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Global Regularity of Solutions for the 3D Non-resistive and Non-diffusive MHD-Boussinesq System with Axisymmetric Data

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Abstract
In this paper, we will show that solutions of the three-dimensional non-resistive and non-diffusive MHD-Boussinesq system are globally regular if the initial data is axisymmetric and the swirl components of the velocity and the magnetic vorticity are zero. Our main result extends previous ones on the three-dimensional non-resistive MHD system and non-diffusive Boussinesq system, and the method used here can also be applied to the magnetic Rayleigh-Bénard convection system.

Keywords Magnetohydrodynamics · Boussinesq · Rayleigh-Bénard convection · Axisymmetric · Global regularity

Mathematics Subject Classification (2020) 35Q35 · 76D03

1 Introduction

In this paper, we consider the global regularity problem for the three-dimensional (3D) magnetohydrodynamics (MHD)-Boussinesq system

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p - \mu \Delta u = h \cdot \nabla h + \rho e_3, \\ \partial_t h + u \cdot \nabla h - h \cdot \nabla u - \nu \Delta h = 0, \\ \partial_t \rho + u \cdot \nabla \rho - \kappa \Delta \rho = 0, \\ \nabla \cdot u = \nabla \cdot h = 0. \end{cases} \tag{1.1}$$

Here $u(t,x),\ h(t,x) \in \mathbb{R}^3$, $p(t,x) \in \mathbb{R}$ and $\rho(t,x) \in \mathbb{R}$ represent the velocity, magnetic field, pressure and temperature fluctuation. The vector $e_3 = (0,0,1)$ is the unit vector in the vertical direction. $\mu \geq 0$, $\nu \geq 0$ and $\kappa \geq 0$ stand for the constant viscosity, magnetic resistivity and thermal diffusivity, respectively. The MHD-Boussinesq system models the convection of an incompressible flow driven by the buoyant effect of a thermal field and the Lorenz force, generated by the magnetic field.

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We say that the MHD-Boussinesq system is non-resistive and non-diffusive, which means $\mu > 0$, but $\nu = \kappa = 0$. Without loss of generality, we set $\mu = 1$ and system (1.1) becomes

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p - \Delta u = h \cdot \nabla h + \rho e_3, \\ \partial_t h + u \cdot \nabla h - h \cdot \nabla u = 0, \\ \partial_t \rho + u \cdot \nabla \rho = 0, \\ \nabla \cdot u = \nabla \cdot h = 0. \end{cases} \tag{1.2}$$

The local well-posedness result of (1.2) can be founded in [22]. However, the global well-posedness is still wildly open even for the Navier-Stokes equations ($h = \rho \equiv 0$), let alone for the system (1.2). In this paper, we will show that a family of axisymmetric solutions to (1.2) are globally as regular as their initial data.

In the following, we will carry out our proof in the cylindrical coordinates (r, θ, z). That is, for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$

$$r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \arctan \frac{x_2}{x_1}, \quad z = x_3.$$

And the axisymmetric solution of system (1.2) is given by

$$u = u^r(t, r, z)e_r + u^\theta(t, r, z)e_\theta + u^z(t, r, z)e_z,$$

$$h = h^r(t, r, z)e_r + h^\theta(t, r, z)e_\theta + h^z(t, r, z)e_z,$$

$$\rho = \rho(t, r, z),$$

where the basis vectors e_r, e_θ, e_z are

$$e_r = (\frac{x_1}{r}, \frac{x_2}{r}, 0), \quad e_\theta = (-\frac{x_2}{r}, \frac{x_1}{r}, 0), \quad e_z = (0, 0, 1).$$

We will prove the global regularity of the following family of axisymmetric solutions

$$u = u^r(t, r, z)e_r + u^z(t, r, z)e_z, \quad h = h^\theta(t, r, z)e_\theta, \quad \rho = \rho(t, r, z). \tag{1.3}$$

Denote

$$\Phi_{k,c}(t) := c\,\underbrace{\exp(\cdots \exp}_{k \text{ times}}(ct)\cdots).$$

More precisely, we have the following theorem.

Theorem 1.1 *Let u_0, h_0 and ρ_0 be all axially symmetric data with $\nabla \cdot u_0 = 0$. Besides, we assume that $u_0^\theta = h_0^r = h_0^z = 0$. If $(u_0, h_0, \rho_0) \in H^3(\mathbb{R}^3)$ and $H_0 := \frac{h_0^\theta}{r} \in L^\infty(\mathbb{R}^3)$, then there exists a unique global solution (u, h, ρ) to the MHD-Boussinesq system (1.2) with data (u_0, h_0, ρ_0), which satisfies*

$$\|(u, h, \rho)(t, \cdot)\|_{H^3}^2 + \int_0^t \|\nabla u(t, \cdot)\|_{H^3}^2 ds \le \Phi_{3,c_0}(t), \tag{1.4}$$

where c_0 is a positive constant depending only on H^3 norms of u_0, h_0, ρ_0 and L^∞ norm of H_0.

Remark 1.2 It is not hard to extend the result of Theorem (1.1) to the case where $\mu > 0$, $\nu \geq 0$ and $\kappa \geq 0$ in (1.1) with the same initial data as that in Theorem (1.1). $\square$

Remark 1.3 When $h^\theta \equiv 0$, the global well-posedness result for the axisymmetric Navier-Stokes-Boussinesq can be found in [2, 17]. While if $\rho \equiv 0$, see [23] for the global well-posedness result for the axisymmetric MHD system. Our main result can be viewed as an extension of those in the above papers. $\square$

Remark 1.4 Define

$$H := \frac{h^\theta}{r}, \quad \Omega := \frac{w^\theta}{r}, \quad w^\theta = \partial_z u^r - \partial_r u^z.$$

The proof of Theorem 1.1 strongly depends on the special structure of the MHD-Boussinesq system in axisymmetric case with zero swirl components of the velocity and the magnetic vorticity. We will show that H and ρ satisfy the same transport equations and Ω satisfies a linear diffusive equation with inhomogeneous terms involving only in H and ρ. See (2.3). Then the $L_t^\infty L_x^2$ norm of Ω will be obtained. This is a key step for us to bootstrap the regularity of u, h and ρ.

