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ON CONORMAL DERIVATIVE PROBLEM FOR PARABOLIC EQUATIONS WITH DINI MEAN OSCILLATION COEFFICIENTS

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Abstract. We show that weak solutions to parabolic equations in divergence form with conormal boundary conditions are continuously differentiable up to the boundary when the leading coefficients have Dini mean oscillation and the lower order coefficients verify certain integrability conditions.

1. **Introduction.** In a recent work [6], the first author *et al.* obtained interior and boundary C^1 estimates for divergence form elliptic equations with conormal boundary conditions, under the assumption that the coefficients and data have Dini mean oscillation. This work was motivated by a question raised by Yanyan Li [17]. In this paper, we consider the corresponding parabolic equations with conormal boundary conditions under the assumption that the coefficients and data have Dini mean oscillation with respect to all the variables, and establish global C^1 estimates.

Let $\Omega_T = (0,T)\times\Omega \subset \mathbb{R}^{n+1}$ be a cylindrical domain, where Ω is a bounded domain in $\mathbb{R}^n$ with $n \geq 1$. We consider a second order parabolic operator P in divergence form

$$Pu = \partial_t u - \sum_{i,j=1}^{n} D_i\left(a^{ij}(t,x)D_j u + a^i(t,x)u\right) + \sum_{i=1}^{n} b^i(t,x)D_i u + c(t,x)u. \qquad (1)$$

Here, the coefficients $\mathbf{A} = \left(a^{ij}\right)_{i,j=1}^n$, $\boldsymbol{a} = \left(a^1,\ldots,a^n\right)$, $\boldsymbol{b} = \left(b^1,\ldots,b^n\right)$, and c are measurable functions defined on $\overline{\Omega_T}$. We assume that the leading coefficients

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$\mathbf{A}=\left(a^{ij}\right)$ are defined on $\mathbb{R}^{n+1}$ and satisfy the uniform parabolicity condition

$$\begin{gathered}\lambda|\xi|^2 \leq \sum_{i,j=1}^n a^{ij}(t,x)\xi^i\xi^j, \quad \left|\sum_{i,j=1}^n a^{ij}(t,x)\xi^i\eta^j\right| \leq \lambda^{-1}|\xi||\eta|, \\ \forall \xi=\left(\xi^1,\ldots,\xi^n\right), \quad \eta=\left(\eta^1,\ldots,\eta^n\right)\in\mathbb{R}^n, \quad \forall (t,x)\in\mathbb{R}^{n+1}\end{gathered} \tag{2}$$

for some positive constant λ.

Throughout the paper, we shall use $X=(t,x)$ to denote a point in $\mathbb{R}^{n+1}=\mathbb{R}\times\mathbb{R}^n$, and $x=(x^1,\ldots,x^n)$ will always be a point in $\mathbb{R}^n$. We also write $Y=(s,y)$, $X_0=(t_0,x_0)$, and $Z=(\tau,z)$, etc. We define the parabolic distance between the points $X=(t,x)$ and $Y=(s,y)$ in $\mathbb{R}^{n+1}$ as

$$|X-Y|=\max(\sqrt{|t-s|},|x-y|).$$

For a domain Ω in $\mathbb{R}^n$, we shall write

$$\Omega_r(x)=\Omega\cap B_r(x)$$

and

$$Q_r^-(X)=Q_r^-(t,x)=(t-r^2,t)\times\Omega_r(x).$$

For any $\varepsilon\in(0,T)$ and $\Omega'\subset\subset\Omega$, We denote $(\varepsilon,T)\times\Omega$ and $(0,T)\times\Omega'$ by $\Omega_{\varepsilon,T}$ and Ω'_T, respectively.

We say that a non-negative measurable function $\omega:(0,1]\to\mathbb{R}$ is a Dini function provided that there are constants $c_1,c_2>0$ such that

$$c_1\omega(t)\leq\omega(s)\leq c_2\omega(t) \tag{3}$$

whenever $0<t/2\leq s\leq t<1$, and

$$\int_0^1\frac{\omega(s)}{s}ds<+\infty.$$

For a function g on $\Omega_T=(0,T)\times\Omega$, we say that g is uniformly Dini continuous if the function $\varrho_g:\mathbb{R}_+\to\mathbb{R}$ defined by

$$\varrho_g(r):=\sup\left\{|g(Y)-g(Y')|: Y,\ Y'\in\Omega_T, |Y-Y'|\leq r\right\}$$

is a Dini function. We write $g\in C^{k,\mathrm{Dini}}$ if $D^\alpha g$ is uniformly Dini continuous for each multi-index α with $|\alpha|\leq k$. We say that a locally integrable function g is of *Dini mean oscillation* over Ω_T and write $g\in\mathrm{DMO}$ if the function $\omega_g:\mathbb{R}_+\to\mathbb{R}$ defined by

$$\omega_g(r):=\sup_{Q_r^-(X)\subset\overline{\Omega}_T}\fint_{Q_r^-(X)}|g(Y)-g_{X,r}|, \quad \left(g_{X,r}:=\fint_{Q_r^-(X)}g\right)$$

is a Dini function. We shall also say that g is of *Dini mean oscillation* in x over Ω_T and write $g\in\mathrm{DMO}_x$ if the function $\omega_g^{\mathrm{x}}:\mathbb{R}_+\to\mathbb{R}$ defined by

$$\omega_g^{\mathrm{x}}(r):=\sup_{Q_r^-(X)\subset\overline{\Omega}_T}\fint_{Q_r(X)}\left|g(s,y)-\bar{g}^{\mathrm{x}}_{x,r}(s)\right|, \quad \left(\bar{g}^{\mathrm{x}}_{x,r}(s):=\fint_{\Omega_r(x)}g(s,\cdot)\right)$$

is a Dini function. In view of the proof on [17, p. 495], we know that ω_g and ω_g^{x} satisfy (3).

The main theorem of this paper is as follows. We refer the reader to Section 2 for the definitions of the function spaces such as $\mathcal{H}_2^1,\mathring{C}^{1/2,1}$, etc.

Theorem 1.1. *Let $q > n+2, \Omega$ have $C^{1,Dini}$ boundary, and the coefficients of P in (1) satisfy the following conditions in addition to (2) : $\mathbf{A} \in \mathrm{DMO}$ and $\boldsymbol{a} \in \mathrm{DMO}$ over Ω_T, $\boldsymbol{b} \in L_q(\Omega_T)$, and $c \in L_q(\Omega_T)$. Let $u \in \mathcal{H}_2^1(\Omega_T)$ be a weak solution of*

$$\begin{cases} Pu = \operatorname{div} \boldsymbol{g} + f & \text{in } \Omega_T, \\ \boldsymbol{A}\nabla u \cdot \boldsymbol{\nu} + \boldsymbol{a}u \cdot \boldsymbol{\nu} + a^0 u = -\boldsymbol{g} \cdot \boldsymbol{\nu} + g^0 & \text{on } (0,T) \times \partial\Omega, \\ u = 0 & \text{on } \{t=0\} \times \Omega, \end{cases}$$

where $\boldsymbol{g} = (g^1, \ldots, g^n) \in \mathrm{DMO}(\Omega_T)$, $f \in L_q(\Omega_T)$, a^0 and g^0 are uniformly Dini continuous in $(0,T) \times \Omega$. Then, we have $u \in \mathring{C}^{1/2,1}(\overline{\Omega_{\varepsilon,T}}) \cap \mathring{C}^{1/2,1}(\overline{\Omega'_T})$ for any $\varepsilon \in (0,T)$ and $\Omega' \subset\subset \Omega$.

Moreover, if we set $g = \boldsymbol{g} \cdot \boldsymbol{\nu} - g^0$ and assume that for any point $(0, x_0) \in \{0\} \times \partial\Omega$, we have the compatibility condition

$$g(0, x_0) = 0, \tag{4}$$

and there exists a Dini function $\varrho_g : (0,1] \to \mathbb{R}$ such that for any $(t,x) \in \Omega_T$ and $|t, x - x_0| \le 1$,

$$|g(t,x)| = |g(t,x) - g(0,x_0)| \le \varrho_g(|t, x - x_0|), \tag{5}$$

then $u \in \mathring{C}^{1/2,1}(\overline{\Omega_T})$.

Remark 1. For general initial data $u(0,\cdot) = u_0 \in C^{1,\mathrm{Dini}}(\Omega)$ satisfying the compatibility condition

$$\boldsymbol{A}(0,\cdot)\nabla u_0 \cdot \boldsymbol{\nu} + \boldsymbol{a}(0,\cdot)u_0 \cdot \boldsymbol{\nu} + a^0(0,\cdot)u_0 = -\boldsymbol{g}(0,\cdot) \cdot \boldsymbol{\nu} + g^0(0,\cdot) \quad \text{on } \partial\Omega,$$

by considering $u - u_0$, we can show that $u \in \mathring{C}^{1/2,1}(\overline{\Omega_T})$ under the additional assumptions that $\mathbf{A}$ and $\boldsymbol{a}$ are uniformly Dini continuous near the corner $\{0\} \times \partial\Omega$.

Remark 2. For divergence form parabolic equations with the homogeneous Dirichlet boundary condition, it was proved in [8] that if the leading coefficients and data are DMO with respect to x only, then any weak solution is in $\mathring{C}^{1/2,1}$ up to the boundary. For the conormal problem, in Theorem 1.1 we require more regularity assumptions on the coefficients and data, which is necessary. In fact, consider the equation

$$u_t - D_1(a_1(t)D_1 u) - D_2(a_2(t)D_2 u) = \operatorname{div} \boldsymbol{g} \quad \text{in } (0,T) \times \mathbb{R}^2_+$$

with the conormal derivative boundary condition, where a_1 and a_2 are bounded from below and above, and $\mathbb{R}^2_+ = \{(x^1, x^2) \in \mathbb{R}^2 : x^2 > 0\}$. It is easily seen that the conormal derivative boundary condition in this case is equivalent to $D_2 u = -g_2/a_2$ on $\{x_2 = 0\}$. Therefore, $D_2 u$ may not be continuous up to the boundary unless g_2/a_2 is continuous. We also note that similar to [12, Theorem 2.4], from the proof below we can see that regularity assumptions on the coefficients and data with respect to the time variable are only required near the boundary.

Elliptic and parabolic equations with Dini mean oscillation coefficients considered in this paper were recently studied in [11, 7, 6, 8]. Using Hölder's inequality, it is easily seen that the Dini mean oscillation condition (or L_1-Dini mean oscillation condition) considered here can be derived from the L_p-Dini mean oscillation condition for any $p > 1$, i.e., the function

$$\omega_{(p)}(r) := \sup_{Q_r^-(X) \subset \overline{\Omega}_T} \left(\fint_{Q_r^-(X)} |g(Y) - g_{X,r}|^p \right)^{\frac{1}{p}}, \quad \left(g_{X,r} := \fint_{Q_r^-(X)} g \right)$$

is a Dini function, which has been used in [17, 15, 4] for the case when $p = 2$. However, whether the converse holds is still unclear.

If we instead consider the function

$$\widehat{\omega}_{(p)}(r) := \sup_{0<s\le r} \omega_{(p)}(s),$$

the proof of [1, Proposition 1.13] shows that $\widehat{\omega}_{(1)}$ is a Dini function implies that $\widehat{\omega}_{(p)}$ are also Dini functions for all $p \in [1,\infty)$, which indicates that the conditions are equivalent for any $p \in [1,\infty)$. Also it is clear that $\widehat{\omega}_{(p)}$ is a Dini function implies that $\omega_{(p)}$ is also a Dini function. However, the converse is not always true as shown in [5, Remark 2.2].

Besides, these Dini mean oscillation conditions are in fact strictly weaker than the uniform Dini continuity condition. See an example in [11, p. 418]. Elliptic and parabolic equations with uniformly Dini continuous coefficients have been well studied. See, for instance, [20, 18, 21, 2] and references therein. The DMO condition considered here is one of the weakest conditions to guarantee the continuous differentiability of solutions. As in [11, 7, 6, 8], the proof of Theorem 1.1 is based on Campanato′s approach, which was used previously, for instance, in [14, 18]. The main step of Campanato′s approach is to show the mean oscillation of Du in balls (or cylinders) vanishes to a certain order as the radii of the balls (or cylinders) go to zero. Here because we only impose the assumption on the L_1 -mean oscillation of the coefficients and data with respect to (t,x), the usual argument based on the L_2 (or L_p for $p>1$) estimates does not work in our case. To this end, we exploit weak type$-(1,1)$ estimates, the proof of which involves a duality argument, as well as the Sobolev estimates for parabolic equations with constant coefficients. See Lemma 2.7 in Section 2. We then adapt Campanato′s idea in the L_p setting for some $p \in (0,1)$. Compared to the Dirichlet case considered in [8], under the conormal boundary condition the proof of the $\mathring{C}^{1/2,1}$ regularity of the solution near the corner $\{t=0\}\times\partial\Omega$ is more intricate. We use a suitable extension argument which relies on the fact that g defined in Theorem 1.1 satisfies the compatibility condition (4) and the uniformly Dini continuous condition (5) at the corner. As the compatibility condition (4) is a pointwise condition, the DMO condition on g is not sufficient for the continuity of Du near the corner $\{0\}\times\partial\Omega$. Therefore, it seems that (5) is also necessary.

The paper is organized as follows. In Section 2, we introduce some notation, and function spaces, and provide some preliminary lemmas. Section 3 is devoted to interior and boundary $\mathring{C}^{1/2,1}$ estimates for the solutions. Throughout the rest of paper, the usual summation convection over repeated indices is assumed. For nonnegative (variable) quantities A and B, we denote $A \lesssim B$ if there exists a generic positive constant C such that $A \le CB$. We may add subscript letters like $A \lesssim_{a,b} B$ to indicate the dependence of the implicit constant C on the parameters a and b.

