COMMUN. MATH. SCI.
Vol. 22, No. 5, pp. 1203–1250

LONG-TIME EXISTENCE OF GEVREY-2 SOLUTIONS TO THE 3D PRANDTL BOUNDARY LAYER EQUATIONS*

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Abstract. For the three dimensional Prandtl boundary layer equations, we will show that for arbitrary M and sufficiently small ϵ, the lifespan of the Gevrey-2 solution is at least of size ϵ^{-M} if the initial data lies in suitable Gevrey-2 spaces with size of ϵ.

Keywords. Long-time existence; tangentially Gevrey-2 solutions; Prandtl equations.

AMS subject classifications. 35Q35; 76D03.

1. Introduction

The purpose of this paper is to show the long-time behavior of small Gevrey-2 solutions to the three dimensional Prandtl boundary layer equations in the domain $\{t>0,(x,y,z)\in\mathbb{R}^3,z>0\}$. The equations read as follows,

$$\begin{cases}\partial_t u+(u\partial_x+v\partial_y+w\partial_z)u+\partial_x p=\partial_z^2 u,\\ \partial_t v+(u\partial_x+v\partial_y+w\partial_z)v+\partial_y p=\partial_z^2 v,\\ \partial_x u+\partial_y v+\partial_z w=0,\\ (u,v,w)\big|_{z=0}=0,\quad \lim\limits_{z\to+\infty}(u,v)=(U(t,x,y),V(t,x,y)),\\ (u,v)\big|_{t=0}=(u_0,v_0),\end{cases}\tag{1.1}$$

where $(U(t,x,y),V(t,x,y))$ and $p(t,x,y)$ are the tangential velocity fields and pressure of the Euler flow, satisfying

$$\begin{cases}\partial_t U+U\partial_x U+V\partial_y U+\partial_x p=0,\\ \partial_t V+U\partial_x V+V\partial_y V+\partial_y p=0.\end{cases}\tag{1.2}$$

The Prandtl equations are degenerate Navier-Stokes equations, which were proposed by Prandtl [26] in 1904 to describe the boundary layer phenomenon. Physically, the Prandtl equations bear underlying instabilities, such as the phenomenon of separation, which is related to the appearance of reverse flow in the boundary layer. See [2, 5–7, 28] and references therein for some instability and separation phenomenon for the boundary layer equations. Reader can see [23] and references therein for more introductions on the boundary layer theory and check [8] for some recent development on this topic.

Compared with the Navier-Stokes equations, the main difference of the Prandtl equations is that there is no time evolution for the vertical velocity w, which can only be recovered from the incompressibility condition $(1.1)_3$. Also since the equations of the tangential velocity have no tangential diffusion, $w\partial_z u$ and $w\partial_z v$ in the advection term will cause one order tangential derivative loss when we perform finite-order energy

*Received: December 06, 2022; Accepted (in revised form): November 06, 2023. Communicated by Yaguang Wang.

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estimates. To show local-in-time well-posedness of the Prandtl equations in Sobolev spaces is not an easy thing.

In two dimensional case, the first rigorous mathematical proof of the local existence in Hölder spaces dates back to Oleinik and Samokhin [23], where the so-called Crocco transform was introduced and the local well-posedness was given under a monotonic assumption in the vertical variable on the initial data. Recently, the local well-posedness result was revisited in [1] and [21] by using direct weighted energy estimates, where a nice change of variable was introduced to overcome the one order derivative loss problem. The global weak solutions under an additional favorable sign condition on the pressure p were given in Xin and Zhang [30]. If the initial data is a small ϵ perturbation around the monotonic shear flow in Sobolev spaces, Xu and Zhang [31] proved that the lifespan of the solutions is of size $\ln \frac{1}{\epsilon}$.

Without monotonicity assumption on the initial data, the related results are much recent. See E and Engquist [4] for a globally ill-posedness result and Gerard-Varet and Dormy [9] for a locally ill-posedness result in Sobolev spaces around non-monotonic outflow. (cf. [8, 11, 16] for some improvement). The result in [9] indicates that local existence in time is possible only in smooth Gevrey regularity class. The first result in this direction is due to Sammartino and Caflisch [27], where the local well-posedness in analytical setting (corresponding to Gevrey-1 class) was established by using the abstract Cauchy-Kowalewski theorem. Later, the analyticity on the normal variable was removed in [20], and Kukavica and Vicol in [13] gave an energy-based proof. When the data is of ϵ size in analytical spaces, authors in [32] showed that the lifespan of the analytical solution is of size $\epsilon^{-4/3}$. This result was extended to an almost global-in-time existence in Ignatova and Vicol [12], and global-in-time existence in Paicu and Zhang [24]. To extend the analyticity results to more generalized Gevrey class is not easy. Under the assumption that the data has only a non-degenerate critical point in the vertical variable for each fixed tangential variable, the local well-posedness of the two dimensional Prandtl equations in Gevrey-7/4 class was proved in Gérard-Varet and Masmoudi [10]. See [18] for extensions to Gevrey-2 class, where the exponent 2 is optimal in view of the instability mechanism indicated in [9]. More recently, the single non-degenerate critical point assumption was removed in Dietert and Gérard-varet [3]. Most recently, global existence of Gevrey-2 small solutions was shown in Wang, Wang and Zhang [29] for the two dimensional Prandtl equations.

Compared with two dimensional case, much less well-posedness results are known for the three dimensional Prandtl equations, which is mainly due to the extra difficulties coming from the secondary flow [22] and the complicated structure arising from the multi-dimensional tangential velocity fields. The tangential velocity fields satisfy a system of nonlinear degenerate parabolic equations (not a scalar equation like the two dimensional case) and are coupled with the normal velocity by the divergence-free constraint. Many cancellation properties observed in 2D case are not enough to overcome the difficulties for the 3D case. New ingredients and cancellations are needed to be introduced in the 3D setting. Actually one of the important open problems in Oleinik and Samokhin [23] is on the well-posedness of the three dimensional Prandtl equations.

In three dimensional case, Liu, Wang and Yang [15] give the local well-posedness result under some flow-structure constraints in addition to the monotonic assumption. Without monotonic assumptions, the above analytical well-posedness results are both valid for the two and three dimensional cases except for the global existence of 3D analytic solutions. See for example [14]. By introducing a tangentially polynomial weight to the energy functional, the global existence of small analytical solutions for the three dimensional axially symmetric Prandtl equations was given in Pan and Xu [25]. To relax the analyticity to more generalized Gevrey smoothness is not easy especially for the three dimensional Prandtl equations. As far as the authors know, only most recently, the local well-posedness result in Gevrey-2 spaces for the three dimensional Prandtl equations was solved in [19] by introducing some new cancellations. Such techniques are used in [17] to prove the global well-posedness of Gevrey-2 solutions for a Prandtl model derived from MHD in the Prandtl-Hartmann regime.

Until now, as far as the authors' knowledge, there isn't any result concerning the long-time behavior of solutions for the three dimensional Prandtl equations in generalized Gevrey spaces rather than the analytical setting. This is the preliminary interest of this paper. The main purpose of this paper is to show that for arbitrary M and sufficiently small ϵ, if we consider the outflow (U,V) in (1.2) is zero, then the lifespan of the Gevrey-2 solution to the three dimensional Prandtl Equations (1.1) is of size ϵ^{-M} if the initial data lie in suitable Gevrey-2 spaces with size ϵ. Also in one of our forthcoming papers, we will show the global existence of Gevrey-2 solutions for the 3D axially symmetric Prandtl equations.

When the outflow $(U,V)\equiv 0$, the Prandtl Equations (1.1) degenerate to

$$\begin{cases} \partial_t u+(u\partial_x+v\partial_y+w\partial_z)u=\partial_z^2 u,\\ \partial_t v+(u\partial_x+v\partial_y+w\partial_z)v=\partial_z^2 v,\\ \partial_x u+\partial_y v+\partial_z w=0,\\ (u,v,w)\big|_{z=0}=0, \quad \lim\limits_{z\to+\infty}(u,v)=(0,0),\\ (u,v)\big|_{t=0}=(u_0,v_0). \end{cases} \tag{1.3}$$

Before stating the main result of this paper, we need to introduce some notations especially for the Gevrey-2 space in which the solution (u,v) lies. We use $\|f(t)\|_{L^p}$ for $1\le p\le+\infty$ to denote the usual spatial L^p norm in $\mathbb{R}^3_+$ and $\|f(t)\|_{L^p(\omega)}$ to denote the weighted L^p norm for $1\le p\le+\infty$ with the weight ω as follows.

$$\|f(t)\|^p_{L^p(\omega)}:=\int_{\mathbb{R}^3_+}|f(t,x,y,z)|^p\omega dxdydz.$$

Definition 1.1. *For $\nu\in(0,1]$, denote $\theta_\nu(t,z):=\exp\left\{\nu\frac{z^2}{8\langle t\rangle}\right\}$. Let $\alpha=(\alpha_1,\alpha_2)\in\mathbb{N}^2$ be the two dimensional multi-index and denote $\partial_h^\alpha=\partial_x^{\alpha_1}\partial_y^{\alpha_2}$. For a function $f(t,x,y,z)$, which is smooth in the tangential variables (x,y), define the weighted Gevrey-σ norm $\|\cdot\|_{G^\sigma_{\tau,\nu}}$ by*

$$\|f\|^2_{G^\sigma_{\tau,\nu}}:=\sum_{j\in\mathbb{N}}\frac{\tau^{2(j+1)}}{(j!)^{2\sigma}}\sup_{|\alpha|=j}\|\theta_\nu\partial_h^\alpha f\|^2_{L^2}=\sum_{j\in\mathbb{N}}\frac{\tau^{2(j+1)}}{(j!)^{2\sigma}}\sup_{|\alpha|=j}\|\partial_h^\alpha f\|^2_{L^2(\theta_{2\nu})}. \tag{1.4}$$

We will look for solutions of (1.1) in the $G^\sigma_{\tau,\nu}$ spaces defined above. We need the initial data satisfying the following compatibility conditions at $z=0$.

$$\partial_z^{2k}(u_0,v_0)\big|_{z=0}=0, \quad \text{for} \quad k=0,1. \tag{1.5}$$

The main result of this paper is the following.

Theorem 1.1. *For any fixed $M\ge 2$, $\tau_0>0$, suppose that $\partial_z^k(u_0,v_0)$ belong to $G^2_{2\tau_0,1}$ for $k=0,1,2,3$, and satisfy the compatibility condition (1.5). Then there exist three* *constants* $c_{\tau_0,M}$*,* $C_{\tau_0,M}$ *and* ϵ_0*, such that for any* $0<\epsilon\leq\epsilon_0$*, if*

$$\sum_{k=0}^{3}\|\partial_z^k(u_0,v_0)\|_{G^2_{2\tau_0,1}}\leq\epsilon, \tag{1.6}$$

then for any $t\in(0,c_{\tau_0,M}\epsilon^{-M}]$*, system* (1.3) *has a unique smooth solution satisfying,*

$$\sum_{k=0}^{3}\langle t\rangle^{\frac{k-1}{2}}\|\partial_z^k(u,v)(t)\|_{G^2_{\frac{1}{2}\tau_0,\frac{1}{2}}}\leq C_{\tau_0,M}\langle t\rangle^{-\frac{10M-1}{8M}}. \tag{1.7}$$

Here, $c_{\tau_0,M}$ *and* $C_{\tau_0,M}$ *are two constants depending only on* τ_0 *and* M*, which are relatively small and large when* M *approaches infinity.*

Some remarks follow.

(1) Actually, the result in Theorem 1.1 can also be easily applied to the Gevrey-σ space $G^\sigma_{\tau,\nu}$ with $\sigma\in[1,2]$. The proof is essentially the same with the case $\sigma=2$. For simplicity, we only show the optimal case $\sigma=2$ and leave the details to the interested reader.

(2) Here in the three dimensional case, we can not obtain a global existence result in Gevrey-2 (even in analytical) spaces as shown in Paicu and Zhang [24] and Wang, Wang and Zhang [29] for the two dimensional Prandtl equations. The reason is that in two dimensional case, by using the divergence-free condition $\partial_x u+\partial_y v=0$ and assuming (u,v) decay fast enough at infinity, we can show that $\int_0^\infty u dy=0$, which indicates that the y anti-derivative of u defined by $\psi(x,y):=-\int_y^\infty u(x,\bar{y})d\bar{y}$ satisfies

$$\psi|_{y=0}=0. \tag{1.8}$$

Then by defining the lifted good unknown

$$\mathfrak{g}:=\partial_y\psi+\frac{z}{2\langle t\rangle}\psi \tag{1.9}$$

to replace the good unknowns $g,\tilde{g}$ in Section 2.2, we can also obtain almost $-5/4$-order time decay for $\mathfrak{g}$. And by solving the ODE (1.9) for y variable with initial data (1.8), we can obtain almost $-5/4$-order time decay of u.

While in three dimensional case, the divergence-free condition involves three components (u,v,w), which results in that we cannot obtain a condition like (1.8) for the z anti-derivative of the tangential velocities (u,v) even assuming the solution decays fast enough at infinity. So good unknowns like $\mathfrak{g}$ can not be defined to obtain (u,v). The decay rate of the lower order norms of the tangential velocities in 3D case can only be obtained from $(g,\tilde{g})$, which is almost power $-3/4$ and is $1/2$-order slower than that of the two dimensional case. This slower decay is not enough to ensure the global existence.

(3) In the model (1.2), the outflow $(U,V)\equiv(0,0)$ is considered for simplicity and convenience of presenting the idea. Actually by making some lifting procedures such as that in [24, page 405], the proof in this paper can be also applied to the case that $(U,V)=\epsilon(U(t),V(t))$, where $\epsilon>0$ is sufficiently small and $(U(t),V(t))$ decays sufficiently fast as $t\to+\infty$. The computation will be more elaborated and complicated. We omit this extension and leave it to the interested reader.

Proof of Theorem 1.1 is a direct consequence of Theorem 2.1 stated in Section 2, which is a more precise and detailed version of our result. We will spare some time to give the proof of Theorem 1.1 when we finish the statement of Theorem 2.1.

2. Notations and detailed statement of the main theorem

2.1. Notations. For $0<\kappa$ and a t-dependent function $\tau=\tau(t)>0$, define

$$M_{j,\kappa}:=\frac{\tau^{j+1}(j+1)^{\kappa}}{(j!)^2}.$$

For a function f and $j\in\mathbb{N}$, define

$$f_{j,\kappa,x}:=M_{j,\kappa}\partial_x^j f,\quad f_{j,\kappa,y}:=M_{j,\kappa}\partial_y^j f,\quad \text{and}\quad f_{\alpha,\kappa}:=M_{|\alpha|,\kappa}\partial_h^\alpha f.$$

Let $\theta_\nu(t,z)$ be the weighted function in Definition 1.1 and we simply denote θ_1 by θ. It is easy to see that for $\alpha,\beta\in\mathbb{R}$, $\theta_{\alpha+\beta}=\theta_\alpha\cdot\theta_\beta$. Now for $\nu\in(0,1]$, define another weighted Gevrey-2 norm $\|\cdot\|_{X_{\tau,\kappa,\nu}}$ as the following.

$$\|f\|_{X_{\tau,\kappa,\nu}}^2=\sum_{j\in\mathbb{N}}\sup_{|\alpha|=j}\|f_{\alpha,\kappa}\theta_\nu\|_{L^2}^2:=\sum_{j\in\mathbb{N}}\sup_{|\alpha|=j}\|f_{\alpha,\kappa}\|_{L^2(\theta_{2\nu})}^2. \tag{2.1}$$

When $\nu=1$, we abbreviate (2.1) as

$$\|f\|_{X_{\tau,\kappa}}^2:=\sum_{j\in\mathbb{N}}\sup_{|\alpha|=j}\|f_{\alpha,\kappa}\|_{L^2(\theta_2)}^2=\|f\|_{X_{\tau,\kappa,1}}^2.$$

Here we remark that the replaced Gevrey-2 norm $\|\cdot\|_{X_{\tau,\kappa,\nu}}$ is more suitable than the Gevrey-2 norm $\|\cdot\|_{G^2_{\tau,\nu}}$ for our later energy estimates and for proof of the theorem. Actually, it is easy to see that $\|\cdot\|_{G^2_{\tau,\nu}}=\|\cdot\|_{X_{\tau,0,\nu}}$.

By using Fourier transform on the tangential variables x and y, we can see that

$$\begin{aligned}&\frac{1}{2}\left(\|f_{j,\kappa,x}\|_{L^2(\theta_{2\nu})}^2+\|f_{j,\kappa,y}\|_{L^2(\theta_{2\nu})}^2\right)\\ &\le\sup_{|\alpha|=j}\|f_{\alpha,\kappa}\|_{L^2(\theta_{2\nu})}^2\le\|f_{j,\kappa,x}\|_{L^2(\theta_{2\nu})}^2+\|f_{j,\kappa,y}\|_{L^2(\theta_{2\nu})}^2.\end{aligned}$$

So the $\|\cdot\|_{X_{\tau,\kappa,\nu}}$ norm defined in (2.1) is equivalent to the following

$$\|f\|_{\widetilde{X}_{\tau,\kappa,\nu}}^2:=\sum_{j\in\mathbb{N}}\left(\|f_{j,\kappa,x}\|_{L^2(\theta_{2\nu})}^2+\|f_{j,\kappa,y}\|_{L^2(\theta_{2\nu})}^2\right).$$

The equivalent norms $\|f\|_{X_{\tau,\kappa}}^2$ and $\|f\|_{\widetilde{X}_{\tau,\kappa}}^2$ will be used alternatively throughout the rest of this paper.

Next, we will give the Gevrey radius $\tau(t)$ in the definition of the Gevrey-2 norm in (2.1), which has a positive lower bound in our constrained time interval.

Choosing of the Gevrey radius τ. For any fixed $\delta\in(0,\frac{1}{100}]$ and $\tau_0>0$, we choose

$$\tau(t):=\tau_0-\lambda\delta^{-1}\sqrt{\epsilon}\tau_0\left(\langle t\rangle^\delta-1\right), \tag{2.2}$$

where λ is a large constant, independent of ϵ and will be determined later. For sufficiently small ϵ, we assume that $\lambda\delta^{-1}\sqrt{\epsilon}<1/2$ and set

$$t\le\left(\frac{1}{2\lambda\delta^{-1}\sqrt{\epsilon}}\right)^{\frac{1}{\delta}}:=T_0. \tag{2.3}$$

Under the constraint (2.3), we can obtain

$$\langle t\rangle^{\delta} \leq \left(1+\frac{1}{2\lambda\delta^{-1}\sqrt{\epsilon}}\right) \leq \frac{1}{\lambda\delta^{-1}\sqrt{\epsilon}}, \tag{2.4}$$

which indicate that

$$\frac{1}{2}\tau_0 \leq \tau(t) \leq \tau_0. \tag{2.5}$$

Taking t derivative of (2.2) shows that

$$\tau'(t) = -\lambda\sqrt{\epsilon}\tau_0\langle t\rangle^{\delta-1}.$$

Denote $\lambda\sqrt{\epsilon}\eta(t) := -\frac{\tau'(t)}{\tau(t)}$, then we have

$$\langle t\rangle^{\delta-1} \leq \eta(t) \leq 2\langle t\rangle^{\delta-1}. \tag{2.6}$$

Next, we will restrict t to be any time according to (2.3).

REMARK 2.1. The constant λ will be chosen to be dependent on δ and τ_0 and will approach infinity as $\delta\to 0$. So there exists a constant c_δ such that

$$T_0 = c_\delta \epsilon^{-\frac{1}{2\delta}}.$$

Throughout the paper, $C_{a,b,c,\ldots}$ denotes a positive constant depending on $a,b,c,\ldots$ which may be different from line to line. Dependence on the initial Gevrey radius τ_0 is default, we will denote C_{τ_0} by C for simplicity. We also apply $A \lesssim_{a,b,c,\cdots} B$ to denote $A \leq C_{a,b,c,\ldots}B$. For a two dimensional multi-index $\alpha=(\alpha_1,\alpha_2)\in\mathbb{N}^2$, we write $\partial_h^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}$ and $\partial_h^k = \{\partial_h^\alpha \,|\, |\alpha|=k\}$. For a norm $\|\cdot\|$, we use $\|(f,g,\cdots)\|$ to denote $\|f\|+\|g\|+\cdots$. For a function $f(t,x,y,z)$ and $1\leq p,q\leq+\infty$, define

$$\|f\|_{L_h^p L_z^q} := \left(\int_0^{+\infty}\left(\int_{\mathbb{R}^2}|f|^p dxdy\right)^{q/p} dz\right)^{1/q}.$$

If $p=q$, we simply write it as $\|f\|_{L^p}$ and besides, if $p=q=2$, we will simply denote it as $\|f\|$. We use $[A,B]=AB-BA$ to denote the commutator of A and B. $\langle\cdot,\cdot\rangle_\omega$ denote weighted L^2 inner product with respect to spatial variables, which means for f and g

$$\langle f,g\rangle_\omega := \int_{\mathbb{R}_+^3} f(x,y,z)g(x,y,z)\omega dxdydz.$$

2.2. Detailed statement of the main theorem. Before presenting the more precise and detailed version of the main theorem, we need to introduce two good unknowns $(g,\tilde{g})$, which are set to control the lower order Gevrey-2 norms of (u,v). Define

$$g := \partial_z u + \frac{z}{2\langle t\rangle}u, \quad \tilde{g} := \partial_z v + \frac{z}{2\langle t\rangle}v,$$

then we have the following theorem.

THEOREM 2.1. *For any fixed $\tau_0>0$, $\delta\in(0,\frac{1}{100}]$, there exist constants c_δ, C_δ and ϵ_0, such that for any $\epsilon\leq\epsilon_0$, if*

$$\|(u,v)(0)\|_{X_{\tau_0,13+\frac{2}{\delta}}} \leq \epsilon, \tag{2.7}$$

$$\|(g,\tilde{g})(0)\|_{X_{\tau_0,12}}+\sqrt{\delta}\|\partial_z(g,\tilde{g})(0)\|_{X_{\tau_0,10}}+\delta\|\partial_z^2(g,\tilde{g})(0)\|_{X_{\tau_0,8}}\leq \epsilon, \tag{2.8}$$

then system (1.3) *has a solution* (u,v,w) *satisfying for any* $t\in\left[0,c_\delta\epsilon^{-\frac{1}{2\delta}}\right]$, *such that*

$$\langle t\rangle^{\frac{1-\delta}{4}}\|(u,v)(t)\|_{X_{\tau,12+\frac{2}{\delta}}}\leq C_\delta\epsilon, \tag{2.9}$$

$$\|(g,\tilde{g})(t)\|_{X_{\tau,12}}+\sqrt{\delta}\langle t\rangle^{1/2}\|\partial_z(g,\tilde{g})(t)\|_{X_{\tau,10}}+\delta\langle t\rangle\|\partial_z^2(g,\tilde{g})(t)\|_{X_{\tau,8}}\leq C_\delta\epsilon\langle t\rangle^{-\frac{5-\delta}{4}}. \tag{2.10}$$

Here ϵ_0, c_δ *and* C_δ *are three constants, depending on* τ_0 *and* δ. *We write* c_δ *and* C_δ *with* δ *subscript to emphasize its dependence on* δ.

REMARK 2.2. This result shows that for any $M>1$, by choosing sufficiently small δ, the lifespan of the solution can be of size ϵ^{-M} if the initial data is of size ϵ. Here, the constant c_δ is small while C_δ is large with respect to δ. Actually in our proof, we will see that

$$\lim_{\delta\to 0}c_\delta=0,\quad \lim_{\delta\to 0}C_\delta=+\infty.$$

This is the main obstacle which prevents us from obtaining the almost global existence of Gevrey-2 solutions.

REMARK 2.3. For the proof of Theorem 2.1, by using the local-wellposedness in [19] and continuity argument, we only need to show a closed a priori estimate in Gevrey-2 spaces, which will be achieved in Section 3.

Proof. **(Proof of Theorem 1.1 based on Theorem 2.1.)** First we verify initial conditions (2.7) and (2.8) in Theorem 2.1 based on the Assumption (1.6) in Theorem 1.1.

Let us choose $\delta=(2M)^{-1}$ in Theorem 2.1. Then

$$\begin{aligned}
\|(u_0,v_0)\|_{X_{\tau_0,13+\frac{2}{\delta}}}&=\left(\sum_{j\in\mathbb{N}}\sup_{|\alpha|=j}M^2_{j,13+4M}\|e^{\frac{z^2}{8}}\partial_h^\alpha(u_0,v_0)\|^2_{L^2(\mathbb{R}^3_+)}\right)^{1/2}\\
&=\left[\sum_{j\in\mathbb{N}}\left(\frac{\tau_0^{(j+1)}(j+1)^{13+4M}}{(j!)^2}\sup_{|\alpha|=j}\|e^{\frac{z^2}{8}}\partial_h^\alpha(u_0,v_0)\|_{L^2(\mathbb{R}^3_+)}\right)^2\right]^{1/2}\quad\text{using (1.4)}\\
&\leq\left[\sum_{j\in\mathbb{N}}\left(\frac{(j+1)^{13+4M}}{2^{j+1}}\right)^2\right]^{1/2}G^2_{2\tau_0,1}\leq C_M\epsilon,\ \text{using (1.6)}.
\end{aligned}$$

Moreover, by using inequalities (3.22) and (3.23) in Lemma 3.22, we obtain that by using (1.6)

$$\begin{aligned}
&\|(g,\tilde{g})(0)\|_{X_{\tau_0,12}}+(2M)^{-1/2}\|\partial_z(g,\tilde{g})(0)\|_{X_{(\tau_0,10}}+(2M)^{-1}\|\partial_z^2(g,\tilde{g})(0)\|_{X_{\tau_0,8}}\\
\leq& C\left(\|\partial_z(u_0,v_0)\|_{X_{\tau_0,12}}+(2M)^{-1/2}\|\partial_z^2(u_0,v_0)\|_{X_{\tau_0,10}}+(2M)^{-1}\|\partial_z^3(u_0,v_0)\|_{X_{\tau_0,8}}\right)\\
\leq& C_M\sum_{k=1}^{3}\|\partial_z^k(u_0,v_0)\|_{G^2_{2\tau_0,1}}\leq C_M\epsilon.
\end{aligned}$$

We have shown that (2.7) and (2.8) are guarantied by replacing ϵ with $\tilde{\epsilon} := C_M \epsilon$. Also from the proof of (A.7), (A.9), (A.10) and (A.11) in the Appendix, we see that

$$\begin{aligned}&\sum_{k=0}^{3} \langle t\rangle^{-\frac{k-1}{2}} \|\partial_z^k (u,v)(t)\|^2_{G^2_{\frac{1}{2}\tau_0,\frac{1}{2}}} = \sum_{k=0}^{3} \langle t\rangle^{\frac{k-1}{2}} \|\partial_z^k (u,v)(t)\|^2_{X_{\frac{1}{2}\tau_0,0,\frac{1}{2}}} \\ &\leq C_{\tau_0,M} \Big(\|(g,\tilde{g})(t)\|_{X_{\tau,12}} + \langle t\rangle^{1/2} \|\partial_z (g,\tilde{g})(t)\|_{X_{\tau,10}} + \langle t\rangle \|\partial_z^2 (g,\tilde{g})(t)\|_{X_{\tau,8}} \Big). \end{aligned} \tag{2.11}$$

Then using the result of Theorem 2.1, we see that there exist three constants $c_{\tau_0,M}$ and $C_{\tau_0,M}$ and ϵ_0, such that, for any $t \in (0, c_{\tau_0,M} \epsilon^{-M}]$, system (1.3) has a unique smooth solution satisfying

$$\sum_{k=0}^{3} \langle t\rangle^{-\frac{k-1}{2}} \|\partial_z^k (u,v)(t)\|^2_{G^2_{\frac{1}{2}\tau_0,\frac{1}{2}}} \leq C_{\tau_0,M} \tilde{\epsilon} \langle t\rangle^{-\frac{5-(2M)^{-1}}{4}} = C_{\tau_0,M} \epsilon \langle t\rangle^{-\frac{10M-1}{8M}}, \tag{2.12}$$

which is (1.7). In the last line of (2.12), we have used (2.11) and (2.10). □

3. Closed a priori estimates: proof to Theorem 2.1

In this section, we will give a closed a priori estimate in Gevrey-2 spaces, which indicates the validity of Theorem 2.1 by combining the local well-posedness results and continuity argument. Our strategy is the following. First, in Section 3.1, we will introduce some auxiliary functions, which originate from the ones in [3] and [19], where similar auxiliary functions are introduced to obtain local well-posedness of Gevrey-2 solutions for the two and three dimensional Prandtl equations. However, here we need some modifications so that they can be applied to obtain the long-time behavior of the solution. Then in Section 3.2, we will introduce some linearly good unknowns, which are some linear combinations of the unknowns, auxiliary functions and their derivatives. They are set to achieve fast decay of lower order Gevrey-2 norms for the unknowns and auxiliary functions. In Section 3.3, we make a priori assumptions on the linearly good unknowns and based on the a priori assumptions, we will give a series of a priori estimates for the unknowns, the auxiliary functions and the linearly good unknowns in Section 3.4. At last, in Section 3.5, by applying the a priori estimates in Section 3.4, we can achieve closed energy estimates in Gevrey-2 spaces in the time interval $[0,T_0]$.

