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GLOBALLY ANALYTICAL SOLUTIONS OF THE COMPRESSIBLE OLDROYD-B MODEL WITHOUT RETARDATION*

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Abstract. In this paper, we prove the global existence of analytical solutions to the compressible Oldroyd-B model without retardation near a nonvacuum equilibrium in $\mathbb{R}^n$ $(n=2,3)$. Zero retardation results in zero dissipation in the velocity equation, which is the main difficulty that prevents us from obtaining the long time well-posedness of solutions. Through dedicated analysis, we find that the linearized equations of this model have damping effects, which ensure the global-in-time existence of small data solutions. However, the nonlinear quadratic terms have one more order derivative than the linear part and no good structure is discovered to overcome this derivative loss problem. So we can only build the result in the analytical energy space rather than Sobolev space with finite order derivatives.

Key words. global analytical solution, Oldroyd-B, without retardation, derivative loss

MSC codes. 76A05, 76D03

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1. Introduction.

The compressible Oldroyd-B model describes the motion of a type of viscoelastic fluid with memory, which is non-Newtonian, and the stress tensor of it is not linearly dependent on the deformation tensor. It is governed by following the conservations of mass and momentum, and the constitutive law.

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \nabla \cdot (\rho u \otimes u) - \mu\omega(\Delta u + \nabla \operatorname{div} u) + \nabla \pi = \operatorname{div} \tau, \\ \nu(\tau_t + u \cdot \nabla \tau + Q(\tau, \nabla u)) + \tau = 2\mu(1-\omega)\mathbb{D}(u), \end{cases} \tag{1.1}$$

where $\rho \in \mathbb{R}$, $u \in \mathbb{R}^n$ $(n=2,3)$, and $\tau \in \mathbb{R}^{n\times n}$ $(n=2,3)$ (a symmetric matrix) are the density, the velocity, and the non-Newtonian part of the stress tensor, respectively. And $\pi \in \mathbb{R}$ is the scalar pressure. The constants $\mu > 0, \nu > 0$, and $\omega \in [0,1]$ are viscosity, relaxation time, and the coupling constant. The bilinear term Q has the following form:

$$Q(\tau, \nabla u) = \tau \mathbb{W}(u) - \mathbb{W}(u)\tau - b(\mathbb{D}(u)\tau + \tau\mathbb{D}(u)),$$

where $b \in [-1,1]$ is a parameter, $\mathbb{D}(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the deformation tensor, and $\mathbb{W}(u) = \frac{1}{2}(\nabla u - (\nabla u)^T)$ is the vorticity tensor.

In this paper, we consider the case $\omega = 0$, which corresponds to the situation of zero retardation as depicted in Oldroyd [21]. For presentation of the physical

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background of this compressible model without retardation, we give a brief derivation. Following [4], a compressible fluid is subject to the following equations:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \nabla \cdot (\rho u \otimes u) = \nabla \cdot \sigma, \end{cases} \tag{1.2}$$

where σ, a symmetric matrix, is the stress tensor and can be decomposed as $\sigma = \tilde{\tau} - \pi \mathrm{Id}$. Here $\tilde{\tau}$ is the tangential part of the stress tensor and $-\pi\mathrm{Id}$ is the normal part. For a Newtonian fluid, $\tilde{\tau}$ depends linearly on ∇u and more precisely

$$\tilde{\tau} = 2\mu \mathbb{D}(u).$$

However, when we consider the non-Newtonian fluid of the Oldroyd-B model, the constitutive law satisfied by $\tilde{\tau}$ is

$$\tilde{\tau} + \nu \frac{\mathcal{D}\tilde{\tau}}{\mathcal{D}t} = 2\mu \left(\mathbb{D}(u) + \tilde{\nu} \frac{\mathcal{D}\mathbb{D}(u)}{\mathcal{D}t} \right), \tag{1.3}$$

where, for a tensor $\tilde{\tau}$,

$$\frac{\mathcal{D}\tilde{\tau}}{\mathcal{D}t} = \partial_t \tilde{\tau} + u \cdot \nabla \tilde{\tau} + Q(\tilde{\tau}, \nabla u). \tag{1.4}$$

In (1.3), ν is the relaxation time, $\tilde{\nu}$ is the retardation time $(0 \le \tilde{\nu} \le \nu)$, and $\mu > 0$ is the dynamical viscosity of the fluid. Fluids of this type have both elastic properties and viscous properties. We divide the Oldroyd-B model into the following three types of mathematical system according to the choice of the retardation time and the relaxation time.

Case $0 = \tilde{\nu} = \nu$: This corresponds to the purely viscous case (compressible Navier–Stokes equation). This model can be found in [15, 16], and rigorous mathematical study on the compressible Navier–Stokes equations has been initiated since then. By now, there is a vast literature concerning well-posedness of the compressible Navier–Stokes equations. Since, in this paper, we consider the non-Newtonian fluid of the Oldroyd-B model, we do not pursue any further references in detail. Readers can refer to papers that cite [15, 16] to see more detailed study on the compressible Navier–Stokes equations.

Case $0 < \tilde{\nu} \le \nu$: If we define

$$\tau := \tilde{\tau} - 2\mu \frac{\tilde{\nu}}{\nu} \mathbb{D}(u),$$

then from (1.3), the second equation of (1.2), and (1.4), we see that

$$\begin{cases} (\rho u)_t + \nabla \cdot (\rho u \otimes u) - \mu \frac{\tilde{\nu}}{\nu} (\Delta u + \nabla \operatorname{div} u) + \nabla \pi = \operatorname{div} \tau, \\ \nu(\tau_t + u \cdot \nabla \tau + Q(\tau, \nabla u)) + \tau = 2\mu \left(1 - \frac{\tilde{\nu}}{\nu}\right) \mathbb{D}(u). \end{cases}$$

This corresponds to system (1.1) with $\omega = \frac{\tilde{\nu}}{\nu} \in (0, 1]$. This model is an extension of the classical incompressible Oldroyd-B system introduced by Oldroyd in [21], which has been studied extensively in the literature.

Case $0 = \tilde{\nu} < \nu$: This is the purely elastic case (the Maxwell model), which corresponds to (1.1) with $\omega = 0$. As far as the author knows, there is little literature that considers this situation. In this paper, we consider this case, and the domain is chosen as $\mathbb{R}^3$. The exact values of the positive constants μ and ν play no essential role in our following analysis. We set $\mu = \nu = 1$ for simplicity. Then system (1.1) becomes

$$
\begin{cases}
\rho_t + \mathrm{div}(\rho u) = 0, & \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla \pi = \mathrm{div}\, \tau, & (x,t) \in \mathbb{R}^n \times (0, +\infty), \\
\tau_t + u \cdot \nabla \tau + \tau + Q(\tau, \nabla u) = 2\mathbb{D}(u), & \\
(\rho, u, \tau)\big|_{t=0} = (\rho_0, u_0, \tau_0), & x \in \mathbb{R}^n.
\end{cases} \tag{1.5}
$$

We will consider that (ρ, u, τ) is a small analytical perturbation around the nonvacuum equilibrium $(1, 0, 0)$. Also, for simplicity, we consider the pressure π satisfying the γ-law and assume that $\pi(\rho) = \frac{1}{\gamma}\rho^\gamma$.

We show the global existence of analytic solutions of system (1.5) in Besov spaces by using the Fourier analysis method. We have the following theorem.

THEOREM 1.1. *Let $\lambda_0 > 0$ be a fixed constant. Assume that the initial data (ρ_0, u_0, τ_0) satisfy $e^{2\lambda_0 \Lambda}(\rho_0 - 1, u_0, \tau_0) \in \dot{B}^{\frac{n}{2}-1} \cap \dot{B}^{\frac{n}{2}}$. Then there exist two positive constants ϵ_0 and C_0, independent of λ_0, such that for any $0 < \epsilon \leq \epsilon_0$, if*

$$
\left\| e^{2\lambda_0 \Lambda}(\rho_0 - 1, u_0, \tau_0) \right\|_{\dot{B}^{\frac{n}{2}-1} \cap \dot{B}^{\frac{n}{2}}} \leq \epsilon \lambda_0,
$$

then system (1.5) *admits a unique solution $(\rho(t), u(t), \tau(t))$ satisfying, for any $t \in (0, +\infty)$,*

$$
\begin{aligned}
&\left\| e^{\lambda_0 \Lambda}(\rho - 1, u, \tau)(t) \right\|_{\dot{B}^{\frac{n}{2}-1} \cap \dot{B}^{\frac{n}{2}}} + \left\| e^{\lambda_0 \Lambda}(\rho - 1, u, \tau) \right\|_{L^1_t(\dot{B}^{\frac{n}{2}})} \\
&\quad \leq C_0 \left\| e^{2\lambda_0 \Lambda}(\rho_0 - 1, u_0, \tau_0) \right\|_{\dot{B}^{\frac{n}{2}-1} \cap \dot{B}^{\frac{n}{2}}}.
\end{aligned}
$$

Remark 1.2. The notation of Besov spaces in Theorem 1.1 can be found in section 2.