2. **Notation and preliminary.**

2.1. **Basic notation.**

We define the standard parabolic cylinder as

$$C_r^-(X) = C_r^-(t,x) = (t-r^2,t)\times B_r(x),$$

where $B_r(x)$ denotes the standard Euclidean ball in $\mathbb{R}^n$. We use $B_r^+(x)$ to denote $B_r(x)\cap\{x^n>0\}$. We also define the double centered cylinder as

$$C_r(X) = \big(t-r^2,t+r^2\big)\times B_r(x).$$

Moreover, C_r^-, Q_r^-, B_r, and B_r^+ denote respectively $C_r^-(X)$, $Q_r^-(X)$, $B_r(x)$, and $B_r^+(x)$ when their center $X=(0,\mathbf{0})$ and $x=\mathbf{0}$.

2.2. **Function spaces.** For any domain $\mathcal{D}\subset\mathbb{R}^{n+1}$ and $p\in(0,\infty]$, we shall denote by $L_p(\mathcal{D})$ the standard Lebesgue space, i.e., the set of all functions for which

$$\|f\|_{L_p(\mathcal{D})} := \left(\int_{\mathcal{D}} |f|^p\right)^{1/p} < \infty \text{ when } p<\infty,$$
$$\|f\|_{L_p(\mathcal{D})} := \operatorname*{ess\,sup}_{\mathcal{D}} |f| < \infty \text{ when } p=\infty.$$

We define the function spaces

$$W_p^{1,2}(\mathcal{D}) = \left\{u : u, \partial_t u, Du, D^2u \in L_p(\mathcal{D})\right\},$$
$$\mathbb{H}_p^{-1}(\mathcal{D}) = \{u : u = \operatorname{div} \boldsymbol{f} + g, \text{ for some } \boldsymbol{f}, g \in L_p(\mathcal{D})\},$$
$$\mathcal{H}_p^1(\mathcal{D}) = \left\{u : u, Du \in L_p(\mathcal{D}), \partial_t u \in \mathbb{H}_p^{-1}(\mathcal{D})\right\},$$

which are equipped with the norms

$$\|u\|_{W_p^{1,2}(\mathcal{D})} := \|u\|_{L_p(\mathcal{D})} + \|Du\|_{L_p(\mathcal{D})} + \|D^2u\|_{L_p(\mathcal{D})} + \|\partial_t u\|_{L_p(\mathcal{D})},$$
$$\|u\|_{\mathbb{H}_p^{-1}(\mathcal{D})} := \inf\left\{\|\boldsymbol{f}\|_{L_p(\mathcal{D})} + \|g\|_{L_p(\mathcal{D})} : u = \operatorname{div}\boldsymbol{f} + g,\ \boldsymbol{f},\ g \in L_p(\mathcal{D})\right\},$$
$$\|u\|_{\mathcal{H}_p^1(\mathcal{D})} := \|u\|_{L_p(\mathcal{D})} + \|Du\|_{L_p(\mathcal{D})} + \|\partial_t u\|_{\mathbb{H}_p^{-1}(\mathcal{D})},$$

respectively. We denote by $C^{0,0}(\overline{\mathcal{D}})$ the space of all continuous functions over $\overline{\mathcal{D}}$. For a constant $\delta\in(0,1]$, we denote

$$[u]_{C^{\delta/2,\delta}(\mathcal{D})} = \sup_{\substack{X,Y\in\mathcal{D}\\ X\neq Y}} \frac{|u(X)-u(Y)|}{|X-Y|^\delta}, \quad \|u\|_{C^{\delta/2,\delta}(\mathcal{D})} = [u]_{C^{\delta/2,\delta}(\mathcal{D})} + \sup_{\mathcal{D}} |u|.$$

By $C^{\delta/2,\delta}(\mathcal{D})$ we denote the space of all functions u for which $\|u\|_{C^{\delta/2,\delta}(\mathcal{D})}<\infty$. Let $\mathring{C}^{1/2,1}(\overline{\mathcal{D}})$ be the set of all functions $u\in C^{1/2,1}(\overline{\mathcal{D}})$ for which $Du\in C^{0,0}(\overline{\mathcal{D}})$ and

$$\frac{|u(t,x)-u(s,x)|}{|t-s|^{1/2}} \to 0 \quad \text{ as } |t-s|\to 0 \text{ for } (t,x),(s,x)\in\overline{\mathcal{D}}.$$

2.3. **Preliminary lemmas.** First, we state Sobolev-Morrey embedding theorems in the parabolic setting. The first one is a special case of [19, Lemma 3.3, §II. p. 80]. For the second lemma, we refer the reader to [16, Lemma 8.1].

Lemma 2.1. *Let $\Omega\subset\mathbb{R}^n$ be a bounded Lipschitz domain. Assume $u\in W_q^{1,2}(\Omega_T)$.*
(i) If $1\le q<(n+2)/2$, then $u\in L_p(\Omega_T)$, where $1/p=1/q-2/(n+2)$, and we have

$$\|u\|_{L_p(\Omega_T)} \le C\|u\|_{W_q^{1,2}(\Omega_T)}.$$

(ii) If $q>(n+2)/2$, then $u\in C^{\alpha/2,\alpha}(\Omega_T)$ and

$$\|u\|_{C^{\alpha/2,\alpha}(\Omega_T)} \le C\|u\|_{W_q^{1,2}(\Omega_T)},$$

where

$$\alpha = \begin{cases} 2-\frac{n+2}{q} & \text{if } q<n+2,\\ 1-\varepsilon \quad \forall\, \varepsilon\in(0,1), & \text{if } q\ge n+2.\end{cases}$$

(iii) If $1\le q<n+2$, then $Du\in L_p(\Omega_T)$, where $1/p=1/q-1/(n+2)$, and we have

$$\|Du\|_{L_p(\Omega_T)} \le C\|u\|_{W_q^{1,2}(\Omega_T)}.$$

(iv) If $q > n+2$, then $Du \in C^{\alpha/2,\alpha}(\Omega_T)$, where $\alpha = 1-(n+2)/q$, and we have

$$\|Du\|_{C^{\alpha/2,\alpha}(\Omega_T)} \le C\|u\|_{W^{1,2}_q(\Omega_T)}.$$

Here, C is a constant depending only on n, q, Ω, and T.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Assume $u \in \mathcal{H}^1_q(\Omega_T)$.*
(i) If $1 \le q < n+2$, then $u \in L_p(\Omega_T)$, where $1/p = 1/q - 1/(n+2)$, and we have

$$\|u\|_{L_p(\Omega_T)} \le C\|u\|_{\mathcal{H}^1_q(\Omega_T)}.$$

(ii) If $q > n+2$, then $u \in C^{\alpha/2,\alpha}(\Omega_T)$, where $\alpha = 1-(n+2)/q$, and we have

$$\|u\|_{C^{\alpha/2,a}(\Omega_T)} \le C\|u\|_{\mathcal{H}^1_q(\Omega_T)}.$$

Here, C is a constant depending only on n, q, Ω, and T.

Next we introduce a trace theorem in the cylindrical domain Ω_T, which is a natural extension of that in a space domain.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and denote $(0,T)\times\partial\Omega$ by $\partial\Omega_T$. Assume $u \in \mathcal{H}^1_q(\Omega_T)$ for $1 < q < +\infty$. Then there exists a bounded linear operator*

$$\mathcal{L} : \mathcal{H}^1_q(\Omega_T) \to L_p(\partial\Omega_T)$$

such that for each $u \in \mathcal{H}^1_q(\Omega_T)$,
(i) If $1 < q < n+2$,

$$\|\mathcal{L}u\|_{L_p(\partial\Omega_T)} \le C\|u\|_{\mathcal{H}^1_q(\Omega_T)}, \quad \forall\, p \in \left[q, \frac{(n+1)q}{n+2-q}\right];$$

(ii) if $q \ge n+2$,

$$\|\mathcal{L}u\|_{L_p(\partial\Omega_T)} \le C\|u\|_{\mathcal{H}^1_q(\Omega_T)}, \quad \forall\, p \in [q, +\infty)$$

with the constant C depending on q, p, n, and Ω_T.

Proof. The proof is a modification of the classical trace theorem in a space domain. See for example, [13, p 274, Theorem 1]. We only consider Case (i) for $1 < q < n+2$, as Case (ii) follows from Case (i) and Hölder's inequality. We also only need to consider the endpoint situation $p = (n+1)q/(n+2-q)$, since the other situations can be obtained by simply using Hölder's inequality.

1. Assume first $u \in C^1(\overline{\Omega_T})$. Let us first suppose that $x_0 \in \partial\Omega$ and $\partial\Omega$ is flat near x_0, lying in the plane $\{x^n = 0\}$. Assume that

$$\partial\Omega \cap B_{1/2}(x_0) = \{x^n = 0\} \cap B_{1/2}(x_0)$$

Take a cutoff function $\zeta \in C^\infty_c(B_1(x_0))$ with $\zeta \ge 0$ in $B_1(x_0)$ and $\zeta \equiv 1$ in $B_{1/2}(x_0)$. Denote by Γ the portion of $\partial\Omega$ within $B_{1/2}(x_0)$.

Set $x' = (x^1, \dots, x^{n-1}) \in \mathbb{R}^{n-1} = \{x^n = 0\}$. Then with $p = (n+1)q/(n+2-q)$,

$$
\begin{aligned}
&\int_0^T \int_\Gamma |u|^p \, dx' dt \le \int_0^T \int_{\{x_n=0\}} \zeta |u|^p \, dx' dt \\
&= -\int_0^T \int_{B_1^+(x_0)} (\zeta |u|^p)_{x^n} \; dxdt \\
&= -\int_0^T \int_{B_1^+(x_0)} |u|^p \zeta_{x^n} + p|u|^{p-1} (\operatorname{sgn} u) u_{x^n} \zeta \, dxdt \\
&\le C \left(\int_0^T \int_{B_1^+(x_0)} |u|^{\frac{(n+2)q}{n+2-q}} \, dxdt \right)^{\frac{n+1}{n+2}} \\
&\quad + C \left(\int_0^T \int_{B_1^+(x_0)} |Du|^q \, dxdt \right)^{1/q} \left(\int_0^T \int_{B_1^+(x_0)} |u|^{(p-1)\frac{q}{q-1}} \, dxdt \right)^{1-1/q} \\
&= C \left(\int_0^T \int_{B_1^+(x_0)} |u|^{\frac{(n+2)q}{n+2-q}} \, dxdt \right)^{\frac{n+1}{n+2}} \\
&\quad + C \left(\int_0^T \int_{B_1^+(x_0)} |Du|^q \, dxdt \right)^{1/q} \left(\int_0^T \int_{B_1^+(x_0)} |u|^{\frac{(n+2)q}{n+2-q}} \, dxdt \right)^{1-1/q} \\
&\le C \|u\|^p_{\mathcal{H}^1_q(\Omega_T)},
\end{aligned}
\tag{6}
$$

where in the fourth and fifth lines of the above inequality, we used Hölder's inequality and in the last line, we used Lemma 2.2.

2. If $x_0 \in \partial\Omega$, but $\partial\Omega$ is not flat near x_0, we as usual flatten the boundary near x_0 to obtain the setting above. We give the details below.

Without loss of generality, under a linear transform of the coordinates, we assume that near x_0, locally $\partial\Omega$ can be expressed by $x^n = \Phi(x')$, where $\Phi(x')$ is a Lipschitz function and $x_0^n = \Phi(x_0')$. We make the following bi-Lipschitz changes of variables

$$
y = (y', y^n) = \mathbf{\Phi}(x) := (x', x^n - \Phi(x'))
$$

and set $\tilde{u}(t, y) := u(t, x)$. It is easy to see that such change of variables maps the neighborhood of $\partial\Omega$ near x_0 to the neighborhood $\{y^n = 0\}$ near $y_0 = (x_0', 0)$ and $\det D\mathbf{\Phi} = 1$. Now set $\Gamma := \mathbf{\Phi}^{-1}(B_{1/2}(y_0) \cap \{y^n = 0\})$.

Applying the estimate (6) and changing variables, we have

$$
\begin{aligned}
&\int_0^T \int_\Gamma |u(t,x)|^p \, dS dt \\
&= \int_0^T \int_{B_{1/2}(y_0) \cap \{y^n = 0\}} |\tilde{u}(t, y', 0)|^p \sqrt{1 + |\Phi_{x'}|^2} \, dy' dt \\
&\le C \int_0^T \int_{B_{1/2}(y_0) \cap \{y^n = 0\}} |\tilde{u}(t, y', 0)|^p \, dy' dt,
\end{aligned}
$$

which is further bounded by

$$
\begin{aligned}
&\le C\left(\int_0^T\int_{B_1^+(y_0)}|\tilde{u}|^{\frac{(n+2)q}{n+2-q}}\,dydt\right)^{\frac{n+1}{n+2}}\\
&\quad+C\left(\int_0^T\int_{B_1^+(y_0)}|D\tilde{u}|^q\,dydt\right)^{1/q}\left(\int_0^T\int_{B_1^+(y_0)}|\tilde{u}|^{\frac{(n+2)q}{n+2-q}}\,dydt\right)^{1-1/q}\\
&\le C\left(\int_0^T\int_{\mathbf{\Phi}^{-1}(B_1^+(y_0))}|u|^{\frac{(n+2)q}{n+2-q}}\,dxdt\right)^{\frac{n+1}{n+2}}\\
&\quad+C\left(\int_0^T\int_{\mathbf{\Phi}^{-1}(B_1^+(y_0))}|Du|^q\,|D\mathbf{\Phi}^{-1}|^q dX\right)^{\frac{1}{q}}\left(\int_0^T\int_{\mathbf{\Phi}^{-1}(B_1^+(y_0))}|u|^{\frac{(n+2)q}{n+2-q}}\,dX\right)^{1-\frac{1}{q}}\\
&\le C\|u\|^p_{\mathcal{H}^1_q(\Omega_T)}.
\end{aligned}
$$

In the above computation, we used the boundedness of $|\Phi_{x'}|$ and $|D\mathbf{\Phi}^{-1}|$.