3.1. Introduction of auxiliary functions. First we introduce the following two auxiliary functions $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ by

$$\begin{cases} \left[\partial_t + (u\partial_x + v\partial_y + w\partial_z) - \partial_z^2\right] \displaystyle\int_z^{+\infty} \mathcal{H} d\bar{z} = \sqrt{\epsilon} \langle t\rangle^{\delta-1} \partial_x w, \\ \mathcal{H}|_{t=0} = 0, \quad \partial_z \mathcal{H}|_{z=0} = 0, \quad \mathcal{H}|_{z\to+\infty} = 0. \end{cases} \tag{3.1}$$

$$\begin{cases} \left[\partial_t + (u\partial_x + v\partial_y + w\partial_z) - \partial_z^2\right] \displaystyle\int_z^{+\infty} \widetilde{\mathcal{H}} d\bar{z} = \sqrt{\epsilon} \langle t\rangle^{\delta-1} \partial_y w, \\ \widetilde{\mathcal{H}}|_{t=0} = 0, \quad \partial_z \widetilde{\mathcal{H}}|_{z=0} = 0, \quad \widetilde{\mathcal{H}}|_{z\to+\infty} = 0. \end{cases} \tag{3.2}$$

The existence of $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ follows the standard linear parabolic theory. These two auxiliary functions are inspired by Dietert and Gérard-Varet [3] and Li-Masmoudi-Yang [19] where similar auxiliary functions are constructed to prove the local well-posedness of the 2D and 3D Prandtl equations in Gevrey-2 spaces. The main differences are the following.

(1) In (3.1) and (3.2), we define the auxiliary functions by $\int_z^{+\infty}\mathcal{H}d\bar{z}$ and $\int_z^{+\infty}\widetilde{\mathcal{H}}d\bar{z}$ instead of $\int_0^z\mathcal{H}d\bar{z}$ and $\int_0^z\widetilde{\mathcal{H}}d\bar{z}$, respectively to ensure that $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ decay fast enough at z infinity.

(2) The time-dependent coefficient on the right-hand side of (3.1) and (3.2) $\sqrt{\epsilon}\langle t\rangle^{\delta-1}$ is specially designed to match with $\eta(t)$ in (2.6), which can ensure closing of Gevrey-2 energy defined for $\mathcal{H}$ and $\widetilde{\mathcal{H}}$.

Remark 3.1. Here we remark that

$$\int_0^{\infty}\mathcal{H}d\bar{z}=0. \tag{3.3}$$

Actually by letting $z=0$ in (3.1) and using the boundary condition of $\partial_z\mathcal{H}$ and w on $z=0$, we can achieve that

$$[\partial_t+(u\partial_x+v\partial_y)]\int_0^{+\infty}\mathcal{H}d\bar{z}=0.$$

Combining the fact that $\mathcal{H}|_{t=0}=0$, we can obtain (3.3) from the above transport equation.

However, as shown in Li-Masmoudi-Yang in [19], the above two auxiliary functions are not enough to show the well posedness of the 3D Prandtl equations in the Gevrey-2 space. More auxiliary functions are needed to seek for new cancellations to overcome the one order derivative loss problem for the 3D much more complicated couple system. We introduce the following other four auxiliary functions.

$$\begin{cases}\mathcal{G}:=\partial_x u+\dfrac{\langle t\rangle^{1-\delta}}{\sqrt{\epsilon}}\partial_z u\displaystyle\int_z^{+\infty}\mathcal{H}d\bar{z}, & \widetilde{\mathcal{G}}:=\partial_y u+\dfrac{\langle t\rangle^{1-\delta}}{\sqrt{\epsilon}}\partial_z u\displaystyle\int_z^{+\infty}\widetilde{\mathcal{H}}d\bar{z},\\ \mathcal{K}:=\partial_x v+\dfrac{\langle t\rangle^{1-\delta}}{\sqrt{\epsilon}}\partial_z v\displaystyle\int_z^{+\infty}\mathcal{H}d\bar{z}, & \widetilde{\mathcal{K}}:=\partial_y v+\dfrac{\langle t\rangle^{1-\delta}}{\sqrt{\epsilon}}\partial_z v\displaystyle\int_z^{+\infty}\widetilde{\mathcal{H}}d\bar{z}.\end{cases}$$

These auxiliary functions are initiated by Li, Masmoudi and Yang in [19], where similar four auxiliary functions are introduced to seek for new cancellations and local well-posedness in Gevrey-2 energy spaces are achieved by combining the aforementioned auxiliary functions $\mathcal{H}$ and $\widetilde{\mathcal{H}}$. Also, to obtain the long-time behavior of the solution, we make some modification to the original auxiliary functions in [19]. These four auxiliary functions will help achieve the $\frac{1}{2}$−order derivative loss of $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ and enable us to close the long-time Gevrey-2 energy of $\mathcal{H}$ and $\widetilde{\mathcal{H}}$. Then the long-time Gevrey-2 energy for the solution (u,v) will be obtained with the help of $\mathcal{H}$ and $\widetilde{\mathcal{H}}$. Here we make a brief introduction for the idea of proof.

Take (3.1) for example. Applying $-\partial_z$ to (3.1), we can obtain, as shown in (4.1), that

$$\begin{aligned}&\left[\partial_t+(u\partial_x+v\partial_y+w\partial_z)-\partial_z^2\right]\mathcal{H}\\ =&\sqrt{\epsilon}\langle t\rangle^{\delta-1}(\partial_x\mathcal{G}+\partial_y\mathcal{K})-(\partial_z\partial_x u+\partial_z\partial_y v)\int_z^{+\infty}\mathcal{H}d\bar{z}+(\partial_x u+\partial_y v)\mathcal{H}\\ :=&\sqrt{\epsilon}\langle t\rangle^{\delta-1}(\partial_x\mathcal{G}+\partial_y\mathcal{K})+\text{l.o.t.}.\end{aligned}\tag{3.4}$$

Here, from the equation of $\mathcal{H}$ and previous results in [3], we consider $\mathcal{H}$ has the same order as $\partial_h(u,v)$, so the term l.o.t doesn't have derivative loss. However, if we view $\partial_x\mathcal{G}$

separately as two terms, then there will be one order derivative loss for $\partial_x^2 u$ in $\partial_x \mathcal{G}$. By computation, we can see that

$$\begin{aligned}
&\left[\partial_t + (u\partial_x + v\partial_y + w\partial_z) - \partial_z^2\right]\partial_x\mathcal{G}\\
=&-\left[\partial_x u\partial_x\mathcal{G} + \partial_x v\partial_y\mathcal{G} + \partial_x w\partial_z\mathcal{G}\right]\\
&+\partial_x\left\{-\left[(\partial_x u)^2 + \partial_x v\partial_y u\right] + \frac{\partial_y v\partial_z u - \partial_y u\partial_z v}{\sqrt{\epsilon}\langle t\rangle^{\delta-1}}\int_z^\infty \mathcal{H}d\bar{z} + \frac{2\partial_z^2 u}{\sqrt{\epsilon}\langle t\rangle^{\delta-1}}\mathcal{H}\right\}\\
:=&\text{terms involving } \partial_h^2(u,v) + \text{l.o.t.}.
\end{aligned}$$

The same is for $\partial_y\mathcal{K}$. Then inserting this into (3.4), we can see that

$$\left[\partial_t + (u\partial_x + v\partial_y + w\partial_z) - \partial_z^2\right]^2\mathcal{H} = \text{terms involving } \partial_h^2(u,v) + \text{l.o.t.}. \tag{3.5}$$

Here $\partial_h^2(u,v)$ have the same order as $\partial_h\mathcal{H}$. The above equation for $\mathcal{H}$ indicates that we can perform energy estimates in Gevrey-σ spaces with $\sigma\in[1,2]$ for $\mathcal{H}$ as indicated in the toy model displayed in Li, Masmoudi and Yang [19]. As $\mathcal{G}$ has the same order as $\partial_h^{-1/2}\mathcal{H}$, so if we define the energy functional of $\mathcal{H},\widetilde{\mathcal{H}}$ as $\|(\mathcal{H},\widetilde{\mathcal{H}})\|_{X_{\tau,\kappa}}$, then correspondingly, we need to define the energy functionals of $\mathcal{G},\widetilde{\mathcal{G}},\mathcal{K},\widetilde{\mathcal{H}}$ as $\|(\mathcal{G},\widetilde{\mathcal{G}},\mathcal{K},\widetilde{\mathcal{H}})\|_{X_{\tau,\kappa+1}}$ and also the energy functionals of u,v as $\|(u,v)\|_{X_{\tau,\kappa+2}}$.

When we obtain from (3.5) the Gevrey-2 energy norm estimate, from (6.4), we see that

$$\left[\partial_t + (u\partial_x + v\partial_y + w\partial_z) - \partial_z^2\right]\left(\partial_x^j u + \frac{\partial_z u}{\sqrt{\epsilon}\langle t\rangle^{\delta-1}}\int_z^\infty \partial_x^{j-1}\mathcal{H}d\bar{z}\right) = \text{l.o.t.}.$$

This equation has no derivative loss. After performing Gevrey-2 norm estimate for the above equation and then combining the Gevrey-2 energy estimates of $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, the Gevrey-2 norm estimate of (u,v) follows.

In order to obtain much long lifespan of the solutions, the equations of (u,v) and $(\mathcal{H},\widetilde{\mathcal{H}})$ are not enough to obtain much faster time decay estimate. Next, we will introduce the following four linearly good unknowns to catch much faster decay to the lower order Gevrey-2 energy of the (u,v) and $(\mathcal{H},\widetilde{\mathcal{H}})$.

3.2. The linearly good unknowns. Inspired by the good unknown in [12] and [24], we define

$$\begin{cases} g := \partial_z u + \dfrac{z}{2\langle t\rangle}u, \quad \tilde{g} := \partial_z v + \dfrac{z}{2\langle t\rangle}v,\\ \mathfrak{H} := \mathcal{H} - \dfrac{z}{2\langle t\rangle}\displaystyle\int_z^\infty \mathcal{H}d\bar{z}, \quad \tilde{\mathfrak{H}} := \widetilde{\mathcal{H}} - \dfrac{z}{2\langle t\rangle}\displaystyle\int_z^\infty \widetilde{\mathcal{H}}d\bar{z}. \end{cases} \tag{3.6}$$

These four linearly good unknowns are set to dig out the sufficiently fast decay rate for the lower order Gevrey-2 norms of the solutions (u,v) and $(\mathcal{H},\widetilde{\mathcal{H}})$, which ensure closing of energy estimates for all the quantities mentioned above in our constrained time $t\in(0,T_0]$. As shown in (2.10), we see that the lower order Gevrey-2 norm of $(g,\tilde{g})$ has a decay rate of almost $-5/4$ order with respect to time, which will induce almost $-3/4$ order decay rate of the lower order Gevrey-2 norm of (u,v), see (3.8) in Lemma 3.1. Based on the almost $-3/4$ order decay of (u,v), we can see that $(\mathfrak{H},\tilde{\mathfrak{H}})$ have the same almost $-3/4$ order decay for the lower order Gevrey-2 norm, which indicates almost $-3/4$ order decay of the lower order Gevrey-2 norm for the auxiliary functions $(\mathcal{H},\widetilde{\mathcal{H}})$. See also (3.8) in Lemma 3.1.

3.3. A prior assumptions. Later for simplification of notations, we denote

$$\boldsymbol{u}=(u,v),\quad \boldsymbol{\mathcal{H}}=(\mathcal{H},\widetilde{\mathcal{H}}),\quad \boldsymbol{\mathcal{G}}=(\mathcal{G},\widetilde{\mathcal{G}},\mathcal{K},\widetilde{\mathcal{K}}),\quad \boldsymbol{g}=(g,\tilde{g}),\quad \boldsymbol{\mathfrak{H}}=(\mathfrak{H},\widetilde{\mathfrak{H}}).$$

We will first make a priori assumptions for the good unknowns as follows. We assume that there exists some constant C_*, (to be determined later), such that

$$\begin{aligned}&\|\boldsymbol{g}(t)\|_{X_{\tau,12}}+\sqrt{\delta}\langle t\rangle^{1/2}\|\partial_z\boldsymbol{g}(t)\|_{X_{\tau,10}}+\delta\langle t\rangle\|\partial_z^2\boldsymbol{g}(t)\|_{X_{\tau,8}}\le C_*\epsilon\langle t\rangle^{-\frac{5-\delta}{4}},\\&\|\boldsymbol{\mathfrak{H}}(t)\|_{X_{\tau,9,7/8}}+\langle t\rangle^{1/2}\|\partial_z\boldsymbol{\mathfrak{H}}(t)\|_{X_{\tau,7,7/8}}\le C_*\epsilon\langle t\rangle^{-\frac{3-\delta}{4}}.\end{aligned}\tag{3.7}$$

Under the a prior Assumption (3.7), we first have the following a priori estimates based on the relations between $\boldsymbol{u}$ and $\boldsymbol{g}$, and between $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathfrak{H}}$ respectively.

LEMMA 3.1. *Under the Assumption (3.7), we have the following a priori estimates. For any $0\le\nu<1$,*

$$\begin{aligned}&\langle t\rangle^{-1/2}\|\boldsymbol{u}\|_{X_{\tau,12,\nu}}+\|\partial_z\boldsymbol{u}\|_{X_{\tau,12,\nu}}+\sqrt{\delta}\langle t\rangle^{\frac12}\left\|\partial_z^2\boldsymbol{u}\right\|_{X_{\tau,10,\nu}}+\delta\langle t\rangle\left\|\partial_z^3\boldsymbol{u}\right\|_{X_{\tau,8,\nu}}\lesssim_\nu C_*\epsilon\langle t\rangle^{-\frac{5-\delta}{4}},\\&\|\boldsymbol{\mathcal{H}}\|_{X_{\tau,9,3/4}}+\langle t\rangle^{1/2}\|\partial_z\boldsymbol{\mathcal{H}}\|_{X_{\tau,7,3/4}}\lesssim C_*\epsilon\langle t\rangle^{-\frac{3-\delta}{4}},\\&\|\boldsymbol{\mathcal{G}}\|_{X_{\tau,9,3/4}}+\sqrt{\delta}\langle t\rangle^{1/2}\|\partial_z\boldsymbol{\mathcal{G}}\|_{X_{\tau,7,3/4}}\lesssim C_*\epsilon\langle t\rangle^{-\frac{3-\delta}{4}}.\end{aligned}\tag{3.8}$$

Sobolev embedding will imply the following finite order L^∞ a priori estimates. For $k\le50$, any $0\le\nu<1$,

$$\begin{aligned}&\langle t\rangle^{-\frac14}\left\|\theta_\nu\partial_h^k\boldsymbol{u}\right\|_{L^\infty}+\sqrt{\delta}\langle t\rangle^{\frac14}\left\|\theta_\nu\partial_h^k\partial_z\boldsymbol{u}\right\|_{L^\infty}+\delta\langle t\rangle^{\frac34}\left\|\theta_\nu\partial_h^k\partial_z^2\boldsymbol{u}\right\|_{L^\infty}\lesssim C_*\epsilon\langle t\rangle^{-\frac{5-\delta}{4}},\\&\langle t\rangle^{-\frac34}\left\|\theta_\nu\partial_h^k w\right\|_{L^\infty}\lesssim C_*\epsilon\langle t\rangle^{-\frac{5-\delta}{4}}.\end{aligned}\tag{3.9}$$

Proof of this lemma will be presented in the Appendix.

Based on the a priori assumptions in (3.7) and the a priori estimates in Lemma 3.1, we can derive a series of estimates as follows, which is based on performing weighted energy estimates to equations of auxiliary functions, the unknowns and the good unknowns.

3.4. A priori estimates of auxiliary functions, the unknowns and the good unknowns. For simplification of notations, let $\kappa=10+\frac{2}{\delta}$ in the following. For auxiliary functions $\boldsymbol{\mathcal{H}}$, we have the following estimate.

PROPOSITION 3.1 (Gevrey-2 estimates of $\boldsymbol{\mathcal{H}}$). *For any fixed $\tau_0>0$, $\delta\in(0,\frac{1}{100}]$, under the assumption of (3.7), for sufficiently small ϵ, there exists a constant C_δ such that for any $t\in(0,T_0]$, we have the following estimate*

$$\begin{aligned}&\langle t\rangle^{\frac{1-\delta}{2}}\|\boldsymbol{\mathcal{H}}(t)\|_{X_{\tau,\kappa}}^2+\delta\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\boldsymbol{\mathcal{H}}(t)\|_{X_{\tau,\kappa}}^2dt+\lambda\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\boldsymbol{\mathcal{H}}(t)\|_{X_{\tau,\kappa+1/2}}^2dt\\&\le C_\delta\frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}\left(\|\boldsymbol{\mathcal{H}}(t)\|_{X_{\tau,\kappa+1/2}}^2+\|\boldsymbol{u}(t)\|_{X_{\tau,\kappa+5/2}}^2\right)dt\\&\quad+C_\delta\frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\left(\|\partial_z\boldsymbol{\mathcal{H}}(t)\|_{X_{\tau,\kappa}}^2+\|\partial_z\boldsymbol{u}(t)\|_{X_{\tau,\kappa+2}}^2\right)dt\\&\quad+C_\delta\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\boldsymbol{\mathcal{G}}(t)\|_{X_{\tau,\kappa+3/2}}^2dt.\end{aligned}\tag{3.10}$$

For auxiliary functions $\boldsymbol{\mathcal{G}}$, we have the following estimate.

Proposition 3.2 (Gevrey-2 estimates of $\boldsymbol{\mathcal{G}}$). *For any fixed* $\tau_0>0$, $\delta\in(0,\frac{1}{100}]$, *under the assumption of* (3.7), *for sufficiently small* ϵ, *there exists a constant* C_δ *such that for any* $t\in(0,T_0]$, *we have the following estimate*

$$
\begin{aligned}
&\langle t\rangle^{\frac{1-\delta}{2}}\|\boldsymbol{\mathcal{G}}(t)\|^2_{X_{\tau,\kappa+1}}+\delta\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\boldsymbol{\mathcal{G}}(t)\|^2_{X_{\tau,\kappa+1}}\,dt\\
&\quad+\lambda\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\boldsymbol{\mathcal{G}}(t)\|^2_{X_{\tau,\kappa+3/2}}\,dt\\
&\le C_\delta\|\boldsymbol{u}(0)\|_{X_{\tau,\kappa+3}}\\
&\quad+C_\delta\frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}\Big(\|\boldsymbol{\mathcal{H}}(t)\|^2_{X_{\tau,\kappa+1/2}}+\|\boldsymbol{\mathcal{G}}(t)\|^2_{X_{\tau,\kappa+3/2}}+\|\boldsymbol{u}(t)\|^2_{X_{\tau,\kappa+5/2}}\Big)\,dt\\
&\quad+C_\delta\frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\Big(\|\partial_z\boldsymbol{\mathcal{H}}(t)\|^2_{X_{\tau,\kappa}}+\|\partial_z\boldsymbol{\mathcal{G}}(t)\|^2_{X_{\tau,\kappa+1}}+\|\partial_z\boldsymbol{u}(t)\|^2_{X_{\tau,\kappa+2}}\Big)\,dt.
\end{aligned}
\tag{3.11}
$$

For the unknown functions $\boldsymbol{u}$, we have the following estimate.

Proposition 3.3 (Gevrey-2 estimates of $\boldsymbol{u}$). *For any fixed* $\tau_0>0$, $\delta\in(0,\frac{1}{100}]$, *under the assumption of* (3.7), *for sufficiently small* ϵ, *there exists a constant* C_δ *such that for any* $t\in(0,T_0]$, *we have the following estimate*

$$
\begin{aligned}
&\langle t\rangle^{\frac{1-\delta}{2}}\|\boldsymbol{u}(t)\|^2_{X_{\tau,\kappa+2}}+\delta\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\boldsymbol{u}(t)\|^2_{X_{\tau,\kappa+2}}\,dt+\lambda\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\boldsymbol{u}(t)\|^2_{X_{\tau,\kappa+5/2}}\,dt\\
&\le C_\delta\|\boldsymbol{u}(0)\|_{X_{\tau,\kappa+2}}+C_\delta C_*^2\|\boldsymbol{\mathcal{H}}\|^2_{X_{\tau,\kappa}}\\
&\quad+C_\delta\frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}\Big(\|\boldsymbol{u}(t)\|^2_{X_{\tau,\kappa+5/2}}+\|\boldsymbol{\mathcal{H}}(t)\|^2_{X_{\tau,\kappa+1/2}}\Big)\,dt\\
&\quad+C_\delta\frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\Big(\|\partial_z\boldsymbol{u}(t)\|^2_{X_{\tau,\kappa+2}}+\|\partial_z\boldsymbol{\mathcal{H}}(t)\|^2_{X_{\tau,\kappa}}\Big)\,dt.
\end{aligned}
\tag{3.12}
$$

Next, we will give the Gevrey-2 estimates of the good unknowns $\boldsymbol{g}$ and $\boldsymbol{\mathfrak{H}}$. Denote

$$\kappa_0=12,\ \kappa_1=10,\ \kappa_2=8,\ \text{and}\ ,\kappa_3=9,\ \kappa_4=7.$$

For $\boldsymbol{g}$, we have the following estimate.

Proposition 3.4 (Gevrey-2 estimates of $\boldsymbol{g}$). *For any fixed* $\tau_0>0$, $\delta\in(0,\frac{1}{100}]$, *under the assumption of* (3.7), *for sufficiently small* ϵ, *there exists a constant* C_δ *such that for any* $t\in(0,T_0]$, *we have the following estimate.*

(i) For the good unknown: $\boldsymbol{g}$,

$$
\begin{aligned}
&\langle t\rangle^{\frac{5-\delta}{2}}\|\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0}}+\delta\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0}}\,dt\\
&\quad+\lambda\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0+1/2}}\,dt\\
&\le C_\delta\|\boldsymbol{g}(0)\|^2_{X_{\tau_0,\kappa_0}}+C_\delta\frac{C_*^2}{\lambda}\epsilon^{3/2}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\boldsymbol{u}(t)\|^2_{X_{\tau,\kappa+2}}\,dt\\
&\quad+C_\delta\frac{C_*^2}{\lambda}\epsilon^{3/2}\int_0^{T_0}\Big(\langle t\rangle^{\frac{3+\delta}{2}}\|\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0+1/2}}+\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0}}\Big)\,dt.
\end{aligned}
\tag{3.13}
$$

(ii) For the first order z-derivative of the good unknown: $\partial_z \boldsymbol{g}$,

$$
\begin{aligned}
&\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z \boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_1}}+\delta\int_0^{T_0}\langle t\rangle^{\frac{7-\delta}{2}}\left\|\partial_z^2 \boldsymbol{g}(t)\right\|^2_{X_{\tau,\kappa_1}}dt\\
&\quad+\lambda\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{7-\delta}{2}}\eta(t)\|\partial_z \boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_1+1/2}}dt\\
&\le C_\delta\|\partial_z \boldsymbol{g}(0)\|^2_{X_{\tau,\kappa_1}}+\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z \boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_1}}dt\\
&\quad+C_\delta\frac{C_*^2}{\lambda}\epsilon^{3/2}\int_0^{T_0}\left(\langle t\rangle^{\frac{3+\delta}{2}}\|\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0+1/2}}+\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z \boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0}}\right)dt. \quad (3.14)
\end{aligned}
$$

(iii) For the second order z-derivative of the good unknown: $\partial_z^2 \boldsymbol{g}$,

$$
\begin{aligned}
&\langle t\rangle^{\frac{9-\delta}{2}}\left\|\partial_z^2 \boldsymbol{g}(t)\right\|^2_{X_{\tau,\kappa_2}}+\delta\int_0^{T}\langle t\rangle^{\frac{9-\delta}{2}}\left\|\partial_z^3 \boldsymbol{g}(t)\right\|^2_{X_{\tau,\kappa_2}}dt\\
&\quad+\lambda\sqrt{\epsilon}\int_0^{T}\langle t\rangle^{\frac{9-\delta}{2}}\eta(t)\left\|\partial_z^2 \boldsymbol{g}(t)\right\|^2_{X_{\tau,\kappa_3+1/2}}dt\\
&\le C_\delta\left\|\partial_z^2 \boldsymbol{g}(0)\right\|^2_{X_{\tau_0,\kappa_2}}+\int_0^{T_0}\langle t\rangle^{\frac{7-\delta}{2}}\left\|\partial_z^2 \boldsymbol{g}(t)\right\|^2_{X_{\tau,\kappa_2}}dt.\\
&\quad+C_\delta\frac{C_*^2}{\lambda}\epsilon^{3/2}\int_0^{T_0}\left(\langle t\rangle^{\frac{3+\delta}{2}}\|\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0+1/2}}+\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z \boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0}}\right.\\
&\qquad\qquad\left.+\langle t\rangle^{\frac{7-\delta}{2}}\left\|\partial_z^2 \boldsymbol{g}(t)\right\|^2_{X_{\tau,\kappa_1}}\right)dt. \quad (3.15)
\end{aligned}
$$

For $\mathfrak{H}$, we have the following estimate.

Proposition 3.5 (Gevrey-2 estimates of $\mathfrak{H}$). *For any fixed $\tau_0>0$, $\delta\in(0,\frac{1}{100}]$, under the assumption of (3.7), for sufficiently small ϵ, there exists a constant C_δ such that for any $t\in(0,T_0]$, we have the following estimate.*

(iv) For the good unknown: $\mathfrak{H}$,

$$
\begin{aligned}
&\langle t\rangle^{\frac{3-\delta}{2}}\|\mathfrak{H}(t)\|^2_{X_{\tau,\kappa_3,7/8}}+\int_0^{T_0}\langle t\rangle^{\frac{3-\delta}{2}}\|\partial_z\mathfrak{H}(t)\|^2_{X_{\tau,\kappa_3,7/8}}dt\\
&\quad+\lambda\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{3-\delta}{2}}\eta(t)\|\mathfrak{H}(t)\|^2_{X_{\tau,\kappa_3+1/2,7/8}}dt\\
&\le C_\delta\frac{C_*^2}{\lambda}\epsilon^{3/2}\int_0^{T_0}\left(\langle t\rangle^{\frac{1+\delta}{2}}\|\mathfrak{H}(t)\|^2_{X_{\tau,\kappa_3+1/2,7/8}}+\langle t\rangle^{\frac{3-\delta}{2}}\|\partial_z\mathfrak{H}(t)\|^2_{X_{\tau,\kappa_3,7/8}}\right.\\
&\quad\left.+\langle t\rangle^{-\frac{1-\delta}{2}}\|\boldsymbol{\mathcal{H}}(t)\|^2_{X_{\tau,\kappa}}\right)dt+\frac{\sqrt{\epsilon}}{\lambda}\int_0^{T_0}\langle t\rangle^{\frac{3+\delta}{2}}\|\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0+1/2}}dt. \quad (3.16)
\end{aligned}
$$

(v) For the first order z-derivative of the good unknown: $\partial_z\mathfrak{H}$,

$$
\begin{aligned}
&\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z\mathfrak{H}(t)\|^2_{X_{\tau,\kappa_4,7/8}}+\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\left\|\partial_z^2\mathfrak{H}(t)\right\|^2_{X_{\tau,\kappa_4,7/8}}dt\\
&\quad+\lambda\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|\partial_z\mathfrak{H}(t)\|^2_{X_{\tau,\kappa_4+1/2,7/8}}dt\\
&\le C_\delta\frac{C_*^2}{\lambda}\epsilon^{3/2}\int_0^{T_0}\left(\langle t\rangle^{\frac{1+\delta}{2}}\|\mathfrak{H}(t)\|^2_{X_{\tau,\kappa_3+1/2,7/8}}+\langle t\rangle^{\frac{3-\delta}{2}}\|\partial_z\mathfrak{H}(t)\|^2_{X_{\tau,\kappa_3,7/8}}\right)dt
\end{aligned}
$$

$$+\frac{\sqrt{\epsilon}}{\lambda}\int_0^{T_0}\langle t\rangle^{\frac{3+\delta}{2}}\|\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0+1/2}}dt+\int_0^{T_0}\langle t\rangle^{\frac{3-\delta}{2}}\|\partial_z\mathfrak{H}(t)\|^2_{X_{\tau,\kappa_4,7/8}}dt. \quad (3.17)$$

3.5. End proof of Theorem 2.1. Based on the a priori estimates from Proposition 3.1 to Proposition 3.5, we can derive the validity of Theorem 2.1. Since the local well-posedness of Gevrey-2 solutions has already been shown in Li-Masnoudi-Yang [19], by continuity argument, we only need to show that under the a priori Assumption (3.7), by choosing suitably large C_*, we can obtain that

$$\begin{aligned}
&\|\boldsymbol{g}(t)\|_{X_{\tau,12}}+\sqrt{\delta}\langle t\rangle^{1/2}\|\partial_z\boldsymbol{g}(t)\|_{X_{\tau,10}}+\delta\langle t\rangle\|\partial_z^2\boldsymbol{g}(t)\|_{X_{\tau,8}}\le\frac{1}{2}C_*\epsilon\langle t\rangle^{-\frac{5-\delta}{4}},\\
&\|\mathfrak{H}(t)\|_{X_{\tau,9,7/8}}+\langle t\rangle^{1/2}\|\partial_z\mathfrak{H}(t)\|_{X_{\tau,7,7/8}}\le\frac{1}{2}C_*\epsilon\langle t\rangle^{-\frac{3-\delta}{4}}.
\end{aligned} \quad (3.18)$$

And to prove the validity of (2.9) and (2.10).

