

Theorem 3.6. *Let $\mathbf{x} \preceq (0, \dots, n-1)$ be an irreducible score, so that $C \cap (0, \infty)^{n \times n} \neq \emptyset$. Then*

$$\lim_{k \rightarrow \infty} M^k = \arg \min_{L \in C} D(L, M^0) = \arg \min_{L \in C} H(L). \quad (3.9)$$

Proof. Note that $\text{dom}(h) = \mathbb{R}_+$, $\text{idom}(h) = (0, \infty)$ and $\text{bdom}(h) = \{0\}$. It is easy to check:

- (1) h is a proper convex function. Moreover, h is closed because $\{x \in \text{dom}(h) : h(x) \leq \alpha\}$ is closed for all $\alpha \in \mathbb{R}$.
- (2) h is Legendre because
 - h is differentiable on $\text{idom}(h)$;
 - $\lim_{t \rightarrow 0+} h'(x + t(y - x))(y - x) = -\infty$ for all $x \in \text{bdom}(f)$ and $y \in \text{idom}(h)$;
 - h is strictly convex on $\text{idom}(h)$.
- (3) h is co-finite because $\lim_{t \rightarrow \infty} \frac{h(tx)}{t} = \infty$ for all $x \neq 0$.
- (4) h is very strictly convex because $h''(x) > 0$ for all $x \in \text{idom}(h)$.

Since C_1, C_2, C_3 are all affine subsets, it suffices to apply [7, Theorem 4.3] to conclude. \square

See also [45] for recent development on the convergence rate of Bregman's iteration under further technical assumptions, which we do not pursue here.

Next we propose a computational scheme inspired by Theorem 3.6. The key is to compute numerically, for each $M \in (0, \infty)^{n \times n}$, its Bregman projection on C_k . We distinguish three cases:

- $k = 1$: We introduce the Lagrange multiplier $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, and set $\Phi(L, \lambda) := D(L, M) + \sum_{i=1}^n \lambda_i \left(\sum_{j=1}^n l_{ij} - 1 \right)$. Differentiating Φ with respect to m_{ij} and λ_i yields $\partial_{l_{ij}} \Phi = \log(l_{ij}) - \log(m_{ij}) + \lambda_i$ and $\partial_{\lambda_i} \Phi = \sum_{j=1}^n l_{ij} - 1$. By setting these to zero, we get

$$\arg \min_{L \in C_1} D(L, M) = \left(\frac{m_{ij}}{\sum_{k=1}^n m_{ik}} \right)_{1 \leq i, j \leq n}. \quad (3.10)$$

- $k = 2$: The same reasoning as in the previous case yields:

$$\arg \min_{L \in C_2} D(L, M) = \left(\frac{m_{ij}}{\sum_{k=1}^n m_{kj}} \right)_{1 \leq i, j \leq n}. \quad (3.11)$$

- $k = 3$: Define $\Phi(L, \lambda) := D(L, M) + \sum_{i=1}^n \lambda_i \left(\sum_{j=1}^n (j-1)l_{ij} - x_i \right)$, and differentiate Φ with respect to m_{ij} and λ_i yields $\partial_{l_{ij}} \Phi = \log(l_{ij}) - \log(m_{ij}) + \lambda_i(j-1)$ and $\partial_{\lambda_i} \Phi = \sum_{j=1}^n (j-1)l_{ij} - x_i$. Setting these to zero yields

$$\arg \min_{L \in C_3} D(L, M) = \left(m_{ij} r_{ij}^{j-1} \right)_{1 \leq i, j \leq n}, \quad (3.12)$$

where r_{ij} is the unique positive root to the polynomial equation $\sum_{j=1}^n (j-1)m_{ij}r^{j-1} - x_i = 0$. Let $f(r) := \sum_{j=1}^n (j-1)m_{ij}r^{j-1} - x_i$. It is easy to see that f is strictly increasing on $[0, \infty)$ with $f(0) \leq 0$ and $\lim_{r \rightarrow \infty} f(r) = +\infty$. Thus, it is easy (and quick) to find a numerical root of f on $[0, \infty)$ by Newton's method.