3. Since $\partial\Omega$ is compact, there exist finitely many points $x_i\in\partial\Omega$ and open subsets $\Gamma_i\subset\partial\Omega$ $(i=1,\ldots,N)$ such that $\partial\Omega=\bigcup_{i=1}^N\Gamma_i$ and

$$\|u\|_{L_p((0,T)\times\Gamma_i)}\le C\|u\|_{\mathcal{H}^1_q(\Omega_T)}\quad(i=1,\ldots,N).$$

Consequently, if we write

$$\mathcal{L}u:=u\,|_{\partial\Omega_T},$$

then

$$\|\mathcal{L}u\|_{L_p(\partial\Omega_T)}\le C\|u\|_{\mathcal{H}^1_q(\Omega_T)} \tag{7}$$

for some appropriate constant C, which does not depend on u.

4. Inequality (7) holds for $u\in C^1(\overline{\Omega_T})$. Assume now $u\in\mathcal{H}^1_q(\Omega_T)$. Then there exist functions $u_m\in C^\infty(\overline{\Omega_T})$ converging to u in $\mathcal{H}^1_q(\Omega_T)$. According to (7), we have

$$\|\mathcal{L}u_m-\mathcal{L}u_l\|_{L_p(\partial\Omega_T)}\le C\,\|u_m-u_l\|_{\mathcal{H}^1_q(\Omega_T)}, \tag{8}$$

so that $\{\mathcal{L}u_m\}_{m=1}^\infty$ is a Cauchy sequence in $L_p(\partial\Omega_T)$. We define

$$\mathcal{L}u:=\lim_{m\to\infty}\mathcal{L}u_m$$

with the limit taken in $L_p(\partial\Omega_T)$. According to (8), this definition does not depend on the particular choice of smooth functions approximating u. The lemma is proved. □

It should be clear that if g is uniformly Dini continuous, then it is of Dini mean oscillation and $\omega_g(r)\le\varrho_g(r)$. It is worthwhile to note that if Ω is such that for any $x\in\overline{\Omega}$,

$$|\Omega_r(x)|\ge A_0r^n,\quad\forall r\in(0,\operatorname{diam}\Omega]\quad(A_0\text{ is a positive constant}) \tag{9}$$

and if g is of Dini mean oscillation, then g is uniformly continuous with a modulus of continuity determined by ω_g.

Lemma 2.4. *Let Ω satisfy the condition (9). If f is uniformly Dini continuous and g is* DMO *in Ω_T, then fg is* DMO *in Ω_T.*

The proof of Lemma 2.4 is similar with that in [6, Lemma 2.6]. Here we omit the details.

Lemma 2.5 (Lemma 2.7 of [11]). *Let ω be a nonnegative bounded function. Suppose there exist $c_1, c_2 > 0$ and $0 < \kappa < 1$ such that*

$$c_1\omega(t) \le \omega(s) \le c_2\omega(t) \quad \text{whenever } \kappa t \le s \le t \text{ and } 0 < t < r.$$

Then, we have

$$\sum_{i=0}^{\infty} \omega\left(\kappa^i r\right) \lesssim \int_0^r \frac{\omega(t)}{t}\,dt.$$

Proof. The proof follows immediately from the comparison principle for Riemann integrals. □

Next we assume that $\mathcal{D} := (a,b) \times \mathcal{B}$ is a cylindrical domain in $\mathbb{R}^{n+1}$, where $\mathcal{B}$ is a bounded and smooth domain in $\mathbb{R}^n$.

Lemma 2.6. *Let $\mathcal{T}$ be a bounded linear operator from $L_2(\mathcal{D})$ to $L_2(\mathcal{D})$. Suppose that there are constants $c > 1$ and $C > 0$ such that for any $Y \in \mathcal{D}$ and $0 < r < 1/2$, we have*

$$\int_{\mathcal{D}\setminus C_{cr}(Y)} |\mathcal{T}b| \le C \int_{C_r^-(Y)\cap\mathcal{D}} |b|$$

whenever $b \in L_2(\mathcal{D})$ is supported in $C_r^-(Y) \cap \mathcal{D}$ and $\int_{\mathcal{D}} b = 0$. Then, for any $f \in L_2(\mathcal{D})$ and $\gamma > 0$, we have

$$|\{X \in \mathcal{D} : |\mathcal{T}f(X)| > \gamma\}| \le \frac{C'}{\gamma}\int_{\mathcal{D}} |f|,$$

where $C' = C'(n, c, C, \mathcal{B})$ is a constant.

Proof. We refer to Stein [22, p. 22], where the proof is based on the Calderón-Zygmund decomposition and the domain is assumed to be the whole space. In our case, we can modify the proof there by using the "dyadic cubes" decomposition of $\mathcal{D}$. See Christ [3, Theorem 11] and [7, Lemma 4.1]. □

Lemma 2.7. *Let $\bar{\mathbf{A}} = (\bar{a}^{ij})$ be a constant matrix satisfying (2). Consider the operator P_0 defined by*

$$P_0 u := \partial_t u - D_i(\bar{a}^{ij} D_j u).$$

For $\boldsymbol{f} = (f^1, \ldots, f^n) \in L_2(\mathcal{D})$, let $u \in \mathcal{H}_2^1(\mathcal{D})$ be the weak solution to

$$\begin{cases} P_0 u = \operatorname{div} \boldsymbol{f} & \text{in } \mathcal{D}, \\ -\bar{\mathbf{A}}\nabla u \cdot \boldsymbol{\nu} = \boldsymbol{f}\cdot\boldsymbol{\nu} & \text{on } (a,b)\times\partial\mathcal{B}, \\ u = 0 & \text{on } \{t=a\}\times\mathcal{B}, \end{cases}$$

where $\boldsymbol{\nu}$ is the unit outward normal vector on $\partial\mathcal{B}$. Then for any $\gamma > 0$, we have

$$|\{X \in \mathcal{D} : |Du(X)| > \gamma\}| \lesssim_{n,\lambda,\mathcal{D},\mathcal{B}} \frac{1}{\gamma}\int_{\mathcal{D}} |\boldsymbol{f}|.$$

Proof. The proof is a modification of [8, Lemma 3.6]. Since the map $\mathcal{T} : \boldsymbol{f} \mapsto Du$ is a bounded linear operator on $L_2(\mathcal{D})$, it suffices to show that $\mathcal{T}$ satisfies the hypothesis of Lemma 2.6. We set $c = 2$. Fixing $Y \in \mathcal{D}$ and $0 < r < 1/2$, let $\boldsymbol{b} \in L_2(\mathcal{D})$ be supported in $C_r^-(Y) \cap \mathcal{D}$ with mean zero. Let $u \in \mathcal{H}_2^1(\mathcal{D})$ be the unique weak solution of

$$\begin{cases} P_0 u = \operatorname{div} \boldsymbol{b} & \text{in } \mathcal{D}, \\ -\bar{\mathbf{A}}\nabla u \cdot \boldsymbol{\nu} = \boldsymbol{b}\cdot\boldsymbol{\nu} & \text{on } (a,b)\times\partial\mathcal{B}, \\ u = 0 & \text{on } \{t=a\}\times\mathcal{B}. \end{cases}$$

For any $R \geq 2r$ such that $\mathcal{D}\backslash C_R(Y) \neq \emptyset$ and $g \in C_c^\infty\left((C_{2R}(Y)\backslash C_R(Y)) \cap \mathcal{D}\right)$, let $v \in \mathcal{H}_2^1(\mathcal{D})$ be a weak solution of

$$\begin{cases} P_0^* v = \operatorname{div} \boldsymbol{g} & \text{in } \mathcal{D}, \\ -\bar{\mathbf{A}}^T \nabla v \cdot \boldsymbol{\nu} = \boldsymbol{g} \cdot \boldsymbol{\nu} & \text{on } (a,b) \times \partial\mathcal{B}, \\ v = 0 & \text{on } \{t = b\} \times \mathcal{B}, \end{cases}$$

where $P_0^* := -\partial_t + D_i(\bar{a}^{ji} D_j)$ is the adjoint operator of P_0. By the definition of weak solutions and the properties of $\boldsymbol{b}$, we have the identity

$$\int_{\mathcal{D}} Du \cdot \boldsymbol{g} = \int_{\mathcal{D}} \boldsymbol{b} \cdot Dv = \int_{C_r^-(Y)\cap\mathcal{D}} \boldsymbol{b} \cdot \left(Dv - \overline{Dv}_{C_r^-(Y)\cap\mathcal{D}} \right).$$

Therefore,

$$\begin{aligned} &\left| \int_{(C_{2R}(Y)\backslash C_R(Y))\cap\mathcal{D}} Du \cdot \boldsymbol{g} \right| \\ &\lesssim \|\boldsymbol{b}\|_{L_1(C_r^-(Y)\cap\mathcal{D})} \|Dv - \overline{(Dv)}_{C_r^-(Y)\cap\mathcal{D}}\|_{L_\infty(C_r^-(Y)\cap\mathcal{D})} \\ &\lesssim \|\boldsymbol{b}\|_{L_1(C_r^-(Y)\cap\mathcal{D})} r [Dv]_{C^{1/2,1}(C_r^-(Y)\cap\mathcal{D})}. \end{aligned} \tag{10}$$

Since $P_0^* v = 0$ in $C_R(Y)\cap\mathcal{D}$ and v satisfies the zero conormal boundary condition on $C_R(Y) \cap ((a,b) \times \partial\mathcal{B})$, we have

$$R[Dv]_{C^{1/2,1}(C_{R/2}^-(Y)\cap\mathcal{D})} \leq C \left(\fint_{C_R^-(Y)\cap\mathcal{D}} |Dv|^2 \right)^{1/2}. \tag{11}$$

See, for instance, [18, Theorem 1.2] or [12, Theorem 2.4]. Since $r \leq R/2$, inserting (11) into (10), we get

$$\begin{aligned} &\left| \int_{(C_{2R}(Y)\backslash C_R(Y))\cap\mathcal{D}} Du \cdot \boldsymbol{g} \right| \\ &\lesssim \|\boldsymbol{b}\|_{L_1(C_r^-(Y)\cap\mathcal{D})} \frac{r}{R} R^{-\frac{n+2}{2}} \|Dv\|_{L_2(C_R^-(Y)\cap\mathcal{D})} \\ &\lesssim \|\boldsymbol{b}\|_{L_1(C_r^-(Y)\cap\mathcal{D})} r R^{-1-\frac{n+2}{2}} \|Dv\|_{L_2(\mathcal{D})} \\ &\lesssim \|\boldsymbol{b}\|_{L_1(C_r^-(Y)\cap\mathcal{D})} r R^{-1-\frac{n+2}{2}} \|\boldsymbol{g}\|_{L_2(\mathcal{D})} \\ &= \|\boldsymbol{b}\|_{L_1(C_r^-(Y)\cap\mathcal{D})} r R^{-1-\frac{n+2}{2}} \|\boldsymbol{g}\|_{L_2((C_{2R}(Y)\backslash C_R(Y))\cap\mathcal{D})}. \end{aligned} \tag{12}$$

Therefore, by the duality, from (12) we get

$$\|Du\|_{L_2((C_{2R}(Y)\backslash C_R(Y))\cap\mathcal{D})} \leqslant r R^{-1-\frac{n+2}{2}} \|\boldsymbol{b}\|_{L_1\left(C_r^-(Y)\cap\mathcal{D}\right)},$$

and thus by Hölder's inequality, we have

$$\|Du\|_{L_1((C_{2R}(Y)\backslash C_R(Y))\cap\mathcal{D})} \lesssim r R^{-1} \|\boldsymbol{b}\|_{L_1\left(C_r^-(Y)\cap\mathcal{D}\right)}. \tag{13}$$

Now let N be the smallest positive integer such that $\mathcal{D} \subset C_{2^{N+1}r}(Y)$. By taking $R = 2r, 4r, \ldots, 2^N r$ in (13), we have

$$\int_{\mathcal{D}\backslash C_{2r}(Y)} |Du| \lesssim \sum_{k=1}^{N} 2^{-k} \|\boldsymbol{b}\|_{L_1(C_r^-(Y)\cap\mathcal{D})} \sim \int_{C_r^-(Y)\cap\mathcal{D}} |\boldsymbol{b}|.$$

Therefore, $\mathcal{T}$ satisfies the hypothesis of Lemma 2.6 and the lemma is proved. $\square$

3. **Proof of Theorem 1.1.** In this section, we first reduce the problem to an equation without lower-order terms and with the standard conormal boundary condition, and derive interior estimates by using the $\mathcal{H}_p^1$ and $W_p^{1,2}$ estimates for divergence form parabolic equations. We then prove the corresponding boundary estimates near the lateral boundary and the bottom boundary of the domain by a modified Campanato argument.