Our proof combines the ideas that in [17] and [23]. Here we outline the main differences. Compared with that in [17], we need to deal with the extra term $\partial_z H$ in (2.3) and later much more estimates on the magnetic filed $h^\theta e_\theta$ are needed, which are nontrivial. Compared with that in [23], in our paper, the $L_t^\infty L_x^2 \cap L_t^2 H_x^1$ of Ω can not be obtained from the system (2.3) due to the appearance of $\frac{\partial_r \rho}{r}$. So the estimate $\|u^r/r\|_{L_t^1 L_x^\infty}$ in [23, Lemma 2.2] is not applicable to us. $\square$

Remark 1.5 This MHD-Boussinesq system (1.1) is closely related to a type of the Rayleigh-Bénard convection, which occurs in a horizontal layer of conductive fluid heated from below, with a presence of a magnetic field. The only difference between the magnetic Rayleigh-Bénard convection system and the MHD-Boussinesq system is that $(1.1)_3$ is replaced by the following equation

$$\partial_t \rho + u \cdot \nabla \rho - \kappa \Delta \rho = u^3.$$

Various physical theories and numerical experiments have been developed to study the magnetic Rayleigh-Bénard convection and related equations. See, for example, [31, 34] and references therein. The result in Theorem 1.1 can also be applied to the following non-resistive and non-diffusive magnetic Rayleigh-Bénard convection system

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p - \Delta u = h \cdot \nabla h + \rho e_3, \\ \partial_t h + u \cdot \nabla h - h \cdot \nabla u = 0, \\ \partial_t \rho + u \cdot \nabla \rho = u^3, \\ \nabla \cdot u = \nabla \cdot h = 0. \end{cases}$$

The proof is essentially the same as that for (1.2) with little difference. We omit the details. $\square$

If the fluid is not affected by the temperature, then our system (1.1) is reduced to the classical MHD system. There already have been many studies and fruitful results related to the well-posedness of the MHD system. Sermange-Temam [33] established the local existence and uniqueness of the solution and particularly the 2D local strong solution was proved to be global. Cao et al. in [5, 6] proved the global regularity of the MHD system for a variety of combinations of partial dissipation and diffusion in 2D space. Lin-Xu-Zhang [28] proved the global well-posedness of classical solutions for 2D non-resistive MHD under the assumption that the initial data is a small perturbation of a nonzero constant magnetic field. See also [32] for similar results. For the 3D case, readers can see [29, 36] for related results. Cai-Lei [4] and He-Xu-Yu [15] proved the global well-posedness of small initial data for the idea (inviscid and non-resistive) MHD system. Lei [23] proved the global regularity of classical solutions to the 3D viscid and non-resistive MHD system with a family of axisymmetric large data. Later, the regularity result on the 3D inviscid and resistive MHD equations with the same structure assumption for the solution was obtained in Hassainia [12]. Li-Pan [27] gave a single-component BKM-type regularity criterion for the inviscid and resistive axially symmetric Hall-MHD system. We also emphasized some partial regularity results and blow up criteria in [10, 13, 14, 24] and references therein.

On the other hand, if the fluid is not affected by the Lorentz force, then our system (1.2) is the classical Boussinesq system without diffusion. Many works and efforts have been made to study the well-posedness of the Cauchy problem for the Boussinesq system. In 2D case, Chae [8] and Hou-Li [19] independently proved the global regularity of solutions to the 2D Boussinesq system. And also Chae [8] considered the case of zero viscosity and non-zero diffusion. See [1, 16] for related results in critical space. For 3D case, Abidi et al. [2] and Hmidi-Rousset [17, 18] proved the global well-posedness of the Cauchy problem for the 3D axisymmetric Boussinesq system without swirl. Readers can see [7, 21] and references therein for more regularity results on the Boussinesq system.

For the full MHD-Boussinesq system, there are also some works concentrated on the global well-posedness of weak and strong solutions. In the 3D case, Larios-Pei [22] proved the local well-posedness results in Sobolev space. Recently, Bian-Pu [3] proved the global regularity of a family of axially symmetric large solutions to the MHDB system without magnetic resistivity and thermal diffusivity under the assumption that the support of the initial thermal fluctuation is away from the z-axis and its projection to the z-axis is compact. In this paper, we will improve the result in [3] by removing the "support set" assumption on the data of the thermal fluctuation. Regarding the MHD-Bénard system, some progress has also been made in 2D and 3D cases. See, e.g., [11, 37–39] and references therein.

Our paper is organized as follows. In Section 2, we reformulate our system in cylindrical coordinates and prove an a priori $L_t^\infty L_x^2$ estimate for Ω. In Section 3, we give the H^1 a priori estimate of the solution. In Section 4, we give the H^2 a priori estimate of the solution and prove Theorem 1.1. Throughout the paper, we use C or c to denote a generic constant which may be different from line to line. We also apply $A \lesssim B$ to denote $A \leq CB$.

2 Reformulation of the System and $L_t^\infty L_x^2$ Estimate of Ω

The axisymmetric MHD-Boussinesq system (1.2) in cylindrical coordinates read

$$
\begin{cases}
\partial_t u^r + (u^r\partial_r + u^z\partial_z)u^r - \dfrac{(u^\theta)^2}{r} + \partial_r P = (h^r\partial_r + h^z\partial_z)h^r - \dfrac{(h^\theta)^2}{r} + (\Delta - \dfrac{1}{r^2})u^r,\\
\partial_t u^\theta + (u^r\partial_r + u^z\partial_z)u^\theta + \dfrac{u^\theta u^r}{r} = (h^r\partial_r + h^z\partial_z)h^\theta + \dfrac{h^r h^\theta}{r} + (\Delta - \dfrac{1}{r^2})u^\theta,\\
\partial_t u^z + (u^r\partial_r + u^z\partial_z)u^z + \partial_z P = (h^r\partial_r + h^z\partial_z)h^z + \Delta u^z + \rho,\\
\partial_t h^r + (u^r\partial_r + u^z\partial_z)h^r - (h^r\partial_r + h^z\partial_z)u^r = 0,\\
\partial_t h^\theta + (u^r\partial_r + u^z\partial_z)h^\theta - (h^r\partial_r + h^z\partial_z)u^\theta + \dfrac{u^\theta h^r}{r} - \dfrac{h^\theta u^r}{r} = 0,\\
\partial_t h^z + (u^r\partial_r + u^z\partial_z)h^z - (h^r\partial_r + h^z\partial_z)u^z = 0,\\
\partial_t \rho + (u^r\partial_r + u^z\partial_z)\rho = 0,\\
\nabla\cdot u = \partial_r u^r + \dfrac{u^r}{r} + \partial_z u^z = 0, \quad \nabla\cdot h = \partial_r h^r + \dfrac{h^r}{r} + \partial_z h^z = 0,
\end{cases}
\tag{2.1}
$$

where the pressure $P = p + \frac{1}{2}|h|^2$ and $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$ is the usual Laplacian operator. By the uniqueness of local solutions, it is easy to see that if the initial data satisfy $u_0^\theta = h_0^r = h_0^z = 0$, then the solution of (2.1) will be the form of (1.3). In this situation, (2.1) can be simplified as

$$
\begin{cases}
\partial_t u^r + (u^r\partial_r + u^z\partial_z)u^r + \partial_r P = -\dfrac{(h^\theta)^2}{r} + (\Delta - \dfrac{1}{r^2})u^r,\\
\partial_t u^z + (u^r\partial_r + u^z\partial_z)u^z + \partial_z P = \Delta u^z + \rho,\\
\partial_t h^\theta + (u^r\partial_r + u^z\partial_z)h^\theta - \dfrac{u^r}{r}h^\theta = 0,\\
\partial_t \rho + (u^r\partial_r + u^z\partial_z)\rho = 0,\\
\dfrac{1}{r}\partial_r(ru^r) + \partial_z u^z = 0.
\end{cases}
\tag{2.2}
$$

