

therefore, $\lambda_1(\mathbf{B} - \omega_0 \mathbf{C}') = \langle \mathbf{V}_1(\mathbf{B}), (\mathbf{B} - \omega_0 \mathbf{C}')(\mathbf{V}_1(\mathbf{B})) \rangle = 2\mu_1 - \omega_0 \xi$ since $\mathbf{C}' \in \mathfrak{C}$. So the strict inequality is asserting that $2\mu_1 - \omega_0 \xi < \lambda_1(\mathbf{B} - \omega_0 \mathbf{C}^*)$. However, the arguments in Lemmas 4 and 5, that apply to \mathbf{C}^* (since $\mathbf{C}^* \in \mathfrak{C}$), show that $\lambda_1(\mathbf{B} - \omega \mathbf{C}^*)$ is a concave function of ω and that the linear function $2\mu_1 - \omega \xi$ is an upper bound for it. This violates the strict inequality in: $\lambda_1(\mathbf{B} - \omega_0 \mathbf{C}') = 2\mu_1 - \omega_0 \xi < \lambda_1(\mathbf{B} - \omega_0 \mathbf{C}^*) \leq 2\mu_1 - \omega_0 \xi$.

Let S denote the two dimensional subspace spanned by $\mathbf{V}_1(\mathbf{B})$ and \mathbf{V} and \mathbf{C}'_S denote the restriction of \mathbf{C}' to this subspace (i.e., \mathbf{C}'_S agrees with \mathbf{C}' for any vector in this subspace and assigns $\mathbf{0}$ to any element outside). Also $\mathbf{V}_1(\mathbf{B} - \omega_0 \mathbf{C}') \in S$ by the definition of \mathbf{V} and $\lambda_1(\mathbf{B} - \omega_0 \mathbf{C}'_S) = \lambda_1(\mathbf{B} - \omega_0 \mathbf{C}')$.

Since $\mathbf{C}' \in \mathfrak{C}$, its restriction to S satisfies $\langle \mathbf{V}_1(\mathbf{B}), \mathbf{C}'_S(\mathbf{V}_1(\mathbf{B})) \rangle = \xi$. Also, $\mathbf{C}'_S \preceq \mathbf{B}/2$ since $\mathbf{C}' \preceq \mathbf{B}/2$. To satisfy the kissing constraint in Properties 2, we can add a component along \mathbf{V} : $\mathbf{C}''_S := \mathbf{C}'_S + \delta \mathbf{V} \otimes \mathbf{V} \in \mathfrak{C}$ for some $\delta \geq 0$. Since $\mathbf{C}''_S \succeq \mathbf{C}'_S$ we have $\lambda_1(\mathbf{B} - \omega_0 \mathbf{C}''_S) \leq \lambda_1(\mathbf{B} - \omega_0 \mathbf{C}'_S)$. Proposition 3 establishes $\mathbf{C}^* \uparrow \mathbf{C}''_S$, a relationship that applies to all $\omega \in [0, 2]$, that guarantees:

$$\lambda_1(\mathbf{B} - \omega_0 \mathbf{C}^*) \leq \lambda_1(\mathbf{B} - \omega_0 \mathbf{C}''_S) \leq \lambda_1(\mathbf{B} - \omega_0 \mathbf{C}'_S).$$

This is in contradiction with the strict inequality $\lambda_1(\mathbf{B} - \omega_0 \mathbf{C}^*) > \lambda_1(\mathbf{B} - \omega_0 \mathbf{C}')$. \square