Set

$$\lambda=C_*^4.$$

For C_* sufficiently large, adding (3.10) in Proposition 3.1 and (3.11) in Proposition 3.2 together, we can obtain that

$$\begin{aligned}
&\langle t\rangle^{\frac{1-\delta}{2}}\|\boldsymbol{\mathcal{H}}(t)\|^2_{X_{\tau,\kappa}}+\delta\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\boldsymbol{\mathcal{H}}(t)\|^2_{X_{\tau,\kappa}}dt\\
&\quad+C_*^4\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\boldsymbol{\mathcal{H}}(t)\|^2_{X_{\tau,\kappa+1/2}}dt\\
&\lesssim_\delta\|u(0)\|^2_{X_{\tau_0,\kappa+3}}+C_*^{-2}\sqrt{\epsilon}\int_0^{T_0}\left(\langle t\rangle^{-\frac{1-\delta}{2}}\|\boldsymbol{u}(t)\|^2_{X_{\tau,\kappa+5/2}}+\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\boldsymbol{u}(t)\|^2_{X_{\tau,\kappa+2}}\right)dt.
\end{aligned}$$

For C_* sufficiently large, combining the above inequality with (3.12) in Proposition 3.3, we can obtain that

$$\begin{aligned}
&\langle t\rangle^{\frac{1-\delta}{2}}\left(\|\boldsymbol{\mathcal{H}}(t)\|^2_{X_{\tau,\kappa}}+\|\boldsymbol{u}(t)\|^2_{X_{\tau,\kappa+2}}\right)\\
&\quad+\delta\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\left(\|\partial_z\boldsymbol{\mathcal{H}}(t)\|^2_{X_{\tau,\kappa}}+\|\partial_z\boldsymbol{u}(t)\|^2_{X_{\tau,\kappa+2}}\right)dt\\
&\quad+C_*^4\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\left(\|\boldsymbol{\mathcal{H}}(t)\|^2_{X_{\tau,\kappa+1/2}}+\|\boldsymbol{u}(t)\|^2_{X_{\tau,\kappa+5/2}}\right)dt\\
&\le C_\delta\|\boldsymbol{u}(0)\|^2_{X_{\tau_0,\kappa+3}}\le C_\delta\epsilon^2.
\end{aligned} \quad (3.19)$$

From the above inequality, estimate (2.9) is proven.

Now multiplying a small constant $c\delta$ to (3.14) and a much smaller constant $c\delta^2$ to (3.15), and adding the resulting equations to (3.13), by letting ϵ be sufficiently small, we can achieve that

$$\begin{aligned}
&\langle t\rangle^{\frac{5-\delta}{2}}\|\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0}}+\delta\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_1}}+\delta^2\langle t\rangle^{\frac{9-\delta}{2}}\left\|\partial_z^2\boldsymbol{g}(t)\right\|^2_{X_{\tau,\kappa_2}}\\
&C_*^4\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0+1/2}}dt\le C_\delta\epsilon^{3/2}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\boldsymbol{u}(t)\|^2_{X_{\tau,\kappa+2}}dt.
\end{aligned} \quad (3.20)$$

Inserting (3.19) into (3.20), for sufficiently small ϵ, we can achieve that for some constant C_δ,

$$\begin{aligned}&\langle t\rangle^{\frac{5-\delta}{2}}\|\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0}}+\delta\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_1}}+\delta^2\langle t\rangle^{\frac{9-\delta}{2}}\left\|\partial_z^2\boldsymbol{g}(t)\right\|^2_{X_{\tau,\kappa_2}}\\&\quad+\lambda\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0+1/2}}dt\le C_\delta\epsilon^2.\end{aligned}\tag{3.21}$$

Then we can obtain (2.10) and the first inequality of (3.18) by letting C_* large enough such that $16C_\delta\le C_*^2$.

At last, from (3.16) and (3.17) in Proposition 3.5, by letting ϵ sufficiently small, we can obtain that

$$\begin{aligned}&\langle t\rangle^{\frac{3-\delta}{2}}\|\boldsymbol{\mathfrak{H}}(t)\|^2_{X_{\tau,\kappa_3,7/8}}+\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z\boldsymbol{\mathfrak{H}}(t)\|^2_{X_{\tau,\kappa_4,7/8}}\\&\le C_\delta\epsilon^{3/2}\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}\|\boldsymbol{\mathcal{H}}(t)\|^2_{X_{\tau,\kappa}}dt+C_*^{-4}\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{3+\delta}{2}}\|\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0+1/2}}dt.\end{aligned}$$

By using (3.19) and (3.21), and remembering (2.6), we can obtain that there exists a constant C_δ such that

$$\langle t\rangle^{\frac{3-\delta}{2}}\|\boldsymbol{\mathfrak{H}}(t)\|^2_{X_{\tau,\kappa_3,7/8}}+\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z\boldsymbol{\mathfrak{H}}(t)\|^2_{X_{\tau,\kappa_4,7/8}}\le C_\delta\epsilon^2,$$

which shows the second inequality of (3.18) by letting C_* large enough.

We still need to show the proofs of Proposition 3.1 to Proposition 3.5. Before that, we give two useful lemmas, which will be frequently used in later estimates.

3.6. Preliminary lemmas. The first one is a weighted Poincáre inequality. We state as follows.

LEMMA 3.2. *Let f be a function belonging to H^1 in z variable, which decays to zero sufficiently fast as $z\to+\infty$. Then for $0\le\nu\le1$ we have*

$$\frac{\nu}{2\langle t\rangle}\|f\|^2_{L^2(\theta_{2\nu})}\le\|\partial_z f\|^2_{L^2(\theta_{2\nu})},\tag{3.22}$$

and

$$\frac{\nu}{4\langle t\rangle}\|f\|^2_{L^2(\theta_{2\nu})}+\frac{\nu^2}{16}\left\|\frac{z}{\langle t\rangle}f\right\|^2_{L^2(\theta_{2\nu})}\le\|\partial_z f\|^2_{L^2(\theta_{2\nu})}.\tag{3.23}$$

Proof of this lemma is similar with [29, Lemma 2.5], where the case $\nu=1$ is handled. The essential idea is to perform integration by parts on z variable. Here we make some details for completeness.

Proof. Since when $\nu=0$, (3.22) and (3.23) are obvious, we will assume that $0<\nu\le1$ in the following. Using integration by parts on z, we have

$$\begin{aligned}&\int_0^\infty f^2(x,y,z)e^{\frac{\nu z^2}{4\langle t\rangle}}dz\\&=-\int_0^\infty z\partial_z\left(f^2(x,y,z)e^{\frac{\nu z^2}{4\langle t\rangle}}\right)dz\\&=-2\int_0^\infty zf(x,y,z)\partial_z f(x,y,z)e^{\frac{\nu z^2}{4\langle t\rangle}}dz-\frac{\nu}{2\langle t\rangle}\int_0^\infty z^2f^2(x,y,z)e^{\frac{\nu z^2}{4\langle t\rangle}}dz.\end{aligned}$$

Then by using Cauchy inequality to the first term of the right-hand side of the above equality, we can obtain that

$$\begin{aligned}&\int_0^\infty f^2(x,y,z)e^{\frac{\nu z^2}{4\langle t\rangle}}\,dz+\frac{\nu}{2\langle t\rangle}\int_0^\infty z^2f^2(x,y,z)e^{\frac{\nu z^2}{4\langle t\rangle}}\,dz\\ \le&\frac{\nu}{2\langle t\rangle}\int_0^\infty z^2f^2(x,y,z)e^{\frac{\nu z^2}{4\langle t\rangle}}\,dz+\frac{2\langle t\rangle}{\nu}\int_0^\infty(\partial_zf)^2(x,y,z)e^{\frac{\nu z^2}{4\langle t\rangle}}\,dz,\end{aligned}$$

and

$$\begin{aligned}&\int_0^\infty f^2(x,y,z)e^{\frac{\nu z^2}{4\langle t\rangle}}\,dz+\frac{\nu}{2\langle t\rangle}\int_0^\infty z^2f^2(x,y,z)e^{\frac{\nu z^2}{4\langle t\rangle}}\,dz\\ \le&\frac{\nu}{4\langle t\rangle}\int_0^\infty z^2f^2(x,y,z)e^{\frac{\nu z^2}{4\langle t\rangle}}\,dz+\frac{4\langle t\rangle}{\nu}\int_0^\infty(\partial_zf)^2(x,y,z)e^{\frac{\nu z^2}{4\langle t\rangle}}\,dz.\end{aligned}$$

Then after absorbing the first term of the right-hand side of the above two inequalities, we can have

$$\frac{\nu}{2\langle t\rangle}\int_0^\infty f^2(x,y,z)e^{\frac{\nu z^2}{4\langle t\rangle}}\,dz\le\int_0^\infty(\partial_zf)^2(x,y,z)e^{\frac{\nu z^2}{4\langle t\rangle}}\,dz, \tag{3.24}$$

and

$$\begin{aligned}&\frac{\nu}{4\langle t\rangle}\int_0^\infty f^2(x,y,z)e^{\frac{\nu z^2}{4\langle t\rangle}}\,dz+\left(\frac{\nu}{4\langle t\rangle}\right)^2\int_0^\infty z^2f^2(x,y,z)e^{\frac{\nu z^2}{4\langle t\rangle}}\,dz\\ \le&\int_0^\infty(\partial_zf)^2(x,y,z)e^{\frac{\nu z^2}{4\langle t\rangle}}\,dz.\end{aligned}\tag{3.25}$$

After integrating (3.24) and (3.25) on x,y, we can obtain (3.23). ∎

Next, we give a lemma to show product estimates of the Gevrey-2 norm.

LEMMA 3.3. *Let $\kappa>0$ and $0<\nu\le1$. For smooth functions f and g, which decay fast enough at z infinity, we have the following product estimates.*

$$\|fg\|^2_{X_{\tau,\kappa,\nu}}\lesssim_{\nu,\kappa}\langle t\rangle^{1/2}\left(\|f\|^2_{X_{\tau,5,\frac{\nu+1}{4}}}\|\partial_zg\|^2_{X_{\tau,\kappa,\frac{\nu+1}{4}}}+\|\partial_zf\|^2_{X_{\tau,\kappa,\frac{\nu+1}{4}}}\|g\|^2_{X_{\tau,5,\frac{\nu+1}{4}}}\right), \tag{3.26}$$

$$\|fg\|^2_{X_{\tau,\kappa,\nu}}\lesssim_{\nu,\kappa}\langle t\rangle^{1/2}\left(\|f\|^2_{X_{\tau,5,\frac{\nu+1}{4}}}\|\partial_zg\|^2_{X_{\tau,\kappa,\frac{\nu+1}{4}}}+\|f\|^2_{X_{\tau,\kappa,\frac{\nu+1}{4}}}\|\partial_zg\|^2_{X_{\tau,5,\frac{\nu+1}{4}}}\right), \tag{3.27}$$

$$\|fg\|^2_{X_{\tau,\kappa,\nu}}\lesssim_{\nu,\kappa}\langle t\rangle^{1/2}\left(\|\partial_zf\|^2_{X_{\tau,5,\frac{\nu+1}{4}}}\|g\|^2_{X_{\tau,\kappa,\frac{\nu+1}{4}}}+\|f\|^2_{X_{\tau,\kappa,\frac{\nu+1}{4}}}\|\partial_zg\|^2_{X_{\tau,5,\frac{\nu+1}{4}}}\right). \tag{3.28}$$

Proof. We only present the proof of (3.26), since the other two are simiar. For simplicty, we write $f_{k,\kappa}$ to denote $f_{k,x,\kappa}$ or $f_{k,y,\kappa}$ and ∂_h^k to denote ∂_x^k or ∂_y^k if no confusion is caused. First, by using Leibniz formula, we see that

$$\begin{aligned}(fg)_{j,\kappa}=&M_{j,\kappa}\partial_h^j(fg)\\ =&\sum_{0\le k\le j}\frac{M_{j,\kappa}}{M_{k,\kappa}M_{j-k,\kappa}}\binom{j}{k}|f_{k,\kappa}||g_{j-k,\kappa}|\\ =&\sum_{0\le k\le j}\frac{1}{\tau}\left(\frac{j+1}{(k+1)(j-k+1)}\right)^\kappa\binom{j}{k}^{-1}|f_{k,\kappa}||g_{j-k,\kappa}|.\end{aligned}$$

Then by using (2.5), we can obtain that

$$|(fg)_{j,\kappa}|\lesssim \sum_{k=0}^{[(j+1)/2]}(k+1)^{-\kappa}|f_{k,\kappa}||g_{j-k,\kappa}|+\sum_{k=[(j+1)/2]+1}^{j}(j-k+1)^{-\kappa}|f_{k,\kappa}||g_{j-k,\kappa}|. \tag{3.29}$$

Using Minkowski inequality and Hölder inequality, we have

$$\begin{aligned}\|(fg)_{j,\kappa}\|_{L^2(\theta_{2\nu})}\le&\sum_{k=0}^{[(j+1)/2]}\|(k+1)^{-\kappa}f_{k,\kappa}g_{j-k,\kappa}\|_{L^2(\theta_{2\nu})}\\&+\sum_{k=[(j+1)/2]+1}^{j}\|(j-k+1)^{-\kappa}f_{k,\kappa}g_{j-k,\kappa}\|_{L^2(\theta_{2\nu})}\\\le&\sum_{k=0}^{[(j+1)/2]}\|(k+1)^{-\kappa}f_{k,\kappa}\|_{L^\infty_hL^2_z(\theta_\nu)}\|\theta_{\frac{\nu}{2}}g_{j-k,\kappa}\|_{L^2_hL^\infty_z}\\&+\sum_{k=[(j+1)/2]+1}^{j}\|f_{k,\kappa}\theta_{\frac{\nu}{2}}\|_{L^2_hL^\infty_z}\|(j-k+1)^{-\kappa}g_{j-k,\kappa}\|_{L^\infty_hL^2_z(\theta_\nu)}.\end{aligned}$$

Then using the following discrete Young's convolution inequality

$$\sum_{j=0}^{\infty}(\sum_{k=0}^{j}a_kb_{j-k})^2\le\left(\sum_{k=0}^{\infty}a_k\right)^2\left(\sum_{k=0}^{\infty}b_k^2\right), \tag{3.30}$$

we can obtain that

$$\begin{aligned}\|fg\|^2_{X_{\tau,\kappa,\nu}}=&\sum_{j\in\mathbb{N}}\|(fg)_{j,\kappa}\|^2_{L^2(\theta_{2\nu})}\\\le&\left(\sum_{k=0}^{\infty}(k+1)^{-\kappa}\|f_{k,\kappa}\|_{L^\infty_hL^2(\theta_\nu)}\right)^2\left(\sum_{k=0}^{\infty}\|\theta_{\nu/2}g_{k,\kappa}\|^2_{L^2_hL^\infty_z}\right)\\&+\left(\sum_{k=0}^{\infty}\|\theta_{\nu/2}f_{k,\kappa}\|^2_{L^2_hL^\infty_z}\right)\left(\sum_{k=1}^{\infty}(k+1)^{-\kappa}\|g_{k,\kappa}\|_{L^\infty_hL^2_v(\theta_\nu)}\right)^2.\end{aligned} \tag{3.31}$$

Before continuing estimates, we give three weighted Sobolev embedding inequalities which will be frequently used later on. For any f, decaying fast enough at z infinity, and $0\le\nu<1$, by using the Sobolev embedding in (x,y) variables, first we have

$$\|f_{k,\kappa}\|_{L^\infty_hL^2_z(\theta_\nu)}\le\sum_{i=0}^{2}\left\|\partial_h^if_{k,\kappa}\right\|_{L^2_hL^2_z(\theta_\nu)}\lesssim\sum_{i=k}^{k+2}(k+1)^4\|f_{i,\kappa}\|_{L^2(\theta_\nu)}. \tag{3.32}$$

Then using the fundamental formula of calculus, Hölder inequality and property of θ, we can obtain that

$$\begin{aligned}&\left\|\theta_{\nu/2}f_{k,\kappa}\right\|_{L^2_hL^\infty_z}\\=&\left\|\theta_{\nu/2}\int_z^\infty\partial_zf_{k,\kappa}d\bar z\right\|_{L^2_hL^\infty_z}\le\left\|\int_z^\infty\theta_{\frac{\nu-1}{4}}\theta_{\frac{\nu+1}{4}}\partial_zf_{k,\kappa}d\bar z\right\|_{L^2_hL^\infty_z}\\\lesssim_\nu&\langle t\rangle^{1/4}\left\|\theta_{\frac{\nu+1}{4}}\partial_zf_{k,\kappa}\right\|_{L^2}.\end{aligned} \tag{3.33}$$

Combining the above two, we can achieve that

$$\begin{aligned}\left\|\theta_{\nu/2}f_{k,\kappa}\right\|_{L_h^\infty L_z^\infty}\lesssim_\nu\langle t\rangle^{1/4}\left\|\theta_{\frac{\nu+1}{4}}\partial_z f_{k,\kappa}\right\|_{L_h^\infty L_z^2}&\\ \lesssim\langle t\rangle^{1/4}\sum_{i=k}^{k+2}(k+1)^4\|\theta_{\frac{\nu+1}{4}}\partial_z f_{i,\kappa}\|_{L^2}.&\end{aligned}$$

Inserting (3.32) and (3.33) into (3.31) and by using discrete Cauchy inequality, we can obtain that

$$\begin{aligned}\|fg\|_{X_{\tau,\kappa,\nu}}^2\lesssim&\langle t\rangle^{1/2}\left(\sum_{k=0}^\infty(k+1)^{-\kappa+4}\|f_{k,\kappa}\|_{L^2(\theta_\nu)}\right)^2\sum_{k=0}^\infty\|\partial_z g_k\|_{L^2(\theta_{\frac{\nu+1}{2}})}^2\\ &+\langle t\rangle^{1/2}\sum_{k=0}^\infty\|\partial_z f_{k,\kappa}\|_{L^2(\theta_{\frac{\nu+1}{2}})}^2\left(\sum_{k=0}^\infty(k+1)^{-\kappa+4}\|g_{k,\kappa}\|_{L^2(\theta_\nu)}\right)^2 dt\\ \lesssim&\langle t\rangle^{1/2}\sum_{k=0}^\infty(k+1)^{-2\kappa+10}\|f_{k,\kappa}\|_{L^2(\theta_{\frac{\nu+1}{2}})}^2\sum_{k=0}^\infty\|\partial_z g_k\|_{L^2(\theta_{\frac{\nu+1}{2}})}^2\\ &+\langle t\rangle^{1/2}\sum_{k=0}^\infty\|\partial_z f_{k,\kappa}\|_{L^2(\theta_{\frac{\nu+1}{2}})}^2\sum_{k=0}^\infty(k+1)^{-2\kappa+10}\|g_{k,\kappa}\|_{L^2(\theta_{\frac{\nu+1}{2}})}^2 dt,\end{aligned}$$

which is (3.26). □

4. Estimates of auxiliary functions $\mathcal{H}$

In this section, we give the proof of Proposition 3.1. We only proceed with the estimate for $\mathcal{H}$, while the estimate for $\widetilde{\mathcal{H}}$ follows along the same lines. First we derive the equation for $\mathcal{H}$.

4.1. Derivation of the equation of $\mathcal{H}$ and its linear estimate. Applying $-\partial_z$ to (3.1) and using the incompressibility, we can have

$$\begin{aligned}&\left[\partial_t+(u\partial_x+v\partial_y+w\partial_z)-\partial_z^2\right]\mathcal{H}\\ =&\sqrt{\epsilon}\langle t\rangle^{\delta-1}(\partial_x^2u+\partial_x\partial_y v)+\partial_z u\partial_x\int_z^{+\infty}\mathcal{H}d\bar z+\partial_z v\partial_y\int_z^{+\infty}\mathcal{H}d\bar z-\partial_z w\mathcal{H}\\ =&\sqrt{\epsilon}\langle t\rangle^{\delta-1}\partial_x(\partial_x u+\frac{\langle t\rangle^{1-\delta}}{\sqrt{\epsilon}}\partial_z u\int_z^{+\infty}\mathcal{H}d\bar z)-\partial_z\partial_x u\int_z^{+\infty}\mathcal{H}d\bar z\\ &+\sqrt{\epsilon}\langle t\rangle^{\delta-1}\partial_y(\partial_x v+\frac{\langle t\rangle^{1-\delta}}{\sqrt{\epsilon}}\partial_z v\int_z^{+\infty}\mathcal{H}d\bar z)-\partial_z\partial_y v\int_z^{+\infty}\mathcal{H}d\bar z+(\partial_x u+\partial_y v)\mathcal{H}\\ =&\sqrt{\epsilon}\langle t\rangle^{\delta-1}(\partial_x\mathcal{G}+\partial_y\mathcal{K})-(\partial_z\partial_x u+\partial_z\partial_y v)\int_z^{+\infty}\mathcal{H}d\bar z+(\partial_x u+\partial_y v)\mathcal{H}\\ :=&\sqrt{\epsilon}\langle t\rangle^{\delta-1}(\partial_x\mathcal{G}+\partial_y\mathcal{K})+H,\end{aligned}\tag{4.1}$$

where H is defined as

$$H:=-(\partial_z\partial_x u+\partial_z\partial_y v)\int_z^{+\infty}\mathcal{H}d\bar z+(\partial_x u+\partial_y v)\mathcal{H}.$$

From Section 4 to Section 6, we set $\kappa=10+\frac{2}{\delta}$ and $M_{j,\kappa}$ is abbreviated to M_j. Also for a function f, f_j denotes $f_{j,x,\kappa}$, $f_{j,y,\kappa}$ or $f_{\alpha,\kappa}$ for $|\alpha|=j$ if no confusion is caused.

From the equation of $\mathcal{H}$ in (4.1), we first have the following linear estimate.

LEMMA 4.1. *Under the assumption of Proposition 3.1, for sufficiently small ϵ, we have the following estimate*

$$
\begin{aligned}
&\langle t\rangle^{\frac{1-\delta}{2}}\|\mathcal{H}(t)\|^2_{X_{\tau,\kappa}}+\delta\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{H}(t)\|^2_{X_{\tau,\kappa}}dt+\lambda\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{H}(t)\|^2_{X_{\tau,\kappa+1/2}}dt\\
\le& C_\delta\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}\|\boldsymbol{\mathcal{G}}\|^2_{X_{\tau,\kappa+3/2}}dt+C_\delta C_*\epsilon\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{H}(t)\|^2_{X_{\tau,\kappa+1/2}}dt\\
&+\frac{C_\delta}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{3-3\delta}{2}}\sum_{j=0}^{\infty}(j+1)^{-1}\|M_j[u\partial_x+v\partial_y+w\partial_z,\partial_x^j]\mathcal{H}\|^2_{L^2(\theta_2)}dt\\
&+\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\sum_{j=0}^{\infty}|\langle H_j,\mathcal{H}_j\rangle_{\theta_2}|dt.
\end{aligned}\tag{4.2}
$$

Proof. Applying $M_j\partial_x^j$ to (4.1) implies that

$$
\begin{aligned}
&\partial_t\mathcal{H}_j+\lambda\sqrt{\epsilon}\eta(t)(j+1)\mathcal{H}_j+(u\partial_x+v\partial_y+w\partial_z)\mathcal{H}_j-\partial_z^2\mathcal{H}_j\\
=&M_j[u\partial_x+v\partial_y+w\partial_z,\partial_x^j]\mathcal{H}+\sqrt{\epsilon}\langle t\rangle^{\delta-1}(\partial_x\mathcal{G}_j+\partial_y\mathcal{K}_j)+H_j.
\end{aligned}\tag{4.3}
$$

Then multiplying (4.3) by $\mathcal{H}_j\theta_2$ and integrating the resulting equation in $\mathbb{R}^3_+$ indicate that

$$
\begin{aligned}
&\left\langle\left[\partial_t+\lambda\sqrt{\epsilon}\eta(t)(j+1)+(u\partial_x+v\partial_y+w\partial_z)-\partial_z^2\right]\mathcal{H}_j,\mathcal{H}_j(t)\right\rangle_{\theta_2}\\
=&\langle H_j,\mathcal{H}_j\rangle_{\theta_2}+\sqrt{\epsilon}\langle t\rangle^{\delta-1}\langle(\partial_x\mathcal{G}_j+\partial_y\mathcal{K}_j),\mathcal{H}_j\rangle_{\theta_2}+\left\langle M_j[u\partial_x+v\partial_y+w\partial_z,\partial_x^j]\mathcal{H},\mathcal{H}_j\right\rangle_{\theta_2}.
\end{aligned}\tag{4.4}
$$

For $\theta_2:=e^{\frac{z^2}{4\langle t\rangle}}$, we have

$$
-\frac{\partial_t\theta_2}{\theta_2}=\frac{z^2}{4\langle t\rangle},\quad -\frac{\partial_z\theta_2}{\theta_2}=-\frac{z}{2\langle t\rangle},\quad -\frac{\partial_z^2\theta_2}{\theta_2}=-\frac{1}{2\langle t\rangle}-\frac{z^2}{4\langle t\rangle}.
$$

Integration by parts indicates that the left-hand side of (4.4) satisfies

$$
\begin{aligned}
&\left\langle\left[\partial_t+\lambda\sqrt{\epsilon}\eta(t)(j+1)+(u\partial_x+v\partial_y+w\partial_z)-\partial_z^2\right]\mathcal{H}_j,\mathcal{H}_j(t)\right\rangle_{\theta_2}\\
=&\frac{1}{2}\frac{d}{dt}\|\mathcal{H}_j(t)\|^2_{L^2(\theta_2)}+\|\partial_z\mathcal{H}_j(t)\|^2_{L^2(\theta_2)}-\frac{1}{4\langle t\rangle}\|\mathcal{H}_j(t)\|^2_{L^2(\theta_2)}\\
&+(j+1)\lambda\sqrt{\epsilon}\eta(t)\|\mathcal{H}_{j,x}(t)\|^2_{L^2(\theta_2)}-\langle\frac{z}{4\langle t\rangle}w,\mathcal{H}_j^2\rangle_{\theta_2}.
\end{aligned}\tag{4.5}
$$

From (3.22) in Lemma 3.2, we have

$$
\|\partial_z\mathcal{H}_j(t)\|^2_{L^2(\theta_2)}\ge\frac{1}{2\langle t\rangle}\|\mathcal{H}_j(t)\|^2_{L^2(\theta_2)}.
$$

Inserting the above inequality and (4.5) into (4.4) implies that

$$
\begin{aligned}
&\frac{d}{dt}\|\mathcal{H}_j(t)\|^2_{L^2(\theta_2)}+\delta\|\partial_z\mathcal{H}_j(t)\|^2_{L^2(\theta_2)}+\frac{1-\delta}{2\langle t\rangle}\|\mathcal{H}_j(t)\|^2_{L^2(\theta_2)}+2(j+1)\lambda\sqrt{\epsilon}\eta(t)\|\mathcal{H}_j(t)\|^2_{L^2(\theta^2)}\\
\le&2\sqrt{\epsilon}\langle t\rangle^{\delta-1}\langle(\partial_x\mathcal{G}_j+\partial_y\mathcal{K}_j),\mathcal{H}_j\rangle_{\theta^2}+\langle\frac{z}{2\langle t\rangle}w,\mathcal{H}_j^2\rangle_{\theta_2}\\
&+2\left\langle M_j[u\partial_x+v\partial_y+w\partial_z,\partial_x^j]\mathcal{H},\mathcal{H}_{j,x}\right\rangle_{\theta^2}+2\langle H_j,\mathcal{H}_j\rangle_{\theta_2}.
\end{aligned}\tag{4.6}
$$

Multiplying (4.6) by $\langle t\rangle^{\frac{1-\delta}{2}}$ and then integrating from 0 to t for any $t\in(0,T_0]$, we can achieve that

$$\begin{aligned}
&\langle t\rangle^{\frac{1-\delta}{2}}\|\mathcal{H}_j(t)\|^2_{L^2(\theta_2)}+\delta\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{H}_j(t)\|^2_{L^2(\theta_2)}dt\\
&\quad+2\lambda\sqrt{\epsilon}(j+1)\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{H}_j(t)\|^2_{L^2(\theta_2)}dt\\
&\le 2\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}|\langle\partial_h\boldsymbol{\mathcal{G}}_j,\mathcal{H}_j\rangle_{\theta_2}|dt+\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\langle\frac{z}{2\langle t\rangle}w,\mathcal{H}_j^2\rangle_{\theta_2}dt\\
&\quad+2\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\left|\left\langle M_j[u\partial_x+v\partial_y+w\partial_z,\partial_x^j]\mathcal{H},\mathcal{H}_j\right\rangle_{\theta_2}\right|dt+2\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}|\langle H_j,\mathcal{H}_j\rangle_{\theta_2}|dt.
\end{aligned}\tag{4.7}$$

Using Cauchy inequality, we can get

The first and third terms of the right-hand side of (4.7)

$$\begin{aligned}
&\le\int_0^{T_0}\frac{\langle t\rangle^{\frac{1-\delta}{2}}}{(j+1)\lambda\sqrt{\epsilon}\eta(t)}\left(2\|M_j[u\partial_x+v\partial_y+w\partial_z,\partial_x^j]\mathcal{H}\|^2_{L^2(\theta_2)}\right)dt\\
&\quad+\sqrt{\epsilon}\int_0^{T_0}(j+1)^{-1}\langle t\rangle^{-\frac{1-\delta}{2}}\|\partial_h\boldsymbol{\mathcal{G}}_j\|^2_{L^2(\theta_2)}dt+(j+1)\lambda\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{H}_j(t)\|^2_{L^2(\theta_2)}dt.
\end{aligned}$$

Noting that $\partial_h\boldsymbol{\mathcal{G}}_j\approx_\delta(j+1)^2\boldsymbol{\mathcal{G}}_{j+1}$ and the a priori estimate in (3.9), then inserting the above inequality into (4.7) and summing the resulting equation over $j\in\mathbb{N}$, we can obtain

$$\begin{aligned}
&\langle t\rangle^{\frac{1-\delta}{2}}\|\mathcal{H}(t)\|^2_{X_{\tau,\kappa}}+\delta\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{H}(t)\|^2_{X_{\tau,\kappa}}dt+\lambda\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{H}(t)\|^2_{X_{\tau,\kappa+1/2}}dt\\
&\le C_\delta\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}\|\boldsymbol{\mathcal{G}}\|^2_{X_{\tau,\kappa+3/2}}dt+C_\delta C_*\epsilon\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}\|\mathcal{H}(t)\|^2_{X_{\tau,\kappa}}dt\\
&\quad+\frac{C_\delta}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{3-3\delta}{2}}\sum_{j=0}^{\infty}(j+1)^{-1}\|M_j[u\partial_x+v\partial_y+w\partial_z,\partial_x^j]\mathcal{H}\|^2_{L^2(\theta_2)}dt\\
&\quad+\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\sum_{j=0}^{\infty}\left|\langle H_j,\mathcal{H}_j\rangle_{\theta_2}\right|dt,
\end{aligned}$$

which is (4.2). ∎

4.2. Estimates of nonlinear terms for $\mathcal{H}$. Now we go to estimate the nonlinear terms on the right-hand side of (4.2), for which we have the following lemma.