We review some literature related to the fluid model of Oldroyd-B type. For the incompressible Oldroyd-B model, Guillopé and Saut [13] gave the local well-posedness of the regular solution and the global existence of small smooth solutions in a smooth open domain with the no-slip boundary condition. Lions and Masmoudi [20] constructed global weak solutions for general initial conditions with the assumptions $b = 0$. Chemin and Masmoudi [4] gave the existence and uniqueness of locally large and globally small solutions in the critical Besov space. Some remarks on the blow-up criteria are shown in Lei, Masmoudi, and Zhou [18]. Readers can also refer to [6, 31, 11] for more global existence results in Besov spaces. If we consider the case $\omega = 0$ but the equation of τ contains a diffusion term $-\Delta\tau$, Elgindi and Rousset [9] proved the global existence of small smooth solutions and large smooth solutions for the case $Q(\tau, \nabla u) \equiv 0$ in 2D. See also Elgindi and Liu [8] for the 3D result. Zhu [28] proved the global existence of small solutions without damping effect (τ is missing in the constitutive law) in Sobolev spaces. See [5, 26] for extensions in critical Besov spaces, respectively. Recently, Zi [30] considered the vanishing viscosity limits of the 3D incompressible Oldroyd-B model in analytical space.

For the compressible Oldroyd-B model, there is relatively less literature. The incompressible limit problems in torus and bounded domain were investigated in Lei [17], and Guillopé, Salloum, and Talhouk [12], respectively for well-prepared data. The case of ill-prepared initial data was studied by Fang and Zi [10]. The global well-posedness and decay rates results in the H^2-framework for the three-dimensional case was given in Zhou, Zhu, and Zi [27]. See Zhu [29] and Pan, Xu, and Zhu [24] for the global existence results in Sobolev space and critical Besov space for the model without damping mechanism. Readers can refer to [19, 25, 14, 23] and references therein for more compressible Oldroyd-type model results.

Throughout the paper, $C_{a,b,c,\ldots}$ denotes a positive constant depending on a, b, $c,\ldots$, which may be different from line to line. We also apply $A \lesssim_{a,b,c,\cdots} B$ to denote $A \leq C_{a,b,c,\ldots} B$. $A \approx_{a,b,c,\cdots} B$ means $A \lesssim_{a,b,c,\cdots} B$ and $B \lesssim_{a,b,c,\cdots} A$. For a norm $\|\cdot\|$, we use $\|(f, g, \cdots)\|$ to denote $\|f\| + \|g\| + \cdots$. For a function $f(t,x)$, $\|f(t)\|_{L^p(\mathbb{R}^n)}$ denotes the usual spacial L^p norm for $1 \leq p \leq +\infty$. Besides, if $p = 2$, we will simply denote $\|f(t)\|_{L^2(\mathbb{R}^n)}$ by $\|f(t)\|$. We use $[A, B] = AB - BA$ to denote the commutator of A and B.

We finish this section with an informal analysis for the linearized system of (1.5). The result in Theorem 1.1 depends heavily on the intrinsic damping effects for the linearized system. The damping effect for τ is obvious from the third equation of (1.5), while the damping effects for the high frequencies and the dissipation effects for the low frequencies of $a := \rho - 1$ and u are hidden in the linear structure. Let us explain this in L^2 space in some detail. The linear system of (1.5) satisfies

$$\begin{cases} a_t + \operatorname{div} u = 0, \\ u_t + \nabla a - \operatorname{div} \tau = 0, \\ \tau_t + \tau - 2\mathbb{D}(u) = 0. \end{cases} \tag{1.6}$$

Damping for τ and div τ. Multiplying $(1.6)_1$ by $2a$, $(1.6)_2$ by $2u$, and $(1.6)_3$ by τ, and then integrating the resulting equations in $\mathbb{R}^n$, we can obtain that

$$\frac{d}{dt}\|(a,u)\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\tau\|_{L^2}^2 + \|\tau\|_{L^2}^2 = 0. \tag{1.7}$$

Applying div to the third equation of (1.6), then multiplying the resulted equation by divτ and integrating it in $\mathbb{R}^n$, we can obtain that

$$\frac{1}{2}\frac{d}{dt}\|\mathrm{div}\tau\|_{L^2}^2 + \|\mathrm{div}\tau\|_{L^2}^2 - \langle \nabla \mathrm{div} u + \Delta u, \mathrm{div}\tau\rangle = 0, \tag{1.8}$$

where $\langle \bullet, \bullet \rangle$ denotes the L^2 inner product in $\mathbb{R}^n$. Multiplying $(1.6)_1$ by $-2\Delta a$ and $(1.6)_2$ by $-2\Delta u$, and then integrating the resulting equations in $\mathbb{R}^n$, we obtain that

$$\frac{d}{dt}\|\nabla(a,u)\|_{L^2}^2 + 2\langle \Delta u, \mathrm{div}\tau\rangle = 0. \tag{1.9}$$

Combining (1.7), (1.8), and (1.9), after cancellation by using integration by parts, we can see that

$$\frac{d}{dt}\|(a,u,\nabla a,\nabla u)\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|(\tau,\mathrm{div}\tau)\|_{L^2}^2 + \|(\tau,\mathrm{div}\tau)\|_{L^2}^2 = 0. \tag{1.10}$$

Damping for ∇a and ∇u. Define

$$\mathfrak{g} = \nabla a - \mathrm{div}\tau.$$

We see from (1.6) that

$$\begin{cases} \mathfrak{g}_t - \mathrm{div}\tau + 2\nabla \operatorname{div} u + \Delta u = 0, \\ u_t + \mathfrak{g} = 0, \\ \operatorname{div} \tau_t + \operatorname{div} \tau - \nabla \operatorname{div} u - \Delta u = 0. \end{cases} \tag{1.11}$$

Multiplying $(1.11)_1$ by u and $(1.11)_2$ by $\mathfrak{g}$, then integrating the resulted equations in $\mathbb{R}^n$ and using integration by parts, we can obtain that

$$\frac{d}{dt}\langle \mathfrak{g}, u\rangle + \|\mathfrak{g}\|_{L^2}^2 + \langle \tau, \nabla u\rangle - 2\|\nabla\cdot u\|_{L^2}^2 - \|\nabla u\|_{L^2}^2 = 0. \tag{1.12}$$

Multiplying $(1.11)_2$ by $\mathrm{div}\tau$ and $(1.11)_3$ by u, then integrating the resulted equations in $\mathbb{R}^n$ and using integration by parts, we can obtain that

$$\frac{d}{dt}\langle u, \mathrm{div}\tau\rangle + \|(\nabla u, \nabla\cdot u)\|_{L^2}^2 - \langle \tau, \nabla u\rangle + \langle \mathfrak{g}, \mathrm{div}\tau\rangle = 0. \tag{1.13}$$

By suitable linear combination of (1.10), (1.12), and (1.13), we can achieve the following:

$$\frac{d}{dt}E(t) + \tilde{E}(t) = 0,$$

where

$$E(t) \approx \|a, u, \tau, \nabla a, \nabla u, \mathrm{div}\tau\|_{L^2}^2, \quad \tilde{E}(t) \approx \|\nabla a, \tau, \nabla u, \mathrm{div}\tau\|_{L^2}^2.$$

This produces the damping effects for τ and high frequencies of a and u, which also indicates the dissipation effects for the low frequencies of a and u.

Remark 1.3. Compared with the compressible Oldroyd-B model with viscosity (with retardation) or even the compressible Navier–Stokes equations, the obvious difference here is the damping effect for the velocity in high frequency rather than the dissipation effect in the viscosity case.

Remark 1.4. Due to the damping effect of τ, we can actually see that the damping effects for the incompressible part and the compressible part of u are the same. Here we give an explanation. Denote $\mathcal{P} := I + (-\Delta)^{-1}\nabla\nabla\cdot$ the projection operator and $\mathcal{P}^\perp := \Delta^{-1}\nabla\nabla\cdot$ its orthogonal complement. Then the incompressible and compressible parts of u are given by

$$\text{The incompressible part of } u = \mathcal{P}u; \quad \text{the compressible part of } u = \mathcal{P}^\perp u.$$

By applying $\mathcal{P}$ and $\mathcal{P}^\perp$ to (1.11), we see that

$$\begin{cases} (\mathcal{P}u)_t + \mathcal{P}\mathfrak{g} = 0, \\ (\mathcal{P}\mathfrak{g})_t + \Delta(\mathcal{P}u) = \mathcal{P}\mathrm{div}\tau, \\ (\mathcal{P}\,\mathrm{div}\,\tau)_t - \Delta(\mathcal{P}u) = -\mathcal{P}\,\mathrm{div}\,\tau, \end{cases} \quad \text{and} \quad \begin{cases} (\mathcal{P}^\perp u)_t + \mathcal{P}^\perp\mathfrak{g} = 0, \\ (\mathcal{P}^\perp\mathfrak{g})_t + 3\Delta(\mathcal{P}^\perp u) = \mathcal{P}^\perp\mathrm{div}\tau, \\ (\mathcal{P}^\perp\,\mathrm{div}\,\tau)_t - 2\Delta(\mathcal{P}^\perp u) = -\mathcal{P}^\perp\,\mathrm{div}\,\tau. \end{cases} \tag{1.14}$$

From our informal analysis for τ as in (1.10), we see that the damping of τ is self-contained. Then the above equations (1.14) show that $\mathcal{P}u$ coupled with $\mathcal{P}\mathfrak{g}$ and $\mathcal{P}\,\mathrm{div}\,\tau$ share the same structure with that of $(\mathcal{P}^\perp u, \mathcal{P}^\perp\mathfrak{g}, \mathcal{P}^\perp\,\mathrm{div}\,\tau)$, which indicates the same damping rate for the incompressible part and compressible part of u.

