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# Global existence and asymptotic behavior of solutions to the Euler equations with time-dependent damping

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## ABSTRACT

We study the isentropic Euler equations with time-dependent damping, given by  $\frac{\mu}{(1+t)^\lambda} \rho u$ . Here,  $\lambda$  and  $\mu$  are two non-negative constants to describe the decay rate of damping with respect to time. We will investigate the global existence and asymptotic behavior of small data solutions to the Euler equations when  $0 < \lambda < 1, 0 < \mu$  in multi-dimensions  $n \geq 1$ . Our strategy of proving the global existence is to convert the Euler system to a time-dependent damped wave equation and use a kind of weighted energy estimate. Investigation to the asymptotic behavior of the solution is based on the detailed analysis to the fundamental solutions of the corresponding linear damped wave equation and it coincides with that of standard results if  $\lambda$  deduces to zero.

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## 1. Introduction

This paper deals with the isentropic Euler equations with time-dependent damping in multi-dimensions:

$$\begin{aligned}\partial_t \rho + \nabla \cdot (\rho u) &= 0, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p &= -\frac{\mu}{(1+t)^\lambda} \rho u, \\ \rho|_{t=0} &= 1 + \rho_0(x), u|_{t=0} = u_0(x),\end{aligned}\tag{1}$$

where  $\rho_0(x) \in \mathbb{R}, u_0(x) \in \mathbb{R}^n$ , supported in  $\{x \in \mathbb{R}^n | |x| \leq R\}$ . Here  $\rho(t, x), u(t, x)$  and  $p(t, x)$  represent the density, fluid velocity and pressure, respectively, and  $\lambda$  and  $\mu$  are two positive constants to describe the decay rate of the damping in time. We assume the fluid is a polytropic gas which means we can assume  $p(\rho) = \frac{1}{\gamma} \rho^\gamma, \gamma > 1$ .

As is well known, when the damping vanishes, shock will form. For the mathematical analysis of finite-time formation of singularities, readers can see Alinhac [1], Chemin [2], Courant–Friedrichs [3], Christodoulou [4], Makino–Ukai–Kawashima [5], Rammaha [6] as well as Chen–Liu [7], Sideris [8, 9] and references therein for more detail.

The Euler equations with non time-decayed damping are

$$\begin{aligned}\partial_t \rho + \nabla \cdot (\rho u) &= 0, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p &= -\kappa \rho u,\end{aligned}\tag{2}$$

where  $\kappa$  is the damping constant and  $1/\kappa$  can be regarded as the relaxation time of some physical fluid. Many authors have proven the global existence and uniqueness of smooth solutions to system (2) with small data. Also, the asymptotic behavior of the smooth solution was studied. For the 1D Euler equations, see Dafermos [10], Hsiao-Liu [11], Huang-Marcati-Pan [12], Nishida [13], Nishihara-Wang-Yang [14] and their references. For the multi-dimensional case, Wang-Yang [15] give the pointwise estimates of the solution by using some energy methods and estimating the Green function to the linearized system. Sideris-Thomases-Wang [16] prove a similar result by using a simpler approach. They both proved that the smooth solution decays in maximum norm to the background state at a rate of  $(1+t)^{-\frac{3}{2}}$  in three dimensions. Sideris-Thomases-Wang also show that the smooth solution has a polynomially decayed lower bound in time while the vorticity will decay exponentially. Tan-Wang [17] and Jiu-Zheng [18] study this problem in the frame of Besov space and obtain similar asymptotic behavior of the solution. Also, see Kong-Wang [19] for extension.

It is natural to ask whether the global solution exists when the damping is decayed and what is the critical decay rate to separate the global existence and the finite-time blow up of solutions with small data. Hou-Witt-Yin in [20] first considered the 3D potential Euler equations with time-dependent damping  $\frac{\mu}{(1+t)^\lambda} \rho u$  and gave a critical point  $(\lambda, \mu) = (1, 0)$  to separate the global existence and finite-time blow up of small data classical solutions. Later, there are many papers concerning the time-decayed damping Euler equations. Readers can see [21–28] for more details in this aspect. This paper deals with the case  $0 < \lambda < 1, 0 < \mu$  in multi-dimensions  $n \geq 1$ . We will obtain the global existence and asymptotic behavior of the solution to system (1). Also we will show that the convergence rate of the solution to the background constant state  $(1, 0)$  will coincide with that one obtained by Wang-Yang [15] and Sideris-Thomases-Wang [16] when  $\lambda = 0$  which indicates that the time-asymptotic behavior of the solution is the diffusion wave of the corresponding linear system. In [26, 27], the author proved that in one dimension, the local classical small data solution will be global if  $\lambda = 1, \mu > 2$  and blow up in finite time if  $\lambda = 1, 0 < \mu \leq 2$  and  $\lambda > 1, \mu > 0$ . Combining with the result in this paper, the author shows that in one dimension, for the Euler equations with time-dependent damping given by the form  $\frac{\mu}{(1+t)^\lambda} \rho u$ ,  $(\lambda, \mu) = (1, 2)$  is the critical point to separate the global existence and finite-time blow up of small data classical solutions. For the two- and three-dimensional Euler system with the same damping, the authors in [20, 23] proved that the corresponding critical points are  $(1, 1)$  and  $(1, 0)$ , respectively. If we use  $(\lambda_n, \mu_n)$  to denote the critical point of separating the global existence and finite-time blow up of small data classical solutions for the  $n$ -dimensional Euler equations with time-dependent damping  $\frac{\mu}{(1+t)^\lambda} \rho u$ , the above results show that  $(\lambda_n, \mu_n) = (1, 3 - n)$  when  $n = 1, 2, 3$ . The proof in [20, 23], using the classical Klainerman vectors' energy method, is skillful and very different from that in the current paper.

The proof of the global existence of the solution is based on the method of weighted energy estimates for the symmetric hyperbolic system by introducing the sound speed as a new variable rather than the density. We will establish some weighted a priori estimates to the solution. The choice of the weight is inspired by the corresponding linear wave equation with effective damping satisfied by the sound speed. See (10). The weight can be found in [29–31]. The local-existence result stated in Kato [32] or Majda [33] and the continuity argument can assure the global existence of the solution.

The estimates of convergence rate of the solution to the background state come from the investigation of the fundamental solutions to the phase function of the corresponding linear wave equation which can be found in Wirth [34]. We will show that  $L^2$  and  $L^\infty$  norms of the solution to system (1) will present a decay estimate similar to that of the corresponding linear dissipative wave equation with the same damping.

Let  $H^l(\mathbb{R}^n)$  be the usual Sobolev space with its norm

$$\|f\|_{H^l} \triangleq \sum_{k=0}^l \sum_{|\alpha|=k} \|\partial_x^\alpha f\|_{L^2},$$

where  $\partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$ . Later for convenience, we will use  $\partial_x^k = \sum_{|\alpha|=k} \partial_x^\alpha$ , use  $\|\cdot\|_p$  to denote  $\|\cdot\|_{L^p}$  and  $\|\cdot\| = \|\cdot\|_2$ .

We state the global existence result as follows.

**Theorem 1.1:** Denote  $B = \frac{(1+\lambda)n}{2} - \delta$ , where  $\delta \in (0, \frac{(1+\lambda)n}{2}]$  can be arbitrarily small. Suppose that  $n \geq 1$ ,  $0 < \lambda < 1$ ,  $0 < \mu$  and  $(\rho_0, u_0) \in H^{s+m}(\mathbb{R}^n)$ , supported in  $\{x \in \mathbb{R}^n | |x| \leq R\}$ , where  $s = [n/2] + 1$  and  $m \geq 2$ . Then there exists a  $\varepsilon_0 = \varepsilon_0(\delta, \lambda, \mu, R)$  such that for any  $0 \leq \varepsilon \leq \varepsilon_0$ , when  $\|(\rho_0, u_0)\|_{H^{s+m}} \leq \varepsilon$ , there exists a unique global classical solution  $(\rho(t, x), u(t, x))$  of (1) satisfying

$$\begin{aligned} & (1+t)^{B+1+\lambda} \left( \|\partial_t \rho(t)\|_{H^{s+m-1}}^2 + \|\partial_x \rho(t)\|_{H^{s+m-1}}^2 + \|\partial_x u(t)\|_{H^{s+m-1}}^2 \right) \\ & + (1+t)^{B+2\lambda} \left( \|\partial_t u(t)\|_{H^{s+m-1}}^2 \right) \\ & + (1+t)^B \left( \|(\rho(t) - 1)\|^2 + \|u(t)\|^2 \right) \\ & \leq C_{\lambda, \mu, \delta, R} \left( \|\rho_0\|_{H^{s+m}}^2 + \|u_0\|_{H^{s+m}}^2 \right). \end{aligned} \quad (3)$$

Throughout this paper, we will denote a generic constant by  $C$  which may be different from line to line.

**Remark 1.1:** From (3), using the Sobolev embedding, for  $k = 0, 1, \dots, m-1$ , we have

$$\begin{aligned} & (1+t)^{\frac{B+1+\lambda}{2}} \left( \|\partial_x^k \partial_t \rho(t)\|_\infty + \|\partial_x^{k+1} \rho(t)\|_\infty + \|\partial_x^{k+1} u(t)\|_\infty \right) \\ & + (1+t)^{\frac{B}{2}+\lambda} \left( \|\partial_x^k \partial_t u(t)\|_\infty \right) \\ & + (1+t)^{\frac{B}{2}} \left( \|(\rho(t) - 1)\|_\infty + \|u(t)\|_\infty \right) \\ & \leq C\varepsilon. \end{aligned} \quad (4)$$

From (4), we see that  $\|(\rho(t) - 1)\|_\infty + \|u(t)\|_\infty \leq C\varepsilon(1+t)^{-\frac{1+\lambda}{4}n+\frac{\delta}{2}}$ . When  $\lambda = 0$ , it decays slower than what we expect for  $(1+t)^{-\frac{n}{2}}$  as shown in [15] and [16]. So next, based on the investigation of the properties to the corresponding linear wave equation (41) of system (1), we have the following further asymptotic behavior of the solution.

**Theorem 1.2:** Define  $k_c \triangleq \frac{1+\lambda}{1-\lambda}(n+1) - n - \frac{2\delta}{1-\lambda}$  and  $m \geq k_c + 2$ . Then under the assumption of Theorem 1.1, we have the following asymptotic behavior of the solution  $(\rho, u)$  in  $L^2$  and  $L^\infty$  norms.

For  $\rho$ :

$$\|\partial_x^k(\rho - 1)\|_\infty \leq C\varepsilon \begin{cases} (1+t)^{-(1-\lambda)\frac{n+k}{2}} & 0 \leq k \leq k_c; \\ (1+t)^{-(1+\lambda)\frac{n+1}{2}+\delta} & k_c \leq k \leq m-2. \end{cases} \quad (5)$$

$$\|\partial_x^k(\rho - 1)\| \leq C\varepsilon \begin{cases} (1+t)^{-(1-\lambda)(\frac{n}{4}+\frac{k}{2})} & 0 \leq k \leq k_c + \frac{n}{2}; \\ (1+t)^{-(1+\lambda)\frac{n+1}{2}+\delta} & k_c + \frac{n}{2} \leq k \leq s+m-2. \end{cases} \quad (6)$$

While for  $u$ , due to the damping, it will decay slower than  $\rho$  by a factor  $(1+t)^\lambda$ . That is

$$\|\partial_x^k u\|_\infty \leq C\varepsilon \begin{cases} (1+t)^{-(1-\lambda)\frac{n+k+1}{2}+\lambda} & 0 \leq k \leq k_c - 1; \\ (1+t)^{-(1+\lambda)\frac{n+1}{2}+\lambda+\delta} & k_c - 1 \leq k \leq m-3. \end{cases} \quad (7)$$

$$\|\partial_x^k u\| \leq C\varepsilon \begin{cases} (1+t)^{-(1-\lambda)(\frac{n}{4}+\frac{k+1}{2})+\lambda} & 0 \leq k \leq k_c + \frac{n}{2} - 1; \\ (1+t)^{-(1+\lambda)\frac{n+1}{2}+\lambda+\delta} & k_c + \frac{n}{2} - 1 \leq k \leq s+m-3. \end{cases} \quad (8)$$

**Remark 1.2:** Noting  $\delta > 0$  can be arbitrarily small, then  $\lim_{\lambda \rightarrow 0} k_c = 1^+$ . From (5) and (7), we have when  $\lambda \rightarrow 0$

$$\|(\rho - 1)(t)\|_\infty \leq C(1+t)^{-\frac{n}{2}}, \|u(t)\|_\infty \leq C(1+t)^{-\frac{n+1}{2}}.$$

This coincides with the decay rate that obtained in the non-decayed damping case.

The paper is organized as follows. In Section 2, we reformulate the Euler equations into a symmetric hyperbolic system. Then based on a fundamental weighted energy inequality in Lemma 2.1 and using detailed weighted energy estimates, we prove the global existence of smooth solutions with small data. In Section 3, by investigating the structure to the linear system of (1), we give the asymptotic behavior of the solution. In Appendix, we give the proof of the fundamental weighted energy inequality.