LEMMA 4.2. *Under the assumption in (3.7), for sufficiently small ϵ, we have the following estimate*

$$\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{3-3\delta}{2}}\sum_{j=0}^{\infty}(j+1)^{-1}\|M_j[u\partial_x+v\partial_y+w\partial_z,\partial_x^j]\mathcal{H}\|^2_{L^2(\theta^2)}dt\\
&\lesssim_\delta\frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}\left(\|\boldsymbol{\mathcal{H}}\|^2_{X_{\tau,\kappa+1/2}}+\|\boldsymbol{u}\|^2_{X_{\tau,\kappa+5/2}}\right)dt\\
&\qquad+\frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\left(\|\partial_z\boldsymbol{\mathcal{H}}\|^2_{X_{\tau,\kappa}}+\|\partial_z\boldsymbol{u}\|^2_{X_{\tau,\kappa+2}}\right)dt,
\end{aligned}\tag{4.8}$$

and

$$\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\sum_{j=0}^{\infty}\left|\langle H_j,\mathcal{H}_j\rangle_{\theta_2}\right|dt\le\frac{\delta}{2}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{H}\|^2_{X_{\tau,\kappa}}dt+C_\delta\frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}\|\mathcal{H}\|^2_{X_{\tau,\kappa}}dt.\tag{4.9}$$

Combining estimates in Lemma 4.1 and Lemma 4.2, we finish the proof of Proposition 3.1. Now we give the proof of Lemma 4.2.

Proof. Recall that

$$
\begin{aligned}
&M_j[u\partial_x+v\partial_y+w\partial_z,\partial_x^j]\mathcal{H}\\
&=-M_j\sum_{k=1}^{j}\binom{j}{k}\left(\partial_x^k u\partial_x^{j-k+1}\mathcal{H}+\partial_x^k v\partial_x^{j-k}\partial_y\mathcal{H}+\partial_x^k w\partial_x^{j-k}\partial_z\mathcal{H}\right)\\
&:=I_j^1+I_j^2+I_j^3.
\end{aligned}
$$

We will estimate I^i $(i=1,2,3)$ term by term.

Estimate of term I^1 and I^2. Since I_1 and I_2 share the same estimate, we only care about term I_1 and handling of term I_2 follows along the same lines. For term I_j^1, noting that when $1\le k\le\left[\frac{j+1}{2}\right]\le j$, we have

$$
\binom{j}{k}^{-1}\le (j+1)^{-1}.
$$

Then similar as derivation of (3.29), we have

$$
\begin{aligned}
|I_j^1|\le&\sum_{k=1}^{[(j+1)/2]}(k+1)^{-\kappa}|u_k|(j-k+1)^{-1}|\partial_x\mathcal{H}_{j-k}|\\
&+\sum_{k=[(j+1)/2]+1}^{j}(j-k+1)^{-\kappa}|u_k||\partial_x\mathcal{H}_{j-k}|.
\end{aligned}\tag{4.10}
$$

By using (4.10), similar derivation as (3.27) in Lemma 3.3, we can obtain that

$$
\begin{aligned}
&\sum_{j\in\mathbb{N}}(j+1)^{-1}\|I_j^1(t)\|^2_{L^2(\theta_2)}dt\\
&\lesssim\langle t\rangle^{1/2}\left(\|\partial_z u\|^2_{X_{\tau,5,1/2}}\|\partial_x\mathcal{H}\|^2_{X_{\tau,\kappa-3/2,1/2}}+\|\partial_z u\|^2_{X_{\tau,\kappa-1/2,1/2}}\|\partial_x\mathcal{H}\|^2_{X_{\tau,5,1/2}}\right)\\
&\lesssim\langle t\rangle^{1/2}\left(\|\partial_z u\|^2_{X_{\tau,5,1/2}}\|\mathcal{H}\|^2_{X_{\tau,\kappa+1/2,1/2}}+\|\partial_z u\|^2_{X_{\tau,\kappa+2,1/2}}\|\mathcal{H}\|^2_{X_{\tau,7,1/2}}\right).
\end{aligned}\tag{4.11}
$$

Using a priori estimates in (3.8), we can obtain that

$$
\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{3-3\delta}{2}}\sum_{j\in\mathbb{N}}(j+1)^{-1}\|(I_j^1,I_j^2)(t)\|^2_{L^2(\theta_2)}dt\\
&\le\frac{C_*^2\epsilon^2}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\left(\langle t\rangle^{-\frac{1-\delta}{2}}\|\mathcal{H}\|^2_{X_{\tau,\kappa+1/2}}+\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z u\|^2_{X_{\tau,\kappa+2}}\right)dt\\
&\le\frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\left(\langle t\rangle^{-\frac{1-\delta}{2}}\|\mathcal{H}\|^2_{X_{\tau,\kappa+1/2}}+\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z u\|^2_{X_{\tau,\kappa+2}}\right)dt.
\end{aligned}\tag{4.12}
$$

Estimate of term I^3. For term I^3, we have

$$
|I_j^3|\lesssim\sum_{k=1}^{[(j+1)/2]}(k+1)^{-\kappa}(j-k+1)^{-1}|w_k||\partial_z\mathcal{H}_{j-k}|\quad+\sum_{k=[(j+1)/2]+1}^{j}(j-k+1)^{-\kappa}|w_k||\partial_z\mathcal{H}_{j-k}|.
$$

Using the above inequality, similar as (4.11) and using the incompressibility, we can obtain

$$
\begin{aligned}
&\sum_{j\in\mathbb{N}}(j+1)^{-1}\|I_j^3(t)\|_{L^2(\theta_2)}^2 dt\\
&\lesssim\langle t\rangle^{1/2}\left(\|\partial_z w\|_{X_{\tau,5,1/2}}^2\|\partial_z\mathcal{H}\|_{X_{\tau,\kappa-3/2,1/2}}^2+\|\partial_z w\|_{X_{\tau,\kappa-1/2,1/2}}^2\|\partial_z\mathcal{H}\|_{X_{\tau,5,1/2}}^2\right)\\
&\lesssim\langle t\rangle^{1/2}\left(\|(u,v)\|_{X_{\tau,7,1/2}}^2\|\partial_z\mathcal{H}\|_{X_{\tau,\kappa}}^2+\|(u,v)\|_{X_{\tau,\kappa+5/2}}^2\|\partial_z\mathcal{H}\|_{X_{\tau,5,1/2}}^2\right).
\end{aligned}\tag{4.13}
$$

Then using the a priori estimates in (3.8) and smallness of ϵ, we can obtain that

$$
\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{3-3\delta}{2}}\sum_{j=0}^{\infty}(j+1)^{-1}\|I_j^3(t)\|_{L^2(\theta^2)}^2 dt\\
&\lesssim\frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\left(\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{H}\|_{X_{\tau,\kappa}}^2+\langle t\rangle^{-\frac{1-\delta}{2}}\|(u,v)\|_{X_{\tau,\kappa+5/2}}^2\right)dt.
\end{aligned}\tag{4.14}
$$

Combining estimates in (4.12) and (4.14), we can obtain (4.8).

Estimate of term involving H_j. Next we estimate term involving H_j. Recall

$$
H_j=M_j\partial_x^j\left(-(\partial_z\partial_x u+\partial_z\partial_y v)\int_z^{+\infty}\mathcal{H}d\bar z+(\partial_x u+\partial_y v)\mathcal{H}\right).
$$

First, by integrating by parts on z and using Hölder inequality, we can have

$$
\begin{aligned}
\left|\langle H_j,\mathcal{H}_j\rangle_{\theta_2}\right|&=\left|\left\langle M_j\partial_x^j\left[(\partial_x u+\partial_y v)\int_z^{\infty}\mathcal{H}d\bar z\right],\partial_z\mathcal{H}_j+\mathcal{H}_j\frac{z}{2\langle t\rangle}\right\rangle_{\theta_2}\right|\\
&\lesssim\left\|M_j\partial_x^j\left[(\partial_x u+\partial_y v)\int_z^{\infty}\mathcal{H}d\bar z\right]\right\|_{L^2(\theta^2)}\|\partial_z H_j\|_{L^2(\theta_2)}.
\end{aligned}
$$

In the last line, we have used (3.23). Then using Cauchy inequality, we have

$$
\begin{aligned}
&\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\sum_{j=0}^{\infty}\left|\langle H_j,\mathcal{H}_j\rangle_{\theta_2}\right|dt\\
&\le\frac{\delta}{2}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{H}\|_{X_{\tau,\kappa}}^2 dt+C_\delta\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\left\|(\partial_x u+\partial_y v)\int_z^{\infty}\mathcal{H}d\bar z\right\|_{X_{\tau,\kappa}}^2 dt.
\end{aligned}
$$

By applying (3.27) in Lemma 3.3 and using the a priori estimate in (3.8), we obtain that

$$
\begin{aligned}
&\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\sum_{j=0}^{\infty}\left|\langle H_j,\mathcal{H}_j\rangle_{\theta_2}\right|dt\\
&\le\frac{\delta}{2}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{H}\|_{X_{\tau,\kappa}}^2 dt+C_\delta\int_0^{T_0}\langle t\rangle^{\frac{2-\delta}{2}}\left(\|\boldsymbol{u}\|_{X_{\tau,7,1/2}}^2\|\mathcal{H}\|_{X_{\tau,\kappa}}^2+\|\boldsymbol{u}\|_{X_{\tau,\kappa+2}}^2\|\mathcal{H}\|_{X_{\tau,5,1/2}}^2\right)dt\\
&\le\frac{\delta}{2}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{H}\|_{X_{\tau,\kappa}}^2 dt+C_\delta C_*^2\epsilon^2\int_0^{T_0}\left(\langle t\rangle^{-\frac{1}{2}}\|\mathcal{H}\|_{X_{\tau,\kappa}}^2+\langle t\rangle^{-\frac{1-2\delta}{2}}\|\boldsymbol{u}\|_{X_{\tau,\kappa+2}}^2\right)dt\\
&\le\frac{\delta}{2}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{H}\|_{X_{\tau,\kappa}}^2 dt+C_\delta\frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}\left(\|\mathcal{H}\|_{X_{\tau,\kappa+1/2}}^2+\|\boldsymbol{u}\|_{X_{\tau,\kappa+5/2}}^2\right)dt,
\end{aligned}
$$

which is (4.9). $\square$

5. Estimates of auxiliary functions $\mathcal{G}$

We only estimate $\mathcal{G}$ since the other three follow along the same lines.

5.1. Derivation of the equation of $\mathcal{G}$ and its linear estimate. From the first equation of (1.1), we can obtain that

$$\partial_t\partial_x u+(u\partial_x+v\partial_y+w\partial_z)\partial_x u-\partial_z^2\partial_x u=-\partial_x w\partial_z u-(\partial_x u)^2-\partial_x v\partial_y u. \tag{5.1}$$

Multiplying $\partial_z u$ to the first equation of (3.1) indicates that

$$\begin{aligned}&\left[\partial_t+(u\partial_x+v\partial_y+w\partial_z)-\partial_z^2\right]\left(\partial_z u\int_z^\infty \mathcal{H}d\bar{z}\right)\\ =&\sqrt{\epsilon}\langle t\rangle^{\delta-1}\partial_x w\partial_z u+[\partial_y v\partial_z u-\partial_y u\partial_z v]\int_z^\infty \mathcal{H}d\bar{z}+2(\partial_z^2 u)\mathcal{H}.\end{aligned} \tag{5.2}$$

Then by multiplying $\langle t\rangle^{1-\delta}\epsilon^{-1/2}$ to (5.2), we can obtain that

$$\begin{aligned}&\left[\partial_t+(u\partial_x+v\partial_y+w\partial_z)-\partial_z^2\right]\left(\frac{\langle t\rangle^{1-\delta}\partial_z u}{\sqrt{\epsilon}}\int_z^\infty \mathcal{H}d\bar{z}\right)-\partial_t\left(\frac{\langle t\rangle^{1-\delta}}{\sqrt{\epsilon}}\right)\left(\partial_z u\int_z^\infty \mathcal{H}d\bar{z}\right)\\ =&\partial_x w\partial_z u+\frac{\langle t\rangle^{1-\delta}[\partial_y v\partial_z u-\partial_y u\partial_z v]}{\sqrt{\epsilon}}\int_z^\infty \mathcal{H}d\bar{z}+\frac{2\langle t\rangle^{1-\delta}(\partial_z^2 u)}{\sqrt{\epsilon}}\mathcal{H}.\end{aligned} \tag{5.3}$$

Then adding (5.1) and (5.3) together implies that

$$\begin{aligned}&\left[\partial_t+(u\partial_x+v\partial_y+w\partial_z)-\partial_z^2\right]\mathcal{G}\\ =&-\left[(\partial_x u)^2+\partial_x v\partial_y u\right]+\frac{\langle t\rangle^{1-\delta}[\partial_y v\partial_z u-\partial_y u\partial_z v]}{\sqrt{\epsilon}}\int_z^\infty \mathcal{H}d\bar{z}+\frac{2\langle t\rangle^{1-\delta}(\partial_z^2 u)}{\sqrt{\epsilon}}\mathcal{H}\\ :=&K^1+K^2+K^3.\end{aligned} \tag{5.4}$$

Lemma 5.1. *Under the assumption in (3.7), for sufficiently small ϵ, we have the following estimate*

$$\begin{aligned}&\langle t\rangle^{\frac{1-\delta}{2}}\|\mathcal{G}(t)\|_{X_{\tau,\kappa+1}}^2+\delta\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{G}(t)\|_{X_{\tau,\kappa+1}}^2dt+\lambda\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{G}(t)\|_{X_{\tau,\kappa+3/2}}^2dt\\ \lesssim_\delta&\|\mathcal{G}(0)\|_{X_{\tau_0,\kappa+1}}^2+C_*\epsilon\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{G}(t)\|_{X_{\tau,\kappa+3/2}}^2dt\\ &+\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{3-3\delta}{2}}\sum_{j=0}^\infty(j+1)\|M_j[u\partial_x+v\partial_y+w\partial_z,\partial_x^j]\mathcal{G}\|_{L^2(\theta_2)}^2dt\\ &+\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{3-3\delta}{2}}\|(K^1,K^2)\|_{X_{\tau,\kappa+1/2}}^2dt+\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\sum_{j=0}^\infty(j+1)^2\left|\left\langle K_j^3,\mathcal{G}_j\right\rangle_{\theta_2}\right|dt.\end{aligned} \tag{5.5}$$

Proof. Applying $M_j\partial_x^j$ to (5.4) and still denoting $\mathcal{G}_{j,x}$ by $\mathcal{G}_j$, we can obtain that

$$\begin{aligned}&\partial_t\mathcal{G}_j+\lambda\sqrt{\epsilon}\eta(t)(j+1)\mathcal{G}_j+(u\partial_x+v\partial_y+w\partial_z)\mathcal{G}_j-\partial_z^2\mathcal{G}_j\\ =&M_j[u\partial_x+v\partial_y+w\partial_z,\partial_x^j]\mathcal{G}+K_j^1+K_j^2+K_j^3.\end{aligned}$$

Performing energy estimates similar as (4.7) and using the a priori estimates in (3.9), we can obtain that

$$\langle t\rangle^{\frac{1-\delta}{2}}(j+1)^2\|\mathcal{G}_j(t)\|_{L^2(\theta_2)}^2+\delta(j+1)^2\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{G}_j(t)\|_{L^2(\theta_2)}^2dt$$

$$
\begin{aligned}
&+\lambda\sqrt{\epsilon}(j+1)^3\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{G}_j(t)\|_{L^2(\theta_2)}^2dt\\
\lesssim_\delta&(j+1)^2\|\mathcal{G}_j(0)\|_{L^2(\theta_2)}^2+C_*\epsilon(j+1)^2\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}\eta(t)\|\mathcal{G}_j(t)\|_{L^2(\theta_2)}^2dt\\
&+\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{3-3\delta}{2}}(j+1)\Big(\|(K_j^1,K_j^2)(t)\|_{L^2(\theta_2)}^2\\
&\qquad\qquad+\|M_j[u\partial_x+v\partial_y+w\partial_z,\partial_x^j]\mathcal{G}\|_{L^2(\theta_2)}^2\Big)dt\\
&+\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}(j+1)^2\left|\langle K_j^3,\mathcal{G}_j\rangle_{L^2(\theta_2)}\right|dt.
\end{aligned}
$$

Summing the above inequality over $j\in\mathbb{N}$ indicates (5.5). ∎

5.2. Estimates of nonlinear terms for $\mathcal{G}$.

LEMMA 5.2. *Under the assumption in (3.7), for sufficiently small ϵ, we have the following estimate*

$$
\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{3-3\delta}{2}}\sum_{j=1}^{\infty}(j+1)\|M_j[u\partial_x+v\partial_y+w\partial_z,\partial_x^j]\mathcal{G}\|_{L^2(\theta_2)}^2dt\\
&+\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{3-3\delta}{2}}\|(K^1,K^2)\|_{X_{\tau,\kappa+1/2}}^2dt+\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\sum_{j=0}^{\infty}(j+1)^2\left|\langle K_j^3,\mathcal{G}_j\rangle_{\theta_2}\right|dt\\
\lesssim_\delta&\frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}\Big(\|\boldsymbol{\mathcal{H}}(t)\|_{X_{\tau,\kappa+1/2}}^2+\|\boldsymbol{\mathcal{G}}(t)\|_{X_{\tau,\kappa+3/2}}^2+\|\boldsymbol{u}(t)\|_{X_{\tau,\kappa+5/2}}^2\Big)dt\\
&+\frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\Big(\|\partial_z\boldsymbol{\mathcal{H}}(t)\|_{X_{\tau,\kappa}}^2+\|\partial_z\boldsymbol{\mathcal{G}}(t)\|_{X_{\tau,\kappa+1}}^2+\|\partial_z\boldsymbol{u}(t)\|_{X_{\tau,\kappa+2}}^2\Big)dt. \qquad (5.6)
\end{aligned}
$$

Combining estimates in Lemma 5.1 and Lemma 5.2, we finish the proof of Proposition 3.2. Now we give the proof of Lemma 5.2.

Proof. First, using Leibniz formula, we see that

$$
\begin{aligned}
&M_j[u\partial_x+v\partial_y+w\partial_z,\partial_x^j]\mathcal{H}\\
=&-M_j\sum_{k=1}^{j}\binom{j}{k}\left(\partial_x^ku\partial_x^{j-k+1}\mathcal{G}+\partial_x^kv\partial_x^{j-k}\partial_y\mathcal{G}+\partial_x^kw\partial_x^{j-k}\partial_z\mathcal{G}\right)\\
:=&L_j^1+L_j^2+L_j^3.
\end{aligned}
$$

Similar as (4.12), using (3.7) and the a priori estimates in Lemma 3.1, for sufficiently small ϵ, we have

$$
\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{3-3\delta}{2}}\sum_{j\in\mathbb{N}}(j+1)\|(L_j^1,L_j^2)(t)\|_{L^2(\theta_2)}^2dt\\
\lesssim_\delta&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{4-3\delta}{2}}\Big(\|\partial_zu\|_{X_{\tau,5,1/2}}^2\|\partial_x\mathcal{G}\|_{X_{\tau,\kappa-1/2,1/2}}^2+\|\partial_zu\|_{X_{\tau,\kappa+1/2,1/2}}^2\|\partial_x\mathcal{G}\|_{X_{\tau,5,1/2}}^2\Big)dt\\
\lesssim_\delta&\frac{C_*^2}{\lambda}\epsilon^{3/2}\int_0^{T_0}\Big(\langle t\rangle^{-\frac{1-\delta}{2}}\|\mathcal{G}\|_{X_{\tau,\kappa+3/2}}^2+\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_zu\|_{X_{\tau,\kappa+2}}^2\Big)dt\\
\lesssim_\delta&\frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\Big(\langle t\rangle^{-\frac{1-\delta}{2}}\|\mathcal{G}\|_{X_{\tau,\kappa+3/2}}^2+\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_zu\|_{X_{\tau,\kappa+2}}^2\Big)dt. \qquad (5.7)
\end{aligned}
$$

Same as (4.13), by using the incompressibility, we can obtain that

$$\begin{aligned}&\sum_{j\in\mathbb{N}}(j+1)\|L^3_j(t)\|^2_{L^2(\theta_2)}dt\\&\lesssim\langle t\rangle^{1/2}\left(\|\partial_z w\|^2_{X_{\tau,5,1/2}}\|\partial_z\mathcal{G}\|^2_{X_{\tau,\kappa-1/2,1/2}}+\|\partial_z w\|^2_{X_{\tau,\kappa+1/2,1/2}}\|\partial_z\mathcal{G}\|^2_{X_{\tau,5,1/2}}\right)\\&\lesssim\langle t\rangle^{1/2}\left(\|\boldsymbol{u}\|^2_{X_{\tau,7,1/2}}\|\partial_z\mathcal{G}\|^2_{X_{\tau,\kappa+1}}+\|\boldsymbol{u}\|^2_{X_{\tau,\kappa+5/2}}\|\partial_z\mathcal{G}\|^2_{X_{\tau,5,1/2}}\right).\end{aligned}\tag{5.8}$$

Then using (5.8) and the a priori estimates in Lemma 3.1, for sufficiently small ϵ, we see that

$$\begin{aligned}&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{3-3\delta}{2}}\sum_{j\in\mathbb{N}}(j+1)\|L^3_j(t)\|^2_{L^2(\theta_2)}dt\\&\le\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{4-3\delta}{2}}\left(\|\boldsymbol{u}\|^2_{X_{\tau,7,1/2}}\|\partial_z\mathcal{G}\|^2_{X_{\tau,\kappa+1}}+\|\boldsymbol{u}\|^2_{X_{\tau,\kappa+5/2}}\|\partial_z\mathcal{G}\|^2_{X_{\tau,5,1/2}}\right)dt\\&\lesssim\frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\left(\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{G}\|^2_{X_{\tau,\kappa+1}}+\langle t\rangle^{-\frac{1-\delta}{2}}\|\boldsymbol{u}\|^2_{X_{\tau,\kappa+5/2}}\right)dt.\end{aligned}\tag{5.9}$$

Estimates of K^1_j. Remembering the representation of K^1 in (5.4) and using the product estimate in (3.28) and a priori estimates (3.8) in Lemma 3.1, we have

$$\begin{aligned}&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{3-3\delta}{2}}\|K^1_j(t)\|^2_{X_{\tau,\kappa+1/2}}dt=\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{3-3\delta}{2}}\|\partial_h\boldsymbol{u}\partial_h\boldsymbol{u}\|^2_{X_{\tau,\kappa+1/2}}dt\\&\lesssim_\delta\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{4-3\delta}{2}}\|\partial_z\partial_h\boldsymbol{u}\|^2_{X_{\tau,5,1/2}}\|\partial_h\boldsymbol{u}\|^2_{X_{\tau,\kappa+1/2,1/2}}dt\\&\lesssim_\delta\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{4-3\delta}{2}}\|\partial_z\boldsymbol{u}\|^2_{X_{\tau,7,1/2}}\|\boldsymbol{u}\|^2_{X_{\tau,\kappa+5/2}}dt\\&\lesssim_\delta\frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}\|\boldsymbol{u}\|^2_{X_{\tau,\kappa+5/2}}dt.\end{aligned}\tag{5.10}$$

Estimates of K^2_j. Remembering the representation of K^2 in (5.4) and using the product estimate in (3.27) and (3.28), we have

$$\begin{aligned}&\|K^2_j(t)\|^2_{X_{\tau,\kappa+1/2}}=\langle t\rangle^{2-2\delta}\epsilon^{-1}\left\|\partial_h\boldsymbol{u}\partial_z\boldsymbol{u}\int_z^\infty\mathcal{H}d\bar{z}\right\|^2_{X_{\tau,\kappa+1/2}}dt\\&\lesssim_\delta\langle t\rangle^{5/2-2\delta}\epsilon^{-1}\left(\left\|\partial_z\left(\partial_z\boldsymbol{u}\int_z^\infty\mathcal{H}d\bar{z}\right)\right\|^2_{X_{\tau,5,1/2}}\|\partial_h\boldsymbol{u}\|^2_{X_{\tau,\kappa+1/2,1/2}}\right.\\&\qquad\left.+\|\partial_z\partial_h\boldsymbol{u}\|^2_{X_{\tau,5,1/2}}\|\partial_z\boldsymbol{u}\int_z^\infty\mathcal{H}\|^2_{X_{\tau,\kappa+1/2,1/2}}\right)\\&\lesssim\langle t\rangle^{3-2\delta}\epsilon^{-1}\left(\|\partial_z^2\boldsymbol{u}\|^2_{X_{\tau,5,3/8}}\|\mathcal{H}\|^2_{X_{\tau,5,3/8}}\|\boldsymbol{u}\|^2_{X_{\tau,\kappa+5/2,1/2}}\right.\\&\qquad\left.+\|\partial_z\boldsymbol{u}\|^2_{X_{\tau,7,3/4}}\left(\|\partial_z\boldsymbol{u}\|^2_{X_{\tau,5,3/8}}\|\mathcal{H}\|^2_{X_{\tau,\kappa+1/2,3/8}}+\|\mathcal{H}\|^2_{X_{\tau,5,3/8}}\|\partial_z\boldsymbol{u}\|^2_{X_{\tau,\kappa+1/2,3/8}}\right)\right).\end{aligned}$$

Then using the a priori estimates in (3.8), for sufficiently small ϵ, we can obtain that

$$\begin{aligned}&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{3-3\delta}{2}}\sum_{j=0}^\infty(j+1)\|K^2_j(t)\|^2_{L^2(\theta^2)}dt\\&\lesssim_\delta\frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\left(\langle t\rangle^{-\frac{1-\delta}{2}}\left(\|\boldsymbol{\mathcal{H}}\|^2_{X_{\tau,\kappa+1/2}}+\|\boldsymbol{u}\|^2_{X_{\tau,\kappa+5/2}}\right)+\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\boldsymbol{u}\|^2_{X_{\tau,\kappa+1/2}}\right)dt.\end{aligned}\tag{5.11}$$

Estimates of K_j^3. First we claim that for $0<\nu\leq 1$ and $\kappa>0$,

$$\sum_{j=0}^{\infty}\|(fg)_{j,\kappa}\|^2_{L^2_hL^1_z(\theta_{2\nu})}\lesssim\|f\|^2_{X_{\tau,5,\nu}}\|g\|^2_{X_{\tau,\kappa,\nu}}+\|f\|^2_{X_{\tau,\kappa,\nu}}\|g\|^2_{X_{\tau,5,\nu}}. \tag{5.12}$$

The proof of (5.12) is essentially the same as that in Lemma 3.3, here we omit the details.