3.1. **Interior estimates.**

Proposition 1. *For any open set $\Omega' \subset\subset \Omega$, we have $u \in \mathring{C}^{1/2,1}\left(\overline{\Omega'_T}\right)$.*

Proof. Since a^{ij} are in VMO in $\overline{\Omega_T}$, by moving the lower-order terms to the right-hand side of the equation, we can show that $u, Du \in L_p(\Omega_T)$ for any $1 < p < \infty$. Indeed, let v solve

$$\begin{cases} v_t - \Delta v = f - b^i D_i u - cu & \text{in } \Omega_T, \\ \nabla v \cdot \boldsymbol{\nu} = 0 & \text{on } (0,T) \times \partial\Omega, \\ v = 0 & \text{on } \{t=0\} \times \Omega. \end{cases} \tag{14}$$

Then $w := u - v$ satisfies

$$\begin{cases} w_t - D_i(a^{ij} D_j w) = D_i h^i & \text{in } \Omega_T, \\ a^{ij} D_j w \nu^i + a^0 u = -h^i \nu^i + g^0 & \text{on } (0,T) \times \partial\Omega, \\ w = 0 & \text{on } \{t=0\} \times \Omega, \end{cases} \tag{15}$$

where

$$h^i := g^i + a^i u + \left(a^{ij} - \delta^{ij}\right) D_j v.$$

As the boundary condition in $(15)_2$ is not the standard conormal boundary condition, we argue as follows.

Since $\partial\Omega \in C^1$, we locally flatten the boundary so that $\boldsymbol{\nu} = -\boldsymbol{e}_n$ and the boundary condition becomes (by an abuse of notation)

$$-\sum_{j=1}^n a^{nj} D_j w = h^n + g^0 - a^0 u \quad \text{on } \Gamma \subset (0,T) \times \{x_n = 0\}.$$

Note that if we set

$$\tilde{h}^n(t,x) = \tilde{h}^n(t,x',x_n) := h^n(t,x',x_n) + g^0(t,x',0) - a^0(t,x',0)u(t,x',0),$$

then we have $D_n \tilde{h}^n = D_n h^n$. Therefore by replacing h^n with $\tilde{h}^n$, the above system (15) becomes (after the transformation)

$$\begin{cases} w_t - D_i(a^{ij} D_j w) = D_i \tilde{h}^i & \text{in } (0,T) \times B_1^+, \\ -\displaystyle\sum_{j=1}^n a^{nj} D_j w = \tilde{h}^n & \text{on } (0,T) \times (\partial B_1^+ \cap \{x^n = 0\}), \\ w = 0 & \text{on } \{t=0\} \times B_1^+, \end{cases} \tag{16}$$

where $\tilde{h}^i = h^i$ for $i = 1, 2, \ldots, n-1$. This reduces the boundary condition to the standard conormal boundary condition. Now we can use the interior and boundary $\mathcal{H}_p^1$ theory for divergence form parabolic equations and bootstrap argument to conclude that $u \in \mathcal{H}_p^1(\Omega_T)$ for any $p \in (1, \infty)$. The details are given as follows.

Below, the default integration domain is always Ω_T if it is not explicitly stated.

Note that $f - b^i D_i u - cu$ in (14) belongs to $L_{q_1} = L_{q_1}(\Omega_T)$, where $1/q_1 = 1/q + 1/2$. By the parabolic L_p estimate, Hölder's inequality, and the parabolic Sobolev embedding (Lemma 2.1), we have

$$\|v\|_{L_{p_1}} + \|Dv\|_{L_{p_1}} \lesssim \|v\|_{W^{1,2}_{q_1}} \lesssim \|f\|_{L_q} + \|\boldsymbol{b}\|_{L_q}\|Du\|_{L_2} + \|c\|_{L_q}\|u\|_{L_2}, \tag{17}$$

where

$$\frac{1}{p_1} := \frac{1}{q_1} - \frac{1}{n+2} = \frac{1}{q} + \frac{1}{2} - \frac{1}{n+2}.$$

Now we estimates $\tilde{\boldsymbol{h}} = (\tilde{h}^1, \ldots, \tilde{h}^n)$. Since there is a boundary term $a^0 u|_{\partial\Omega_T}$ in $\tilde{h}^n$, we need to use the trace theorem in Lemma 2.3 to bound it. Initially we have $u \in \mathcal{H}^1_2(\Omega_T)$. By choosing $p = 2(n+1)/n$ in Lemma 2.3, we have

$$\|u\|_{L_{\frac{2(n+1)}{n}}(\partial\Omega_T)} \le C\|u\|_{\mathcal{H}^1_2(\Omega_T)}. \tag{18}$$

Therefore, by choosing $\tilde{p}_1 = \min\left\{p_1, 2(n+1)/n\right\} > 2$, from (17), (18), and Hölder's inequality, we see that $\tilde{\boldsymbol{h}} = (\tilde{h}^1, \ldots, \tilde{h}^n)$ satisfies

$$\begin{aligned}\|\tilde{\boldsymbol{h}}\|_{L_{\tilde{p}_1}} \lesssim& \|\boldsymbol{g}, g^0\|_{L_\infty} + \|u\|_{L_{\tilde{p}_1}} + \|u\|_{L_{\tilde{p}_1}(\partial\Omega_T)} + \|Dv\|_{L_{\tilde{p}_1}} \\ \lesssim& \|\boldsymbol{g}, g^0\|_{L_\infty} + \|u\|_{L_{\frac{2(n+2)}{n}}} + \|u\|_{L_{\frac{2(n+1)}{n}}(\partial\Omega_T)} + \|Dv\|_{L_{p_1}} \\ \lesssim& \|\boldsymbol{g}, g^0\|_{L_\infty} + \|u\|_{\mathcal{H}^1_2} + \|f\|_{L_q}.\end{aligned}$$

Then we apply the parabolic L_p estimate (see, for instance, [12]) to w and get

$$\|w\|_{\mathcal{H}^1_{\tilde{p}_1}} \le C\left(\|f\|_{L_q} + \|\boldsymbol{g}, g^0\|_{L_\infty} + \|u\|_{\mathcal{H}^1_2}\right),$$

where C is a constant depending only on n, λ, q, Ω, $\partial\Omega$, $\|\mathbf{A}, \boldsymbol{a}, a^0\|_{L_\infty}$, and $\|\boldsymbol{b}, c\|_{L_q}$. Therefore, we have $u, Du \in L_{\tilde{p}_1}$ and

$$\|u\|_{L_{\tilde{p}_1}} + \|Du\|_{L_{\tilde{p}_1}} \lesssim \|f\|_{L_q} + \|\boldsymbol{g}, g^0\|_{L_\infty} + \|u\|_{\mathcal{H}^1_2}.$$

Feeding it back to the equations (14) and (16) (i.e., bootstrapping), we eventually get $u, Du \in L_p$ for any $1 < p < \infty$, and

$$\|u\|_{L_p} + \|Du\|_{L_p} \le C\left(\|f\|_{L_q} + \|\boldsymbol{g}, g^0\|_{L_\infty} + \|u\|_{\mathcal{H}^1_2}\right)$$

as claimed, with C also depending on p.

It then follows from the equation of u that $u \in \mathcal{H}^1_p$ for any $p \in (n+2, q)$ and thus by the Sobolev embedding (Lemma 2.2), we particularly have $u \in C^{\alpha/2,\alpha}(\Omega_T)$ for $\alpha = 1 - (n+2)/p \in (0,1)$. Recall that v solves (14) with $f - b^i D_i u - cu \in L_p(\Omega_T)$ for $p \in (n+2, q)$. By the parabolic L_p theory for non-divergence form parabolic equations and the Sobolev embedding (Lemma 2.1), we find $Dv \in C^{\delta/2,\delta}(\Omega_T)$ with $\delta = 1 - (n+2)/p$. Therefore, by Lemma 2.4, we see that $\tilde{\boldsymbol{h}} \in$ DMO in Ω_T.

In summary, $w = u - v$ is a weak solution of (15), where $\boldsymbol{h} \in$ DMO, and $\omega_{\boldsymbol{h}}$ is completely determined by the given data (namely, n, λ, Ω, T, $\omega_{\mathbf{A}}$, q, $\|f, \boldsymbol{b}, c\|_{L_q}$, $\omega_{\boldsymbol{a}}$, $\omega_{\boldsymbol{g}}$ and $\|\boldsymbol{g}, g^0, a^0\|_{L_\infty}$. By the interior estimates in [8, Theorem 3.1], we find that $w \in \mathring{C}^{1/2,1}(\overline{\Omega'_T})$ and $\|w\|_{\mathring{C}^{1/2,1}(\overline{\Omega'_T})}$ is bounded by a constant C depending only on the above mentioned given data, and Ω'. Since $v \in C^{(1+\delta)/2,1+\delta}(\overline{\Omega'_T})$, we see that $u \in \mathring{C}^{1/2,1}(\overline{\Omega'_T})$. □

Remark 3. In the proof above, to apply the $\mathcal{H}^1_p$ and $W^{1,2}_p$ estimates for parabolic equations, instead of the DMO condition, we only require a^{ij} to be VMO with respect to the x variable. See, for instance, [16]. For the interior estimate in [8, Theorem 3.1], the leading coefficients and data only need to be in DMO_x.

3.2. **Boundary estimates.** Next, we turn to $\mathring{C}^{1/2,1}$ estimate near the lateral boundary and the bottom boundary. Under a volume preserving mapping of locally flattening boundary

$$y = \mathbf{\Phi}(x) = \left(\Phi^1(x), \ldots, \Phi^n(x)\right), \quad (\det D\mathbf{\Phi} = 1),$$

let $\tilde{u}(t,y) = u(t,x)$, which satisfies

$$\tilde{u}_t - D_i\left(\tilde{a}^{ij}D_j\tilde{u} + \tilde{a}^i\tilde{u}\right) + \tilde{b}^iD_i\tilde{u} + \tilde{c}\tilde{u} = \operatorname{div}\tilde{\boldsymbol{g}} + \tilde{f}$$

with

$$\begin{aligned} \tilde{a}^{ij}(t,y) &= D_l\Phi^i(x)D_k\Phi^j(x)a^{kl}(t,x), \qquad & \tilde{a}^i(t,y) &= D_l\Phi^i(x)a^l(t,x),\\ \tilde{b}^i(t,y) &= D_l\Phi^i(x)b^l(t,x), & \tilde{c}(t,y) &= c(t,x),\\ \tilde{g}^i(t,y) &= D_l\Phi^i(x)g^l(t,x), & \tilde{f}(t,y) &= f(t,x). \end{aligned}$$

Without loss of generality, we assume that the above equation is satisfied in $(-16,0)\times\mathcal{B}$, with a smooth domain $\mathcal{B}\subset\mathbb{R}^n$ satisfying $B_4^+(0)\subset\mathcal{B}\subset B_5^+(0)$. By Lemma 2.4, we see that the coefficients and data satisfy the same conditions in $(-16,0)\times\mathcal{B}$. Now write $u=v+w$, where v and w are as in the proof of Proposition 1 with Ω_T replaced by $(-16,0)\times\mathcal{B}$. By the global parabolic L_p estimate and Sobolev embedding $v\in C^{(1+\delta)/2,1+\delta}((-16,0)\times\mathcal{B})$, it is sufficient to show that w is $\mathring{C}^{1/2,1}$ near the flat lateral boundary and bottom boundary. Since a^0 and g^0 are uniformly Dini continuous, it is easily seen that (4) and (5) are satisfied with $\tilde{h}^n$ in place of g, where $\tilde{h}^n$ is defined in Section 1. Thus we are reduced to prove the following proposition. Hereafter, for any $\overline{X}\in\partial\mathbb{R}^{n+1}_+=\{x^n=0\}$, we write

$$Q_r^-(\overline{X}) = C_r^-\cap\{x^n>0\}+\overline{X}, \quad \Delta_r^-(\overline{X}) = C_r^-\cap\{x^n=0\}+\overline{X}.$$

Proposition 2. *Assume that* $\mathbf{A}:=(a^{ij})\in\mathrm{DMO}\left(Q_4^-\right)$ *and* $\boldsymbol{g}\in\mathrm{DMO}\left(Q_4^-\right)$. *Let* $u\in\mathcal{H}_2^1\left(Q_4^-\right)$ *be a weak solution of*

$$\begin{cases} u_t - D_i(a^{ij}D_ju) = \operatorname{div}\boldsymbol{g} & \text{in } Q_4^-,\\ -\boldsymbol{A}\nabla u\cdot\boldsymbol{e}_n = \boldsymbol{g}\cdot\boldsymbol{e}_n & \text{on } \Delta_4^-,\\ u=0 & \text{on } \{t=-16\}\times B_4^+. \end{cases} \tag{19}$$

Then $u\in\mathring{C}^{1/2,1}\left(\overline{Q_1^-}\right)$.

Moreover, if we assume that g^n *satisfies the compatibility condition*

$$g^n(-16,\bar{x}_0)=0, \quad \forall\bar{x}_0\in B_4\cap\{x_n=0\}, \tag{20}$$

and there exists a Dini function $\varrho_{g^n}:(0,1]\to\mathbb{R}$ *such that* $\forall\ (t,x)\in Q_4^-\cap C_1(-16,\bar{x}_0)$,

$$|g^n(t,x)| = |g^n(t,x)-g^n(-16,\bar{x}_0)| \le \varrho_{g^n}(|t+16,x-\bar{x}_0|), \tag{21}$$

then $u\in\mathring{C}^{1/2,1}(\overline{(-16,0)\times B_1^+})$.

The rest of this subsection is devoted to the proof of Proposition 2. We shall derive an a priori estimate of the modulus of continuity of Du by assuming that u is in $C^{1/2,1}(\overline{(-16,0)\times B_3^+})$. The general case follows from a standard approximation argument.

We present a series of lemmas that will provide key estimates for the proof of Proposition 2.

