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Global Gevrey-2 solutions of the 3D axially symmetric Prandtl equations

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In this paper, we prove the global existence of small Gevrey-2 solutions to the 3D axially symmetric Prandtl equations. The index 2 is the optimal index for well-posedness result in smooth Gevrey function spaces for data without monotonic assumptions. The novelty of our paper lies in two aspects: one is the tangentially weighted energy construction to match the r weight in the incompressibility and the other is introducing of the new linearly good unknowns to obtain the fast decay of the lower order Gevrey-2 norms of the solutions and auxiliary functions.

Keywords: Global existence; Gevrey-2 solutions; axially symmetric; Prandtl equations.

Mathematics Subject Classification 2020: 35Q35, 76D03

1. Introduction

In this paper, we focus on study on the well-posedness of the initial-boundary value problem to the three-dimensional axially symmetric Prandtl equations in $\Omega := \{t > 0, r > 0, z > 0\}$, which read as follows:

$$\begin{cases} \partial_t \tilde{u} + (\tilde{u}\partial_r + \tilde{v}\partial_z)\tilde{u} + \partial_r p = \partial_z^2 \tilde{u}, \\ \partial_r(r\tilde{u}) + \partial_z(r\tilde{v}) = 0, \\ (\tilde{u}, \tilde{v})|_{z=0} = 0, \quad \tilde{u}|_{r=0} = 0, \quad \lim_{z\to+\infty} \tilde{u} = U(t, r), \\ \tilde{u}|_{t=0} = \tilde{u}_0(r, z), \end{cases} \tag{1.1}$$

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where $U(t, r)$ and $p(t, r)$ are the tangential velocity field and pressure of the Euler flow, satisfying

$$\begin{cases} \partial_t U + U\partial_r U + \partial_r p = 0 \quad \text{in } \Omega, \\ U|_{r=0} = 0. \end{cases}$$

The Prandtl equations was proposed by Prandtl [38] in 1904 in order to explain the boundary layer phenomenon, mismatch of the no-slip boundary condition of the Navier–Stokes equations with that of the Euler equation on a solid boundary when the viscosity of the Navier–Stokes equations approaches to zero. While the 3D axially symmetric Prandtl equations is a model, which deals with a flow past a body of revolution at rest. Here, r represents the distance to the symmetric axis, which we name by horizontal variable and z is the vertical variable. $\tilde{u}$ and $\tilde{v}$ are the horizontally radial component and vertical component of the velocity in cylindrical coordinates. This model (1.1) was introduced in [34, Chap. 3] and see Loitsyansky [28] or Pan and Xu [36] for a derivation. The 3D axially symmetric Prandtl equations has been studied rigorously in mathematics in [34, Chap. 3] for stationary case and [34, Chap. 4] for nonstationary case, where some well-posedness results are given based on the von Mises transformation and Crocco transformation.

The boundary layer equations (1.1) is a degenerate parabolic equations and no tangential diffusion for the tangential velocity. Also, the vertical velocity can only be recovered from the divergence-free condition, which results in one order derivative loss in the tangential equation $(1.1)_1$ for the term $\tilde{v}\partial_z\tilde{u}$. Such degeneration and derivative loss make the Prandtl equations bear underlying instability, such as the phenomenon of separation of boundary layer, induced by the appearance of reverse flow. See [4, 13–15, 40] and references therein for some recent results in this aspect.

So far, the well-posedness study on the Prandtl equations has proceeded twofold: well-posedness in Sobolev spaces under Oleinik's monotonic assumption and well-posedness in Gevrey smooth spaces without monotonicity. The pioneering works in these two aspects are attributed to Oleinik and Samokhin [34], where the local well-posedness in Sobolev spaces for the two-dimensional case was proved by using Crocco transform, and Sammartino and Caflisch [39], where the local well-posedness in analytical spaces was established by using the abstract Cauchy–Kowalewski theorem. Recently, by using a nice cancellation technique, the local well-posedness in Sobolev spaces was revisited by two groups of researchers independently, see [1, 32]. Under an additional favorable sign condition on the pressure, the global in-time weak solution was shown in Xin and Zhang [44]. We also mention [26], where the local well-posedness result in Sobolev spaces for the three-dimensional Prandtl equations was given under some constraints on the flow structure in addition to the monotonicity.

Without monotonicity assumption, the boundary layer separation may occur and ill-posedness in Sobolev spaces is expected. Recently, there have been a lot of studies in this aspect, here we mention E-Engquist [6], where a globally ill-posedness result was given and Gérard-Varet and Dormy [8], where the locally ill-posedness was presented. See some extensions and improvements in [12, 17, 27]. The result in [8] indicates that the optimal index for well-posedness result in smooth Gevrey function spaces is 2, which was proved in [5] for the two-dimensional case and most recently in [24] for the three-dimensional case without structure assumption. Here, we mention [11, 22, 25, 29] and references therein for some related well-posedness results in analytical spaces and Gevrey class spaces.

In this paper, we consider the long time behavior of the solution to system (1.1) in the optimal Gevrey-2 spaces in the sense of the instability results considered in [8]. We will consider the simplified case that the outflow $U \equiv 0$. The general case for sufficiently small and fast-decay in time outflow can also be addressed with a lifting trick. Then system (1.1) is simplified to

$$\begin{cases} \partial_t \tilde{u} + (\tilde{u}\partial_r + \tilde{v}\partial_z)\tilde{u} - \partial_z^2 \tilde{u} = 0, \\ \partial_r(r\tilde{u}) + \partial_z(r\tilde{v}) = 0, \\ (\tilde{u}, \tilde{v})|_{z=0} = 0, \quad \tilde{u}|_{r=0} = 0, \quad \lim\limits_{z\to+\infty} \tilde{u} = 0, \\ \tilde{u}|_{t=0} = \tilde{u}_0. \end{cases} \tag{1.2}$$

Actually there already are some long-time existence results in Gevrey class spaces for the 2D and 3D Prandtl equations. See [18, 35–37, 43, 45]. Here, we mention reference [43], where the global existence of small Gevrey-2 solutions was proved for 2D Prandtl equations and [37], where nearly almost existence of Gevrey-2 solutions for 3D Prandtl equations was given.

All the results mentioned above are closely related to the vanishing viscosity limits of the Navier–Stokes equation with a high Reynolds number. Without the boundary layer effect, the mathematical theory of vanishing viscosity limit now is satisfactory and rather complete. See for example [2, 20, 31, 41]. If the boundary effect appears, the situation is much more complicated and some of results are much more recent. Kato [21] gave a necessary and sufficient condition to state the vanishing viscosity holds if and only if the total dissipation of the energy at the boundary with a width of $O(\nu)$ vanishes as the viscosity $\nu \to 0$. Recently, the vanishing viscosity was derived in Maekawa [30] by assuming that the initial vorticity is away from the boundary in Sobolev spaces for the two-dimensional case, which was extended to the three-dimensional case in [7]. See also a related result in [23], where the vanishing viscosity stands for data only analytical near the boundary. Readers can refer to [3, 33, 42] and references therein for more information on this topic. At last, we mention some results on the stability of Prandtl expansion to the stationary and nonstationary Navier–Stokes equations in [9, 10, 16, 19] and references therein.

2. Reformulation of Our Problem and the Main Theorem

2.1. *Reformulation of the equations*

Since $\tilde{u}|_{r=0} = 0$, there isn't singularity for the quantity $\tilde{u}^r/r$ at $r = 0$. Set the new unknowns

$$(u, v) := \left(\frac{\tilde{u}}{r}, \tilde{v}\right), \tag{2.1}$$

which after direct calculation from (1.2), satisfy

$$\begin{cases} \partial_t u + (ur\partial_r + v\partial_z)u - \partial_z^2 u + u^2 = 0, \\ r\partial_r u + 2u + \partial_z v = 0, \\ (u, v)|_{z=0} = 0, \quad \lim\limits_{z\to+\infty} u = 0, \\ u|_{t=0} = u_0(r, z). \end{cases} \tag{2.2}$$

We will state our main result in the framework of the reformulated system (2.2). Before that, we need to give some notations.

2.2. *Notations*

For $\kappa \in \mathbb{R}_+$, $j \in \mathbb{N}$ and a time-dependent function $\tau(t)$, we define

$$M_{j,\kappa} := \frac{\tau(t)^{j+1}(j+1)^\kappa}{(j!)^2}.$$

Later, for simplicity, we will abbreviate $\tau(t)$ by τ if no ambiguity is caused.

For a function f and some weighted function $\omega(t, r, z)$, denote the spacial L^2 norm by

$$\|f(t)\|_{L^2}^2 := \int_0^\infty \int_0^\infty f^2 drdz \quad \text{and} \quad \|f\|_{L^2(\omega)}^2 := \int_0^\infty \int_0^\infty f^2 \omega drdz.$$

Now, for $\nu \in \mathbb{R}$, let $\theta_\nu := \exp(\frac{\nu z^2}{8\langle t\rangle})$ and simply denote θ_1 by θ. It is easy to see that $\theta_{\alpha+\beta} = \theta_\alpha \cdot \theta_\beta$ for $\alpha, \beta \in \mathbb{R}$. For $j \in \mathbb{N}$, denote

$$[r]^j = r^j + r^{(j-1)_+},$$

where $(j-1)_+ = \max\{j-1, 0\}$.

Let

$$f_{j,\kappa} := M_{j,\kappa}[r]^j \partial_r^j f.$$

Then, for $0 \le \nu \le 1$, we define the weighted Gevrey-2 norm as follows:

$$\|f\|_{X_{\tau,\kappa,\nu}}^2 = \sum_{j\in\mathbb{N}} \|M_{j,\kappa}[r]^j \partial_r^j f \theta_\nu\|_{L^2}^2 = \sum_{j\in\mathbb{N}} \|f_{j,\kappa}\|_{L^2(\theta_{2\nu})}^2,$$

where when $\nu = 1$, we simply denote $\|f\|_{X_{\tau,\kappa,1}}$ by $\|f\|_{X_{\tau,\kappa}}^2$.

Next, we choose the Gevrey-2 radius $\tau(t)$ as following, which has a lower positive bound for any $t \in (0, +\infty)$.

Setting of the Gevrey-2 radius.

For any fixed $\delta \in (0, 1/50]$ and $\tau_0 > 0$, define

$$\tau(t) := \tau_0 - \lambda\delta^{-1}\sqrt{\epsilon}\tau_0 \left(1 - \langle t\rangle^{-\delta}\right), \tag{2.3}$$

where $\langle t\rangle := (1 + t)$ and λ is an absolutely large constant, which is independent of ϵ, and will be determined later. We can choose sufficiently small ϵ such that $\lambda\delta^{-1}\sqrt{\epsilon} \le 1/2$. Then we can have, for any $t > 0$,

$$\frac{1}{2}\tau_0 \le \tau(t) \le \tau_0. \tag{2.4}$$

Direct computation shows that

$$\dot{\tau}(t) := -\lambda\sqrt{\epsilon}\tau_0\langle t\rangle^{-1-\delta}.$$

Then we denote that

$$\lambda\sqrt{\epsilon}\eta(t) = -\frac{\dot{\tau}(t)}{\tau(t)} = \frac{\lambda\sqrt{\epsilon}\tau_0\langle t\rangle^{-1-\delta}}{\tau(t)}.$$

Then, by using (2.4), we see that

$$\langle t\rangle^{-1-\delta} \le \eta(t) \le 2\langle t\rangle^{-1-\delta}. \tag{2.5}$$

2.3. *The main theorem*

Before stating the main theorem, we introduce a linearly good unknowns g, which can give a fast decay rate to the lower order Gevrey-2 norms of u. Define

$$g := u - \frac{z}{2\langle t\rangle}\int_z^\infty u\,d\bar{z},$$

then we have the following theorem. Below we use g_0 to denote the initial data of g.

Theorem 2.1. *Assume that the initial data satisfies the following compatibility conditions at* $z = 0$:

$$\partial_z^{2k}u_0|_{z=0} = 0, \quad \textit{for } k = 0, 1, 2, \quad \textit{and} \quad \int_0^{+\infty} u_0(r, z)dz = 0. \tag{2.6}$$

For any fixed $\tau_0 > 0$, $\delta \in (0, \frac{1}{50}]$, *there exist constants* ϵ_0 *and* C, *such that for any* $\epsilon \le \epsilon_0$, *if*

$$\|u_0\|_{X_{\tau_0,17}} + \sum_{k=0}^{3}\delta^{\frac{k}{2}}\|\partial_z^k g_0\|_{X_{\tau_0,11-2k}} \le \epsilon,$$

then system (2.2) *has a unique solution* (u, v) *satisfying for any* $t > 0$,

$$\langle t\rangle^{\frac{1-\delta}{4}}\|u(t)\|_{X_{\tau,16}} + \langle t\rangle^{\frac{5-\delta}{4}}\sum_{k=0}^{3}(\delta\langle t\rangle)^{\frac{k}{2}}\|\partial_z^k g(t)\|_{X_{\tau,11-2k}} \le C\epsilon. \tag{2.7}$$

Throughout the paper, $C_{a,b,c,\ldots}$ denotes a positive constant depending on a, b, c, $\ldots$, which may be different from line to line. Dependence on the initial Gevrey radius τ_0 and the fixed constant δ is default, we will denote $C_{\tau_0,\delta}$ by C for simplicity. We also apply $A \lesssim_{a,b,c,\ldots} B$ to denote $A \leq C_{a,b,c,\ldots}B$. For a norm $\|\cdot\|$, we use $\|(f,g,\ldots)\|$ to denote $\|f\| + \|g\| + \cdots$. For a function $f(t,r,z)$ and $1 \leq p,q \leq +\infty$, define

$$\|f(t)\|_{L^p_r L^q_z} := \left(\int_0^{+\infty}\left(\int_0^{+\infty}|f|^p dr\right)^{q/p} dz\right)^{1/q}.$$

If $p = q$, we simply write it as $\|f(t)\|_{L^p}$ and besides, if $p = q = 2$, we will simply denote it as $\|f(t)\|$. We use $[A,B] = AB - BA$ to denote the commutator of A and B. $\langle\cdot,\cdot\rangle_\omega$ denote weighted L^2 inner product with respect to spacial variables, which means for f and g,

$$\langle f,g\rangle_\omega := \int_0^{+\infty}\int_0^{+\infty} f(r,z)g(r,z)\omega dr dz.$$

3. Proof of the Main Theorem

3.1. *Introduction of auxiliary functions*

First, we introduce the following auxiliary function $\mathcal{A}$ by

$$\begin{cases} \left[\partial_t + (ur\partial_r + v\partial_z) - \partial_z^2\right]\displaystyle\int_z^{+\infty}\mathcal{A}d\bar{z} = \sqrt{\epsilon}\langle t\rangle^{-\delta-1}r\partial_r v, \\ \mathcal{A}|_{t=0} = 0, \quad \partial_z\mathcal{A}|_{z=0} = 0, \quad \mathcal{A}|_{z\to+\infty} = 0. \end{cases} \tag{3.1}$$

The existence of $\mathcal{A}$ follows the standard linear parabolic theory. This auxiliary function is inspired by Dietert and Gérard-Varet [5] and Li *et al.* [24], where a similar auxiliary function is constructed to prove the local well-posedness of the 2D and 3D Prandtl equations in Gevrey-2 spaces. The main difference is the following.

(1) In (3.1), The purpose that we define the auxiliary function by $\int_z^{+\infty}\mathcal{A}d\bar{z}$ instead of $\int_0^z \mathcal{A}d\bar{z}$ is to ensure the solution $\mathcal{A}$ decays fast enough when z approaches infinity.
(2) The time-dependent coefficient $\sqrt{\epsilon}\langle t\rangle^{-\delta-1}$ on the right hand of (3.1) is specially designed to match with $\eta(t)$ in (2.5), which can ensure closing of Gevrey-2 energy defined for $\mathcal{A}$. Also, there is a r weight for the tangential derivative ∂_r, which is caused by our transform of unknowns in (2.1).

Remark 3.1. Here, we remark that

$$\int_0^{\infty}\mathcal{A}d\bar{z} = 0. \tag{3.2}$$

Actually by letting $z = 0$ in (3.1) and using that the boundary condition of $\partial_z \mathcal{A}$ and u, v on $z = 0$, we can achieve that

$$\partial_t \int_0^{+\infty} \mathcal{A} d\bar{z} = 0.$$

By using the fact that $\mathcal{A}|_{t=0} = 0$, the above transport equation indicates (3.2).

Then define

$$\varphi_j = [r]^j \partial_r^j u + \frac{\langle t \rangle^{1+\delta} \partial_z u}{\sqrt{\epsilon}} \int_z^{\infty} [r]^{j-1} \partial_r^{j-1} \mathcal{A} d\bar{z}.$$

From (6.4), we see that

$$[\partial_t + (ur\partial_r + v\partial_z) - \partial_z^2]\varphi_j = \text{l.o.t.},$$

where l.o.t. represent terms, which have no derivative loss for the equation. Gevrey-2 norm estimate for the above equation is easy. So after performing Gevrey-2 energy estimates for $\mathcal{A}$, we can obtain the Gevrey-2 norm estimate of the solution u.