Denote $H := \frac{h^\theta}{r}$ and $\Omega := \frac{w^\theta}{r}$. From (2.2), we can get

$$
\begin{cases}
\partial_t \Omega + u\cdot\nabla\Omega = (\Delta + \dfrac{2}{r}\partial_r)\Omega - \partial_z H^2 - \dfrac{\partial_r \rho}{r},\\
\partial_t H + u\cdot\nabla H = 0,\\
\partial_t \rho + u\cdot\nabla\rho = 0.
\end{cases}
\tag{2.3}
$$

First we have the following Proposition.

Proposition 2.1 *Let* (u, h, ρ) *be a smooth solution of* (2.2), *then we have*
(1) *for* $p \in [1, \infty]$ *and* $t \in \mathbb{R}_+$, *we have*

$$
\|(H(t), \rho(t))\|_{L^p} \le \|(H_0, \rho_0)\|_{L^p}; \tag{2.4}
$$

(2) *for* $u_0, h_0, \rho_0 \in L^2$ *and* $t \in \mathbb{R}_+$, *we have*

$$
\|(u(t), h(t))\|_{L^2}^2 + \int_0^t \|\nabla u(s)\| ds \le C_0(1+t)^2, \tag{2.5}
$$

where C_0 depends only on $\|(u_0, h_0)\|_{L^2}$ and $\|\rho_0\|_{L^2}$.

Proof of Proposition 2.1

Proof The estimate in (2.4) is classical for the transport equation with finite p. While if $p = \infty$, it is just the maximum principle. For the estimate in (2.5), we proceed the standard L^2 inner product estimate of system (1.2). Then we have

$$\frac{1}{2}\frac{d}{dt}\|(u(t), h(t))\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 \le \|u(t)\|_{L^2}\|\rho(t)\|_{L^2}. \tag{2.6}$$

This indicates that

$$\frac{d}{dt}\|(u(t), h(t))\|_{L^2} \le 2\|\rho(t)\|_{L^2}.$$

Integration on time indicates that

$$\begin{aligned}\|(u(t), h(t))\|_{L^2} &\le \|(u_0, h_0)\|_{L^2} + 2\int_0^t \|\rho(\tau)\|_{L^2} d\tau \\ &\le \|(u_0, h_0)\|_{L^2} + 2\|\rho_0\|_{L^2} t.\end{aligned}$$

Inserting this into (2.6) and integration on time, we have

$$\begin{aligned}&\frac{1}{2}\|(u(t), h(t))\|_{L^2}^2 + \int_0^t \|\nabla u(s)\|_{L^2}^2 ds \\ \le\ &\frac{1}{2}\|(u_0, h_0)\|_{L^2}^2 + \big(\|(u_0, h_0)\|_{L^2} + 2\|\rho_0\|_{L^2} t\big)\|\rho_0\|_{L^2} t.\end{aligned}$$

This gives (2.5). □

Based on Proposition 2.1, we have the following Proposition which gives the a priori $L_t^\infty L_x^2$ estimate of Ω.

Proposition 2.2 *Suppose (u, h, ρ) be the smooth solution of (1.2) with initial data (u_0, h_0, ρ_0) satisfying assumptions in Theorem 1.1, then we have, for $t \in \mathbb{R}_+$,*

$$\|\Omega(t)\|_{L^2} \le \Phi_{1, c_0}(t), \tag{2.7}$$

where c_0 is a positive constant depending only on H^2 norms of u_0, h_0, ρ_0 and L^∞ norm of H_0.

Before proving Proposition 2.2, we collect some useful estimates and identities.

Lemma 2.3 (Proposition 3.1, 3.2 and Lemma 3.3 of [17]) *Denote $\mathcal{L} = (\Delta + \frac{2}{r}\partial_r)^{-1}\frac{\partial_r}{r}$ and $\tilde{\mathcal{L}} = (\Delta + \frac{2}{r}\partial_r)^{-1}\frac{\partial_z}{r}$. Suppose $\rho \in H^2(\mathbb{R}^3)$ be axisymmetric, then for every $p \in [2, +\infty)$, there exists an absolute constant $C_p > 0$ such that*

$$\|\mathcal{L}\rho\|_{L^p} \le C_p\|\rho\|_{L^p}, \quad \|\tilde{\mathcal{L}}\rho\|_{L^p} \le C_p\|\rho\|_{L^p}. \tag{2.8}$$

Moreover, for any smooth axisymmetric function f, we have the identity

$$\mathcal{L}\partial_r f = \frac{f}{r} - \mathcal{L}\Big(\frac{f}{r}\Big) - \partial_z\tilde{\mathcal{L}} f. \tag{2.9}$$

Lemma 2.4 *For $1 < p < +\infty$, there exists an absolute constant $C_p > 0$ such that*

$$\|\nabla \frac{u^r}{r}\|_{L^p} \le C_p \|\Omega\|_{L^p}. \tag{2.10}$$

The proof of this lemma can be founded in many literatures, such as [23, A.5 on page 3213], [9, Lemma 2.3] or [30, Proposition 2.5].

Proof of Proposition 2.2

Proof Applying $\mathcal{L}$ to $(2.3)_3$, we get

$$\partial_t \mathcal{L}\rho + u \cdot \nabla \mathcal{L}\rho = -[\mathcal{L}, u \cdot \nabla]\rho, \tag{2.11}$$

where $[A, B] = AB - BA$ is the commutator.