Lemma 3.1. *Let $\bar{\mathbf{A}} := (\bar{a}^{ij})$ be a constant matrix satisfying (2). Consider the operator P_0 defined by*

$$P_0 u := \partial_t u - D_i(\bar{a}^{ij} D_j u).$$

Let $u \in \mathcal{H}^1_2$ be a weak solution of

$$\begin{cases} P_0 u = 0 & \text{in } Q_r^-, \\ \bar{\mathbf{A}}\nabla u \cdot \boldsymbol{e}_n = \boldsymbol{c}\cdot \boldsymbol{e}_n & \text{on } \Delta_r^-, \end{cases} \tag{22}$$

where $\boldsymbol{c} \in \mathbb{R}^n$ is a constant vector. Then, for any constant vector $\boldsymbol{q} \in \mathbb{R}^n$, we have

$$r[Du]_{C^{1/2,1}(Q^-_{r/2})} \le C \left(\fint_{Q_r^-} |Du - \boldsymbol{q}|^{1/2} \right)^2, \tag{23}$$

where C is a constant depending only on n and λ.

Proof. We set $r = 1$. The general case follows from the scaling. First we make a linear transformation in the space-variable to reduce the conormal boundary condition in (22) to the Neumann boundary condition. Let $\mathbf{O}y = x$, where $\mathbf{O}$ is the following invertible matrix

$$\mathbf{O} = \left[\begin{array}{ccc|c} & & & a^{n1} \\ & \mathbf{E}_{n-1} & & \vdots \\ & & & a^{n,n-1} \\ \hline 0 & \cdots & 0 & a^{nn} \end{array}\right],$$

where $\mathbf{E}_{n-1}$ is the $n-1$ dimensional unit matrix. Note that this linear transformation maps $\{x \mid x^n = 0\}$ to $\{y \mid y^n = 0\}$. The inverse matrix $\mathbf{O}^{-1}$ of $\mathbf{O}$ is given by

$$\mathbf{O}^{-1} = \left[\begin{array}{ccc|c} & & & -a^{n1}/a^{nn} \\ & \mathbf{E}_{n-1} & & \vdots \\ & & & -a^{n,n-1}/a^{nn} \\ \hline 0 & \cdots & 0 & 1/a^{nn} \end{array}\right].$$

Define $\tilde{u}(y) := u(\mathbf{O}y) = u(x)$. By a direct calculation, we see that $\tilde{u}(y)$ satisfies

$$\begin{cases} \partial_t \tilde{u} - D_i(\tilde{a}^{ij} D_j \tilde{u}) = 0 & \text{in } \tilde{Q}_1^-, \\ D_n \tilde{u} = \boldsymbol{c}\cdot \boldsymbol{e}_n & \text{on } \Delta_1^-, \end{cases}$$

where $(\tilde{a}^{ij}) = \mathbf{O}^{-1}\bar{\mathbf{A}}\mathbf{O}^{-T}$, and $\tilde{Q}_1^- = \mathbf{O}^{-1}(Q_1^-)$. By using a covering argument, for simplicity, we may assume in the proof below that $\tilde{Q}_1^- = Q_1^-$.

First using the result in [8, Lemma 4.15] (reverse the time there), for any $q^n \in \mathbb{R}$, we have

$$[D_n \tilde{u}]_{C^{1/2,1}(Q^-_{1/2})} \le C \left(\fint_{Q_1^-} |D_n \tilde{u} - q^n|^{1/2} \right)^2. \tag{24}$$

Then for any $q^k \in \mathbb{R}$ and $k = 1, 2, \ldots, n-1$, $v^k := D_k \tilde{u} - q^k$ satisfies

$$\begin{cases} \partial_t v^k - D_i(\tilde{a}^{ij} D_j v^k) = 0 & \text{in } Q_1^-, \\ D_n v^k = 0 & \text{on } \Delta_1^-. \end{cases}$$

By using the $W^{1,2}_p$ estimate for parabolic equations with constant coefficients and the zero Neumann boundary condition, and the Sobolev embedding in Lemma 2.1, we have, for $\gamma \in (0,1/2]$, and $\theta \in [1/2, 1-\gamma]$,

$$\|v^k\|_{L_\infty(Q^-_\theta)} + \gamma [v^k]_{C^{1/2,1}(Q^-_\theta)} \le C\gamma^{-\frac{n+2}{2}} \|v^k\|_{L_2(Q^-_{\theta+\gamma})}, \tag{25}$$

where C is independent of θ and γ. Also it is easily seen that

$$\|v^k\|_{L_2(Q^-_{\theta+\gamma})} \le \|v^k\|^{3/4}_{L_\infty(Q^-_{\theta+\gamma})} \|v^k\|^{1/4}_{L_{1/2}(Q^-_{\theta+\gamma})}. \tag{26}$$

We define

$$M_\theta := \|v^k\|_{L_\infty(Q^-_\theta)} + [v^k]_{C^{1/2,1}(Q^-_\theta)}.$$

Inserting (26) into (25), we get

$$M_\theta \le C\gamma^{-1-\frac{n+2}{2}} M^{3/4}_{\theta+\gamma} \|v^k\|^{1/4}_{L_{1/2}(Q^-_1)}.$$

Now we choose $\theta_0 = 1/2$ and $\theta_i = \theta_{i-1} + 1/2^{i+2}$, $\gamma = 1/2^{i+3}$ for $i \ge 1$. Then the above inequality indicates that

$$M_{\theta_i} \le C 2^{(1+\frac{n+2}{2})(i+2)} M^{3/4}_{\theta_{i+1}} \|v^k\|^{1/4}_{L_{1/2}(Q^-_1)} \quad \text{for } i \ge 0.$$

By iteration on i, we have

$$M_{\theta_0} \le \Big(C\|v^k\|^{1/4}_{L_{1/2}(Q^-_1)} \Big)^{\sum_{k=0}^{i}(3/4)^k} 2^{(1+\frac{n+2}{2})\sum_{k=0}^{i}(k+2)(3/4)^k} M^{(3/4)^{i+1}}_{\theta_{i+1}}.$$

Note that $\theta_i \nearrow 3/4$ as $i \to +\infty$ and from (25), $M_{3/4}$ is bounded. Then letting $i \to +\infty$ from the above inequality, we have

$$M_{\theta_0} \le C\|v^k\|_{L_{1/2}(Q^-_1)}.$$

From the above inequality, we have

$$[D_k\tilde{u}]_{C^{1/2,1}(Q^-_{1/2})} = [v^k]_{C^{1/2,1}(Q^-_{1/2})} \le C\|v^k\|_{L_{1/2}(Q^-_1)} \le C\left(\fint_{Q^-_1} \left| D_k\tilde{u} - q^k \right|^{1/2} \right)^2. \tag{27}$$

Combining (24), (27), going back to the original variable x, and scaling, we get (23). The lemma is proved. □

Suppose that u is the weak solution of (19). Define

$$\tilde{u}(t,x) = \begin{cases} u(t,x) & (t,x) \in Q^-_4, \\ 0 & (t,x) \in (-17,-16] \times B^+_4. \end{cases}$$

Then $\tilde{u}$ is defined in $(-17,0) \times B^+_4$. Later on, when talking about u and $\tilde{u}$, we automatically take their domains to be Q^-_4 and $(-17,0) \times B^+_4$, respectively. For simplification of notation, we denote $(-17,0) \times B^+_4$ by $\tilde{Q}^-_4$.

For $X \in \tilde{Q}^-_4$ and $r > 0$, define

$$\phi(X,r) := \inf_{\boldsymbol{q}\in\mathbb{R}^n} \left(\fint_{C^-_r(X)\cap\tilde{Q}^-_4} |D\tilde{u} - \boldsymbol{q}|^{\frac{1}{2}} \right)^2$$

and choose a vector $\boldsymbol{q}_{X,r} \in \mathbb{R}^n$ satisfying

$$\phi(X,r) := \left(\fint_{C^-_r(X)\cap\tilde{Q}^-_4} |D\tilde{u} - \boldsymbol{q}_{X,r}|^{\frac{1}{2}} \right)^2. \tag{28}$$

Lemma 3.2. *Let $\beta \in (0,1)$. For any $\bar{X}_0 = (t_0, \bar{x}_0) \in (-16, 0) \times (B_3 \cap \{x^n = 0\})$ and for $0 < \rho \le r \le 1/2$, we have*

$$\phi\left(\bar{X}_0, \rho\right) \lesssim_{n,\lambda,\beta} \left(\frac{\rho}{r}\right)^{\beta} \phi\left(\bar{X}_0, r\right) + [t_0 < -16 + 4r^2]\tilde{\varrho}_{g^n}(2\rho) + C\|Du\|_{L_\infty(Q^-_{2r}(\bar{X}_0)\cap Q^-_4)}\tilde{\omega}_{\mathbf{A}}(2\rho) + C\tilde{\omega}_{\boldsymbol{g}}(2\rho),$$

where $\tilde{\omega}_\bullet$ and $\tilde{\varrho}_\bullet$ are Dini functions given by (37). Here, we used the Iverson bracket notation, i.e., $[P] = 1$ if P is true and $[P] = 0$ otherwise.

Proof. We fix a smooth set $\mathcal{B} \subset \mathbb{R}^n$ satisfying $B^+_{2/3}(0) \subset \mathcal{B} \subset B^+_{3/4}(0)$ and denote

$$\mathcal{B}_r\left(\bar{x}_0\right) = r\mathcal{B} + \bar{x}_0.$$

We divide the proof into two cases.
Case 1: $t_0 - 4r^2 \ge -16$. Then $Q^-_{2r}\left(\bar{X}_0\right) \cap Q^-_4 = Q^-_{2r}\left(\bar{X}_0\right)$, $\tilde{u} = u$ in $Q^-_{2r}\left(\bar{X}_0\right)$, and

$$\phi(\bar{X}_0, r) := \inf_{\boldsymbol{q}\in\mathbb{R}^n}\left(\fint_{Q^-_r(\bar{X}_0)} |Du - \boldsymbol{q}|^{\frac{1}{2}}\right)^2.$$

Define

$$\bar{a}^{ij} = \fint_{Q^-_{2r}(\bar{X}_0)} a^{ij}(t,x)\,dxdt, \quad \bar{g}^i = \fint_{Q^-_{2r}(\bar{X}_0)} g^i(t,x)\,dxdt.$$

Next we decompose $u = v + w$ as follows. Let $w \in \mathcal{H}^1_2$ be the weak solution of the problem

$$\begin{cases} \partial_t w - D_i\left(\bar{a}^{ij}D_j w\right) = D_i\hat{g}^i & \text{in } (t_0 - 4r^2, t_0) \times \mathcal{B}_{2r}\left(\bar{x}_0\right), \\ -\bar{a}^{ij}D_j w\nu^i = \hat{g}^i\nu^i & \text{on } (t_0 - 4r^2, t_0) \times \partial\mathcal{B}_{2r}\left(\bar{x}_0\right), \\ w = 0 & \text{on } \{t = t_0 - 4r^2\} \times \mathcal{B}_{2r}\left(\bar{x}_0\right), \end{cases}$$

where

$$\hat{g}^i := (a^{ij} - \bar{a}^{ij})D_j u + (g^i - \bar{g}^i).$$

Then $v := u - w$ satisfies

$$\begin{cases} v_t - D_i\left(\bar{a}^{ij}D_j v\right) = 0 & \text{in } Q^-_r(\bar{X}_0), \\ -\bar{a}^{nj}D_j v = \bar{g}^n & \text{on } \Delta^-_r(\bar{X}_0). \end{cases}$$

We apply a modified and scaled version of Lemma 2.7 to w to find that for any $\gamma > 0$,

$$\left|\left\{X \in Q^-_r(\bar{X}_0) : |Dw(X)| > \gamma\right\}\right| \lesssim \frac{1}{\gamma}\left(\|Du\|_{L_\infty(Q^-_{2r}(\bar{X}_0))}\int_{Q^-_{2r}(\bar{X}_0)} |\mathbf{A} - \overline{\mathbf{A}}| + \int_{Q^-_{2r}(\bar{X}_0)} |\boldsymbol{g} - \bar{\boldsymbol{g}}|\right),$$

where we also used

$$Q^-_r(\bar{X}_0) \subset (t_0 - 4r^2, t_0) \times \mathcal{B}_{2r}(\bar{x}_0) \subset Q^-_{2r}(\bar{X}_0).$$

Then, using Hölder's inequality, we have

$$\left(\fint_{Q^-_r(\bar{X}_0)} |Dw|^{1/2}\right)^2 \lesssim \omega_{\mathbf{A}}(2r)\|Du\|_{L_\infty(Q^-_{2r}(\bar{X}_0))} + \omega_{\boldsymbol{g}}(2r). \tag{29}$$

Let $0<\kappa<1/2$ be a number to be fixed later. Note that we have

$$\left(\fint_{Q^-_{\kappa r}(\bar X_0)}\left|Dv-\overline{Dv}_{Q^-_{\kappa r}(\bar X_0)}\right|^{1/2}\right)^2\le N\kappa r[Dv]_{C^{1/2,1}(Q^-_{\kappa r}(\bar X_0))}.$$