Applying $-\partial_z$ to (3.1), we can obtain, as shown in (4.1), that

$$[\partial_t + (ur\partial_r + v\partial_z) - \partial_z^2]\mathcal{A} = \sqrt{\epsilon}\langle t \rangle^{-\delta-1} r\partial_r \mathcal{B} + \text{l.o.t.}, \tag{3.3}$$

where

$$\mathcal{B} := r\partial_r u + \frac{\langle t \rangle^{1+\delta}}{\sqrt{\epsilon}} \partial_z u \int_z^{+\infty} \mathcal{A} d\bar{z}.$$

If we consider $\mathcal{A}$ have the same order as $\partial_r u$, as indicated in previous results in [5], then there will be one order derivative loss for $(r\partial_r)^2 u$ and $r\partial_r \int_z^\infty \mathcal{A} d\bar{z}$ in $r\partial_r \mathcal{B}$. So we cannot view $r\partial_r \mathcal{B}$ separately as two terms. If we see $\mathcal{B}$ as a whole to be a new auxiliary function, it satisfies

$$[\partial_t + (ur\partial_r + v\partial_z) - \partial_z^2] r\partial_r \mathcal{B} = \text{terms involving } (r\partial_r)^2 u + \text{l.o.t.}$$

Then inserting this into (3.3), we can see that

$$[\partial_t + (ur\partial_r + v\partial_z) - \partial_z^2]^2 \mathcal{A} = \text{terms involving } (r\partial_r)^2 u + \text{l.o.t.}$$

Here, $(r\partial_r)^2 u$ has the same order with $r\partial_r \mathcal{A}$. The auxiliary function $\mathcal{B}$ helps to achieve the $\frac{1}{2}$-order derivative loss of $\mathcal{A}$. The above equation for $\mathcal{A}$ indicates that we can perform Gevrey-2 energy functional for $\mathcal{A}$ as indicated in the toy model displayed in Li *et al.* [24]. So if we define the energy functional of $\mathcal{A}$ as $\|\mathcal{A}\|_{X_{\tau,\kappa}}$, then correspondingly, we need to define the energy functionals of u by $\|u\|_{X_{\tau,\kappa+2}}$ and $\mathcal{B}$ by $\|\mathcal{B}\|_{X_{\tau,\kappa+1}}$, which has the same order as $(r\partial_r)^{1/2} u$.

In order to obtain the global existence of the solutions, the equations of u and $\mathcal{A}$ are not enough to obtain enough time decay estimate. Next, we will introduce the following two linearly good unknowns to catch much faster decay to the lower order Gevrey-2 energy of the u and $\mathcal{A}$.

3.2. *The linearly good unknowns*

Inspired by the linearly good unknown in [18, 35], we define

$$g := u - \frac{z}{2\langle t\rangle}\int_z^\infty u d\bar{z}, \quad \mathcal{G} := \mathcal{A} - \frac{z}{2\langle t\rangle}\int_z^\infty \mathcal{A} d\bar{z}.$$

These two linearly good unknowns are set to dig out the sufficiently fast decay rate for the lower order Gevrey-2 norms of the solution u and the auxiliary function $\mathcal{A}$, which ensure the closing of energy estimates for all the quantities mentioned above. As shown in (2.7), we see that the lower order Gevrey-2 norm of g has a decay rate of almost $-5/4$ order with respect to time, which will induce the same decay rate of the lower order Gevrey-2 norm of u, see (3.5) in Lemma 3.1. Based on the lower order energy decay of u, we can see that $\mathcal{G}$ have the same almost $-5/4$ order decay for the lower order Gevrey-2 norm, which indicates the same decay of the lower order Gevrey-2 norm for the auxiliary functions $\mathcal{A}$. See also (3.5) in Lemma 3.1.

3.3. *A prior assumptions*

We will first make *a priori* assumptions for the linearly good unknowns as follows. We assume that

$$\begin{aligned}&\sum_{k=0}^{3}(\delta\langle t\rangle)^{k/2}\|\partial_z^k g(t)\|_{X_{\tau,11-2k}} + \sum_{k=0}^{1}(\delta\langle t\rangle)^{k/2}\|\partial_z^k \mathcal{G}(t)\|_{X_{\tau,7-2k,7/8}}\\ &\quad\le \lambda^{1/4}\epsilon\langle t\rangle^{-\frac{5-\delta}{4}}.\end{aligned} \tag{3.4}$$

Here, λ is the *to-be-determined* constant in (2.3).

Under the *a priori* assumption (3.4), we first have the following *a priori* estimates based on the relations between u and g, and between $\mathcal{A}$ and $\mathcal{G}$, respectively.

Lemma 3.1. *Under the assumption* (3.4), *we have the following a priori estimates. For any* $0 \le \nu < 1$,

$$\begin{aligned}&\sum_{k=0}^{3}(\delta\langle t\rangle)^{k/2}\|\partial_z^k u(t)\|_{X_{\tau,11-2k,\nu}} \lesssim_\nu \lambda^{1/4}\epsilon\langle t\rangle^{-\frac{5-\delta}{4}},\\ &\sum_{k=0}^{1}(\delta\langle t\rangle)^{k/2}\|\partial_z^k \mathcal{A}(t)\|_{X_{\tau,7-2k,3/4}} + \sum_{k=0}^{1}(\delta\langle t\rangle)^{k/2}\|\partial_z^k \mathcal{B}(t)\|_{X_{\tau,7-2k,3/4}}\\ &\quad\lesssim \lambda^{1/4}\epsilon\langle t\rangle^{-\frac{5-\delta}{4}}.\end{aligned} \tag{3.5}$$

The following finite-order L^∞ a priori estimate is direct consequences of Sobolev embedding. For $j \le 10$, any $0 \le \nu < 1$,

$$\|\theta_\nu r^j\partial_r^j v\|_{L^\infty} + \sum_{k=0}^{2}(\delta\langle t\rangle)^{\frac{k+1}{2}}\left\|\theta_\nu r^j\partial_r^j\partial_z^k u\right\|_{L^\infty} \lesssim \lambda^{1/4}\epsilon\langle t\rangle^{-\frac{4-\delta}{4}}. \tag{3.6}$$

Proof of this lemma is very similar to that in [37, Lemma 3.2]. Here, we omit the details. The details are left to the interested reader.

Based on the *a priori* assumptions in (3.4) and the *a priori* estimates in Lemma 3.1, we can derive a series of estimates as follows, which is based on performing weighted energy estimates to the equations of auxiliary functions, the unknowns and the linearly good unknowns.

3.4. *A priori estimates*

For simplification of notations, let $\kappa = 14$ in the following. For the auxiliary function $\mathcal{A}$, we have the following estimate.

Proposition 3.1 (Gevrey-2 Estimates of $\mathcal{A}$). *For any fixed* $\tau_0 > 0$, $\delta \in (0, \frac{1}{50}]$, *under the assumption of* (3.4), *for sufficiently small* ϵ, *there exits a constant* C *such that for any* $t \in (0, T]$, *we have the following estimate*:

$$\begin{aligned}
&\langle t\rangle^{\frac{1-\delta}{2}} \|\mathcal{A}(t)\|^2_{X_{\tau,\kappa}} + \delta \int_0^T \langle t\rangle^{\frac{1-\delta}{2}} \|\partial_z \mathcal{A}(t)\|^2_{X_{\tau,\kappa}} dt \\
&\quad + \lambda\sqrt{\epsilon} \int_0^T \langle t\rangle^{\frac{1-\delta}{2}} \eta(t) \, \|\mathcal{A}(t)\|^2_{X_{\tau,\kappa+1/2}} \, dt \\
&\le C\lambda^{-1/2}\sqrt{\epsilon} \int_0^T \langle t\rangle^{\frac{1-\delta}{2}} \eta(t)(\|\mathcal{A}(t)\|^2_{X_{\tau,\kappa+1/2}} + \|u(t)\|^2_{X_{\tau,\kappa+5/2}}) dt \\
&\quad + C\lambda^{-1/2}\sqrt{\epsilon} \int_0^T \langle t\rangle^{\frac{1-\delta}{2}} (\|\partial_z \mathcal{A}(t)\|^2_{X_{\tau,\kappa}} + \|\partial_z u(t)\|^2_{X_{\tau,\kappa+2}}) dt \\
&\quad + C\sqrt{\epsilon} \int_0^T \langle t\rangle^{\frac{1-\delta}{2}} \eta(t) \|\mathcal{B}(t)\|^2_{X_{\tau,\kappa+3/2}} dt.n
\end{aligned} \tag{3.7}$$

For the auxiliary function $\mathcal{B}$, we have the following estimate.

Proposition 3.2 (Gevrey-2 Estimates of $\mathcal{B}$). *For any fixed* $\tau_0 > 0$, $\delta \in (0, \frac{1}{50}]$, *under the assumption of* (3.4), *for sufficiently small* ϵ, *there exits a constant* C *such that for any* $t \in (0, T]$, *we have the following estimate*:

$$\begin{aligned}
&\langle t\rangle^{\frac{1-\delta}{2}} \|\mathcal{B}(t)\|^2_{X_{\tau,\kappa+1}} + \delta \int_0^T \langle t\rangle^{\frac{1-\delta}{2}} \|\partial_z \mathcal{B}(t)\|^2_{X_{\tau,\kappa+1}} dt \\
&\quad + \lambda\sqrt{\epsilon} \int_0^T \langle t\rangle^{\frac{1-\delta}{2}} \eta(t) \|\mathcal{B}(t)\|^2_{X_{\tau,\kappa+3/2}} dt \\
&\le C\|u(0)\|_{X_{\tau,\kappa+3}} + C\lambda^{-1/2}\sqrt{\epsilon} \int_0^T \langle t\rangle^{\frac{1-\delta}{2}} (\eta(t)\|\mathcal{A}(t)\|^2_{X_{\tau,\kappa+1/2}} + \|\partial_z \mathcal{A}(t)\|^2_{X_{\tau,\kappa}}) dt \\
&\quad + C\lambda^{-1/2}\sqrt{\epsilon} \int_0^T \langle t\rangle^{\frac{1-\delta}{2}} (\eta(t)\|\mathcal{B}(t)\|^2_{X_{\tau,\kappa+3/2}} + \|\partial_z \mathcal{B}(t)\|^2_{X_{\tau,\kappa+1}}) dt \\
&\quad + C\lambda^{-1/2}\sqrt{\epsilon} \int_0^T \langle t\rangle^{\frac{1-\delta}{2}} (\eta(t)\|u(t)\|^2_{X_{\tau,\kappa+5/2}} + \|\partial_z u(t)\|^2_{X_{\tau,\kappa+2}}) dt.
\end{aligned} \tag{3.8}$$

For the unknown functions u, we have the following estimate.

Proposition 3.3 (Gevrey-2 Estimates of u). *For any fixed $\tau_0 > 0$, $\delta \in (0, \frac{1}{50}]$, under the assumption of* (3.4), *for sufficiently small ϵ, there exits a constant C such that for any $t \in (0, T]$, we have the following estimate*:

$$\begin{aligned}
&\langle t\rangle^{\frac{1-\delta}{2}}\|u(t)\|^2_{X_{\tau,\kappa+2}} + \delta\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z u(t)\|^2_{X_{\tau,\kappa+2}}dt \\
&\quad + \lambda\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|u(t)\|^2_{X_{\tau,\kappa+5/2}}dt \\
&\leq C\|u(0)\|_{X_{\tau,\kappa+2}} + C\lambda^{1/2}\sqrt{\epsilon}\|\mathcal{A}\|^2_{X_{\tau,\kappa}} \\
&\quad + C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)(\|u(t)\|^2_{X_{\tau,\kappa+5/2}} + \|\mathcal{A}(t)\|^2_{X_{\tau,\kappa+1/2}})dt \\
&\quad + C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}(\|\partial_z u(t)\|^2_{X_{\tau,\kappa+2}} + \|\partial_z\mathcal{A}(t)\|^2_{X_{\tau,\kappa}})dt. \qquad (3.9)
\end{aligned}$$

Next, we will give the Gevrey-2 estimates of the good unknowns g and $\mathcal{G}$. Denote

$$\kappa_0 = 11, \quad \kappa_1 = 9, \quad \kappa_2 = 7, \quad \text{and} \quad \kappa_3 = 5.$$

For g, we have the following estimate.

Proposition 3.4 (Gevrey-2 Estimates of g). *For any fixed $\tau_0 > 0$, $\delta \in (0, \frac{1}{50}]$, under the assumption of* (3.4), *for sufficiently small ϵ, there exits a constant C such that for any $t \in (0, T]$, we have the following estimate.*

(i) *For the good unknown*: g,

$$\begin{aligned}
&\langle t\rangle^{\frac{5-\delta}{2}}\|g(t)\|^2_{X_{\tau,\kappa_0}} + \delta\int_0^T \langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g(t)\|^2_{X_{\tau,\kappa_0}}dt \\
&\quad + \lambda\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|g(t)\|^2_{X_{\tau,\kappa_0+1/2}}dt \\
&\leq C\|g(0)\|^2_{X_{\tau_0,\kappa_0}} + C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|u(t)\|^2_{X_{\tau,\kappa+5/2}}dt \\
&\quad + C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g(t)\|^2_{X_{\tau,\kappa_0}}dt. \qquad (3.10)
\end{aligned}$$

(ii) *For the first-order z-derivative of the good unknown*: $\partial_z g$,

$$\begin{aligned}
&\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z g(t)\|^2_{X_{\tau,\kappa_1}} + \delta\int_0^T \langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z^2 g(t)\|^2_{X_{\tau,\kappa_1}}dt \\
&\quad + \lambda\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{7-\delta}{2}}\eta(t)\|\partial_z g(t)\|^2_{X_{\tau,\kappa_1+1/2}}dt
\end{aligned}$$

$$\le C\,\|\partial_z g(0)\|^2_{X_{\tau,\kappa_1}} + \int_0^T \langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g(t)\|^2_{X_{\tau,\kappa_1}}dt$$

$$+ C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g(t)\|^2_{X_{\tau,\kappa_0}}dt. \tag{3.11}$$

(iii) *For the second-order z-derivative of the good unknown*: $\partial_z^2 g$,

$$\langle t\rangle^{\frac{9-\delta}{2}}\|\partial_z^2 g(t)\|^2_{X_{\tau,\kappa_2}} + \delta\int^T \langle t\rangle^{\frac{9-\delta}{2}}\|\partial_z^3 g(t)\|^2_{X_{\tau,\kappa_2}}dt$$

$$+ \lambda\sqrt{\epsilon}\int^T \langle t\rangle^{\frac{9-\delta}{2}}\eta(t)\|\partial_z^2 g(t)\|^2_{X_{\tau,\kappa_2+1/2}}dt$$

$$\le C\|\partial_z^2 g(0)\|^2_{X_{\tau_0,\kappa_2}} + \int_0^T \langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z^2 g(t)\|^2_{X_{\tau,\kappa_2}}dt$$

$$+ C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z^2 g(t)\|^2_{X_{\tau,\kappa_1}}dt. \tag{3.12}$$

(iv) *For the third-order z-derivative of the good unknown*: $\partial_z^3 g$,

$$\langle t\rangle^{\frac{11-\delta}{2}}\|\partial_z^3 g(t)\|^2_{X_{\tau,\kappa_3}} + \delta\int^T \langle t\rangle^{\frac{11-\delta}{2}}\|\partial_z^4 g(t)\|^2_{X_{\tau,\kappa_3}}dt$$

$$+ \lambda\sqrt{\epsilon}\int^T \langle t\rangle^{\frac{11-\delta}{2}}\eta(t)\|\partial_z^3 g(t)\|^2_{X_{\tau,\kappa_3+1/2}}dt$$

$$\le C\|\partial_z^3 g(0)\|^2_{X_{\tau_0,\kappa_3}} + \int_0^T \langle t\rangle^{\frac{9-\delta}{2}}\|\partial_z^2 g(t)\|^2_{X_{\tau,\kappa_3}}dt.$$

$$+ C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{9-\delta}{2}}\|\partial_z^3 g(t)\|^2_{X_{\tau,\kappa_2}}dt. \tag{3.13}$$

For $\mathcal{G}$, we have the following estimate.

Proposition 3.5 (Gevrey-2 Estimates of $\mathcal{G}$). *For any fixed $\tau_0 > 0$, $\delta \in (0, \frac{1}{50}]$, under the assumption of* (3.4), *for sufficiently small ϵ, there exits a constant C such that for any $t \in (0, T]$, we have the following estimate.*

(v) *For the good unknown*: $\mathcal{G}$,

$$\langle t\rangle^{\frac{5-\delta}{2}}\|\mathcal{G}(t)\|^2_{X_{\tau,\kappa_2,1-\delta/2}} + \delta\int_0^T \langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z\mathcal{G}(t)\|^2_{X_{\tau,\kappa_2,1-\delta/2}}dt$$

$$+ \lambda\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|\mathcal{G}(t)\|^2_{X_{\tau,\kappa_2+1/2,1-\delta/2}}dt$$

$$\le C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T (\langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|\mathcal{G}(t)\|^2_{X_{\tau,\kappa_2+1/2,1-\delta/2}}$$

$$
\begin{aligned}
&+\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z\mathcal{G}(t)\|^2_{X_{\tau,\kappa_2,1-\delta/2}})dt\\
&+C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T(\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{A}(t)\|^2_{X_{\tau,\kappa}})dt\\
&+C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T\langle t\rangle^{\frac{5-\delta}{2}}(\eta(t)\|g(t)\|^2_{X_{\tau,\kappa_0+1/2}}+\|\partial_z g(t)\|^2_{X_{\tau,\kappa_0}})dt.
\end{aligned}
\tag{3.14}
$$

(vi) *For the first-order z-derivative of the good unknown*: $\partial_z\mathcal{G}$,

$$
\begin{aligned}
&\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z\mathcal{G}(t)\|^2_{X_{\tau,\kappa_3,1-\delta/2}}+\delta\int_0^T\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z^2\mathcal{G}(t)\|^2_{X_{\tau,\kappa_3,1-\delta/2}}dt\\
&\quad+\lambda\sqrt{\epsilon}\int_0^T\langle t\rangle^{\frac{7-\delta}{2}}\eta(t)\|\partial_z\mathcal{G}(t)\|^2_{X_{\tau,\kappa_3+1/2,1-\delta/2}}dt\\
&\le C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T(\langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|\mathcal{G}(t)\|^2_{X_{\tau,\kappa_2+1/2,1-\delta/2}}\\
&\quad+\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z\mathcal{G}(t)\|^2_{X_{\tau,\kappa_2,1-\delta/2}})dt\\
&\quad+C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T\langle t\rangle^{\frac{5-\delta}{2}}(\eta(t)\|g(t)\|^2_{X_{\tau,\kappa_0+1/2}}+\|\partial_z g(t)\|^2_{X_{\tau,\kappa_0}})dt.
\end{aligned}
\tag{3.15}
$$

3.5. *Proof of the main theorem*

Based on the *a priori* estimates from Propositions 3.1 to 3.5, we can derive the validity of the main theorem. Since the local well-posedness of Gevrey-2 solutions has already been shown in 2D case in Dietert and Gérard-Varet [5] and in 3D case in Li *et al.* [24], by continuity argument, we only need to show that under the *a priori* assumption, by choosing suitably large λ, we can obtain that

$$
\begin{aligned}
&\sum_{k=0}^{3}(\delta\langle t\rangle)^{k/2}\|\partial_z^k g(t)\|_{X_{\tau,11-2k}}+\sum_{k=0}^{1}(\delta\langle t\rangle)^{k/2}\|\partial_z^k\mathcal{G}(t)\|_{X_{\tau,7-2k,7/8}}\\
&\quad\le\frac{1}{2}\lambda^{1/4}\epsilon\langle t\rangle^{-\frac{5-\delta}{4}}.
\end{aligned}
\tag{3.16}
$$

And to prove the validity of (2.7).