Denote $L := \Omega - \mathcal{L}\rho$. Subtracting (2.11) from $(2.3)_1$, we have

$$\partial_t L + u \cdot \nabla L - (\Delta + \frac{2}{r}\partial_r)L = [\mathcal{L}, u \cdot \nabla]\rho - \partial_z H^2. \tag{2.12}$$

Taking L^2 inner product of (2.12), using integration by parts and divergence-free condition of u, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|L(t)\|_{L^2}^2 + \|\nabla L(t)\|_{L^2}^2 \\
\le &\int_{\mathbb{R}^3} \mathcal{L}(u \cdot \nabla\rho) L dx - \int_{\mathbb{R}^3} u \cdot \nabla(\mathcal{L}\rho) L dx - \int_{\mathbb{R}^3} \partial_z H^2 L dx \\
\le &\int_{\mathbb{R}^3} \mathcal{L}(u \cdot \nabla\rho) L dx + \int_{\mathbb{R}^3} (\mathcal{L}\rho) u \cdot \nabla L dx + \int_{\mathbb{R}^3} H^2 \partial_z L dx \\
:= &I_1 + I_2 + I_3.
\end{aligned}$$

Next we will estimate I_i $(i = 1, 2, 3)$ term by term. For I_1, first we make some computation on $\mathcal{L}(u \cdot \nabla\rho)$.

$$\begin{aligned}
\mathcal{L}(u \cdot \nabla\rho) &= \mathcal{L}(\nabla \cdot (u\rho)) \\
&= \mathcal{L}\Big(\partial_r(u^r\rho) + \frac{1}{r}(u^r\rho) + \partial_z(u^z\rho)\Big).
\end{aligned}$$

From (2.9), we have

$$\begin{aligned}
\mathcal{L}(u \cdot \nabla\rho) =& \mathcal{L}\partial_r(u^r\rho) + \mathcal{L}\Big(\frac{u^r\rho}{r}\Big) + \mathcal{L}\partial_z(u^z\rho) \\
=& \frac{u^r}{r}\rho - \partial_z\tilde{\mathcal{L}}(u^r\rho) + \partial_z\mathcal{L}(u^z\rho),
\end{aligned}$$

where we have used the fact that ∂_z is commutated with $\mathcal{L}$.

Then, using integration by parts, we get

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^3} \frac{u^r}{r}\rho L dx + \int_{\mathbb{R}^3} \tilde{\mathcal{L}}(u^r\rho)\partial_z L dx - \int_{\mathbb{R}^3} \mathcal{L}(u^z\rho)\partial_z L dx \\
&= I_1^1 + I_1^2 + I_1^3.
\end{aligned}$$

Using Hölder inequality, Sobolev embedding and (2.10), we have

$$
\begin{aligned}
|I_1^1| &\leq \|\frac{u^r}{r}\|_{L^6}\|\rho\|_{L^3}\|L\|_{L^2}\\
&\leq \|\nabla\frac{u^r}{r}\|_{L^2}\|\rho\|_{L^3}\|L\|_{L^2}\\
&\leq \|\Omega\|_{L^2}\|\rho\|_{L^3}\|L\|_{L^2}\\
&\leq (\|L\|_{L^2}+\|\mathcal{L}\rho\|_{L^2})\|\rho\|_{L^3}\|L\|_{L^2}.
\end{aligned}
$$

Using (2.8), (2.4) and Sobolev embedding, we have

$$
\begin{aligned}
|I_1^1| &\leq C(\|L\|_{L^2}+\|\rho\|_{L^2})\|\rho\|_{L^3}\|L\|_{L^2}\\
&\leq C\|\rho_0\|_{L^3}\|L\|_{L^2}^2+C\|\rho_0\|_{L^2}\|\rho_0\|_{L^3}\|L\|_{L^2}\\
&\leq C\|\rho_0\|_{H^2}\|L\|_{L^2}^2+C\|\rho_0\|_{H^2}^2\|L\|_{L^2}\\
&\leq C\big(\|\rho_0\|_{H^2}+1\big)\|L\|_{L^2}^2+C\|\rho_0\|_{H^2}^4.
\end{aligned}
$$

From (2.8), Proposition 2.1 and using Hölder inequality, Young inequality, we have

$$
\begin{aligned}
&|I_1^2|+|I_1^3|\\
&\leq \Big(\|\tilde{\mathcal{L}}(u^r\rho)\|_{L^2}+\|\mathcal{L}(u^r\rho)\|_{L^2}\Big)\|\partial_z L\|_{L^2}\\
&\leq C\|u^r\rho\|_{L^2}\|\partial_z L\|_{L^2}\\
&\leq C\|\rho_0\|_{L^\infty}\|u\|_{L^2}\|\partial_z L\|_{L^2}\\
&\leq C\|\rho_0\|_{L^\infty}^2\|u\|_{L^2}^2+\frac{1}{4}\|\partial_z L\|_{L^2}^2\\
&\leq C_0(1+t)^2+\frac{1}{4}\|\partial_z L\|_{L^2}^2,
\end{aligned}
$$

where C_0 is a positive constant depending only on H^2 norms of u_0, h_0, ρ_0 and L^∞ norm of H_0. Also, the same techniques as above imply

$$
\begin{aligned}
&|I^2|+|I^3|\\
&\leq \Big(\|(\mathcal{L}\rho)u\|_{L^2}+\|H^2\|_{L^2}\Big)\|\nabla L\|_{L^2}\\
&\leq \Big(\|\mathcal{L}\rho\|_{L^3}\|u\|_{L^6}+\|H\|_{L^\infty}\|H\|_{L^2}\Big)\|\nabla L\|_{L^2}\\
&\leq \Big(\|\rho\|_{L^3}\|\nabla u\|_{L^2}+\|H_0\|_{L^\infty}\|H_0\|_{L^2}\Big)\|\nabla L\|_{L^2}\\
&\leq \Big(\|\rho_0\|_{L^3}\|\nabla u\|_{L^2}+\|H_0\|_{L^\infty}\|h_0\|_{H^2}\Big)^2+\frac{1}{4}\|\nabla L\|_{L^2}^2\\
&\leq C_0\Big(1+\|\nabla u\|_{L^2}^2\Big)+\frac{1}{4}\|\nabla L\|_{L^2}^2.
\end{aligned}
$$

The above estimates indicate that

$$
\begin{aligned}
&\frac{d}{dt}\|L(t)\|_{L^2}^2+\|\nabla L(t)\|_{L^2}^2\\
&\leq C_0\Big(1+\|\nabla u\|_{L^2}^2\Big)+C_0(1+t)^2\\
&\quad+C\big(\|\rho_0\|_{H^2}+1\big)\|L\|_{L^2}^2+C\|\rho_0\|_{H^2}^4.
\end{aligned}
$$

Gronwall inequality indicates that

$$
\|L(t)\|_{L^2}^2+\int_0^t\|\nabla L(s)\|_{L^2}^2ds\leq\Phi_{1,c_0}(t).
$$

Then we have

$$
\begin{aligned}
\|\Omega(t)\|_{L^2}&\leq\|L\|_{L^2}+\|\mathcal{L}\rho\|_{L^2}\\
&\leq\|L\|_{L^2}+C\|\rho\|_{L^2}\\
&\leq\|L\|_{L^2}+\|\rho_0\|_{L^2}\leq\Phi_{1,c_0}(t).
\end{aligned}
$$

This proves Proposition 2.2 and (2.7) is valid. □

3 H^1 Estimate of the Solution

In this section, we give a prior H^1 estimate for the solution of system 2.2. We have the following Proposition.

