

Figure 29: [Test Case 3] (Left) Reference Darcy Velocity Field. (Right) Perturbed Velocity Field.

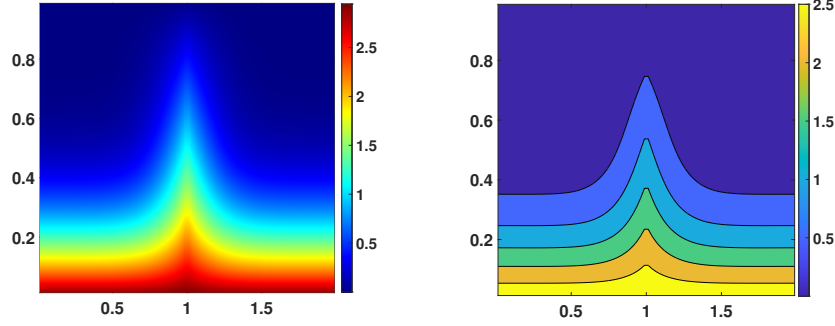


Figure 30: [Test Case 3] Reference Concentration Field. (Left) Heat Map. (Right) Contour Map.

mesh size  $h = 1/40$  and on a very fine temporal mesh  $\Delta t = T/800$  with  $T = 5$ . The data assimilation algorithm is performed over the same spatial mesh size and on a coarser temporal filtering steps  $\Delta t_{\text{Filter}} = T/N t_{\text{Filter}}$  where  $N t_{\text{Filter}} = 50$ , with the same assumption on the observational operator as in Test Case 2. Since concentration transport is driven by the Darcy velocity, we assume that there are some uncertainties in the process of solving the Darcy's flow system to increase the complexity of state estimation. Thus, we add a small perturbation  $\xi_{\text{Darcy}} = 0.0001\epsilon_D$  to the reference Darcy velocities with  $\epsilon_{\text{Darcy}} \sim N(0, I)$  and input the disturbed velocity to the forward solver. The reference Darcy velocity field and one example of the perturbed velocity field are shown in Figure 29 on a very coarse mesh for the purpose of illustration. We can see that even with a small amount of noise, the perturbed Darcy field is significantly chaotic, which makes the task of state estimation challenging.

Similar to Test Case 2, we aim to investigate the performance of the United Filter under various levels of perturbation. Thus, we consider two types of disturbed noise  $\omega$  in (14):  $\tilde{\omega}^1 = 0.001\sqrt{\Delta t_{\text{Filter}}}\tilde{\epsilon}^1$  and  $\tilde{\omega}^2 = 0.1\sqrt{\Delta t_{\text{Filter}}}\tilde{\epsilon}^2$ , where  $\tilde{\epsilon}^i \sim \mathcal{N}(0, I_l)$  for  $i = 1, 2$ . Regarding the setting for the parameter estimation, as the exact values are small, we apply the assimilation algorithm to approximate  $\rho_i = 1/d_i, i = 1, 2$  and  $\rho_\gamma = 1/\alpha_\gamma$ . We choose the number of Direct Filter particles to be  $M = 40$ , and the number of iterations in the United Filter to be  $R = 4$ .

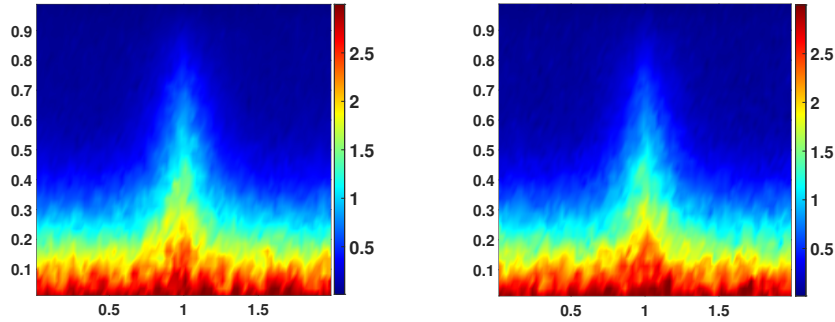


Figure 31: [Test Case 3] Heat map of estimated concentration field by the United Filter: (Left) With noise  $\tilde{\omega}^1$ . (Right) With noise  $\tilde{\omega}^2$ .