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A Liouville theorem of Navier-Stokes equations with two periodic variables

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ABSTRACT

In this paper, we study the Liouville theorem of the stationary Navier-Stokes equations in $\mathbb{R}^3$. When the solution is periodic in two variables, we can prove that actually the solution is trivial (constant vector) under the assumption that one component of the velocity, vanishing at infinity, has finite Dirichlet integral and the other two components can have some growth with respect to the distance to the origin.

1. Introduction

In this paper, we consider the stationary Navier-Stokes (NS) equations

$$\begin{cases} u\cdot\nabla u+\nabla p-\Delta u=0, \\ \nabla\cdot u=0, \end{cases} \tag{1.1}$$

in an unbounded domain $\Omega\in\mathbb{R}^3$ with finite Dirichlet integral

$$\int_\Omega |\nabla u|^2 dx<+\infty. \tag{1.2}$$

Here $u(x)\in\mathbb{R}^3$, $p(x)\in\mathbb{R}$ represent the velocity vector and the scalar pressure respectively.

The main aim of our paper is to study the Liouville type theorem of the stationary NS equations (1.1) with (1.2). Such kind of solutions are referred as D-solutions. The study is partly motivated by the related Liouville problem of the stationary Navier-Stokes equations, which has attracted much attention in recent years and is still far from being fully understood. See for example [1–3,5,7,9–13] and the reference therein.

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In addition to the assumption that u vanishes at infinity, the 2-d problem is solved by Gilbarg and Weinberger [8]. However, for the 3 dimensional problem, it is not even known if a general D-solution has any definite decay rate comparing with the distance function near infinity. The following is a list of vanishing results with extra integral or decay assumptions for the solution u. Galdi [7] Theorem X.9.5 proved that if u is a homogeneous D-solution in the $\mathbb{R}^3$ and $u \in L^{9/2}(\mathbb{R}^3)$ vanishes at infinity, then $u = 0$. This result was improved by a log factor in Chae and Wolf [5]. In [3], Chae proved that homogeneous D-solutions in $\mathbb{R}^3$ are 0 if also $\Delta u \in L^{6/5}(\mathbb{R}^3)$. This condition scales the same way as $\|\nabla u\|_2$. Seregin [12] proved that homogeneous D-solutions in $\mathbb{R}^3$ is 0 if $u \in L^6(\mathbb{R}^3) \cap BMO^{-1}$. In a recent paper [11], Kozono etc showed that homogeneous D-solutions in $\mathbb{R}^3$ are 0 if either the vorticity $w = w(x)$ decays faster than $c/|x|^{5/3}$ at infinity, or the velocity u decays like $c/|x|^{2/3}$ with c being a small number. Most recently, in paper [1,2], the authors prove a series of Liouville theorems under some different assumptions, where the most significant two results are vanishing of the solution when u is an axially symmetric z-periodic solutions or u is a solution in a slab with non-slip boundary condition. See also [4,14,15] for more results in this aspect.

We use $\mathbf{S}$ to denote the 1 dimensional periodic domain $[0, 1]$. Our main result is the following:

Theorem 1.1. *Let u be a smooth solution to the problem*

$$\begin{cases} u \cdot \nabla u + \nabla p - \Delta u = 0, & in \quad \mathbf{S}^2 \times \mathbb{R}^1, \\ \nabla \cdot u = 0, & in \quad \mathbf{S}^2 \times \mathbb{R}^1, \\ u(x) = u(x_1 + 1, x_2, x_3) = u(x_1, x_2 + 1, x_3), & in \quad \mathbf{S}^2 \times \mathbb{R}^1, \end{cases} \tag{1.3}$$

with finite Dirichlet integral only on the vertical component of u

$$\int_{\mathbf{S}^2 \times \mathbb{R}} |\nabla u^3(x)|^2 dx < +\infty. \tag{1.4}$$

Then $u \equiv (c_1, c_2, 0)$, where c_1, c_2 are two constants, provided that

i) $|\nabla^k u^h| \le C(1 + |x^3|)^{1/4}$ for $k = 0, 1$ and $|\nabla^2 u^h| \le C(1 + |x^3|)^{1/2}$;

ii) $\lim\limits_{x_3 \to \infty} u^3 = 0$ uniformly for $(x_1, x_2) \in \mathbf{S}^2$.

Here $u^h = (u^1, u^2)$ denote the horizontal components of the velocity.