## 2. Global existence

In this section, first we reformulate system (1) to a symmetric system. Remember  $c = \sqrt{P'(\rho)} = \rho^{\frac{\gamma-1}{2}}$ . First, we transform (1) into the following system

$$\begin{aligned} \frac{2}{\gamma-1} \partial_t c + c \nabla \cdot u + \frac{2}{\gamma-1} u \nabla \cdot c &= 0, \\ \partial_t u + u \cdot \nabla u + \frac{2}{\gamma-1} c \nabla c + \frac{\mu}{(1+t)^\lambda} u &= 0, \\ c|_{t=0} &= 1 + c_0(x), u|_{t=0} = u_0(x), \end{aligned}$$

where  $c_0(x) \in \mathbb{R}$ , supported in  $\{x \in \mathbb{R}^n | |x| \leq R\}$ .

Let  $v = \frac{2}{\gamma-1}(c-1)$ , then  $(v, u)$  satisfies

$$\begin{aligned} \partial_t v + \nabla \cdot u &= -u \cdot \nabla v - \frac{\gamma-1}{2} v \nabla \cdot u, \\ \partial_t u + \nabla v + \frac{\mu}{(1+t)^\lambda} u &= -u \cdot \nabla u - \frac{\gamma-1}{2} v \nabla v, \\ v|_{t=0} &= v_0(x), u|_{t=0} = u_0(x), \end{aligned} \tag{9}$$

where  $v_0(x) = \frac{2}{\gamma-1} c_0(x)$ .

### 2.1. A fundamental weighted energy inequality

From (9), we have

$$\partial_{tt} v - \Delta v + \frac{\mu}{(1+t)^\lambda} \partial_t v = Q(v, u), \tag{10}$$

where

$$\begin{aligned} Q(v, u) &= \frac{\mu}{(1+t)^\lambda} \left( -u \cdot \nabla v - \frac{\gamma-1}{2} v \nabla \cdot u \right) \\ &\quad - \partial_t \left( u \cdot \nabla v - \frac{\gamma-1}{2} v \nabla \cdot u \right) + \nabla \cdot \left( u \cdot \nabla u + \frac{\gamma-1}{2} v \nabla v \right). \end{aligned}$$

In the following, we will obtain a fundamental weighted energy inequality about (10). This technique comes from [29–31]. Introduce the weight

$$e^{2\psi}, \quad \psi(t, x) = a \frac{|x|^2}{(1+t)^{1+\lambda}},$$

where  $a = \frac{(1+\lambda)\mu}{8} \left(1 - \frac{\delta}{(1+\lambda)n}\right)$  and  $\delta$  is described in Theorem 1.1. For simplification of notation, we denote

$$J(t; g) = \int_{\mathbb{R}^n} e^{2\psi} g^2(t, x) \, dx, \quad J_\psi(t; g) = \int_{\mathbb{R}^n} e^{2\psi} (-\psi_t) g^2(t, x) \, dx.$$

**Lemma 2.1:** Denote  $B = \frac{(1+\lambda)n}{2} - \delta$ , where  $\delta \in (0, \frac{(1+\lambda)n}{2}]$  can be arbitrarily small. Then Equation (10) has the following weighted energy inequality.

$$\begin{aligned} & (1+t)^{B+1+\lambda} [J(t; v_t) + J(t; |\partial_x v|)] + (1+t)^B J(t; v) \\ & + \int_0^t (1+\tau)^{B+1+\lambda} [J_\psi(\tau; v_\tau) + J_\psi(\tau; |\nabla v|)] \, d\tau \\ & + \int_0^t [(1+\tau)^{B+1} J(\tau, v_\tau) + (1+\tau)^{B+\lambda} J(\tau, |\nabla v|)] \, d\tau \\ & + \int_0^t [(1+\tau)^B J_\psi(\tau, v) + (1+\tau)^{B-1} J(\tau, v)] \, d\tau \\ & \leq C \|(v_0, u_0)\|_{H^1}^2 + CG(t), \end{aligned}$$

where

$$G(t) \triangleq \int_0^t \int_{\mathbb{R}^n} e^{2\psi} \{[(1+\tau)^{B+1+\lambda} v_\tau + (1+\tau)^{B+\lambda} v] Q(v, u)\} \, dx \, d\tau,$$

and  $C$  depends on  $\lambda, \mu, \delta, R$ .

**Proof:** We give the proof of Lemma 2.1 in Appendix. ■

## 2.2. Some a priori weighted energy estimates

Define the weighted Sobolev space  $H_\psi^{s+m}(\mathbb{R}^n)$  as

$$H_\psi^{s+m}(\mathbb{R}^n) = \{f | e^\psi \partial_x^k f \in L^2(\mathbb{R}^n), 0 \leq k \leq s+m, k \in \mathbb{N}\}$$

with its norm

$$\|f\|_{H_\psi^{s+m}} = \sum_{k=0}^{s+m} \left( \int_{\mathbb{R}^n} (\partial_x^k f)^2 e^{2\psi} \, dx \right)^{\frac{1}{2}}.$$

We define the weighted energy as follows

$$\begin{aligned} E_{s+m}^\psi(T) =: & \sup_{0 < t < T} \left\{ (1+t)^{B+1+\lambda} \left( \|v_t(t)\|_{H_\psi^{s+m-1}}^2 + \|\partial_x v(t)\|_{H_\psi^{s+m-1}}^2 + \|\partial_x u(t)\|_{H_\psi^{s+m-1}}^2 \right) \right. \\ & \left. + (1+t)^B \left( \|v(t)\|_{L_\psi^2}^2 + \|u(t)\|_{L_\psi^2}^2 \right) \right\}^{\frac{1}{2}}. \end{aligned}$$

Because the initial data is compactly supported, we have

$$\begin{aligned} E_{s+m}^\psi(0) &\leq C \left( \|(v_t(0), \partial_x v(0), \partial_x u(0))\|_{H^{s+m-1}} + \|(v(0), u(0))\| \right) \\ &\leq C \|(v_0, u_0)\|_{H^{s+m}} \leq C\varepsilon. \end{aligned}$$

In the following, we will estimate  $(v, u)$  under the a priori assumption

$$E_{s+m}^\psi(t) \leq M\varepsilon, \quad (11)$$

where  $M$ , independent of  $\varepsilon$ , will be determined later. By choosing  $M$  large and  $\varepsilon$  sufficient small, we will prove

$$E_{s+m}^\psi(t) \leq \frac{1}{2}M\varepsilon. \quad (12)$$

We will first obtain some a priori estimates for the first-order derivatives. Then the higher derivatives will be handled in a similar way. From (11), by Sobolev embedding, we have

$$\begin{aligned} &\left\{ (1+t)^{\frac{B}{2}} \|(v(t), u(t))\|_{L^\infty} + (1+t)^{\frac{B+1+\lambda}{2}} \|(v_t(t), \partial_x v(t), \partial_x u(t))\|_{L^\infty} \right\} \\ &\leq CE_{[n/2]+2}^\psi(t) \leq CM\varepsilon. \end{aligned} \quad (13)$$

From (9), we see

$$\begin{aligned} &(1+t)^{B/2+\lambda} \|u_t(t)\|_{L^\infty} \\ &\leq (1+t)^{B/2+\lambda} \left\{ \|\nabla v(t)\|_{L^\infty} + (1+t)^{-\lambda} \|u(t)\|_{L^\infty} + \|u \cdot \nabla u\|_{L^\infty} + \|v \nabla v\|_{L^\infty} \right\} \\ &\leq CM\varepsilon. \end{aligned} \quad (14)$$

Now for simplification of notation in our later computation, we introduce

$$\begin{aligned} &P_0(t) \\ &\triangleq P(t; v, u, v_t, u_t, \partial_x v, \partial_x u) \\ &\triangleq (1+t)^{B+1+\lambda} [J_\psi(t; v_t) + J_\psi(t; |\partial_x v|) + J_\psi(t; |\partial_x u|)] \\ &\quad + (1+t)^{B+1} [J(t; v_t) + J(t; |\partial_x u|)] + (1+t)^{B+\lambda} [J(t; |u_t|) + J(t; |\partial_x v|)] \\ &\quad + (1+t)^B [J_\psi(t; v) + J_\psi(t; |u|)] \\ &\quad + (1+t)^{B-\lambda} J(t; |u|) + (1+t)^{B-1} J(t; v). \end{aligned} \quad (15)$$

In the following, we will give three estimates for the first-order derivatives under the a priori assumption, which can reflect our idea of proving the (12).

*Estimate 1*

**Lemma 2.2:** *Under the a priori assumption (11), we have the following estimate*

$$\begin{aligned} G(t) &\leq CM\varepsilon(1+t)^{B+1+\lambda} [J(t; |v_t|) + J(t; |\partial_x v|)] \\ &\quad + C\varepsilon \left( E_1^\psi(0) \right)^2 + CM\varepsilon \int_0^t P_0(\tau) d\tau, \end{aligned}$$

where  $P_0(\tau)$  is defined in (15).

**Proof:** Remember the formulation of  $Q(v, u)$  we have

$$\begin{aligned}
 G(t) = & \int_0^t \int_{\mathbb{R}^n} e^{2\psi} \left( (1+\tau)^{B+1+\lambda} v_\tau + (1+\tau)^{B+\lambda} v \right) \\
 & \times \left\{ \underbrace{\frac{\mu}{(1+\tau)^\lambda} \left( -u \cdot \nabla v - \frac{\gamma-1}{2} v \nabla \cdot u \right)}_{I_1} \right. \\
 & \underbrace{- \left( \partial_\tau (u \cdot \nabla v) + \frac{\gamma-1}{2} \partial_\tau (v \nabla \cdot u) \right)}_{I_2} \\
 & \left. + \underbrace{\left( \nabla \cdot (u \cdot \nabla u) + \frac{\gamma-1}{2} \nabla \cdot (v \nabla v) \right)}_{I_3} \right\} dx d\tau.
 \end{aligned}$$

Now we estimate  $I_i (i = 1, 2, 3)$  term by term under the a priori estimate (11).

Using (13), (14) and the Cauchy–Schwartz inequality, we have

$$\begin{aligned}
 I_1 = & \mu \int_0^t \int_{\mathbb{R}^n} e^{2\psi} (1+\tau)^{B+1} v_\tau \left( -u \cdot \nabla v - \frac{\gamma-1}{2} v \nabla \cdot u \right) dx d\tau \\
 & + \mu \int_0^t \int_{\mathbb{R}^n} (1+\tau)^B e^{2\psi} v \left( -u \cdot \nabla v - \frac{\gamma-1}{2} v \nabla \cdot u \right) dx d\tau \\
 \leq & C \|(1+\tau)^{\frac{1+\lambda}{2}} \nabla v(\tau)\|_\infty \int_0^t \left[ (1+\tau)^{B+1} J(\tau; v_\tau) + (1+\tau)^{B-\lambda} J(\tau; |u|) \right] d\tau \\
 & + C \|v(\tau)\|_\infty \int_0^t \left[ (1+\tau)^{B+1} J(\tau; v_\tau) + (1+\tau)^{B+1} J(\tau; |\nabla u|) \right] d\tau \\
 & + C \|v(\tau)\|_\infty \int_0^t \left[ (1+\tau)^{B-\lambda} J(\tau; |u|) + (1+\tau)^{B+\lambda} J(\tau; |\nabla v|) \right] d\tau \\
 & + C \|v(\tau)\|_\infty \int_0^t \left[ (1+\tau)^{B-1} J(\tau; v) + (1+\tau)^{B-1} J(\tau; |\nabla u|) \right] d\tau \\
 \leq & CM\varepsilon \int_0^t P_0(\tau) d\tau.
 \end{aligned} \tag{16}$$

And

$$\begin{aligned}
 I_2 = & - \int_0^t \int_{\mathbb{R}^n} e^{2\psi} \left( (1+\tau)^{B+1+\lambda} v_\tau + (1+\tau)^{B+\lambda} v \right) \\
 & \times \left\{ \left( u_\tau \cdot \nabla v + \frac{\gamma-1}{2} v_\tau \nabla \cdot u \right) \right.
 \end{aligned}$$



$$\begin{aligned}
 & \left. + \underbrace{(u \cdot \nabla v_\tau)}_{I_{2,1}} + \underbrace{\left( \frac{\gamma-1}{2} v \nabla \cdot u_\tau \right)}_{I_{2,2}} \right\} dx d\tau \\
 & \leq CM\varepsilon \int_0^t P_0(\tau) d\tau + I_{2,1} + I_{2,2}.
 \end{aligned}$$

Next, we use integration by parts to estimate  $I_{2,1}$  and  $I_{2,2}$ .