Combining the above inequality and (23) of Lemma 3.1, we have

$$\left(\fint_{Q^-_{\kappa r}(\bar X_0)}\left|Dv-\overline{Dv}_{Q^-_{\kappa r}(\bar X_0)}\right|^{1/2}\right)^2\le C_0\kappa\left(\fint_{Q^-_{r}(\bar X_0)}|Dv-\boldsymbol{q}|^{1/2}\right)^2,\quad\forall\,\boldsymbol{q}\in\mathbb{R}^n,\tag{30}$$

where C_0 is an absolute constant depending only on n and λ. By using the decomposition $u=v+w$, we obtain from (30) that

$$\begin{aligned}&\left(\fint_{Q^-_{\kappa r}(\bar X_0)}\left|Du-\overline{Dv}_{Q^-_{\kappa r}(\bar X_0)}\right|^{1/2}\right)^2\\&\le 2\left(\fint_{Q^-_{\kappa r}(\bar X_0)}\left|Dv-\overline{Dv}_{Q^-_{\kappa r}(\bar X_0)}\right|^{1/2}\right)^2+2\left(\fint_{Q^-_{\kappa r}(\bar X_0)}|Dw|^{1/2}\right)^2\\&\le C_0\kappa\left(\fint_{Q^-_{r}(\bar X_0)}|Du-\boldsymbol{q}|^{1/2}\right)^2+C_0(1+\kappa^{-2(n+2)})\left(\fint_{Q_{r}(\bar X_0)}|Dw|^{1/2}\right)^2.\end{aligned}\tag{31}$$

Since $\boldsymbol{q}\in\mathbb{R}^n$ is arbitrary, by using (29), we thus obtain

$$\begin{aligned}\phi\left(\bar X_0,\kappa r\right)\le&C_0\kappa\phi\left(\bar X_0,r\right)\\&+CC_0\left(1+\kappa^{-2(n+2)}\right)\left(\omega_{\mathbf{A}}(2r)\|Du\|_{L_\infty(Q^-_{2r}(\bar X_0))}+\omega_{\boldsymbol{g}}(2r)\right).\end{aligned}\tag{32}$$

Case 2: $t_0-4r^2<-16$. Recall that from (19) u is a weak solution of the following problem

$$\begin{cases}\partial_t u-D_i\left(a^{ij}D_ju\right)=\operatorname{div}\boldsymbol{g} & \text{in } (-16,t_0)\times\mathcal{B}_{2r}\left(\bar x_0\right),\\ -a^{nj}D_ju=g^n & \text{on } (-16,t_0)\times\left(B_r\left(\bar x_0\right)\cap\{x^n=0\}\right),\\ u=0 & \text{on } \{t=-16\}\times\mathcal{B}_{2r}\left(\bar x_0\right).\end{cases}$$

We take the even extensions of a^{ij} and $\boldsymbol{g}$ with respect to $\{t=-16\}$ and still denote them by a^{ij} and $\boldsymbol{g}$. We note that by the triangle inequality, the DMO condition is preserved under the even extension. Denote

$$\tilde g^i=\chi_{\{t>-16\}}(g^i-\bar g^i),\ i=1,\ldots,n-1,\quad \tilde g^n=\chi_{\{t>-16\}}g^n,$$

where

$$\bar g^i=\fint_{Q^-_{2r}(\bar X_0)}g^i(t,x)\,dxdt.$$

Then we see that $\tilde u$ is a weak solution of the following equation

$$\begin{cases}\partial_t\tilde u-D_i\left(a^{ij}D_j\tilde u\right)=D_i\tilde g^i & \text{in } (-17,t_0)\times\mathcal{B}_{2r}\left(\bar x_0\right),\\ -a^{nj}D_j\tilde u=\tilde g^n & \text{on } (-17,t_0)\times\left(B_r\left(\bar x_0\right)\cap\{x^n=0\}\right),\\ \tilde u=0 & \text{on } \{t=-17\}\times\mathcal{B}_{2r}\left(\bar x_0\right).\end{cases}$$

Define

$$\bar a^{ij}=\fint_{Q^-_{2r}(\bar X_0)}a^{ij}(t,x)\,dxdt.$$

We then decompose $\tilde{u} = \tilde{v} + \tilde{w}$ as follows. Let $\tilde{w} \in \mathcal{H}_2^1$ be the weak solution of the problem

$$\begin{cases} \partial_t \tilde{w} - D_i \left(\bar{a}^{ij} D_j \tilde{w}\right) = D_i \left((a^{ij} - \bar{a}^{ij}) D_j \tilde{u}\right) + D_i \tilde{g}^i & \text{in } (t_0 - 4r^2, t_0) \times \mathcal{B}_{2r}(\bar{x}_0), \\ -\bar{a}^{ij} D_j \tilde{w} \nu^i = (a^{ij} - \bar{a}^{ij}) D_j \tilde{u} \nu^i + \tilde{g}^i \nu^i & \text{on } (t_0 - 4r^2, t_0) \times \partial \mathcal{B}_{2r}(\bar{x}_0), \\ \tilde{w} = 0 & \text{on } \{t = t_0 - 4r^2\} \times \mathcal{B}_{2r}(\bar{x}_0). \end{cases}$$

Here we note that when $t_0 > -16$ and $0 < r \le 1/2$, we have $t_0 - 4r^2 > -17$. Then $\tilde{v} := \tilde{u} - \tilde{w}$ satisfies

$$\begin{cases} \tilde{v}_t - D_i \left(\bar{a}^{ij} D_j \tilde{v}\right) = 0 & \text{in } Q_r^-(\bar{X}_0), \\ -\bar{a}^{nj} D_j \tilde{v} = 0 & \text{on } \Delta_r^-(\bar{X}_0). \end{cases}$$

Again we apply Lemma 2.7 to $\tilde{w}$ to find that for any $\gamma > 0$,

$$\begin{aligned} &\left|\left\{X \in Q_r^-(\bar{X}_0) : |D\tilde{w}(X)| > \gamma\right\}\right| \\ &\lesssim \frac{1}{\gamma}\left(\|D\tilde{u}\|_{L_\infty(Q_{2r}^-(\bar{X}_0))} \int_{Q_{2r}^-(\bar{X}_0)} |\mathbf{A} - \overline{\mathbf{A}}| + \int_{Q_{2r}^-(\bar{X}_0)} |\tilde{g}^i|\right). \end{aligned}$$

If we go back to the original solution u, recall the definition of $\tilde{g}^i$ and use (21), we have

$$\begin{aligned} &\left|\left\{X \in Q_r^-(\bar{X}_0) : |D\tilde{w}(X)| > \gamma\right\}\right| \\ &\lesssim \frac{1}{\gamma}\|Du\|_{L_\infty(Q_{2r}^-(\bar{X}_0) \cap Q_4^-)} \omega_{\mathbf{A}}(2r) + \sum_{i=1}^{n-1} \omega_{g^i}(2r) + \varrho_{g^n}(2r). \end{aligned}$$

Then, using Hölder's inequality, we have

$$\left(\fint_{Q_r^-(\bar{X}_0)} |D\tilde{w}|^{1/2}\right)^2 \lesssim \omega_{\mathbf{A}}(2r) \|Du\|_{L_\infty(Q_{2r}^-(\bar{X}_0) \cap Q_4^-)} + \sum_{i=1}^{n-1} \omega_{g^i}(2r) + \varrho_{g^n}(2r). \tag{33}$$

Also similar to (30) in Case 1, we have

$$\left(\fint_{Q_{\kappa r}^-(\bar{X}_0)} \left|D\tilde{v} - \overline{D\tilde{v}}_{Q_{\kappa r}^-(\bar{X}_0)}\right|^{1/2}\right)^2 \le C_0 \kappa \left(\fint_{Q_r^-(\bar{X}_0)} |D\tilde{v} - \boldsymbol{q}|^{1/2}\right)^2, \quad \forall\, \boldsymbol{q} \in \mathbb{R}^n. \tag{34}$$

Combining (33) and (34), similar to (31), we have

$$\begin{aligned} &\left(\fint_{Q_{\kappa r}^-(\bar{X}_0)} \left|D\tilde{u} - \overline{D\tilde{v}}_{Q_{\kappa r}^-(\bar{X}_0)}\right|^{1/2}\right)^2 \\ &\le 2\left(\fint_{Q_{\kappa r}^-(\bar{X}_0)} \left|D\tilde{v} - \overline{D\tilde{v}}_{Q_{\kappa r}^-(\bar{X}_0)}\right|^{1/2}\right)^2 + 2\left(\fint_{Q_{\kappa r}^-(\bar{X}_0)} |D\tilde{w}|^{1/2}\right)^2 \\ &\le C_0 \kappa \left(\fint_{Q_r^-(\bar{X}_0)} |D\tilde{u} - \boldsymbol{q}|^{1/2}\right)^2 + C_0(1 + \kappa^{-2(n+2)}) \left(\fint_{Q_r(\bar{X}_0)} |D\tilde{w}|^{1/2}\right)^2. \end{aligned}$$

Since $\boldsymbol{q} \in \mathbb{R}^n$ is arbitrary, by using (33), we thus obtain

$$\begin{aligned} &\phi\left(\bar{X}_0, \kappa r\right) \le C_0 \kappa \phi\left(\bar{X}_0, r\right) \\ &\quad + CC_0\left(1 + \kappa^{-2(n+2)}\right)\left(\omega_{\mathbf{A}}(2r) \|Du\|_{L_\infty(Q_{2r}^-(\bar{X}_0) \cap Q_4^-)} + \omega_{\boldsymbol{g}}(2r) + \varrho_{g^n}(2r)\right). \end{aligned} \tag{35}$$

Combining (32) in Case 1 and (35) in Case 2, we have

$$\phi\left(\bar{X}_0,\kappa r\right)\le C_0\kappa\phi\left(\bar{X}_0,r\right)+CC_0\left(1+\kappa^{-2(n+2)}\right)$$
$$\cdot\left(\omega_{\mathbf{A}}(2r)\|Du\|_{L_\infty(Q^-_{2r}(\bar{X}_0)\cap Q^-_4)}+\omega_{\boldsymbol{g}}(2r)+\varrho_{g^n}(2r)[t_0<-16+4r^2]\right).$$

For $\forall\ \beta\in(0,1)$, let $\kappa\in(0,1/2)$ be sufficiently small so that $C_0\kappa\le\kappa^\beta$. Then, we obtain

$$\phi\left(\bar{X}_0,\kappa r\right)\le\kappa^\beta\phi\left(\bar{X}_0,r\right)$$
$$+C\left(\omega_{\mathrm{A}}(2r)\|Du\|_{L_\infty(Q^-_{2r}(\bar{X}_0)\cap Q^-_4)}+\omega_{\boldsymbol{g}}(2r)+\varrho_{g^n}(2r)[t_0<-16+4r^2]\right).$$

Note that $\kappa^\beta<1$. By iterating, for $j=0,1,2,\ldots$, we get

$$\phi(\bar{X}_0,\kappa^j r)\le\kappa^{j\beta}\phi(\bar{X}_0,r)+C[t_0<-16+4r^2]\sum_{i=1}^{j}\kappa^{(i-1)\beta}\varrho_{g^n}(2\kappa^{j-i}r)$$
$$+C\|Du\|_{L_\infty(Q^-_{2r}(\bar{X}_0)\cap Q^-_4)}\sum_{i=1}^{j}\kappa^{(i-1)\beta}\omega_{\mathbf{A}}(2\kappa^{j-i}r)+C\sum_{i=1}^{j}\kappa^{(i-1)\beta}\omega_{\boldsymbol{g}}(2\kappa^{j-i}r).$$

Therefore, we have

$$\begin{aligned}\phi(\bar{X}_0,\kappa^j r)\le&\kappa^{j\beta}\phi(\bar{X}_0,r)+C[t_0<-16+4r^2]\tilde{\varrho}_{g^n}(2\kappa^j r)\\&+C\|Du\|_{L_\infty(Q_{2r}(\bar{X}_0)\cap Q^-_4)}\tilde{\omega}_{\mathbf{A}}(2\kappa^j r)+C\tilde{\omega}_{\boldsymbol{g}}(2\kappa^j r),\end{aligned}\tag{36}$$

where the Dini function $\tilde{f}_\bullet$ (for $f=\omega$ or ϱ) is given by

$$\tilde{f}_\bullet(t)=\sum_{i=1}^{\infty}\kappa^{i\beta}\left(f_\bullet\left(\kappa^{-i}t\right)\left[\kappa^{-i}t\le1\right]+f_\bullet(1)\left[\kappa^{-i}t>1\right]\right).\tag{37}$$

We remark that $\tilde{\omega}_\bullet(t)$ and $\tilde{\varrho}_\bullet(t)$ are Dini functions. See [4, Lemma 1].