For λ sufficiently large, adding (3.7) in Proposition 3.1 and (3.8) in Proposition 3.2 together, we can obtain that

$$
\begin{aligned}
&\langle t\rangle^{\frac{1-\delta}{2}}\|\mathcal{A}(t)\|^2_{X_{\tau,\kappa}}+\delta\int_0^T\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{A}(t)\|^2_{X_{\tau,\kappa}}dt\\
&\quad+\lambda\sqrt{\epsilon}\int_0^T\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{A}(t)\|^2_{X_{\tau,\kappa+1/2}}dt
\end{aligned}
$$

$$\begin{aligned}&\lesssim \|u(0)\|^2_{X_{\tau_0,\kappa+3}} + \lambda^{-1/2}\sqrt{\epsilon}\int_0^T (\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|u(t)\|^2_{X_{\tau,\kappa+5/2}}\\ &\quad + \langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z u(t)\|^2_{X_{\tau,\kappa+2}})dt.\end{aligned}$$

For λ sufficiently large, combining the above inequality with (3.9) in Proposition 3.3, we can obtain that

$$\begin{aligned}&\langle t\rangle^{\frac{1-\delta}{2}}(\|\mathcal{A}(t)\|^2_{X_{\tau,\kappa}} + \|u(t)\|^2_{X_{\tau,\kappa+2}}) + \delta\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}(\|\partial_z\mathcal{A}(t)\|^2_{X_{\tau,\kappa}}\\ &\quad + \|\partial_z u(t)\|^2_{X_{\tau,\kappa+2}})dt + \lambda\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)(\|\mathcal{A}(t)\|^2_{X_{\tau,\kappa+1/2}}\\ &\quad + \|u(t)\|^2_{X_{\tau,\kappa+5/2}})dt\\ &\le C\|u(0)\|^2_{X_{\tau_0,\kappa+3}} \le C\epsilon^2. \end{aligned} \tag{3.17}$$

From the above inequality, estimate of u in (2.7) is proven.

Now, multiplying a small constant $\frac{1}{2}\delta$ to (3.11), $\frac{1}{4}\delta^2$ to (3.12) and $\frac{1}{8}\delta^3$ to (3.13), and adding the resulted equations to (3.10), by letting ϵ be sufficiently small, we can achieve that

$$\begin{aligned}&\langle t\rangle^{\frac{5-\delta}{2}}\|g(t)\|^2_{X_{\tau,\kappa_0}} + \delta\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z g(t)\|^2_{X_{\tau,\kappa_1}}\\ &\quad + \delta^2\langle t\rangle^{\frac{9-\delta}{2}}\|\partial_z^2 g(t)\|^2_{X_{\tau,\kappa_2}} + \delta^3\langle t\rangle^{\frac{11-\delta}{2}}\|\partial_z^3 g(t)\|^2_{X_{\tau,\kappa_3}}\\ &\quad + \lambda\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|g(t)\|^2_{X_{\tau,\kappa_0+1/2}}dt + \delta\int_0^T\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g(t)\|^2_{X_{\tau,\kappa_0}}dt\\ &\le \|g(0)\|^2_{X_{\tau_0,11}} + \delta\|\partial_z g(0)\|^2_{X_{\tau,9}} + \delta^2\|\partial_z^2 g(0)\|^2_{X_{\tau,7}} + \delta^3\|\partial_z^3 g(0)\|^2_{X_{\tau,5}}\\ &\quad + C\sqrt{\epsilon}\int_0^T\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z u(t)\|^2_{X_{\tau,\kappa+2}}dt. \end{aligned} \tag{3.18}$$

Inserting (3.17) into (3.18), for sufficiently small ϵ, we can achieve that for some constant C,

$$\begin{aligned}&\langle t\rangle^{\frac{5-\delta}{2}}\|g(t)\|^2_{X_{\tau,\kappa_0}} + \delta\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z g(t)\|^2_{X_{\tau,\kappa_1}}\\ &\quad + \delta^2\langle t\rangle^{\frac{9-\delta}{2}}\|\partial_z^2 g(t)\|^2_{X_{\tau,\kappa_2}} + \delta^3\langle t\rangle^{\frac{11-\delta}{2}}\|\partial_z^3 g(t)\|^2_{X_{\tau,\kappa_3}}\\ &\quad + \lambda\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|g(t)\|^2_{X_{\tau,\kappa_0+1/2}}dt + \delta\int_0^T\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g(t)\|^2_{X_{\tau,\kappa_0}}dt \le C\epsilon^2. \end{aligned} \tag{3.19}$$

Then we can obtain estimate of g in (2.7) and the first one of (3.16) by letting λ large enough such that $16C \le \sqrt{\lambda}$.

At last, from (3.14) and (3.15) in Proposition 3.5, by letting ϵ is sufficiently small, we can obtain that

$$\begin{aligned}
&\langle t\rangle^{\frac{5-\delta}{2}}\|\mathcal{G}(t)\|^2_{X_{\tau,\kappa_2,7/8}}+\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z\mathcal{G}(t)\|^2_{X_{\tau,\kappa_3,7/8}}\\
&\quad\le C\sqrt{\epsilon}\int_0^T\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{A}(t)\|^2_{X_{\tau,\kappa}}dt+\lambda^{-1}\sqrt{\epsilon}\int_0^T\langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|g(t)\|^2_{X_{\tau,\kappa_0+1/2}}dt\\
&\qquad+C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T\langle t\rangle^{\frac{5-\delta}{2}}(\eta(t)\|g(t)\|^2_{X_{\tau,\kappa_0+1/2}}+\|\partial_z g(t)\|^2_{X_{\tau,\kappa_0}})dt. \qquad (3.20)
\end{aligned}$$

By using (3.17) and (3.19), and remembering (2.5), we can obtain, from (3.20), there exists a constant C such that

$$\langle t\rangle^{\frac{5-\delta}{2}}\|\mathcal{G}(t)\|^2_{X_{\tau,\kappa_2,7/8}}+\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z\mathcal{G}(t)\|^2_{X_{\tau,\kappa_3,7/8}}\le C\epsilon^2,$$

which shows the second one of (3.16) by letting λ large enough.

We still need to show the proof of Propositions 3.1 to 3.5. Before that, we give two useful lemmas, which will be frequently used in later estimates.

3.6. *Preliminary lemmas*

Lemma 3.2. *Let f be a smooth enough function in r variable and belong to H^1 in z variable, which decays to zero sufficiently fast as $z\to+\infty$. Then we have for $0\le\nu\le 1$,*

$$\frac{\nu}{2\langle t\rangle}\|f\|^2_{L^2(\theta_{2\nu})}\le\|\partial_z f\|^2_{L^2(\theta_{(2\nu)}} \qquad (3.21)$$

and

$$\frac{\nu}{4\langle t\rangle}\|f\|^2_{L^2(\theta_{2\nu})}+\frac{\nu^2}{16}\left\|\frac{z}{\langle t\rangle}f\right\|^2_{L^2(\theta_{2\nu})}\le\|\partial_z f\|^2_{L^2(\theta_{2\nu})}. \qquad (3.22)$$

See [37, Lemma 3.8] for the proof. The essential idea is to perform integration by parts on z variable. Here, we omit the details.

Next, we give a lemma to show the Gevrey-2 norm product estimate.

Lemma 3.3. *For smooth functions f and g, which decay fast enough at z infinity, we have the following product estimates*:

$$\|fg\|^2_{X_{\tau,\kappa,\nu}}\lesssim\langle t\rangle^{1/2}(\|f\|^2_{X_{\tau,3,\frac{\nu+1}{4}}}\|\partial_z g\|^2_{X_{\tau,\kappa,\frac{\nu+1}{4}}}+\|g\|^2_{X_{\tau,3,\frac{\nu+1}{4}}}\|\partial_z f\|^2_{X_{\tau,\kappa,\frac{\nu+1}{4}}}), \qquad (3.23)$$

$$\|fg\|^2_{X_{\tau,\kappa,\nu}}\lesssim\langle t\rangle^{1/2}(\|f\|^2_{X_{\tau,3,\frac{\nu+1}{4}}}\|\partial_z g\|^2_{X_{\tau,\kappa,\frac{\nu+1}{4}}}+\|\partial_z g\|^2_{X_{\tau,3,\frac{\nu+1}{4}}}\|f\|^2_{X_{\tau,\kappa,\frac{\nu+1}{4}}}), \qquad (3.24)$$

$$\|fg\|^2_{X_{\tau,\kappa,\nu}}\lesssim\langle t\rangle^{1/2}(\|\partial_z f\|^2_{X_{\tau,3,\frac{\nu+1}{4}}}\|g\|^2_{X_{\tau,\kappa,\frac{\nu+1}{4}}}+\|\partial_z g\|^2_{X_{\tau,3,\frac{\nu+1}{4}}}\|f\|^2_{X_{\tau,\kappa,\frac{\nu+1}{4}}}). \qquad (3.25)$$

Proof. We only present the proof of (3.23), since the other two are similar. For simplicity, we use $f_{j,\kappa}$ to denote $M_{j,\kappa}[r]^j\partial_r^j f$ if no confusion is caused. First, we see that

$$\begin{aligned}
(fg)_{j,\kappa} &= M_{j,\kappa}[r]^j\partial_r^j(fg)\\
&= \sum_{0\le k\le j}\binom{j}{k}\frac{M_{j,\kappa}}{M_{k,\kappa}M_{j-k,\kappa}}M_{k,\kappa}M_{j-k,\kappa}[r]^j\partial_r^k f\partial_r^{j-k}g\\
&= \sum_{0\le k\le j}\frac{1}{\tau}\left(\frac{j+1}{(k+1)(j-k+1)}\right)^{\kappa}\binom{j}{k}^{-1}M_{k,\kappa}M_{j-k,\kappa}[r]^j|\partial_r^k f||\partial_r^{j-k}g|.
\end{aligned}$$

Then by using (2.5), we can obtain that

$$\begin{aligned}
|(fg)_{j,\kappa}| \lesssim{}& \sum_{k=0}^{[(j+1)/2]}(k+1)^{-\kappa}M_{k,\kappa}r^k|\partial_r^k f||g_{j-k,\kappa}|\\
&+\sum_{k=[(j+1)/2]+1}^{j}(j-k+1)^{-\kappa}|f_{k,\kappa}|M_{j-k,\kappa}r^{j-k}|\partial_r^{j-k}g|.
\end{aligned}$$

First, using Minkowski inequality and then Hölder inequality, we obtain

$$\begin{aligned}
&\|(fg)_{j,\kappa}\|_{L^2(\theta_{2\nu})}\\
&\quad\le \sum_{k=0}^{[(j+1)/2]}\|(k+1)^{-\kappa}M_{k,\kappa}r^k\partial_r^k f g_{j-k,\kappa}\|_{L^2(\theta_{2\nu})}\\
&\qquad+\sum_{k=[(j+1)/2]+1}^{j}\|(j-k+1)^{-\kappa}f_{k,\kappa}M_{j-k,\kappa}r^{j-k}\partial_r^{j-k}g\|_{L^2(\theta_{2\nu})}\\
&\quad\le \sum_{k=0}^{[(j+1)/2]}\|(k+1)^{-\kappa}M_{k,\kappa}r^k\partial_r^k f\|_{L_r^\infty L^2(\theta_\nu)}\|\theta_{\frac{\nu}{2}}g_{j-k,\kappa}\|_{L_r^2L_z^\infty}\\
&\qquad+\sum_{k=[(j+1)/2]+1}^{j}\|f_{k,\kappa}\theta_{\frac{\nu}{2}}\|_{L_r^2L_z^\infty}\|(j-k+1)^{-\kappa}M_{j-k,\kappa}r^{j-k}\partial_r^{j-k}g\|_{L_r^\infty L_z^2(\theta_\nu)}.
\end{aligned}$$

Then using the following discrete Young's convolution inequality:

$$\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}a_kb_{j-k}\right)^2\le\left(\sum_{k=0}^{\infty}a_k\right)^2\left(\sum_{k=0}^{\infty}b_k^2\right), \tag{3.26}$$

squaring (3.26) and summing the resulted equation over $j \in \mathbb{N}$, we can achieve that

$$
\begin{aligned}
\|fg\|^2_{X_{\tau,\kappa,\nu}} &= \sum_{j\in\mathbb{N}} \|(fg)_{j,\kappa}\|^2_{L^2(\theta_{2\nu})} \\
&\le \left(\sum_{k=0}^{\infty}(k+1)^{-\kappa}\|M_{k,\kappa} r^k\partial_r^k f\|_{L^\infty_r L^2(\theta_\nu)}\right)^2 \left(\sum_{k=0}^{\infty}\|\theta_{\nu/2} g_{k,\kappa}\|^2_{L^2_r L^\infty_z}\right) \\
&\quad + \left(\sum_{k=0}^{\infty}\|\theta_{\nu/2} f_{k,\kappa}\|^2_{L^2_r L^\infty_z}\right)\left(\sum_{k=1}^{\infty}(k+1)^{-\kappa}\|M_{k,\kappa} r^k\partial_r^k g\|_{L^\infty_r L^2_v(\theta_\nu)}\right)^2 .
\end{aligned}
\tag{3.27}
$$

Before continuing estimates, we give three weighted Sobolev embedding inequalities which will be frequently used later on. For any f, decaying fast enough at z infinity, for $0 \le \nu < 1$, we have

$$
\begin{aligned}
\|M_{k,\kappa} r^k \partial_r^k f\|_{L^\infty_r L^2_z(\theta_\nu)} &\lesssim \sum_{i=0}^{1}\|M_{k,\kappa}\partial_r^i(r^k\partial_r^k f)\|_{L^2_r L^2_z(\theta_\nu)} \\
&\lesssim (k+1)^2(\|f_{k,\kappa}\|_{L^2(\theta_\nu)} + \|f_{k+1,\kappa}\|_{L^2(\theta_\nu)}).
\end{aligned}
$$

Here, we have used one-dimensional Sobolev embedding in r direction. Also, Hölder inequality in the vertical direction indicates that

$$
\begin{aligned}
\|\theta_{\nu/2} f_{k,\kappa}\|_{L^2_r L^\infty_z} &= \left\|\theta_{\nu/2}\int_z^\infty \partial_z f_{k,\kappa} d\bar{z}\right\|_{L^2_r L^\infty_z} \\
&\lesssim \left\|\int_z^\infty \theta_{\frac{\nu-1}{4}}\theta_{\frac{\nu+1}{4}}\partial_z f_{k,\kappa} d\bar{z}\right\|_{L^2_r L^\infty_z} \\
&\lesssim_\nu \langle t\rangle^{1/4}\left\|\theta_{\frac{\nu+1}{4}}\partial_z f_{k,\kappa}\right\|_{L^2}.
\end{aligned}
\tag{3.28}
$$

Combining results in (3.28) and (3.28), we immediately obtain that

$$
\begin{aligned}
&\|\theta_{\nu/2} M_{k,\kappa} r^k\partial_r^k f\|_{L^\infty_r L^\infty_z} \\
&\quad\lesssim_\nu \langle t\rangle^{1/4}\|\theta_{\frac{\nu+1}{4}}\partial_z f_{k,\kappa}\|_{L^\infty_r L^2_z} \\
&\quad\lesssim \langle t\rangle^{1/4}(k+1)^2(\|\partial_z f_{k,\kappa}\|_{L^2(\theta_{\frac{\nu+1}{2}})} + \|\partial_z f_{k+1,\kappa}\|_{L^2(\theta_{\frac{\nu+1}{2}})}).
\end{aligned}
\tag{3.29}
$$

Inserting (3.28) and (3.28) into (3.27) and by using discrete Cauchy inequality, we can obtain that

$$
\begin{aligned}
\|fg\|^2_{X_{\tau,\kappa,\nu}} &\lesssim \langle t\rangle^{1/2}\left(\sum_{k=0}^{\infty}(k+1)^{-\kappa+2}\|f_{k,\kappa}\|_{L^2(\theta_\nu)}\right)^2\sum_{k=0}^{\infty}\|\partial_z g_k\|^2_{L^2(\theta_{\frac{\nu+1}{2}})} \\
&\quad + \langle t\rangle^{1/2}\sum_{k=0}^{\infty}\|\partial_z f_{k,\kappa}\|^2_{L^2(\theta_{\frac{\nu+1}{2}})}\left(\sum_{k=0}^{\infty}(k+1)^{-\kappa+2}\|g_{k,\kappa}\|_{L^2(\theta_\nu)}\right)^2
\end{aligned}
$$

$$\lesssim \langle t\rangle^{1/2}\sum_{k=0}^{\infty}(k+1)^{-2\kappa+6}\|f_{k,\kappa}\|^2_{L^2(\theta_{\frac{\nu+1}{2}})}\sum_{k=0}^{\infty}\|\partial_z g_k\|^2_{L^2(\theta_{\frac{\nu+1}{2}})}$$

$$+\langle t\rangle^{1/2}\sum_{k=0}^{\infty}\|\partial_z f_{k,\kappa}\|^2_{L^2(\theta_{\frac{\nu+1}{2}})}\sum_{k=0}^{\infty}(k+1)^{-2\kappa+6}\|g_{k,\kappa}\|^2_{L^2(\theta_{\frac{\nu+1}{2}})},$$

which is (3.23). □

In the next five sections, we give *a priori* estimates from Propositions 3.1 to 3.5.

