \\ &= \psi(t) \sum_{j=0}^{\infty} \delta^j (1 - \delta) \mu_+\left(\frac{a_k - a_{k+j+1}}{4}\right), \end{aligned}$$