

where $R^\varepsilon = R_2^\varepsilon + \operatorname{div}(R_1^\varepsilon) + R_3^\varepsilon + R_4^\varepsilon$ and C_1 is some positive constant independent of ε . Therefore,

$$\lim_{\varepsilon \rightarrow 0} \left\| \frac{G(I + \varepsilon h) - G(I)}{\varepsilon} - (\xi, \eta) \right\|_X = \lim_{\varepsilon \rightarrow 0} \|(\tilde{\xi}, \tilde{\eta})\|_X = 0.$$

This establishes the Gâteaux differentiability of G at I in the direction h , with $G'(I)h = (\xi, \eta)$. \square

Now, we establish a result on the differentiability of the cost functional:

Theorem 4.1 (On the Gâteaux-differentiability of the Cost Functional). *Let Assumptions (A1)-(A10) hold. Then the cost functional J is Gâteaux differentiable at any $I \in \mathcal{U}_{ad}$. Furthermore, for any direction $h \in L^\infty(0, T)$, the Gâteaux derivative of J at I in the direction h is given by:*

$$\langle J'(I), h \rangle = \int_0^T \left[\alpha \int_{\mathbb{R}^N} \xi(t, x) dx + 2\beta I(t)h(t) \right] dt + \gamma \int_{\mathbb{R}^N} \xi(T, x) dx \quad (45)$$

where (ξ, η) is the solution to the linearized system (35).

Proof. Let $I \in \mathcal{U}_{ad}$ and define the functional $L_I : L^\infty(0, T) \rightarrow \mathbb{R}$ by:

$$\langle L_I, h \rangle = \int_0^T \left[\alpha \int_{\mathbb{R}^N} \xi(t, x) dx + 2\beta I(t)h(t) \right] dt + \gamma \int_{\mathbb{R}^N} \xi(T, x) dx \quad (46)$$

where (ξ, η) solves the linearized system (35) with the direction h . Now let $h_1, h_2 \in L^\infty(0, T)$ and $\lambda \in \mathbb{R}$, and let us consider $h = h_1 + \lambda h_2$, with (ξ_1, η_1) and (ξ_2, η_2) are the solutions to the linearized system corresponding to directions h_1 and h_2 respectively. By the linearity of system (35), the solution (ξ, η) corresponding to h satisfies $\xi = \xi_1 + \lambda \xi_2$ and $\eta = \eta_1 + \lambda \eta_2$. Consequently:

$$\begin{aligned} \langle L_I, h_1 + \lambda h_2 \rangle &= \int_0^T \left[\alpha \int_{\mathbb{R}^N} (\xi_1 + \lambda \xi_2) dx + 2\beta I(t)(h_1(t) + \lambda h_2(t)) \right] dt + \gamma \int_{\mathbb{R}^N} (\xi_1 + \lambda \xi_2)(T, x) dx \\ &= \langle L_I, h_1 \rangle + \lambda \langle L_I, h_2 \rangle. \end{aligned}$$

Therefore L_I is linear. Furthermore, for any $h \in L^\infty(0, T)$, by using the estimates from Lemma 4.1 we obtain:

$$|\langle L_I, h \rangle| \leq \alpha \int_0^T \int_{\mathbb{R}^N} |\xi(t, x)| dx dt + 2\beta \|I\|_{L^\infty} \|h\|_{L^\infty} T + \gamma \int_{\mathbb{R}^N} |\xi(T, x)| dx \leq C_1 \|h\|_{L^\infty(0, T)},$$

where C_1 is a positive constant depending on α, β, γ, T , and the bounds from Lemma 4.1. Therefore L_I is continuous.

To show that L_I represents the Gâteaux derivative of J , let $h \in L^\infty(0, T)$, $\varepsilon > 0$, and let define $I^\varepsilon = I + \varepsilon h$ and let $(p^\varepsilon, d^\varepsilon) = G(I^\varepsilon)$ and $(p, d) = G(I)$. Let define $\Phi_I(\varepsilon) = \frac{J(I^\varepsilon) - J(I)}{\varepsilon}$, we have:

$$\begin{aligned} \Phi_I(\varepsilon) &= \int_0^T \left[\alpha \int_{\mathbb{R}^N} \frac{p^\varepsilon - p}{\varepsilon} dx + \beta \frac{(I^\varepsilon)^2 - I^2}{\varepsilon} \right] dt + \gamma \int_{\mathbb{R}^N} \frac{p^\varepsilon(T, x) - p(T, x)}{\varepsilon} dx \\ &= \int_0^T \left[\alpha \int_{\mathbb{R}^N} \xi^\varepsilon dx + \beta(2Ih + \varepsilon h^2) \right] dt + \gamma \int_{\mathbb{R}^N} \xi^\varepsilon(T, x) dx \end{aligned}$$

where $\xi^\varepsilon = (p^\varepsilon - p)/\varepsilon$. By Lemma 4.1, we have $\xi^\varepsilon \rightarrow \xi$ in X as $\varepsilon \rightarrow 0$, where ξ is the solution to the linearized system (35). Therefore:

$$\lim_{\varepsilon \rightarrow 0} \Phi_I(\varepsilon) = \int_0^T \left[\alpha \int_{\mathbb{R}^N} \xi(t, x) dx + 2\beta I(t)h(t) \right] dt + \gamma \int_{\mathbb{R}^N} \xi(T, x) dx = \langle L_I, h \rangle. \quad (47)$$

This establishes that J is Gâteaux differentiable at I with derivative $J'(I) = L_I$, which complete the proof. \square