Proposition 3.1 *Suppose (u,h,ρ) be the smooth solution of (1.2) with initial data (u_0,h_0,ρ_0) satisfying assumptions in Theorem 1.1, then we have, for $t\in\mathbb{R}_+$,*

$$
\|(\nabla u(t),\nabla h(t),\nabla\rho(t))\|_{L^2}^2+\int_0^t\|\nabla^2u(s)\|_{L^2}^2ds\leq\Phi_{2,c_0}(t),\tag{3.1}
$$

where c_0 is a positive constant depending only on H^2 norms of u_0,h_0,ρ_0 and L^∞ norm of H_0.

3.1 $L_t^\infty L^2\cap L_t^2H^1$ Estimate of ∇u

In cylindrical coordinates, the vorticity of the swirl-free axisymmetric velocity u is given by $w=\nabla\times u=w^\theta e_\theta$ and w^θ satisfies

$$
\partial_t w^\theta+u\cdot\nabla w^\theta-(\Delta-\frac{1}{r^2})w^\theta-\frac{u^r}{r}w^\theta=-\partial_z\frac{(h^\theta)^2}{r}-\partial_r\rho.
$$

Performing the standard L^2 inner product, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w^\theta\|_{L^2}^2+\|\nabla w^\theta\|_{L^2}^2+\|\frac{w^\theta}{r}\|_{L^2}^2\\
&\leq\int_{\mathbb{R}^3}\frac{u^r}{r}(w^\theta)^2dx-\int_{\mathbb{R}^3}\partial_z\frac{(h^\theta)^2}{r}w^\theta dx-\int_{\mathbb{R}^3}\partial_r\rho w^\theta dx\\
&:=I_1+I_2+I_3.
\end{aligned}
$$

We estimate I_i $(i = 1, 2, 3)$ separately. Hölder inequality and Gagliardo-Nirenberg interpolation inequality imply that

$$\begin{aligned}
I_1 &\le \|u^r\|_{L^3} \Big\| \frac{w^\theta}{r} \Big\|_{L^2} \|w^\theta\|_{L^6} \\
&\le \|u^r\|_{L^3} \big\|\Omega\big\|_{L^2} \|\nabla w^\theta\|_{L^2} \\
&\le C\|u^r\|_{L^3}^2 \big\|\Omega\big\|_{L^2}^2 + \frac{1}{4}\|\nabla w^\theta\|_{L^2}^2 \\
&\le C\|u^r\|_{L^2}\|\nabla u^r\|_{L^2} \big\|\Omega\big\|_{L^2}^2 + \frac{1}{4}\|\nabla w^\theta\|_{L^2}^2,
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}^3} \frac{(h^\theta)^2}{r} \partial_z w^\theta dx \\
&\le \|H\|_{L^\infty} \|h^\theta\|_{L^2} \|\nabla w^\theta\|_{L^2} \\
&\le C\|H\|_{L^\infty}^2 \|h^\theta\|_{L^2}^2 + \frac{1}{4}\|\nabla w^\theta\|_{L^2}^2.
\end{aligned}$$

Also

$$\begin{aligned}
I_3 &= -2\pi \int_{\mathbb{R}} \int_0^\infty \partial_r \rho w^\theta r dr dz \\
&= 2\pi \int_{\mathbb{R}} \int_0^\infty \rho \partial_r (w^\theta r) dr dz \\
&= 2\pi \int_{\mathbb{R}} \int_0^\infty \rho \partial_r w^\theta r dr dz + \int_{\mathbb{R}^3} \rho \frac{w^\theta}{r} dx \\
&\le \|\rho\|_{L^2} \|\nabla w^\theta\|_{L^2} + \|\rho\|_{L^2} \Big\| \frac{w^\theta}{r} \Big\|_{L^2} \\
&\le C\|\rho\|_{L^2}^2 + \frac{1}{4}\Big(\|\nabla w^\theta\|_{L^2}^2 + \Big\| \frac{w^\theta}{r} \Big\|_{L^2}^2 \Big).
\end{aligned}$$

The above estimates and Proposition 2.1, Proposition 2.2 indicate that

$$\begin{aligned}
&\frac{d}{dt}\|w^\theta\|_{L^2}^2 + \|\nabla w^\theta\|_{L^2}^2 + \Big\| \frac{w^\theta}{r} \Big\|_{L^2}^2 \\
&\le C\|u^r\|_{L^2}\|\nabla u^r\|_{L^2} \big\|\Omega\big\|_{L^2}^2 + C\|H\|_{L^\infty}^2 \|h\|_{L^2}^2 + C\|\rho\|_{L^2}^2 \\
&\le C_0(1+t)\Phi_{1,c_0}(t)\|\nabla u^r\|_{L^2} + C_0\|H_0\|_{L^\infty}^2 (1+t)^2 + C\|\rho_0\|_{L^2}^2.
\end{aligned}$$

Integration on time implies that

$$\begin{aligned}
&\|w^\theta(t)\|_{L^2}^2 + \int_0^t \|\nabla w^\theta(s)\|_{L^2}^2 ds + \int_0^t \Big\| \frac{w^\theta}{r}(s) \Big\|_{L^2}^2 ds \\
&\le \Phi_{1,c_0}(t).
\end{aligned} \tag{3.2}$$

Using the identity $\nabla \times \nabla \times u = -\Delta u + \nabla \nabla \cdot u$ and divergence-free condition of u, we have

$$\nabla u = \nabla(-\Delta)^{-1} \nabla \times w = \nabla(-\Delta)^{-1} \nabla \times (w^\theta e_\theta). \tag{3.3}$$

Calderón-Zygmund theorem implies that for any $1 < p < +\infty$, we have

$$\|\nabla u(t)\|_{L^p} \leq C_p \|w^\theta(t)\|_{L^p}, \qquad \|\nabla^2 u(t)\|_{L^p} \leq C_p \left(\|\nabla w^\theta(t)\|_{L^p} + \left\| \frac{w^\theta(t)}{r} \right\|_{L^p} \right). \tag{3.4}$$

From (3.2) and (3.4), we see that

$$\|\nabla u(t)\|_{L^2}^2 + \int_0^t \|\nabla^2 u(s)\|_{L^2}^2 ds \leq \Phi_{1,c_0}(t). \tag{3.5}$$

In order to bootstrap our energy estimates, we need the $L_t^1 L^\infty$ estimate of u. Before getting that, we first perform the $L_t^\infty L^4$ estimates of h^θ and w^θ.