For term K^3, we decompose it as the following.

$$\begin{aligned}K_j^3=&2\frac{\langle t\rangle^{1-\delta}}{\sqrt{\epsilon}}\partial_z^2u\mathcal{H}_j+2\frac{\langle t\rangle^{1-\delta}}{\sqrt{\epsilon}}\sum_{1\leq k\leq j}\frac{M_j}{M_kM_{j-k}}\binom{j}{k}\partial_z^2u_k\mathcal{H}_{j-k}\\:=&K^3_{j,\mathrm{low}}+K^3_{j,\mathrm{other}}.\end{aligned}$$

Then by using Cauchy inequality and the a priori estimates in Lemma 3.1, we have

$$\begin{aligned}&\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\sum_{j=0}^{\infty}(j+1)^2\left|\langle K^3_{j,\mathrm{low}},\mathcal{G}_j\rangle_{\theta_2}\right|dt\\\leq&\frac{C_\delta}{\lambda\epsilon^{3/2}}\int_0^{T_0}\langle t\rangle^{\frac{7-7\delta}{2}}\|\partial_z^2u\|^2_{L^\infty}\|\mathcal{H}\|^2_{X_{\tau,\kappa+1/2}}dt+\frac{\lambda\sqrt{\epsilon}}{2}\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}\|\mathcal{G}\|^2_{X_{\tau,\kappa+3/2}}dt\\\leq&C_\delta\frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}\|\mathcal{H}\|^2_{X_{\tau,\kappa+1/2}}dt+\frac{\lambda\sqrt{\epsilon}}{2}\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}\|\mathcal{G}\|^2_{X_{\tau,\kappa+3/2}}dt.\end{aligned}\tag{5.13}$$

By using integration by parts on z, we have that

$$\begin{aligned}|\langle K^3_{j,\mathrm{other}},\mathcal{G}_j\rangle_{\theta_2}|\leq&2\frac{\langle t\rangle^{1-\delta}}{\sqrt{\epsilon}}\left|\left\langle\sum_{1\leq k\leq j}\frac{M_j}{M_kM_{j-k}}\binom{j}{k}\partial_zu_k\partial_z\mathcal{H}_{j-k},\mathcal{G}_j\right\rangle_{\theta_2}\right|\\&+2\frac{\langle t\rangle^{1-\delta}}{\sqrt{\epsilon}}\left|\left\langle\sum_{1\leq k\leq j}\frac{M_j}{M_kM_{j-k}}\binom{j}{k}\partial_zu_k\mathcal{H}_{j-k},\partial_z\mathcal{G}_j+\frac{z}{2\langle t\rangle}\mathcal{G}_j\right\rangle_{\theta_2}\right|.\end{aligned}$$

Using Hölder inequality and (3.33), (3.23), we can obtain that

$$\begin{aligned}&\sum_{j=0}^{\infty}(j+1)^2|\langle K_{3,j,\mathrm{other}},\mathcal{G}_j\rangle_{\theta^2}|\\\lesssim&2\frac{\langle t\rangle^{1-\delta}}{\sqrt{\epsilon}}\sum_{j=0}^{\infty}(j+1)^2\left\|\sum_{1\leq k\leq j}\frac{M_j}{M_kM_{j-k}}\binom{j}{k}\partial_zu_k\partial_z\mathcal{H}_{j-k}\right\|_{L^2_hL^1_z(\theta_{3/2})}\langle t\rangle^{1/4}\|\partial_z\mathcal{G}_j\|_{L^2(\theta)}\\&+2\frac{\langle t\rangle^{1-\delta}}{\sqrt{\epsilon}}\sum_{j=0}^{\infty}(j+1)^2\left\|\sum_{1\leq k\leq j}\frac{M_j}{M_kM_{j-k}}\binom{j}{k}\partial_zu_k\mathcal{H}_{j-k}\right\|_{L^2(\theta_2)}\|\partial_z\mathcal{G}_j\|_{L^2_z(\theta_2)}.\end{aligned}\tag{5.14}$$

The same as before, it is easy to see that

$$\begin{aligned}\left|\sum_{1\leq k\leq j}\frac{M_j}{M_kM_{j-k}}\binom{j}{k}\partial_zu_k\partial_z\mathcal{H}_{j-k}\right|\lesssim&\sum_{k=1}^{[(j+1)/2]}(k+1)^{-\kappa}|\partial_zu_k|(j-k+1)^{-1}|\partial_z\mathcal{H}_{j-k}|\\&+\sum_{k=[(j+1)/2]+1}^{j}(j-k+1)^{-\kappa}|\partial_zu_k||\partial_z\mathcal{H}_{j-k}|.\end{aligned}$$

Using Minkowski inequality, Sobolev embedding, discrete young inequality in (3.30), and a priori estimates in (3.8) and (3.9), we have

$$\begin{aligned}&\sum_{j=0}^{\infty}(j+1)^2\left\|\sum_{1\le k\le j}\frac{M_j}{M_kM_{j-k}}\binom{j}{k}\partial_z u_k\partial_z\mathcal{H}_{j-k}\right\|^2_{L^2_hL^1_z(\theta_{3/2})}\\ \lesssim&\|\partial_z u_k\|^2_{X_{\tau,5,3/4}}\|\partial_z\mathcal{H}\|^2_{X_{\tau,\kappa,3/4}}+\|\partial_z u\|^2_{X_{\tau,\kappa+1,3/4}}\|\partial_z\mathcal{H}\|^2_{X_{\tau,5,3/4}}\\ \lesssim&C_*^2\epsilon^2\langle t\rangle^{-\frac{5-3\delta}{2}}\left(\|\partial_z\mathcal{H}\|^2_{X_{\tau,\kappa}}+\|\partial_z u\|^2_{X_{\tau,\kappa+2}}\right),\end{aligned}\tag{5.15}$$

and similar as the proof of Lemma 3.3, we have

$$\begin{aligned}&\sum_{j=0}^{\infty}(j+1)^2\left\|\sum_{1\le k\le j}\frac{M_j}{M_kM_{j-k}}\binom{j}{k}\partial_z u_k\mathcal{H}_{j-k}\right\|^2_{L^2(\theta_{3/2})}\\ \lesssim&\langle t\rangle^{1/2}\|\partial_z u_k\|^2_{X_{\tau,5,7/16}}\|\partial_z\mathcal{H}\|^2_{X_{\tau,\kappa,7/16}}+\langle t\rangle^{1/2}\|\partial_z u\|^2_{X_{\tau,\kappa+1,7/16}}\|\partial_z\mathcal{H}\|^2_{X_{\tau,5,7/16}}\\ \lesssim&C_*^2\epsilon^2\langle t\rangle^{-\frac{4-3\delta}{2}}\left(\|\partial_z\mathcal{H}\|^2_{X_{\tau,\kappa}}+\|\partial_z u\|^2_{X_{\tau,\kappa+2}}\right).\end{aligned}\tag{5.16}$$

By using Young inequality to (5.14) and inserting (5.13) and (5.16) into the resulting inequality, we can obtain that

$$\begin{aligned}&\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\sum_{j=0}^{\infty}(j+1)^2|\langle K^3_{j,\mathrm{other}},\mathcal{G}_j\rangle_{\theta^2}|dt\\ \le&\frac{\delta}{4}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{G}\|^2_{X_{\tau,\kappa+1}}dt\\ &+\frac{C_\delta}{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{6-5\delta}{2}}\sum_{j=0}^{\infty}(j+1)^2\left\|\sum_{1\le k\le j}\frac{M_j}{M_kM_{j-k}}\binom{j}{k}\partial_z u_k\partial_z\mathcal{H}_{j-k}\right\|^2_{L^2_hL^1_z(\theta^{3/2})}dt\\ &+\frac{C_\delta}{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{5-5\delta}{2}}\sum_{j=0}^{\infty}(j+1)^2\left\|\sum_{1\le k\le j}\frac{M_j}{M_kM_{j-k}}\binom{j}{k}\partial_z u_k\mathcal{H}_{j-k}\right\|^2_{L^2(\theta_2)}dt\\ \le&\frac{\delta}{4}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{G}\|_{X_{\tau,\kappa+1}}dt+C_\delta C_*^2\epsilon\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\left(\|\partial_z\mathcal{H}\|^2_{X_{\tau,\kappa}}+\|\partial_z u\|^2_{X_{\tau,\kappa+2}}\right)dt.\end{aligned}\tag{5.17}$$

Combining the estimates in (5.7), (5.9), (5.10), (5.11) and (5.17), we can obtain (5.6). □

6. Estimates of the unknowns u

We only derive estimates of u since the estimate of v follows along the same lines. Due to the fact that the equation of u_j have one order derivative less, direct Gevrey-2 energy estimates on the equations of u_j do not work. Instead, we will use another alternative quantity ψ_j, defined as

$$\psi_j:=u_j+\frac{\langle t\rangle^{1-\delta}\partial_z u}{\sqrt{\epsilon}}M_j\int_z^{\infty}\partial_x^{j-1}\mathcal{H}d\bar{z},\tag{6.1}$$

to perform energy estimates, which has no derivative loss. Combining estimates of $\mathcal{H}_{j-1}$ and ψ_j, we can achieve estimates of u_j. Then Proposition 3.3 follows.

6.1. The equation of an equivalent quantity and its linear estimate. First, we derive the equation of ψ_j. From the first equation of (1.1), we can obtain that

$$\begin{aligned}
&\partial_t\partial_x^j u+(u\partial_x+v\partial_y+w\partial_z)\partial_x^j u-\partial_z^2\partial_x^j u\\
=&[u\partial_x+v\partial_y+w\partial_z,\partial_x^j]u\\
=&-\partial_x^j w\partial_z u-\sum_{k=1}^{j}\binom{j}{k}\left(\partial_x^k u\partial_x^{j-k+1}u+\partial_x^k v\partial_x^{j-k}\partial_y u\right)-\sum_{k=1}^{j-1}\binom{j}{k}\partial_x^k w\partial_x^{j-k}\partial_z u\\
:=&-\partial_x^j w\partial_z u+O^1+O^2+O^3.
\end{aligned}\tag{6.2}$$

Also by applying $\partial_z u\partial_x^{j-1}$ to the first equation of (3.1), we can obtain that

$$\begin{aligned}
&\left[\partial_t+(u\partial_x+v\partial_y+w\partial_z)-\partial_z^2\right]\left(\partial_z u\int_z^\infty \partial_x^{j-1}\mathcal{H}d\bar z\right)\\
=&\sqrt{\epsilon}\langle t\rangle^{\delta-1}\partial_x^j w\partial_z u-\partial_z u\sum_{k=1}^{j-1}\binom{j-1}{k}\left(\partial_x^k u\int_z^\infty \partial_x^{j-k}\mathcal{H}d\bar z+\partial_x^k v\int_z^\infty \partial_x^{j-1-k}\partial_y\mathcal{H}d\bar z\right)\\
&+\partial_z u\sum_{k=1}^{j-1}\binom{j-1}{k}\partial_x^k w\partial_x^{j-1-k}\mathcal{H}+[\partial_y v\partial_z u-\partial_y u\partial_z v]\int_z^\infty \partial_x^{j-1}\mathcal{H}d\bar z+2(\partial_z^2 u)\partial_x^{j-1}\mathcal{H}.
\end{aligned}$$

Dividing the above equation by $\sqrt{\epsilon}\langle t\rangle^{\delta-1}$, we can obtain that

$$\begin{aligned}
&\left[\partial_t+(u\partial_x+v\partial_y+w\partial_z)-\partial_z^2\right]\left(\frac{\langle t\rangle^{1-\delta}\partial_z u}{\sqrt{\epsilon}}\int_z^\infty \partial_x^{j-1}\mathcal{H}d\bar z\right)\\
&-\partial_t\left(\frac{\langle t\rangle^{1-\delta}}{\sqrt{\epsilon}}\right)\left(\partial_z u\int_z^\infty \partial_x^{j-1}\mathcal{H}d\bar z\right)\\
=&\partial_x^j w\partial_z u-\frac{\langle t\rangle^{1-\delta}\partial_z u}{\sqrt{\epsilon}}\sum_{k=1}^{j-1}\binom{j-1}{k}\left(\partial_x^k u\int_z^\infty \partial_x^{j-k}\mathcal{H}d\bar z+\partial_x^k v\int_z^\infty \partial_x^{j-1-k}\partial_y\mathcal{H}d\bar z\right)\\
&+\frac{\langle t\rangle^{1-\delta}\partial_z u}{\sqrt{\epsilon}}\sum_{k=1}^{j-1}\binom{j-1}{k}\partial_x^k w\partial_x^{j-1-k}\mathcal{H}\\
&+\frac{[\partial_y v\partial_z u-\partial_y u\partial_z v]}{\sqrt{\epsilon}\langle t\rangle^{\delta-1}}\int_z^\infty \partial_x^{j-1}\mathcal{H}d\bar z+\frac{2(\partial_z^2 u)}{\sqrt{\epsilon}\langle t\rangle^{\delta-1}}\partial_x^{j-1}\mathcal{H}\\
:=&\partial_x^j w\partial_z u+\sum_{i=1}^5 P^i.
\end{aligned}\tag{6.3}$$

Then adding (6.3) and (6.2) together implies that

$$\begin{aligned}
&\left[\partial_t+(u\partial_x+v\partial_y+w\partial_z)-\partial_z^2\right]\left(\partial_x^j u+\frac{\partial_z u}{\sqrt{\epsilon}\langle t\rangle^{\delta-1}}\int_z^\infty \partial_x^{j-1}\mathcal{H}d\bar z\right)\\
=&\partial_t\left(\frac{1}{\sqrt{\epsilon}\langle t\rangle^{\delta-1}}\right)\left(\partial_z u\int_z^\infty \partial_x^{j-1}\mathcal{H}d\bar z\right)+\sum_{i=1}^3 O^i+\sum_{i=1}^5 P^i.
\end{aligned}\tag{6.4}$$

Remembering the definition of ψ_j in (6.1), by multiplying M_j to (6.4), we can obtain that

$$\begin{aligned}
&\left[\partial_t+\lambda\sqrt{\epsilon}(j+1)+(u\partial_x+v\partial_y+w\partial_z)-\partial_z^2\right]\psi_j\\
=&M_j\partial_t\left(\frac{\langle t\rangle^{1-\delta}}{\sqrt{\epsilon}}\right)\left(\partial_z u\int_z^\infty \partial_x^{j-1}\mathcal{H}d\bar z\right)+\sum_{i=1}^3 O_j^i+\sum_{i=1}^5 P_j^i,
\end{aligned}\tag{6.5}$$

where $O^i_j := M_j O^i$ and $P^i_j := M_j P^i$.

There is no derivative loss for the Equation (6.5). For $\alpha \geq 0$, denote

$$\|\psi\|^2_{X_{\tau,\kappa+\alpha}} := \sum_{j=0}^{\infty} (j+1)^{2\alpha} \|\psi_j\|^2_{L^2(\theta_2)}.$$

We have the following linear estimate.

LEMMA 6.1. *Under the assumption in (3.7), for sufficiently small ϵ, we have the following estimate*

$$\begin{aligned}
&\langle t\rangle^{\frac{1-\delta}{2}} \|\psi(t)\|^2_{X_{\tau,\kappa+2}} + \delta \int_0^{T_0} \langle t\rangle^{\frac{1-\delta}{2}} \|\partial_z \psi(t)\|^2_{X_{\tau,\kappa+2}} dt + \lambda\sqrt{\epsilon} \int_0^{T_0} \langle t\rangle^{\frac{1-\delta}{2}} \eta(t) \|\psi(t)\|^2_{X_{\tau,\kappa+5/2}} dt \\
&\lesssim_\delta \|u(0)\|^2_{X_{\tau_0,\kappa+2}} + C_* \epsilon \int_0^{T_0} \langle t\rangle^{\frac{1-\delta}{2}} \eta(t) \|\psi(t)\|^2_{X_{\tau,\kappa+5/2}} dt \\
&\quad + \frac{1}{\lambda\sqrt{\epsilon}} \int_0^{T_0} \langle t\rangle^{\frac{3-3\delta}{2}} \sum_{j=0}^{\infty} (j+1)^3 \left\| (\sum_{i=1}^{3} O^i_j + \sum_{i=1}^{5} P^i_j) \right\|^2_{L^2(\theta_2)} dt \\
&\quad + \int_0^{T_0} \langle t\rangle^{\frac{1-3\delta}{2}} \sum_{j=0}^{\infty} (j+1)^4 \left\langle M_j \frac{\partial_z u}{\sqrt{\epsilon}} \int_z^{\infty} \partial_x^{j-1} \mathcal{H} d\bar{z}, \psi_j \right\rangle_{\theta_2} dt. \qquad (6.6)
\end{aligned}$$

Proof. Performing energy estimates for (6.5) similar as (4.7) and using Cauchy inequality, we can have

$$\begin{aligned}
&\langle t\rangle^{\frac{1-\delta}{2}} (j+1)^4 \|\psi_j(t)\|^2_{L^2(\theta_2)} + \delta (j+1)^4 \int_0^{T_0} \langle t\rangle^{\frac{1-\delta}{2}} \|\partial_z \psi_j(t)\|^2_{L^2(\theta_2)} dt \\
&\quad + \lambda\sqrt{\epsilon} (j+1)^5 \int_0^{T_0} \langle t\rangle^{\frac{1-\delta}{2}} \eta(t) \|\psi_j(t)\|^2_{L^2(\theta_2)} dt \\
&\lesssim_\delta (j+1)^4 \int_0^{T_0} \langle t\rangle^{\frac{1-\delta}{2}} \langle \frac{z}{2\langle t\rangle} w, \psi_j^2(t) \rangle_{\theta^2} dt + \frac{1}{\lambda\sqrt{\epsilon}} \int_0^{T_0} \langle t\rangle^{\frac{3-3\delta}{2}} (j+1)^3 \|O^i_j, P^i_j\|^2_{L^2(\theta_2)} dt \\
&\quad + (j+1)^4 \int_0^{T_0} \langle t\rangle^{\frac{1-3\delta}{2}} \left\langle M_j \frac{\partial_z u}{\sqrt{\epsilon}} \int_z^{\infty} \partial_x^{j-1} \mathcal{H} d\bar{z}, \psi_j \right\rangle_{\theta_2} dt. \qquad (6.7)
\end{aligned}$$

Then summing (6.7) over $j \in \mathbb{N}$ and using (3.9) to bound the first term of the right-hand side, we can achieve (6.6). ∎

6.2. Estimates of nonlinear terms.

LEMMA 6.2. *Under the assumption in (3.7), for sufficiently small ϵ, we have the following estimate*

$$\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}} \int_0^{T_0} \langle t\rangle^{\frac{3-3\delta}{2}} \sum_{j=0}^{\infty} (j+1)^3 \left\| (\sum_{i=1}^{3} O^i_j + \sum_{i=1}^{5} P^i_j) \right\|^2_{L^2(\theta_2)} dt \\
&\quad + \int_0^{T_0} \langle t\rangle^{\frac{1-3\delta}{2}} \sum_{j=0}^{\infty} (j+1)^4 \left\langle M_j \frac{\partial_z u}{\sqrt{\epsilon}} \int_z^{\infty} \partial_x^{j-1} \mathcal{H} d\bar{z}, \psi_j \right\rangle_{\theta_2} dt \\
&\leq C_\delta \frac{C_*^2}{\lambda} \sqrt{\epsilon} \int_0^{T_0} \langle t\rangle^{-\frac{1-\delta}{2}} \left(\|\boldsymbol{u}(t)\|^2_{X_{\tau,\kappa+5/2}} + \|\boldsymbol{\mathcal{H}}(t)\|^2_{X_{\tau,\kappa+1/2}} \right) dt \\
&\quad + C_\delta \frac{C_*^2}{\lambda} \sqrt{\epsilon} \int_0^{T_0} \langle t\rangle^{\frac{1-\delta}{2}} \left(\|\partial_z \boldsymbol{u}(t)\|^2_{X_{\tau,\kappa+2}} + \|\partial_z \boldsymbol{\mathcal{H}}(t)\|^2_{X_{\tau,\kappa}} \right) dt + \frac{\lambda\sqrt{\epsilon}}{3} \int_0^{T_0} \langle t\rangle^{-\frac{1-\delta}{2}} \|\psi\|^2_{X_{\tau,\kappa+5/2}} dt. \qquad (6.8)
\end{aligned}$$

Proof. First, using Hölder inequality, a priori estimates in (3.9) and noting that $(j+1)^2 M_j \approx_\delta M_{j-1}$, we have

$$\begin{aligned}
&\int_0^{T_0} \langle t\rangle^{\frac{1-3\delta}{2}} \sum_{j=0}^{\infty}(j+1)^4 \left\langle \frac{\partial_z u}{\sqrt{\epsilon}} \int_z^\infty M_j \partial_x^{j-1}\mathcal{H} d\bar{z}, \psi_j \right\rangle_{\theta^2} dt \\
\leq & \int_0^{T_0} \langle t\rangle^{\frac{1-3\delta}{2}} \sum_{j=0}^{\infty}(j+1)^2 \left\| \frac{\partial_z u}{\sqrt{\epsilon}} \right\|_{L_h^\infty L_z^2(\theta^2)} \left\| \int_z^\infty \mathcal{H}_{j-1} d\bar{z} \right\|_{L_h^2 L_z^\infty} \|\psi_j\|_{L^2(\theta_2)} dt \\
\lesssim_\delta & \int_0^{T_0} \langle t\rangle^{\frac{3-6\delta}{4}} \sum_{j=0}^{\infty}(j+1)^2 \left\| \frac{\partial_z u}{\sqrt{\epsilon}} \right\|_{L_h^\infty L_z^2(\theta^2)} \|\mathcal{H}_{j-1}\|_{L^2(\theta_2)} \|\psi_j\|_{L^2(\theta_2)} dt \\
\leq & C_\delta \frac{C_*^2\sqrt{\epsilon}}{\lambda} \int_0^{T_0} \langle t\rangle^{-\frac{1-\delta}{2}} \|\mathcal{H}\|^2_{X_{\tau,\kappa+1/2}} dt + \frac{\lambda\sqrt{\epsilon}}{3} \int_0^{T_0} \langle t\rangle^{-\frac{1-\delta}{2}} \|\psi\|^2_{X_{\tau,\kappa+5/2}} dt.
\end{aligned} \tag{6.9}$$

Estimates of O_j^1, O_j^2. Since O_j^1 and O_j^2 share the same estimate, we only care about O_j^1. Noting that

$$\begin{aligned}
|O_j^1| \leq & \sum_{k=1}^{[(j+1)/2]} (k+1)^{-\kappa}(j-k+1)^{-1}|u_k||\partial_h u_{j-k}| \\
&+ \sum_{k=[(j+1)/2]+1}^{j} (j-k+1)^{-\kappa}|u_k||\partial_h u_{j-k}|,
\end{aligned}$$

then similar as product estimates in (3.26) to (3.28) and using the a priori estimates in Lemma 3.1, we have

$$\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}} \int_0^{T_0} \langle t\rangle^{\frac{3-3\delta}{2}} \sum_{j=0}^{\infty}(j+1)^3 \|(O_j^1, O_j^2)(t)\|^2_{L^2(\theta_2)} dt \\
\lesssim & \frac{1}{\lambda\sqrt{\epsilon}} \int_0^{T_0} \langle t\rangle^{\frac{4-3\delta}{2}} \left(\|\partial_z u\|^2_{X_{\tau,5,1/2}} \|\partial_x u\|^2_{X_{\tau,\kappa+1/2,1/2}} + \|\partial_z u\|^2_{X_{\tau,\kappa+3/2,1/2}} \|\partial_x u\|^2_{X_{\tau,5,1/2}} \right) dt \\
\lesssim_\delta & \frac{C_*^2 \epsilon^{3/2}}{\lambda} \int_0^{T_0} \left(\langle t\rangle^{-\frac{1-\delta}{2}} \|u\|^2_{X_{\tau,\kappa+5/2}} + \langle t\rangle^{\frac{1-\delta}{2}} \|\partial_z u\|^2_{X_{\tau,\kappa+2}} \right) dt.
\end{aligned} \tag{6.10}$$

Estimates of O_j^3. Noting that

$$\begin{aligned}
|O^3| \lesssim_\delta & \sum_{k=1}^{[(j+1)/2]} (k+1)^{-\kappa}(j-k+1)^{-1}|w_k||\partial_z u_{j-k}| \\
&+ \sum_{k=[(j+1)/2]+1}^{j-1} (k+1)^{-1}(j-k+1)^{-\kappa}|w_k||\partial_z u_{j-k}|,
\end{aligned}$$

then similar as product estimates in (3.26) to (3.28), and using incompressibility and the a priori estimates in Lemma 3.1, we have

$$\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}} \int_0^{T_0} \langle t\rangle^{\frac{3-3\delta}{2}} \sum_{j=0}^{\infty}(j+1)^3 \|O_j^3(t)\|^2_{L^2(\theta_2)} dt \\
\lesssim & \frac{1}{\lambda\sqrt{\epsilon}} \int_0^{T_0} \langle t\rangle^{\frac{4-3\delta}{2}} \left(\|\partial_z w\|^2_{X_{\tau,5,1/2}} \|\partial_z u\|^2_{X_{\tau,\kappa+1/2,1/2}} + \|\partial_z w\|^2_{X_{\tau,\kappa+1/2,1/2}} \|\partial_z u\|^2_{X_{\tau,5,1/2}} \right) dt
\end{aligned}$$

$$\lesssim \frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{4-3\delta}{2}}\left(\|\boldsymbol{u}\|^2_{X_{\tau,7,1/2}}\|\partial_z u\|^2_{X_{\tau,\kappa+1/2}}+\|\boldsymbol{u}\|^2_{X_{\tau,\kappa+5/2}}\|\partial_z u\|^2_{X_{\tau,5,1/2}}\right)dt$$
$$\lesssim_\delta \frac{C_*^2\epsilon^{3/2}}{\lambda}\int_0^{T_0}\left(\langle t\rangle^{-\frac{1-\delta}{2}}\|u\|^2_{X_{\tau,\kappa+5/2}}+\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z u\|^2_{X_{\tau,\kappa+2}}\right)dt. \tag{6.11}$$

Estimates of P_j^1, P_j^2. Since P_j^1 and P_j^2 share the same estimate, we only care about P_j^1. For term P_j^1, using the a priori estimates in (3.9) to obtain that

$$|P_j^1|\lesssim_\delta C_*\sqrt{\epsilon}\langle t\rangle^{-\frac{2+3\delta}{4}}\sum_{k=1}^{[(j+1)/2]}(k+1)^{-\kappa}(j-k+1)^{-3}|u_k|\left|\int_z^\infty \partial_h\mathcal{H}_{j-k-1}d\bar{z}\right|$$
$$C_*\sqrt{\epsilon}\langle t\rangle^{-\frac{2+3\delta}{4}}\sum_{k=[(j+1)/2]+1}^{j-1}(j-k+1)^{-\kappa}(k+1)^{-2}|u_k|\left|\int_z^\infty \partial_h\mathcal{H}_{j-k-1}d\bar{z}\right|.$$

Similar as product estimates in (3.26) to (3.28) and using the a priori estimates in Lemma 3.1, we have

$$\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{3-3\delta}{2}}\sum_{j=0}^\infty(j+1)^3\|(P_j^1,P_j^2)(t)\|^2_{L^2(\theta^2)}dt$$
$$\lesssim_\delta \frac{C_*^2\sqrt{\epsilon}}{\lambda}\int_0^{T_0}\langle t\rangle\left(\|u\|^2_{X_{\tau,5,1/2}}\|\partial_h\mathcal{H}\|^2_{X_{\tau,\kappa-3/2,1/2}}+\|u\|^2_{X_{\tau,\kappa-1/2,1/2}}\|\partial_h\mathcal{H}\|^2_{X_{\tau,5,1/2}}\right)dt$$
$$\lesssim_\delta \frac{C_*^4\epsilon^{5/2}}{\lambda}\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}\langle t\rangle^{\delta}\left(\|\mathcal{H}\|^2_{X_{\tau,\kappa+1/2}}+\|u\|^2_{X_{\tau,\kappa+5/2}}\right)dt$$
$$\lesssim_\delta \frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}\left(\|\mathcal{H}\|^2_{X_{\tau,\kappa+1/2}}+\|u\|^2_{X_{\tau,\kappa+5/2}}\right)dt, \tag{6.12}$$

where in the last line, we have used (2.3), which indicates that $\langle t\rangle^\delta\sqrt{\epsilon}\lesssim_\delta 1$.

Estimates of P_j^3. For term P_j^3, using the a priori estimates in Lemma (3.1) to obtain that

$$|P_j^3|\lesssim_\delta C_*\sqrt{\epsilon}\langle t\rangle^{-\frac{2+3\delta}{4}}\sum_{k=1}^{[(j+1)/2]}(k+1)^{-\kappa}(j-k+1)^{-3}|w_k||\mathcal{H}_{j-k-1}|$$
$$C_*\sqrt{\epsilon}\langle t\rangle^{-\frac{2+3\delta}{4}}\sum_{k=[(j+1)/2]+1}^{j-1}(j-k+1)^{-\kappa}(k+1)^{-2}|w_k||\mathcal{H}_{j-k-1}|.$$

Similar as product estimates in (3.26) to (3.28), using incompressibility and the a priori estimates in Lemma 3.1, we have

$$\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{3-3\delta}{2}}\sum_{j=0}^\infty(j+1)^3\|P_j^3(t)\|^2_{L^2(\theta_2)}dt$$
$$\lesssim \frac{C_*^2\sqrt{\epsilon}}{\lambda}\int_0^{T_0}\langle t\rangle\left(\|\partial_z w\|^2_{X_{\tau,5,1/2}}\|\mathcal{H}\|^2_{X_{\tau,\kappa-3/2,1/2}}+\|\partial_z w\|^2_{X_{\tau,\kappa-1/2,1/2}}\|\mathcal{H}\|^2_{X_{\tau,5,1/2}}\right)dt$$
$$\lesssim_\delta \frac{C_*^2\sqrt{\epsilon}}{\lambda}\int_0^{T_0}\langle t\rangle\left(\|\boldsymbol{u}\|^2_{X_{\tau,7,1/2}}\|\mathcal{H}\|^2_{X_{\tau,\kappa+1/2}}+\|\boldsymbol{u}\|^2_{X_{\tau,\kappa+5/2,1/2}}\|\mathcal{H}\|^2_{X_{\tau,5,1/2}}\right)dt$$

$$
\begin{aligned}
&\lesssim \frac{C_*^4\epsilon^{5/2}}{\lambda}\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}\langle t\rangle^{\delta}\left(\|\mathcal{H}\|^2_{X_{\tau,\kappa+1/2}}+\|\boldsymbol{u}\|^2_{X_{\tau,\kappa+5/2}}\right)dt\\
&\lesssim \frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}\left(\|\mathcal{H}\|^2_{X_{\tau,\kappa+1/2}}+\|\boldsymbol{u}\|^2_{X_{\tau,\kappa+5/2}}\right)dt, \qquad (6.13)
\end{aligned}
$$

where in the last line of the above inequality, we have used (2.4).