4. Estimate of the Auxiliary Function $\mathcal{A}$

In this section, we give the proof of Proposition 3.1. First, we derive the equation for $\mathcal{A}$.

4.1. *Derivation of the equation of $\mathcal{A}$ and its linear estimate*

Taking $-\partial_z$ of (3.1) and using the incompressibility, we can have

$$\begin{aligned}
&[\partial_t+(ur\partial_r+v\partial_z)-\partial_z^2]\mathcal{A}\\
&\quad=\sqrt{\epsilon}\langle t\rangle^{-\delta-1}r\partial_r(r\partial_r u+2u)+\partial_z ur\partial_r\int_z^{\infty}\mathcal{A}d\bar z+(r\partial_r u+2u)\mathcal{A}\\
&\quad=\sqrt{\epsilon}\langle t\rangle^{-\delta-1}r\partial_r\left(r\partial_r u+\frac{\langle t\rangle^{1+\delta}}{\sqrt{\epsilon}}\partial_z u\int_z^{\infty}\mathcal{A}d\bar z+2u\right)\\
&\qquad-r\partial_r\partial_z u\int_z^{\infty}\mathcal{A}d\bar z+(r\partial_r u+2u)\\
&\quad:=\sqrt{\epsilon}\langle t\rangle^{-\delta-1}\left(r\partial_r\mathcal{B}+2r\partial_r u\right)+H,
\end{aligned}\tag{4.1}$$

where

$$H:=-r\partial_r\partial_z u\int_z^{\infty}\mathcal{A}d\bar z+(r\partial_r u+2u).$$

From Secs. 4 to 6, we set $\kappa=14$ and $M_{j,\kappa}$ is abbreviated to M_j. Also for a function f, f_j denotes $f_{j,\kappa}$ for simplicity.

From the equation of $\mathcal{A}$ in (4.1), we first have the following linear estimate.

Lemma 4.1. *Under the assumption of Proposition* 3.1, *for sufficiently small* ϵ, *we have the following estimate*:

$$\langle t\rangle^{\frac{1-\delta}{2}}\|\mathcal{A}(t)\|^2_{X_{\tau,\kappa}}+\delta\int_0^T\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{A}(t)\|^2_{X_{\tau,\kappa}}dt$$

$$+\lambda\sqrt{\epsilon}\int_0^T\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{A}(t)\|^2_{X_{\tau,\kappa+1/2}}dt$$

$$
\begin{aligned}
&\le C\lambda^{-1}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\,\|(\boldsymbol{\mathcal{B}},u)\|^2_{X_{\tau,\kappa+3/2}}\,dt \\
&\quad + C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{A}(t)\|^2_{X_{\tau,\kappa+1/2}}dt \\
&\quad + \frac{C}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}}\sum_{j=0}^{\infty}(j+1)^{-1}\|M_j[[r]^j\partial_r^j,(ur\partial_r+v\partial_z)]\mathcal{A}\|^2_{L^2(\theta_2)}dt \\
&\quad + \int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\sum_{j=0}^{\infty}|\langle H_j,\mathcal{A}_j\rangle_{\theta_2}|dt.
\end{aligned} \tag{4.2}
$$

Proof. Applying $M_j[r]^j\partial_r^j$ to (4.1) implies that

$$
\begin{aligned}
&[\partial_t+\lambda\sqrt{\epsilon}\eta(t)(j+1)+(ur\partial_r+v\partial_z)-\partial_z^2]\mathcal{A}_j \\
&\quad = -M_j[[r]^j\partial_r^j,(ur\partial_r+v\partial_z)]\mathcal{A}+\sqrt{\epsilon}\langle t\rangle^{-\delta-1}(r\partial_r\mathcal{B}+2r\partial_r u)_j+H_j.
\end{aligned} \tag{4.3}
$$

Here, by using the Leibniz formula and direct computation, we have

$$
[[r]^j\partial_r^j,ur\partial_r]\mathcal{A}=[r]^j\sum_{k=1}^{j}\partial_r^k u\partial_r^{j-k}(r\partial_r\mathcal{A})+ur^{j-1}\partial_r^j\mathcal{A} \tag{4.4}
$$

and

$$
[[r]^j\partial_r^j,v\partial_z]\mathcal{A}=[r]^j\sum_{k=1}^{j}\partial_r^k v\partial_r^{j-k}(\partial_z\mathcal{A}). \tag{4.5}
$$

For $\theta_2 := e^{\frac{z^2}{4\langle t\rangle}}$, we have the following equalities, which will be frequently used in later derivation:

$$
\begin{aligned}
-\frac{\partial_t\theta_2}{\theta_2}&=\frac{z^2}{4\langle t\rangle}, \\
-\frac{\partial_z\theta_2}{\theta_2}&=-\frac{z}{2\langle t\rangle}, \\
-\frac{\partial_z^2\theta_2}{\theta_2}&=-\frac{1}{2\langle t\rangle}-\frac{z^2}{4\langle t\rangle}.
\end{aligned}
$$

Taking inner product of (4.3) with $\mathcal{A}_j\theta_2$, we can obtain that

$$
\begin{aligned}
&\langle[\partial_t+\lambda\sqrt{\epsilon}\eta(t)(j+1)+(ur\partial_r+v\partial_z)-\partial_z^2]\mathcal{A}_j,\mathcal{A}_j\rangle_{\theta_2} \\
&\quad = -\langle M_j[[r]^j\partial_r^j,(ur\partial_r+v\partial_z)]\mathcal{A},\mathcal{A}_j\rangle_{\theta_2} \\
&\qquad + \sqrt{\epsilon}\langle t\rangle^{-\delta-1}\langle(r\partial_r\mathcal{B}+2r\partial_r u)_j,\mathcal{A}_j\rangle_{\theta_2}+\langle H_j,\mathcal{A}_j\rangle_{\theta_2}.
\end{aligned}
$$

Integration by parts indicates that the left hand of (4.6) satisfies

$$\begin{aligned}
&\langle[\partial_t + \lambda\sqrt{\epsilon}\eta(t)(j+1) + (ur\partial_r + v\partial_z) - \partial_z^2]\mathcal{A}_j, \mathcal{A}_j(t)\rangle_{\theta_2}\\
&\quad = \frac{1}{2}\frac{d}{dt}\|\mathcal{A}_j(t)\|^2_{L^2(\theta_2)} + \|\partial_z\mathcal{A}_j(t)\|^2_{L^2(\theta_2)} - \frac{1}{4\langle t\rangle}\|\mathcal{A}_j(t)\|^2_{L^2(\theta_2)}\\
&\qquad + (j+1)\lambda\sqrt{\epsilon}\eta(t)\|\mathcal{A}_j(t)\|^2_{L^2(\theta_2)} + \left\langle \frac{1}{2}u - \frac{z}{4\langle t\rangle}v, \mathcal{A}_j^2\right\rangle_{\theta_2}.
\end{aligned} \tag{4.6}$$

Inserting (4.6) into (4.6), we can obtain

$$\begin{aligned}
&\frac{d}{dt}\|\mathcal{A}_j(t)\|^2_{L^2(\theta_2)} + 2\|\partial_z\mathcal{A}_j(t)\|^2_{L^2(\theta_2)} - \frac{1}{2\langle t\rangle}\|\mathcal{A}_j(t)\|^2_{L^2(\theta_2)}\\
&\qquad + 2(j+1)\lambda\sqrt{\epsilon}\eta(t)\|\mathcal{A}_{j,\kappa}(t)\|^2_{L^2(\theta_2)} + \left\langle u - \frac{z}{2\langle t\rangle}v, \mathcal{A}_j^2\right\rangle_{\theta_2}\\
&\quad = -2\langle M_j[[r]^j\partial_r^j, (ur\partial_r + v\partial_z)]\mathcal{A}, \mathcal{A}_j\rangle_{\theta_2}\\
&\qquad + 2\sqrt{\epsilon}\langle t\rangle^{-\delta-1}\langle(r\partial_r\mathcal{B} + 2r\partial_r u)_j, \mathcal{A}_j\rangle_{\theta_2} + 2\langle H_j, \mathcal{A}_j\rangle_{\theta_2}.
\end{aligned} \tag{4.7}$$

Using (3.21) in Lemma 3.2, we have

$$2\|\partial_z\mathcal{A}_j(t)\|^2_{L^2(\theta_2)} \geq \delta\|\partial_z\mathcal{A}_j(t)\|^2_{L^2(\theta_2)} + \frac{2-\delta}{2\langle t\rangle}\|\mathcal{A}_j(t)\|^2_{L^2(\theta_2)}.$$

Inserting this into (4.7), we can get

$$\begin{aligned}
&\frac{d}{dt}\|\mathcal{A}_j(t)\|^2_{L^2(\theta_2)} + \delta\|\partial_z\mathcal{A}_j(t)\|^2_{L^2(\theta_2)} + \frac{1-\delta}{2\langle t\rangle}\|\mathcal{A}_j(t)\|^2_{L^2(\theta_2)}\\
&\qquad + 2(j+1)\lambda\sqrt{\epsilon}\eta(t)\|\mathcal{A}_j(t)\|^2_{L^2(\theta_2)}\\
&\quad = -\left\langle u - \frac{z}{2\langle t\rangle}v, \mathcal{A}_j^2\right\rangle_{\theta_2} - 2\langle M_j[[r]^j\partial_r^j, (ur\partial_r + v\partial_z)]\mathcal{A}, \mathcal{A}_j\rangle_{\theta_2}\\
&\qquad + 2\sqrt{\epsilon}\langle t\rangle^{-\delta-1}\langle(r\partial_r\mathcal{B} + 2r\partial_r u)_j, \mathcal{A}_j\rangle_{\theta_2} + 2\langle H_j, \mathcal{A}_j\rangle_{\theta_2}.
\end{aligned}$$

By using the *a priori* estimates (3.6) in Lemma 3.1, we can easily obtain that

$$\left\|u - \frac{z}{2\langle t\rangle}v\right\|_{L^\infty} \leq C\lambda^{1/4}\delta^{-1/2}\epsilon\langle t\rangle^{-\frac{6-\delta}{4}} \leq C\lambda^{-1/2}\sqrt{\epsilon}\eta(t).$$

Here, we have chosen ϵ sufficiently small compared to δ and λ^{-1}. Then we have

$$\begin{aligned}
&\frac{d}{dt}\|\mathcal{A}_j(t)\|^2_{L^2(\theta_2)} + \delta\|\partial_z\mathcal{A}_j(t)\|^2_{L^2(\theta_2)}\\
&\qquad + \frac{1-\delta}{2\langle t\rangle}\|\mathcal{A}_j(t)\|^2_{L^2(\theta_2)} + 2(j+1)\lambda\sqrt{\epsilon}\eta(t)\|\mathcal{A}_j(t)\|^2_{L^2(\theta_2)}
\end{aligned}$$

$$\begin{aligned}&\le C\lambda^{-1/2}\sqrt{\epsilon}\eta(t)\|\mathcal{A}_j(t)\|^2_{L^2(\theta_2)} - 2\langle M_j[[r]^j\partial_r^j,(ur\partial_r+v\partial_z)]\mathcal{A},\mathcal{A}_j\rangle_{\theta_2}\\&\quad+2\sqrt{\epsilon}\langle t\rangle^{-\delta-1}\langle(r\partial_r\mathcal{B}+2r\partial_r u)_j,\mathcal{A}_j\rangle_{\theta_2}+2\langle H_j,\mathcal{A}_j\rangle_{\theta_2}.\end{aligned}\tag{4.8}$$

Multiplying (4.8) by $\langle t\rangle^{\frac{1-\delta}{2}}$ and then integrating from 0 to t for any $t\in[0,T]$, we can achieve that

$$\begin{aligned}&\langle t\rangle^{\frac{1-\delta}{2}}\|\mathcal{A}_j(t)\|^2_{L^2(\theta_2)}+\delta\int_0^T\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{A}_j(t)\|^2_{L^2(\theta_2)}dt\\&\qquad+2(j+1)\lambda\sqrt{\epsilon}\int_0^T\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{A}_j(t)\|^2_{L^2(\theta_2)}dt\\&\quad\le 2\sqrt{\epsilon}\int_0^T\langle t\rangle^{-\frac{1+3\delta}{2}}|\langle(r\partial_r\mathcal{B}+2r\partial_r u)_j,\mathcal{A}_j\rangle_{\theta_2}|dt\\&\qquad+C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T\eta(t)\langle t\rangle^{\frac{1-\delta}{2}}\|\mathcal{A}_j(t)\|^2_{L^2(\theta_2)}dt\\&\qquad+2\int_0^T\langle t\rangle^{\frac{1-\delta}{2}}|\langle M_j[[r]^j\partial_r^j,(ur\partial_r+v\partial_z)]\mathcal{A},\mathcal{A}_j\rangle_{\theta_2}|dt\\&\qquad+2\int^T\langle t\rangle^{\frac{1-\delta}{2}}|\langle H_j,\mathcal{A}_j\rangle_{\theta_2}|dt.\end{aligned}$$

Using Cauchy inequality, we can get

$$\begin{aligned}&\text{The first and third terms of the right hand of (4.9)}\\&\quad\le C\int_0^T\frac{\langle t\rangle^{\frac{1-\delta}{2}}}{(j+1)\lambda\sqrt{\epsilon}\eta(t)}(2\|M_j[[r]^j\partial_r^j,ur\partial_r+v\partial_z]\mathcal{A}\|^2_{L^2(\theta_2)})dt\\&\qquad+C\frac{\sqrt{\epsilon}}{\lambda}\int_0^T(j+1)^{-1}\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|(r\partial_r\mathcal{B}+2r\partial_r u)_j\|^2_{L^2(\theta_2)}dt\\&\qquad+(j+1)\lambda\sqrt{\epsilon}\int_0^T\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{A}_j(t)\|^2_{L^2(\theta_2)}dt.\end{aligned}$$

Using the *a priori* estimate in (3.6), inserting the above inequality into (4.9) and summing the resulted equation over $j\in\mathbb{N}$, we can obtain

$$\begin{aligned}&\langle t\rangle^{\frac{1-\delta}{2}}\|\mathcal{A}(t)\|^2_{X_{\tau,\kappa}}+\delta\int_0^T\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{A}(t)\|^2_{X_{\tau,\kappa}}dt+\lambda\sqrt{\epsilon}\int_0^T\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{A}(t)\|^2_{X_{\tau,\kappa+1/2}}dt\\&\quad\le C\frac{\sqrt{\epsilon}}{\lambda}\int_0^T\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|(\mathcal{B},u)\|^2_{X_{\tau,\kappa+3/2}}dt+C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{A}(t)\|^2_{X_{\tau,\kappa}}dt\end{aligned}$$

$$+\frac{C}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}}\sum_{j=0}^{\infty}(j+1)^{-1}\|M_j[[r]^j\partial_r, ur\partial_r+v\partial_z]\mathcal{A}\|^2_{L^2(\theta_2)}dt$$

$$+\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\sum_{j=0}^{\infty}|\langle H_j,\mathcal{A}_j\rangle_{\theta_2}|dt,$$

which is (4.2). Here, we have used the fact that

$$\|r\partial_r\mathcal{B}\|^2_{X_{\tau,\kappa-1/2}}\lesssim\|\mathcal{B}\|^2_{X_{\tau,\kappa+3/2}}\quad\text{and}\quad\|r\partial_r u\|^2_{X_{\tau,\kappa-1/2}}\lesssim\|u\|^2_{X_{\tau,\kappa+3/2}}.\qquad\square$$

4.2. *Estimates of the nonlinear terms*

Now, we go to estimate the nonlinear terms on the right hand of (4.2), we have the following lemma.

Lemma 4.2. *Under the assumption in* (3.4), *we have the following estimate*:

$$\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}}\sum_{j=0}^{\infty}(j+1)^{-1}\|M_j[[r]^j\partial_r^j, ur\partial_r+v\partial_z]\mathcal{A}\|^2_{L^2(\theta_2)}dt$$
$$\lesssim\lambda^{-1/2}\sqrt{\epsilon}\int_0^T\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)(\|\mathcal{A}\|^2_{X_{\tau,\kappa+1/2}}+\|u\|^2_{X_{\tau,\kappa+5/2}})dt$$
$$+\lambda^{-1/2}\sqrt{\epsilon}\int_0^T\langle t\rangle^{\frac{1-\delta}{2}}(\|\partial_z\mathcal{A}(t)\|^2_{X_{\tau,\kappa}}+\|\partial_z u(t)\|^2_{X_{\tau,\kappa+2}})dt \tag{4.9}$$

and

$$\int_0^T\langle t\rangle^{\frac{1-\delta}{2}}\sum_{j=0}^{\infty}|\langle H_j,\mathcal{A}_j\rangle_{\theta_2}|dt$$
$$\le\frac{\delta}{2}\int_0^T\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{A}\|^2_{X_{\tau,\kappa}}dt+\frac{\lambda}{2}\sqrt{\epsilon}\int_0^T\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{A}\|^2_{X_{\tau,\kappa}}dt$$
$$+C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)(\|\mathcal{A}\|^2_{X_{\tau,\kappa+1/2}}+\|u\|^2_{X_{\tau,\kappa+5/2}})dt.$$

Combining estimates in Lemmas 4.1 and 4.2, we finish the proof of Proposition 3.1. Now, we give the proof of Lemma 4.2.