3.2 $L_t^\infty L^4$ Estimate of h^θ and w^θ

Performing L^4 inner product of h^θ and using Hölder inequality, Gagliardo-Nirenberg interpolation inequality, we see that

$$\begin{aligned}
\frac{d}{dt} \|h^\theta(t)\|_{L^4}^4 &\leq 4 \int_{\mathbb{R}^3} \frac{u^r}{r} (h^\theta)^4 dx \\
&\leq 4 \|H\|_{L^\infty} \int_{\mathbb{R}^3} |u^r| (h^\theta)^3 dx \\
&\leq 4 \|H_0\|_{L^\infty} \|u^r\|_{L^4} \|h^\theta\|_{L^4}^3 \\
&\leq C \|H_0\|_{L^\infty} \|\nabla u^r\|_{L^2}^{3/4} \|u^r\|_{L^2}^{1/4} \|h^\theta\|_{L^4}^3.
\end{aligned}$$

Integration on time implies that

$$\|h^\theta(t)\|_{L^4} \leq \Phi_{1,c_0}(t). \tag{3.6}$$

Next performing the standard L^4 inner product of the w^θ equation, we have

$$\begin{aligned}
&\frac{1}{4} \frac{d}{dt} \|w^\theta\|_{L^4}^4 + \frac{3}{4} \|\nabla |w^\theta|^2\|_{L^2}^2 + \left\| \frac{|w^\theta|^2}{r} \right\|_{L^2}^2 \\
&\leq \int_{\mathbb{R}^3} \frac{u^r}{r} (w^\theta)^4 dx - \int_{\mathbb{R}^3} \partial_z \frac{(h^\theta)^2}{r} (w^\theta)^3 dx - \int_{\mathbb{R}^3} \partial_r \rho (w^\theta)^3 dx \\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

By the Hölder inequality, Gagliardo-Nirenberg interpolation inequality and Young inequality, we have

$$
\begin{aligned}
I_1 &\le \|u^r\|_{L^4}\Big\|\frac{w^\theta}{r}\Big\|_{L^2}\|(w^\theta)^3\|_{L^4}\\
&\le C\|u^r\|_{L^2}^{1/4}\|\nabla u^r\|_{L^2}^{3/4}\big\|\Omega\big\|_{L^2}\|(w^\theta)^2\|_{L^6}^{3/2}\\
&\le C\|u^r\|_{L^2}^{1/4}\|\nabla u^r\|_{L^2}^{3/4}\big\|\Omega\big\|_{L^2}\|\nabla(w^\theta)^2\|_{L^2}^{3/2}\\
&\le C\|u^r\|_{L^2}\|\nabla u^r\|_{L^2}^{3}\big\|\Omega\big\|_{L^2}^{4}+\frac{1}{8}\|\nabla(w^\theta)^2\|_{L^2}^{2}.
\end{aligned}
$$

Also, Hölder inequality and Young inequality imply

$$
\begin{aligned}
I_2 &= \int_{\mathbb{R}^3}\frac{(h^\theta)^2}{r}\partial_z(w^\theta)^3dx\\
&=3\int_{\mathbb{R}^3}\frac{(h^\theta)^2}{r}(w^\theta)^2\partial_z w^\theta dx\\
&\le C\|H\|_{L^\infty}\|h^\theta\|_{L^4}\|w^\theta\partial_z w^\theta\|_{L^2}\|w^\theta\|_{L^4}\\
&\le C\|H_0\|_{L^\infty}^4\|h^\theta\|_{L^4}^4+\frac{1}{8}\|\partial_z(w^\theta)^2\|_{L^2}^2+\|w^\theta\|_{L^4}^4,
\end{aligned}
$$

and the same, we have

$$
\begin{aligned}
I_3 &= -2\pi\int_{\mathbb{R}}\int_0^\infty \partial_r\rho(w^\theta)^3 rdrdz\\
&=2\pi\int_{\mathbb{R}}\int_0^\infty \rho\partial_r\big((w^\theta)^3r\big)drdz\\
&=6\pi\int_{\mathbb{R}}\int_0^\infty \rho(w^\theta)^2\partial_r w^\theta rdrdz+\int_{\mathbb{R}^3}\rho\frac{(w^\theta)^3}{r}dx\\
&\le C\|\rho\|_{L^\infty}\|\nabla(w^\theta)^2\|_{L^2}\|w^\theta\|_{L^2}+\|\rho\|_{L^\infty}\Big\|\frac{(w^\theta)^2}{r}\Big\|_{L^2}\|w^\theta\|_{L^2}\\
&\le C\|\rho\|_{L^\infty}^2\|w^\theta\|_{L^2}^2+\frac{1}{4}\|\nabla(w^\theta)^2\|_{L^2}^2+\frac{1}{4}\Big\|\frac{(w^\theta)^2}{r}\Big\|_{L^2}^2.
\end{aligned}
$$

Using (3.5), (3.6) and Proposition 2.1, the above inequalities imply

$$
\begin{aligned}
&\frac{d}{dt}\|w^\theta\|_{L^4}^4+\|\nabla|w^\theta|^2\|_{L^2}^2+\Big\|\frac{|w^\theta|^2}{r}\Big\|_{L^2}^2\\
&\le C\|w^\theta\|_{L^4}^4+C\|u^r\|_{L^2}\|\nabla u^r\|_{L^2}^3\big\|\Omega\big\|_{L^2}^4+C\|H_0\|_{L^\infty}^4\|h^\theta\|_{L^4}^4+C\|\rho\|_{L^\infty}^2\|w^\theta\|_{L^2}^2\\
&\le C\|w^\theta\|_{L^4}^4+\Phi_{1,c_0}(t).
\end{aligned}
$$

Gronwall inequality implies that

$$
\|w^\theta(t)\|_{L^4}^4+\int_0^t\|\nabla|w^\theta(s)|^2\|_{L^2}^2ds+\int_0^t\Big\|\frac{(w^\theta)^2}{r}(s)\Big\|_{L^2}^2ds\le\Phi_{1,c_0}(t).
$$

The above inequality implies that

$$\|\nabla u(t)\|_{L^4} \leq \Phi_{1,c_0}(t). \tag{3.7}$$

Next we give a crucial estimate for bootstrapping the regularity of the solution.

