

Some key theoretical results behind the Gröbner walk are stated below. Detailed proofs and additional context may be found in Chapters 1 and 2 of [10].

Theorem 2. *For an ideal $I \triangleleft R$, the following sets are in one-to-one correspondence:*

$$\left\{ \begin{array}{c} \text{in}_{<}(I), \\ < \text{ is a term order} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{marked Gröbner bases} \\ \text{of } I \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{full-dimensional} \\ \text{cones of } \mathbb{G}(I) \end{array} \right\}$$

In our setting, a *marked* Gröbner basis is a reduced Gröbner basis with the leading terms identified (formally, each g is encoded as a pair (g, x^α) , where $x^\alpha = \text{in}_{<}(g)$). The first correspondence in Theorem 2 is immediate, whilst the second correspondence is a consequence of [13, Theorem 1.11]: marked Gröbner bases encode the defining integer vectors of an H-description of the corresponding cone.

Lower-dimensional cones in $\mathbb{G}(I)$ correspond to generalized initial ideals $\text{in}_\omega(I)$, where ω is any weight vector in the relative interior of said cone. Generically, such ideals are “almost monomial”, and may be retrieved with the help of the following lemma:

Lemma 3. *Let $G_{<}$ be a marked Gröbner basis of I with regards to $<$ and $\omega \in \mathbb{R}_{\geq 0}^n$ be a weight vector on the boundary of the corresponding cone in $\mathbb{G}(I)$. The set*

$$\text{in}_\omega(G_{<}) = \{\text{in}_\omega(g), g \in G_{<}\}$$

is a marked Gröbner basis of $\text{in}_\omega(I)$ with respect to $<$.

At every step of the Gröbner walk, a basis of this form is converted with Buchberger’s algorithm and then lifted to the basis of I corresponding to the adjacent full-dimensional cone, which corresponds to $(<_t)_\omega$, i.e. the refinement of the target ordering $<_t$ by ω .

Lemma 4. *Let $M = \{m_1, \dots, m_r\}$ be the marked Gröbner basis of $\text{in}_\omega(I)$ with respect to the refinement ordering $(<_t)_\omega$. Then*

$$G := \{m_1 - \overline{m}_1^{G_{<}}, \dots, m_r - \overline{m}_r^{G_{<}}\}$$

is a Gröbner basis of I with respect to $(<_t)_\omega$ where $\overline{f}^{G_{<}}$ denotes the normal form of f with respect to the basis $G_{<}$.

This process of subsequent passing to the generalized initial ideal and lifting to the adjacent basis is repeated until the target basis is computed.

3. FUNCTIONALITY

Our implementation of the Gröbner walk ships with `OSCAR` since version 1.2.0, thus it suffices to load `OSCAR`. There is a straightforward interface through the function `groebner_walk`.

Example 5. Continuing from example Example 1, we can calculate a Gröbner basis of the ideal

$$I = \langle y^4 + x^3 - x^2 + x, x^4 \rangle \triangleleft \mathbb{Q}[x, y]$$

with respect to $<_{\text{lex}}$ by starting from a Gröbner basis for the *graded reverse lexicographic ordering* $<_{\text{degrevlex}}$. Since $<_{\text{degrevlex}}$ is the default internal ordering of any polynomial ring in `OSCAR`, it suffices to call the Gröbner walk in the following way.