Our paper is arranged as follows. In section 2, we give an a priori estimate to solutions of system (1.5). Then the proof of Theorem 1.1 is obtained in section 3.

2. The a priori estimate. In this section, we give an a priori estimate for system (1.5). First we introduce some notation and the functional spaces that we use. See [1].

Notation.

- For a function $f \in \mathcal{S}'$ (the dual space of the Schwartz space), denote by $\hat{f}$ or $\mathcal{F}(f)$ the Fourier transform of f, and by $\check{f}$ or $\mathcal{F}^{-1}(f)$ the inverse Fourier transform of f.
- Denote $\sqrt{-\Delta}$ by Λ and for any $s \in \mathbb{R}$, $\Lambda^s = (-\Delta)^{\frac{s}{2}}$.
- For functions $f, g \in L^2$, denote the L^2 inner product by $(f|g)$, namely,

$$(f|g) = \int_{\mathbb{R}^3} f\bar{g}dx.$$

- Denote by $\mathcal{Z}'(\mathbb{R}^n)$ the dual space of

$$\mathcal{Z}(\mathbb{R}^n) := \{f \in \mathcal{S}(\mathbb{R}^n) : \partial^\alpha \hat{f}(0) = 0, \forall \alpha \in (\mathbb{N} \cup 0)^n\}.$$

Littlewood–Paley decomposition. Next, we need a Littlewood–Paley decomposition. There exist two radial smooth functions $\varphi(x)$, $\chi(x)$ supported in the annulus $\mathcal{C} = \{\xi \in \mathbb{R}^n : 3/4 \le |\xi| \le 8/3\}$ and the ball $B = \{\xi \in \mathbb{R}^n : |\xi| \le 4/3\}$, respectively, such that

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

The homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low-frequency cut-off operators $\dot{S}_j$ are defined for all $j \in \mathbb{Z}$ by

$$\dot{\Delta}_j u = \varphi(2^{-j}D)f, \quad \dot{S}_j f = \sum_{k \le j-1} \dot{\Delta}_k f = \chi(2^{-j}D)f.$$

Let us now turn to the definition of the main functional spaces and norms that will come into play in our paper.

DEFINITION 2.1. *Let s be a real number and (p, r) be in $[1, \infty]^2$. The homogeneous Besov space $\dot{B}^s_{p,r}$ consists of those distributions $u \in \mathcal{Z}'(\mathbb{R}^n)$ such that*

$$\|u\|_{\dot{B}^s_{p,r}} \triangleq \left(\sum_{j \in \mathbb{Z}} 2^{jsr} \|\dot{\Delta}_j u\|^r_{L^p} \right)^{\frac{1}{r}} < \infty.$$

Also, we introduce the hybrid Besov space since our analysis will be performed at different frequencies. See [7, Definition 2.1], for example.

DEFINITION 2.2. *Let $s, \sigma \in \mathbb{R}$. The hybrid Besov space $\dot{B}^{s,\sigma}$ is defined by*

$$\dot{B}^{s,\sigma} \triangleq \{f \in \mathcal{Z}'(\mathbb{R}^n) : \|f\|_{\dot{B}^{s,\sigma}} < \infty\},$$

with

$$\begin{aligned} \|f\|_{\dot{B}^{s,\sigma}} &\triangleq \sum_{k \le k_0} 2^{ks} \|\dot{\Delta}_k f\|_{L^2} + \sum_{k > k_0} 2^{k\sigma} \|\dot{\Delta}_k f\|_{L^2} \\ &= \|f\|^{\ell}_{\dot{B}^s_{2,1}} + \|f\|^h_{\dot{B}^\sigma_{2,1}}, \end{aligned}$$

where k_0 is a fixed suitably large constant to be defined.

Remark 2.3. We note the following:

- If $\sigma = s$, $\dot{B}^{s,s}$ is the usual Besov space $\dot{B}^s_{2,1}$.
- If $\sigma < s$, $\dot{B}^{s,\sigma} = \dot{B}^s_{2,1} \cap \dot{B}^\sigma_{2,1}$.

In the case where u depends on the time variable, we consider the space-time mixed spaces as follows:

$$\|u\|_{L^q_T\dot{B}^{s,\sigma}} := \big\|\|u(t,\cdot)\|_{\dot{B}^{s,\sigma}}\big\|_{L^q(0,T)}.$$

In addition, we introduce another space-time mixed space, which is usually referred to as Chemin–Lerner space. The definition is given by

$$\|u\|_{\tilde{L}^q_T\dot{B}^{s,\sigma}} \triangleq \sum_{k\le k_0} 2^{ks}\left\|\|\dot{\Delta}_k u(t)\|_{L^2}\right\|_{L^q(0,T)} + \sum_{k>k_0} 2^{k\sigma}\left\|\|\dot{\Delta}_k u(t)\|_{L^2}\right\|_{L^q(0,T)}.$$

The index T will be omitted if $T=+\infty$. It is easy to check that $\tilde{L}^1_T\dot{B}^{s,\sigma} = L^1_T\dot{B}^{s,\sigma}$ and $\tilde{L}^q_T\dot{B}^{s,\sigma} \subseteq L^q_T\dot{B}^{s,\sigma}$ for $q>1$.

In this paper, we also need the following time-weighted hybrid Besov norm. The more general case was introduced by Paicu and Zhang in [22].

Definition 2.4. *Let $\theta(t)\in L^1_{\text{loc}}(\mathbb{R}_+)$ be a positive function. Define*

$$\|f\|_{L^1_{T,\theta(t)}(\dot{B}^{s,\sigma})} = \int_0^T \theta(t)\|f(t)\|_{\dot{B}^{s,\sigma}}\,dt.$$

Let $\theta(t)\in C^1[0,+\infty)$ with $\theta(0)=0$ be a nondecreasing function. Denote

$$\Phi(t,\xi) = (2\lambda_0 - \lambda\theta(t))|\xi|,$$

where λ is a suitably large constant and will be determined later. For a function f, define

$$f_\Phi(t,x) = \mathcal{F}^{-1}_{\xi\to x}\left(e^{\Phi(t,\xi)}\hat{f}(t,\xi)\right) = e^{\Phi(t,\Lambda)}f(t,x). \tag{2.1}$$

In particular,

$$f_\Phi(0,x) = e^{2\lambda_0\Lambda}f(0,x), \text{ and } \dot{\Delta}_k f_\Phi(t,x) = \mathcal{F}^{-1}_{\xi\to x}\left(\varphi(2^{-k}\xi)e^{\Phi(t,\xi)}\hat{f}(t,\xi)\right).$$

Later for convenience and simplification of notation, we use $f_{\Phi,k}$ to denote $\dot{\Delta}_k f_\Phi$.

Now our a priori estimate is stated in the following proposition.

Proposition 2.5. *Let $\lambda_0>0$ be a fixed constant. Assume that (ρ,u,τ) is a solution of system* (1.5)*, with the initial data (ρ_0,u_0,τ_0) satisfying $e^{2\lambda_0\Lambda}(\rho_0-1,u_0,\tau_0)\in \dot{B}^{n/2-1,n/2}$. Then there exist two constants ϵ_0 and C, such that for any $t>0$, if $\|a\|_{L^\infty}\le\epsilon_0$, then we have*

$$\begin{aligned}&\|(a_\Phi,u_\Phi,\tau_\Phi)(t)\|_{\dot{B}^{n/2-1,n/2}} + \lambda\|(a_\Phi,u_\Phi,\tau_\Phi)\|_{L^1_{t,\dot{\theta}}(\dot{B}^{n/2,n/2+1})} + \|(a_\Phi,u_\Phi,\tau_\Phi)\|_{L^1_t(\dot{B}^{n/2}_{2,1})}\\ &\le C\bigg(\|e^{2\lambda_0\Lambda}(a_0,u_0,\tau_0)\|_{\dot{B}^{n/2-1,n/2}}\\ &\quad+\int_0^t \|(a_\Phi,u_\Phi,\tau_\Phi)(s)\|_{\dot{B}^{n/2}_{2,1}}\|(a_\Phi,u_\Phi,\tau_\Phi)(s)\|_{B^{n/2,n/2+1}_{2,1}}ds\bigg).\end{aligned} \tag{2.2}$$

Here $a:=\rho-1$, $a_0:=\rho_0-1$, and the constant C is independent of λ_0.

2.1. The linearized problem and its estimates. In this section, we linearize system (1.5) and give its linear a priori estimates.

Set $\rho = 1 + a$; then we rewrite (1.5) into

$$\left\{\begin{array}{ll} a_t + \operatorname{div} u = F, & \\ u_t + \nabla a - \operatorname{div} \tau = G, & (x,t) \in \mathbb{R}^3 \times (0,+\infty), \\ \tau_t + \tau - 2\mathbb{D}(u) = H, & \\ (a,u,\tau)\big|_{t=0} = (a_0,u_0,\tau_0), & x \in \mathbb{R}^3, \end{array}\right. \tag{2.3}$$

where

$$\begin{aligned} &F := -\nabla \cdot (au), \\ &G := -u \cdot \nabla u + [1-(1+a)^{\gamma-2}]\nabla a - \frac{a}{1+a}\operatorname{div}\tau, \\ &H := -u\cdot\nabla\tau - Q(\tau,\nabla u). \end{aligned} \tag{2.4}$$

Then, we have the following a priori estimate for the linearized system (2.3).