Proof. Recall that from (4.4) and (4.5), we have

$$M_j[[r]^j\partial_r^j, ur\partial_r+v\partial_z]\mathcal{A}$$
$$=M_j[r]^j\sum_{k=1}^{j}\partial_r^k u\partial_r^{j-k}(r\partial_r\mathcal{A})+M_j ur^{j-1}\partial_r^j\mathcal{A}+M_j[r]^j\sum_{k=1}^{j}\partial_r^k v\partial_r^{j-k}(\partial_z\mathcal{A})$$
$$:=I_j^1+I_j^2+I_j^3.\qquad\square$$

We will estimate I_j^i $(i=1,2,3)$ term by term.

Estimate of term I_j^1

Noting that when $1 \le k \le [\frac{j+1}{2}] \le j$, we have

$$\binom{j}{k}^{-1} \le (j+1)^{-1}.$$

Then similar as derivation of (3.26), we have

$$\begin{aligned} |I_j^1| \lesssim & \sum_{k=1}^{[(j+1)/2]} (k+1)^{-\kappa} r^k M_k |\partial_r^k u| (j-k+1)^{-1} |(r\partial_r \mathcal{A})_{j-k}| \\ & + \sum_{k=[(j+1)/2]+1}^{j} (j-k+1)^{-\kappa} |u_k| |M_{j-k} r^{j-k} \partial_r^{j-k} (r\partial_r \mathcal{A})|. \end{aligned} \tag{4.10}$$

By using (4.10) to replace (3.26), similar derivation as (3.24) in Lemma 3.3, we can obtain that

$$\begin{aligned} & \sum_{j\in\mathbb{N}} (j+1)^{-1} \|I_j^1(t)\|_{L^2(\theta_2)}^2 \\ & \quad \lesssim \langle t\rangle^{1/2} (\|\partial_z u\|_{X_{\tau,3,1/2}}^2 \|r\partial_r \mathcal{A}\|_{X_{\tau,\kappa-3/2,1/2}}^2 + \|\partial_z u\|_{X_{\tau,\kappa-1/2,1/2}}^2 \|r\partial_r \mathcal{A}\|_{X_{\tau,3,1/2}}^2) \\ & \quad \lesssim \langle t\rangle^{1/2} (\|\partial_z u\|_{X_{\tau,3,1/2}}^2 \|\mathcal{A}\|_{X_{\tau,\kappa+1/2,1/2}}^2 + \|\partial_z u\|_{X_{\tau,\kappa+2,1/2}}^2 \|\mathcal{A}\|_{X_{\tau,5,1/2}}^2). \end{aligned} \tag{4.11}$$

Here, we have used the estimate that for $\tilde{\kappa} > 0$ and $0 \le \nu \le 1$,

$$\|r\partial_r \mathcal{A}\|_{X_{\tau,\tilde{\kappa},\nu}}^2 \le \|\mathcal{A}\|_{X_{\tau,\tilde{\kappa}+2,\nu}}^2.$$

Using the *a priori* estimates in (3.5) and smallness of ϵ, we can obtain that

$$\begin{aligned} & \frac{1}{\lambda\sqrt{\epsilon}} \int_0^T \langle t\rangle^{\frac{3+\delta}{2}} \sum_{j\in\mathbb{N}} (j+1)^{-1} \|I_j^1(t)\|_{L^2(\theta_2)}^2 dt \\ & \qquad \le \lambda^{-1/2}\sqrt{\epsilon} \int_0^T (\langle t\rangle^{\frac{1-\delta}{2}} \eta(t) \|\mathcal{A}\|_{X_{\tau,\kappa+1/2}}^2 + \langle t\rangle^{\frac{1-\delta}{2}} \|\partial_z u\|_{X_{\tau,\kappa+2}}^2) dt. \end{aligned} \tag{4.12}$$

Estimate of term I_j^2

This is direct. Using *a priori* estimates in (3.6) and smallness of ϵ, we can obtain that

$$\begin{aligned} & \frac{1}{\lambda\sqrt{\epsilon}} \int_0^T \langle t\rangle^{\frac{3+\delta}{2}} \sum_{j\in\mathbb{N}} (j+1)^{-1} \|I_j^2(t)\|_{L^2(\theta_2)}^2 dt \\ & \quad \le \frac{1}{\lambda\sqrt{\epsilon}} \int_0^T \|u\|_{L^\infty}^2 \langle t\rangle^{\frac{3+\delta}{2}} \|\mathcal{A}\|_{X_{\tau,\kappa-1/2}}^2 dt \\ & \quad \lesssim \lambda^{-1/2}\sqrt{\epsilon} \int^T \langle t\rangle^{\frac{1-\delta}{2}} \eta(t) \|\mathcal{A}\|_{X_{\tau,\kappa+1/2}}^2 dt. \end{aligned} \tag{4.13}$$

Estimate of term I_j^3

For term I_j^3, we have

$$
\begin{aligned}
|I_3| \lesssim & \sum_{k=1}^{[(j+1)/2]} (k+1)^{-\kappa}(j-k+1)^{-1}|M_k r^k \partial_r^k v||\partial_z \mathcal{A}_{j-k}| \\
& + \sum_{k=[(j+1)/2]+1}^{j} (j-k+1)^{-\kappa}|v_k||M_{j-k} r^{j-k}\partial_r^{j-k}\partial_z \mathcal{A}|.
\end{aligned}
$$

Similar as (4.11) and using the incompressibility

$$
\begin{aligned}
&\sum_{j\in\mathbb{N}}(j+1)^{-1}\|I_j^3(t)\|^2_{L^2(\theta_2)}dt \\
&\quad \lesssim \langle t\rangle^{1/2}(\|\partial_z v\|^2_{X_{\tau,3,1/2}}\|\partial_z\mathcal{A}\|^2_{X_{\tau,\kappa-3/2,1/2}} + \|\partial_z v\|^2_{X_{\tau,\kappa-1/2,1/2}}\|\partial_z\mathcal{A}\|^2_{X_{\tau,3,1/2}}) \\
&\quad \lesssim \langle t\rangle^{1/2}(\|u\|^2_{X_{\tau,5,1/2}}\|\partial_z\mathcal{A}\|^2_{X_{\tau,\kappa}} + \|u\|^2_{X_{\tau,\kappa+5/2}}\|\partial_z\mathcal{A}\|^2_{X_{\tau,3,1/2}}).
\end{aligned} \tag{4.14}
$$

Then using the *a priori* estimates in (3.5), we can obtain that

$$
\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}}\sum_{j=0}^{\infty}(j+1)^{-1}\|I_j^3(t)\|^2_{L^2(\theta_2)}dt \\
&\qquad \lesssim \lambda^{-1/2}\sqrt{\epsilon}\int_0^T (\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{A}\|^2_{X_{\tau,\kappa}} + \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|u\|^2_{X_{\tau,\kappa+5/2}})dt.
\end{aligned} \tag{4.15}
$$

Combining estimates in (4.12), (4.13) and (4.15), we obtain (4.9).

Estimate of term involving H_j

Next, we estimate term involving H_j. Recall

$$
H_j = M_j[r]^j\partial_r^j\left(-r\partial_r\partial_z u\int_z^{+\infty}\mathcal{A}d\bar{z} + (r\partial_r u + 2u)\mathcal{A}\right).
$$

First, by integrating by parts on z, we can have

$$
\begin{aligned}
&|\langle H_j, \mathcal{A}_j\rangle_{\theta_2}| \\
&\quad = \left|\left\langle M_j[r]^j\partial_r^j\left[r\partial_r u\int_z^{\infty}\mathcal{A}d\bar{z}\right], \partial_z\mathcal{A}_j + \mathcal{A}_j\frac{z}{2\langle t\rangle}\right\rangle_{\theta_2}\right| \\
&\qquad + |\langle M_j[r]^j\partial_r^j(2u\mathcal{A}), \mathcal{A}_j\rangle_{\theta_2}| \\
&\quad \lesssim \left\|\left[r\partial_r u\int_z^{\infty}\mathcal{A}d\bar{z}\right]_j\right\|_{L^2(\theta_2)}\|\partial_z\mathcal{A}_j\|_{L^2(\theta_2)} + \|[u\mathcal{A}]_j\|_{L^2(\theta_2)}\|\mathcal{A}_j\|_{L^2(\theta_2)}.
\end{aligned}
$$

Here, we have used (3.21) and (3.22) in Lemma 3.2. Then using Cauchy inequality, we have

$$\begin{aligned}
&2\int_0^T \langle t\rangle^{\frac{1-\delta}{2}} \sum_{j=0}^{\infty} \langle H_j, \mathcal{A}_j\rangle_{\theta_2} dt \\
&\quad \le \frac{\delta}{2}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}} \|\partial_z \mathcal{A}\|^2_{X_{\tau,\kappa}} dt + \frac{2}{\delta}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}} \left\| r\partial_r u \int_z^\infty \mathcal{A} d\bar{z}\right\|^2_{X_{\tau,\kappa}} dt \\
&\qquad + \frac{\lambda}{2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}} \eta(t)\|\mathcal{A}\|^2_{X_{\tau,\kappa+1/2}} dt + \frac{C}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}} \|u\mathcal{A}\|^2_{X_{\tau,\kappa-1/2}} dt.
\end{aligned}$$

Using product estimates in (3.23) to (3.25), and the *a priori* estimate in (3.5), we obtain that

$$\begin{aligned}
&\int_0^T \langle t\rangle^{\frac{1-\delta}{2}} \sum_{j=0}^{\infty} \langle H_j, \mathcal{A}_j\rangle_{\theta_2} dt \\
&\quad \le \frac{\delta}{2}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}} \|\partial_z \mathcal{A}\|^2_{X_{\tau,\kappa}} dt + \frac{\lambda}{2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}} \eta(t)\|\mathcal{A}\|^2_{X_{\tau,\kappa+1/2}} dt \\
&\qquad + \frac{C}{\delta}\int_0^T \langle t\rangle^{\frac{2-\delta}{2}} (\|r\partial_r u\|^2_{X_{\tau,3,1/2}} \|\mathcal{A}\|^2_{X_{\tau,\kappa}} + \|r\partial_r u\|^2_{X_{\tau,\kappa,1/2}} \|\mathcal{A}\|^2_{X_{\tau,3,1/2}}) dt \\
&\qquad + \frac{C}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{4+\delta}{2}} (\|\partial_z u\|^2_{X_{\tau,3,1/2}} \|\mathcal{A}\|^2_{X_{\tau,\kappa+1/2}} \\
&\qquad + \|u\|^2_{X_{\tau,\kappa+1/2,1/2}} \|\partial_z \mathcal{A}\|^2_{X_{\tau,3,1/2}}) dt \\
&\quad \le \frac{\delta}{2}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}} \|\partial_z \mathcal{A}\|^2_{X_{\tau,\kappa}} dt + \frac{\lambda}{2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}} \eta(t)\|\mathcal{A}\|^2_{X_{\tau,\kappa+1/2}} dt \\
&\qquad + C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}} \eta(t)(\|\mathcal{A}\|^2_{X_{\tau,\kappa+1/2}} + \|u\|^2_{X_{\tau,\kappa+5/2}}) dt.
\end{aligned}$$

This is (4.10). Here, we have chosen sufficiently small ϵ. For example we can set

$$\epsilon \le \delta^{20}, \quad \epsilon \le \lambda^{-10}.$$

5. Estimates of the Auxiliary Function $\mathcal{B}$

In this section, we give the proof of Proposition 3.2.

5.1. *Derivation of the equation for $\mathcal{B}$ and its linear estimate*

By applying $r\partial_r$ to $(2.2)_1$, we can obtain that

$$[\partial_t + (ur\partial_r + v\partial_z) - \partial_z^2](r\partial_r u) = -r\partial_r v\partial_z u - (r\partial_r u)^2 - 2ur\partial_r u. \tag{5.1}$$

Multiplying $\partial_z u$ to $(3.1)_1$ and using the equation satisfying by $\partial_z u$,

$$[\partial_t + (ur\partial_r + v\partial_z) - \partial_z^2](\partial_z u) = 0,$$

we can obtain that

$$[\partial_t + (ur\partial_r + v\partial_z) - \partial_z^2]\left(\partial_z u \int_z^{+\infty} \mathcal{A} d\bar{z}\right)$$
$$= \sqrt{\epsilon}\langle t\rangle^{-\delta-1} r\partial_r v\partial_z u + 2\partial_z^2 u\mathcal{A}.$$

By multiplying the above equality by $\frac{\langle t\rangle^{1+\delta}}{\sqrt{\epsilon}}$, we can obtain that

$$[\partial_t + (ur\partial_r + v\partial_z) - \partial_z^2]\left(\frac{\langle t\rangle^{1+\delta}}{\sqrt{\epsilon}}\partial_z u \int_z^{+\infty} \mathcal{A} d\bar{z}\right)$$
$$= r\partial_r v\partial_z u + \frac{2\langle t\rangle^{1+\delta}}{\sqrt{\epsilon}}\partial_z^2 u\mathcal{A} + \frac{(1+\delta)}{\sqrt{\epsilon}}\langle t\rangle^{\delta}\partial_z u \int_z^{+\infty} \mathcal{A} d\bar{z}. \tag{5.2}$$

Adding (5.1) and (5.2) together implies that

$$[\partial_t + (ur\partial_r + v\partial_z) - \partial_z^2]\mathcal{B}$$
$$= \frac{2\langle t\rangle^{1+\delta}}{\sqrt{\epsilon}}\partial_z^2 u\mathcal{A} + \frac{(1+\delta)}{\sqrt{\epsilon}}\langle t\rangle^{\delta}\partial_z u \int_z^{+\infty} \mathcal{A} d\bar{z} - (r\partial_r u)^2 - 2ur\partial_r u$$
$$:= \sum_{i=1}^{4} K^i. \tag{5.3}$$

Lemma 5.1. *Under the assumption in* (3.4), *we have the following estimate*, *there exists a constant C such that*

$$\langle t\rangle^{\frac{1-\delta}{2}}\|\mathcal{B}(t)\|^2_{X_{\tau,\kappa+1}} + \delta\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{B}(t)\|^2_{X_{\tau,\kappa+1}}dt$$
$$+ \lambda\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{B}(t)\|^2_{X_{\tau,\kappa+3/2}}dt$$
$$\lesssim \|\mathcal{B}(0)\|^2_{X_{\tau_0,\kappa+1}} + \lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{B}(t)\|^2_{X_{\tau,\kappa+3/2}}dt$$
$$+ \frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}}\sum_{j=0}^{\infty}(j+1)\|M_j[[r]^j\partial_r^j, ur\partial_r + v\partial_z]\mathcal{B}\|^2_{L^2(\theta_2)}dt.$$
$$+ \frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}}\|(K^2, K^3, K^4)\|^2_{X_{\tau,\kappa+1/2}}dt$$
$$+ \int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\sum_{j=0}^{\infty}(j+1)^2|\langle K_j^1, \mathcal{B}_j\rangle_{L^2(\theta_2)}|dt. \tag{5.4}$$

Proof. Applying $M_j[r]^j\partial_r^j$ to (5.2), we can obtain that

$$\partial_t \mathcal{B}_j + \delta\sqrt{\epsilon}\eta(t)(j+1)\mathcal{B}_j + (ur\partial_r + v\partial_z)\mathcal{B}_j - \partial_z^2\mathcal{B}_j$$
$$= M_j[[r]^j\partial_r^j, ur\partial_r + v\partial_z]\mathcal{B} + \sum_{i=1}^{4} K_j^i.$$

Performing the energy estimates as (4.9), we can obtain that

$$\begin{aligned}
&\langle t\rangle^{\frac{1-\delta}{2}}(j+1)^2\|\mathcal{B}_j(t)\|^2_{L^2(\theta_2)} + \delta(j+1)^2\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{B}_j(t)\|^2_{L^2(\theta_2)}dt\\
&\quad + \lambda\sqrt{\epsilon}(j+1)^3\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{B}_j(t)\|^2_{L^2(\theta_2)}dt\\
&\leq (j+1)^2\|\mathcal{B}_j(0)\|^2_{L^2(\theta_2)} + \lambda^{-1/2}\sqrt{\epsilon}(j+1)^3\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{B}_j(t)\|^2_{L^2(\theta_2)}dt\\
&\quad + \frac{C}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}}(j+1)(\|(K_j^2, K_j^3, K_j^4)(t)\|^2_{L^2(\theta^2)}\\
&\quad + \|M_j[[r]^j\partial_r^j, ur\partial_r + v\partial_z]\mathcal{B}\|^2_{L^2(\theta_2)})dt\\
&\quad + \int_0^T \langle t\rangle^{\frac{1-\delta}{2}}(j+1)^2|\langle K_j^1, \mathcal{B}_j\rangle_{L^2(\theta_2)}|dt.
\end{aligned}$$

