

arbitrary pair (j_-, j_+) . We can see that each ellipsoid can be chosen one after another and disjointly, for each critical point of the function $\pi_{2,1,1}|_{S_{0,c_G}}$ corresponding to each vertex v of degree at least 3.

We can put an ellipsoid of the standard form centered at some point and containing the point $(\pi_{2,1,1}(v_0) - \epsilon_{v_0,+,j}, a_{v_0,+,j}) \in F_{D_{\{S_{0,c_G}\},1}}((\pi_{2,1,1}(v_0) + \epsilon_{v_0,-,j}, a_{v_0,-,j}) \in F_{D_{\{S_{0,c_G}\},1}})$ in the interior in such a way that the boundary $\overline{D_{j'}} - D_{j'}$ of its closure contains a point $p_{j'} \in S_{0,c_G}$ with $\pi_{2,1,1}(p_{j'}) = \pi_{2,1,1}(v_0)$. We also do in such a way that the intersection $(\overline{D_{j'}} - D_{j'}) \cap \overline{D_{\{S_{0,c_G}\}}}$ is mapped by $\pi_{2,1,1}$ into \mathbb{R} as an injective function with the minimum (resp. maximum) $\pi_{2,1,1}(v_0)$. Remember that the relations $0 < \epsilon_{v_0,-,j}, \epsilon_{v_0,+,j} < \epsilon' < \epsilon$ are satisfied and that the numbers $\epsilon_{v_0,-,j_1}$ and $\epsilon_{v_0,-,j_2}$ ($\epsilon_{v_0,+,j_1}$ and $\epsilon_{v_0,+,j_2}$) are mutually disjoint for distinct numbers j_1 and j_2 and sufficiently small. Remember also that the relation $a_{v_0,-,j_1} < a_{v_0,-,j_2}$ holds for an arbitrary pair (j_1, j_2) with $j_1 < j_2$, that the relation $a_{v_0,+,j_1} < a_{v_0,+,j_2}$ holds for an arbitrary pair (j_1, j_2) with $j_1 < j_2$, and that the relations $a_{v_0,-,j_-} < a_{v_0,+,j_+}$ and $a_{v_0} - \epsilon' < a_{v_0,-,j_-}, a_{v_0,+,j_+} < a_{v_0} + \epsilon'$ hold for an arbitrary pair (j_-, j_+) . We can see that each ellipsoid can be chosen one after another, disjointly, and disjoint from the previous ellipsoids, for each critical point of the function $\pi_{2,1,1}|_{S_{0,c_G}}$ corresponding to each vertex v_0 of degree 1.

We put ellipsoids $D_{j'}$ of the standard form one after another to have our desired result. \square

Proposition 1. *In the present inductive procedure, for a step putting an ellipsoid $D_{j'}$ centered at the point $x_{j'} := (\pi_{2,1,1}(v) \pm \epsilon_{v,\pm,j}, a_{v,\pm,j})$ or $x_{j'} := (\pi_{2,1,1}(v_0) \pm \epsilon_{v_0,\pm,j}, a_{v_0,\pm,j})$, we abuse the notation from Definition 4.*

First, $D_{j'}$ is $(\mathcal{S}, D_{\mathcal{S}})$ -connected.

In addition, we assume at least one of the following in a step in the procedure.

- *At least one ellipsoid centered at a point of the form $(\pi_{2,1,1}(v) \pm \epsilon_{v,\pm,j_0}, a_{v,\pm,j_0})$ has been put in the case v is of degree at least 3.*
- *At least one ellipsoid centered at a point of the form $(\pi_{2,1,1}(v_0) \pm \epsilon_{v_0,\pm,j_0}, a_{v_0,\pm,j_0})$ has been put in the case v_0 is of degree 1.*

Under this additional assumption, the pointed set $(D_j, x_{j'})$ is $(\mathcal{S}, D_{\mathcal{S}}, A_{D_{\mathcal{S}}})$ -PLS and it is not $(\mathcal{S}, D_{\mathcal{S}}, A_{D_{\mathcal{S}}})$ -PS.

Proof. The first part follows from assumptions on a sufficiently small positive number $\epsilon_0 > 0$, another sufficiently small one $\epsilon > 0$, and sufficiently small ones with the relations $0 < \epsilon_{v,-,j}, \epsilon_{v,+,j}, \epsilon_{v_0,-,j}, \epsilon_{v_0,+,j} < \epsilon' < \epsilon$.

We see the additional part. The set $\overline{D_{j'}}$ contains a point of the boundary $\overline{D_{j'}} - D_{j'}$ such that the value of the projection $\pi_{2,1,1}$ there is $\pi_{2,1,1}(v)$ or $\pi_{2,1,1}(v_0)$. The set $\overline{D_{j'}}$ contains no point from $F_{D_{\mathcal{S}},2}$ by considering the local shape of the curve S_{0,c_G} . From the additional assumption, there exists at least one point different from this such that the value of the projection $\pi_{2,1,1}$ there is $\pi_{2,1,1}(v)$ or $\pi_{2,1,1}(v_0)$ in $F_{D_{\mathcal{S}},1}$. We can easily check that the pointed set $(D_j, x_{j'})$ is $(\mathcal{S}, D_{\mathcal{S}}, A_{D_{\mathcal{S}}})$ -PLS and it is not $(\mathcal{S}, D_{\mathcal{S}}, A_{D_{\mathcal{S}}})$ -PS. \square

We present another example as our new result.

Theorem 5. *We also abuse the notation from Definition 4 here. In our steps here, for each pair of ellipsoids of the standard form centered at $(\pi_{2,1,1}(v_0) -$*