PROPOSITION 2.6. *Assume that (a,u,τ) is a solution of the linearized system (2.3), with the initial data (a_0,u_0,τ_0) satisfying $e^{2\lambda_0\Lambda}(a_0,u_0,\tau_0) \in \dot{B}^{n/2-1,n/2}$. Then there exists a uniform constant C, such that for any $t>0$, we have*

$$\begin{aligned} &\|(a_\Phi,u_\Phi,\tau_\Phi)(t)\|_{\dot{B}^{n/2-1,n/2}} + \lambda\|(a_\Phi,u_\Phi,\tau_\Phi)\|_{L^1_{t,\dot\theta}(\dot{B}^{n/2,n/2+1})} + \|(a_\Phi,u_\Phi,\tau_\Phi)\|_{L^1_t(\dot{B}^{n/2}_{2,1})} \\ &\quad \le C\left(\|e^{2\lambda_0\Lambda}(a_0,u_0,\tau_0)\|_{\dot{B}^{n/2-1,n/2}} + \|(F_\Phi,G_\Phi,H_\Phi)\|_{L^1_t(\dot{B}^{n/2-1,n/2})}\right). \end{aligned} \tag{2.5}$$

Proof. From (2.3), we see that $(a_\Phi,u_\Phi,\tau_\Phi)$ satisfy the following equations:

$$\left\{\begin{array}{l} \partial_t a_{\Phi,k} + \lambda\dot\theta(t)\Lambda a_{\Phi,k} + \operatorname{div} u_{\Phi,k} = F_{\Phi,k}, \\ \partial_t u_{\Phi,k} + \lambda\dot\theta(t)\Lambda u_{\Phi,k} + \nabla a_{\Phi,k} - \operatorname{div}\tau_{\Phi,k} = G_{\Phi,k}, \\ \partial_t \tau_{\Phi,k} + \lambda\dot\theta(t)\Lambda \tau_{\Phi,k} + \tau_{\Phi,k} - 2\mathbb{D}u_{\Phi,k} = H_{\Phi,k}. \end{array}\right. \tag{2.6}$$

Performing the L^2 inner product of $(2.6)_{1,2,3}$ with $(2a_{\Phi,k}, 2u_{\Phi,k}, \tau_{\Phi,k})$, respectively, we can obtain that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(2\|a_{\Phi,k}\|^2_{L^2} + 2\|u_{\Phi,k}\|^2_{L^2} + \|\tau_{\Phi,k}\|^2_{L^2}\right) \\ &\quad + \lambda\dot\theta(t)\left(2\|\Lambda^{1/2}a_{\Phi,k}\|^2_{L^2} + 2\|\Lambda^{1/2}u_{\Phi,k}\|^2_{L^2} + \|\Lambda^{1/2}\tau_{\Phi,k}\|^2_{L^2}\right) + \|\tau_{\Phi,k}\|^2_{L^2} \\ &\quad + 2(a_{\Phi,k}|\mathrm{div}u_{\Phi,k}) + 2(\nabla a_{\Phi,k}|u_{\Phi,k}) - 2(\mathrm{div}\tau_{\Phi,k}|u_{\Phi,k}) - 2(\tau_{\Phi,k}|\mathbb{D}u_{\Phi,k}) \\ &\quad = 2(a_{\Phi,k}|F_{\Phi,k}) + 2(G_{\Phi,k}|u_{\Phi,k}) + (\tau_{\Phi,k}|H_{\Phi,k}). \end{aligned}$$

By using the symmetry of τ and integration by parts, we see that terms on the third line of the above equality are cancelled. Then we can obtain that

$$\begin{aligned} &\frac{d}{dt}\left(2\|a_{\Phi,k}\|^2_{L^2} + 2\|u_{\Phi,k}\|^2_{L^2} + \|\tau_{\Phi,k}\|^2_{L^2}\right) \\ &\quad + 2\lambda\dot\theta(t)\left(2\|\Lambda^{1/2}a_{\Phi,k}\|^2_{L^2} + 2\|\Lambda^{1/2}u_{\Phi,k}\|^2_{L^2} + \|\Lambda^{1/2}\tau_{\Phi,k}\|^2_{L^2}\right) + 2\|\tau_{\Phi,k}\|^2_{L^2} \\ &\quad = 4(a_{\Phi,k}|F_{\Phi,k}) + 4(G_{\Phi,k}|u_{\Phi,k}) + 2(\tau_{\Phi,k}|H_{\Phi,k}). \end{aligned} \tag{2.7}$$

Estimate in high frequencies. We apply $\Lambda^{-1}\nabla$ to $(2.6)_1$, Λ^{-1} to $(2.6)_2$, and $\Lambda^{-1}\mathrm{div}$ to $(2.6)_3$ to obtain that

(2.8)
$$\begin{cases} \partial_t\Lambda^{-1}\nabla a_{\Phi,k} + \lambda\dot{\theta}(t)\nabla a_{\Phi,k} + \Lambda^{-1}\nabla \operatorname{div} u_{\Phi,k} = \Lambda^{-1}\nabla F_{\Phi,k}, \\ \partial_t\Lambda^{-1}u_{\Phi,k} + \lambda\dot{\theta}(t)u_{\Phi,k} + \Lambda^{-1}\nabla a_{\Phi,k} - \Lambda^{-1}\operatorname{div}\tau_{\Phi,k} = \Lambda^{-1}G_{\Phi,k}, \\ \partial_t\Lambda^{-1}\mathrm{div}\tau_{\Phi,k} + \lambda\dot{\theta}(t)\mathrm{div}\tau_{\Phi,k} + \Lambda^{-1}\mathrm{div}\tau_{\Phi,k} + (\Lambda - \nabla\Lambda^{-1}\mathrm{div})u_{\Phi,k} = \Lambda^{-1}\mathrm{div}H_{\Phi,k}. \end{cases}$$

Multiplying $(2.8)_1$ by $\Lambda^{-1}u_{\Phi,k}$ and $(2.8)_2$ by $\Lambda^{-1}\nabla a_{\Phi,k}$, then integrating over $\mathbb{R}^n$, we obtain that

$$\begin{aligned} &\partial_t(\Lambda^{-1}\nabla a_{\Phi,k}|\Lambda^{-1}u_{\Phi,k}) + 2\lambda\dot{\theta}(t)(\nabla a_{\Phi,k}|\Lambda^{-1}u_{\Phi,k}) - \|\Lambda^{-1}\operatorname{div} u_{\Phi,k}\|_{L^2}^2 \\ &\quad + \|a_{\Phi,k}\|_{L^2}^2 - (\Lambda^{-1}\nabla a_{\Phi,k}|\Lambda^{-1}\operatorname{div}\tau_{\Phi,k}) \\ &= (\Lambda^{-1}\nabla F_{\Phi,k}|\Lambda^{-1}u_{\Phi,k}) + (\Lambda^{-1}\nabla a_{\Phi,k}|\Lambda^{-1}G_{\Phi,k}), \end{aligned} \tag{2.9}$$

where we have used integration by parts to obtain that

$$(\Lambda^{-1}\nabla \operatorname{div} u_{\Phi,k}|\Lambda^{-1}u_{\Phi,k}) = -\|\Lambda^{-1}\operatorname{div} u_{\Phi,k}\|_{L^2}^2.$$

Multiplying $(2.8)_2$ by $\Lambda^{-1}\mathrm{div}\tau_{\Phi,k}$ and $(2.8)_3$ by $\Lambda^{-1}u_{\Phi,k}$, then integrating over $\mathbb{R}^n$, we obtain that

$$\begin{aligned} &\partial_t(\Lambda^{-1}u_{\Phi,k}|\Lambda^{-1}\mathrm{div}\tau_{\Phi,k}) + 2\lambda\dot{\theta}(t)(\Lambda^{-1}u_{\Phi,k}|\mathrm{div}\tau_{\Phi,k}) \\ &\quad + (\Lambda^{-1}\nabla a_{\Phi,k}|\Lambda^{-1}\operatorname{div}\tau_{\Phi,k}) - \|\Lambda^{-1}\mathrm{div}\tau_{\Phi,k}\|_{L^2}^2 \\ &\quad + (\Lambda^{-1}u_{\Phi,k}|\Lambda^{-1}\operatorname{div}\tau_{\Phi,k}) + \|u_{\Phi,k}\|_{L^2}^2 + \|\Lambda^{-1}\mathrm{div}u_{\Phi,k}\|_{L^2}^2 \\ &= (\Lambda^{-1}G_{\Phi,k}|\Lambda^{-1}\mathrm{div}\tau_{\Phi,k}) + (\Lambda^{-1}u_{\Phi,k}|\Lambda^{-1}\mathrm{div}H_{\Phi,k}). \end{aligned} \tag{2.10}$$