Summing the above inequality over $j\in\mathbb{N}$ indicates (5.4). $\square$

5.2. *Estimates of nonlinear terms of $\mathcal{B}$*

Lemma 5.2. *Under the assumption in* (3.4), *for sufficiently small* ϵ, *we have the following estimate*:

$$\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\sum_{j=1}^{\infty}(j+1)\|M_j[[r]^j\partial_r^j, ur\partial_r + v\partial_z]\mathcal{B}\|^2_{L^2(\theta_2)}dt\\
&\quad + \frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}}\|(K^2, K^2, K^4)\|^2_{X_{\tau,\kappa+1/2}}dt\\
&\quad + \int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\sum_{j=0}^{\infty}(j+1)^2|\langle K_j^1, \mathcal{B}_j\rangle_{\theta_2}|dt\\
&\leq \frac{\lambda\sqrt{\epsilon}}{2}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{B}\|^2_{X_{\tau,\kappa+3/2}}dt + \frac{\delta}{2}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{B}\|^2_{X_{\tau,\kappa+3/2}}dt\\
&\quad + C\lambda^{-1/2}\sqrt{\epsilon}\int_0^{T_0} \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)(\|\mathcal{A}(t)\|^2_{X_{\tau,\kappa+1/2}} + \|\mathcal{B}(t)\|^2_{X_{\tau,\kappa+3/2}}
\end{aligned}$$

$$
\begin{aligned}
&+ \|u(t)\|^2_{X_{\tau,\kappa+5/2}})dt + C\lambda^{-1/2}\sqrt{\epsilon}\int_0^{T_0}\langle t\rangle^{\frac{1-\delta}{2}}(\|\partial_z\mathcal{A}(t)\|^2_{X_{\tau,\kappa}} \\
&+ \|\partial_z\mathcal{B}(t)\|^2_{X_{\tau,\kappa+1}} + \|\partial_z u(t)\|^2_{X_{\tau,\kappa+2}})dt.
\end{aligned} \tag{5.5}
$$

Proof. First, using Leibniz formula and the same as (4.4) and (4.5), we have

$$
\begin{aligned}
M_j[[r]^j\partial_r^j, ur\partial_r + v\partial_z]\mathcal{B} \;=\; & M_j[r]^j\sum_{k=1}^{j}\partial_r^k u\partial_r^{j-k}(r\partial_r\mathcal{B}) + M_j ur^{j-1}\partial_r^j\mathcal{B} \\
& + M_j[r]^j\sum_{k=1}^{j}\partial_r^k v\partial_r^{j-k}(\partial_z\mathcal{B}) \\
:= \;& L_j^1 + L_j^2 + L_j^3.
\end{aligned}
$$

□

Estimate of term L_j^1

Almost the same as (4.12), using (3.5) in Lemma 3.1, we have

$$
\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T\langle t\rangle^{\frac{3+\delta}{2}}\sum_{j\in\mathbb{N}}(j+1)\|L_j^1(t)\|^2_{L^2(\theta_2)}dt \\
&\quad\le \frac{1}{\lambda\sqrt{\epsilon}}\int_0^T\langle t\rangle^{\frac{4+\delta}{2}}(\|\partial_z u\|^2_{X_{\tau,3,1/2}}\|r\partial_r\mathcal{B}\|^2_{X_{\tau,\kappa-1/2,1/2}} \\
&\qquad + \|\partial_z u\|^2_{X_{\tau,\kappa+1/2,1/2}}\|r\partial_r\mathcal{B}\|^2_{X_{\tau,3,1/2}})dt \\
&\quad\lesssim \lambda^{-1/2}\sqrt{\epsilon}\int_0^T(\langle t\rangle^{-\frac{1+3\delta}{2}}\|\mathcal{B}\|^2_{X_{\tau,\kappa+3/2}} + \langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z u\|^2_{X_{\tau,\kappa+2}})dt.
\end{aligned} \tag{5.6}
$$

Estimate of term L_j^2

This is direct. Using *a priori* estimates in (3.6) in Lemma 3.1, we can obtain that

$$
\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T\langle t\rangle^{\frac{3+\delta}{2}}\sum_{j\in\mathbb{N}}(j+1)\|L_j^2(t)\|^2_{L^2(\theta_2)}dt \\
&\quad\le \frac{1}{\lambda\sqrt{\epsilon}}\int_0^T\|u\|^2_{L^\infty}\langle t\rangle^{\frac{3+\delta}{2}}\|\mathcal{B}\|^2_{X_{\tau,\kappa+1/2}}dt \\
&\quad\lesssim \lambda^{-1/2}\sqrt{\epsilon}\int_0^T\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{B}\|^2_{X_{\tau,\kappa+3/2}}dt.
\end{aligned}
$$

Estimate of term L_j^3

Almost the same as (4.14), (4.15) and using the *a priori* estimates (3.5) in Lemma 3.1, we can obtain that

$$\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}}\sum_{j\in\mathbb{N}}(j+1)\|L_j^3(t)\|^2_{L^2(\theta_2)}dt\\
&\quad\leq \frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{4+\delta}{2}}(\|u\|^2_{X_{\tau,5,1/2}}\|\partial_z\mathcal{B}\|^2_{X_{\tau,\kappa+1}}+\|u\|^2_{X_{\tau,\kappa+5/2}}\|\partial_z\mathcal{B}\|^2_{X_{\tau,3,1/2}})dt\\
&\quad\lesssim \lambda^{-1/2}\sqrt{\epsilon}\int_0^T(\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\mathcal{B}\|^2_{X_{\tau,\kappa+1}}+\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|u\|^2_{X_{\tau,\kappa+5/2}})dt. \qquad (5.7)
\end{aligned}$$

Estimates of K^2

Remembering the representation of K^2, then from (3.24) in Lemma 3.3 and using the *a priori* estimates in (3.5), we have

$$\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}}\|K^2(t)\|^2_{X_{\tau,\kappa+1/2}}dt\\
&\quad\lesssim \frac{1}{\lambda\epsilon^{3/2}}\int_0^T \langle t\rangle^{\frac{4+5\delta}{2}}(\|\partial_z u\|^2_{X_{\tau,3,1/2}}\|\mathcal{A}\|^2_{X_{\tau,\kappa+1/2,1/2}}\\
&\qquad+\|\partial_z u\|^2_{X_{\tau,\kappa+1/2,1/2}}\|\mathcal{A}\|^2_{X_{\tau,3,1/2}})dt\\
&\quad\lesssim \lambda^{-1/2}\sqrt{\epsilon}\int_0^T(\langle t\rangle^{-\frac{1+3\delta}{2}}\|\mathcal{A}\|^2_{X_{\tau,\kappa+1/2}}+\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z u\|^2_{X_{\tau,\kappa+2}})dt. \qquad (5.8)
\end{aligned}$$

Estimates of K^3

From the product estimate (3.25) in Lemma 3.3 and using the *a priori* estimates in (3.5), we have

$$\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}}\|K^3(t)\|^2_{X_{\tau,\kappa+1/2}}dt\\
&\quad\lesssim \frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{4+\delta}{2}}\|\partial_z r\partial_r u\|^2_{X_{\tau,3,1/2}}\|r\partial_r u\|^2_{X_{\tau,\kappa+1/2,1/2}}dt\\
&\quad\lesssim \frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{4+\delta}{2}}\|\partial_z u\|^2_{X_{\tau,5,1/2}}\|u\|^2_{X_{\tau,\kappa+5/2}}dt\\
&\quad\lesssim \lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|u\|^2_{X_{\tau,\kappa+5/2}}dt. \qquad (5.9)
\end{aligned}$$

Estimates of K^4

Also from the product estimate (3.25) in Lemma 3.3 and using the *a priori* estimates in (3.5), we have

$$\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}}\sum_{j=0}^{\infty}(j+1)\|K_j^4(t)\|^2_{L^2(\theta^2)}dt\\
&\quad\lesssim \frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{4+\delta}{2}}(\|\partial_z u\|^2_{X_{\tau,3,1/2}}\|r\partial_r u\|^2_{X_{\tau,\kappa+1/2,1/2}}\\
&\qquad+\|u\|^2_{X_{\tau,\kappa+1/2,1/2}}\|\partial_z r\partial_r u\|^2_{X_{\tau,3,1/2}})dt\\
&\quad\lesssim \frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{4+\delta}{2}}\|\partial_z u\|^2_{X_{\tau,5,1/2}}\|u\|^2_{X_{\tau,\kappa+5/2}}dt\\
&\quad\lesssim \lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|u\|^2_{X_{\tau,\kappa+5/2}}dt. \qquad (5.10)
\end{aligned}$$

Estimates of K^1

For term K^1, we decompose it as the following:

$$\begin{aligned}
K_j^1 &= 2\frac{\langle t\rangle^{1+\delta}}{\sqrt{\epsilon}}\partial_z^2 u\mathcal{A}_j + 2M_j\frac{\langle t\rangle^{1+\delta}}{\sqrt{\epsilon}}[r]^j\sum_{1\le k\le j}\binom{j}{k}\partial_z^2\partial_r^k u\partial_r^{j-k}\mathcal{A}\\
&:= K^1_{j,\mathrm{low}} + K^1_{j,\mathrm{other}}.
\end{aligned}$$

Then by using Cauchy inequality and *a priori* estimates in (3.6) in Lemma 3.1, we have

$$\begin{aligned}
&\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\sum_{j=0}^{\infty}(j+1)^2|\langle K^1_{j,\mathrm{low}},\mathcal{B}_j\rangle_{\theta_2}|dt\\
&\quad\le \frac{C}{\lambda\epsilon^{3/2}}\int_0^T \langle t\rangle^{\frac{7+5\delta}{2}}\|\partial_z^2 u\|^2_{L^\infty}\|\mathcal{A}\|^2_{X_{\tau,\kappa+1/2}}dt + \frac{\lambda\sqrt{\epsilon}}{2}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{B}\|^2_{X_{\tau,\kappa+3/2}}dt\\
&\quad\le C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{A}\|^2_{X_{\tau,\kappa+1/2}}dt + \frac{\lambda\sqrt{\epsilon}}{2}\int_0^T \langle t\rangle^{\frac{1+\delta}{2}}\eta(t)\|\mathcal{B}\|^2_{X_{\tau,\kappa+3/2}}dt. \qquad (5.11)
\end{aligned}$$

By using integration by parts on z, we have that

$$\begin{aligned}
&|\langle K^1_{j,\mathrm{other}},\mathcal{B}_j\rangle_{\theta_2}|\\
&\quad\le 2\frac{\langle t\rangle^{1+\delta}}{\sqrt{\epsilon}}\left|\left\langle M_j[r]^j\sum_{1\le k\le j}\binom{j}{k}\partial_z\partial_r^k u\partial_r^{j-k}\partial_z\mathcal{A},\mathcal{B}_j\right\rangle_{\theta_2}\right|\\
&\qquad+2\frac{\langle t\rangle^{1+\delta}}{\sqrt{\epsilon}}\left|\left\langle M_j[r]^j\sum_{1\le k\le j}\binom{j}{k}\partial_z\partial_r^k u\partial_r^{j-k}\mathcal{A},\partial_z\mathcal{B}_j+\frac{z}{2\langle t\rangle}\mathcal{B}_j\right\rangle_{\theta_2}\right|.
\end{aligned}$$

Using Hölder inequality and (3.28), (3.22) in Lemma 3.2, we can obtain that

$$
\begin{aligned}
&\sum_{j=0}^{\infty}(j+1)^2|\langle K^1_{j,\mathrm{other}},\mathcal{B}_j\rangle_{\theta_2}|\\
&\quad\lesssim 2\frac{\langle t\rangle^{1+\delta}}{\sqrt{\epsilon}}\sum_{j=0}^{\infty}(j+1)^2\left\|M_j[r]^j\sum_{1\le k\le j}\binom{j}{k}\partial_z\partial_r^k u\partial_r^{j-k}\partial_z\mathcal{A}\right\|_{L^2_rL^1_z(\theta_{3/2})}\\
&\qquad\times\langle t\rangle^{1/4}\|\partial_z\mathcal{B}_j\|_{L^2(\theta)}+2\frac{\langle t\rangle^{1+\delta}}{\sqrt{\epsilon}}\sum_{j=0}^{\infty}(j+1)^2\\
&\qquad\times\left\|M_j[r]^j\sum_{1\le k\le j}\binom{j}{k}\partial_z\partial_r^k u\partial_r^{j-k}\mathcal{A}\right\|_{L^2(\theta_2)}\|\partial_z\mathcal{B}_j\|_{L^2_z(\theta_2)}.
\end{aligned}\tag{5.12}
$$

The same as derivation of (3.26) as before, it is easy to see that

$$
\begin{aligned}
&\left|M_j[r]^j\sum_{1\le k\le j}\binom{j}{k}\partial_z\partial_r^k u\partial_r^{j-k}\partial_z\mathcal{A}\right|\\
&\quad\lesssim\sum_{k=1}^{[(j+1)/2]}(k+1)^{-\kappa}M_kr^j|\partial_z\partial_r^k u|(j-k+1)^{-1}|\partial_z\mathcal{A}_{j-k}|\\
&\qquad+\sum_{k=[(j+1)/2]+1}^{j}M_{j-k}(j-k+1)^{-\kappa}|\partial_z u_k||r^{j-k}\partial_r^{j-k}\partial_z\mathcal{A}|.
\end{aligned}
$$

Using Minkowski inequality, Sobolev embedding, discrete young inequality in (3.26), and *a priori* estimates in (3.5) and (3.6), we have

$$
\begin{aligned}
&\sum_{j=0}^{\infty}(j+1)^2\left\|[r]^j\sum_{1\le k\le j}\binom{j}{k}\partial_z\partial_r^k u\partial_r^{j-k}\partial_z\mathcal{A}\right\|^2_{L^2_hL^1_z(\theta_{3/2})}\\
&\quad\lesssim\|\partial_z u_k\|^2_{X_{\tau,3,3/4}}\|\partial_z\mathcal{A}\|^2_{X_{\tau,\kappa,3/4}}+\|\partial_z u\|^2_{X_{\tau,\kappa+1,3/4}}\|\partial_z\mathcal{A}\|^2_{X_{\tau,3,3/4}}\\
&\quad\lesssim\lambda^{1/2}\epsilon^2\langle t\rangle^{-\frac{7-\delta}{2}}(\|\partial_z\mathcal{A}\|^2_{X_{\tau,\kappa}}+\|\partial_z u\|^2_{X_{\tau,\kappa+2}})
\end{aligned}\tag{5.13}
$$

and similar as the proof of Lemma 3.3, we have

$$
\begin{aligned}
&\sum_{j=0}^{\infty}(j+1)^2\left\|[r]^j\sum_{1\le k\le j}\binom{j}{k}\partial_z\partial_r^k u\partial_r^{j-k}\mathcal{A}\right\|^2_{L^2(\theta_2)}\\
&\quad\lesssim\langle t\rangle^{1/2}\|\partial_z u_k\|^2_{X_{\tau,3,1/2}}\|\partial_z\mathcal{A}\|^2_{X_{\tau,\kappa,1/2}}+\langle t\rangle^{1/2}\|\partial_z u\|^2_{X_{\tau,\kappa+1,1/2}}\|\partial_z\mathcal{A}\|^2_{X_{\tau,3,1/2}}\\
&\quad\lesssim\lambda^{1/2}\epsilon^2\langle t\rangle^{-\frac{6-\delta}{2}}(\|\partial_z\mathcal{A}\|^2_{X_{\tau,\kappa}}+\|\partial_z u\|^2_{X_{\tau,\kappa+2}}).
\end{aligned}\tag{5.14}
$$

By using Young inequality to (5.12) and inserting (5.13) and (5.14) into the resulted inequality, we can obtain that

$$\begin{aligned}
&\int_0^T \langle t\rangle^{\frac{1-\delta}{2}} \sum_{j=0}^{\infty}(j+1)^2 |\langle K^3_{j,\text{other}}, \mathcal{B}_j\rangle_{\theta_2}| dt \\
&\quad \le \frac{\delta}{4}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}} \|\partial_z \mathcal{B}\|^2_{X_{\tau,\kappa+1}} dt \\
&\qquad + \frac{C}{\epsilon\delta}\int_0^T \langle t\rangle^{\frac{6+3\delta}{2}} \sum_{j=0}^{\infty}(j+1)^2 \left\| [r]^j \sum_{1\le k\le j} \binom{j}{k} \partial_z \partial_r^k u \partial_r^{j-k}\partial_z \mathcal{A}\right\|^2_{L^2_h L^1_z(\theta_{3/2})} dt \\
&\qquad + \frac{C}{\epsilon\delta}\int_0^T \langle t\rangle^{\frac{5+3\delta}{2}} \sum_{j=0}^{\infty}(j+1)^2 \left\| [r]^j \sum_{1\le k\le j} \binom{j}{k} \partial_z \partial_r^k u \partial_r^{j-k} \mathcal{A}\right\|^2_{L^2(\theta_2)} dt \\
&\quad \le \frac{\delta}{4}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}} \|\partial_z \mathcal{B}\|_{X_{\tau,\kappa+1}} dt + \lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}} (\|\partial_z \mathcal{A}\|^2_{X_{\tau,\kappa}} \\
&\qquad + \|\partial_z u\|^2_{X_{\tau,\kappa+2}}) dt.
\end{aligned} \tag{5.15}$$

Combining estimates in (5.6)–(5.11) and (5.15), we can obtain (5.5) in Lemma 5.2.

6. Estimates of the Solution u

Due to the fact that the equation of u_j has one order derivative loss, direct Gevrey-2 energy estimates on the equations of u_j do not work. Instead, we will use another alternative quantity φ_j, defined as

$$\varphi_j := M_j\left([r]^j\partial_r^j u + \frac{\langle t\rangle^{1+\delta}\partial_z u}{\sqrt{\epsilon}}\int_z^{\infty}[r]^{j-1}\partial_r^{j-1}\mathcal{A} d\bar{z}\right), \tag{6.1}$$

to perform energy estimates, which has no derivative loss. Combining estimates of $\mathcal{A}_{j-1}$ and φ_j, we can achieve estimates of u_j. Then Proposition 3.3 follows.

