

where the constant c is independent of A and n (but does depend on the dimension of $\text{Ran}(P)$). We use that $\int_1^\infty t^{s-2} dt < \infty$ for $s < 1$ to bound the second term. Thus, there exists a constant $C(s)$ such that

$$\int_n^{n+1} \|(A+x)^{-1}\|^s dx \leq C(s)$$

for all maximally dissipative A , where $C(s)$ is independent of n and $\text{Ran}(P)$. Using that $\sum_{n \in \mathbb{Z}} \frac{1}{1+(|n|-1)^2} < \infty$, we can therefore bound the right hand side of (9) for all $|z| > 1$. If $|z| < 1$, we drop a minus sign in both norms in (9) and use that $i\widehat{F}_z$ and $i\widehat{F}_z^{-1}$ are dissipative. Repeating the arguments from above yields the bound for all $|z| < 1$.

The case $x^{(i)} = y^{(j)}$ is significantly easier: We can directly take $\alpha = \omega_x$ and do not need β . The operators F_z and \widehat{F}_z act on \mathbb{C}^3 , while all other estimates still hold. The integrals in (9) can be similarly bounded by Lemma 1. \square

We state a simplified version of Lemma 3.1 from [3], which we use to bound the integrals in the proof of Theorem 2.

Lemma 1. *Let \mathcal{H} be a separable Hilbert space, A be a maximally dissipative operator with strictly positive imaginary part and $M_1, M_2 : \mathcal{H} \rightarrow \mathcal{H}$ be Hilbert-Schmidt operators. Then there exists a constant c independent of A , M_1 and M_2 such that for any $t > 0$:*

$$|\{x \in \mathbb{R} \text{ s.t. } \|M_1(A+x)^{-1}M_2\| > t\}| \leq c \|M_1\|_{HS} \|M_2\|_{HS} \frac{1}{t},$$

where $|\cdot|$ denotes the Lebesgue measure.

4.2 The boundary of a box

We want to define a "box" Λ_L for any size $L \in \mathbb{N}^2$, whose sides have lengths L_1 and L_2 , such that the Quantum Walker is unable to cross the boundary of Λ_L . Restricting the Walker to some box Λ_L is achieved by changing the coin matrix at specific lattice sites on the boundary of Λ_L . In other words, we want to obtain unitary operators $U_\omega^{(L)} = U_\omega^{\Lambda_L} \oplus U_\omega^{\Lambda_L^C}$ and subspaces $\mathcal{H}_L \oplus \mathcal{H}_L^C = \mathcal{H}$ such that \mathcal{H}_L , respectively \mathcal{H}_L^C , are invariant under $U_\omega^{\Lambda_L}$, respectively $U_\omega^{\Lambda_L^C}$. Note that we call a subspace $\mathcal{H}' \subset \mathcal{H}$ invariant under U if $U\mathcal{H}' \subset \mathcal{H}'$. Recalling the definition of $\mathcal{H}^{j,k}$ (4), we use that C_0 induces a fully localized Quantum Walk, see section 2, and define the invariant subspaces:

$$\mathcal{H}_L = \bigoplus_{\substack{-L_1 \leq j \leq L_1-1 \\ -L_2 \leq k \leq L_2-1 \\ j+k > -L_1-L_2}} \mathcal{H}^{j,k}, \quad \mathcal{H}_L^C = \mathcal{H} \setminus \mathcal{H}_L.$$

The choice $j+k > -L_1-L_2$ is not necessary, but simplifies the structure of Λ_L , see Figure 3. We call the number of Γ_A -vertices in Λ_L the volume of Λ_L , that is:

$$\text{vol}(\Lambda_L) = 4L_1L_2 - 1 \quad \text{and} \quad |L| = \sqrt{L_1^2 + L_2^2}. \quad (10)$$

To obtain a Quantum Walk such that these two subspaces are invariant, we need to change the coin matrix at specific lattice sites from C to C_0 . In particular, we use the coin matrix C_0 at all Γ_B sites in

$$\begin{aligned} \Gamma_{C_0}^{(L)} = \Big\{ & |j, k\rangle \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ s.t. } \left(-L_1 \leq j \leq L_1-1, k = L_2-1 \right) \text{ or} \\ & \left(-L_1+1 \leq j \leq L_1, k = -L_2-1 \right) \text{ or } \left(j = L_1, -L_2 \leq k \leq L_2-2 \right) \\ & \text{or } \left(j = -L_1, -L_2 \leq k \leq L_2-2 \right) \Big\}. \end{aligned}$$