Adding (2.9) and (2.10) together indicates that

$$\begin{aligned} &\partial_t\left\{(\Lambda^{-1}\nabla a_{\Phi,k}|\Lambda^{-1}u_{\Phi,k}) + (\Lambda^{-1}u_{\Phi,k}|\Lambda^{-1}\mathrm{div}\tau_{\Phi,k})\right\} \\ &\quad + 2\lambda\dot{\theta}(t)\left\{(\nabla a_{\Phi,k}|\Lambda^{-1}u_{\Phi,k}) + (\Lambda^{-1}u_{\Phi,k}|\mathrm{div}\tau_{\Phi,k})\right\} + \|a_{\Phi,k}\|_{L^2}^2 + \|u_{\Phi,k}\|_{L^2}^2 \\ &\quad + (\Lambda^{-1}u_{\Phi,k}|\Lambda^{-1}\operatorname{div}\tau_{\Phi,k}) - \|\Lambda^{-1}\mathrm{div}\tau_{\Phi,k}\|_{L^2}^2 \\ &= (\Lambda^{-1}\nabla F_{\Phi,k}|\Lambda^{-1}u_{\Phi,k}) + (\Lambda^{-1}\nabla a_{\Phi,k}|\Lambda^{-1}G_{\Phi,k}) \\ &\quad + (\Lambda^{-1}G_{\Phi,k}|\Lambda^{-1}\mathrm{div}\tau_{\Phi,k}) + (\Lambda^{-1}u_{\Phi,k}|\Lambda^{-1}\mathrm{div}H_{\Phi,k}). \end{aligned} \tag{2.11}$$

From Bernstein's inequality, it is easy to see that

$$\frac{3}{4}2^k\|\dot{\Delta}_k f\|_{L^2} \le \|\Lambda\dot{\Delta}_k f\|_{L^2} \le \frac{8}{3}2^k\|\dot{\Delta}_k f\|_{L^2}. \tag{2.12}$$

For $k \ge k_0$, define

$$\widetilde{\mathcal{E}}_k^2 = 2\|a_{\Phi,k}\|_{L^2}^2 + 2\|u_{\Phi,k}\|_{L^2}^2 + \|\tau_{\Phi,k}\|_{L^2}^2 + (\Lambda^{-1}\nabla a_{\Phi,k}|\Lambda^{-1}u_{\Phi,k}) + (\Lambda^{-1}u_{\Phi,k}|\Lambda^{-1}\mathrm{div}\tau_{\Phi,k}),$$

and

$$\mathcal{E}_k^2 = \|a_{\Phi,k}\|_{L^2}^2 + \|u_{\Phi,k}\|_{L^2}^2 + \|\tau_{\Phi,k}\|_{L^2}^2.$$

We choose $k_0 = 3$; then by using (2.12), we can see that

$$\frac{1}{2}\mathcal{E}_k^2 \le \widetilde{\mathcal{E}}_k^2 \le 3\mathcal{E}_k^2.$$

Adding (2.7) and (2.11) together, and using the Hölder inequality, the Cauchy inequality, and (2.12), we can obtain that

$$\frac{d}{dt}\widetilde{\mathcal{E}}_k^2+\frac{1}{4}\lambda\dot{\theta}(t)2^k\widetilde{\mathcal{E}}_k^2+\frac{1}{8}\widetilde{\mathcal{E}}_k^2\le C\|(F_{\Phi,k},G_{\Phi,k},H_{\Phi,k})\|_{L^2}\widetilde{\mathcal{E}}_k,$$

where C is a uniform constant, independent of λ_0. Then from the above inequality, we have

$$\frac{d}{dt}\widetilde{\mathcal{E}}_k+\frac{1}{8}\lambda\dot{\theta}(t)2^k\widetilde{\mathcal{E}}_k+\frac{1}{16}\widetilde{\mathcal{E}}_k\le C\|(F_{\Phi,k},G_{\Phi,k},H_{\Phi,k})\|_{L^2}. \tag{2.13}$$

Multiplying (2.13) by $2^{\frac{n}{2}k}$, integrating the resulting equation from 0 to t with the time variable, and then summing over $k_0<k\in\mathbb{N}$, we can achieve that

$$\begin{aligned}&\|(a_\Phi,u_\Phi,\tau_\Phi)(t)\|^h_{\dot{B}^{n/2}_{2,1}}+\lambda\int_0^t\dot{\theta}(\tau)\|(a_\Phi,u_\Phi,\tau_\Phi)(s)\|^h_{\dot{B}^{n/2+1}_{2,1}}ds\\&\quad+\int_0^t\|(a_\Phi,u_\Phi,\tau_\Phi)(s)\|^h_{\dot{B}^{n/2}_{2,1}}ds\\&\le C\int_0^t\|(F_\Phi,G_\Phi,H_\Phi)(s)\|^h_{\dot{B}^{n/2}_{2,1}}ds.\end{aligned} \tag{2.14}$$

Estimate in low frequencies. We apply ∇ to $(2.6)_1$ and div to $(2.6)_3$ to obtain that

$$\begin{cases}\partial_t\nabla a_{\Phi,k}+\lambda\dot{\theta}(t)\nabla\Lambda a_{\Phi,k}+\nabla\operatorname{div}u_{\Phi,k}=\nabla F_{\Phi,k},\\ \partial_t u_{\Phi,k}+\lambda\dot{\theta}(t)\Lambda u_{\Phi,k}+\nabla a_{\Phi,k}-\operatorname{div}\tau_{\Phi,k}=G_{\Phi,k},\\ \partial_t\mathrm{div}\tau_{\Phi,k}+\lambda\dot{\theta}(t)\mathrm{div}\Lambda\tau_{\Phi,k}+\mathrm{div}\tau_{\Phi,k}-(\Delta+\nabla\mathrm{div})u_{\Phi,k}=\mathrm{div}H_{\Phi,k}.\end{cases} \tag{2.15}$$

Multiplying $(2.15)_1$ by $u_{\Phi,k}$ and $(2.15)_2$ by $\nabla a_{\Phi,k}$, then integrating over $\mathbb{R}^n$, we obtain that

$$\begin{aligned}&\partial_t(\nabla a_{\Phi,k}|u_{\Phi,k})+2\lambda\dot{\theta}(t)(\nabla\Lambda a_{\Phi,k}|u_{\Phi,k})-\|\operatorname{div}u_{\Phi,k}\|^2_{L^2}\\&+\|\nabla a_{\Phi,k}\|^2_{L^2}-(\nabla a_{\Phi,k}|\operatorname{div}\tau_{\Phi,k})=(\nabla F_{\Phi,k}|u_{\Phi,k})+(\nabla a_{\Phi,k}|G_{\Phi,k}),\end{aligned} \tag{2.16}$$

where we have used integration by parts to obtain that

$$(\nabla\operatorname{div}u_{\Phi,k}|u_{\Phi,k})=-\|\operatorname{div}u_{\Phi,k}\|^2_{L^2}.$$

Multiplying $(2.15)_2$ by $\mathrm{div}\tau_{\Phi,k}$ and $(2.15)_3$ by $u_{\Phi,k}$, then integrating over $\mathbb{R}^n$, we obtain that

$$\begin{aligned}&\partial_t(u_{\Phi,k}|\mathrm{div}\tau_{\Phi,k})+2\lambda\dot{\theta}(t)(\Lambda u_{\Phi,k}|\mathrm{div}\tau_{\Phi,k})+(\nabla a_{\Phi,k}|\operatorname{div}\tau_{\Phi,k})-\|\mathrm{div}\tau_{\Phi,k}\|^2_{L^2}\\&\quad+(u_{\Phi,k}|\operatorname{div}\tau_{\Phi,k})+\|\Lambda u_{\Phi,k}\|^2_{L^2}+\|\mathrm{div}u_{\Phi,k}\|^2_{L^2}=(G_{\Phi,k}|\mathrm{div}\tau_{\Phi,k})+(u_{\Phi,k}|\mathrm{div}H_{\Phi,k}).\end{aligned} \tag{2.17}$$

Adding (2.16) and (2.17) together indicates that

$$\begin{aligned}&\partial_t\left\{(\nabla a_{\Phi,k}|u_{\Phi,k})+(u_{\Phi,k}|\mathrm{div}\tau_{\Phi,k})\right\}\\&\quad+2\lambda\dot{\theta}(t)\left\{(\nabla\Lambda a_{\Phi,k}|u_{\Phi,k})+(\Lambda u_{\Phi,k}|\mathrm{div}\tau_{\Phi,k})\right\}+\|\Lambda a_{\Phi,k}\|^2_{L^2}+\|\Lambda u_{\Phi,k}\|^2_{L^2}\\&\quad+(u_{\Phi,k}|\operatorname{div}\tau_{\Phi,k})-\|\mathrm{div}\tau_{\Phi,k}\|^2_{L^2}\\&=(\nabla F_{\Phi,k}|u_{\Phi,k})+(\nabla a_{\Phi,k}|G_{\Phi,k})+(G_{\Phi,k}|\mathrm{div}\tau_{\Phi,k})+(u_{\Phi,k}|\mathrm{div}H_{\Phi,k}).\end{aligned} \tag{2.18}$$