6.1. *The equation of φ_j and its linear estimate*

Applying $[r]^j\partial_r^j$ to the first equation of (2.2), we can obtain that

$$\begin{aligned}
&[\partial_t + (ur\partial_r + v\partial_z) - \partial_z^2]([r]^j\partial_r^j u) \\
&\quad = -[[r]^j\partial_r^j, ur\partial_r + v\partial_z]u - [r]^j\partial_r^j(u^2) \\
&\quad = -[r]^j\partial_r^j v\partial_z u - [r]^j\sum_{k=1}^{j}\binom{j}{k}\partial_r^k u\partial_r^{j-k}(r\partial_r u) - ur^{j-1}\partial_r^j u
\end{aligned}$$

$$
\begin{aligned}
&- [r]^j \sum_{k=1}^{j-1} \binom{j}{k} \partial_r^k v \partial_r^{j-k} \partial_z u - [r]^j \partial_r^j (u^2)\\
&:= -[r]^j \partial_r^j v \partial_z u + O^1 + O^2 + O^3 + O^4.
\end{aligned} \tag{6.2}
$$

Also applying $\frac{\langle t\rangle^{\delta+1}}{\sqrt{\epsilon}} \partial_z u [r]^{j-1} \partial_r^{j-1}$ to the first equation of (3.1), then we can obtain that

$$
\begin{aligned}
&[\partial_t + (ur\partial_r + v\partial_z) - \partial_z^2] \left(\frac{\langle t\rangle^{1+\delta}}{\sqrt{\epsilon}} \partial_z u \int_z^\infty [r]^{j-1} \partial_r^{j-1} \mathcal{A} d\bar{z} \right)\\
&= [r]^j \partial_r^j v \partial_z u + (j-1)[r]^{j-1} \partial_r^{j-1} v \partial_z u\\
&\quad - \frac{\langle t\rangle^{1+\delta}}{\sqrt{\epsilon}} \partial_z u [r]^{j-1} \sum_{k=1}^{j-1} \binom{j-1}{k} \partial_r^k u \int_z^\infty \partial_r^{j-1-k} (r\partial_r \mathcal{A}) d\bar{z}\\
&\quad - \frac{\langle t\rangle^{1+\delta}}{\sqrt{\epsilon}} \partial_z u u \int_z^\infty r^{j-2} \partial_r^{j-1} \mathcal{A} d\bar{z} + \frac{\langle t\rangle^{1+\delta}}{\sqrt{\epsilon}} \partial_z u [r]^{j-1}\\
&\quad \times \sum_{k=1}^{j-1} \binom{j-1}{k} \partial_r^k v \partial_r^{j-1-k} \mathcal{A} + \frac{2\langle t\rangle^{1+\delta}}{\sqrt{\epsilon}} (\partial_z^2 u) [r]^{j-1} \partial_r^{j-1} \mathcal{A}\\
&\quad + (1+\delta) \frac{\langle t\rangle^{\delta}}{\sqrt{\epsilon}} \partial_z u \int_z^\infty [r]^{j-1} \partial_r^{j-1} \mathcal{A} d\bar{z}\\
&:= [r]^j \partial_r^j v \partial_z u + \sum_{i=1}^{6} P^i.
\end{aligned} \tag{6.3}
$$

Then add (6.3) and (6.2) together implies that

$$
\begin{aligned}
&[\partial_t + (ur\partial_r + v\partial_z) - \partial_z^2] \left([r]^j \partial_r^j u + \frac{\langle t\rangle^{1+\delta} \partial_z u}{\sqrt{\epsilon}} \int_z^\infty [r]^{j-1} \partial_r^{j-1} \mathcal{A} d\bar{z} \right)\\
&= \sum_{i=1}^{4} O^i + \sum_{i=1}^{6} P^i.
\end{aligned} \tag{6.4}
$$

Multiplying (6.4) by M_j, we can obtain that

$$
[\partial_t + \lambda\sqrt{\epsilon}(j+1) + (ur\partial_r + v\partial_z) - \partial_z^2] \varphi_j = \sum_{i=1}^{4} O_j^i + \sum_{i=1}^{6} P_j^i, \tag{6.5}
$$

where $O_j^i := M_j O^i$ and $P_j^i := M_j P^i$.

There is no derivative loss for Eq. (6.5). For $\alpha \geq 0$, denote

$$
\|\varphi\|_{X_{\tau,\kappa+\alpha}}^2 := \sum_{j=0}^{\infty} (j+1)^{2\alpha} \|\varphi_j\|_{L^2(\theta_2)}^2.
$$

We have the following linear estimate.

Lemma 6.1. *Under the assumption in* (3.4), *for sufficiently small* ϵ, *we have the following estimate*:

$$
\begin{aligned}
&\langle t\rangle^{\frac{1-\delta}{2}}\|\varphi(t)\|^2_{X_{\tau,\kappa+2}} + \delta\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\varphi(t)\|^2_{X_{\tau,\kappa+2}}dt \\
&\quad + \lambda\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\varphi(t)\|^2_{X_{\tau,\kappa+5/2}}dt \\
&\lesssim \|u(0)\|^2_{X_{\tau_0,\kappa+2}} + \lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\varphi(t)\|^2_{X_{\tau,\kappa+5/2}}dt \\
&\quad + \frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}}\sum_{j=0}^{\infty}(j+1)^3\left\|\left(\sum_{i=1}^4 O_j^i + \sum_{i=1}^6 P_j^i\right)\right\|^2_{L^2(\theta_2)}dt. \qquad (6.6)
\end{aligned}
$$

Proof. Performing energy estimates for (6.5) similar as (4.9) and using Cauchy inequality, we can have

$$
\begin{aligned}
&\langle t\rangle^{\frac{1-\delta}{2}}(j+1)^4\|\varphi_j(t)\|^2_{L^2(\theta_2)} + \delta(j+1)^4\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\varphi_j(t)\|^2_{L^2(\theta_2)}dt \\
&\quad + \lambda\sqrt{\epsilon}(j+1)^5\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\varphi_j(t)\|^2_{L^2(\theta_2)}dt \\
&\lesssim (j+1)^4\|u_j(0)\|^2_{L^2(\theta_2)} \\
&\quad + \lambda^{1/2}\epsilon(j+1)^4\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\varphi_j(t)\|^2_{L^2(\theta_2)}dt \\
&\quad + \frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}}(j+1)^3\|O_j^i, P_j^i\|^2_{L^2(\theta_2)}dt. \qquad (6.7)
\end{aligned}
$$

Then summing (6.7) over $j \in \mathbb{N}$, we can achieve (6.6). □

6.2. *Estimates of the nonlinear terms*

Lemma 6.2. *Under the assumption in* (3.4), *for sufficiently small* ϵ, *we have the following estimate*:

$$
\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}}\sum_{j=0}^{\infty}(j+1)^3\left\|\left(\sum_{i=1}^4 O_j^i + \sum_{i=1}^6 P_j^i\right)\right\|^2_{L^2(\theta_2)}dt \\
&\quad\lesssim \lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)(\|u(t)\|^2_{X_{\tau,\kappa+5/2}} + \|\mathcal{A}(t)\|^2_{X_{\tau,\kappa+1/2}})dt \\
&\qquad + \lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}(\|\partial_z u(t)\|^2_{X_{\tau,\kappa+2}} + \|\partial_z\mathcal{A}(t)\|^2_{X_{\tau,\kappa}})dt. \qquad (6.8)
\end{aligned}
$$

Proof. We estimate O^i and P^i term by term. $\square$

Estimates of O^1_j

Similar as (3.26), noting that

$$|O^1_j| \leq \sum_{k=1}^{[(j+1)/2]} (k+1)^{-\kappa}(j-k+1)^{-1} M_k r^k \partial_r^k u||(r\partial_r u)_{j-k}| + \sum_{k=[(j+1)/2]+1}^{j} (j-k+1)^{-\kappa}|u_k||M_{j-k} r^{j-k}\partial_r^{j-k}(r\partial_r u)|.$$

then similar as product estimates in (3.24) and using the *a priori* estimates (3.5) in Lemma 3.1, we have

$$\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}} \sum_{j=0}^{\infty}(j+1)^3\|O^1_j(t)\|^2_{L^2(\theta_2)}dt \\
&\quad\lesssim \frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{4+\delta}{2}}(\|\partial_z u\|^2_{X_{\tau,3,1/2}}\|r\partial_r u\|^2_{X_{\tau,\kappa+1/2,1/2}} \\
&\qquad + \|\partial_z u\|^2_{X_{\tau,\kappa+3/2,1/2}}\|r\partial_r u\|^2_{X_{\tau,3,1/2}})dt \\
&\quad\lesssim \lambda^{-1/2}\sqrt{\epsilon}\int_0^T (\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|u\|^2_{X_{\tau,\kappa+5/2}} + \langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z u\|^2_{X_{\tau,\kappa+2}})dt. \qquad (6.9)
\end{aligned}$$

Estimate of term O^2_j

This is direct. Using *a priori* estimates (3.5) in Lemma 3.1, we can obtain that

$$\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}} \sum_{j\in\mathbb{N}}(j+1)^3\|O^2_j(t)\|^2_{L^2(\theta_2)}dt \\
&\quad\lesssim \frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \|u\|^2_{L^\infty}\langle t\rangle^{\frac{3+\delta}{2}}\|u\|^2_{X_{\tau,\kappa+3/2}}dt \\
&\quad\lesssim \lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|u\|^2_{X_{\tau,\kappa+5/2}}dt. \qquad (6.10)
\end{aligned}$$

Estimate of term O^3_j

Noting that

$$|O^3_j| \lesssim \sum_{k=1}^{[(j+1)/2]} (k+1)^{-\kappa}(j-k+1)^{-1}|M_k r^k\partial_r^k v||\partial_z u_{j-k}| + \sum_{k=[(j+1)/2]+1}^{j-1} (k+1)^{-1}(j-k+1)^{-\kappa}|v_k||M_{j-k}r^{j-k}\partial_r^{j-k}\partial_z u|, \qquad (6.11)$$

then similar as product estimates in (3.24), using the *a priori* estimates (3.5) in Lemma 3.1 and incompressibility, we have

$$\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}}\sum_{j=0}^{\infty}(j+1)^3\|O_j^3(t)\|_{L^2(\theta_2)}^2 dt\\
&\quad\lesssim \frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{4+\delta}{2}}(\|\partial_z v\|_{X_{\tau,3,1/2}}^2\|\partial_z u\|_{X_{\tau,\kappa+1/2,1/2}}^2\\
&\qquad+\|\partial_z v\|_{X_{\tau,\kappa+1/2,1/2}}^2\|\partial_z u\|_{X_{\tau,3,1/2}}^2)dt\\
&\quad\lesssim \lambda^{-1/2}\sqrt{\epsilon}\int_0^T(\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|u\|_{X_{\tau,\kappa+5/2}}^2+\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z u\|_{X_{\tau,\kappa+2}}^2)dt. \qquad (6.12)
\end{aligned}$$

Estimate of term O_j^4

Using product estimates in (3.24) and using the *a priori* estimates (3.5) in Lemma 3.1, we have

$$\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}}\sum_{j=0}^{\infty}(j+1)^3\|O_j^4(t)\|_{L^2(\theta_2)}^2 dt\\
&\quad\lesssim \frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{4+\delta}{2}}(\|\partial_z u\|_{X_{\tau,3,1/2}}^2\|u\|_{X_{\tau,\kappa+1/2,1/2}}^2\\
&\qquad+\|\partial_z u\|_{X_{\tau,\kappa+3/2,1/2}}^2\|u\|_{X_{\tau,3,1/2}}^2)dt\\
&\quad\lesssim \lambda^{-1/2}\sqrt{\epsilon}\int_0^T(\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|u\|_{X_{\tau,\kappa+5/2}}^2+\langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z u\|_{X_{\tau,\kappa+2}}^2)dt. \qquad (6.13)
\end{aligned}$$

Estimate of term P_j^1

This is direct. Using *a priori* estimates in (3.6) in Lemma 3.1 and incompressibility, we can obtain that

$$\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}}\sum_{j\in\mathbb{N}}(j+1)^3\|P_j^1(t)\|_{L^2(\theta_2)}^2 dt\\
&\quad\le \frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \|\partial_z u\|_{L^\infty}^2\langle t\rangle^{\frac{3+\delta}{2}}\|v\|_{X_{\tau,\kappa-1/2}}^2 dt\\
&\quad\lesssim \lambda^{-1/2}\sqrt{\epsilon}\int_0^T\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|u\|_{X_{\tau,\kappa+5/2}}^2 dt. \qquad (6.14)
\end{aligned}$$

Estimate of term P_j^2

For term P_j^2, using the *a priori* estimates in (3.6) to obtain that

$$
\begin{aligned}
|P_j^2| &\lesssim \lambda^{1/4}\delta^{-1}\sqrt{\epsilon}\langle t\rangle^{-\frac{4-5\delta}{4}} \\
&\quad\times \sum_{k=1}^{[(j+1)/2]} (k+1)^{-\kappa}(j-k+1)^{-3}|M_k r^k \partial_r^k u| \left|\int_z^\infty (r\partial_r\mathcal{A})_{j-1-k}d\bar{z}\right| \\
&\quad+ \lambda^{1/4}\sqrt{\epsilon}\delta^{-1}\langle t\rangle^{-\frac{4-5\delta}{4}} \sum_{k=[(j+1)/2]+1}^{j-1} (j-k+1)^{-\kappa}(k+1)^{-2}|u_k| \\
&\quad\times \left|\int_z^\infty M_{j-1-k} r^{j-1-k}\partial_r^{j-1-k}(r\partial_r\mathcal{A})d\bar{z}\right|.
\end{aligned}
$$

Similar as product estimates in (3.24) and using the *a priori* estimates (3.5) in Lemma 3.1, we have

$$
\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}} \sum_{j=0}^\infty (j+1)^3 \|P_j^2(t)\|_{L^2(\theta_2)}^2 dt \\
&\quad\lesssim \lambda^{-1/2}\delta^{-2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{3\delta}(\|u\|_{X_{\tau,3,1/2}}^2 \|r\partial_r\mathcal{A}\|_{X_{\tau,\kappa-3/2,1/2}}^2 \\
&\qquad+ \|u\|_{X_{\tau,\kappa-1/2,1/2}}^2 \|r\partial_r\mathcal{A}\|_{X_{\tau,3,1/2}}^2)dt \\
&\quad\lesssim \lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)(\|\mathcal{A}\|_{X_{\tau,\kappa+1/2}}^2 + \|u\|_{X_{\tau,\kappa+5/2}}^2)dt. \qquad (6.15)
\end{aligned}
$$

Estimate of term P_j^3

This is direct. Using *a priori* estimates (3.6) in Lemma 3.1, we can obtain that

$$
\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int^T \langle t\rangle^{\frac{3+\delta}{2}} \sum_{j\in\mathbb{N}} (j+1)^3 \|P_j^3(t)\|_{L^2(\theta_2)}^2 dt \\
&\quad\lesssim \frac{1}{\lambda\epsilon^{3/2}}\int^T \langle t\rangle^{\frac{7+5\delta}{2}} \|u\partial_z u\|_{L^\infty}^2 \left\|\int_z^\infty \mathcal{A}d\bar{z}\right\|_{X_{\tau,\kappa-1/2}}^2 dt \\
&\quad\lesssim \lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{A}\|_{X_{\tau,\kappa}}^2 dt. \qquad (6.16)
\end{aligned}
$$

Estimate of term P_j^4

For term P_j^4, using the *a priori* estimates in (3.6) to obtain that

$$|P_j^4| \lesssim \lambda^{1/4}\delta^{-1}\sqrt{\epsilon}\langle t\rangle^{-\frac{4-5\delta}{4}} \sum_{k=1}^{[(j+1)/2]} (k+1)^{-\kappa}(j-k+1)^{-3}|M_k r^k \partial_r^k v||\mathcal{A}_{j-1-k}|$$

$$+ \lambda^{1/4}\delta^{-1}\sqrt{\epsilon}\langle t\rangle^{-\frac{4-5\delta}{4}} \sum_{k=[(j+1)/2]+1}^{j-1} (j-k+1)^{-\kappa}(k+1)^{-2}|v_k|$$

$$\times |M_{j-1-k} r^{j-1-k}\partial_r^{j-1-k}\mathcal{A}|.$$

Similar as product estimates in (3.24), using incompressibility and the *a priori* estimates in (3.5), we have

$$\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}} \sum_{j=0}^{\infty}(j+1)^3\|P_j^4(t)\|^2_{L^2(\theta_2)}dt$$

$$\lesssim \lambda^{-1/2}\delta^{-2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{3\delta}(\|\partial_z v\|^2_{X_{\tau,3,1/2}}\|\mathcal{A}\|^2_{X_{\tau,\kappa-3/2,1/2}}$$

$$+ \|\partial_z v\|^2_{X_{\tau,\kappa-1/2,1/2}}\|\mathcal{A}\|^2_{X_{\tau,3,1/2}})dt$$

$$\lesssim \lambda^{-1/2}\delta^{-2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{3\delta}(\|u\|^2_{X_{\tau,5,1/2}}\|\mathcal{A}\|^2_{X_{\tau,\kappa+1/2}} + \|u\|^2_{X_{\tau,\kappa+5/2}}\|\mathcal{A}\|^2_{X_{\tau,3,1/2}})dt$$

$$\lesssim \lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)(\|\mathcal{A}\|^2_{X_{\tau,\kappa+1/2}} + \|u\|^2_{X_{\tau,\kappa+5/2}})dt. \tag{6.17}$$

Estimate of term P_j^5 and P_j^6

This is direct. Using *a priori* estimates in (3.5), we can obtain that

$$\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}} \sum_{j\in\mathbb{N}}(j+1)^3\|P_j^5(t)\|^2_{L^2(\theta_2)}dt$$

$$\leq \frac{1}{\lambda\epsilon^{3/2}}\int_0^T \|\partial_z^2 u\|^2_{L^\infty}\langle t\rangle^{\frac{7+5\delta}{2}}\|\mathcal{A}\|^2_{X_{\tau,\kappa-1/2}}dt$$

$$\lesssim \lambda^{-1/2}\delta^{-3}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{A}\|^2_{X_{\tau,\kappa+1/2}}dt \tag{6.18}$$

and

$$\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{3+\delta}{2}} \sum_{j\in\mathbb{N}}(j+1)^3\|P_j^6(t)\|_{L^2(\theta_2)}^2 dt \\
&\quad\le \frac{1}{\lambda\epsilon^{3/2}}\int_0^T \|\partial_z u\|_{L^\infty}^2 \langle t\rangle^{\frac{3+5\delta}{2}} \left\|\int_z^\infty \mathcal{A} d\bar{z}\right\|_{X_{\tau,\kappa-1/2}}^2 dt \\
&\quad\lesssim \lambda^{-1/2}\delta^{-2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{A}\|_{X_{\tau,\kappa+1/2}}^2 dt. \qquad (6.19)
\end{aligned}$$

Combining estimates in (6.9), (6.10) and (6.12)–(6.19), we can achieve (6.8) in Lemma 6.2.