3.3 $L_t^1 L^\infty$ Estimate of ∇u

Applying $\nabla \times$ to $(1.2)_1$, we have

$$\partial_t w - \Delta w = -\nabla \times [u \cdot \nabla u - h \cdot \nabla h - \rho e_3]. \tag{3.8}$$

For a H^1 vector function f, we have

$$(\nabla \times f) \times f = f \cdot \nabla f - \frac{1}{2}\nabla |f|^2.$$

Then we have

$$\nabla \times (f \cdot \nabla f) = \nabla \times [(\nabla \times f) \times f].$$

Inserting this into (3.8), we have

$$\partial_t w - \Delta w = -\nabla \times [(\nabla \times u) \times u - (\nabla \times h) \times h - \rho e_3].$$

Then we can write it as

$$\begin{aligned} w &= e^{t\Delta} w_0 - \int_0^t e^{(t-s)\Delta} (\nabla \times [(\nabla \times u) \times u - (\nabla \times h) \times h - \rho e_3]) ds \\ &= e^{t\Delta} w_0 - \int_0^t e^{(t-s)\Delta} \nabla \times [(\nabla \times u) \times u] ds \\ &\quad + \int_0^t e^{(t-s)\Delta} \nabla \times [(\nabla \times h) \times h] ds + \int_0^t e^{(t-s)\Delta} \nabla \times [\rho e_3] ds. \end{aligned}$$

By a direct computation, if $h = h^\theta e_\theta$, we can get

$$\nabla \times [(\nabla \times h) \times h] = -2\frac{h^\theta}{r} \partial_z h^\theta e_\theta = -\partial_z (H h^\theta e_\theta).$$

Then we have

$$\begin{aligned} w = e^{t\Delta} w_0 &- \int_0^t e^{(t-s)\Delta} \nabla \times [(\nabla \times u) \times u] ds \\ &- \int_0^t e^{(t-s)\Delta} \partial_z (H h^\theta e_\theta) ds + \int_0^t e^{(t-s)\Delta} \nabla \times [\rho e_3] ds. \end{aligned}$$

Then by using (3.7), the $L_t^s L_x^q$ $(1 < s, q < +\infty)$ estimates for the parabolic equation of singular integral and potentials (see, for example, [25, 35]) give that

$$
\begin{aligned}
&\|\nabla w\|_{L^2([0,t],L^4(\mathbb{R}^3))}\\
\lesssim&\|\nabla w_0\|_{L^4(\mathbb{R}^3)}t^{1/2} + \|(\nabla\times u)\times u\|_{L^2([0,t],L^4(\mathbb{R}^3))}\\
&+\|Hh^\theta\|_{L^2([0,t],L^4(\mathbb{R}^3))} + \|\rho\|_{L^2([0,t],L^4(\mathbb{R}^3))}\\
\lesssim&\|\nabla w_0^\theta\|_{L^4(\mathbb{R}^3)}t^{1/2} + \|u\|_{L^\infty([0,t],L^\infty(\mathbb{R}^3))}\|\nabla\times u\|_{L^2([0,t],L^4(\mathbb{R}^3))}\\
&+\|H\|_{L^\infty([0,t],L^\infty(\mathbb{R}^3))}\|h^\theta\|_{L^2([0,t],L^4(\mathbb{R}^3))} + \|\rho_0\|_{L^2([0,t],L^4(\mathbb{R}^3))}\\
\lesssim&\|\nabla w_0\|_{L^4(\mathbb{R}^3)}t^{1/2} + \|u\|^{1/7}_{L^\infty([0,t],L^2(\mathbb{R}^3))}\|\nabla u\|^{6/7}_{L^\infty([0,t],L^4(\mathbb{R}^3))}\|\nabla u\|_{L^2([0,t],L^4(\mathbb{R}^3))}\\
&+\|H\|_{L^\infty([0,t],L^\infty(\mathbb{R}^3))}\|h^\theta\|_{L^2([0,t],L^4(\mathbb{R}^3))} + \|\rho_0\|_{L^2([0,t],L^4(\mathbb{R}^3))}\\
\le&\Phi_{1,c_0}(t).
\end{aligned}
$$

This, combining with (3.3), implies

$$\|\nabla^2 u\|_{L^2([0,t],L^4(\mathbb{R}^3))} \le C\|\nabla w\|_{L^2([0,t],L^4(\mathbb{R}^3))} \le \Phi_{1,c_0}(t).$$

Then by using Hölder inequality and Gagliardo-Nirenberg interpolation inequality, we have

$$
\begin{aligned}
\|\nabla u\|_{L^1([0,t],L^\infty(\mathbb{R}^3))} \le& \int_0^t \|\nabla u(s)\|^{1/4}_{L^4}\|\nabla^2 u(s)\|^{3/4}_{L^4}ds\\
\le&\|\nabla u(s)\|^{1/4}_{L^\infty[0,t],L^4(\mathbb{R}^3)}\Big(\int_0^t \|\nabla^2 u(s)\|^2_{L^4}ds\Big)^{3/8}(\int_0^t ds)^{5/8}\\
\le&\Phi_{1,c_0}(t).
\end{aligned}
\tag{3.9}
$$

Remark 3.2 In cylindrical coordinates, for the axially symmetric velocity u, a direct computation indicates that

$$|\nabla u| \approx |\tilde{\nabla}(u^r, u^\theta, u^z)| + \left|\left(\frac{u^r}{r}, \frac{u^\theta}{r}\right)\right|, \tag{3.10}$$

where $\tilde{\nabla} = (\partial_r, \partial_z)$. From (3.9) and (3.10), we can also have

$$\left\|\frac{u^r}{r}\right\|_{L^1([0,t],L^\infty(\mathbb{R}^3))} \le \Phi_{1,c_0}(t). \qquad \square \tag{3.11}$$

Next we will use $L_t^1 L^\infty$ estimate of ∇u to bootstrap the regularity of the solution.