Estimates of P_j^4. Using the a prior estimates (3.9) and (3.22), we can obtain that

$$
\begin{aligned}
\|P_j^4\|_{L^2(\theta^2)}\leq&\left\|\frac{[\partial_y v\partial_z u-\partial_y u\partial_z v]}{\sqrt{\epsilon}\langle t\rangle^{\delta-1}}\right\|_{L^\infty}\left\|M_j\int_z^\infty\partial_x^{j-1}\mathcal{H}d\bar{z}\right\|_{L^2(\theta_2)}\\
\lesssim_\delta& C_*^2\epsilon^{3/2}\langle t\rangle^{-\frac{2+\delta}{2}}(j+1)^{-2}\|\mathcal{H}_j\|_{L^2(\theta_2)}.
\end{aligned}
$$

Then it is easy to see that

$$
\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{3-3\delta}{2}}\sum_{j=0}^{\infty}(j+1)^3\|P_j^4(t)\|^2_{L^2(\theta_2)}dt\lesssim_\delta \frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}\|\mathcal{H}\|^2_{X_{\tau,\kappa}}dt. \qquad (6.14)
$$

Estimates of P_j^5. Using the a prior estimates (3.9), we can obtain that

$$
\|P_j^5\|_{L^2(\theta_2)}\leq\left\|\frac{\partial_z^2 u}{\sqrt{\epsilon}\langle t\rangle^{\delta-1}}\right\|_{L^\infty}\left\|M_j\partial_x^{j-1}\mathcal{H}\right\|_{L^2(\theta_2)}\lesssim_\delta \quad C_*\sqrt{\epsilon}\langle t\rangle^{-\frac{4+3\delta}{4}}(j+1)^{-2}\|\mathcal{H}_{j-1}\|_{L^2(\theta_2)}.
$$

Then it is easy to see that

$$
\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{3-3\delta}{2}}\sum_{j=0}^{\infty}(j+1)^3\|P_j^5(t)\|^2_{L^2(\theta^2)}dt\lesssim_\delta \frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}\|\mathcal{H}\|^2_{X_{\tau,\kappa}}dt. \qquad (6.15)
$$

Combining estimates in (6.9), (6.10), (6.11), (6.12), (6.13), (6.14) and (6.15), we obtain (6.8). ∎

Proof. **(Proof of Proposition 3.3.)** From Lemma 6.1 and Lemma 6.2, we have achieved that

$$
\begin{aligned}
&\langle t\rangle^{\frac{1-\delta}{2}}\|\psi(t)\|^2_{X_{\tau,\kappa+2}}+\delta\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\psi(t)\|^2_{X_{\tau,\kappa+2}}dt\\
&\quad+\lambda\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\psi(t)\|^2_{X_{\tau,\kappa+5/2}}dt\\
&\lesssim_\delta\|u(0)\|^2_{X_{\tau_0,\kappa+2}}+\frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{-\frac{1-\delta}{2}}\left(\|\boldsymbol{u}(t)\|^2_{X_{\tau,\kappa+5/2}}+\|\boldsymbol{\mathcal{H}}(t)\|^2_{X_{\tau,\kappa+1/2}}\right)dt\\
&\quad+\frac{C_*^2}{\lambda}\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\left(\|\partial_z\boldsymbol{u}(t)\|^2_{X_{\tau,\kappa+2}}+\|\partial_z\boldsymbol{\mathcal{H}}(t)\|^2_{X_{\tau,\kappa}}\right)dt. \qquad (6.16)
\end{aligned}
$$

Besides, from the definition of ψ_j in (6.1) and using (3.8) in Lemma 3.8, we see that, for $\tilde{\kappa}>0$,

$$
\begin{aligned}
\|u(t)\|^2_{X_{\tau,\tilde{\kappa}+2}}\lesssim&\|\psi(t)\|^2_{X_{\tau,\tilde{\kappa}+2}}+\langle t\rangle^{5/2-2\delta}\epsilon^{-1}\|\partial_z u\|^2_{L^\infty}\|\mathcal{H}(t)\|^2_{X_{\tau,\tilde{\kappa}}}\\
\lesssim&\|\psi(t)\|^2_{X_{\tau,\tilde{\kappa}+2}}+C_*^2\epsilon\|\mathcal{H}(t)\|^2_{X_{\tau,\tilde{\kappa}}}. \qquad (6.17)
\end{aligned}
$$

Similarly, we can obtain that

$$
\|\partial_z u(t)\|^2_{X_{\tau,\tilde{\kappa}+2}}\lesssim_\delta\|\partial_z\psi(t)\|^2_{X_{\tau,\tilde{\kappa}+2}}+C_*^2\epsilon\|\partial_z\mathcal{H}(t)\|^2_{X_{\tau,\tilde{\kappa}}}. \qquad (6.18)
$$

Inserting (6.17) and (6.18) into (6.16) and by letting ϵ sufficiently small, we can achieve (3.12) in Proposition 3.3. ∎

7. Estimates of the good unknowns $\boldsymbol{g}$

In this section, we focus on Gevrey-2 estimates of the linearly good unknowns $\boldsymbol{g}$ and its $z-$ derivatives up to second order. It will induce faster decay rate for low order Gevrey-2 energy of the unknowns $\boldsymbol{u}$ as displayed in (3.8).

Below we set

$$\kappa_0=12,\quad \kappa_1=10,\quad \kappa_2=8.$$

7.1. Estimates of $\boldsymbol{g}$.

LEMMA 7.1. *Under the assumption of (3.7), for sufficiently small ϵ, there exists a constant C_δ such that for any $t\in(0,T_0]$, we have the following estimate.*

$$\begin{aligned}&\langle t\rangle^{\frac{5-\delta}{2}}\|\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0}}+\delta\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0}}dt+\lambda\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0+1/2}}dt\\&\le C_\delta\|\boldsymbol{g}(0)\|^2_{X_{\tau_0,\kappa_0}}+C_\delta\frac{C_*^2}{\lambda}\epsilon^{3/2}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\boldsymbol{u}(t)\|^2_{X_{\tau,\kappa+2}}dt\\&\quad+C_\delta\frac{C_*^2}{\lambda}\epsilon^{3/2}\int_0^{T_0}\Big(\langle t\rangle^{\frac{3+\delta}{2}}\|\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0+1/2}}+\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0}}\Big)dt.\end{aligned}\tag{7.1}$$

Proof. We only show estimates of g, since $\tilde g$ follows along the same lines. By direct computation, we see that g satisfies the following

$$\begin{cases}\partial_t g+\dfrac{1}{\langle t\rangle}g-\partial_z^2 g=-(u\partial_x+v\partial_y+w\partial_z)g+\dfrac{wu}{2\langle t\rangle}+(\partial_z u\partial_y v-\partial_z v\partial_y u),\\ \qquad\qquad\qquad :=\displaystyle\sum_{i=1}^6 R^i,\\ \partial_z g\big|_{z=0}=0,\quad \displaystyle\lim_{z\to+\infty}g=0,\quad g\big|_{t=0}=g_0=\partial_z u_0+\dfrac{z}{2}u_0.\end{cases}\tag{7.2}$$

Now applying $M_{j,\kappa_0}\partial_x^j$ to the first equation of (7.2) and denoting f_{j,x,κ_0} by f_{j,κ_0} for a function f, we can obtain that

$$\partial_t g_{j,\kappa_0}+\lambda\sqrt{\epsilon}\eta(t)(j+1)g_{j,\kappa_0}-\partial_z^2 g_{j,\kappa_0}+\frac{1}{\langle t\rangle}g_{j,\kappa_0}=\sum_{i=1}^6 R^i_{j,\kappa_0}.$$

Performing energy estimates similar as (4.7), we have

$$\begin{aligned}&\langle t\rangle^{\frac{5-\delta}{2}}\|g_{j,\kappa_0}(t)\|^2_{L^2(\theta_2)}+\delta\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g_{j,\kappa_0}(t)\|^2_{L^2(\theta_2)}dt\\&+\lambda\sqrt{\epsilon}(j+1)\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|g_{j,\kappa_0}(t)\|^2_{L^2(\theta_2)}dt\\&\lesssim\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{7-3\delta}{2}}(j+1)^{-1}\sum_{i=1}^6\|R^i_{j,\kappa_0}(t)\|^2_{L^2(\theta_2)}dt.\end{aligned}$$

Summing the above inequality over $j\in\mathbb{N}$, we can obtain that

$$\begin{aligned}&\langle t\rangle^{\frac{5-\delta}{2}}\|g(t)\|^2_{X_{\tau,\kappa_0}}+\delta\textstyle\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g(t)\|^2_{X_{\tau,\kappa_0}}dt+\lambda\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|g(t)\|^2_{X_{\tau,\kappa_0+1/2}}dt\\&\lesssim_\delta\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{7-3\delta}{2}}(j+1)^{-1}\sum_{j=0}^\infty\sum_{i=1}^6\|R^i_{j,\kappa_0}(t)\|^2_{L^2(\theta_2)}dt.\end{aligned}$$

Before we proceed to estimate the nonlinear terms, we make the following claim.

Claim. For $0<\tilde\kappa\in\mathbb{R}$, $0\le\nu\le1$ and $1<m\in\mathbb{N}$, we have the following embedding in Gevrey-2 spaces.

$$\|\partial_h f\|^2_{X_{\tau,\tilde\kappa,\nu}}\lesssim_m \|f\|^{\frac{2}{m}}_{X_{\tau,\tilde\kappa+2m,\nu}}\|f\|^{2-\frac{2}{m}}_{X_{\tau,\tilde\kappa,\nu}}. \tag{7.3}$$

Proof. Using Gagliardo-Nirenberg inequality in (x,y) variables, we have

$$\begin{aligned}\|\partial_h f_{j,\tilde\kappa}\|_{L^2(\theta_{2\nu})}&\lesssim\|\partial_h^m f_{j,\tilde\kappa}\|^{\frac{1}{m}}_{L^2(\theta_{2\nu})}\|f_{j,\tilde\kappa}\|^{1-\frac{1}{m}}_{L^2(\theta_{2\nu})}\\&\lesssim_m\left((j+m+1)^{2m}\|f_{j+m,\tilde\kappa}\|\right)^{\frac{1}{m}}_{L^2(\theta_{2\nu})}\|f_{j,\tilde\kappa}\|^{1-\frac{1}{m}}_{L^2(\theta_{2\nu})}.\end{aligned}\tag{7.4}$$

Then using discrete Hölder inequality, we can obtain that

$$\begin{aligned}\|\partial_h f\|^2_{X_{\tau,\tilde\kappa,\nu}}&=\sum_{j=0}^{\infty}\|\partial_h f_{j,\tilde\kappa}\|_{L^2(\theta_{2\nu})}\\&\lesssim\left(\sum_{j=0}^{\infty}(j+m+1)^{2m}\|f_{j+m,\tilde\kappa}\|^2_{L^2(\theta_{2\nu})}\right)^{\frac{1}{m}}\left(\sum_{j=0}^{\infty}\|f_{j,\tilde\kappa}\|^2_{L^2(\theta_{2\nu})}\right)^{1-\frac{1}{m}}\\&\lesssim\|f\|^{\frac{2}{m}}_{X_{\tau,\tilde\kappa+2m,\nu}}\|f\|^{2-\frac{2}{m}}_{X_{\tau,\tilde\kappa,\nu}},\end{aligned}$$

which is (7.3). ∎

Estimate of term R_j^1 and R_j^2. By using (3.23) and the definition of g, we have the fact that

$$\|\partial_h g\|^2_{X_{\tau,\kappa_0-1/2,1/2}}\lesssim\|\partial_z\partial_h u\|^2_{X_{\tau,\kappa_0-1/2,1/2}}.$$

Using the product estimate in (3.27) and the above inequality, we can have

$$\begin{aligned}&\sum_{j=0}^{\infty}(j+1)^{-1}\|(R^1,R_j^2)_{j,\kappa_0}(t)\|^2_{L^2(\theta_2)}\\&\lesssim\langle t\rangle^{1/2}\left(\|\partial_z u\|^2_{X_{\tau,5,1/2}}\|\partial_h g\|^2_{X_{\tau,\kappa_0-1/2,1/2}}+\|\partial_z u\|^2_{X_{\tau,\kappa_0-1/2,1/2}}\|\partial_h g\|^2_{X_{\tau,5,1/2}}\right)\\&\lesssim\langle t\rangle^{1/2}\left(\|\partial_z u\|^2_{X_{\tau,5,1/2}}\|\partial_z\partial_h u\|^2_{X_{\tau,\kappa_0-1/2,1/2}}+\|\partial_z u\|^2_{X_{\tau,\kappa_0-1/2,1/2}}\|g\|^2_{X_{\tau,7,1/2}}\right).\end{aligned}\tag{7.5}$$

From (7.3), we have

$$\|\partial_h\partial_z u\|^2_{X_{\tau,\kappa_0-1/2,1/2}}\lesssim\|\partial_z u\|^{\frac{2(m-1)}{m}}_{X_{\tau,\kappa_0-1/2,1/2}}\|\partial_z u\|^{\frac{2}{m}}_{X_{\tau,\kappa_0-1/2+2m,1/2}}.$$

Inserting the above inequality into (7.5) and using the a priori estimates in (3.8), we can obtain that

$$\begin{aligned}&\sum_{j=0}^{\infty}(j+1)^{-1}\|(R^1,R_j^2)_{j,\kappa_0}(t)\|^2_{L^2(\theta_2)}\\&\lesssim\langle t\rangle^{1/2}\left(\|\partial_z u\|^2_{X_{\tau,5,1/2}}\|\partial_z u\|^{\frac{2(m-1)}{m}}_{X_{\tau,\kappa_0-1/2,1/2}}\|\partial_z u\|^{\frac{2}{m}}_{X_{\tau,\kappa_0-1/2+2m,1/2}}+\|\partial_z u\|^2_{X_{\tau,\kappa_0-1/2,1/2}}\|g\|^2_{X_{\tau,7,1/2}}\right)\\&\lesssim C_*^2\epsilon^2\langle t\rangle^{-\frac{4-\delta}{2}}\left(\|g\|^{\frac{2(m-1)}{m}}_{X_{\tau,\kappa_0-1/2}}\|\partial_z u\|^{\frac{2}{m}}_{X_{\tau,\kappa_0-1/2+2m}}+\|g\|^2_{X_{\tau,\kappa_0-1/2}}\right).\end{aligned}$$

Here we have used the fact that

$$\|\partial_z u\|_{X_{\tau,\kappa_0-1/2,1/2}} \lesssim \|g\|^2_{X_{\tau,\kappa_0-1/2}}.$$

Then by using Young inequality, we can obtain that

$$\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{7-3\delta}{2}}\sum_{j=0}^{\infty}(j+1)^{-1}\|(R^1,R^2)_{j,\kappa_0}(t)\|^2_{L^2(\theta^2)}dt\\
\lesssim&\frac{C_\delta C_*^2\epsilon^{3/2}}{\lambda}\int_0^{T_0}\langle t\rangle^{\frac{3-2\delta}{2}}\left(\|g\|^{\frac{2(m-1)}{m}}_{X_{\tau,\kappa_0-1/2}}\|\partial_z u\|^{\frac{2}{m}}_{X_{\tau,\kappa_0-1/2+2m}}+\|g\|^2_{X_{\tau,\kappa_0-1/2}}\right)dt\\
\lesssim&\frac{C_*^2\epsilon^{3/2}}{\lambda}\int_0^{T_0}\langle t\rangle^{\left(\frac{3-2\delta}{2}-\frac{1-\delta}{2m}\right)\frac{m}{m-1}}\|g\|^2_{X_{\tau,\kappa_0-1/2}}dt\\
&+\frac{C_*^2\epsilon^{3/2}}{\lambda}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z u\|^2_{X_{\tau,\kappa_0-1/2+2m}}dt+\frac{C_*^2\epsilon^{3/2}}{\lambda}\int_0^{T_0}\langle t\rangle^{\frac{3-2\delta}{2}}\|g\|^2_{X_{\tau,\kappa_0-1/2}}dt. \qquad (7.6)
\end{aligned}$$

By choosing $m:=\left[\frac{1}{\delta}\right]$, we can have

$$\left(\frac{3-2\delta}{2}-\frac{1-\delta}{2m}\right)\frac{m}{m-1}\le\frac{3+\delta}{2},\qquad \kappa_0-1/2+2m\le\kappa+2.$$

Then inserting the above into (7.6) implies that

$$\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{7-3\delta}{2}}\sum_{j=0}^{\infty}(j+1)^{-1}\|(R^1,R^2)_{j,\kappa_0}(t)\|^2_{L^2(\theta^2)}dt\\
\lesssim&\frac{C_*^2\epsilon^{3/2}}{\lambda}\int_0^{T_0}\langle t\rangle^{\frac{3+\delta}{2}}\|g\|^2_{X_{\tau,\kappa_0+1/2}}dt+\frac{C_*^2\epsilon^{3/2}}{\lambda}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z u\|^2_{X_{\tau,\kappa+2}}dt. \qquad (7.7)
\end{aligned}$$

Estimate of term R_j^3. Using the product estimate in (3.27) and the incompressibility, we can obtain that

$$\begin{aligned}
&\sum_{j=0}^{\infty}(j+1)^{-1}\|R^3_{j,\kappa_0}(t)\|^2_{L^2(\theta_2)}\\
\lesssim&\langle t\rangle^{1/2}\left(\|\partial_z w\|^2_{X_{\tau,5,1/2}}\|\partial_z g\|^2_{X_{\tau,\kappa_0-1/2,1/2}}+\|\partial_z w\|^2_{X_{\tau,\kappa_0-1/2,1/2}}\|\partial_z g\|^2_{X_{\tau,5,1/2}}\right)\\
\lesssim&\langle t\rangle^{1/2}\left(\|\boldsymbol{u}\|^2_{X_{\tau,7,1/2}}\|\partial_z g\|^2_{X_{\tau,\kappa_0-1/2,1/2}}+\|\partial_h \boldsymbol{u}\|^2_{X_{\tau,\kappa_0-1/2,1/2}}\|\partial_z g\|^2_{X_{\tau,5,1/2}}\right). \qquad (7.8)
\end{aligned}$$

From (7.3), we have

$$\|\partial_h\boldsymbol{u}\|^2_{X_{\tau,\kappa_0-1/2,1/2}}\lesssim\|\boldsymbol{u}\|^{\frac{2(m-1)}{m}}_{X_{\tau,\kappa_0-1/2,1/2}}\|\boldsymbol{u}\|^{\frac{2}{m}}_{X_{\tau,\kappa_0-1/2+2m,1/2}}.$$

Inserting the above inequality into (7.8) and using the a priori estimates in (3.8), we can obtain that

$$\begin{aligned}
&\sum_{j=0}^{\infty}(j+1)^{-1}\|R^3_{j,\kappa_0}(t)\|^2_{L^2(\theta^2)}\\
\lesssim&\langle t\rangle^{1/2}\left(\|\boldsymbol{u}\|^{\frac{2(m-1)}{m}}_{X_{\tau,\kappa_0-1/2,1/2}}\|\boldsymbol{u}\|^{\frac{2}{m}}_{X_{\tau,\kappa_0-1/2+2m,1/2}}\|\partial_z g\|^2_{X_{\tau,5}}+\|\boldsymbol{u}\|^2_{X_{\tau,7,1/2}}\|\partial_z g\|^2_{X_{\tau,\kappa_0}}\right)
\end{aligned}$$

$$\lesssim C_*^2\epsilon^2\left(\langle t\rangle^{-\frac{4-\delta}{2}}\|g\|_{X_{\tau,\kappa_0-1/2}}^{\frac{2(m-1)}{m}}\|\partial_z \boldsymbol{u}\|_{X_{\tau,\kappa_0-1/2+2m}}^{\frac{2}{m}}+\langle t\rangle^{-\frac{2-\delta}{2}}\|\partial_z g\|_{X_{\tau,\kappa_0}}^2\right),$$

where in the last line, we have used the fact that

$$\|\boldsymbol{u}\|_{X_{\tau,\kappa_0-1/2,1/2}}\lesssim\langle t\rangle^{1/2}\|\boldsymbol{g}\|_{X_{\tau,\kappa_0-1/2}},\quad \|\boldsymbol{u}\|_{X_{\tau,\kappa_0-1/2,1/2}}\lesssim\langle t\rangle^{1/2}\|\partial_z\boldsymbol{u}\|_{X_{\tau,\kappa_0-1/2,1/2}}.$$

Then the rest is the same as estimate of (7.7), we can obtain that

$$\begin{aligned}&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{7-3\delta}{2}}\sum_{j=0}^{\infty}(j+1)^{-1}\|(R^3_{j,\kappa_0}(t)\|_{L^2(\theta^2)}^2dt\\ &\lesssim\frac{C_*^2\epsilon^{3/2}}{\lambda}\int_0^{T_0}\left(\langle t\rangle^{\frac{3+\delta}{2}}\|g\|_{X_{\tau,\kappa_0+1/2}}^2+\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g\|_{X_{\tau,\kappa_0}}^2\right)dt+\frac{C_*^2\epsilon^{3/2}}{\lambda}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z u\|_{X_{\tau,\kappa+2}}^2dt.\end{aligned}\tag{7.9}$$

Estimate of term R_j^4. Using the product estimate in (3.27), the incompressibility and (3.22), we can obtain that

$$\begin{aligned}&\sum_{j=0}^{\infty}(j+1)^{-1}\|R^4_{j,\kappa_0}(t)\|_{L^2(\theta_2)}^2\\ &\lesssim\langle t\rangle^{1/2}\left(\|\partial_z w\|_{X_{\tau,5,1/2}}^2\|\langle t\rangle^{-1}u\|_{X_{\tau,\kappa_0-1/2,1/2}}^2+\|\partial_z w\|_{X_{\tau,\kappa_0-1/2,1/2}}^2\|\langle t\rangle^{-1}u\|_{X_{\tau,5,1/2}}^2\right)\\ &\lesssim\langle t\rangle^{1/2}\left(\|\partial_z w\|_{X_{\tau,5,1/2}}^2\|\partial_z^2 u\|_{X_{\tau,\kappa_0-1/2,1/2}}^2+\|\partial_z w\|_{X_{\tau,\kappa_0-1/2,1/2}}^2\|\partial_z^2 u\|_{X_{\tau,5,1/2}}^2\right)\\ &\lesssim\langle t\rangle^{1/2}\left(\|\boldsymbol{u}\|_{X_{\tau,7,1/2}}^2\|\partial_z g\|_{X_{\tau,\kappa_0-1/2,3/4}}^2+\|\partial_h\boldsymbol{u}\|_{X_{\tau,\kappa_0-1/2,1/2}}^2\|\partial_z g\|_{X_{\tau,5}}^2\right),\end{aligned}$$

where in the last line, we have used the fact that

$$\|\partial_z^2 u\|_{X_{\tau,\kappa_0-1/2,1/2}}\lesssim\|\partial_z g\|_{X_{\tau,\kappa_0-1/2,3/4}},\quad \|\partial_z^2 u\|_{X_{\tau,5,1/2}}\lesssim\|\partial_z g\|_{X_{\tau,5}}.$$

Then the rest is the same as (7.8).

Estimate of term R_j^5 and R_j^6. Using the product estimate in (3.27), we can have

$$\begin{aligned}&\sum_{j=0}^{\infty}(j+1)^{-1}\|(R^5,R^6)_{j,\kappa_0}(t)\|_{L^2(\theta^2)}^2\\ &\lesssim\langle t\rangle^{1/2}\left(\|\partial_h\boldsymbol{u}\|_{X_{\tau,5,1/2}}^2\|\partial_z^2\boldsymbol{u}\|_{X_{\tau,\kappa_0-1/2,1/2}}^2+\|\partial_h\boldsymbol{u}\|_{X_{\tau,\kappa_0-1/2,1/2}}^2\|\partial_z^2\boldsymbol{u}\|_{X_{\tau,5,1/2}}^2\right)\\ &\lesssim\langle t\rangle^{1/2}\left(\|\boldsymbol{u}\|_{X_{\tau,7,1/2}}^2\|\partial_z g\|_{X_{\tau,\kappa_0-1/2,3/4}}^2+\|\partial_h\boldsymbol{u}\|_{X_{\tau,\kappa_0-1/2,1/2}}^2\|\partial_z g\|_{X_{\tau,5}}^2\right).\end{aligned}$$

Then the rest is the same as (7.8).

From (7.7) and (7.9), we can obtain (7.1) in Lemma 7.1. ∎

7.2. Estimates of $\partial_z g$.

LEMMA 7.2. *Under the assumption of (3.7), for sufficiently small ϵ, there exists a constant C_δ such that for any $t\in(0,T_0]$, we have the following estimate.*

$$\begin{aligned}&\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z\boldsymbol{g}(t)\|_{X_{\tau,\kappa_1}}^2+\delta\int_0^{T_0}\langle t\rangle^{\frac{7-\delta}{2}}\left\|\partial_z^2\boldsymbol{g}(t)\right\|_{X_{\tau,\kappa_1}}^2dt\\ &+\lambda\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{7-\delta}{2}}\eta(t)\|\partial_z\boldsymbol{g}(t)\|_{X_{\tau,\kappa_1+1/2}}^2dt\end{aligned}$$

$$\begin{aligned}&\le C_\delta\|\partial_z\boldsymbol{g}(0)\|^2_{X_{\tau,\kappa_1}}+\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_1}}dt\\&\quad+C_\delta\frac{C_*^2}{\lambda}\epsilon^{3/2}\int_0^{T_0}\Big(\langle t\rangle^{\frac{3+\delta}{2}}\|\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0+1/2}}+\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0}}\Big)dt.\end{aligned}\tag{7.10}$$

Proof. Now applying $M_{j,\kappa_1}\partial_z\partial_x^j$ to the first equation of (7.2) and denoting f_{j,x,κ_1} by f_{j,κ_1} for a function f, we can obtain that

$$\left[\partial_t+\lambda\sqrt{\epsilon}(j+1)-\partial_z^2+\frac{1}{\langle t\rangle}\right]\partial_z g_{j,\kappa_1}=\sum_{i=1}^6\partial_z R^i_{j,\kappa_1}.$$

Performing space variable energy estimates, we can have

$$\begin{aligned}&\frac{d}{dt}\|\partial_z g_{j,\kappa_1}(t)\|^2_{L^2(\theta_2)}+\delta\|\partial_z^2 g_{j,\kappa_1}(t)\|^2_{L_2(\theta_2)}+\frac{5-\delta}{2\langle t\rangle}\|\partial_z g_{j,\kappa_1}(t)\|^2_{L^2(\theta_2)}\\&\quad+2(j+1)\lambda\sqrt{\epsilon}\eta(t)\|\partial_z g_{j,\kappa_1}(t)\|^2_{L^2(\theta_2)}\\&\le 2\left|\left\langle\sum_{i=1}^6\partial_z R^i_{j,\kappa_1},\partial_z g_{j,\kappa_1}\right\rangle_{\theta_2}\right|.\end{aligned}$$

Multiplying the above equality by $\langle t\rangle^{\frac{7-\delta}{2}}$ and using integration by parts for the right-hand side of the above inequality, and then integrating the resulting equation from 0 to t for any $t\in(0,T_0]$, we can achieve that

$$\begin{aligned}&\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z g_{j,\kappa_1}(t)\|^2_{L^2(\theta_2)}+\delta\int_0^{T_0}\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z^2 g_{j,\kappa_1}(t)\|^2_{L^2(\theta_2)}dt\\&\quad+2\lambda\sqrt{\epsilon}(j+1)\int_0^{T_0}\langle t\rangle^{\frac{7-\delta}{2}}\eta(t)\|\partial_z g_{j,\kappa_1}(t)\|^2_{L^2(\theta_2)}dt\\&\le\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g_{j,\kappa_1}(t)\|^2_{L^2(\theta_2)}dt+\int_0^{T_0}\langle t\rangle^{\frac{7-\delta}{2}}\left|\left\langle\sum_{i=1}^6 R^i_{j,\kappa_1},\partial_z^2 g_{j,\kappa_1}+\frac{z}{2\langle t\rangle}\partial_z g_{j,\kappa_1}\right\rangle_{\theta_2}\right|dt.\end{aligned}$$

By using Cauchy inequality and (3.23) to the right-hand side of above inequality, and then summing the resulting equations over $j\in\mathbb{N}$, we can obtain that

$$\begin{aligned}&\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z g(t)\|^2_{X_{\tau,\kappa_1}}+\delta\int_0^{T_0}\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z^2 g(t)\|^2_{X_{\tau,\kappa_1}}dt\\&\quad+\lambda\sqrt{\epsilon}(j+1)\int_0^{T_0}\langle t\rangle^{\frac{7-\delta}{2}}\eta(t)\|\partial_z g(t)\|^2_{X_{\tau,\kappa_1}}dt\\&\le\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g(t)\|^2_{X_{\tau,\kappa_1}}dt+C_\delta\int_0^{T_0}\langle t\rangle^{\frac{7-\delta}{2}}\sum_{i=1}^6\|R^i\|^2_{X_{\tau,\kappa_1}}dt.\end{aligned}\tag{7.11}$$

□

LEMMA 7.3. *We have the following estimates*

$$\sum_{i=1}^6\|R^i\|^2_{X_{\tau,\kappa_1}}\lesssim\langle t\rangle^{3/2}\Big(\|g\|^2_{X_{\tau,7}}\|\partial_z g\|^2_{X_{\tau,\kappa_1+2}}+\|\partial_z g\|^2_{X_{\tau,7}}\|g\|^2_{X_{\tau,\kappa_1+2}}\Big).\tag{7.12}$$

Proof. Proof of this Lemma involves repeated use of (3.22) and (3.23) in Lemma 3.2, product estimates in (3.26) to (3.28) in Lemma 3.3 and the relation between $\boldsymbol{u}$ and $\boldsymbol{g}$. Since it is a routine estimate, we omit the details.

