

$$\mathbf{A}\mathbf{u} = \mathbf{f} \equiv \begin{bmatrix} \mathbf{A}_{11} & & \mathbf{A}_{15} \\ & \mathbf{A}_{22} & \mathbf{A}_{25} \\ \mathbf{A}_{51} & \mathbf{A}_{52} & \mathbf{A}_{55} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_5 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_5 \end{bmatrix}. \quad (8)$$

Here \mathbf{u} is the solution and \mathbf{f} is the body load on the interior of Ω , split into the two subdomains and shared boundary. Because an elliptic PDO is a local operator there is no interaction through \mathbf{A} on \mathbf{f}_1 by \mathbf{u}_2 or \mathbf{f}_2 by \mathbf{u}_1 , so $\mathbf{A} = 0$ in those submatrices. However, both \mathbf{u}_1 and \mathbf{u}_2 , are local to the shared boundary where \mathbf{u}_5 lies, so they both interact with the body load there, \mathbf{f}_5 , through \mathbf{A} . To interpret the first two block rows, note that

$$\mathbf{A}_{11}\mathbf{u}_1 + \mathbf{A}_{15}\mathbf{u}_5 = \mathbf{f}_1 \iff \mathbf{u}_1 = -\mathbf{A}_{11}^{-1}\mathbf{A}_{15}\mathbf{u}_5 + \mathbf{A}_{11}^{-1}\mathbf{f}_1 \quad (9)$$

This is similar to our solution operator formulation shown in (4), with the part of the subdomain boundary on $\partial\Omega$ now folded into \mathbf{u}_1 . Next, let $\tilde{\mathbf{u}}_1 = \mathbf{A}_{11}^{-1}\mathbf{f}_1$ and $\tilde{\mathbf{u}}_2 = \mathbf{A}_{22}^{-1}\mathbf{f}_2$. If we supply these vectors in place of $\mathbf{u}_1, \mathbf{u}_2$ in (8) and set $\mathbf{u}_5 = \mathbf{0}$, then the equation still holds with the same body loads \mathbf{f}_1 and \mathbf{f}_2 . Thus $\tilde{\mathbf{u}}_1$ and $\tilde{\mathbf{u}}_2$ are the particular solutions to our PDE on the interiors of $\Omega^{(1)}$, and $\Omega^{(2)}$ with $\mathbf{u}_5 = \mathbf{0}$. Now consider an upper triangular matrix \mathbf{U} that satisfies

$$\underbrace{\begin{bmatrix} \mathbf{I} & & \mathbf{A}_{11}^{-1}\mathbf{A}_{15} \\ & \mathbf{I} & \mathbf{A}_{22}^{-1}\mathbf{A}_{25} \\ & & \mathbf{I} \end{bmatrix}}_{\mathbf{U}} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_5 \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{u}}_1 \\ \tilde{\mathbf{u}}_2 \\ \mathbf{u}_5 \end{bmatrix}. \quad (10)$$

This linear system can be confirmed with (9). \mathbf{U} decouples our solutions $\mathbf{u}_1, \mathbf{u}_2$ into the particular solutions $\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2$ which are derived from our global Dirichlet BC without the subdomain-only (shared face) Dirichlet BC, and \mathbf{u}_5 which is that subdomain-only boundary condition. Thus \mathbf{U}^{-1} collects both components of our solutions - in effect it represents the end of solving the merged system, where we use \mathbf{u}_3 and \mathbf{u}_5 to get \mathbf{u}_1 , and \mathbf{u}_4 and \mathbf{u}_5 to get \mathbf{u}_2 . Next, let \mathbf{L} be a lower triangular matrix defined such that

$$\underbrace{\begin{bmatrix} & \mathbf{I} & \\ & & \mathbf{I} \\ \mathbf{A}_{51}\mathbf{A}_{11}^{-1} & \mathbf{A}_{52}\mathbf{A}_{22}^{-1} & \mathbf{I} \end{bmatrix}}_{\mathbf{L}} \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \underbrace{\mathbf{f}_5 - \mathbf{A}_{51}\mathbf{A}_{11}^{-1}\mathbf{f}_1 - \mathbf{A}_{52}\mathbf{A}_{22}^{-1}\mathbf{f}_2}_{\tilde{\mathbf{f}}_5} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_5 \end{bmatrix}. \quad (11)$$