$$\begin{aligned}
 I_{2,1} &= - \int_0^t \int_{\mathbb{R}^n} e^{2\psi} \left( (1+\tau)^{B+1+\lambda} v_\tau + (1+\tau)^{B+\lambda} v \right) (u \cdot \nabla v_\tau) dx d\tau \\
 &= -\frac{1}{2} \int_0^t \int_{\mathbb{R}^n} e^{2\psi} (1+\tau)^{B+1+\lambda} u \cdot \nabla v_\tau^2 dx d\tau - \int_0^t \int_{\mathbb{R}^n} e^{2\psi} (1+\tau)^{B+\lambda} v u \cdot \nabla v_\tau dx d\tau \\
 &= \frac{1}{2} \int_0^t \int_{\mathbb{R}^n} e^{2\psi} (1+\tau)^{B+1+\lambda} v_\tau^2 (\nabla \cdot u + \underbrace{2u \cdot \nabla \psi}_{I_{2,1}^1}) dx d\tau \\
 &\quad + \int_0^t \int_{\mathbb{R}^n} e^{2\psi} (1+\tau)^{B+\lambda} v_\tau (v \nabla \cdot u + u \cdot \nabla v + \underbrace{2vu \cdot \nabla \psi}_{I_{2,1}^2}) dx d\tau \\
 &\leq CM\varepsilon \int_0^t P_0(\tau) d\tau + I_{2,1}^1 + I_{2,1}^2.
 \end{aligned} \tag{17}$$

Noting (A1), we have

$$|\nabla \psi| \leq C(1+t)^{-\frac{\lambda}{2}} (-\psi_t)^{\frac{1}{2}}. \tag{18}$$

Using Cauchy–Schwartz inequality, we obtain

$$\begin{aligned}
 & |I_{2,1}^1| + |I_{2,1}^2| \\
 & \leq C \|(1+\tau)^{\frac{1+\lambda}{2}} v_\tau\|_{L^\infty} \int_0^t \left[ (1+\tau)^{B+1} J(\tau; v_\tau) + (1+\tau)^B J_\psi(\tau; |u|) \right] dx d\tau \\
 & \leq C \|(1+\tau)^{\frac{1+\lambda}{2}} v_\tau\|_{L^\infty} \int_0^t \left[ (1+\tau)^{B-1} J(\tau; v) + (1+\tau)^B J_\psi(\tau; |u|) \right] dx d\tau \\
 & \leq CM\varepsilon \int_0^t P_0(\tau) d\tau.
 \end{aligned} \tag{19}$$

Combining (17) and (19), we have

$$|I_{2,1}| \leq CM\varepsilon \int_0^t P_0(\tau) d\tau.$$

For  $I_{2,2}$ , we have

$$\begin{aligned}
 I_{2,2} &= \int_0^t \int_{\mathbb{R}^n} e^{2\psi} \left( (1+\tau)^{B+1+\lambda} v_\tau + (1+\tau)^{B+\lambda} v \right) \left( \frac{\gamma-1}{2} v \nabla \cdot u_\tau \right) dx d\tau \\
 &= C \underbrace{\int_0^t \int_{\mathbb{R}^n} e^{2\psi} (1+\tau)^{B+\lambda} v^2 \nabla \cdot u_\tau dx d\tau}_{I_{2,2}^1}
 \end{aligned}$$

$$+ C \underbrace{\int_0^t \int_{\mathbb{R}^n} e^{2\psi} (1+\tau)^{B+1+\lambda} v_\tau v \nabla \cdot u_\tau \, dx \, d\tau}_{I_{2,2}^2}.$$

Using integration by parts, (18) and Cauchy–Schwartz inequality, we have

$$\begin{aligned} I_{2,2}^1 &= -C \int_0^t \int_{\mathbb{R}^n} e^{2\psi} (1+\tau)^{B+\lambda} u_\tau \cdot (2v \nabla \cdot v + 2v^2 \nabla \psi) \, dx \, d\tau \\ &\leq CM\varepsilon \int_0^t P_0(\tau) \, d\tau. \end{aligned}$$

For the estimate of  $I_{2,2}^2$ , we use (9) and integration by parts. From (9), we have

$$\begin{aligned} \nabla \cdot u &= -\frac{v_t + u \cdot \nabla v}{1 + \frac{\gamma-1}{2}v}, \\ \nabla \cdot u_t &= -\frac{v_{tt} + u_t \cdot \nabla v + u \cdot \nabla v_t}{1 + \frac{\gamma-1}{2}v} + \frac{\frac{\gamma-1}{2}v_t(v_t + u \cdot \nabla v)}{(1 + \frac{\gamma-1}{2}v)^2}. \end{aligned} \quad (20)$$

From (13), we have

$$\frac{1}{1 + \frac{\gamma-1}{2}v} \leq \frac{1}{1 - CM\varepsilon} \leq 2. \quad (21)$$

Inserting (20) and (21) into  $I_{2,2}^2$ , we have

$$\begin{aligned} I_{2,2}^2 &= \int_0^t \int_{\mathbb{R}^n} e^{2\psi} (1+\tau)^{B+1+\lambda} v_\tau v \nabla \cdot u_\tau \, dx \, d\tau \\ &= \int_0^t \int_{\mathbb{R}^n} e^{2\psi} (1+\tau)^{B+1+\lambda} v_\tau v \left\{ -\frac{v_{\tau\tau} + u_\tau \cdot \nabla v + u \cdot \nabla v_\tau}{1 + \frac{\gamma-1}{2}v} \right. \\ &\quad \left. + \frac{\frac{\gamma-1}{2}v_\tau(v_\tau + u \cdot \nabla v)}{(1 + \frac{\gamma-1}{2}v)^2} \right\} \, dx \, d\tau \\ &\leq \underbrace{-\int_0^t \int_{\mathbb{R}^n} e^{2\psi} (1+\tau)^{B+1+\lambda} \frac{v v_\tau v_{\tau\tau}}{1 + \frac{\gamma-1}{2}v} \, dx \, d\tau}_{I_{2,2}^{2,1}} \\ &\quad - \underbrace{\int_0^t \int_{\mathbb{R}^n} e^{2\psi} (1+\tau)^{B+1+\lambda} \frac{v v_\tau u \cdot \nabla v_\tau}{1 + \frac{\gamma-1}{2}v} \, dx \, d\tau}_{I_{2,2}^{2,2}} \\ &\quad + CM\varepsilon \int_0^t P_0(\tau) \, d\tau. \end{aligned} \quad (22)$$

Using integration by parts in time and (21), we have

$$\begin{aligned}
 I_{2,2}^{2,1} &= - \int_0^t \int_{\mathbb{R}^n} e^{2\psi} (1+\tau)^{B+1+\lambda} \frac{v(v_\tau^2)_\tau}{2 \left(1 + \frac{\gamma-1}{2} v\right)} dx d\tau \\
 &= - \int_{\mathbb{R}^n} e^{2\psi} (1+\tau)^{B+1+\lambda} \frac{v v_\tau^2}{2 \left(1 + \frac{\gamma-1}{2} v\right)} \Big|_{\tau=0}^{\tau=t} dx \\
 &\quad + \int_0^t \int_{\mathbb{R}^n} e^{2\psi} v_\tau^2 \left\{ (1+\tau)^{B+1+\lambda} \partial_\tau \left( \frac{v}{2 \left(1 + \frac{\gamma-1}{2} v\right)} \right) \right. \\
 &\quad \left. + (B+1+\lambda)(1+\tau)^{B+\lambda} \frac{v}{2 \left(1 + \frac{\gamma-1}{2} v\right)} \right. \\
 &\quad \left. + 2\psi_t (1+\tau)^{B+1+\lambda} \frac{v}{2 \left(1 + \frac{\gamma-1}{2} v\right)} \right\} dx d\tau \\
 &\leq CM\varepsilon(1+t)^{B+1+\lambda} J(t; v_t) + C\varepsilon \left( E_1^\psi(0) \right)^2 + CM\varepsilon \int_0^t P_0(\tau) d\tau.
 \end{aligned} \tag{23}$$

And

$$\begin{aligned}
 I_{2,2}^{2,2} &= - \int_0^t \int_{\mathbb{R}^n} e^{2\psi} (1+\tau)^{B+1+\lambda} \frac{v}{2 \left(1 + \frac{\gamma-1}{2} v\right)} u \cdot \nabla v_\tau^2 dx d\tau \\
 &= \int_0^t \int_{\mathbb{R}^n} e^{2\psi} (1+\tau)^{B+1+\lambda} v_\tau^2 \left\{ u \cdot \nabla \left( \frac{v}{2 \left(1 + \frac{\gamma-1}{2} v\right)} \right) \right. \\
 &\quad \left. + \frac{v}{2 \left(1 + \frac{\gamma-1}{2} v\right)} \nabla \cdot u \right. \\
 &\quad \left. + \frac{v}{1 + \frac{\gamma-1}{2} v} u \nabla \psi \right\} dx d\tau \\
 &\leq CM\varepsilon \int_0^t P_0(\tau) d\tau.
 \end{aligned} \tag{24}$$

From (22), (23) and (24), we have

$$I_{2,2}^2 \leq CM\varepsilon(1+t)^{B+1+\lambda} J(t; v_t) + C\varepsilon \left( E_1^\psi(0) \right)^2 + CM\varepsilon \int_0^t P_0(\tau) d\tau.$$

Summing all the estimates about  $I_2$ , we can get

$$I_2 \leq CM\varepsilon(1+t)^{B+1+\lambda} J(t; v_t) + C\varepsilon \left( E_1^\psi(0) \right)^2 + CM\varepsilon \int_0^t P_0(\tau) d\tau. \tag{25}$$

The estimate of  $I_3$  will be essentially the same with  $I_2$ , actually, we can get

$$I_3 \leq CM\varepsilon(1+t)^{B+1+\lambda} J(t; |\partial_x v|) + C\varepsilon \left( E_1^\psi(0) \right)^2 + CM\varepsilon \int_0^t P_0(\tau) d\tau. \tag{26}$$

Combining the estimates (16), (25) and (26), we get the estimate

$$\begin{aligned} G(t) &\leq CM\varepsilon(1+t)^{B+1+\lambda} [J(t; |v_t|) + J(t; |\partial_x v|)] \\ &\quad + C\varepsilon \left(E_1^\psi(0)\right)^2 + CM\varepsilon \int_0^t P_0(\tau) d\tau, \end{aligned}$$

which finishes the proof of the lemma. ■

Combining Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned} &(1+t)^{B+1+\lambda} [J(t; v_t) + J(t; |\nabla v|)] + (1+t)^B J(t; v) \\ &\quad + \int_0^t (1+\tau)^{B+1+\lambda} [J_\psi(\tau; v_\tau) + J_\psi(\tau; |\nabla v|)] d\tau \\ &\quad + \int_0^t [(1+\tau)^{B+1} J(\tau, v_\tau) + (1+\tau)^{B+\lambda} J(\tau, |\nabla v|)] d\tau \\ &\quad + \int_0^t [(1+\tau)^B J_\psi(\tau, v) + (1+\tau)^{B-1} J(\tau, v)] d\tau \\ &\leq C \left(E_1^\psi(0)\right)^2 + CM\varepsilon \int_0^t P_0(\tau) d\tau. \end{aligned} \tag{27}$$

#### Estimate 2

Next, we consider the weighted  $L^2$  norm of  $(v, u)$ . We have the following lemma.

**Lemma 2.3:** *Under the a priori assumption (11), we have the following estimate*

$$\begin{aligned} &(1+t)^B [J(t; v) + J(t; |u|)] \\ &\quad + \int_0^t (1+\tau)^B J_\psi(\tau; |u|) d\tau + \int_0^t (1+\tau)^{B-\lambda} J(\tau; |u|) d\tau \\ &\quad - C \int_0^t [(1+\tau)^B J_\psi(\tau; v) + (1+\tau)^{B-1} J(\tau; v)] d\tau \\ &\leq C (J(0; v) + J(0; |u|)) + CM\varepsilon \int_0^t P_0(\tau) d\tau, \end{aligned} \tag{28}$$

where  $P_0(\tau)$  is defined in (15).

**Proof:** Multiplying (9) by  $(K+t)^B e^{2\psi} u$  yields

$$\begin{aligned} &\partial_t \left[ (K+t)^B \frac{e^{2\psi}}{2} |u|^2 \right] + e^{2\psi} (K+t)^B (-\psi_t) |u|^2 \\ &\quad \left( \frac{\mu(K+t)^B}{(1+t)^\lambda} - \frac{B(K+t)^{B-1}}{2} \right) e^{2\psi} |u|^2 + (K+t)^B e^{2\psi} u \cdot \nabla v \\ &= (K+t)^B e^{2\psi} u \cdot (-u \cdot \nabla u - \frac{\gamma-1}{2} v \nabla v). \end{aligned} \tag{29}$$

Integrating (29) on  $\mathbb{R}^n \times [0, t]$  and choosing  $K$  be large, we can get

$$\begin{aligned}
 & (K+t)^B J(t; |u|) + \int_0^t (K+\tau)^B J_\psi(\tau; |u|) \, d\tau + \int_0^t (K+\tau)^{B-\lambda} J(\tau; |u|) \, d\tau \\
 & \quad + C \underbrace{\int_0^t \int_{\mathbb{R}^n} e^{2\psi} (K+\tau)^B u \cdot \nabla v \, dx \, d\tau}_{I_1} \\
 & \leq C \underbrace{\int_0^t \int_{\mathbb{R}^n} (K+\tau)^B e^{2\psi} u \cdot \left( -u \cdot \nabla u - \frac{\gamma-1}{2} v \nabla v \right) \, dx \, d\tau}_{I_2} + C J(0; |u|). \quad (30)
 \end{aligned}$$

We come to estimate  $I_1$  and  $I_2$ .