For $k \le k_0$, define

$$\widetilde{\mathcal{E}}_k^2 = 2\|a_{\Phi,k}\|_{L^2}^2 + 2\|u_{\Phi,k}\|_{L^2}^2 + \|\tau_{\Phi,k}\|_{L^2}^2 + \frac{3}{8}2^{-k}(\nabla a_{\Phi,k}|u_{\Phi,k}) + \frac{3}{8}2^{-k}(u_{\Phi,k}|\mathrm{div}\tau_{\Phi,k}),$$

and

$$\mathcal{E}_k^2 = \|a_{\Phi,k}\|_{L^2}^2 + \|u_{\Phi,k}\|_{L^2}^2 + \|\tau_{\Phi,k}\|_{L^2}^2.$$

We choose $k_0 = 3$; then by using (2.12), we can see that

$$\frac{1}{2}\mathcal{E}_k^2 \le \widetilde{\mathcal{E}}_k^2 \le 3\mathcal{E}_k^2.$$

Multiplying (2.18) by $\frac{3}{8}2^{-k}$, adding the resulted equation to (2.7), and using the Hölder inequality, the Cauchy inequality, and (2.12), we can obtain that

$$\frac{d}{dt}\widetilde{\mathcal{E}}_k^2 + \frac{1}{4}\lambda\dot{\theta}(t)2^k\widetilde{\mathcal{E}}_k^2 + \frac{1}{8}2^k\widetilde{\mathcal{E}}_k^2 \le C\|(F_{\Phi,k}, G_{\Phi,k}, H_{\Phi,k})\|_{L^2}\widetilde{\mathcal{E}}_k,$$

where C is a uniform constant, independent of λ_0. Then from the above inequality, we have

$$\frac{d}{dt}\widetilde{\mathcal{E}}_k + \frac{1}{8}\lambda\dot{\theta}(t)2^k\widetilde{\mathcal{E}}_k + \frac{1}{16}2^k\widetilde{\mathcal{E}}_k \le C\|(F_{\Phi,k}, G_{\Phi,k}, H_{\Phi,k})\|_{L^2}. \tag{2.19}$$

Multiplying (2.19) by $2^{(n/2-1)k}$, integrating the resulted equation from 0 to t with the time variable, and then summing over $\mathbb{N} \ni k \le k_0$, we can achieve that

$$\begin{aligned}
&\|(a_\Phi, u_\Phi, \tau_\Phi)(t)\|^{\ell}_{\dot{B}^{n/2-1}_{2,1}} + \lambda\int_0^t \dot{\theta}(s)\|(a_\Phi, u_\Phi, \tau_\Phi)(s)\|^{\ell}_{\dot{B}^{n/2}_{2,1}}ds \\
&\quad + \int_0^t \|(a_\Phi, u_\Phi, \tau_\Phi)(s)\|^{\ell}_{\dot{B}^{n/2}_{2,1}}ds \\
&\le C\int_0^t \|(F_\Phi, G_\Phi, H_\Phi)(s)\|^{\ell}_{\dot{B}^{n/2-1}_{2,1}}ds. \qquad (2.20)
\end{aligned}$$

Proof of the a priori estimate in (2.5). By adding (2.14) and (2.20), we can achieve the a priori estimate in (2.5). ∎

2.2. Estimates of nonlinear terms. Using the Bony decomposition, we have the following lemma.

LEMMA 2.7. *Let f_Φ be defined as in* (2.1). *For $s \in (-n/2, n/2]$ and $(f_\Phi, g_\Phi) \in \dot{B}^{n/2}_{2,1} \times \dot{B}^s_{2,1}$, there exists a positive constant C, depending on s, such that the following product estimate holds:*

$$\|(fg)_\Phi\|_{\dot{B}^s_{2,1}} \le C\|f_\Phi\|_{\dot{B}^{n/2}_{2,1}}\|g_\Phi\|_{\dot{B}^s_{2,1}}. \tag{2.21}$$

The proof of Lemma 2.7 is postponed to Appendix A. By letting $s = n/2$ in (2.21), we have the following corollary.

COROLLARY 2.8. *Let f_Φ be defined as in* (2.1) *and $f_\Phi \in \dot{B}^{n/2}_{2,1}$; there exists a positive constant C, such that for any $k \in \mathbb{N}/\{0\}$, the following estimate holds:*

$$\|(f^k)_\Phi\|_{\dot{B}^{n/2}_{2,1}} \le C^k\|f_\Phi\|^k_{\dot{B}^{n/2}_{2,1}}. \tag{2.22}$$

Achievement of (2.22) is via k-time use of (2.21) with $s = n/2$.

Now, we use product estimates in (2.21) and (2.22) to give estimates of nonlinear terms F_Φ, G_Φ, and H_Φ. We have the following estimate.

LEMMA 2.9. *Let F, G, and H be defined as in* (2.4). *There exist a constant ϵ_0 and C such that if*

$$\|a(t)\|_{\dot{B}^{n/2}_{2,1}} \leq \epsilon_0 < 1,$$

then for $s \in (-n/2, n/2]$, we have

$$\|(F_\Phi, G_\Phi, H_\Phi)\|_{\dot{B}^{s}_{2,1}} \leq C\|(a_\Phi, u_\Phi, \tau_\Phi)\|_{\dot{B}^{n/2}_{2,1}}\|(a_\Phi, u_\Phi, \tau_\Phi)\|_{\dot{B}^{s+1}_{2,1}}. \tag{2.23}$$

Proof. We first deal with the $\dot{B}^{n/2}_{2,1}$ norm of

$$1-(1+a)^{\gamma-2} \quad \text{and} \quad \frac{a}{1+a}.$$

By using Taylor expansion, we see that

$$1-(1+a)^{\gamma-2} = -\sum_{k=1}^{\infty} C_{k,\nu} a^k, \quad \text{with} \quad C_{k,\nu} = \frac{(\gamma-2)(\gamma-3)\cdots(\gamma-2-k+1)}{k!}.$$

It is easy to see that there exists a constant C such that $C_{\gamma,k} \leq C^k$. Using (2.22), we can obtain that

$$\begin{aligned}
\left\|(1-(1+a)^{\gamma-2})_\Phi\right\|_{\dot{B}^{n/2}_{2,1}} &\leq \sum_{k=1}^{\infty} C_{k,\nu}\left\|(a^k)_\Phi\right\|_{\dot{B}^{n/2}_{2,1}} \leq \sum_{k=1}^{\infty} C^k \left\|a_\Phi\right\|^k_{\dot{B}^{n/2}_{2,1}} \\
&\leq \frac{C\left\|a_\Phi\right\|_{\dot{B}^{n/2}_{2,1}}}{1-C\left\|a_\Phi\right\|_{\dot{B}^{n/2}_{2,1}}} \leq C\left\|a_\Phi\right\|_{\dot{B}^{n/2}_{2,1}},
\end{aligned} \tag{2.24}$$

provided that $C\epsilon_0 \leq 1/2$. The same is true for $a/(1+a)$. From the representation formula of F, G, and H, by using (2.21) and the above estimate (2.24), we have that

$$\begin{aligned}
\|(F_\Phi, G_\Phi, H_\Phi)\|_{\dot{B}^{s}_{2,1}} &\leq C\|(a_\Phi, u_\Phi, \tau_\Phi)\|_{\dot{B}^{n/2}_{2,1}}\|(\nabla a_\Phi, \nabla u_\Phi, \nabla\tau_\Phi)\|_{\dot{B}^{s}_{2,1}} \\
&\leq C\|(a_\Phi, u_\Phi, \tau_\Phi)\|_{\dot{B}^{n/2}_{2,1}}\|(a_\Phi, u_\Phi, \tau_\Phi)\|_{\dot{B}^{s+1}_{2,1}},
\end{aligned}$$

which is (2.23). ∎

Inserting (2.23) into (2.5), we finish the proof of Proposition 2.5.

3. Proof of Theorem 1.1. Inserting (2.23) into (2.5), we see that there exist constant ϵ_0 and C such that if $\|a\|_{L^\infty} \leq \epsilon_0$, we can obtain the following a priori estimate for system (2.3) with F, G, and H being given in (2.4):

$$\begin{aligned}
&\|(a_\Phi, u_\Phi, \tau_\Phi)\|_{\dot{B}^{n/2-1,n/2}} + \lambda\|(a_\Phi, u_\Phi, \tau_\Phi)\|_{L^1_{t,\dot{\theta}}(\dot{B}^{n/2,n/2+1})} + \|(a_\Phi, u_\Phi, \tau_\Phi)\|_{L^1_t(\dot{B}^{n/2}_{2,1})} \\
&\leq C\bigg(\|e^{2\lambda_0\Lambda}(a_0, u_0, \tau_0)\|_{\dot{B}^{n/2-1,n/2}} \\
&\quad + \int_0^t \|(a_\Phi, u_\Phi, \tau_\Phi)(s)\|_{(\dot{B}^{n/2}_{2,1})}\|(a_\Phi, u_\Phi, \tau_\Phi)(s)\|_{(B^{n/2,n/2+1}_{2,1})} ds\bigg).
\end{aligned} \tag{3.1}$$

First, we approximate (2.3) by a sequence of ordinary differential equations by the classical Friedrichs method; see [3] or [1, Chapter 10], for instance.