Then by using the a priori estimates in (3.8) and (7.12), we see that

$$\int_0^{T_0} \langle t\rangle^{\frac{7-\delta}{2}} \sum_{i=1}^{6} \|R^i\|^2_{X_{\tau,\kappa_1}} dt \lesssim C_*^2\epsilon^2 \int_0^{T_0} \left(\langle t\rangle^{\frac{3+\delta}{2}} \|g\|^2_{X_{\tau,\kappa_0+1/2}} + \langle t\rangle^{\frac{5-\delta}{2}} \|\partial_z g\|^2_{X_{\tau,\kappa_0}} \right) dt.$$

Inserting the above inequality into (7.11), we can obtain (7.10) in Lemma 7.2. □

7.3. Estimates of $\partial_z^2 \boldsymbol{g}$.

LEMMA 7.4. *Under the assumption of (3.7), for sufficiently small ϵ, there exists a constant C_δ such that for any $t\in(0,T_0]$, we have the following estimate.*

$$\begin{aligned}
&\langle t\rangle^{\frac{9-\delta}{2}} \left\|\partial_z^2 \boldsymbol{g}(t)\right\|^2_{X_{\tau,\kappa_2}} + \delta \int_0^{T_0} \langle t\rangle^{\frac{9-\delta}{2}} \left\|\partial_z^3 \boldsymbol{g}(t)\right\|^2_{X_{\tau,\kappa_2}} dt \\
&\quad + \lambda\sqrt{\epsilon} \int_0^{T_0} \langle t\rangle^{\frac{9-\delta}{2}} \eta(t) \left\|\partial_z^2 \boldsymbol{g}(t)\right\|^2_{X_{\tau,\kappa_3+1/2}} dt \\
&\le C_\delta \left\|\partial_z^2 \boldsymbol{g}(0)\right\|^2_{X_{\tau_0,\kappa_2}} + \int_0^{T_0} \langle t\rangle^{\frac{7-\delta}{2}} \left\|\partial_z^2 \boldsymbol{g}(t)\right\|^2_{X_{\tau,\kappa_2}} dt. \\
&\quad + C_\delta \frac{C_*^2}{\lambda} \epsilon^{3/2} \int_0^{T_0} \Big(\langle t\rangle^{\frac{3+\delta}{2}} \|\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0+1/2}} + \langle t\rangle^{\frac{5-\delta}{2}} \|\partial_z \boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0}} \\
&\qquad + \langle t\rangle^{\frac{7-\delta}{2}} \left\|\partial_z^2 \boldsymbol{g}(t)\right\|^2_{X_{\tau,\kappa_1}} \Big) dt. \qquad (7.13)
\end{aligned}$$

Proof. Now applying $M_{j,\kappa_2}\partial_z^2\partial_x^j$ to the first equation of (7.2) and denoting f_{j,x,κ_2} by f_{j,κ_2} for a function f, we can obtain that

$$\left[\partial_t + \lambda\sqrt{\epsilon}(j+1) - \partial_z^2 + \frac{1}{\langle t\rangle}\right] \partial_z^2 g_{j,\kappa_2} = \sum_{i=1}^{6} \partial_z^2 R^i_{j,\kappa_2}.$$

Performing spacial energy estimates, we can have

$$\begin{aligned}
&\frac{d}{dt}\|\partial_z^2 g_{j,\kappa_2}(t)\|^2_{L^2(\theta_2)} + \delta\|\partial_z^3 g_{j,\kappa_2}(t)\|^2_{L^2(\theta_2)} + \frac{5-\delta}{2\langle t\rangle}\|\partial_z^2 g_{j,\kappa_2}(t)\|^2_{L^2(\theta_2)} \\
&\quad + 2(j+1)\lambda\sqrt{\epsilon}\eta(t)\|\partial_z^2 g_{j,\kappa_2}(t)\|^2_{L^2(\theta_2)} \\
&\le 2\left|\left\langle \sum_{i=1}^{6} \partial_z^2 R^i_{j,\kappa_2}, \partial_z^2 g_{j,\kappa_2} \right\rangle_{\theta_2}\right|.
\end{aligned}$$

Multiplying the above equality by $\langle t\rangle^{\frac{9-\delta}{2}}$ and then integrating the resulting equation from 0 to t for any $t\in(0,T_0]$, we can achieve that

$$\begin{aligned}
&\langle t\rangle^{\frac{9-\delta}{2}} \|\partial_z^2 g_{j,\kappa_2}(t)\|^2_{L^2(\theta_2)} + \delta \int_0^{T_0} \langle t\rangle^{\frac{9-\delta}{2}} \|\partial_z^3 g_{j,\kappa_2}(t)\|^2_{L^2(\theta_2)} dt \\
&\quad + 2\lambda\sqrt{\epsilon}(j+1) \int_0^{T_0} \langle t\rangle^{\frac{9-\delta}{2}} \eta(t) \|\partial_z^2 g_{j,\kappa_2}(t)\|^2_{L^2(\theta_2)} dt \\
&\le \int_0^{T_0} \langle t\rangle^{\frac{7-\delta}{2}} \|\partial_z^2 g_{j,\kappa_2}(t)\|^2_{L^2(\theta_2)} dt + \int_0^{T_0} \langle t\rangle^{\frac{9-\delta}{2}} \left|\left\langle \sum_{i=1}^{6} \partial_z^2 R^i_{j,\kappa_2}, \partial_z^2 g_{j,\kappa_2} \right\rangle_{\theta_2}\right| dt.
\end{aligned}$$

By using Cauchy inequality and (3.23) to the right-hand side of above inequality, and then summing the resulting equations over $j\in\mathbb{N}$, we can obtain that

$$\begin{aligned}&\langle t\rangle^{\frac{9-\delta}{2}}\|\partial_z^2 g(t)\|_{X_{\tau,\kappa_2}}^2+\delta\int_0^{T_0}\langle t\rangle^{\frac{9-\delta}{2}}\|\partial_z^3 g(t)\|_{X_{\tau,\kappa_2}}^2 dt\\&\quad+\lambda\sqrt{\epsilon}(j+1)\int_0^{T_0}\langle t\rangle^{\frac{9-\delta}{2}}\eta(t)\|\partial_z^2 g(t)\|_{X_{\tau,\kappa_2}}^2 dt\\&\le\int_0^{T_0}\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z^2 g(t)\|_{X_{\tau,\kappa_2}}^2 dt+C_\delta\int_0^{T_0}\langle t\rangle^{\frac{11-\delta}{2}}\sum_{i=1}^{6}\|\partial_z^2 R^i\|_{X_{\tau,\kappa_2}}^2 dt.\end{aligned}\tag{7.14}$$

Similar as Lemma 7.3, we have the following lemma. ∎

LEMMA 7.5. *We have the following estimates*

$$\begin{aligned}&\sum_{i=1}^{6}\|\partial_z^2 R^i\|_{X_{\tau,\kappa_2}}^2\\&\lesssim\langle t\rangle^{3/2}\Big(\|g\|_{X_{\tau,7}}^2\|\partial_z^3 g\|_{X_{\tau,\kappa_2}}^2+\|\partial_z g\|_{X_{\tau,7}}^2\|\partial_z^2 g\|_{X_{\tau,\kappa_2+2}}^2+\|\partial_z^2 g\|_{X_{\tau,7}}^2\|\partial_z g\|_{X_{\tau,\kappa_2+2}}^2\Big).\end{aligned}\tag{7.15}$$

Proof. Proof of this lemma also involves repeated use of (3.22) and (3.23) in Lemma 3.2, product estimates in (3.26) to (3.28) in Lemma 3.3 and the relation between $\boldsymbol{u}$ and $\boldsymbol{g}$. Since it is a routine estimate, we omit the details.

Then by using the a priori estimates in (3.8) and (7.15), we see that

$$\begin{aligned}&\int_0^{T_0}\langle t\rangle^{\frac{11-\delta}{2}}\sum_{i=1}^{6}\|\partial_z^2 R^i\|_{X_{\tau,\kappa_1}}^2 dt\\&\lesssim C_*^2\epsilon^2\int_0^{T_0}\Big(\langle t\rangle^{\frac{9-\delta}{2}}\|\partial_z^3 g\|_{X_{\tau,\kappa_2}}^2+\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g\|_{X_{\tau,\kappa_0}}^2+\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z^2 g\|_{X_{\tau,\kappa_1}}^2\Big)dt.\end{aligned}$$

Inserting the above inequality into (7.14), we can obtain (7.13) in Lemma 7.4. ∎

8. Estimates of the good unknowns $\mathfrak{H}$

After we obtain the faster decay rate for low order Gevrey-2 energy of the unknowns $\boldsymbol{u}$ through the linearly good unknowns $\boldsymbol{g}$, in this section, we focus on the Gevrey-2 estimates of the linearly good unknowns $\mathfrak{H}$ and its $z-$ derivative. It will induce faster decay rate for low order Gevrey-2 energy of the auxiliary functions $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{G}}$ as displayed in (3.8).

Below we set

$$\kappa_3=9,\quad \kappa_4=7.$$

8.1. Estimates of $\mathfrak{H}$.

LEMMA 8.1. *Under the assumption of (3.7), for sufficiently small ϵ, there exists a constant C_δ such that for any $t\in(0,T_0]$, we have the following estimate.*

$$\begin{aligned}&\langle t\rangle^{\frac{3-\delta}{2}}\|\mathfrak{H}(t)\|_{X_{\tau,\kappa_3,7/8}}^2+\int_0^{T_0}\langle t\rangle^{\frac{3-\delta}{2}}\|\partial_z\mathfrak{H}(t)\|_{X_{\tau,\kappa_3,7/8}}^2 dt\\&\quad+\lambda\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{3-\delta}{2}}\eta(t)\|\mathfrak{H}(t)\|_{X_{\tau,\kappa_3+1/2,7/8}}^2 dt\end{aligned}$$

$$
\begin{aligned}
\leq & C_\delta \frac{C_*^2}{\lambda}\epsilon^{3/2}\int_0^{T_0}\Big(\langle t\rangle^{\frac{1+\delta}{2}}\|\boldsymbol{\mathfrak{H}}(t)\|_{X_{\tau,\kappa_3+1/2,7/8}}^2+\langle t\rangle^{\frac{3-\delta}{2}}\|\partial_z\boldsymbol{\mathfrak{H}}(t)\|_{X_{\tau,\kappa_3,7/8}}^2 \\
&+\langle t\rangle^{-\frac{1-\delta}{2}}\|\boldsymbol{\mathcal{H}}(t)\|_{X_{\tau,\kappa}}^2\Big)dt+\frac{\sqrt{\epsilon}}{\lambda}\int_0^{T_0}\langle t\rangle^{\frac{3+\delta}{2}}\|\boldsymbol{g}(t)\|_{X_{\tau,\kappa_0+1/2}}^2 dt. \qquad (8.1)
\end{aligned}
$$

Proof. We only show estimates for $\mathfrak{H}$, since that of $\widetilde{\mathfrak{H}}$ follows along the same lines. First we derive the equation satisfied by $\mathfrak{H}$. By multiplying $\frac{z}{2\langle t\rangle}$ to the first equation of (3.1), we see that

$$
\begin{aligned}
&\left[\partial_t-\partial_z^2\right]\frac{z}{2\langle t\rangle}\int_z^{+\infty}\mathcal{H}d\bar{z}-\frac{1}{\langle t\rangle}\mathfrak{H} \\
=&-\frac{z}{2\langle t\rangle}(u\partial_x+v\partial_y+w\partial_z)\int_z^{+\infty}\mathcal{H}d\bar{z}+\frac{z\sqrt{\epsilon}}{2\langle t\rangle}\langle t\rangle^{\delta-1}\partial_x w. \qquad (8.2)
\end{aligned}
$$

Then subtracting (8.2) from (4.1), we have

$$
\begin{aligned}
\left[\partial_t-\partial_z^2+\frac{1}{\langle t\rangle}\right]\mathfrak{H}=&\sqrt{\epsilon}\langle t\rangle^{\delta-1}(\partial_x^2 u+\partial_{xy}^2 v)-\frac{z\sqrt{\epsilon}}{2\langle t\rangle}\langle t\rangle^{\delta-1}\partial_x w-\frac{z}{2\langle t\rangle}w\mathcal{H}+(\partial_x u+\partial_y v)\mathcal{H} \\
&+(\partial_z u\partial_x+\partial_z v\partial_y)\int_z^{+\infty}\mathcal{H}d\bar{z}+\frac{z}{2\langle t\rangle}(u\partial_x+v\partial_y)\int_z^{+\infty}\mathcal{H}d\bar{z} \\
:=&Q^1+Q^2+Q^3, \qquad (8.3)
\end{aligned}
$$

with

$$
\partial_z\mathfrak{H}\big|_{z=0}=0,\quad \lim_{z\to+\infty}\mathfrak{H}=0.
$$

Now applying $M_{j,\kappa_3}\partial_x^j$ to (8.3) and denoting f_{j,x,κ_3} by f_{j,κ_3} for a function f, we can obtain that

$$
\partial_t\mathfrak{H}_{j,\kappa_3}+\lambda\sqrt{\epsilon}\eta(t)(j+1)\mathfrak{H}_{j,\kappa_3}-\partial_z^2\mathfrak{H}_{j,\kappa_3}+\frac{1}{\langle t\rangle}\mathfrak{H}_{j,\kappa_3}=\sum_{i=1}^3 Q_{j,\kappa_3}^i.
$$

Similar as (4.5), for $0<\nu<1$, we can have

$$
\begin{aligned}
&\left\langle\left[\partial_t+\lambda\sqrt{\epsilon}\eta(t)(j+1)-\partial_z^2\right]\mathfrak{H}_{j,\kappa_3},\mathfrak{H}_{j,\kappa_3}(t)\right\rangle_{\theta_{2\nu}} \\
=&\frac{1}{2}\frac{d}{dt}\|\mathfrak{H}_{j,\kappa_3}(t)\|_{L^2(\theta_{2\nu})}^2+\|\partial_z\mathfrak{H}_{j,\kappa_3}(t)\|_{L^2(\theta_{2\nu})}^2+\frac{4-\nu}{4\langle t\rangle}\|\mathfrak{H}_{j,\kappa_3}(t)\|_{L^2(\theta_{2\nu})}^2 \\
&+\frac{\nu-\nu^2}{8}\left\|\frac{z}{\langle t\rangle}\mathfrak{H}_{j,\kappa_3}(t)\right\|_{L^2(\theta_{2\nu})}^2+(j+1)\lambda\sqrt{\epsilon}\eta(t)\|\mathfrak{H}_{j,\kappa_3}(t)\|_{L^2(\theta_{2\nu})}^2.
\end{aligned}
$$

Using (3.22) in Lemma 3.2, we have

$$
\begin{aligned}
&2\left\langle\left[\partial_t+\lambda\sqrt{\epsilon}\eta(t)(j+1)-\partial_z^2\right]\mathfrak{H}_{j,\kappa_3},\mathfrak{H}_{j,\kappa_3}(t)\right\rangle_{\theta_{2\nu}} \\
\geq&\frac{d}{dt}\|\mathfrak{H}_{j,\kappa_3}(t)\|_{L^2(\theta_{2\nu})}^2+\|\partial_z\mathfrak{H}_{j,\kappa_3}(t)\|_{L^2(\theta_{2\nu})}^2+\frac{2}{\langle t\rangle}\|\mathfrak{H}_{j,\kappa_3}(t)\|_{L^2(\theta_{2\nu})}^2 \\
&+2(j+1)\lambda\sqrt{\epsilon}\eta(t)\|\mathfrak{H}_{j,\kappa_3}(t)\|_{L^2(\theta_{2\nu})}^2.
\end{aligned}
$$

Performing energy estimates as before and using Cauchy inequality, we have

$$\begin{aligned}&\langle t\rangle^{\frac{3-\delta}{2}}\|\mathfrak{H}_{j,\kappa_3}(t)\|^2_{L^2(\theta_{2\nu})}+\int_0^{T_0}\langle t\rangle^{\frac{3-\delta}{2}}\|\partial_z\mathfrak{H}_{j,\kappa_3}(t)\|^2_{L^2(\theta_{2\nu})}dt\\&\quad+\lambda\sqrt{\epsilon}(j+1)\int_0^{T_0}\langle t\rangle^{\frac{3-\delta}{2}}\eta(t)\|\mathfrak{H}_{j,\kappa_3}(t)\|^2_{L^2(\theta_{2\nu})}dt\\&\lesssim\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{5-3\delta}{2}}(j+1)^{-1}\sum_{i=1}^{3}\|Q^i_{j,\kappa_3}\|^2_{L^2(\theta_{2\nu})}dt.\end{aligned}$$

Letting $\nu=7/8$ and summing the above inequality over $j\in\mathbb{N}$, we can achieve that

$$\begin{aligned}&\langle t\rangle^{\frac{3-\delta}{2}}\|\mathfrak{H}(t)\|^2_{X_{\tau,\kappa_3,7/8}}+\int_0^{T_0}\langle t\rangle^{\frac{3-\delta}{2}}\|\partial_z\mathfrak{H}(t)\|^2_{X_{\tau,\kappa_3,7/8}}dt\\&\quad+\lambda\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{3-\delta}{2}}\eta(t)\|\mathfrak{H}(t)\|^2_{X_{\tau,\kappa_3+1/2,7/8}}dt\\&\le\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{5-3\delta}{2}}\sum_{i=1}^{3}\|Q^i\|^2_{X_{\tau,\kappa_3-1/2,7/8}}dt.\end{aligned}\tag{8.4}$$

Estimates of Q^1. This is direct. By using incompressibility and (3.23), we have

$$\begin{aligned}&\|Q^1\|^2_{X_{\tau,\kappa_3-1/2,7/8}}\lesssim\epsilon\langle t\rangle^{2\delta-2}\|\partial_h^2\boldsymbol{u}\|^2_{X_{\tau,\kappa_3-1/2,7/8}}\\&\lesssim\epsilon\langle t\rangle^{2\delta-2}\|\boldsymbol{u}\|^2_{X_{\tau,\kappa_3+7/2,7/8}}\lesssim\epsilon\langle t\rangle^{2\delta-1}\|g\|^2_{X_{\tau,\kappa_0+1/2}}.\end{aligned}\tag{8.5}$$

From this, we can obtain

$$\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{5-3\delta}{2}}\|Q^1\|^2_{X_{\tau,\kappa_3-1/2,7/8}}dt\lesssim\frac{\sqrt{\epsilon}}{\lambda}\int_0^{T_0}\langle t\rangle^{\frac{3+\delta}{2}}\|\boldsymbol{g}\|^2_{X_{\tau,\kappa_0+1/2}}dt.\tag{8.6}$$

Estimates of Q^2. By using (3.23) in Lemma 3.2, (3.26) to (3.28) in Lemma 3.3, incompressibility, and the a priori estimates in Lemma 3.1, we have

$$\begin{aligned}&\|Q^2\|^2_{X_{\tau,\kappa_3-1/2,\nu}}\\&\lesssim\|\partial_z(w\mathcal{H})\|^2_{X_{\tau,\kappa_3-1/2,\nu}}+\|\partial_h\boldsymbol{u}\mathcal{H}\|^2_{X_{\tau,\kappa_3-1/2,\nu}}\\&\lesssim\langle t\rangle^{1/2}\left(\|\partial_h\boldsymbol{u}\|^2_{X_{\tau,5,\frac{\nu+1}{4}}}\|\partial_z\mathcal{H}\|^2_{X_{\tau,\kappa_3-1/2,\frac{\nu+1}{4}}}+\|\partial_h\boldsymbol{u}\|^2_{X_{\tau,\kappa_3-1/2,\frac{\nu+1}{4}}}\|\partial_z\mathcal{H}\|^2_{X_{\tau,5,\frac{\nu+1}{4}}}\right)\\&\lesssim C_*^2\epsilon^2\left(\langle t\rangle^{-\frac{2-\delta}{2}}\|\partial_z\mathfrak{H}\|^2_{X_{\tau,\kappa_3,\nu}}+\langle t\rangle^{-\frac{2-\delta}{2}}\|g\|^2_{X_{\tau,\kappa_0+1/2}}\right).\end{aligned}\tag{8.7}$$

Here in the last line of the above inequality, we have used the fact

$$\|\partial_z\mathcal{H}\|_{X_{\tau,\kappa_3-1/2,\frac{\nu+1}{4}}}\lesssim\|\partial_z\mathfrak{H}\|^2_{X_{\tau,\kappa_3-1/2,\nu}},\quad\|\partial_h\boldsymbol{u}\|^2_{X_{\tau,\kappa_3-1/2,\frac{\nu+1}{4}}}\lesssim\|\boldsymbol{g}\|^2_{X_{\tau,\kappa_3+3/2}}.$$

From this, we can obtain

$$\begin{aligned}&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{5-3\delta}{2}}\|Q^2\|^2_{X_{\tau,\kappa_3-1/2,7/8}}dt\\&\lesssim C_\delta\frac{C_*^2}{\lambda}\epsilon^{3/2}\int_0^{T_0}\left(\langle t\rangle^{\frac{3-\delta}{2}}\|\partial_z\mathfrak{H}(t)\|^2_{X_{\tau,\kappa_3,7/8}}+\langle t\rangle^{\frac{3+\delta}{2}}\|\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0+1/2}}\right)dt.\end{aligned}\tag{8.8}$$

Estimates of Q^3. Using (3.23) in Lemma 3.2 and product estimates (3.26) to (3.28) in Lemma 3.3, we have

$$\begin{aligned}
&\|Q^3\|^2_{X_{\tau,\kappa_3-1/2,\nu}}\\
\lesssim&\left\|\partial_z(u,v)\int_z^\infty \partial_h\mathcal{H}d\bar{z}\right\|^2_{X_{\tau,\kappa_3-1/2,\nu}}+\left\|\partial_z\left((u,v)\int_z^\infty \partial_h\mathcal{H}d\bar{z}\right)\right\|^2_{X_{\tau,\kappa_3-1/2,\nu}}\\
\lesssim&\langle t\rangle^{1/2}\left(\|\partial_z u\|^2_{X_{\tau,5,\frac{\nu+1}{4}}}\|\partial_h\mathcal{H}\|^2_{X_{\tau,\kappa_3-1/2,\frac{\nu+1}{4}}}+\|\partial_z u\|^2_{X_{\tau,\kappa_3-1/2,\frac{\nu+1}{4}}}\|\partial_h\mathcal{H}\|^2_{X_{\tau,5,\frac{\nu+1}{4}}}\right). \quad (8.9)
\end{aligned}$$

Then using the a priori estimates in (3.8) in Lemma 3.1 and (7.3) in Claim (7.3), we can obtain that

$$\begin{aligned}
\|Q^3\|^2_{X_{\tau,\kappa_3-1/2,\nu}}\lesssim&\langle t\rangle^{1/2}\left(\|g\|^2_{X_{\tau,5}}\|\mathcal{H}\|^{\frac{2(m-1)}{m}}_{X_{\tau,\kappa_3-1/2,\frac{\nu+1}{4}}}\|\mathcal{H}\|^{\frac{2}{m}}_{X_{\tau,\kappa_3-1/2+2m,\frac{\nu+1}{4}}}+\|g\|^2_{X_{\tau,\kappa_0}}\|\mathfrak{H}\|^2_{X_{\tau,7,\nu}}\right)\\
\lesssim&C_*^2\epsilon^2\left(\langle t\rangle^{-\frac{4-\delta}{2}}\|\mathfrak{H}\|^{\frac{2(m-1)}{m}}_{X_{\tau,\kappa_3,\nu}}\|\mathcal{H}\|^{\frac{2}{m}}_{X_{\tau,\kappa}}+\langle t\rangle^{-\frac{2-\delta}{2}}\|g\|^2_{X_{\tau,\kappa_0}}\right). \quad (8.10)
\end{aligned}$$

Here in the last line of the above inequality, we have used the fact

$$\|\mathcal{H}\|_{X_{\tau,\kappa_3-1/2,\frac{\nu+1}{4}}}\lesssim\|\mathfrak{H}\|^2_{X_{\tau,\kappa_3-1/2,\nu}},\quad \kappa_3-1/2+2m\le\kappa,$$

by our choice of $m=\left[\delta^{-1}\right]$.

Then by using Young inequality and (8.10), we can obtain that

$$\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^{T_0}\langle t\rangle^{\frac{5-3\delta}{2}}\|Q^3\|^2_{X_{\tau,\kappa_3-1/2,7/8}}dt\\
\lesssim&\frac{C_*^2\epsilon^{3/2}}{\lambda}\int_0^{T_0}\left(\langle t\rangle^{\left(\frac{1-2\delta}{2}+\frac{1-\delta}{2m}\right)\frac{m}{m-1}}\|\mathfrak{H}\|^2_{X_{\tau,\kappa_3,7/8}}+\langle t\rangle^{-\frac{1-\delta}{2}}\|\mathcal{H}\|^2_{X_{\tau,\kappa}}+\langle t\rangle^{\frac{3+\delta}{2}}\|g\|^2_{X_{\tau,\kappa_0}}\right)dt\\
\lesssim&\frac{C_*^2\epsilon^{3/2}}{\lambda}\int_0^{T_0}\left(\langle t\rangle^{\frac{1+\delta}{2}}\|\mathfrak{H}\|^2_{X_{\tau,\kappa_3,7/8}}+\langle t\rangle^{-\frac{1-\delta}{2}}\|\mathcal{H}\|^2_{X_{\tau,\kappa}}+\langle t\rangle^{\frac{3+\delta}{2}}\|g\|^2_{X_{\tau,\kappa_0}}\right)dt. \quad (8.11)
\end{aligned}$$

Here in the last line of (8.11), by our choice of m, we have

$$\left(\frac{1-2\delta}{2}+\frac{1-\delta}{2m}\right)\frac{m}{m-1}\le\frac{1+\delta}{2}.$$

Inserting (8.6), (8.8) and (8.11) into (8.4), we can obtain (8.1) in Lemma 8.1. □

8.2. Estimates of $\partial_z\mathfrak{H}$.

LEMMA 8.2. *Under the assumption of (3.7), for sufficiently small ϵ, there exists a constant C_δ such that for any $t\in(0,T_0]$, we have the following estimate.*

$$\begin{aligned}
&\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z\mathfrak{H}(t)\|^2_{X_{\tau,\kappa_4,7/8}}+\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\left\|\partial_z^2\mathfrak{H}(t)\right\|^2_{X_{\tau,\kappa_4,7/8}}dt\\
&+\lambda\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|\partial_z\mathfrak{H}(t)\|^2_{X_{\tau,\kappa_4+1/2,7/8}}dt\\
\le&C_\delta\frac{C_*^2}{\lambda}\epsilon^{3/2}\int_0^{T_0}\left(\langle t\rangle^{\frac{1+\delta}{2}}\|\mathfrak{H}(t)\|^2_{X_{\tau,\kappa_3+1/2,7/8}}+\langle t\rangle^{\frac{3-\delta}{2}}\|\partial_z\mathfrak{H}(t)\|^2_{X_{\tau,\kappa_3,7/8}}\right)dt
\end{aligned}$$

$$+\frac{\sqrt{\epsilon}}{\lambda}\int_0^{T_0}\langle t\rangle^{\frac{3+\delta}{2}}\|\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0+1/2}}dt+\int_0^{T_0}\langle t\rangle^{\frac{3-\delta}{2}}\|\partial_z\mathfrak{H}(t)\|^2_{X_{\tau,\kappa_4,7/8}}dt. \tag{8.12}$$

Proof. Now applying $M_{j,\kappa_4}\partial_z\partial_x^j$ to (8.3) and denoting f_{j,x,κ_4} by f_{j,κ_4} for a function f, we can obtain that

$$\partial_t\partial_z\mathfrak{H}_{j,\kappa_4}+\lambda\sqrt{\epsilon}\eta(t)(j+1)\partial_z\mathfrak{H}_{j,\kappa_4}-\partial_z^2\partial_z\mathfrak{H}_{j,\kappa_4}+\frac{1}{\langle t\rangle}\partial_z\mathfrak{H}_{j,\kappa_4}=\sum_{i=1}^3\partial_z Q^i_{j,\kappa_4}. \tag{8.13}$$

Similar as (4.5), we can have

$$\begin{aligned}&\left\langle\left[\partial_t+\lambda\delta\sqrt{\epsilon}\eta(t)(j+1)-\partial_z^2\right]\partial_z\mathfrak{H}_{j,\kappa_4},\partial_z\mathfrak{H}_{j,\kappa_4}(t)\right\rangle_{\theta_{2\nu}}\\
=&\frac{1}{2}\frac{d}{dt}\|\partial_z\mathfrak{H}_{j,\kappa_4}(t)\|^2_{L^2(\theta_{2\nu})}+\|\partial_z^2\mathfrak{H}_{j,\kappa_4}(t)\|^2_{L^2(\theta_{2\nu})}\\
&+\tfrac{4-\nu}{4\langle t\rangle}\|\partial_z\mathfrak{H}_{j,\kappa_4}(t)\|^2_{L^2(\theta_{2\nu})}+\tfrac{\nu-\nu^2}{8}\left\|\tfrac{z}{\langle t\rangle}\partial_z\mathfrak{H}_{j,\kappa_4}(t)\right\|^2_{L^2(\theta_{2\nu})}+(j+1)\lambda\sqrt{\epsilon}\eta(t)\|\partial_z\mathfrak{H}_{j,\kappa_4}(t)\|^2_{L^2(\theta_{2\nu})}.\end{aligned}$$

Using the inequality (3.22) in Lemma 3.2, we have

$$\begin{aligned}&2\left\langle\left[\partial_t+\lambda\delta\sqrt{\epsilon}\eta(t)(j+1)-\partial_z^2\right]\partial_z\mathfrak{H}_{j,\kappa_4},\partial_z\mathfrak{H}_{j,\kappa_4}(t)\right\rangle_{\theta_{2\nu}}\\
\geq&\frac{d}{dt}\|\partial_z\mathfrak{H}_{j,\kappa_4}(t)\|^2_{L^2(\theta_{2\nu})}+\|\partial_z^2\mathfrak{H}_{j,\kappa_4}(t)\|^2_{L^2(\theta_{2\nu})}+\frac{2}{\langle t\rangle}\|\partial_z\mathfrak{H}_{j,\kappa_4}(t)\|^2_{L^2(\theta_{2\nu})}\\
&+2(j+1)\lambda\sqrt{\epsilon}\eta(t)\|\partial_z\mathfrak{H}_{j,\kappa_4}(t)\|^2_{L^2(\theta_{2\nu})}.\end{aligned}$$

Performing space and time energy estimates to (8.13) as before and using integration by parts for $\partial_z Q^i$ with $z-$ variable and then Cauchy inequality, we have

$$\begin{aligned}&\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z\mathfrak{H}_{j,\kappa_4}(t)\|^2_{L^2(\theta_{2\nu})}+\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z^2\mathfrak{H}_{j,\kappa_4}(t)\|^2_{L^2(\theta_{2\nu})}dt\\
&+\lambda\sqrt{\epsilon}(j+1)\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|\partial_z\mathfrak{H}_{j,\kappa_4}(t)\|^2_{L^2(\theta_{2\nu})}dt\\
\lesssim&\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\sum_{i=1}^3\|Q^i_{j,\kappa_4}\|^2_{L^2(\theta_{2\nu})}dt+\int_0^{T_0}\langle t\rangle^{\frac{3-\delta}{2}}\|\partial_z\mathfrak{H}_{j,\kappa_4}(t)\|^2_{L^2(\theta_{2\nu})}dt.\end{aligned}$$

Letting $\nu=7/8$ and summing the above inequality over $j\in\mathbb{N}$, we can achieve that

$$\begin{aligned}&\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z\mathfrak{H}(t)\|^2_{X_{\tau,\kappa_4,7/8}}+\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z^2\mathfrak{H}(t)\|^2_{X_{\tau,\kappa_4,7/8}}dt\\
&+\lambda\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|\partial_z^2\mathfrak{H}(t)\|^2_{X_{\tau,\kappa_4,7/8}}dt\\
\lesssim&\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\sum_{i=1}^3\|Q^i\|^2_{X_{\tau,\kappa_4,7/8}}dt+\int_0^{T_0}\langle t\rangle^{\frac{3-\delta}{2}}\|\partial_z\mathfrak{H}(t)\|^2_{X_{\tau,\kappa_4,7/8}}dt.\end{aligned} \tag{8.14}$$

Estimates of Q^1. This is direct. Same as (8.5), we have

$$\|Q^1\|^2_{X_{\tau,\kappa_4,7/8}}\lesssim\epsilon\langle t\rangle^{2\delta-2}\|\partial_h^2\boldsymbol{u}\|^2_{X_{\tau,\kappa_4,7/8}}\lesssim\epsilon\langle t\rangle^{2\delta-2}\|\boldsymbol{u}\|^2_{X_{\tau,\kappa_4+4,7/8}}\lesssim\epsilon\langle t\rangle^{2\delta-1}\|\boldsymbol{g}\|^2_{X_{\tau,\kappa_0+1/2}}.$$

From this, we can obtain

$$\begin{aligned}\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\|Q^1\|^2_{X_{\tau,\kappa_4,7/8}}dt\lesssim&\epsilon\int_0^{T_0}\langle t\rangle^{\frac{3+\delta}{2}}\langle t\rangle^{\delta}\|\boldsymbol{g}\|^2_{X_{\tau,\kappa_0+1/2}}dt\\
\lesssim&\frac{\sqrt{\epsilon}}{\lambda}\int_0^{T_0}\langle t\rangle^{\frac{3+\delta}{2}}\|\boldsymbol{g}\|^2_{X_{\tau,\kappa_0+1/2}}dt.\end{aligned} \tag{8.15}$$

Here in the last line, we have used (2.4).