Proof of Theorem 1. Theorem 2 implies $\lambda_1(\mathbf{B}^* - \omega \mathbf{C}^*) = \lambda_1(\mathbf{B} - \omega \mathbf{C}^*) \leq \lambda_1(\mathbf{B} - \omega \mathbf{C})$ for $\omega \in [0, 2]$ that establishes $C(\omega) \geq \lambda_{\max}(\mathcal{A}(\omega))$ based on (14). The covariance Σ_k converges to $\mathbf{0}$ according to the rate given by $\lambda_{\max}(\mathcal{A}(\omega))$ and the expected error at each step is the trace of Σ_k : $\mathbb{E}[\|\varepsilon_k\|^2] = \text{tr } \Sigma_k$. When i is drawn independently and identically distributed at each step of (1), the geometric rate of convergence is bound by $C(\omega)$ for every $\omega \in [0, 2]$. \square

4.4 Perron-Frobenius Theory For Positive Linear Maps

The superoperator \mathcal{A} defined in (7) plays the role of the iteration matrix — whose spectrum provides convergence analysis in classical iterative methods [Saad, 2003] — for randomized iterations. In this section we discuss the theoretical foundations that provide necessary properties on the spectrum of \mathcal{A} in the covariance analysis we have seen.

Recall the superoperator \mathcal{A} , for a fixed ω , denotes a linear map over the space of $n \times n$ matrices as:

$$\mathcal{A}(X) = \frac{1}{m} \sum_{i=1}^m (\mathbf{I} - \omega \mathbf{P}_i) X (\mathbf{I} - \omega \mathbf{P}_i).$$

Since orthogonal projection is a symmetric operator, for any symmetric positive semi-definite matrix \mathbf{X} the operation $(\mathbf{I} - \omega \mathbf{P}_i) \mathbf{X} (\mathbf{I} - \omega \mathbf{P}_i)$ preserves its positivity [Bhatia, 2009]. Hence the superoperator \mathcal{A} is a *positive linear map*, leaving the cone of symmetric positive semi-definite matrices invariant.

The spectra of positive linear maps on general (noncommutative) matrix algebras was studied in [Evans and Høegh-Krohn, 1978] that generalized the Perron-Frobenius theorem to this context. The spectral radius of a positive linear map is attained by an eigenvalue for which there exists an eigenvector that is positive semi-definite (see Theorem 6.5 in [Wolf, 2012]). The notion of *irreducibility* for positive linear maps guarantees that the eigenvalue is simple and the corresponding eigenvector is well-defined (up to a sign). What is more is that the eigenvector can be chosen to be a positive definite matrix. This guarantees that the power iterations in (7) converge along this positive definite matrix with the corresponding simple eigenvalue giving the rate of convergence.

For a system of equations in $\mathbf{A}\mathbf{x} = \mathbf{b}$, we examine the irreducibility of its corresponding superoperator \mathcal{A} for any given relaxation value ω . The criteria for irreducibility of positive linear maps was developed in [Farenick, 1996] and involve invariant subspaces. A collection S of (closed) subspaces of the vector space of $n \times n$ matrices is called trivial if it only contains $\{\mathbf{0}\}$ and the space itself. Given a bounded linear operator \mathbf{M} , let $\text{Lat}(\mathbf{M})$ denote the invariant subspace lattice of \mathbf{M} . The following theorem is a specialization of a more general result in [Farenick, 1996] (see Theorem 2) to our superoperator.

Theorem 3 (Irreducibility of the superoperator \mathcal{A}). *The positive linear map \mathcal{A} is irreducible if and only if, $\bigcap_{i=1}^m \text{Lat}(\mathbf{I} - \omega \mathbf{P}_i)$ is trivial.*

Based on this theorem, we establish the equivalence of the irreducibility of \mathcal{A} , in the sense of positive linear maps, to a geometric notion of irreducibility defined for alternating projections (2) that is inherently a geometric approach to solving a system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$. We recall the Frobenius notion of irreducibility for symmetric matrices. Such a matrix \mathbf{M} is called irreducible if it can not be transformed to block diagonal form by a permutation matrix Π :

$$\mathbf{M} = \Pi \begin{bmatrix} \mathbf{M}' & \mathbf{0} \\ \mathbf{0} & \mathbf{M}'' \end{bmatrix} \Pi^{-1},$$