Let $L^2_k(\mathbb{R}^n)$ denote the set of $L^2(\mathbb{R}^n)$ functions spectrally supported in the annulus

$$\mathcal{C}_k := \{\xi \in \mathbb{R}^n : k^{-1} \le |\xi| \le k\}.$$

Define $\dot{\mathbb{E}}_k : L^2 \Rightarrow L^2_k$ to be the Friedrichs projector by

$$\mathcal{F}\dot{\mathbb{E}}_k U(\xi) := \mathbf{1}_{\mathcal{C}_k}\mathcal{F}U(\xi) \quad \text{for all} \quad \xi \in \mathbb{R}^n.$$

Consider the following ODE approximate system:

$$\begin{aligned} &\frac{d}{dt}\begin{pmatrix} a \\ u \\ \tau \end{pmatrix} = \dot{\mathbb{E}}_k \begin{pmatrix} -\operatorname{div} - \nabla\cdot(au) \\ -\nabla a + \operatorname{div}\tau - u\cdot\nabla u + [1-(1+a)^{\gamma-2}]\nabla a - \frac{a}{1+a}\operatorname{div}\tau \\ -\tau + 2\mathbb{D}(u) - u\cdot\nabla\tau - Q(\tau,\nabla u) \end{pmatrix}, \\ &\begin{pmatrix} a \\ u \\ \tau \end{pmatrix}_{|t=0} = \dot{\mathbb{E}}_k \begin{pmatrix} a_0 \\ u_0 \\ \tau_0 \end{pmatrix}. \end{aligned} \tag{3.2}$$

Solutions of (3.2) are represented by (a^k, u^k, τ^k). Define the solution space by

$$\tilde{L}^2_k(\mathbb{R}^n) := \{(a^k, u^k, \tau^k)\big|\inf_{x\in\mathbb{R}^n} a > -1\}.$$

Note that if $\|a_0\|_{\dot{B}^{n/2}_{2,1}}$ is small, then $1+\dot{\mathbb{E}}_k a_0$ is positive for large k. Then the initial data of (3.2) are in $\tilde{L}^2_k(\mathbb{R}^n)$. Thanks to the low-frequency cut-off of the operator $\dot{\mathbb{E}}_k$, all the Sobolev norms are equivalent. For fixed k, solving the ODE system (3.2), there exists a time maximal existing time T_k such that

$$(a^k, u^k, \tau^k) \in C^1([0,T_k); \tilde{L}^2_k(\mathbb{R}^n)).$$

For the obtained solution (a^k, u^k, τ^k), we define $\theta_k(t)$ as the solution of the following ODE problem:

$$\dot{\theta}_k(t) = \|(a^k_{\Phi_k}, u^k_{\Phi_k}, \tau^k_{\Phi_k})\|_{\dot{B}^{n/2}_{2,1}}, \quad \text{with} \quad \theta_k(0)=0, \quad \Phi_k(t,\xi) = (2\lambda_0 - \lambda\theta_k(t))|\xi|. \tag{3.3}$$

Since the Fourier transform of (a^k, u^k, τ^k) is compactly supported, the right-hand side of the above ODE is Lipschitz with respect to θ_k. Then (3.3) has a unique solution on $[0, T_k)$.

Next we note that the initial data (a_0, u_0, τ_0) of system 2.3 are analytic, with $e^{2\lambda_0\Lambda}(a_0, u_0, \tau_0)$ lying in $\dot{B}^{\frac{n}{2}-1,\frac{n}{2}}$. It is also obvious that

$$\|e^{2\lambda_0\Lambda}(a^k_0, u^k_0, \tau^k_0)\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{2}}} \le \|e^{2\lambda_0\Lambda}(a_0, u_0, \tau_0)\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{2}}} \le \epsilon_0\lambda_0.$$

Now we define T^*_k to be

$$T^*_k := \sup\left\{t \in [0,T_k) : \theta_k(t) \le \frac{\lambda_0}{\lambda}, \quad \text{and} \quad \|(a^k, u^k, \tau^k)\|_{\dot{B}^{n/2-1,n/2}_{2,1}} \le M\epsilon_0\right\},$$

where λ and M are two constants to be determined later.

Next, we will show that by choosing suitably large λ and M, we can obtain that $T^*_k = T_k$. We will use the continuity argument and (3.1) to show this.

Performing energy estimates almost the same as (3.1) on system (3.2), we can obtain that for any $t \in [0, T_k^*)$, we have

$$
\begin{aligned}
&\|(a_\Phi^k, u_\Phi^k, \tau_\Phi^k)(t)\|_{\dot{B}^{n/2-1,n/2}} + \lambda\|(a_\Phi^k, u_\Phi^k, \tau_\Phi^k)\|_{L^1_{t,\dot\theta}(\dot{B}^{n/2,n/2+1})} + \|(a_\Phi^k, u_\Phi^k, \tau_\Phi^k)\|_{L^1_t(\dot{B}^{n/2})} \\
&\quad \le C\bigg(\|e^{2\lambda_0\Lambda}(a_0^k, u_0^k, \tau_0^k)\|_{\dot{B}^{n/2-1,n/2}} \\
&\qquad + \int_0^t \|(a_\Phi^k, u_\Phi^k, \tau_\Phi^k)(s)\|_{(\dot{B}^{n/2}_{2,1})}\|(a_\Phi^k, u_\Phi^k, \tau_\Phi^k)(s)\|_{(B^{n/2,n/2+1}_{2,1})} ds\bigg) \\
&\quad \le C_0\epsilon_0\lambda_0 + C_0\int_0^t \dot\theta(s)\|(a_\Phi^k, u_\Phi^k, \tau_\Phi^k)(s)\|_{(B^{n/2,n/2+1}_{2,1})} ds.
\end{aligned} \tag{3.4}
$$

Now, we first choose $\lambda = 2C_0$; then from (3.4), we have

$$
\begin{aligned}
&\|(a_\Phi^k, u_\Phi^k, \tau_\Phi^k)\|_{\dot{B}^{n/2-1,n/2}} + C_0\|(a_\Phi^k, u_\Phi^k, \tau_\Phi^k)\|_{L^1_{t,\dot\theta}(\dot{B}^{n/2,n/2+1})} + \|(a_\Phi^k, u_\Phi^k, \tau_\Phi^k)\|_{L^1_t(\dot{B}^{n/2})} \\
&\quad \le C_0\epsilon_0\lambda_0.
\end{aligned} \tag{3.5}
$$

Now, we choose $M := 2C_0\lambda_0$ and ϵ_0 sufficiently small such that

$$C_0\epsilon_0 \le \frac{1}{2C_0} = \frac{1}{2\lambda}.$$

Then from (3.5), we can obtain that for any $t \in [0, T_k^*)$,

$$\theta_k(t) \le \frac{\lambda_0}{2\lambda}, \quad \text{and} \quad \|(a^k, u^k, \tau^k)\|_{\dot{B}^{n/2-1,n/2}_{2,1}} \le \frac{M}{2}\epsilon_0.$$

By continuity, we can see that $T_k^* = T_k$ and the estimate (3.5) is valid for any $t \in [0, T_k)$. Also by the same continuity argument, we can see that system (3.2) has a global time solution which satisfies, for any $t \in [0, +\infty)$, the estimate (3.5) and, for any $t \in [0, +\infty)$,

$$\theta_k(t) \le \frac{\lambda_0}{\lambda}.$$

Thanks to the uniform bound in (3.5) and the uniformly low bound for the analytic radius $2\lambda_0 - \lambda\theta(t) \ge \lambda_0$, one can deduce, by a compactness argument, that there exists a unique solution (a, u, τ) to system (2.3) with the same bound as in (3.5). The details are omitted. See, for example, [1, Chapter 10].

Appendix A. Proof of Lemma 2.7. We introduce the Bony decomposition [2] to perform nonlinear estimates in Besov space. The paraproduct between f and g is defined by

$$T_f g := \sum_{k\in\mathbb{Z}} \dot{S}_{k-1} f \dot\Delta_k g,$$

and the remainder is given by

$$R(f,g) := \sum_{|k-j|\le 1} \dot\Delta_k f \dot\Delta_j g.$$

Then for $f, g \in \mathcal{Z}'(\mathbb{R}^3)$, we have $fg = T_f g + T_g f + R(f,g)$.