Note that for any ρ satisfying $0<\rho\le r$, if we set j to be the integer satisfying $\kappa^{j+1}<\rho/r\le\kappa^j$. Then by (36) we get

$$\begin{aligned}\phi\left(\bar{X}_0,\rho\right)&\le\kappa^{-\beta}\left(\frac{\rho}{r}\right)^\beta\phi\left(\bar{X}_0,\kappa^{-j}\rho\right)+C[t_0<-16+4r^2]\tilde{\varrho}_{g^n}(2\rho)\\&\quad+C\|Du\|_{L_\infty(Q^-_{2r}(\bar{X}_0)\cap Q^-_4)}\tilde{\omega}_{\mathbf{A}}(2\rho)+C\tilde{\omega}_{\boldsymbol{g}}(2\rho)\\&\le\kappa^{-\beta-2(n+2)}\left(\frac{\rho}{r}\right)^\beta\phi\left(\bar{X}_0,r\right)+C[t_0<-16+4r^2]\tilde{\varrho}_{g^n}(2\rho)\\&\quad+C\|Du\|_{L_\infty(Q^-_{2r}(\bar{X}_0)\cap Q^-_4)}\tilde{\omega}_{\mathbf{A}}(2\rho)+C\tilde{\omega}_{\boldsymbol{g}}(2\rho)\end{aligned}$$

and

$$\begin{aligned}\phi\left(\bar{X}_0,\rho\right)\lesssim&\left(\frac{\rho}{r}\right)^\beta\phi\left(\bar{X}_0,r\right)+[t_0<-16+4r^2]\tilde{\varrho}_{g^n}(2\rho)\\&+C\|Du\|_{L_\infty(Q^-_{2r}(\bar{X}_0)\cap Q^-_4)}\tilde{\omega}_{\mathbf{A}}(2\rho)+C\tilde{\omega}_{\boldsymbol{g}}(2\rho).\end{aligned}$$

The lemma is proved. □

Lemma 3.3. *Let $\beta\in(0,1)$. For any $X_0\in(-16,0)\times B^+_3$ and $0<\rho\le r\le1/5$, we have*

$$\begin{aligned}\phi\left(X_0,\rho\right)\lesssim_{n,\lambda,\beta}&\rho^\beta r^{-\beta-n-2}\|Du\|_{L_1(C^-_{3r}(X_0)\cap Q^-_4)}+[t_0<-16+16r^2]\hat{\varrho}_{g^n}(\rho)\\&+\|Du\|_{L_\infty(C^-_{5r}(X_0)\cap Q^-_4)}\hat{\omega}_{\mathbf{A}}(\rho)+\hat{\omega}_{\boldsymbol{g}}(\rho),\end{aligned}\tag{38}$$

where $\hat{f}_\bullet(t)$ (for $f=\omega$ or ϱ) is a Dini function defined by

$$\hat{f}_\bullet(t) := \sup_{s\in[t,1]}\left(\frac{t}{s}\right)^\beta \tilde{f}_\bullet(s) \quad (0<t\le 1). \tag{39}$$

Proof. In this proof, we shall denote

$$X_0=(t_0,x_0)=\left(t_0,x_0^1,x_0^2,\ldots,x_0^n\right) \text{ and } \bar{X}_0=(t_0,\bar{x}_0)=\left(t_0,x_0^1,\ldots,x_0^{n-1},0\right).$$

First, we note that for $\theta\le 5$, $Q^-_{\theta r}\left(\bar{X}_0\right)\cap\tilde{Q}^-_4=Q^-_{\theta r}\left(\bar{X}_0\right)$ and

$$\phi\left(\bar{X}_0,\theta r\right)\le\left(\fint_{Q^-_{\theta r}(\bar{X}_0)}|D\tilde{u}|^{\frac12}\right)^2\lesssim_{n,\theta} r^{-n-2}\|Du\|_{L_1\left(Q^-_{\theta r}(\bar{X}_0)\cap Q^-_4\right)}. \tag{40}$$

There are three possibilities.

(i) $\rho\le r\le x_0^n$. Since $B_r(x_0)\subset B_4^+$, we observe that $\phi(X_0,\rho)$ is equal to that introduced in [8, Section 3.2], which satisfies [8, (3.15)]. Also as (40), we have

$$\phi(X_0,r)\le\left(\fint_{C^-_r(\bar{X}_0)\cap Q^-_4}|Du|^{\frac12}\right)^2\lesssim r^{-n-2}\|Du\|_{L_1\left(C^-_r(X_0)\cap Q^-_4\right)}.$$

Thus, by the same argument as in deriving [8, (3.15)], we have

$$\begin{aligned}\phi(X_0,\rho)&\lesssim\left(\frac{\rho}{r}\right)^\beta\phi(X_0,r)+\|Du\|_{L_\infty\left(C^-_r(X_0)\cap Q^-_4\right)}\tilde{\omega}_{\mathbf{A}}(\rho)+\tilde{\omega}_{\boldsymbol{g}}(\rho)\\&\lesssim\left(\frac{\rho}{r}\right)^\beta r^{-n-2}\|Du\|_{L_1\left(C^-_r(X_0)\cap Q^-_4\right)}+\|Du\|_{L_\infty\left(C^-_r(X_0)\cap Q^-_4\right)}\tilde{\omega}_{\mathbf{A}}(\rho)+\tilde{\omega}_{\boldsymbol{g}}(\rho).\end{aligned} \tag{41}$$

Note that for [8, (3.15)], we only need $\mathbf{A}$ and $\boldsymbol{g}$ to be in DMO with respect to x.

(ii) $x_0^n\le\rho\le r$. Since $C^-_\rho(X_0)\cap\tilde{Q}^-_4\subset Q^-_{2\rho}(\bar{X}_0)\cap\tilde{Q}^-_4$, we have

$$\begin{aligned}\phi(X_0,\rho)&=\left(\fint_{C^-_\rho(X_0)\cap\tilde{Q}^-_4}\left|D\tilde{u}-\boldsymbol{q}_{X_0,\rho}\right|^{\frac12}\right)^2\le\left(\fint_{C^-_\rho(X_0)\cap\tilde{Q}^-_4}\left|D\tilde{u}-\boldsymbol{q}_{\bar{X}_0,2\rho}\right|^{\frac12}\right)^2\\&\le C\left(\fint_{Q^-_{2\rho}(\bar{X}_0)\cap\tilde{Q}^-_4}\left|D\tilde{u}-\boldsymbol{q}_{\bar{X}_0,2\rho}\right|^{\frac12}\right)^2=C\phi\left(\bar{X}_0,2\rho\right).\end{aligned} \tag{42}$$

Therefore, by Lemma 3.2 and (40), we have

$$\begin{aligned}\phi(X_0,\rho)\lesssim&\left(\frac{2\rho}{2r}\right)^\beta\phi\left(\bar{X}_0,2r\right)+[t_0<-16+16r^2]\tilde{\varrho}_{g^n}(4\rho)\\&+\|Du\|_{L_\infty\left(Q^-_{4r}(\bar{X}_0)\cap Q^-_4\right)}\tilde{\omega}_{\mathbf{A}}(4\rho)+\tilde{\omega}_{\boldsymbol{g}}(4\rho)\\\lesssim&\rho^\beta r^{-\beta-n-2}\|Du\|_{L_1\left(Q^-_{2r}(\bar{X}_0)\cap Q^-_4\right)}+[t_0<-16+16r^2]\hat{\varrho}_{g^n}(\rho)\\&+\|Du\|_{L_\infty\left(Q^-_{4r}(\bar{X}_0)\cap Q^-_4\right)}\hat{\omega}_{\mathbf{A}}(\rho)+\hat{\omega}_{\boldsymbol{g}}(\rho),\end{aligned} \tag{43}$$

where we used the fact

$$\tilde{\omega}_\bullet(4\rho)\leqslant 4^\beta\hat{\omega}_\bullet(\rho),\quad\tilde{\varrho}_\bullet(4\rho)\leqslant 4^\beta\hat{\varrho}_\bullet(\rho)$$

in the last step since $4\rho\le 1$.

(iii) $\rho \le x_0^n \le r$. Take $R = x_0^n$. Since $C_R^-(X_0) \subset Q_{2R}^-(\bar{X}_0)$, we have (42) with R in place of ρ. Therefore, by [8, (3.15)] and Lemma 3.2, we get

$$\begin{aligned}
\phi(X_0,\rho) &\lesssim \left(\frac{\rho}{R}\right)^{\beta} \phi(X_0,R) + \|Du\|_{L_\infty\left(C_R^-(X_0)\cap Q_4^-\right)}\tilde{\omega}_{\mathbf{A}}(\rho) + \tilde{\omega}_{\boldsymbol{g}}(\rho) \\
&\lesssim \left(\frac{\rho}{R}\right)^{\beta} \phi(\bar{X}_0,2R) + \|Du\|_{L_\infty\left(C_R^-(X_0)\cap Q_4^-\right)}\tilde{\omega}_{\mathbf{A}}(\rho) + \tilde{\omega}_{\boldsymbol{g}}(\rho) \\
&\lesssim \left(\frac{\rho}{R}\right)^{\beta} \left\{ \left(\frac{2R}{2r}\right)^{\beta} \phi(\bar{X}_0,2r) + [t_0 < -16+16r^2]\tilde{\varrho}_{g^n}(4R) \right. \\
&\qquad \left. + \|Du\|_{L_\infty\left(Q_{4r}^-(\bar{X}_0)\cap Q_4^-\right)}\tilde{\omega}_{\mathbf{A}}(4R) + \tilde{\omega}_{\boldsymbol{g}}(4R)\right\} \\
&\qquad + \|Du\|_{L_\infty\left(Q_{2R}(\bar{X}_0)\cap Q_4^-\right)}\tilde{\omega}_{\mathbf{A}}(\rho) + \tilde{\omega}_{\boldsymbol{g}}(\rho).
\end{aligned}$$

Notice that from (39) for $f = \omega$ or ϱ, we find

$$\left(\frac{\rho}{R}\right)^{\beta} \tilde{f}_{\bullet}(4R) \leqslant 4^{\beta}\hat{f}_{\bullet}(\rho), \quad \tilde{f}_{\bullet}(\rho) \le \hat{f}_{\bullet}(\rho).$$

Therefore, we have

$$\begin{aligned}
\phi(X_0,\rho) \lesssim& \rho^{\beta} r^{-\beta-n-2}\|Du\|_{L_1\left(Q_{2r}^-(\bar{X}_0)\cap Q_4^-\right)} \\
&+ \|Du\|_{L_\infty\left(Q_{4r}^-(\bar{X}_0)\cap Q_4^-\right)}\hat{\omega}_{\mathbf{A}}(\rho) + [t_0 < -16+16r^2]\hat{\varrho}_{g^n}(\rho) + \hat{\omega}_{\boldsymbol{g}}(\rho).
\end{aligned} \tag{44}$$

We have thus covered all three possible cases and obtained bounds for $\phi(X_0,\rho)$, namely, (41), (43) and (44). Notice that $|X_0 - \bar{X}_0| = x_0^n \le r$ in Cases (ii) and (iii). Therefore, we have for any $\theta \in (0,4]$,

$$Q_{\theta r}^-(\bar{X}_0) \cap Q_4^- \subset C_{(\theta+1)r}^-(X_0) \cap Q_4^-,$$

and (38) follows immediately. We note that $\hat{\omega}_{\bullet}$ and $\hat{\varrho}_{\bullet}$ are Dini functions. See [7]. □

Lemma 3.4. *We have*

$$\begin{aligned}
\|Du\|_{L_\infty\left((-16,0)\times B_2^+\right)} \le& C\|Du\|_{L_1(Q_4^-)} \\
&+ C\int_0^1 \frac{\hat{\omega}_{\boldsymbol{g}}(t)}{t}dt + C\int_0^1 \frac{\hat{\varrho}_{g^n}(t)}{t}\,dt,
\end{aligned} \tag{45}$$

where $C > 0$ is a constant depending only on $n, \lambda,$ and $\omega_{\mathbf{A}}$.

Proof. For $X = (t,x) \in \tilde{Q}_4^-$ and $0 < r \le 1/5$, let $\{\boldsymbol{q}_{X,2^{-k}r}\}_{k=0}^{\infty}$ be a sequence of vectors in $\mathbb{R}^n$ as given in (28). Since we have

$$\left|\boldsymbol{q}_{X,r} - \boldsymbol{q}_{X,r/2}\right|^{\frac12} \le \left|D\tilde{u}(Y) - \boldsymbol{q}_{X,r}\right|^{\frac12} + \left|D\tilde{u}(Y) - \boldsymbol{q}_{X,r/2}\right|^{\frac12},$$

by taking average over $Y \in C_{r/2}^-(X) \cap \tilde{Q}_4^-$ and then taking squares, we obtain

$$\left|\boldsymbol{q}_{X,r} - \boldsymbol{q}_{X,r/2}\right| \le 4^{n+2}2\phi(X,r) + 2\phi(X,r/2).$$

Then, by iteration, we get

$$\left|\boldsymbol{q}_{X,2^{-k}r} - \boldsymbol{q}_{X,r}\right| \le C\sum_{j=0}^{k} \phi\left(X, 2^{-j}r\right). \tag{46}$$

Therefore, by taking $k \to \infty$ in (46), using (38) and Lemma 2.5, we obtain

$$\begin{aligned}\left|D\tilde{u}(X) - \boldsymbol{q}_{X,r}\right| \lesssim & r^{-n-2}\|Du\|_{L_1\left(C^-_{3r}(X)\cap Q^-_4\right)} + \|Du\|_{L_\infty\left(C^-_{5r}(X)\cap Q^-_4\right)}\int_0^r \frac{\hat{\omega}_{\mathbf{A}}(t)}{t}\,dt\\ &+\int_0^r \frac{\hat{\omega}_{\boldsymbol{g}}(t)}{t}dt + [t<-16+16r^2]\int_0^r \frac{\hat{\varrho}_{g^n}(t)}{t}\,dt.\end{aligned}$$

By averaging the obvious inequality

$$\left|\boldsymbol{q}_{X,r}\right|^{\frac12} \le \left|D\tilde{u}(Y) - \boldsymbol{q}_{X,r}\right|^{\frac12} + |D\tilde{u}(Y)|^{\frac12}$$

over $Y \in C^-_r(X)\cap \tilde{Q}^-_4$ and taking square, we get

$$\left|\boldsymbol{q}_{X,r}\right| \le 2\phi(X,r) + 2\left(\fint_{C_r(X)\cap\tilde{Q}_4} |D\tilde{u}|^{\frac12}\right)^2.$$

Combining these together and using

$$\phi(X,r) \le \left(\fint_{C^-_r(X)\cap\tilde{Q}^-_4} |D\tilde{u}|^{\frac12}\right)^2 \lesssim r^{-n-2}\|Du\|_{L_1\left(C^-_r(X)\cap Q^-_4\right)},$$

we obtain

$$\begin{aligned}|D\tilde{u}(X)| \lesssim & r^{-n-2}\|Du\|_{L_1(C^-_{3r}(X)\cap Q^-_4)} + \|Du\|_{L_\infty\left(C^-_{5r}(X)\cap Q^-_4\right)}\int_0^r \frac{\hat{\omega}_{\mathbf{A}}(t)}{t}\,dt\\ &+\int_0^r \frac{\hat{\omega}_{\boldsymbol{g}}(t)}{t}\,dt + [t<-16+16r^2]\int_0^r \frac{\hat{\varrho}_{g^n}(t)}{t}\,dt.\end{aligned}$$