Using integration by parts and (9), we have

$$\begin{aligned}
 I_1 &= - \int_0^t \int_{\mathbb{R}^n} (K+\tau)^B e^{2\psi} v (\nabla \cdot u + 2u \cdot \nabla \psi) \, dx \, d\tau \\
 &= \int_0^t \int_{\mathbb{R}^n} (K+\tau)^B e^{2\psi} v \left( v_\tau + u \cdot \nabla v + \frac{\gamma-1}{2} v \nabla \cdot u - 2u \cdot \nabla \psi \right) \, dx \, d\tau \\
 &= \underbrace{\int_0^t \int_{\mathbb{R}^n} \left\{ \partial_\tau \left[ (K+\tau)^B e^{2\psi} \frac{v^2}{2} \right] + (K+\tau)^B e^{2\psi} (-\psi_\tau) v^2 - B/2 (K+\tau)^{B-1} e^{2\psi} v^2 \right\} \, dx \, d\tau}_{I_{1,1}} \\
 & \quad + \underbrace{\int_0^t \int_{\mathbb{R}^n} (K+\tau)^B e^{2\psi} v \left( u \cdot \nabla v + \frac{\gamma-1}{2} v \nabla \cdot u \right) \, dx \, d\tau}_{I_{1,2}} \\
 & \quad - 2 \underbrace{\int_0^t \int_{\mathbb{R}^n} (K+\tau)^B e^{2\psi} v u \cdot \nabla \psi \, dx \, d\tau}_{I_{1,3}}. \quad (31)
 \end{aligned}$$

We see that

$$\begin{aligned}
 I_{1,1} &= 1/2 (K+t)^B J(t; v) - C J(0; v) \\
 & \quad + \int_0^t (K+\tau)^B J_\psi(\tau; v) \, d\tau - B/2 \int_0^t (K+\tau)^{B-1} J(\tau; v) \, d\tau.
 \end{aligned}$$

Using (13) and (14), we have

$$\begin{aligned}
 |I_{1,2}| &\leq \|(K+t)^{(1+\lambda)/2} \partial_x v(t)\|_{L^\infty} \int_0^t [(K+t)^{B-\lambda} J(\tau; u) + (K+\tau)^{B-1} J(\tau; v)] \, d\tau \\
 & \quad + \|(K+t)^{(B+1)/2} \partial_x u(t)\|_{L^\infty} \int_0^t (K+t)^{B-1} J(\tau; v) \, d\tau \\
 &\leq CM\varepsilon \int_0^t P_0(\tau) \, d\tau.
 \end{aligned}$$

Using Cauchy–Schwartz inequality and (A1), we obtain

$$|I_{1,3}| \leq \nu \int_0^t (K+t)^{B-\lambda} J(\tau; |u|) \, d\tau + C_\nu \int_0^t (K+t)^B J_\psi(\tau; \nu) \, d\tau, \quad (32)$$

where  $\nu$  is a small number. Combing (31)–(32), we have

$$\begin{aligned} I_1 &\geq 1/2(K+t)^B J(t; \nu) - CJ(0; \nu) \\ &\quad + \int_0^t (K+\tau)^B J_\psi(\tau; \nu) \, d\tau - B/2 \int_0^t (K+\tau)^{B-1} J(\tau; \nu) \, d\tau \\ &\quad - \nu \int_0^t (K+t)^{B-\lambda} J(\tau; |u|) \, d\tau - C_\nu \int_0^t (K+t)^B J_\psi(\tau; \nu) \, d\tau \\ &\quad - CM\varepsilon \int_0^t P_0(\tau) \, d\tau. \end{aligned} \quad (33)$$

$I_2$  can be done the same with  $I_{1,2}$ , so, we have

$$|I_2| \leq CM\varepsilon \int_0^t P_0(\tau) \, d\tau. \quad (34)$$

Combining (30), (33) and (34), by choosing  $\nu$  small, we have

$$\begin{aligned} &(K+t)^B [J(t; \nu) + J(t; |u|)] \\ &\quad + \int_0^t (K+\tau)^B J_\psi(\tau; |u|) \, d\tau + \int_0^t (K+\tau)^{B-\lambda} J(\tau; |u|) \, d\tau \\ &\quad - C \int_0^t [(K+t)^B J_\psi(\tau; \nu) + (K+t)^{B-1} J(\tau; \nu)] \, d\tau \\ &\leq C(J(0; \nu) + J(0; |u|)) + CM\varepsilon \int_0^t P_0(\tau) \, d\tau. \end{aligned}$$

This finishes the proof of Lemma 2.3. ■

### Estimate 3

Next we consider the weighted  $L^2$  norm of  $(\partial_x v, \partial_x u)$ . We have the following lemma.

**Lemma 2.4:** *Under the a priori assumption (11), we have the following estimate*

$$\begin{aligned} &(1+t)^{B+1+\lambda} [J(t; |\partial_x v|) + J(t; |\partial_x u|)] \\ &\quad + \int_0^t (1+\tau)^{B+1+\lambda} J_\psi(\tau; |\partial_x u|) \, d\tau + \int_0^t (1+\tau)^{B+1} J(\tau; |\partial_x u|) \, d\tau \\ &\quad - C \int_0^t [(1+\tau)^{B+1+\lambda} J_\psi(\tau; |\partial_x v|) + (1+\tau)^{B+\lambda} J(\tau; |\partial_x v|)] \, d\tau \\ &\leq C \left( E_1^\psi(0) \right)^2 + CM\varepsilon \int_0^t P_0(\tau) \, d\tau, \end{aligned} \quad (35)$$

where  $P_0(\tau)$  is defined in (15).

**Proof:** By differentiating (9) with respect to  $x$  and then multiplying it with  $(K+t)^{B+1+\lambda}e^{2\psi}\partial_x u$ , we get

$$\begin{aligned} & \partial_t \left[ (K+t)^{B+1+\lambda} e^{2\psi} \frac{|\partial_x u|^2}{2} \right] + e^{2\psi} (K+t)^{B+1+\lambda} (-\psi_t) |\partial_x u|^2 \\ & + \left[ \left( \mu \frac{(K+t)^\lambda}{(1+t)^\lambda} - \frac{B+1+\lambda}{2} (K+t)^{\lambda-1} \right) (K+t)^{B+1} e^{2\psi} |\partial_x u|^2 \right] \\ & + (K+t)^{B+1+\lambda} e^{2\psi} \partial_x u \cdot \nabla \partial_x v \\ & = (K+t)^{B+1+\lambda} e^{2\psi} \partial_x u \cdot \partial_x \left( -u \cdot \nabla u - \frac{\gamma-1}{2} v \nabla v \right). \end{aligned} \quad (36)$$

By choosing  $K$  large and integrating (36) on  $\mathbb{R}^n \times [0, t]$ , we can obtain the estimate (35) by using the same trick as that in the proof of Lemma 2.3. We omit the detail.  $\blacksquare$

#### Estimate 4

From (9), we can easily have

$$\begin{aligned} & \int_0^t (1+\tau)^{B+\lambda} J(\tau; |u_\tau|) d\tau \\ & \leq \int_0^t (1+\tau)^{B+\lambda} J(\tau; |\partial_x v|) d\tau + \int_0^t (1+\tau)^{B-\lambda} J(\tau; |u|) d\tau + CM_\varepsilon \int_0^t P_0(\tau) d\tau. \end{aligned} \quad (37)$$

Adding (27) to  $v_1 \cdot (28) + v_2 \cdot (35) + v_3 \cdot (37)$ , where  $v_1, v_2 \ll 1$  and  $v_3 \ll v_1$ , we can get

$$\begin{aligned} & \left( E_1^\psi(t) \right)^2 + \int_0^t P_0(\tau) d\tau \\ & \leq \left( E_1^\psi(0) \right)^2 + CM_\varepsilon \int_0^t P_0(\tau) d\tau. \end{aligned} \quad (38)$$

### 2.3. Proof of Theorem 1.1

**Proof:** Actually, the estimate (38) can be obtained for higher derivatives. Define  $P_k(t)$  the same as  $P_0$ .

$$\begin{aligned} P_k(t) & \triangleq P(t; \partial_x^k v, \partial_x^k u, \partial_x^k v_t, \partial_x^k u_t, \partial_x \partial_x^k v, \partial_x \partial_x^k u) \\ & \triangleq (1+t)^{B+1+\lambda} \left[ J_\psi(t; \partial_x^k v_t) + J_\psi(t; |\partial_x \partial_x^k v|) + J_\psi(t; |\partial_x \partial_x^k u|) \right] \\ & + (1+t)^{B+1} \left[ J(t; \partial_x^k v_t) + J(t; |\partial_x \partial_x^k v|) \right] + (1+t)^{B+\lambda} \left[ J(t; |\partial_x^k u_t|) + J(t; |\partial_x \partial_x^k v|) \right] \\ & + (1+t)^B \left[ J_\psi(t; \partial_x^k v) + J_\psi(t; |\partial_x^k u|) \right] \\ & + (1+t)^{B-\lambda} J(t; |\partial_x^k u|) + (1+t)^{B-1} J(t; \partial_x^k v). \end{aligned}$$

For any  $l \in \mathbb{N}$ , differentiating (10)  $l$  times with respect to space variable  $x$ , we have

$$\partial_{tt} \partial_x^l v - \Delta \partial_x^l v + \frac{\mu}{(1+t)^\lambda} \partial_t \partial_x^l v = \partial_x^l Q(v, u).$$

Then do the same estimate as shown in *Estimate 1*. We can get the following similar estimate to (27)

$$\begin{aligned}
 & (1+t)^{B+1+\lambda} \left[ J(t; \partial_x^l v_t) + J(t; |\partial_x \partial_x^l v|) \right] \\
 & + \int_0^t (1+\tau)^{B+1+\lambda} \left[ J_\psi(\tau; \partial_x^l v_\tau) + J_\psi(\tau; |\partial_x \partial_x^l v|) \right] d\tau \\
 & + \int_0^t (1+\tau)^{B+1} J(\tau; \partial_x^l v_\tau) + (1+\tau)^{B+\lambda} J(\tau; |\partial_x \partial_x^l v|) d\tau \\
 & + \int_0^t (1+\tau)^B J_\psi(\tau, \partial_x^l v) + (1+\tau)^{B-1} J(\tau; \partial_x^l v) d\tau \\
 & \leq CE_{l+1}^\psi(0) + CM\varepsilon \sum_{k=0}^l \int_0^t P_k(\tau) d\tau.
 \end{aligned} \tag{39}$$

Differentiating (9)<sub>2</sub>  $l$  times with respect to  $x$ , multiplying it by  $(K+t)^B e^{2\psi} \partial_x^l u$  and doing the same estimate as in *Estimate 2*, we have

$$\begin{aligned}
 & (1+t)^B [J(t; \partial_x^l v) + J(t; |\partial_x^l u|)] \\
 & + \int_0^t (1+\tau)^B J_\psi(\tau; |\partial_x^l u|) d\tau + \int_0^t (1+\tau)^{B-\lambda} J(\tau; |\partial_x^l u|) d\tau \\
 & - C \int_0^t \left[ (1+\tau)^B J_\psi(\tau; \partial_x^l v) + (1+\tau)^{B-1} J(\tau; \partial_x^l v) \right] d\tau \\
 & \leq CE_l^\psi(0) + CM\varepsilon \sum_{k=0}^{l-1} \int_0^t P_k(\tau) d\tau.
 \end{aligned}$$

Differentiating (9)<sub>2</sub>  $l+1$  times with respect to  $x$ , multiplying it by  $(K+t)^{B+1+\lambda} e^{2\psi} \partial_x^{l+1} u$  and doing the same estimate as in *Estimate 3*, we have

$$\begin{aligned}
 & (1+t)^{B+1+\lambda} [J(t; |\partial_x \partial_x^l v|) + J(t; |\partial_x \partial_x^l u|)] \\
 & + \int_0^t (1+\tau)^{B+1+\lambda} J_\psi(\tau; |\partial_x \partial_x^l u|) d\tau + \int_0^t (1+\tau)^{B+1} J(\tau; |\partial_x \partial_x^l u|) d\tau \\
 & - C \int_0^t \left[ (1+\tau)^{B+1+\lambda} J_\psi(\tau; |\partial_x \partial_x^l v|) + (1+\tau)^{B+\lambda} J(\tau; |\partial_x \partial_x^l v|) \right] d\tau \\
 & \leq CE_{l+1}^\psi(0) + CM\varepsilon \sum_{k=0}^l \int_0^t P_k(\tau) d\tau.
 \end{aligned}$$