We note that $\Phi(t,\xi)$ satisfies $\Phi(t,\xi) \le \Phi(t,\eta) + \Phi(t,\xi-\eta)$ for $\xi, \eta \in \mathbb{R}^3$. Using [30, Lemma 6.1], we have the following paraproduct estimate:

- If $s \in \mathbb{R}$, then there exists a positive constant C, depending on s, such that for $(f_\Phi, g_\Phi) \in L^\infty \times \dot{B}^s_{2,1}$, we have

$$\|(T_f g)_\Phi\|_{\dot{B}^s_{2,1}} \le C\|f_\Phi\|_{L^\infty}\|g_\Phi\|_{\dot{B}^s_{2,1}}. \tag{A.1}$$

- If $s_1 \in \mathbb{R}$ and $s_2 < 0$, then there exists a positive constant C, depending on s_1 and s_2, such that for $(f_\Phi, g_\Phi) \in \dot{B}^{s_2}_{\infty,\infty} \times \dot{B}^{s_1}_{2,1}$, we have

$$\|(T_f g)_\Phi\|_{\dot{B}^{s_1+s_2}_{2,1}} \le C\|f_\Phi\|_{\dot{B}^{s_2}_{\infty,\infty}}\|g_\Phi\|_{\dot{B}^{s_1}_{2,1}}. \tag{A.2}$$

- If $s_1 + s_2 > 0$, then there exists a positive constant C, depending on s_1 and s_2, such that for $(f_\Phi, g_\Phi) \in \dot{B}^{s_1}_{2,\infty} \times \dot{B}^{s_2}_{2,1}$, we have

$$\|(R(f,g))_\Phi\|_{\dot{B}^{s_1+s_2-n/2}_{2,1}} \le C\|f_\Phi\|_{\dot{B}^{s_1}_{2,\infty}}\|g_\Phi\|_{\dot{B}^{s_2}_{2,1}}. \tag{A.3}$$

Now we come to prove (2.21). By using the interpolation $\dot{B}^{n/p}_{p,1} \hookrightarrow L^\infty$ for $p \in [1,\infty]$ and (A.1), we see that

$$\|(T_f g)_\Phi\|_{\dot{B}^s_{2,1}} \le C\|f_\Phi\|_{L^\infty}\|g_\Phi\|_{\dot{B}^s_{2,1}} \le C\|f_\Phi\|_{\dot{B}^{n/2}_{2,1}}\|g_\Phi\|_{\dot{B}^s_{2,1}}. \tag{A.4}$$

When $s = n/2$, the same as (A.4), we have

$$\|(T_g f)_\Phi\|_{\dot{B}^{n/2}_{2,1}} \le C\|g_\Phi\|_{L^\infty}\|f_\Phi\|_{\dot{B}^{n/2}_{2,1}} \le C\|f_\Phi\|_{\dot{B}^{n/2}_{2,1}}\|g_\Phi\|_{\dot{B}^s_{2,1}}.$$

When $s < n/2$, by choosing $s_1 = n/2$ and $s_2 = s - n/2$ in (A.2) and the interpolation $\dot{B}^s_{2,1} \hookrightarrow \dot{B}^{s-n/2}_{\infty,\infty}$, we see that

$$\|(T_g f)_\Phi\|_{\dot{B}^s_{2,1}} \le C\|g_\Phi\|_{\dot{B}^{s-n/2}_{\infty,\infty}}\|f_\Phi\|_{\dot{B}^{n/2}_{2,1}} \le C\|f_\Phi\|_{\dot{B}^{n/2}_{2,1}}\|g_\Phi\|_{\dot{B}^s_{2,1}}.$$

By choosing $s_1 = n/2$ and $s_2 = s$ in (A.3) and the interpolation $\dot{B}^{3/p}_{p,1} \hookrightarrow \dot{B}^{3/p}_{p,\infty}$, we see that

$$\|(R(f,g))_\Phi\|_{\dot{B}^s_{2,1}} \le C\|f_\Phi\|_{\dot{B}^{n/2}_{2,\infty}}\|g_\Phi\|_{\dot{B}^s_{2,1}} \le C\|f_\Phi\|_{\dot{B}^{n/2}_{2,1}}\|g_\Phi\|_{\dot{B}^s_{2,1}}. \tag{A.5}$$

Combining the estimates in (A.4) and (A.5), we obtain (2.21).

Data availability statement. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflict of interest statement. The authors declare that they have no conflict of interest.

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REFERENCES

[1] H. Bahouri, J. Y. Chemin, and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren Math. Wiss. 343, Springer, Heidelberg, 2011.

[2] J. M. Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires [Symbolic calculus and propagation of singularities for nonlinear partial differential equations]*, Ann. Sci. École Norm. Sup. (4), 14 (1981), pp. 209–246.

[3] J.-Y. Chemin, I. Gallagher, and M. Paicu, *Global regularity for some classes of large solutions to the Navier-Stokes equations*, Ann. of Math. (2), 173 (2011), pp. 983–1012.

[4] J.-Y. Chemin and N. Masmoudi, *About lifespan of regular solutions of equations related to viscoelastic fluids*, SIAM J. Math. Anal., 33 (2001), pp. 84–112, https://doi.org/10.1137/S0036141099359317.

[5] Q. Chen and X. Hao, *Global well-posedness in the critical Besov spaces for the incompressible Oldroyd-B model without damping mechanism*, J. Math. Fluid Mech., 21 (2019), 42.

[6] Q. Chen and C. Miao, *Global well-posedness of viscoelastic fluids of Oldroyd type in Besov spaces*, Nonlinear Anal., 68 (2008), pp. 1928–1939.

[7] R. Danchin, *Global existence in critical spaces for compressible Navier-Stokes equations*, Invent. Math., 141 (2000), pp. 579–614.

[8] T. M. Elgindi and J. Liu, *Global wellposedness to the generalized Oldroyd type models in* $\mathbb{R}^3$, J. Differential Equations, 259 (2015), pp. 1958–1966.

[9] T. M. Elgindi and F. Rousset, *Global regularity for some Oldroyd-B type models*, Comm. Pure Appl. Math., 68 (2015), pp. 2005–2021.

[10] D. Fang and R. Zi, *Incompressible limit of Oldroyd-B fluids in the whole space*, J. Differential Equations, 256 (2014), pp. 2559–2602.

[11] D. Fang and R. Zi, *Global solutions to the Oldroyd-B model with a class of large initial data*, SIAM J. Math. Anal., 48 (2016), pp. 1054–1084, https://doi.org/10.1137/15M1037020.

[12] C. Guillopé, Z. Salloum, and R. Talhouk, *Regular flows of weakly compressible viscoelastic fluids and the incompressible limit*, Discrete Contin. Dyn. Syst. Ser. B, 14 (2010), pp. 1001–1028.

[13] C. Guillopé and J. C. Saut, *Existence results for the flow of viscoelastic fluids with a differential constitutive law*, Nonlinear Anal., 15 (1990), pp. 849–869.

[14] X. Hu and D. Wang, *Global existence for the multi-dimensional compressible viscoelastic flows*, J. Differential Equations, 250 (2011), pp. 1200–1231.

[15] H. Lamb, *Hydrodynamics*, reprint of the 1932 sixth edition, with a foreword by R. A. Caflisch, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1993.

[16] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, translated from the Russian by J. B. Sykes and W. H. Reid, Course of Theoretical Physics 6, Pergamon Press, London, Paris, Frankfurt; Addison-Wesley, Reading, MA, 1959.

[17] Z. Lei, *Global existence of classical solutions for some Oldroyd-B model via the incompressible limit*, Chinese Ann. Math. Ser. B, 27 (2006), pp. 565–580.

[18] Z. Lei, N. Masmoudi, and Y. Zhou, *Remarks on the blowup criteria for Oldroyd models*, J. Differential Equations, 248 (2010), pp. 328–341.

[19] Z. Lei and Y. Zhou, *Global existence of classical solutions for the two-dimensional Oldroyd model via the incompressible limit*, SIAM J. Math. Anal., 37 (2005), pp. 797–814, https://doi.org/10.1137/040618813.

[20] P. L. Lions and N. Masmoudi, *Global solutions for some Oldroyd models of non-Newtonian flows*, Chinese Ann. Math. Ser. B, 21 (2000), pp. 131–146.

[21] J. G. Oldroyd, *Non-Newtonian effects in steady motion of some idealized elastico-viscous liquids*, Proc. Roy. Soc. London Ser. A, 245 (1958), pp. 278–297.

[22] M. Paicu and P. Zhang, *Global solutions to the* 3*-D incompressible anisotropic Navier-Stokes system in the critical spaces*, Comm. Math. Phys., 307 (2011), 713C759.

[23] X. Pan and J. Xu, *Global existence and optimal decay estimates of the compressible viscoelastic flows in* L^p *critical spaces*, Discrete Contin. Dyn. Syst., 39 (2019), pp. 2021–2057.

[24] X. Pan, J. Xu, and Y. Zhu, *Global existence in critical spaces for non Newtonian compressible viscoelastic flows*, J. Differential Equations, 331 (2022), pp. 162–191.

[25] J. Qian and Z. Zhang, *Global well-posedness for compressible viscoelastic fluids near equilibrium*, Arch. Ration. Mech. Anal., 198 (2010), pp. 835–868.

[26] X. Zhai, *Global solutions to the* n*-dimensional incompressible Oldroyd-B model without damping mechanism*, J. Math. Phys., 62 (2021), 021503.

[27] Z. Zhou, C. Zhu, and R. Zi, *Global well-posedness and decay rates for the three dimensional compressible Oldroyd-B model*, J. Differential Equations, 265 (2018), pp. 1259–1278.

[28] Y. Zhu, *Global small solutions of* 3*D incompressible Oldroyd-B model without damping mechanism*, J. Funct. Anal., 274 (2018), pp. 2039–2060.

[29] Y. Zhu, *Global classical solutions of* 3*D compressible viscoelastic system near equilibrium*, Calc. Var. Partial Differential Equations, 61 (2022), 21.

[30] R. Zi, *Vanishing viscosity limit of the* 3*D incompressible Oldroyd-B model*, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 38 (2021), pp. 1841–1867.

[31] R. Zi, D. Fang, and T. Zhang, *Global solution to the incompressible Oldroyd-B model in the critical* L^p *framework: The case of the non-small coupling parameter*, Arch. Ration. Mech. Anal., 213 (2014), pp. 651–687.