Now, taking the supremum for $X \in C^-_r(X_0)\cap Q^-_4$, where $X_0 \in (-16,0)\times B^+_3$ and $r \le 1/5$, we have

$$\begin{aligned}&\|Du\|_{L_\infty(C^-_r(X_0)\cap Q^-_4)}\\ &\le Cr^{-n-2}\|Du\|_{L_1(C^-_{4r}(X_0)\cap Q^-_4)} + C\|Du\|_{L_\infty(C^-_{6r}(X_0)\cap Q^-_4)}\int_0^r \frac{\hat{\omega}_{\mathbf{A}}(t)}{t}\,dt\\ &+\int_0^r \frac{\hat{\omega}_{\boldsymbol{g}}(t)}{t}\,dt + \int_0^r \frac{\hat{\varrho}_{g^n}(t)}{t}\,dt.\end{aligned}$$

We fix $r_0 < 1/5$ such that for any $0 < r \le r_0$,

$$C\int_0^r \frac{\hat{\omega}_{\mathbf{A}}(t)}{t}\,dt \le \frac{1}{3^{n+2}}.$$

Then, we have for any $X_0 \in (-16,0)\times B^+_3$ and $0 < r \le r_0$ that

$$\begin{aligned}&\|Du\|_{L_\infty(C^-_r(X_0)\cap Q^-_4)}\\ &\le 3^{-n-2}\|Du\|_{L_\infty(C^-_{6r}(X_0)\cap Q^-_4)} + Cr^{-n-2}\|Du\|_{L_1(C^-_{4r}(X_0)\cap Q^-_4)}\\ &+\int_0^r \frac{\hat{\omega}_{\boldsymbol{g}}(t)}{t}\,dt + \int_0^r \frac{\hat{\varrho}_{g^n}(t)}{t}\,dt.\end{aligned}$$

For $k = 1,2,\ldots$, denote $r_k = 3 - 2^{1-k}$. Note that $r_{k+1} - r_k = 2^{-k}$ for $k \ge 1$ and $r_1 = 2$. For $X_0 \in (-16,0)\times B^+_{r_k}$ and $r = 2^{-k-3}$, we have $C^-_{6r}(X_0)\cap Q^-_4 \subset$

$(-16,0)\times B^+_{r_{k+1}}$. We take k_0 sufficiently large such that $2^{-k_0-3}\le r_0$. It then follows that for any $k\ge k_0$,

$$\begin{aligned}\|Du\|_{L_\infty\left((-16,0)\times B^+_{r_k}\right)} \le& C2^{k(n+2)}\|Du\|_{L_1(Q_4^-)}+3^{-n-2}\|Du\|_{L_\infty\left((-16,0)\times B^+_{r_{k+1}}\right)}\\ &+C\int_0^1\frac{\hat\omega_{\boldsymbol g}(t)}{t}\,dt+\int_0^1\frac{\hat\varrho_{g^n}(t)}{t}\,dt.\end{aligned}$$

By multiplying the above by $3^{-k(n+2)}$ and then summing over $k\ge k_0$, we reach

$$\begin{aligned}&\sum_{k=k_0}^\infty 3^{-k(n+2)}\|Du\|_{L_\infty\left((-16,0)\times B^+_{r_k}\right)}\\ &\le C\|Du\|_{L_1(Q_4^-)}+\sum_{k=k_0}^\infty 3^{-(k+1)(n+2)}\|Du\|_{L_\infty\left((-16,0)\times B^+_{r_{k+1}}\right)}\\ &\quad+C\int_0^1\frac{\hat\omega_{\boldsymbol g}(t)}{t}\,dt+\int_0^1\frac{\hat\varrho_{g^n}(t)}{t}\,dt.\end{aligned}$$

Since we assume that $u\in C^{1/2,1}(\overline{(-16,0)\times B_3^+})$, the summations on both sides are convergent and we obtain (45) after absorbing the summation on the right-hand side to the left-hand side. □

Remark 4. By the same proof as that in Lemma 3.4, we have

$$\|Du\|_{L_\infty(Q_3^-)}\le C\|Du\|_{L_1(Q_4^-)}+C\int_0^1\frac{\hat\omega_{\boldsymbol g}(t)}{t}\,dt.$$

Lemma 3.5. *Let $\beta\in(0,1)$. For any $X_0\in(-16,0)\times B_3^+$ and $0<r\le 1/5$, we have*

$$\begin{aligned}&\left|Du(X_0)-\boldsymbol q_{X_0,r}\right|\\ &\lesssim_{n,\lambda,\beta} r^\beta\|Du\|_{L_1(C^-_{3/5}(X_0)\cap Q_4^-)}+\|Du\|_{L_\infty(C_1^-(X_0)\cap Q_4^-)}\int_0^r\frac{\hat\omega_{\mathbf A}(t)}{t}\,dt\\ &\quad+\int_0^r\frac{\hat\omega_{\boldsymbol g}(t)}{t}\,dt+[t_0<-15]\int_0^r\frac{\hat\varrho_{g^n}(t)}{t}\,dt.\end{aligned}$$

Proof. Let $\left\{\boldsymbol q_{X_0,2^{-k}r}\right\}_{k=0}^\infty\in\mathbb R^n$ be as in the proof of Lemma 3.4. By taking $k\to\infty$ in (46), we get

$$\left|Du\left(X_0\right)-\boldsymbol q_{X_0,r}\right|\lesssim\sum_{k=0}^\infty\left|\boldsymbol q_{X_0,2^{-k}r}-\boldsymbol q_{X_0,2^{-k-1}}\right|\lesssim\sum_{k=0}^\infty\phi\left(X_0,2^{-k}r\right).$$

Note that by taking $2^{-k}r$ and $1/5$ in place of ρ and r in (38), we have

$$\begin{aligned}\phi(X_0,2^{-k}r)\lesssim& 2^{-k\beta}r^\beta\|Du\|_{L_1(C^-_{3/5}(X_0)\cap Q_4^-)}+\|Du\|_{L_\infty(C_1^-(X_0)\cap Q_4^-)}\hat\omega_{\mathbf A}(2^{-k}r)\\ &+\hat\omega_{\boldsymbol g}(2^{-k}r)+[t_0<-15]\hat\varrho_{g^n}(2^{-k}r).\end{aligned}\tag{47}$$

Therefore, the lemma follows from Lemma 2.5. □

Proof of Proposition 2. Now we are ready to give the $\mathring C^{1/2,1}$ estimate of u. We only prove that $u\in\mathring C^{1/2,1}(\overline{(-16,0)\times B_1^+})$ under the additional assumptions (20) and (21) besides $\mathbf A$ and $\boldsymbol g$ are in $\mathrm{DMO}(Q_4^-)$ as the other case is simpler, and can be proved in the same way.

For any $X, Y \in (-16,0)\times B_1^+$, we have

$$|Du(X)-Du(Y)| \le \left|Du(X)-\boldsymbol{q}_{X,r}\right| + \left|\boldsymbol{q}_{X,r}-\boldsymbol{q}_{Y,r}\right| + \left|Du(Y)-\boldsymbol{q}_{Y,r}\right|.$$

In the case when $|X-Y|<1/2$, we set $r=2|X-Y|$ and apply Lemma 3.5 to get

$$\begin{aligned}
&\left|Du(X)-\boldsymbol{q}_{X,r}\right| + \left|Du(Y)-\boldsymbol{q}_{Y,r}\right|\\
&\lesssim r^\beta \|Du\|_{L_1\left((-16,0)\times B_2^+\right)} + \|Du\|_{L_\infty\left((-16,0)\times B_2^+\right)} \int_0^r \frac{\hat\omega_{\mathbf{A}}(t)}{t}\,dt\\
&\quad + \int_0^r \frac{\hat\omega_{\boldsymbol{g}}(t)}{t}\,dt + \int_0^r \frac{\hat\varrho_{g^n}(t)}{t}\,dt.
\end{aligned}$$

We take the average over $Z \in C_r^-(X)\cap C_r^-(Y)\cap \tilde Q_4^-$ of the inequality

$$\left|\boldsymbol{q}_{X,r}-\boldsymbol{q}_{Y,r}\right|^{\frac12} \le \left|D\tilde u(Z)-\boldsymbol{q}_{X,r}\right|^{\frac12} + \left|D\tilde u(Z)-\boldsymbol{q}_{Y,r}\right|^{\frac12},$$

take the square, and apply Lemma 3.3 to get

$$\begin{aligned}
\left|\boldsymbol{q}_{X,r}-\boldsymbol{q}_{Y,r}\right| &\lesssim \phi(X,r)+\phi(Y,r)\\
&\lesssim r^\beta \|Du\|_{L_1\left((-16,0)\times B_2^+\right)} + \|Du\|_{L_\infty\left((-16,0)\times B_2^+\right)} \int_0^r \frac{\hat\omega_{\mathbf{A}}(t)}{t}\,dt\\
&\quad + \int_0^r \frac{\hat\omega_{\boldsymbol{g}}(t)}{t}\,dt + \int_0^r \frac{\hat\varrho_{g^n}(t)}{t}\,dt.
\end{aligned}$$

Combining these inequalities together and using Lemma 3.4, we obtain

$$\begin{aligned}
&|Du(X)-Du(Y)|\\
&\lesssim \|Du\|_{L_1\left((-16,0)\times B_2^+\right)}|X-Y|^\beta\\
&\quad + \left(\|Du\|_{L_1(Q_4^-)} + \int_0^1 \frac{\hat\omega_{\boldsymbol{g}}(t)}{t}\,dt + \int_0^1 \frac{\hat\varrho_{g^n}(t)}{t}\,dt\right) \int_0^{2|X-Y|} \frac{\hat\omega_{\mathbf{A}}(t)}{t}\,dt\\
&\quad + \int_0^{2|X-Y|} \frac{\hat\omega_{\boldsymbol{g}}(t)}{t}\,dt + \int_0^{2|X-Y|} \frac{\hat\varrho_{g^n}(t)}{t}\,dt.
\end{aligned} \tag{48}$$

In the case when $|X-Y|\ge 1/2$, we use

$$|Du(X)-Du(Y)| \le 2\|Du\|_{L_\infty\left((-16,0)\times B_1^+\right)}.$$

Applying Lemma 3.4 still gives (48).

Finally, by almost the same proof as that of (3.22) in [8], one gets

$$\frac{|u(t,x)-u(s,x)|}{|t-s|^{1/2}} \to 0 \ \text{ as } |t-s|\to 0 \ \text{ for } (t,x),\ (s,x)\in \overline{(-16,0)\times B_1^+}. \tag{49}$$

Here we make a simple explanation since now $x\in \overline{B_1^+}$ which is slightly different from the case $x\in\overline{B_1}$. For $X=(t,x)$, we separate the discussion into two cases.

(i) $x\in B_1^+$. For fixed $(t,x)\in(-16,0)\times B_1^+$, we have $B_r(x)\subset B_1$ for $r<x^n$. By exactly the same argument of proving (3.22) in [8], we have

$$r^{-1}|u(t-r^2,x)-u(t,x)| \le C\left(\operatorname{osc}_{C_r^-(X)} Du + \omega_{\mathbf{A}Du}(r) + \omega_{\boldsymbol{g}}(r)\right) \tag{50}$$

for sufficiently small r. The right-hand of the above inequality goes to zero as $r\to 0$.

(ii) $x = (x', 0) \in \partial B_1^+ \cap \{x^n = 0\}$. For any $0 < r < 1/2$, we set $x_{2r} = (x', 2r)$. Then

$$\begin{aligned}
&r^{-1}|u(t-r^2,x) - u(t,x)| \\
&\le r^{-1}|u(t-r^2,x) - u(t-r^2,x_{2r}) + u(t,x_{2r}) - u(t,x)| \\
&\quad + r^{-1}|u(t-r^2,x_{2r}) - u(t,x_{2r})|.
\end{aligned}$$

Using the mean value formula and (50), for sufficiently small r, we have

$$\begin{aligned}
&r^{-1}|u(t-r^2,x) - u(t,x)| \\
&\le 2\left(|D_n u(t-r^2,\tilde{x}_{2r}) - D_n u(t,\hat{x}_{2r})|\right) + C\left(\operatorname{osc}_{C_r^-(t,x_{2r})} Du + \omega_{\mathbf{A}Du}(r) + \omega_{\boldsymbol{g}}(r)\right) \\
&\lesssim \operatorname{osc}_{Q_{3r}^-(X)} Du + \omega_{\mathbf{A}Du}(r) + \omega_{\boldsymbol{g}}(r),
\end{aligned} \tag{51}$$

where $\tilde{x}_{2r}$ and $\hat{x}_{2r}$ lie on the line segment connecting x and x_{2r}.

Combining (50) and (51), we get (49). Then we finish the proof of Proportion 2 and that of Theorem 1.1. $\square$

Remark 5. The estimate (48) together with the definition of $\hat{\omega}_\bullet(t)$ in (39) shows that in the case when $\mathbf{A}$ and $\boldsymbol{g}$ are $C^{\alpha/2,\alpha}$ functions with $\alpha \in (0,1)$, by choosing $\beta \in (\alpha, 1)$, Du is a $C^{\alpha/2,\alpha}$ function. In short, we recover the classical Schauder estimates for divergence form parabolic equations with the conormal boundary condition.

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