Remark 1.1. The assumption ii) and smoothness of u indicate that u^3 is bounded. Under the finite Dirichlet integral condition on u, i.e. $\int_{\mathbf{S}^2 \times \mathbb{R}} |\nabla u(x)|^2 dx < +\infty$, by following the proof of Theorem 4.1 in [6], there exists a constant vector u_0 such that $\lim\limits_{x_3 \to \infty} u = u_0$ uniformly for $(x_1, x_2) \in \mathbf{S}^2$. So it can be deduced that the velocity is bounded and constancy of it can be obtained by applying Theorem 1.1 under additional assumption ii). However, the assumption (1.4) in Theorem 1.1 only imposes the finite Dirichlet integral on u^3. So it seems reasonable to only assume boundedness on u^3.

Remark 1.2. Here for the pointwise L^∞ assumption on the solution u, we only need the vanishing of component u^3 at infinity, while the L^∞ norm of the horizontal components $u^h = (u^1, u^2)$ can even be increasing by order $1/4$ of the distance to the origin.

Remark 1.3. Compared the result here to that in [2] for the periodic case. We emphasize that although we assume the solution to be periodic in two variables, however we do not have the axially symmetric assumption and the horizontal components of u are not necessary bounded.

This paper is organized as follows. In Section 2, we give the proof of Theorem 1.1. Two key lemmas are presented. One shows that u^3 belong to $L^2(\mathbf{S}^2 \times \mathbb{R})$ and the other indicates that the supremum of the oscillation of p in a finitely long pipe of length R will be controlled by order $1/2$ of the length R.

Throughout the paper, we use C to denote a generic constant which may be different from line to line. We also apply $A \lesssim B$ to denote $A \leq CB$. For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we write $x = (x_h, x_3)$ for simplicity. Also sometimes we denote $u = (u^1, u^2, u^3)$ by (u^h, u^3). The symbol ∂_i stands for $\frac{\partial}{\partial x_i}$, for $i = 1, 2, 3$ and ∂_h stand for ∂_1, ∂_2.

2. Proof of Theorem 1.1

2.1. Estimates of $\boldsymbol{p}$ in a finite long pipe

In this subsection, we will show that the supremum of the oscillation of p in a finitely long pipe of length R will be controlled by order $1/2$ of the length R. That is the following

Lemma 2.1. *Under the assumptions of Theorem 1.1, for fixed $R > 1$, it follows that*

$$\sup_{x_h \in \mathbf{S}^2, x_3 \in [R, 2R]} |p(x) - p(0_h, R)| \leq CR^{1/2}, \tag{2.1}$$

where $0_h = (0, 0)$.

Proof. Horizontal derivatives' estimates for $\boldsymbol{p}$

From the first two equations of $(1.3)_1$, we can see that

$$\partial_h p = \Delta u^h - u \cdot \nabla u^h.$$

Then the assumption i) in Theorem 1.1 indicates that

$$|\partial_h p| \leq C(1 + |x_3|)^{1/2}. \tag{2.2}$$

An estimate for the horizontal integral of $\boldsymbol{p}$

Now we integrate the third equation of $(1.3)_1$ on $\mathbf{S}^2$ and use integration by parts. Then we can obtain

$$\begin{aligned}
\partial_3 \int_{\mathbf{S}^2} p dx_h &= \int_{\mathbf{S}^2} (\Delta u^3 - u \cdot \nabla u^3) dx_h \\
&= \int_{\mathbf{S}^2} \Big(\sum_{i=1}^{3} \partial_i^2 u^3 - (u^h \partial_h + u^3 \partial_3) u^3 \Big) dx_h \\
&= \int_{\mathbf{S}^2} \big(\partial_3^2 u^3 + u^3 \partial_h u^h - u^3 \partial_3 u^3 \big) dx_h.
\end{aligned} \tag{2.3}$$

At the last line of (2.3), we have used the fact that u is periodic in x_h and all the boundary terms cancel when we do integration by parts. Using the divergence-free condition, we get

$$\begin{aligned}
\partial_3 \int_{\mathbf{S}^2} p dx_h &= \partial_3^2 \int_{\mathbf{S}^2} u^3 dx_h - 2 \int_{\mathbf{S}^2} u^3 \partial_3 u^3 dx_h \\
&= \partial_3^2 \int_{\mathbf{S}^2} u^3 dx_h - \partial_3 \int_{\mathbf{S}^2} (u^3)^2 dx_h.
\end{aligned} \tag{2.4}$$