From (9)<sub>2</sub>, we also have

$$\begin{aligned}
 & \int_0^t (1+\tau)^{B+\lambda} J(\tau; |\partial_x^l u_\tau|) d\tau \\
 & \leq \int_0^t (1+\tau)^{B+\lambda} J(\tau; |\partial_x \partial_x^l v|) d\tau + \int_0^t (1+\tau)^{B-\lambda} J(\tau; |\partial_x^l u|) d\tau \\
 & + CM\varepsilon \sum_{k=0}^l \int_0^t P_k(\tau) d\tau.
 \end{aligned} \tag{40}$$



Adding (39)–(40) together the same as (38) for  $l$  from 0 to  $s+m-1$  together. Then there exists a  $C_0$ , depending on  $\lambda, \mu, \delta$  and  $R$  such that

$$\begin{aligned} & \left(E_{s+m}^{\psi}(t)\right)^2 + \sum_{k=0}^{s+m-1} \int_0^t P_k(\tau) \, d\tau \\ & \leq C \left(E_{s+m}^{\psi}(0)\right)^2 + CM\varepsilon \sum_{k=0}^{s+m-1} \int_0^t P_k(\tau) \, d\tau \\ & \leq C_0^2 \varepsilon^2 + C_0 M\varepsilon \sum_{k=0}^{s+m-1} \int_0^t P_k(\tau) \, d\tau. \end{aligned}$$

Let  $M = 2C_0$  and  $C_0 M\varepsilon < 1$ . Then we obtain

$$E_{s+m}^{\psi}(t) \leq \frac{1}{2} M\varepsilon.$$

The local existence of symmetrizable hyperbolic equations has been proved by using the fixed point theorem. In order to get the global existence of the system, we only need a priori estimate. Based on the a priori estimate (11), we yield (12). This finishes the proof of Theorem 1.1. ■

### 3. Asymptotic behavior of the solution

In this section, we restudy the decay rate of the smooth solution to system (1) and obtain the asymptotic behavior of the solution to the background state  $(1, 0)^T$ . For this purpose, we consider the linear equation of (10)

$$w_{tt} - \Delta w + \frac{\mu}{(1+t)^\lambda} w_t = f. \quad (41)$$

This equation involves in the study of the Cauchy problem to the wave equation with time-dependent damping.

$$\begin{aligned} w_{tt} - \Delta w + \frac{\mu}{(1+t)^\lambda} w_t &= f, \\ v(0, x) &= w_0(x), \quad w_t(0, x) = w_1(x), \end{aligned} \quad (42)$$

where  $0 < \lambda < 1, 0 < \mu$ .

#### 3.1. Decay rate of solutions to the linear wave equation

If we apply the partial Fourier transform with respect to the space variables to (42), we can get

$$\begin{aligned} \hat{w}_{tt} + |\xi|^2 \hat{w} + \frac{\mu}{(1+t)^\lambda} \hat{w}_t &= \hat{f}, \\ \hat{w}(0, \xi) &= \hat{w}_0(\xi), \quad \hat{w}_t(0, \xi) = \hat{w}_1(\xi), \end{aligned} \quad (43)$$

where  $\xi$  is the frequency parameter. The solution of (43) can be represented by the sum of the solution of the following two equations

$$\begin{aligned} \hat{w}_{tt}^1 + |\xi|^2 \hat{w}^1 + \frac{\mu}{(1+t)^\lambda} \hat{w}_t^1 &= 0, \\ \hat{w}(0, \xi) &= \hat{w}_0(\xi), \quad \hat{w}_t(0, \xi) = \hat{w}_1(\xi), \end{aligned} \quad (44)$$

and

$$\begin{aligned}\hat{w}_{tt}^2 + |\xi|^2 \hat{w}^2 + \frac{\mu}{(1+t)^\lambda} \hat{w}_t^2 &= \hat{f}, \\ \hat{w}(0, x) &= 0, \quad \hat{w}_t(0, x) = 0.\end{aligned}\tag{45}$$

The solution of (44) can be represented in the form

$$\hat{w}^1(t, \xi) = \Phi_1(t, \xi) \hat{w}_0(\xi) + \Phi_2(t, \xi) \hat{w}_1(\xi).$$

And the solution of (45) can be represented in the form

$$\hat{w}^2(t, \xi) = \int_0^t \Phi_2(t - \tau, \xi) \hat{f}(\xi) d\xi.$$

$\Phi_1(t, \xi)$  and  $\Phi_2(t, \xi)$  are the fundamental solutions of (44) satisfying

$$\begin{aligned}\Phi_1(0, \xi) &= 1, \quad \partial_t \Phi_1(0, \xi) = 0, \\ \Phi_2(0, \xi) &= 0, \quad \partial_t \Phi_2(0, \xi) = 1.\end{aligned}$$

If we set

$$K_i(t, x) = \mathcal{F}_{\xi \rightarrow x}^{-1}[\Phi_i(t, \xi)] \quad i = 1, 2.$$

Then the solution of (43) can be represented by

$$\begin{aligned}w(t, x) &= (K_1(t, \cdot) * w_0(\cdot))(x) + (K_2(t, \cdot) * w_1(\cdot))(x) \\ &\quad + \int_0^t (K_2(t - \tau, \cdot) * f(\cdot))(x) d\tau.\end{aligned}\tag{46}$$

The asymptotic behavior of  $\Phi_i(t, \xi) (i = 1, 2)$  with respect to time has already been investigated carefully in Wirth [34]. Here we will give the most important theorem there for our use. Before that, we need to introduce some notations.

In the space  $(t, \xi) \in (\mathbb{R}^+ \times \mathbb{R}^n)$ , denote  $Z_1, Z_2$  and  $Z_3$  as follows.

$$\begin{aligned}Z_1 &= \left\{ (t, \xi) : |\xi| \leq \frac{1}{4} \frac{\mu}{(1+t)^\lambda} \right\}, \\ Z_2 &= \left\{ (t, \xi) : |\xi| \geq \frac{1}{4} \frac{\mu}{(1+t)^\lambda} \right\} \cap \{(t, \xi) : |\xi| \leq 1\}, \\ Z_3 &= \left\{ (t, \xi) : |\xi| \geq \frac{1}{4} \frac{\mu}{(1+t)^\lambda} \right\} \cap \{(t, \xi) : |\xi| \geq 1\}.\end{aligned}$$

**Lemma 3.1 (Theorem 17 of [34]):** *There exists a constant  $C_0$ , depending on  $\lambda, \mu$  such that  $\Phi_i$  has the following estimates in different zones*

When  $(t, \xi) \in Z_1$ ,

$$|\Phi_i(t, \xi)| \leq C_0 e^{-C_0 |\xi|^2 (1+t)^{1-\lambda}}.\tag{47}$$

When  $(t, \xi) \in Z_2$ ,

$$|\Phi_i(t, \xi)| \leq C_0 e^{-C_0 (1+t)^{1-\lambda}} e^{C_0 (1-|\xi|^2)(1+t_\xi)^{1-\lambda}},\tag{48}$$

where  $t_\xi$  is the upper boundary line of the zone  $Z_1$ , which means

$$|\xi| = \frac{1}{4} \frac{\mu}{(1+t_\xi)^\lambda}.$$

When  $(t, \xi) \in Z_3$ ,

$$|\Phi_i(t, \xi)| \leq C_0 e^{-C_0 (1+t)^{1-\lambda}}.$$

**Proof:** This is a direct result of Theorem 17 in Wirth [34]. We omit the detail. ■

Next, we give two integral properties of  $\Phi_i$  in the phase space  $\xi$ .

**Lemma 3.2:**

$$\|\xi^\alpha \Phi_i(t, \xi)\|_{L^1(Z_j)} \leq C(1+t)^{-(1-\lambda)(n/2+|\alpha|/2)}, \quad (49)$$

$$\|\xi^\alpha \Phi_i(t, \xi)\|_{L^2(Z_j)} \leq C(1+t)^{-(1-\lambda)(n/4+|\alpha|/2)}, \quad (50)$$

where  $i, j = 1, 2$ .

**Proof:** In the zone  $Z_1$ , using (47), we have

$$\begin{aligned} & \|\xi^\alpha \Phi_i(t, \xi)\|_{L^1(Z_1)} \\ & \leq \int_{\xi \in Z_1} |\xi|^\alpha |\Phi_i(t, \xi)| \, d\xi \\ & \leq C \int_0^\infty |\xi|^{n-1+\alpha} e^{-C_0|\xi|^2(1+t)^{1-\lambda}} \, d|\xi| \\ & \leq C(1+t)^{-(1-\lambda)(n/2+|\alpha|/2)} \int_0^\infty s^{n-1+|\alpha|} e^{-C_0s^2} \, ds \\ & \leq C(1+t)^{-(1-\lambda)(n/2+|\alpha|/2)}. \end{aligned}$$

Integration in  $Z_2$  will use the representation of  $t\xi$ . From (47), we have  $1+t\xi = (\frac{4}{\mu}|\xi|)^{-\frac{1}{\lambda}}$ . Then using (48), we have

$$\begin{aligned} & \|\xi^\alpha \Phi_i(t, \xi)\|_{L^1(Z_2)} \\ & \leq C_0 e^{-C_0(1+t)^{1-\lambda}} \int_{\xi \in Z_2} |\xi|^\alpha e^{C_0(1-|\xi|^2)\left(\frac{4}{\mu}|\xi|\right)^{-\frac{1-\lambda}{\lambda}}} \, d\xi \\ & \leq C e^{-C_0(1+t)^{1-\lambda}} \int_{\frac{\mu}{4}(1+t)^{-\lambda}}^1 |\xi|^{n-1+|\alpha|} e^{C_0\left(\frac{4}{\mu}|\xi|\right)^{-\frac{1-\lambda}{\lambda}}} e^{-C|\xi|^2-\frac{1-\lambda}{\lambda}} \, d|\xi|. \end{aligned}$$

Noting that

$$\begin{aligned} & |\xi|^{n-1+|\alpha|} e^{-C|\xi|^2-\frac{1-\lambda}{\lambda}} \\ & = \left(|\xi|^{2-\frac{1-\lambda}{\lambda}}\right)^{\frac{n-1+|\alpha|}{2}} e^{-C|\xi|^2-\frac{1-\lambda}{\lambda}} |\xi|^{\frac{1-\lambda}{2\lambda}(n-1+|\alpha|)} \\ & \leq C|\xi|^{\frac{1-\lambda}{2\lambda}(n-1+|\alpha|)}. \end{aligned}$$

Then we get

$$\begin{aligned} & \|\xi^\alpha \Phi_i(t, \xi)\|_{L^1(Z_2)} \\ & \leq C e^{-C_0(1+t)^{1-\lambda}} \int_{\frac{\mu}{4}(1+t)^{-\lambda}}^1 |\xi|^{\frac{1-\lambda}{2\lambda}(n-1+|\alpha|)} e^{C_0\left(\frac{4}{\mu}|\xi|\right)^{-\frac{1-\lambda}{\lambda}}} \, d|\xi| \end{aligned}$$

$$\begin{aligned}
&= Ce^{-C_0(1+t)^{1-\lambda}} \int_{\left(\frac{4}{\mu}\right)^{-\frac{1-\lambda}{\lambda}}}^{(1+t)^{1-\lambda}} s^{-\frac{1}{2}(n-1+|\alpha|)-\frac{1}{1-\lambda}} e^{C_0 s} ds \quad \left(s = \left(\frac{4}{\mu}|\xi|\right)^{-\frac{1-\lambda}{\lambda}}\right) \\
&\leq Ce^{-C_0(1+t)^{1-\lambda}} \underbrace{\int_{\left(\frac{4}{\mu}\right)^{-\frac{1-\lambda}{\lambda}}}^{(1+t)^{1-\lambda}} s^{-\frac{1}{2}(n+|\alpha|)} e^{C_0 s} ds}_I.
\end{aligned} \tag{51}$$

$I$  can be dealt with integration by parts

$$\begin{aligned}
I &= \frac{1}{C_0} s^{-\frac{1}{2}(n+|\alpha|)} e^{C_0 s} \Big|_{s=\left(\frac{4}{\mu}\right)^{-\frac{1-\lambda}{\lambda}}}^{s=(1+t)^{1-\lambda}} \\
&\quad - \frac{n+|\alpha|}{2C_0} \int_{\left(\frac{4}{\mu}\right)^{-\frac{1-\lambda}{\lambda}}}^{(1+t)^{1-\lambda}} s^{-\frac{1}{2}(n+|\alpha|)-1} e^{C_0 s} ds \\
&\leq C(1+t)^{-\frac{1-\lambda}{2}(n+|\alpha|)} e^{C_0(1+t)^{1-\lambda}}.
\end{aligned}$$

Inserting this into (51), we obtain that

$$\|\xi^\alpha \Phi_i(t, \xi)\|_{L^1(Z_2)} \leq C(1+t)^{-(1-\lambda)(n/2+|\alpha|/2)}.$$

The proof of (50) will be essentially the same as (49). We omit the detail.