3.4 $L_t^\infty L^p$ Estimate of $\nabla\rho$ and ∇h

Applying ∇ to the third equation of (1.2), we have

$$\partial_t \nabla\rho + u\cdot\nabla\nabla\rho = -\nabla u\cdot\nabla\rho.$$

We can have for $1 \le p \le +\infty$,

$$\|\nabla\rho(t)\|_{L^p} \le \|\nabla\rho_0\|_{L^p} + C\int_0^t \|\nabla u\|_{L^\infty}\|\nabla\rho(s)\|_{L^p}ds.$$

Using the estimate (3.9), Gronwall inequality indicates that

$$\|\nabla\rho(t)\|_{L^p} \le \Phi_{2,c_0}(t). \tag{3.12}$$

For the estimate of ∇h, first we write the second equation of (1.2) as

$$\partial_t h + u\cdot\nabla h = \frac{u^r}{r}h.$$

Applying ∇ to the above equality, we have

$$\partial_t\nabla h + u\cdot\nabla\nabla h = -\nabla u\cdot\nabla h + \frac{u^r}{r}\nabla h + \nabla u^r He_\theta + (\nabla\frac{1}{r})u^r h.$$

Noting

$$(\nabla\frac{1}{r})u^r h = -\frac{1}{r^2}e_r u^r h = -\frac{u^r}{r}He_r\otimes e_\theta,$$

and, as (3.10), $|H| = |\frac{h^\theta}{r}| \lesssim |\nabla h|$, we have, for $1\le p\le+\infty$,

$$\begin{aligned}\|\nabla h(t)\|_{L^p} \le \|\nabla h_0\|_{L^p} &+ C\int_0^t \|(\nabla u, u^r/r)\|_{L^\infty}\|\nabla h(s)\|_{L^p}ds\\ &+ C\int_0^t \|(\nabla u, u^r/r)\|_{L^\infty}\|H(s)\|_{L^p}ds.\end{aligned}$$

Also using the estimates (3.9) and (3.11), Gronwall inequality indicates that

$$\|\nabla h(t)\|_{L^p} \le \Phi_{2,c_0}(t). \tag{3.13}$$

Combining the estimates in (3.5), (3.12) and (3.13), we finish the proof of Proposition 3.1 and (3.1) is valid.

4 H^3 Estimate of the Solution and Proof of Theorem 1.1

In this section, we give a prior H^3 estimate for the solution of system 1.2. We have the following Proposition.

Proposition 4.1 *Suppose (u, h, ρ) be the smooth solution of (1.2) with initial data (u_0, h_0, ρ_0) satisfying assumptions in Theorem 1.1, then we have, for $t\in\mathbb{R}_+$,*

$$\|(\nabla^3 u(t), \nabla^3 h(t), \nabla^3\rho(t))\|_{L^2}^2 + \int_0^t \|\nabla^4 u(s)\|_{L^2}^2 ds \le \Phi_{3,c_0}(t),$$

where c_0 is a positive constant depending only on H^3 norms of u_0, h_0, ρ_0 and L^∞ norm of H_0.

Before proving this lemma, we give a commutator estimates for a triple product.

Lemma 4.2 *Let $m \in \mathbb{N}$, $m \geq 2$, and $f, g, k \in C_0^\infty(\mathbb{R}^3)$. Then the following estimate holds*:

$$\left|\int_{\mathbb{R}^3} [\nabla^m,\ f \cdot \nabla] g \nabla^m k dx\right| \leq C\ \|\nabla^m (f, g, k)\|_{L^2}^2 \|\nabla (f,\ g)\|_{L^\infty}.$$

Proof Proof of this lemma is a direct consequence of Hölder's inequality and the commutator estimate by Kato-Ponce [20]. See also [26, Lemma 2.3]. □

Proof of Proposition 4.1 Apply ∇^3 to $(1.2)_{1,2,3}$ to derive that

$$\begin{cases} \partial_t \nabla^3 u + u \cdot \nabla\nabla^3 u + \nabla\nabla^3 p - \Delta\nabla^3 u \\ \qquad = h \cdot \nabla\nabla^3 h + \nabla^3(\rho e_3) - [\nabla^3, u \cdot \nabla]u + [\nabla^3, h \cdot \nabla]h, \\ \partial_t \nabla^3 h + u \cdot \nabla\nabla^3 h - h \cdot \nabla\nabla^3 u = -[\nabla^3, u \cdot \nabla]h + [\nabla^3, h \cdot \nabla]u, \\ \partial_t \nabla^3 \rho + u \cdot \nabla\nabla^3 \rho = -[\nabla^3, u \cdot \nabla]\rho. \end{cases} \tag{4.1}$$

Performing the L^2 energy estimate of (4.1), noting that

$$\int_{\mathbb{R}^3} h \cdot \nabla\nabla^3 h \cdot \nabla^3 u dx + \int_{\mathbb{R}^3} h \cdot \nabla\nabla^3 u \cdot \nabla^3 h dx = 0,$$

we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|\nabla^3(u, h, \rho)(t, \cdot)\right\|_{L^2}^2 + \left\|\nabla^4 u(t, \cdot)\right\|_{L^2}^2 \\ &\quad = -\int_{\mathbb{R}^3} [\nabla^3, u \cdot \nabla]u \nabla^3 u dx + \int_{\mathbb{R}^3} [\nabla^3, h \cdot \nabla]h \nabla^3 u dx - \int_{\mathbb{R}^3} [\nabla^3, u \cdot \nabla]h \nabla^3 h dx \\ &\qquad + \int_{\mathbb{R}^3} [\nabla^3, h \cdot \nabla]u \nabla^3 h dx - \int_{\mathbb{R}^3} [\nabla^3, u \cdot \nabla]\rho \nabla^3 \rho dx + \int_{\mathbb{R}^3} \nabla^3(\rho e_3) \nabla^3 u dx. \end{aligned}$$

By Lemma 4.2, the above equation implies

$$\begin{aligned} &\frac{d}{dt}\left\|\nabla^3(u, h, \rho)(t, \cdot)\right\|_{L^2}^2 + \left\|\nabla^4 u(t, \cdot)\right\|_{L^2}^2 \\ &\lesssim \|\nabla^3(u, h, \rho)(t, \cdot)\|_{L^2}^2 \left(\|\nabla(u, h, \rho)(t, \cdot)\|_{L^\infty} + 1\right). \end{aligned}$$

Using Gronwall inequality, we can obtain that

$$\begin{aligned} &\left\|\nabla^3(u, h, \rho)(t, \cdot)\right\|_{L^2}^2 + \int_0^t \|\nabla^4 u(s)\|_{L^2}^2 ds \\ &\lesssim \left\|\nabla^3(u_0, h_0, \rho_0)\right\|_{L^2}^2 \exp\{\int_0^t \left(\|\nabla(u, h, \rho)(s, \cdot)\|_{L^\infty} + 1\right) ds\} \lesssim \Phi_{3, c_0}(t). \quad \square \end{aligned}$$

Proof of Theorem 1.1 Combining Proposition 2.1, Proposition 3.1 and Proposition 4.1, we can get the a priori estimate (1.4). Then the local existence and uniqueness theorem in [22] and the a priori estimate (1.4) together prove Theorem 1.1. □

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