Estimates of Q^2. Similar as estimate in (8.7), we can obtain that

$$\|Q^2\|^2_{X_{\tau,\kappa_3,7/8}} \lesssim C_*^2\epsilon^2\left(\langle t\rangle^{-\frac{2-\delta}{2}}\|\partial_z\mathfrak{H}\|^2_{X_{\tau,\kappa_4,7/8}}+\langle t\rangle^{-\frac{2-\delta}{2}}\|g\|^2_{X_{\tau,\kappa_0+1/2}}\right).$$

From this, we can obtain

$$\begin{aligned}
&\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\|Q^2\|^2_{X_{\tau,\kappa_4,7/8}}dt\\
\lesssim& C_*^2\epsilon^2\int_0^{T_0}\left(\langle t\rangle^{\frac{3}{2}}\|\partial_z\boldsymbol{\mathfrak{H}}(t)\|^2_{X_{\tau,\kappa_4,7/8}}+\langle t\rangle^{\frac{3}{2}}\|\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0+1/2}}\right)dt\\
\lesssim& \frac{C_*^2}{\lambda}\epsilon^{3/2}\int_0^{T_0}\left(\langle t\rangle^{\frac{3-\delta}{2}}\|\partial_z\boldsymbol{\mathfrak{H}}(t)\|^2_{X_{\tau,\kappa_4,7/8}}+\langle t\rangle^{\frac{3+\delta}{2}}\|\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0+1/2}}\right)dt. \qquad (8.16)
\end{aligned}$$

Estimates of Q^3. Similar as estimate in (8.9) and using the a priori estimate in (3.8), we can obtain that

$$\begin{aligned}
\|Q^3\|^2_{X_{\tau,\kappa_4,7/8}} \lesssim& \langle t\rangle^{1/2}\left(\|\partial_z u\|^2_{X_{\tau,5,\frac{15}{32}}}\|\partial_h\mathcal{H}\|^2_{X_{\tau,\kappa_4,\frac{15}{32}}}+\|\partial_z u\|^2_{X_{\tau,\kappa_4,\frac{15}{32}}}\|\partial_h\mathcal{H}\|^2_{X_{\tau,5,\frac{15}{32}}}\right)\\
\lesssim& \langle t\rangle^{1/2}\left(\|\partial_z u\|^2_{X_{\tau,5,\frac{15}{32}}}\|\mathcal{H}\|^2_{X_{\tau,\kappa_3,\frac{15}{32}}}+\|\partial_z u\|^2_{X_{\tau,\kappa_4,\frac{15}{32}}}\|\mathcal{H}\|^2_{X_{\tau,7,\frac{15}{32}}}\right)\\
\lesssim& C_*^2\epsilon^2\left(\langle t\rangle^{-\frac{4-\delta}{2}}\|\mathfrak{H}\|^2_{X_{\tau,\kappa_3}}+\langle t\rangle^{-\frac{2-\delta}{2}}\|\boldsymbol{g}\|^2_{X_{\tau,\kappa_0+1/2}}\right).
\end{aligned}$$

From this, we can obtain

$$\begin{aligned}
&\int_0^{T_0}\langle t\rangle^{\frac{5-\delta}{2}}\|Q^3\|^2_{X_{\tau,\kappa_4,7/8}}dt\\
\lesssim& C_*^2\epsilon^2\int_0^{T_0}\left(\langle t\rangle^{\frac{1}{2}}\|\boldsymbol{\mathfrak{H}}(t)\|^2_{X_{\tau,\kappa_3,7/8}}+\langle t\rangle^{\frac{3}{2}}\|\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0+1/2}}\right)dt\\
\lesssim& \frac{C_*^2}{\lambda}\epsilon^{3/2}\int_0^{T_0}\left(\langle t\rangle^{\frac{1+\delta}{2}}\|\boldsymbol{\mathfrak{H}}(t)\|^2_{X_{\tau,\kappa_3,7/8}}+\langle t\rangle^{\frac{3+\delta}{2}}\|\boldsymbol{g}(t)\|^2_{X_{\tau,\kappa_0+1/2}}\right)dt. \qquad (8.17)
\end{aligned}$$

Inserting (8.15), (8.16) and (8.17) into (8.14), we can obtain (8.12) in Lemma 8.2. □

Acknowledgments. X. Pan is supported by National Natural Science Foundation of China (No. 11801268, No. 12031006) and the Fundamental Research Funds for the Central Universities of China (No. NS2023039). C. J. Xu is supported by National Natural Science Foundation of China (No. 12031006) and the Fundamental Research Funds for the Central Universities of China.

Appendix. Estimates in Lemma 3.1.

Proof. We only show estimates for u, $\mathcal{H}$ and $\mathcal{G}$ since the others are completely the same.

By the definition of g and boundary condition for u, we have

$$\begin{cases}\partial_z u+\dfrac{z}{2\langle t\rangle}u=g,\\ u\big|_{z=0}=0.\end{cases} \qquad (A.1)$$

Solving this ODE, we get

$$u(t,x_h,z)=\exp\left(-\frac{z^2}{4\langle t\rangle}\right)\int_0^z g(t,x_h,\bar z)\exp\left(\frac{\bar z^2}{4\langle t\rangle}\right)d\bar z.$$

For any $0\le\nu<1$, by multiplying the above equality with $\theta_\nu M_{k,\kappa}$ and taking ∂_h^k, we have

$$\theta_\lambda M_{k,\kappa}\partial_h^k u=\theta_{\lambda-1}(z)\int_0^z \theta(\bar z)M_{k,\kappa}\partial_h^k g(\bar z)\exp\left(\frac{1}{8\langle t\rangle}(\bar z^2-z^2)\right)d\bar z. \tag{A.2}$$

Using the fact that for any $\beta\ge 0$,

$$\sup_{z\geqq 0}\zeta^\beta e^{-\zeta^2}\le C_\beta,$$

we have

$$\left|\left(\frac{z}{\sqrt{\langle t\rangle}}\right)^\beta\theta_{\nu-1}\right|\le C_{\nu,\beta}.$$

Moreover, by considering $0\le\zeta\le 1$ and $\zeta>1$, it is not hard to check that

$$e^{-\zeta^2}\int_0^\zeta e^{\bar\zeta^2}d\bar\zeta\le\frac{2}{1+\zeta}.$$

Then a change of variable indicates that

$$\int_0^z\exp\left(\frac{1}{4\langle t\rangle}(\bar z^2-z^2)\right)d\bar z\le\frac{C}{1+\zeta}\sqrt{\langle t\rangle}.$$

In (A.2), by using Hölder inequality on z, we have

$$\begin{aligned}\left|\theta_\nu M_{k,\kappa}\partial_h^k u\right| &\le\theta_{\nu-1}\|\theta M_{k,\kappa}\partial_h^k g\|_{L^2_z}\left(\int_0^z\exp\left(\frac{1}{4\langle t\rangle}(\bar z^2-z^2)\right)d\bar z\right)^{1/2}\\ &\lesssim\theta_{\nu-1}\|\theta M_{k,\kappa}\partial_h^k g\|_{L^2_z}\langle t\rangle^{1/4}(1+\zeta)^{-1/2}\\ &\lesssim\theta_{\nu-1}\|\theta M_{k,\kappa}\partial_h^k g\|_{L^2_z}\langle t\rangle^{1/4}.\end{aligned}\tag{A.3}$$

Also, from the definition of $\mathfrak{H}$ in (3.6) and the fact in (3.3), we see that

$$\begin{cases}\partial_z\left(\int_z^\infty\mathcal{H}\bar z\right)+\frac{z}{2\langle t\rangle}\left(\int_z^\infty\mathcal{H}\bar z\right)=-\mathfrak{H},\\ \int_z^\infty\mathcal{H}\bar z\Big|_{z=0}=0.\end{cases}\tag{A.4}$$

Similar as (A.3), we can have for $0<\nu<7/8$

$$\left|\theta_\nu M_{k,\kappa}\partial_h^k\int_z^\infty\mathcal{H}d\bar z\right|\lesssim\theta_{\nu-7/8}\|\theta_{7/8}M_{k,\kappa}\partial_h^k\mathfrak{H}\|_{L^2_z}\langle t\rangle^{1/4}.\tag{A.5}$$

From (A.3), by using the a priori Assumption (3.7), it is easy to see that

$$\langle t\rangle^{-1/2}\left\|\theta_\nu M_{k,\kappa}\partial_h^k u\right\|_{L^2}\lesssim\langle t\rangle^{-1/2}\|\theta_{\nu-1}\|_{L^2_z}\|\theta M_{k,\kappa}\partial_h^k g\|_{L^2}\langle t\rangle^{1/4}$$

$$\lesssim \|\theta M_{k,\kappa}\partial_h^k g\|_{L^2} \lesssim C_*\epsilon\langle t\rangle^{-\frac{5-\delta}{4}}. \tag{A.6}$$

By squaring (A.6) and summing the resulting equation over $k\in\mathbb{N}$, we obtain that

$$\langle t\rangle^{-1}\|u\|^2_{X_{\tau,12,\nu}} \lesssim \|g\|^2_{X_{\tau,12}} \lesssim C_*\epsilon\langle t\rangle^{-\frac{5-\delta}{2}}. \tag{A.7}$$

By using (A.5) and same as (A.7), we have, for $0<\nu<7/8$

$$\langle t\rangle^{-1}\left\|\int_z^\infty \mathcal{H}d\bar z\right\|^2_{X_{\tau,9,\nu}} \lesssim \|\mathfrak{H}\|^2_{X_{\tau,9,7/8}} \lesssim C_*\epsilon\langle t\rangle^{-\frac{3-\delta}{2}}. \tag{A.8}$$

Applying $\theta_\nu\partial_h^k$, $\theta_\nu\partial_z\partial_h^k$ and $\theta_\nu\partial_z^2\partial_h^k$ to the first equation of (A.1) respectively give that

$$\begin{aligned}
&\theta_\nu\partial_z\partial_h^k u = \theta_\nu\partial_h^k g - \theta_\nu\frac{z}{2\langle t\rangle}\partial_h^k u,\\
&\theta_\nu\partial_z^2\partial_h^k u = \theta_\nu\partial_z\partial_h^k g - \theta_\nu\frac{1}{2\langle t\rangle}\partial_h^k u - \theta_\nu\frac{z}{2\langle t\rangle}\partial_z\partial_h^k u,\\
&\theta_\nu\partial_z^3\partial_h^k u = \theta_\nu\partial_z^2\partial_h^k g - \theta_\nu\frac{1}{\langle t\rangle}\partial_h^k\partial_z u - \theta_\nu\frac{z}{2\langle t\rangle}\partial_z^2\partial_h^k u.
\end{aligned}$$

From the above equalities, we can easily deduce that

$$\|\partial_z u\|_{X_{\tau,12,\nu}} \lesssim \|g\|_{X_{\tau,12,\nu}} + \langle t\rangle^{-1/2}\|u\|_{X_{\tau,12,\frac{\nu+1}{2}}} \lesssim_\nu C_*\epsilon\langle t\rangle^{-\frac{5-\delta}{4}}, \tag{A.9}$$

$$\begin{aligned}
\sqrt{\delta}\langle t\rangle^{1/2}\|\partial_z^2 u\|_{X_{\tau,10,\nu}} &\lesssim \sqrt{\delta}\langle t\rangle^{1/2}\|\partial_z g\|_{X_{\tau,10,\nu}} + \sqrt{\delta}\langle t\rangle^{-1/2}\|u\|_{X_{\tau,10,\nu}}\\
&\quad + \sqrt{\delta}\|\partial_z u\|_{X_{\tau,10,\frac{\nu+1}{2}}} \lesssim_\nu C_*\epsilon\langle t\rangle^{-\frac{5-\delta}{4}},
\end{aligned} \tag{A.10}$$

$$\delta\langle t\rangle\|\partial_z^3 u\|_{X_{\tau,8,\nu}} \lesssim \delta\langle t\rangle\|\partial_z^2 g\|_{X_{\tau,8,\nu}} + \delta\|\partial_z u\|_{X_{\tau,8,\nu}} + \delta\langle t\rangle^{1/2}\|\partial_z^2 u\|_{X_{\tau,8,\frac{\nu+1}{2}}} \lesssim_\nu C_*\epsilon\langle t\rangle^{-\frac{5-\delta}{4}}. \tag{A.11}$$

The above estimates in (A.7), (A.9), (A.10) and (A.11) indicate the first inequality in (3.8).

Applying $\theta_\nu\partial_h^k$ and $\theta_\nu\partial_z\partial_h^k$ to the first equation of (A.4) respectively give that

$$\begin{aligned}
&\theta_\nu\partial_h^k\mathcal{H} = \theta_\nu\partial_h^k\mathfrak{H} + \theta_\nu\frac{z}{2\langle t\rangle}\partial_h^k\int_z^\infty \mathcal{H}d\bar z,\\
&\theta_\nu\partial_z\partial_h^k\mathcal{H} = \theta_\nu\partial_h^k\partial_z\mathfrak{H} + \frac{\theta_\nu}{2\langle t\rangle}\partial_h^k\int_z^\infty \mathcal{H}d\bar z - \frac{z}{2\langle t\rangle}\theta_\nu\partial_h^k\mathcal{H}.
\end{aligned}$$

From the above equalities and by using (A.8), we can easily deduce that

$$\|\mathcal{H}\|_{X_{\tau,\kappa,3/4}} \lesssim \|\mathfrak{H}\|_{X_{\tau,\kappa,3/4}} + \langle t\rangle^{-1/2}\left\|\int_z^\infty \mathcal{H}d\bar z\right\|_{X_{\tau,\kappa,\frac{13}{16}}} \lesssim_\nu C_*\epsilon\langle t\rangle^{-\frac{3-\delta}{4}}, \tag{A.12}$$

$$\begin{aligned}
\langle t\rangle^{1/2}\|\partial_z\mathcal{H}\|_{X_{\tau,\kappa,3/4}} &\lesssim \langle t\rangle^{1/2}\|\partial_z\mathfrak{H}\|_{X_{\tau,\kappa,3/4}} + \langle t\rangle^{-1/2}\left\|\int_z^\infty \mathcal{H}d\bar z\right\|_{X_{\tau,\kappa,3/4}} + \|\mathcal{H}\|_{X_{\tau,\kappa,\frac{13}{16}}}\\
&\lesssim_\nu C_*\epsilon\langle t\rangle^{-\frac{3-\delta}{4}}.
\end{aligned} \tag{A.13}$$

The above estimates in (A.12) and (A.13) indicate the second inequality in (3.8).

For the estimate of $\mathcal{G}$, we need to use a product estimate in (3.27) in Lemma (3.3). From the representation of $\mathcal{G}$, we have that

$$\|\mathcal{G}\|_{X_{\tau,9,3/4}} \lesssim \|\partial_x u\|_{X_{\tau,9,3/4}} + \frac{\langle t\rangle^{1-\delta}}{\sqrt{\epsilon}}\left\|\partial_z u\int_z^\infty \mathcal{H}d\bar z\right\|_{X_{\tau,9,3/4}}$$

$$\lesssim \|u\|_{X_{\tau,11,3/4}} + \frac{\langle t\rangle^{5/4-\delta}}{\sqrt{\epsilon}}\left(\|\partial_z u\|_{X_{\tau,5,3/4}}\|\mathcal{H}\|_{X_{\tau,9,3/4}} + \|\partial_z u\|_{X_{\tau,9,3/4}}\|\mathcal{H}\|_{X_{\tau,5,3/4}}\right)$$
$$\lesssim C_*\epsilon\langle t\rangle^{-\frac{3-\delta}{4}} + C_*^2\epsilon^{3/2}\langle t\rangle^{-\frac{3+2\delta}{4}} \lesssim C_*\epsilon\langle t\rangle^{-\frac{3-\delta}{4}}, \tag{A.14}$$

by letting ϵ small such that $C_*\sqrt{\epsilon}\leq 1$. Also by using (3.27), we can see that

$$\sqrt{\delta}\langle t\rangle^{1/2}\|\mathcal{G}\|_{X_{\tau,7,3/4}}$$
$$\lesssim \sqrt{\delta}\langle t\rangle^{1/2}\|\partial_x\partial_z u\|_{X_{\tau,7,3/4}} + \sqrt{\delta}\frac{\langle t\rangle^{3/2-\delta}}{\sqrt{\epsilon}}\left(\left\|\partial_z^2 u\int_z^\infty \mathcal{H}d\bar{z}\right\|_{X_{\tau,7,3/4}} + \|\partial_z u\mathcal{H}\|_{X_{\tau,7,3/4}}\right)$$
$$\lesssim \sqrt{\delta}\langle t\rangle^{1/2}\|\partial_z u\|_{X_{\tau,9,3/4}}$$
$$+\sqrt{\delta}\frac{\langle t\rangle^{7/4-\delta}}{\sqrt{\epsilon}}\left(\left\|\partial_z^2 u\right\|_{X_{\tau,5,3/4}}\|\mathcal{H}\|_{X_{\tau,7,3/4}} + \left\|\partial_z^2 u\right\|_{X_{\tau,7,3/4}}\|\mathcal{H}\|_{X_{\tau,5,3/4}}\right) \lesssim C_*\epsilon\langle t\rangle^{-\frac{3-\delta}{4}}. \tag{A.15}$$

Combining estimates in (A.14) and (A.15), we obtain the third line of (3.8).

Now, we use Sobolev embedding and (3.8) to show the low order estimates in (3.9). First by using Sobolev embedding in x,y directions and (3.8), we can easily obtain that for $0\leq k\leq 51$ and $0\leq \nu<1$, we have

$$\langle t\rangle^{-\frac{1}{2}}\left\|\theta_\nu\partial_h^k(u,v)\right\|_{L_h^\infty L_z^2} + \left\|\theta_\nu\partial_h^k\partial_z(u,v)\right\|_{L_h^\infty L_z^2}$$
$$+\sqrt{\delta}\langle t\rangle^{\frac{1}{2}}\left\|\theta_\nu\partial_z^2\partial_h^k(u,v)\right\|_{L_h^\infty L_z^2} + \delta\langle t\rangle\left\|\theta_\nu\partial_z^3\partial_h^k(u,v)\right\|_{L_h^\infty L_z^2} \lesssim C_*\epsilon\langle t\rangle^{-\frac{5-\delta}{4}}. \tag{A.16}$$

Also for any f which decays fast enough at z infinity, we have

$$|\theta_\nu(z)f(z)| = \theta_\nu(z)\left|\int_z^\infty \partial_z f d\bar{z}\right| \leq \left|\int_z^\infty \theta_{\frac{\nu-1}{2}}(\bar{z})\theta_{\frac{\nu+1}{2}}(\bar{z})\partial_z f d\bar{z}\right|$$
$$\leq \left\|\theta_{\frac{\nu-1}{2}}\right\|_{L_z^2}\left\|\theta_{\frac{\nu+1}{2}}(z)\partial_z f\right\|_{L_z^2} \lesssim_\nu \langle t\rangle^{1/4}\left\|\theta_{\frac{\nu+1}{2}}(z)\partial_z f\right\|_{L_z^2}.$$

This indicates that for $0\leq\nu<1$, we have

$$\|\theta_\nu(z)f(z)\|_{L^\infty} \lesssim \langle t\rangle^{1/4}\left\|\theta_{\frac{\nu+1}{2}}(z)\partial_z f\right\|_{L_h^\infty L_z^2}. \tag{A.17}$$

Combining (A.16) and (A.17), we see that

$$\langle t\rangle^{-\frac{1}{4}}\left\|\theta_\nu\partial_h^k(u,v)\right\|_{L^\infty} + \sqrt{\delta}\langle t\rangle^{\frac{1}{4}}\left\|\theta_\nu\partial_z\partial_h^k(u,v)\right\|_{L^\infty} + \delta\langle t\rangle^{\frac{3}{4}}\left\|\theta_\nu\partial_z^2\partial_h^k(u,v)\right\|_{L^\infty} \lesssim C_*\epsilon\langle t\rangle^{-\frac{5-\delta}{4}}. \tag{A.18}$$

Also from (A.17) and incompressibility, we have

$$\langle t\rangle^{-3/4}\|\theta_\nu\partial_h^k w\|_{L^\infty} \lesssim \langle t\rangle^{-1/2}\left\|\theta_{\frac{\nu+1}{2}}(z)\partial_z\partial_h^k w\right\|_{L_h^\infty L_z^2}$$
$$\lesssim \langle t\rangle^{-1/2}\left\|\theta_{\frac{\nu+1}{2}}(z)\partial_h^{k+1}\boldsymbol{u}\right\|_{L_h^\infty L_z^2} \lesssim C_*\epsilon\langle t\rangle^{-\frac{5-\delta}{4}}. \tag{A.19}$$

Combining (A.18) and (A.19), we can obtain (3.9) in Lemma 3.1. ∎

REFERENCES

[1] R. Alexandre, Y.G. Wang, C.J. Xu, and T. Yang, *Well-posedness of the Prandtl equation in Sobolev spaces*, J. Amer. Math. Soc., 28(3):745–784, 2015. 1

[2] A.L. Dalibard and N. Masmoudi, *Separation for the stationary Prandtl equation*, Publ. Math. Inst. Hautes Études Sci., 130:187–297, 2019. 1

[3] H. Dietert and D. Gérard-Varet, *Well-posedness of the Prandtl equations without any structural assumption*, Ann. PDE, 5(1):8, 2019. 1, 3, 3.1, 3.1

[4] W. E and B. Engquist, *Blowup of solutions of the unsteady Prandtl's equation*, Commun. Pure Appl. Math., 50(12):1287–1293, 1997. 1

[5] E. Grenier, *On the nonlinear instability of Euler and Prandtl equations*, Commun. Pure Appl. Math., 53(9):1067–1091, 2000. 1

[6] E. Grenier, Y. Guo, and T.T. Nguyen, *Spectral instability of characteristic boundary layer flows*, Duke Math. J., 165(16):3085–3146, 2016. 1

[7] E. Grenier and T.T. Nguyen, *L^∞ instability of Prandtl layers*, Ann. PDE, 5(2):18, 2019. 1

[8] Y. Guo and T. Nguyen, *A note on Prandtl boundary layers*, Commun. Pure Appl. Math., 64(10):1416–1438, 2011. 1

[9] D. Gérard-Varet and E. Dormy, *On the ill-posedness of the Prandtl equation*, J. Amer. Math. Soc., 23(2):591–609, 2010. 1

[10] D. Gerard-Varet and N. Masmoudi, *Well-posedness for the Prandtl system without analyticity or monotonicity*, Ann. Sci. Éc. Norm. Supér., 48(6):1273–1325, 2015. 1

[11] D. Gérard-Varet and T. Nguyen, *Remarks on the ill-posedness of the Prandtl equation*, Asymptot. Anal., 77(1-2):71–88, 2012. 1

[12] M. Ignatova and V. Vicol, *Almost global existence for the Prandtl boundary layer equations*, Arch. Ration. Mech. Anal., 220(2):809–848, 2016. 1, 3.2

[13] I. Kukavica and V. Vicol, *On the local existence of analytic solutions to the Prandtl boundary layer equations*, Commun. Math. Sci., 11(1):269–292, 2013. 1

[14] X. Lin and T. Zhang, *Almost global existence for the 3D Prandtl boundary layer equations*, Acta Appl. Math., 169:383–410, 2020. 1

[15] C.J. Liu, Y.G. Wang, and T. Yang, *A well-posedness theory for the Prandtl equations in three space variables*, Adv. Math., 308:1074–1126, 2017. 1

[16] C.J. Liu and T. Yang, *Ill-posedness of the Prandtl equations in Sobolev spaces around a shear flow with general decay*, J. Math. Pures Appl., 108(2):150–162, 2017. 1

[17] W.X. Li, R. Xu, and T. Yang, *Global well-posedness of a Prandtl model from MHD in Gevrey function spaces*, Acta Math. Sci. Ser. B (Engl. Ed.), (6):2343–2366, 2022. 1

[18] W.X. Li and T. Yang, *Well-posedness in Gevrey function spaces for the Prandtl equations with non-degenerate critical points*, J. Eur. Math. Soc., 22(3):717–775, 2020. 1

[19] W.X. Li, N. Masmoudi, and T. Yang, *Well-posedness in Gevrey function space for 3D Prandtl equations without structural assumption*, Commun. Pure Appl. Math., 75(8):1755–1797, 2022. 1, 2.3, 3, 3.1, 3.1, 3.1, 3.5

[20] M.C. Lombardo, M. Cannone, and M. Sammartino, *Well-posedness of the boundary layer equations*, SIAM J. Math. Anal., 35(4):987–1004, 2003. 1

[21] N. Masmoudi and T.K. Wong, *Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods*, Commun. Pure Appl. Math., 68(10):1683–1741, 2015. 1

[22] F.K. Moore, *Three-dimensional boundary layer theory*, Adv. Appl. Mech., 4:159–228, 1956. 1

[23] O.A. Oleinik and V.N. Samokhin, *Mathematical Models in Boundary Layer Theory*, Chapman & Hall/CRC, Boca Raton, FL, 15, 1999. 1

[24] M. Paicu and P. Zhang, *Global existence and the decay of solutions to the Prandtl system with small analytic data*, Arch. Ration. Mech. Anal., 241(1):403–446, 2021. 1, 1, 1, 3.2

[25] X. Pan and C.J. Xu, *Global tangentially analytical solutions of the 3D axially symmetric Prandtl equations*, Chinese Ann. Math. Ser. B, to appear. 1

[26] L. Prandtl, *Über Flüssigleitsbewegung bei sehr kleiner Reibung*, Verhandlung des III Intern. Math. Kongresses, Heidelberg: 484–491, 1904. 1

[27] M. Sammartino and R.E. Caflisch, *Zero viscosity limit for analytic solutions, of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations*, Commun. Math. Phys., 192(2):433–461, 1998. 1

[28] W. Shen, Y. Wang, and Z. Zhang, *Boundary layer separation and local behavior for the steady Prandtl equation*, Adv. Math., 389:107896, 2021. 1

[29] C. Wang, Y. Wang, and P. Zhang, *On the global small solution of 2-D Prandtl system with initial data in the optimal Gevrey class*, Adv. Math., 440:109517, 2024. 1, 1, 3.6

[30] Z. Xin and L. Zhang, *On the global existence of solutions to the Prandtl's system*, Adv. Math., 181(1):88–133, 2004. 1

[31] C.J. Xu and X. Zhang, *Long time well-posedness of Prandtl equations in Sobolev space*, J. Differ. Equ., 263(12):8749–8803, 2017. 1

[32] P. Zhang and Z. Zhang, *Long time well-posedness of Prandtl system with small and analytic initial data*, J. Funct. Anal., 270(7):2591–2615, 2016. 1