Combining the estimates of  $\Phi_i(t, \xi)$  in Lemma 3.1 and Lemma 3.2, we have the following estimates to the solution of (42). ■

**Lemma 3.3:** Assume that for a function  $g(x) \in L^1 \cap H^{s+l}(\mathbb{R}^n)$ , where  $s = [n/2] + 1$  and  $l \geq 0$ . Then we have

$$\|\partial_x^k(K_i(t, \cdot) * g(\cdot))\|_\infty \leq C(1+t)^{-(1-\lambda)\frac{n+k}{2}} \|g\|_1 + Ce^{-C_0(1+t)^{1-\lambda}} \|\partial_x^{s+k} g\|, \tag{52}$$

where  $k \in \mathbb{N}$  and  $0 \leq k \leq l$ .

$$\|\partial_x^k(K_i(t, \cdot) * g(\cdot))\| \leq C(1+t)^{-(1-\lambda)(\frac{n}{4}+\frac{k}{2})} \|g\|_1 + Ce^{-C_0(1+t)^{1-\lambda}} \|\partial_x^k g\|, \tag{53}$$

where  $k \in \mathbb{N}$  and  $0 \leq k \leq s+l$ .

**Proof:** Using Lemma 3.1, Lemma 3.2 and Hölder inequality, we have

$$\begin{aligned}
&\|\partial_x^k(K_i(t, \cdot) * g(\cdot))\|_\infty \\
&\leq \| |\xi|^k \hat{K}_i(t, \xi) \hat{g}(\xi) \|_1 \\
&= \left( \int_{\xi \in Z_1 \cup Z_2} + \int_{\xi \in Z_3} \right) |\xi|^k |\Phi_i(t, \xi)| |\hat{g}(\xi)| d\xi \\
&\leq \|\hat{g}\|_\infty \int_{\xi \in Z_1 \cup Z_2} |\xi|^k |\Phi_i(t, \xi)| d\xi + \int_{\xi \in Z_3} \|\Phi_i(t, \xi)\|_\infty |\xi|^k |\hat{g}(\xi)| d\xi \\
&\leq \|\hat{g}\|_\infty \int_{\xi \in Z_1 \cup Z_2} |\xi|^k |\Phi_i(t, \xi)| d\xi
\end{aligned}$$

$$\begin{aligned}
 & + Ce^{-C_0(1+t)^{1-\lambda}} \left( \int_{|\xi| \geq 1} |\xi|^{-2([n/2]+1)} d\xi \right)^{\frac{1}{2}} \left( \int_{|\xi| \geq 1} |\xi|^{2([n/2]+1+k)} |\hat{g}|^2 d\xi \right)^{\frac{1}{2}} \\
 & \leq C(1+t)^{-(1-\lambda)\frac{n+k}{2}} \|g\|_1 + Ce^{-C_0(1+t)^{1-\lambda}} \|\partial_x^{k+s} g\|.
 \end{aligned}$$

This proves (52).

Using Plancherel equality, Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned}
 & \|\partial_x^k (K_i(t, \cdot) * g(\cdot))\|^2 \\
 & = \| |\xi|^k \hat{K}_i(t, \xi) \hat{g}(\xi) \|^2 \\
 & = \left( \int_{\xi \in Z_1 \cup Z_2} + \int_{\xi \in Z_3} \right) |\xi|^{2k} |\Phi_i(t, \xi)|^2 |\hat{g}|^2 d\xi \\
 & \leq \|\hat{g}\|_\infty^2 \int_{\xi \in Z_1 \cup Z_2} |\xi|^{2k} |\Phi_i(t, \xi)|^2 d\xi + \int_{\xi \in Z_3} \|\Phi_i(t, \xi)\|_\infty^2 |\xi|^{2k} |\hat{g}|^2 d\xi \\
 & \leq \|g\|_1^2 \int_{\xi \in Z_1 \cup Z_2} |\xi|^{2k} |\Phi_i(t, \xi)|^2 d\xi + Ce^{-2C_0(1+t)^{1-\lambda}} \int_{\xi \in Z_3} |\xi|^{2k} |\hat{g}|^2 d\xi \\
 & \leq C(1+t)^{-(1-\lambda)(\frac{n}{2}+k)} \|g\|_1^2 + Ce^{-2C_0(1+t)^{1-\lambda}} \|\partial_x^k g\|^2.
 \end{aligned}$$

This proves (53). ■

### 3.2. Asymptotic behavior of $v$

From (10) and (46), we have

$$\begin{aligned}
 v(t, x) & = (K_1(t, \cdot) * v_0(\cdot))(x) + (K_2(t, \cdot) * v_1(\cdot))(x) \\
 & \quad + \int_0^t (K_2(t - \tau, \cdot) * Q(v, u)(\tau, \cdot))(x) d\tau,
 \end{aligned} \tag{54}$$

where

$$v_1(x) = \partial_t v|_{t=0} = -\nabla \cdot u_0 - u_0 \cdot \nabla v_0 - \frac{\gamma-1}{2} v_0 \nabla \cdot u_0.$$

Before we do estimates on  $v$ , we first estimate the nonlinear term  $Q(v, u)$  under the help of Theorem 1.1. Remember that

$$\begin{aligned}
 Q(v, u) & = \frac{\mu}{(1+t)^\lambda} \left( -u \cdot \nabla v - \frac{\gamma-1}{2} v \nabla \cdot u \right) \\
 & \quad - \partial_t \left( u \cdot \nabla v - \frac{\gamma-1}{2} v \nabla \cdot u \right) + \nabla \cdot \left( u \cdot \nabla u + \frac{\gamma-1}{2} v \nabla v \right).
 \end{aligned} \tag{55}$$

**Proposition 3.1:** Assume  $f, g \in H^l(\mathbb{R}^n)$ . Then for  $|\alpha| \leq l$ , we have

$$\|\partial_x^\alpha (fg)\| \leq C_l \left( \|f\|_\infty \|\partial_x^l g\| + \|g\|_\infty \|\partial_x^l f\| \right).$$

Proof of the proposition can be found in many literatures. See [35] for example.

**Lemma 3.4:** Under the assumption of Theorem 1.1, we have the following estimates for  $Q$ .

$$\|Q(v, u)(t, \cdot)\|_1 \leq C\varepsilon^2(1+t)^{-B-\frac{1+\lambda}{2}}.$$

And

$$\|\partial_x^k Q(v, u)(t, \cdot)\| \leq C\varepsilon^2(1+t)^{-B-\frac{1+\lambda}{2}},$$

where  $k = 0, 1, 2, \dots, s+m-2$ .

**Proof:** This is a direct computation of the combination of Leibniz formula, Hölder inequality, Proposition 3.1 and (4) in Theorem 1.1. ■

**Lemma 3.5:** Suppose that  $a > 1$  and  $a \geq b > 0$ . Then there exists a constant  $C$  such that for all  $t \geq 0$ ,

$$\int_0^t (1+t-\tau)^{-a}(1+\tau)^{-b} d\tau \leq C(1+t)^{-b}.$$

**Proof:** This is just a direct computation

$$\begin{aligned} & \int_0^t (1+t-\tau)^{-a}(1+\tau)^{-b} d\tau \\ &= \left( \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t \right) (1+t-\tau)^{-a}(1+\tau)^{-b} d\tau \\ &\leq C(1+t)^{-a} \int_0^{\frac{t}{2}} (1+\tau)^{-b} d\tau + C(1+t)^{-b} \int_{\frac{t}{2}}^t (1+t-\tau)^{-a} d\tau \\ &\leq C(1+t)^{-a+1-b} + C(1+t)^{-b} \\ &\leq C(1+t)^{-b}. \end{aligned}$$
■

$L^\infty$  estimate

From (54), using (52), for  $k = 0, 1, \dots, m-2$

$$\begin{aligned} & \|\partial_x^k v\|_\infty \\ &\leq C(1+t)^{-(1-\lambda)\frac{n+k}{2}} \|(v_0, v_1)\|_1 + Ce^{-C_0(1+t)^{1-\lambda}} \|\partial_x^{s+k}(v_0, v_1)\| \\ &\quad + C \int_0^t (1+t-\tau)^{-(1-\lambda)\frac{n+k}{2}} \|Q(v, u)\|_1 d\tau \\ &\quad + C \int_0^t e^{-C_0(1+t-\tau)^{1-\lambda}} \|\partial_x^{s+k} Q(v, u)\| d\tau \\ &\leq C(1+t)^{-(1-\lambda)\frac{n+k}{2}} \|(v_0, u_0)\|_{H^{s+m-1}} \\ &\quad + C\varepsilon^2 \int_0^t (1+t-\tau)^{-(1-\lambda)\frac{n+k}{2}} (1+\tau)^{-B-\frac{1+\lambda}{2}} d\tau. \end{aligned}$$

Noting  $B = \frac{(1+\lambda)n}{2} - \delta$ , when  $(1-\lambda)\frac{n+k}{2} = B + \frac{1+\lambda}{2}$ , we have

$$k = k_c = \frac{1+\lambda}{1-\lambda}(n+1) - n - \frac{2\delta}{1-\lambda}.$$

Case 1:  $0 \leq k \leq k_c$

Using Lemma 3.5 and the initial data assumption, we have

$$\begin{aligned}
 & \|\partial_x^k v\|_\infty \\
 & \leq C\varepsilon(1+t)^{-(1-\lambda)\frac{n+k}{2}} + C\varepsilon^2(1+t)^{-(1-\lambda)\frac{n+k}{2}} \\
 & \leq C\varepsilon(1+t)^{-(1-\lambda)\frac{n+k}{2}}.
 \end{aligned} \tag{56}$$

Case 2:  $k_c \leq k \leq m-2$

$$\begin{aligned}
 & \|\partial_x^k v\|_\infty \\
 & \leq C\varepsilon(1+t)^{-(1-\lambda)\frac{n+k}{2}} + C\varepsilon^2(1+t)^{-B-\frac{1+\lambda}{2}} \\
 & \leq C(1+t)^{-(1+\lambda)\frac{n+1}{2}+\delta}.
 \end{aligned}$$

$L^2$  estimate

From (54), using (53), for  $k = 0, 1, \dots, s+m-2$

$$\begin{aligned}
 & \|\partial_x^k v\| \\
 & \leq C(1+t)^{-(1-\lambda)(\frac{n}{4}+\frac{k}{2})} \|(v_0, v_1)\|_1 + Ce^{-C_0(1+t)^{1-\lambda}} \|\partial_x^k(v_0, v_1)\| \\
 & \quad + C \int_0^t (1+t-\tau)^{-(1-\lambda)(\frac{n}{4}+\frac{k}{2})} \|Q(v, u)\|_1 d\tau \\
 & \quad + C \int_0^t e^{-C_0(1+t-\tau)^{1-\lambda}} \|\partial_x^k Q(v, u)\| d\tau \\
 & \leq C(1+t)^{-(1-\lambda)(\frac{n}{4}+\frac{k}{2})} \|(v_0, u_0)\|_{H^{s+m-1}} \\
 & \quad + C\varepsilon^2 \int_0^t (1+t-\tau)^{-(1-\lambda)(\frac{n}{4}+\frac{k}{2})} (1+\tau)^{-B-\frac{1+\lambda}{2}} d\tau.
 \end{aligned}$$

When  $(1-\lambda)(\frac{n}{4}+\frac{k}{2}) = B + \frac{1+\lambda}{2}$ , we have

$$k = k_c + \frac{n}{2}.$$

Case 1:  $0 \leq k \leq k_c + \frac{n}{2}$

Using Lemma 3.5, we have

$$\begin{aligned}
 & \|\partial_x^k v\| \\
 & \leq C\varepsilon(1+t)^{-(1-\lambda)(\frac{n}{4}+\frac{k}{2})} + C\varepsilon^2(1+t)^{-(1-\lambda)(\frac{n}{4}+\frac{k}{2})} \\
 & \leq C\varepsilon(1+t)^{-(1-\lambda)(\frac{n}{4}+\frac{k}{2})}.
 \end{aligned}$$

Case 2:  $k_c + \frac{n}{2} \leq k \leq m-2$

$$\begin{aligned}
 & \|\partial_x^k v\| \\
 & \leq C\varepsilon(1+t)^{-(1-\lambda)(\frac{n}{4}+\frac{k}{2})} + C\varepsilon^2(1+t)^{-B-\frac{1+\lambda}{2}} \\
 & \leq C\varepsilon(1+t)^{-(1+\lambda)\frac{n+1}{2}+\delta}.
 \end{aligned} \tag{57}$$

Combining the estimates (56)–(57), we proved (5) and (6).

