

Note that the level lines $\{(x_1, x_2) \in \mathbb{R}^2 : 2U(x_1) + x_2^2 = r^2\}$ for all $0 < |r| < (2\vartheta)^{-1/2}$ correspond to $T(r)$ -periodic solutions

$$\begin{aligned} \hat{x}_{1,0}(t, r) &\equiv r \operatorname{sn} \left(\frac{t}{\sqrt{k_r^2 + 1}}, k_r \right) \sqrt{k_r^2 + 1}, \\ \hat{x}_{2,0}(t, r) &\equiv r \operatorname{cn} \left(\frac{t}{\sqrt{k_r^2 + 1}}, k_r \right) \operatorname{dn} \left(\frac{t}{\sqrt{k_r^2 + 1}}, k_r \right), \\ T(r) &\equiv 4\mathcal{K}(k_r) \sqrt{k_r^2 + 1}, \quad \nu(r) \equiv \frac{2\pi}{T(r)}, \end{aligned} \quad (8)$$

where $\operatorname{sn}(u, k)$, $\operatorname{cn}(u, k)$, $\operatorname{dn}(u, k)$ are the Jacoby elliptic functions, $\mathcal{K}(k)$ is the complete elliptic integral of the first kind, and $k_r \in (0, 1)$ is a root of the equation $(k_r + k_r^{-1})^{-2} = \vartheta r^2/2$. Define auxiliary 2π -periodic functions

$$X_1(\varphi, r) \equiv \hat{x}_{1,0} \left(\frac{\varphi}{\nu(r)}, r \right), \quad X_2(\varphi, r) \equiv \hat{x}_{2,0} \left(\frac{\varphi}{\nu(r)}, r \right). \quad (9)$$

It can easily be checked that

$$\begin{aligned} \nu(r) \partial_\varphi X_1 &= X_2, \quad \nu(r) \partial_\varphi X_2 = -U(X_1), \\ 2U(X_1) + X_2^2 &\equiv r^2, \quad \det \frac{\partial(X_1, X_2)}{\partial(\varphi, r)} \equiv \begin{vmatrix} \partial_\varphi X_1 & \partial_\varphi X_2 \\ \partial_r X_1 & \partial_r X_2 \end{vmatrix} \equiv \frac{r}{\nu(r)}. \end{aligned}$$

Hence, the transformation (9) is invertible for all $0 < |r| < (2\vartheta)^{-1/2}$ and $\varphi \in [0, 2\pi)$. Then, by applying Itô's formula [39, §4.2], it can be shown that in the variables (r, φ) system (7) takes the form (1) with $\mu(t) \equiv t^{-1/4}$, $s_0 = 3/2$, $s_i = 0$,

$$\begin{aligned} a_i(r, \varphi, S, t) &\equiv t^{-\frac{n}{4}} a_{i,n}(r, \varphi, S) + t^{-\frac{n}{2}} a_{i,2p}^\varepsilon(r, \varphi, S), \quad i \in \{1, 2\}, \\ \alpha_{i,j}(r, \varphi, S, t) &\equiv t^{-\frac{p}{4}} \alpha_{i,j,p}(r, \varphi, S), \quad i, j \in \{1, 2\}, \end{aligned} \quad (10)$$

where

$$\begin{aligned} a_{1,n}(r, \varphi, S) &\equiv \frac{X_2(\varphi, r)}{r} (Q(S)X_1(\varphi, r) + \mathcal{P}(S)X_2(\varphi, r)), \\ a_{1,2p}^\varepsilon(r, \varphi, S) &\equiv \frac{\varepsilon^2 \mathcal{B}^2(S)U(X_1(\varphi, r))}{r^3}, \\ a_{2,n}(r, \varphi, S) &\equiv -\frac{\nu(r) \partial_r X_1(\varphi, r)}{r} (Q(S)X_1(\varphi, r) + \mathcal{P}(S)X_2(\varphi, r)), \\ a_{2,2p}^\varepsilon(r, \varphi, S) &\equiv -\frac{\varepsilon^2 \mathcal{B}^2(S) \nu(r)}{2r} \partial_r \left(\frac{\partial_r X_1(\varphi, r) X_2(\varphi, r)}{r} \right), \\ \alpha_{1,1,p}(r, \varphi, S) &\equiv \frac{\mathcal{B}(S)X_2(\varphi, r)}{r}, \\ \alpha_{2,1,p}(r, \varphi, S) &\equiv -\frac{\mathcal{B}(S) \nu(r) \partial_r X_1(\varphi, r)}{r}, \end{aligned}$$

and $\alpha_{1,2,p}(r, \varphi, S) \equiv \alpha_{2,2,p}(r, \varphi, S) \equiv 0$. Note that $0 < \nu(r) < 1$ for all $0 < |r| < (2\vartheta)^{-1/2}$. Hence, there exist $\kappa, \varkappa \in \mathbb{Z}_+$ and $0 < |r_0| < (2\vartheta)^{-1/2}$ such that the condition (6) holds.

If $\mathcal{G}(x, y, S) \equiv 0$ and $\varepsilon = 0$, then $r(t) \equiv \varrho_0$ and $\varphi(t) \equiv \nu(\varrho_0)t + \phi_0$ with arbitrary constants ϱ_0 and ϕ_0 . Numerical analysis of the system with $\mathcal{G}(x, y, S) \not\equiv 0$ and $\varepsilon = 0$ shows that, depending on the values of the perturbation parameters, two cases are possible. In the first case, the amplitude of the solutions either tends to zero or grows until it reaches the boundary of the considered domain, depending on the sign of \mathcal{Q}_0 (see Fig. 1, a). In the second case, the solutions with an asymptotically constant amplitude $r(t) \approx r_0$ may appear (see Fig. 1, b). The emergence of such solutions is associated with the capture of the system into resonance. If $\varepsilon \neq 0$, the stochastic perturbations may disrupt such a behaviour (see Fig. 1, c).