Fixed $x_3 \in [R, 2R]$, we integrate (2.4) from R to x_3, then we get

$$
\begin{aligned}
& \Big| \int_{\mathbf{S}^2} p(x_h, x_3) - p(x_h, R) dx_h \Big| \\
= & \Big| \int_R^{x_3} \partial_3^2 \int_{\mathbf{S}^2} u^3 dx_h d\tilde{x}_3 - \int_R^{x_3} \partial_3 \int_{\mathbf{S}^2} |u^3|^2 dx_h d\tilde{x}_3 \Big| \\
= & \Big| \int_{\mathbf{S}^2} (\partial_3 u^3(x_h, x_3) - \partial_3 u^3(x_h, R)) dx_h \\
& - \int_{\mathbf{S}^2} (u^3)^2(x_h, x_3) - (u^3)^2(x_h, R) dx_h \Big| \\
= & \Big| \int_{\mathbf{S}^2} (-\partial_h u^h(x_h, x_3) + \partial_h u^h(x_h, R)) dx_h \\
& - \int_{\mathbf{S}^2} (u^3)^2(x_h, x_3) - (u^3)^2(x_h, R) dx_h \Big| \\
\le & 2 \sum_{x_h \in \mathbf{S}^2, x_3 \in [R, 2R]} (|\partial_h u^h| + |u^3|^2) \int_{\mathbf{S}^2} dx_h \\
\lesssim & R^{1/4}, \qquad \text{for} \quad R > 1,
\end{aligned} \tag{2.5}
$$

where we have used the derivative assumption i) in Theorem 1.1.

Proof of (2.1)

By using mean value theorem of integrals, from (2.5), there exists a $x_h(x_3)$ such that

$$
|p(x_h(x_3), x_3) - p(x_h(x_3), R)| \lesssim R^{1/4}. \tag{2.6}
$$

Then for any $x \in \mathbf{S}^2 \times [R, 2R]$, we have

$$
\begin{aligned}
& |p(x) - p(0_h, R)| \\
= & |p(x_h, x_3) - p(x_h(x_3), x_3) \\
& + p(x_h(x_3), x_3) - p(x_h(x_3), R) + p(x_h(x_3), R) - p(0_h, R)| \\
\le & |p(x_h, x_3) - p(x_h(x_3), x_3)| \\
& + |p(x_h(x_3), x_3) - p(x_h(x_3), R)| + |p(x_h(x_3), R) - p(0_h, R)| \\
\lesssim & |\partial_h p(x_h^*, x_3)(x_h - x_h(x_3))| + |\partial_h p(x_h^\dagger, R) x_h(x_3)| + R^{1/4}.
\end{aligned} \tag{2.7}
$$

At the last line of (2.7), we have used the mean value theorem of differentials and (2.6). Then the horizontal derivatives estimates (2.2) for p indicates that

$$
|p(x_h, x_3) - p(0_h, R)| \lesssim (1 + |x_3|)^{1/2} + R^{1/4} \lesssim R^{1/2},
$$

for $x_3 \in [R, 2R]$ and $R > 1$. $\square$

2.2. L^2 boundedness of u^3 and trivialness of u

Before finishing the proof of Theorem 1.1, we need a L^2 boundedness of u^3.

Lemma 2.2. *Under the assumptions of Theorem 1.1 it follows that*

$$\|u^3\|_{L^2(\mathbf{S}^2\times\mathbb{R}^1)} \lesssim 1.$$

Proof. Integrating the divergence free condition $(1.3)_2$ in $\mathbf{S}^2$ about x_h and using the periodicity of u in x_h, we can obtain

$$\partial_3 \int\limits_{\mathbf{S}^2} u^3 dx_h = -\int\limits_{\mathbf{S}^2} (\partial_1 u^1 + \partial_2 u^2) dx_h = 0.$$

So the above equality indicates that $\int_{\mathbf{S}^2} u^3 dx_h = Const$. Moreover since $\lim\limits_{x_3\to\infty} u^3 = 0$, it is easy to see

$$\int\limits_{\mathbf{S}^2} u^3 dx_h = 0.$$

Now we can use Poincare inequality to prove that $u^3 \in L^2(\mathbf{S}^2\times\mathbb{R})$. Actually

$$\begin{aligned}
\|u^3\|^2_{L^2(\mathbf{S}^2\times\mathbb{R})} &= \int\limits_{\mathbb{R}}\int\limits_{\mathbf{S}^2} |u^3|^2 dx_h dx_3 \\
&= \int\limits_{\mathbb{R}}\int\limits_{\mathbf{S}^2} \Big|u^3 - \frac{1}{|\mathbf{S}^2|}\int u^3 dx_h\Big|^2 dx_h dx_3 \\
&\lesssim |\mathrm{diam}(\mathbf{S}^2)|^2 \int\limits_{\mathbb{R}}\int\limits_{\mathbf{S}^2} |\partial_h u^3|^2 dx_h dx_3 \\
&\lesssim \|\nabla u^3\|^2_{L^2(\mathbf{S}^2\times\mathbb{R})},
\end{aligned}$$

where $\mathrm{diam}(\mathbf{S}^2)$ is the diameter of domain $\mathbf{S}^2$. $\square$