### 3.3. Asymptotic behavior of $u$

Denote  $u = (u^1, \dots, u^n)$ . From (9), differentiating it  $k$  time in  $x$ , we have

$$\partial_t \partial_x^k u^i + \frac{\mu}{(1+t)^\lambda} \partial_x^k u^i = -\partial_x^k \left( \partial_i v + u^j \partial_j u^i + \frac{\gamma-1}{2} v \partial_i v \right). \quad (58)$$

$L^\infty$  estimate

From (58), we have

$$\frac{d}{dt} \left[ e^{\frac{\mu}{1-\lambda}(1+t)^{1-\lambda}} \partial_x^k u^i \right] = -e^{\frac{\mu}{1-\lambda}(1+t)^{1-\lambda}} \partial_x^k \left( \partial_i v + u^j \partial_j u^i + \frac{\gamma-1}{2} v \partial_i v \right). \quad (59)$$

Integrating (59) from 0 to  $t$ , we obtain

$$\begin{aligned} e^{\frac{\mu}{1-\lambda}(1+t)^{1-\lambda}} \partial_x^k u^i &= e^{\frac{\mu}{1-\lambda}} \partial_x^k u_0^i(x) \\ &\quad - \int_0^t e^{\frac{\mu}{1-\lambda}(1+\tau)^{1-\lambda}} \partial_x^k \left( \partial_i v + u^j \partial_j u^i + \frac{\gamma-1}{2} v \partial_i v \right) d\tau. \end{aligned} \quad (60)$$

From the estimates of  $v$  (5) and (6) and (4), we have

$$\begin{aligned} & \left| -\partial_x^k \left( \partial_i v + u^j \partial_j u^i + \frac{\gamma-1}{2} v \partial_i v \right) \right| \\ & \leq \begin{cases} (1+t)^{-(1-\lambda)\frac{n+k+1}{2}} & 0 \leq k \leq k_c - 1, \\ (1+t)^{-\frac{1+\lambda}{2}(n+1)+\delta} & k_c - 1 < k \leq m-3. \end{cases} \end{aligned} \quad (61)$$

Then from (60) and (61), we have

Case 1:  $0 \leq k \leq k_c - 1$

$$\begin{aligned} \|\partial_x^k u^i\|_\infty &\leq C\epsilon e^{-\frac{\mu}{1-\lambda}(1+t)^{1-\lambda}} \\ &\quad + e^{-\frac{\mu}{1-\lambda}(1+t)^{1-\lambda}} \underbrace{\int_0^t (1+\tau)^{-(1-\lambda)\frac{n+k+1}{2}+\lambda} e^{\frac{\mu}{1-\lambda}(1+\tau)^{1-\lambda}} d\tau}_I. \end{aligned} \quad (62)$$

Using integration by parts, it is easy to see

$$\begin{aligned} I &= C \int_0^t (1+\tau)^{-(1-\lambda)\frac{n+k+1}{2}+\lambda} d e^{\frac{\mu}{1-\lambda}(1+\tau)^{1-\lambda}} \\ &\leq C(1+t)^{-(1-\lambda)\frac{n+k+1}{2}+\lambda} e^{\frac{\mu}{1-\lambda}(1+t)^{1-\lambda}}. \end{aligned}$$

So, from (62), we get

$$\|\partial_x^k u^i\|_\infty \leq C\epsilon(1+t)^{-(1-\lambda)\frac{n+k+1}{2}+\lambda}. \quad (63)$$

Case 2:  $k_c - 1 \leq k \leq m-3$

$$\begin{aligned} \|\partial_x^k u^i\|_\infty &\leq C\epsilon e^{-\frac{\mu}{1-\lambda}(1+t)^{1-\lambda}} \\ &\quad + e^{-\frac{\mu}{1-\lambda}(1+t)^{1-\lambda}} \int_0^t (1+\tau)^{-(1+\lambda)\frac{n+1}{2}+\delta} e^{\frac{\mu}{1-\lambda}(1+\tau)^{1-\lambda}} d\tau \\ &\leq C\epsilon(1+t)^{-(1+\lambda)\frac{n+1}{2}+\delta}. \end{aligned} \quad (64)$$



### $L^2$ estimate

Multiplying (58) by  $\partial_x^k u^i$ , then integrating it on  $\mathbb{R}^n$  and using Hölder inequality, we get

$$\begin{aligned} & \frac{d}{dt} \|\partial_x^k u^i\|^2 + \frac{\mu}{(1+t)^\lambda} \|\partial_x^k u^i\|^2 \\ &= - \int_{\mathbb{R}^n} \partial_x^k u^i \partial_x^k \left( \partial_i v + u^j \partial_j u^i + \frac{\gamma-1}{2} v \partial_i v \right) dx \\ &\leq \frac{\mu}{2(1+t)^\lambda} \|\partial_x^k u^i\|^2 + C(1+t)^\lambda \|\partial_x^k \partial_i v\|^2 \\ &\quad + C(1+t)^\lambda \left( \|\partial_x^k (u^j \partial_j u^i)\|^2 + \|\partial_x^k (v \partial_i v)\|^2 \right) \\ &\leq \frac{\mu}{2(1+t)^\lambda} \|\partial_x^k u^i\|^2 + C(1+t)^\lambda \|\partial_x^k \partial_i v\|^2 \\ &\quad + C\varepsilon^4 (1+t)^{-2B-1}. \end{aligned}$$

Form (5) and (6), we have

$$\begin{aligned} & \|\partial_x^k \partial_i v\| \\ &\leq \begin{cases} C\varepsilon(1+t)^{-(1-\lambda)(\frac{n}{4}+\frac{k+1}{2})} & 0 \leq k \leq k_c + \frac{n}{2} - 1, \\ C\varepsilon(1+t)^{-(1+\lambda)\frac{n+1}{2}+\delta} & k_c + \frac{n}{2} - 1 < k \leq s + m - 3. \end{cases} \end{aligned}$$

Case 1:  $0 \leq k \leq k_c + \frac{n}{2} - 1$

$$\begin{aligned} & \frac{d}{dt} \|\partial_x^k u^i\|^2 + \frac{\mu}{2(1+t)^\lambda} \|\partial_x^k u^i\|^2 \\ &\leq C\varepsilon^2 (1+t)^{-(1-\lambda)(\frac{n}{2}+k+1)+\lambda}. \end{aligned}$$

Then we have

$$\begin{aligned} & \frac{d}{dt} \left( \|\partial_x^k u^i\|^2 e^{\frac{\mu}{2(1-\lambda)}(1+t)^{1-\lambda}} \right) \\ &\leq C\varepsilon^2 \int_0^t (1+\tau)^{-(1-\lambda)(\frac{n}{2}+k+1)+\lambda} e^{\frac{\mu}{2(1-\lambda)}(1+\tau)^{1-\lambda}} d\tau. \end{aligned} \quad (65)$$

Integrating (65) from 0 to  $t$ , we obtain

$$\|\partial_x^k u^i\| \leq C\varepsilon(1+t)^{-(1-\lambda)(\frac{n}{4}+\frac{k+1}{2})+\lambda}. \quad (66)$$

Case 2: when  $k_c + \frac{n}{2} - 1 < k \leq s + m - 3$

$$\begin{aligned} & \frac{d}{dt} \|\partial_x^k u^i\|^2 + \frac{\mu}{2(1+t)^\lambda} \|\partial_x^k u^i\|^2 \\ &\leq C\varepsilon^2 (1+t)^{-(1+\lambda)(n+1)+\lambda+2\delta}. \end{aligned}$$

The same as (65), we have

$$\|\partial_x^k u^i\| \leq C\varepsilon(1+t)^{-(1+\lambda)\frac{n+1}{2}+\lambda+\delta}. \quad (67)$$

Combining (63), (64), (66) and (67), we proved (7) and (8).

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## Appendix

**Proof of Lemma 2.1:** Remember

$$\psi(t, x) = a \frac{|x|^2}{(1+t)^{1+\lambda}}, \quad a = \frac{(1+\lambda)\mu}{8} \left(1 - \frac{\delta}{(1+\lambda)n}\right).$$

Then

$$\begin{aligned} \psi_t &= -(1+\lambda) \frac{a|x|^2}{(1+t)^{2+\lambda}} = -\frac{1+\lambda}{1+t} \psi, \\ \nabla \psi &= \frac{2ax}{(1+t)^{1+\lambda}}, \quad \Delta \psi = \frac{2an}{(1+t)^{1+\lambda}}. \end{aligned}$$

And

$$\begin{aligned} \frac{|\nabla \psi|^2}{-\psi_t} &= \frac{4a}{1+\lambda} \frac{1}{(1+t)^\lambda} = \frac{1}{2} \left(1 - \frac{\delta}{(1+\lambda)n}\right) \frac{\mu}{(1+t)^\lambda}, \\ \Delta \psi &= \left[ \frac{(1+\lambda)n}{4} - \frac{\delta}{4} \right] \frac{\mu}{(1+t)^{1+\lambda}}. \end{aligned} \tag{A1}$$

Multiplying (10) by  $e^{2\psi} \partial_t v$  and  $e^{2\psi} v$ , we have

$$\begin{aligned} &\partial_t \left[ \frac{e^{2\psi}}{2} ((\partial_t v)^2 + |\nabla v|^2) \right] - \nabla \cdot (e^{2\psi} \partial_t v \nabla v) \\ &\quad + e^{2\psi} \left( \frac{\mu}{(1+t)^\lambda} - \frac{|\nabla \psi|^2}{-\psi_t} - \psi_t \right) (\partial_t v)^2 + \underbrace{\frac{e^{2\psi}}{-\psi_t} |\psi_t \nabla v - \partial_t v \nabla \psi|^2}_{I_1} \\ &= e^{2\psi} \partial_t v Q(v, u), \end{aligned} \tag{A2}$$

and

$$\begin{aligned} & \partial_t \left[ e^{2\psi} \left( v \partial_t v + \frac{\mu}{2(1+t)^\lambda} v^2 \right) \right] - \nabla \cdot (e^{2\psi} v \nabla v) \\ & + e^{2\psi} \left\{ |\nabla v|^2 + \left( -\psi_t + \frac{\lambda}{2(1+t)} \right) \frac{\mu}{(1+t)^\lambda} v^2 + \underbrace{2v \nabla \psi \cdot \nabla v}_{I_2} - 2\psi_t v \partial_v - (\partial_t v)^2 \right\} \\ & = e^{2\psi} v Q(v, u). \end{aligned} \quad (\text{A3})$$

We estimate  $I_1, I_2$  as follows:

$$\begin{aligned} I_1 & \geq \frac{e^{2\psi}}{-\psi_t} \left( (1 - \delta_1) \psi_t^2 |\nabla v|^2 - (1/\delta_1 - 1) v_t^2 |\nabla \psi|^2 \right) \\ & = e^{2\psi} \left\{ (1 - \delta_1)(-\psi_t) |\nabla v|^2 - \frac{1}{2} \left( 1 - \frac{\delta}{(1+\lambda)n} \right) (1/\delta_1 - 1) \frac{\mu}{(1+t)^\lambda} v_t^2 \right\}. \end{aligned} \quad (\text{A4})$$

Choosing  $\delta_1$  close to 1 such that

$$\frac{1}{2} \left( 1 - \frac{\delta}{(1+\lambda)n} \right) (1/\delta_1 - 1) \leq \frac{\delta}{2(1+\lambda)n},$$

and

$$\begin{aligned} I_2 & = 4e^{2\psi} v \nabla v \cdot \nabla \psi - e^{2\psi} \nabla v^2 \cdot \nabla \psi \\ & = 4e^{2\psi} v \nabla v \cdot \nabla \psi - \nabla \cdot (e^{2\psi} v^2 \nabla \psi) + 2e^{2\psi} v^2 |\nabla \psi|^2 + e^{2\psi} (\Delta \psi) v^2. \end{aligned} \quad (\text{A5})$$

Then inserting (A4) and (A5) into (A2) and (A3), we have

$$\begin{aligned} & \partial_t \left[ \frac{e^{2\psi}}{2} ((\partial_t v)^2 + |\nabla v|^2) \right] - \nabla \cdot (e^{2\psi} \partial_t v \nabla v) \\ & + e^{2\psi} \left\{ \left( \frac{\mu}{2(1+t)^\lambda} - \psi_t \right) (v_t)^2 + (1 - \delta_1)(-\psi_t) |\nabla v|^2 \right\} \\ & \leq e^{2\psi} \partial_t v Q(v, u), \end{aligned} \quad (\text{A6})$$

and

$$\begin{aligned} & \partial_t \left[ e^{2\psi} \left( v \partial_t v + \frac{\mu}{2(1+t)^\lambda} v^2 \right) \right] - \nabla \cdot \{ (e^{2\psi} (v \nabla v + v^2 \nabla \psi)) \} \\ & + e^{2\psi} \left\{ \underbrace{|\nabla v|^2 + 4v \nabla \psi \cdot \nabla v + \left( \frac{\mu}{(1+t)^\lambda} (-\psi_t) + 2|\nabla \psi|^2 \right) v^2}_{I_3} \right. \\ & \left. + \left( \lambda + \frac{(1+\lambda)n}{2} - 2\delta \right) \frac{\mu}{2(1+t)^{1+\lambda}} v^2 - 2\psi_t v v_t - v_t^2 \right\} \\ & = e^{2\psi} v Q(v, u). \end{aligned} \quad (\text{A7})$$

