

defined by the formal sum

$$\alpha \star_\tau \beta = \alpha \cup \beta + \sum_{\substack{d \in \mathbf{Eff}_{\neq 0}, \\ 0 \leq \ell \leq N-1}} \langle \alpha, \beta, \phi^\ell \rangle_{0,3,d} \phi_\ell e^{\langle \tau, d \rangle} \quad (\diamond)$$

where  $\mathbf{Eff} \subseteq H_2(X, \mathbb{Z})$  is the set of effective curve classes.

The quantum product reduces to the classical cup product when  $\langle \tau, d \rangle \rightarrow -\infty$  for non-zero effective class  $d \in \mathbf{Eff}_{\neq 0}$ . When  $X$  is Fano, the degree axiom of Gromov-Witten invariants [12] implies that  $\langle \alpha, \beta, \gamma \rangle_{0,3,d} = 0$  when  $\deg \alpha + \deg \beta + \deg \gamma \neq \langle c_1, d \rangle$ . Therefore, the sum on the right-hand side of  $(\diamond)$  is finite.

All the genus-0 Gromov-Witten invariants can be encoded within a connection, known as the Dubrovin connection.

**Definition 2.3** (Quantum Differential Equations [2, Chapter 10]). *Let  $X$  be a Fano manifold. Let  $B$  be the trivial vector bundle with fibre  $H^*(X)$  over  $H^2(X) \times \mathbb{C}^\times$ . Set the coordinate  $(\tau, z) = (\sum_{j=1}^{b^2(X)} t_j \phi_j, z) \in H^2(X) \times \mathbb{C}^\times$ . The **Dubrovin connection** on  $B$  can be defined by:*

$$\begin{aligned} \nabla_{\partial_{t_j}} \varphi &= \frac{1}{z} \phi_j \star_\tau \varphi, \\ \nabla_{z \partial_z} \varphi &= -\frac{1}{z} c_1(X) \star_\tau \varphi + \mu(\varphi), \end{aligned}$$

where  $\mu : H^*(X) \rightarrow H^*(X)$ ; the Hodge grading operator, is a linear map defined by

$$\mu(\phi_\ell) = \frac{1}{2}(\deg \phi_\ell - \dim X) \phi_\ell,$$

and  $\varphi \in H^*(X)$  is regarded as a constant section of the trivial bundle.

The Dubrovin connection is a flat connection. Its fundamental solution along the  $\tau$ -direction, i.e., sections satisfying  $\nabla_{\partial_{t_j}} (L(\tau, z) \alpha) = 0$  for all  $j = 1, \dots, b^2(X)$ , can be given by

$$L(\tau, z) \alpha := e^{-\tau/z} \alpha - \sum_{\substack{d \in \mathbf{Eff}_{\neq 0} \\ 0 \leq \ell \leq \dim H^*(X) - 1}} \left\langle \phi^\ell, \frac{e^{-\tau/z} \alpha}{z + \psi} \right\rangle_{0,2,d} e^{\langle \tau, d \rangle} \phi_\ell,$$

where the second argument of the coefficients is expanded as

$$\frac{1}{z + \psi} = \sum_{k=0}^{\infty} (-1)^k z^{-(k+1)} \psi^k.$$

By the linearity of the Gromov-Witten invariants, the coefficients are a sum of the Gromov-Witten invariants with gravitational descendants.