Trivialness of u

Now let $\phi = \phi(x_3)$ be a smooth cut-off function satisfying

$$\begin{cases} \phi(x_3) = 1, & |x_3| \in [0,1], \\ \phi(x_3) = 0, & |x_3| \geq 2, \\ \ \ 0 \leq \phi \leq 1, & \forall x_3 \in \mathbb{R}, \end{cases}$$

with ϕ' and ϕ'' being bounded. And we set $\phi_R(x_3) = \phi\left(\frac{x_3}{R}\right)$ with $R > 1$. Testing the first equation of (1.3)

$$u\cdot\nabla u + \nabla p - \Delta u = 0$$

with $u\phi_R$, we achieve that

$$\int\limits_{\mathbf{S}^2\times\mathbb{R}} u\phi_R \Delta u dx = \int\limits_{\mathbf{S}^2\times\mathbb{R}} u\phi_R \left(u\cdot\nabla u + \nabla(p - p(0_h, R))\right) dx.$$

Direct integration by parts imply

$$
\begin{aligned}
&\int\limits_{\mathbf{S}^2\times\mathbb{R}} |\nabla u|^2\phi_R dx - \frac{1}{2}\int\limits_{\mathbf{S}^2\times\mathbb{R}} |u|^2\Delta\phi_R dx\\
=&\frac{1}{2}\int\limits_{\mathbf{S}^2\times\mathbb{R}} |u|^2 u\cdot\nabla\phi_R dx + \int\limits_{\mathbf{S}^2\times\mathbb{R}} (p-p(0_h,R))u\cdot\nabla\phi_R dx\\
=&\frac{1}{2}\int\limits_{\mathbf{S}^2\times\mathbb{R}} |u|^2 u^3\partial_3\phi_R dx + \int\limits_{\mathbf{S}^2\times\mathbb{R}} (p-p(0_h,R))u^3\partial_3\phi_R dx.
\end{aligned}
$$

Then by using Hölder inequality, it follows that

$$
\begin{aligned}
&\int\limits_{\mathbf{S}^2\times\mathbb{R}} |\nabla u|^2\phi_R dx\\
\lesssim&\int\limits_{\mathbf{S}^2\times[R,2R]} |u|^2\cdot|\partial_3^2\phi_R|dx + \int\limits_{\mathbf{S}^2\times[R,2R]} \big(|u|^2+|p-p(0_h,R)|\big)|u^3\partial_3\phi_R|dx\\
\lesssim&\frac{\sup\limits_{x\in\mathbf{S}^2\times[R,2R]}|u|^2}{R^2}\int\limits_{\mathbf{S}^2\times[R,2R]} dx\\
&+\frac{\sup\limits_{x\in\mathbf{S}^2\times[R,2R]}\big(|u|^2+|p-p(0_h,R)|\big)}{R}\|u^3\|_{L^2(\mathbf{S}^2\times[R,2R])}\|1\|_{L^2(\mathbf{S}^2\times[R,2R])}\\
\lesssim&\frac{\sup\limits_{x\in\mathbf{S}^2\times[R,2R]}|u|^2}{R} + \frac{\sup\limits_{x\in\mathbf{S}^2\times[R,2R]}\big(|u|^2+|p-p(0_h,R)|\big)}{R^{1/2}}\|u^3\|_{L^2(\mathbf{S}^2\times[R,2R])}\\
:=&I_1+I_2.
\end{aligned}
\tag{2.8}
$$

Using our assumption on u, when $x\in\mathbf{S}^2\times[R,2R]$,

$$
|u|\lesssim(1+|x_3|)^{1/4}\lesssim R^{1/4}.
$$

Then, I_1 satisfies

$$
I_1\lesssim\frac{R^{1/2}}{R}\lesssim R^{-1/2}\to 0 \quad \text{as} \quad R\to+\infty. \tag{2.9}
$$

By using Lemma 2.1, it follows that

$$
I_2\lesssim\frac{R^{1/2}}{R^{1/2}}\|u^3\|_{L^2(\mathbf{S}^2\times[R,2R])} = \|u^3\|_{L^2(\mathbf{S}^2\times[R,2R])}\to 0 \quad \text{as} \quad R\to+\infty, \tag{2.10}
$$

where we have used Lemma 2.2.

Combining (2.8), (2.9) and (2.10), we can get

$$
\int\limits_{\mathbf{S}^2\times\mathbb{R}} |\nabla u|^2 dx = 0,
$$

by letting $R\to\infty$, which means u is constant vector. Recalling u^3 vanishes at the far field, we deduce that u has the form $u=(c_1,c_2,0)$. This finishes the proof of **Theorem 1.1**.

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