By (A1), we have

$$\begin{aligned} I_3 & = |\nabla v|^2 + 4v \nabla \psi \cdot \nabla v + 4 \frac{1 - \delta / (2(1+\lambda)n)}{1 - \delta / ((1+\lambda)n)} v^2 |\nabla \psi|^2 \\ & \geq (1 - \delta_2) |\nabla v|^2 + 4 \left( \frac{1 - \delta / (2(1+\lambda)n)}{1 - \delta / ((1+\lambda)n)} - 1/\delta_2 \right) |\nabla \psi|^2 v^2. \end{aligned}$$

Choosing  $\delta_2$  close to 1 can assure that for some  $\delta_3, \delta_4 > 0$ , we have

$$\begin{aligned} I_3 & \geq \delta_3 (|\nabla v|^2 + |\nabla \psi|^2 v^2) \\ & \geq \delta_4 \left( |\nabla v|^2 + \frac{\mu}{(1+t)^\lambda} (-\psi_t) v^2 \right). \end{aligned} \quad (\text{A8})$$

Inserting (A8) into (A7), we get

$$\begin{aligned}
 & \partial_t \left[ e^{2\psi} \left( v \partial_t v + \frac{\mu}{2(1+t)^\lambda} v^2 \right) \right] - \nabla \cdot \{ (e^{2\psi} (v \nabla v + v^2 \nabla \psi)) \} \\
 & + e^{2\psi} \left\{ \delta_4 \left( |\nabla v|^2 + \frac{\mu}{(1+t)^\lambda} (-\psi_t) v^2 \right) \right. \\
 & \left. + \left( \lambda + \frac{(1+\lambda)n}{2} - 2\delta \right) \frac{\mu}{2(1+t)^{1+\lambda}} v^2 - 2\psi_t v v_t - v_t^2 \right\} \\
 & \leq e^{2\psi} v Q(v, u).
 \end{aligned} \tag{A9}$$

Integrating (A9) on  $\mathbb{R}^n$ , we have

$$\begin{aligned}
 & \frac{d}{dt} \int_{\mathbb{R}^n} e^{2\psi} \left( v \partial_t v + \frac{\mu}{2(1+t)^\lambda} v^2 \right) dx \\
 & + \int_{\mathbb{R}^n} e^{2\psi} \left\{ \delta_4 \left( |\nabla v|^2 + \frac{\mu}{(1+t)^\lambda} (-\psi_t) v^2 \right) + \left( \lambda + \frac{(1+\lambda)n}{2} - 2\delta \right) \frac{\mu}{2(1+t)^{1+\lambda}} v^2 \right. \\
 & \left. - 2\psi_t v v_t - v_t^2 \right\} dx \\
 & \leq \int_{\mathbb{R}^n} e^{2\psi} v Q(v, u) dx.
 \end{aligned} \tag{A10}$$

To absorb the negative term  $-v_t^2$ , we integrate (A6) in  $\mathbb{R}^n$  and multiply it by  $(K+t)^\lambda$ , where  $K$  is a sufficiently large constant. Then we get

$$\begin{aligned}
 & \frac{d}{dt} \left[ (K+t)^\lambda \int_{\mathbb{R}^n} \frac{e^{2\psi}}{2} (v_t^2 + |\nabla v|^2) dx \right] - \frac{\lambda}{(K+t)^{1-\lambda}} \int_{\mathbb{R}^n} \frac{e^{2\psi}}{2} (v_t^2 + |\nabla v|^2) dx \\
 & + \int_{\mathbb{R}^n} e^{2\psi} \left\{ \left( \frac{\mu}{2} - \psi_t (K+t)^\lambda \right) v_t^2 + (1-\delta_1) (-\psi_t) (K+t)^\lambda |\nabla v|^2 \right\} \\
 & \leq (K+t)^\lambda \int_{\mathbb{R}^n} e^{2\psi} \partial_t v Q(v, u) dx.
 \end{aligned} \tag{A11}$$

Now adding  $v \cdot$ (A10) to (A11), where  $v$  is a sufficient small constant, we get

$$\begin{aligned}
 & \frac{d}{dt} \int_{\mathbb{R}^n} e^{2\psi} \left\{ \frac{(K+t)^\lambda}{2} (v_t^2 + |\nabla v|^2) + \underbrace{v v v_t}_{I_4} + \frac{v\mu}{2(1+t)^\lambda} v^2 \right\} dx \\
 & + \int_{\mathbb{R}^n} e^{2\psi} \left\{ \left( \frac{\mu}{2} - v + (-\psi_t) (K+t)^\lambda - \frac{\lambda}{2(K+t)^{1-\lambda}} \right) v_t^2 \right. \\
 & + \left( (1-\delta_1) (-\psi_t) (K+t)^\lambda + \delta_4 v - \frac{\lambda}{2(K+t)^{1-\lambda}} \right) |\nabla v|^2 \\
 & + \delta_4 v \frac{\mu}{(1+t)^\lambda} (-\psi_t) v^2 + \left( \lambda + \frac{(1+\lambda)n}{2} - \frac{\delta}{2} \right) \frac{v\mu}{2(1+t)^{1+\lambda}} v^2 \\
 & \left. - 2v \psi_t v v_t \right\} dx \\
 & \leq (K+t)^\lambda \int_{\mathbb{R}^n} e^{2\psi} \partial_t v Q(v, u) dx + v \int_{\mathbb{R}^n} e^{2\psi} v Q(v, u) dx.
 \end{aligned}$$

The terms  $I_4$  and  $I_5$  can be absorbed by the other positive terms by applying the smallness of  $v$  and largeness of  $K$  and the following Cauchy-Schwartz inequality

$$\begin{aligned}
 |I_4| & \leq \frac{(K+t)^\lambda}{4} v_t^2 + \frac{v^2}{(K+t)^\lambda} v^2 \\
 & \leq \frac{(K+t)^\lambda}{4} v_t^2 + \frac{v^2}{(1+t)^\lambda} v^2,
 \end{aligned} \tag{A12}$$

and

$$\begin{aligned} |I_5| &\leq \frac{\nu\delta_4}{2}(-\psi_t)\frac{\mu}{(1+t)^\lambda}v^2 + \frac{2\nu}{\mu\delta_4}(-\psi_t)(1+t)^\lambda v_t^2 \\ &\leq \frac{\nu\delta_4}{2}(-\psi_t)\frac{\mu}{(1+t)^\lambda}v^2 + \frac{2\nu}{\mu\delta_4}(-\psi_t)(K+t)^\lambda v_t^2. \end{aligned}$$

Denote

$$E(t) = \int_{\mathbb{R}^n} e^{2\psi} \left\{ \frac{(K+t)^\lambda}{2} (v_t^2 + |\nabla v|^2) + \underbrace{\nu v v_t}_{I_4} + \frac{\nu\mu}{2(1+t)^\lambda} v^2 \right\} dx.$$

When  $\nu$  is sufficiently small and  $K$  is large, using (A12), there exists a small constant  $c_\delta$  such that

$$\begin{aligned} &\frac{d}{dt}E(t) + \int_{\mathbb{R}^n} \left( \lambda + \frac{(1+\lambda)n}{2} - \frac{\delta}{2} \right) \frac{\nu\mu}{2(1+t)^{1+\lambda}} v^2 dx \\ &\quad + c_\delta \int_{\mathbb{R}^n} e^{2\psi} \left\{ (1 + (-\psi_t)(K+t)^\lambda) v_t^2 \right. \\ &\quad \left. + (1 + (-\psi_t)(K+t)^\lambda) |\nabla v|^2 \right. \\ &\quad \left. + (1+t)^{-\lambda}(-\psi_t)v^2 \right\} dx d\tau \\ &\leq (K+t)^\lambda \int_{\mathbb{R}^n} e^{2\psi} \partial_t \nu Q(v, u) dx + \int_{\mathbb{R}^n} e^{2\psi} \nu Q(v, u) dx. \end{aligned} \quad (\text{A13})$$

Using (A12), we see that

$$\begin{aligned} &\frac{1}{4}(K+t)^\lambda [J(t; v_t) + J(t; |\nabla v|)] + \left( \frac{\nu\mu}{2} - v^2 \right) (1+t)^{-\lambda} J(t; v) \\ &\leq E(t) \\ &\leq \frac{3}{4}(K+t)^\lambda [J(t; v_t) + J(t; |\nabla v|)] + \left( \frac{\nu\mu}{2} + v^2 \right) (1+t)^{-\lambda} J(t; v). \end{aligned} \quad (\text{A14})$$

Multiplying (A13) by  $(K+t)^{B+\lambda}$  and using (A14), we have

$$\begin{aligned} &\frac{d}{dt} [(K+t)^{B+\lambda} E(t)] \\ &\quad + \left( c_\delta - \frac{3}{4}(B+\lambda)(K+t)^{\lambda-1} \right) (K+t)^{B+\lambda} [J(t; v_t) + J(t; |\nabla v|)] \\ &\quad + \frac{(K+t)^{B+\lambda}}{2(1+t)^{1+\lambda}} \left( \left( \lambda + \frac{(1+\lambda)n}{2} - \frac{\delta}{2} \right) \nu\mu - 2(B+\lambda) \left( \frac{\nu\mu}{2} + v^2 \right) \right) J(t; v) \\ &\quad + c_\delta (K+t)^{B+2\lambda} [J_\psi(v_t) + J_\psi(t; |\nabla v|)] + c_\delta (K+t)^B J_\psi(t; v) \\ &\leq \int_{\mathbb{R}^n} e^{2\psi} [(K+t)^{B+2\lambda} v_t + \nu(K+t)^{B+\lambda} v] Q(v, u) dx. \end{aligned} \quad (\text{A15})$$

Integrating (A15) over  $[0, t]$ , we can show that by choosing small  $\nu$  and large  $K$ , there exists a constant  $C_0$  depending on  $\lambda, \mu, \delta, R$  such that

$$\begin{aligned} &(K+t)^{B+2\lambda} [J(t; v_t) + J(t; |\nabla v|)] + (K+t)^B J(t; v) \\ &\quad + \int_0^t (K+\tau)^{B+\lambda} [J(\tau; v_\tau) + J(\tau; |\nabla v|)] d\tau \\ &\quad + \int_0^t (K+\tau)^{B+2\lambda} [J_\psi(\tau; v_\tau) + J_\psi(\tau; |\nabla v|)] d\tau \\ &\quad + \int_0^t [(K+\tau)^B J_\psi(\tau; v) + (K+\tau)^{B-1} J(\tau; v)] d\tau \\ &\leq C \|(\nu(0), v_t(0), \partial_x v(0))\| \\ &\quad + C \int_0^t \int_{\mathbb{R}^n} e^{2\psi} \left\{ (K+\tau)^{B+2\lambda} v_\tau + (K+\tau)^{B+\lambda} v \right\} Q(v, u) dx d\tau. \end{aligned} \quad (\text{A16})$$

Considering that  $\int_0^t (K + \tau)^{B+\lambda} [J(\tau; v_\tau) + J(\tau; |\nabla v|)] \, d\tau$  has been estimated, we multiply (A11) by  $(K + t)^{B+1}$  and integrate it over  $[0, t]$  to obtain

$$\begin{aligned}
 & (K + t)^{B+1+\lambda} (J(t; v_t) + J(t; |\nabla v|)) \\
 & - \int_0^t (\lambda + B + 1)(K + \tau)^{B+\lambda} [J(\tau; v_\tau) + J(\tau; |\nabla v|)] \, d\tau \\
 & + \int_0^t (K + \tau)^{B+1+\lambda} [J_\psi(\tau; v_\tau) + J_\psi(\tau; |\nabla v|)] \, d\tau \\
 & + \int_0^t (K + \tau)^{B+1} J(\tau; v_\tau) \, d\tau \\
 & \leq C \|(v_t(0), \partial_x v(0))\| + C \int_0^t \int_{\mathbb{R}^n} (K + \tau)^{B+1+\lambda} e^{2\psi} \partial_\tau v Q(v, u) \, dx \, d\tau.
 \end{aligned} \tag{A17}$$

For small  $v$ , adding  $v \cdot$  (A17) to (A16), we have

$$\begin{aligned}
 & (K + t)^{B+1+\lambda} [J(t; v_t) + J(t; |\nabla v|)] + (K + t)^B J(t; v) \\
 & + \int_0^t (K + \tau)^{B+1+\lambda} [J_\psi(\tau; v_\tau) + J_\psi(\tau; |\nabla v|)] \, d\tau \\
 & + \int_0^t [(K + \tau)^{B+1} J(\tau; v_\tau) + (K + \tau)^{B+\lambda} J(\tau; |\nabla v|)] \, d\tau \\
 & + \int_0^t [(K + \tau)^B J_\psi(\tau, v) + (K + \tau)^{B-1} J(\tau, v)] \, d\tau \\
 & \leq C \|(v(0), v_t(0), \partial_x v(0))\| \\
 & + C \int_0^t \int_{\mathbb{R}^n} e^{2\psi} \{((K + \tau)^{B+1+\lambda} v_\tau + (K + \tau)^{B+\lambda} v) Q(v, u)\} \, dx \, d\tau.
 \end{aligned}$$

This proves Lemma 2.1. ■