Proof of Proposition 3.3. From Lemmas 6.1 and 6.2, we have achieved that

$$\begin{aligned}
&\langle t\rangle^{\frac{1-\delta}{2}}\|\varphi(t)\|_{X_{\tau,\kappa+2}}^2 + \delta\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z\varphi(t)\|_{X_{\tau,\kappa+2}}^2 dt \\
&\quad + \lambda\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\varphi(t)\|_{X_{\tau,\kappa+5/2}}^2 dt \\
&\lesssim \|u(0)\|_{X_{\tau_0,\kappa+2}}^2 + \lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{-\frac{1-\delta}{2}}(\|u(t)\|_{X_{\tau,\kappa+5/2}}^2 + \|\mathcal{A}(t)\|_{X_{\tau,\kappa+1/2}}^2) dt \\
&\quad + \lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}(\|\partial_z u(t)\|_{X_{\tau,\kappa+2}}^2 + \|\partial_z\mathcal{A}(t)\|_{X_{\tau,\kappa}}^2) dt. \qquad (6.20)
\end{aligned}$$

Besides, from the definition of φ_j in (6.1) and using (3.5) in Lemma 3.5, we see that, for $\tilde{\kappa} > 0$,

$$\begin{aligned}
\|u(t)\|_{X_{\tau,\tilde{\kappa}+2}}^2 &\lesssim \|\varphi(t)\|_{X_{\tau,\tilde{\kappa}+2}}^2 + \langle t\rangle^{3+2\delta}\epsilon^{-1}\|\partial_z u\|_{L^\infty}^2\|\mathcal{A}(t)\|_{X_{\tau,\tilde{\kappa}}}^2 \\
&\lesssim \|\varphi(t)\|_{X_{\tau,\tilde{\kappa}+2}}^2 + \lambda^{1/2}\epsilon\|\mathcal{A}(t)\|_{X_{\tau,\tilde{\kappa}}}^2. \qquad (6.21)
\end{aligned}$$

Similarly, we can obtain that

$$\|\partial_z u(t)\|_{X_{\tau,\tilde{\kappa}+2}}^2 \lesssim \|\partial_z\varphi(t)\|_{X_{\tau,\tilde{\kappa}+2}}^2 + \lambda^{1/2}\epsilon\|\partial_z\mathcal{A}(t)\|_{X_{\tau,\tilde{\kappa}}}^2. \qquad (6.22)$$

Inserting (6.21) and (6.22) into (6.20) and by letting ϵ is sufficiently small, we can achieve (2.7) in Proposition 3.3. □

7. Estimate for the Linearly Good Unknown g

In this section, we focus on the Gevrey-2 estimates of the linearly good unknowns g and its z-derivatives up to the third-order. It will induce faster decay rate for low order Gevrey-2 energy of the unknowns u as displayed in (3.5).

Below we set

$$\kappa_0 = 11, \quad \kappa_1 = 9, \quad \kappa_2 = 7, \quad \text{and} \quad \kappa_3 = 5.$$

7.1. *Estimates of g*

Lemma 7.1. *Under the assumption of* (3.4), *for sufficiently small* ϵ, *there exits a constant* C *such that for any* $t \in (0, T]$, *we have the following estimate*:

$$\begin{aligned}
&\langle t\rangle^{\frac{5-\delta}{2}}\|g(t)\|^2_{X_{\tau,\kappa_0}} + \delta\int_0^T \langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g(t)\|^2_{X_{\tau,\kappa_0}}dt\\
&\quad + \lambda\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|g(t)\|^2_{X_{\tau,\kappa_0+1/2}}dt\\
&\quad \le C\|g(0)\|^2_{X_{\tau_0,\kappa_0}} + \lambda^{-1/2}\epsilon^{1/2}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\|\partial_z u(t)\|^2_{X_{\tau,\kappa+2}}dt\\
&\quad + C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T (\langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|g(t)\|^2_{X_{\tau,\kappa_0+1/2}} + \langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g(t)\|^2_{X_{\tau,\kappa_0}})dt.
\end{aligned} \tag{7.1}$$

Proof. By a direct computation from the equation of u in (2.2), the equation of g satisfies

$$\begin{cases}
[\partial_t + \dfrac{1}{\langle t\rangle} - \partial_z^2]g\\
\qquad = -(ru\partial_r + v\partial_z)g - u^2 - \dfrac{v}{2\langle t\rangle}\partial_z\left(z\displaystyle\int_z^\infty u d\bar{z}\right)\\
\qquad\quad + \dfrac{z}{\langle t\rangle}u\displaystyle\int_z^\infty u d\bar{z} - \dfrac{z}{2\langle t\rangle}\displaystyle\int_z^\infty u^2 d\bar{z} + \dfrac{z}{\langle t\rangle}\displaystyle\int_z^\infty \partial_z uv d\bar{z},\\
\qquad := \displaystyle\sum_{i=1}^7 R^i,\\
g|_{z=0} = 0, \quad \displaystyle\lim_{z\to+\infty} g = 0, \quad g|_{t=0} = g_0.
\end{cases} \tag{7.2}$$

Here, we remark that under the compatibility condition (2.6),

$$R^i|_{z=0} = 0, \quad \partial_z^2 R^i|_{z=0} = 0 \quad \text{for } i = 1, 2, \ldots, 7, \quad \partial_z^2 g|_{z=0} = \partial_z^4 g|_{z=0} = 0.$$

As before, define

$$M_{j,\kappa_0} := \frac{\tau(t)^{j+1}(j+1)^{\kappa_0}}{(j!)^2} \quad \text{and} \quad g_{j,\kappa_0} := M_{j,\kappa_0}[r]^j\partial_r^j g.$$

Applying $M_{j,\kappa_0}[r]^j\partial_r^j$ to (7.2) to deduce that g_{j,κ_0} satisfies

$$\left[\partial_t + \lambda\sqrt{\epsilon}\eta(j+1) - \partial_z^2 + \frac{1}{\langle t\rangle}\right]g_{j,\kappa_0} = \sum_{i=1}^7 R^i_{j,\kappa_0}$$

with the initial and boundary condition satisfying

$$g_{j,\kappa_0}|_{z=0} = 0, \quad \lim_{z\to+\infty} g_{j,\kappa_0} = 0, \quad g_{j,\kappa_0}|_{t=0} = g_{0,j,\kappa_0}.$$

Performing spacial energy estimates similar as that in (4.9) and using Cauchy inequality, we can have

$$\begin{aligned}&\langle t\rangle^{\frac{5-\delta}{2}}\|g_{j,\kappa_0}(t)\|^2_{L^2(\theta_2)}+\delta\int_0^T\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g_{j,\kappa_0}(t)\|^2_{L^2(\theta_2)}dt\\&\quad+(j+1)\lambda\sqrt{\epsilon}\int_0^T\eta(t)\langle t\rangle^{\frac{5-\delta}{2}}\|g_{j,\kappa_0}(t)\|^2_{L^2(\theta_2)}dt\\&\le\|g_{j,\kappa_0}(0)\|^2_{L^2(\theta_2)}+\frac{C}{\lambda\sqrt{\epsilon}}\int_0^T\langle t\rangle^{\frac{7+\delta}{2}}(j+1)^{-1}\left\|\sum_{i=1}^7 R^i_{j,\kappa_0}\right\|^2dt.\end{aligned}\tag{7.3}$$

By summing the above inequality (7.3) over $j\in\mathbb{N}$, we can achieve that

$$\begin{aligned}&\langle t\rangle^{\frac{5-\delta}{2}}\|g(t)\|^2_{X_{\tau,\kappa_0}}+\delta\int_0^T\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g(t)\|^2_{X_{\tau,\kappa_0}}dt\\&\quad+\lambda\sqrt{\epsilon}\int_0^T\eta(t)\langle t\rangle^{\frac{5-\delta}{2}}\|g(t)\|^2_{X_{\tau,\kappa_0+1/2}}dt\\&\lesssim\|g(0)\|^2_{X_{\tau,\kappa_0}}+\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T\langle t\rangle^{\frac{7+\delta}{2}}\sum_{i=1}^7\|R^i\|^2_{X_{\tau,\kappa_0-1/2}}dt.\end{aligned}\tag{7.4}$$

□

Estimates of R^1_j, R^2_j, R^4_j and R^7_j

In these terms, there are r derivatives on g or u (hiding in the incompressibility for v). We view them as derivative-loss terms for the linear part of the equation of g. First, we give the following lemma and corollary concerning on handing of these term.

Lemma 7.2. *For some smooth function $f(r,z)$ decaying sufficiently fast at r infinity and satisfying $r^3\partial_r^3 f\in L^2(\omega(z))$ and $f\in L^2(\omega(z))$, then we have*

$$\|r\partial_r f\|_{L^2(\omega(z))}\le C\|r^3\partial_r^3 f\|^{1/3}_{L^2(\omega(z))}\|f\|^{2/3}_{L^2(\omega(z))}+\|f\|_{L^2(\omega(z))}.\tag{7.5}$$

We leave the proof of this lemma to Appendix A.

Corollary 7.1. *For any κ, $\nu\in\mathbb{R}$, we have*

$$\|r\partial_r f\|^2_{X_{\tau,\kappa,\nu}}\lesssim\|f\|^2_{X_{\tau,\kappa,\nu}}+\|f\|^{2/3}_{X_{\tau,\kappa+6,\nu}}\|f\|^{4/3}_{X_{\tau,\kappa,\nu}}.\tag{7.6}$$

Proof. First, by direct computation, we have

$$[r]^j\partial_r^j(r\partial_r f)=r\partial_r([r]^j\partial_r^j f)+r^{j-1}\partial_r^j f.$$

Then direct weighted L^2 estimates indicate that

$$\|[r]^j\partial_r^j(r\partial_r f)\|_{L^2(\theta_{2\nu})}\le\|r\partial_r([r]^j\partial_r^j f)\|_{L^2(\theta_{2\nu})}+\|r^{j-1}\partial_r^j f\|_{L^2(\theta_{2\nu})}.\tag{7.7}$$

By using (7.5) in Lemma 7.2, we can have

$$\begin{aligned}
&\|r\partial_r([r]^j\partial_r^j f)\|_{L^2(\theta_{2\nu})}\\
&\quad\lesssim \|[r]^j\partial_r^j f\|_{L^2(\theta_{2\nu})} + \|r^3\partial_r^3([r]^j\partial_r^j f)\|_{L^2(\theta_{2\nu})}^{1/3}\|[r]^j\partial_r^j f\|_{L^2(\theta_{2\nu})}^{2/3}\\
&\quad\lesssim \|[r]^j\partial_r^j f\|_{L^2(\theta_{2\nu})} + \left(\sum_{k=0}^{3}(j+1)^{3-k}\|([r]^{j+k}\partial_r^{j+k} f)\|_{L^2(\theta_{2\nu})}\right)^{1/3}\\
&\qquad\times\|[r]^j\partial_r^j f\|_{L^2(\theta_{2\nu})}^{2/3},
\end{aligned}\tag{7.8}$$

where at the last line of the above inequality, we have used Leibniz formula.

Inserting (7.8) into (7.7) and multiplying the resulted equation by $M_{j,\kappa}$, we can achieve that

$$\|(r\partial_r f)_{j,\kappa}\|_{L^2(\theta_{2\nu})} \lesssim \|f_{j,\kappa}\|_{L^2(\theta_{2\nu})} + \left(\sum_{k=0}^{3}\|f_{j+k,\kappa+6}\|_{L^2(\theta_{2\nu})}\right)^{1/3}\|f_{j,\kappa}\|_{L^2(\theta_{2\nu})}^{2/3}.$$

Squaring the above equation, summing the resulted equation over $j\in\mathbb{N}$, and using the discrete Hölder inequality, we can achieve that

$$\|r\partial_r f\|_{X_{\tau,\kappa,\nu}}^2 \lesssim \|f\|_{X_{\tau,\kappa,\nu}}^2 + \|f\|_{X_{\tau,\kappa+6,\nu}}^{2/3}\|f\|_{X_{\tau,\kappa,\nu}}^{4/3},$$

which is (7.6). □

Estimate of term R_j^1

By using (3.25) in Lemma 3.3, we can obtain that

$$\begin{aligned}
&\sum_{j\in\mathbb{N}}(j+1)^{-1}\|R_{j,\kappa_0}^1(t)\|_{L^2(\theta_2)}^2\\
&\quad\lesssim \langle t\rangle^{1/2}(\|\partial_z u\|_{X_{\tau,3,1/2}}^2\|r\partial_r g\|_{X_{\tau,\kappa_0-1/2,1/2}}^2 + \|u\|_{X_{\tau,\kappa_0-1/2,1/2}}^2\|r\partial_r\partial_z g\|_{X_{\tau,3,1/2}}^2)\\
&\quad\lesssim \langle t\rangle^{1/2}(\|\partial_z u\|_{X_{\tau,3,1/2}}^2\|r\partial_r u\|_{X_{\tau,\kappa_0-1/2,1/2}}^2 + \|g\|_{X_{\tau,\kappa_0-1/2}}^2\|\partial_z g\|_{X_{\tau,5}}^2).
\end{aligned}$$

At the last line, by using (3.22) and the definition of g, we have the fact that

$$\|r\partial_r g\|_{X_{\tau,\kappa_0-1/2,1/2}}^2 \lesssim \|r\partial_r u\|_{X_{\tau,\kappa_0-1/2,1/2}}^2.$$

Then by using (7.6), we have

$$\begin{aligned}
&\sum_{j\in\mathbb{N}}(j+1)^{-1}\|R_{j,\kappa_0}^1(t)\|_{L^2(\theta_2)}^2\\
&\quad\lesssim \langle t\rangle^{1/2}(\|\partial_z g\|_{X_{\tau,5}}^2\|g\|_{X_{\tau,\kappa_0-1/2}}^2 + (\|u\|_{X_{\tau,\kappa_0,1/2}}^2\\
&\qquad+\|u\|_{X_{\tau,\kappa_0+11/2,1/2}}^{2/3}\|u\|_{X_{\tau,\kappa_0,1/2}}^{4/3})\|\partial_z g\|_{X_{\tau,3}}^2)\\
&\quad\lesssim \langle t\rangle^{1/2}(\|g\|_{X_{\tau,\kappa_0}}^2\|\partial_z g\|_{X_{\tau,5}}^2 + \|u\|_{X_{\tau,\kappa_0+11/2,1/2}}^{2/3}\|u\|_{X_{\tau,\kappa_0,1/2}}^{4/3}\|\partial_z g\|_{X_{\tau,3}}^2).
\end{aligned}$$

Then using the *a priori* estimates (3.5) in Lemma 3.1, (3.21) in Lemma 3.2 and Young inequality, we can obtain that

$$\begin{aligned}&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{7+\delta}{2}}\sum_{j=0}^{\infty}(j+1)^{-1}\|R_j^1(t)\|^2_{L^2(\theta_2)}dt\\&\quad\lesssim \lambda^{-1/2}\sqrt{\epsilon}\int_0^T(\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|u\|^2_{X_{\tau,\kappa+5/2}}+\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g\|^2_{X_{\tau,\kappa_0}})dt. \qquad (7.9)\end{aligned}$$

Estimate of term R_j^2

By using (3.24) in Lemma 3.3 and the incompressibility, we can obtain that

$$\begin{aligned}&\sum_{j\in\mathbb{N}}(j+1)^{-1}\|R^2_{j,\kappa_0}(t)\|^2_{L^2(\theta_2)}\\&\quad\lesssim\langle t\rangle^{1/2}(\|\partial_z v\|^2_{X_{\tau,3,1/2}}\|\partial_z g\|^2_{X_{\tau,\kappa_0-1/2,1/2}}+\|\partial_z v\|^2_{X_{\tau,\kappa_0-1/2,1/2}}\|\partial_z g\|^2_{X_{\tau,3,1/2}})\\&\quad\lesssim\langle t\rangle^{1/2}(\|u\|^2_{X_{\tau,5,1/2}}\|\partial_z g\|^2_{X_{\tau,\kappa_0}}+\|(r\partial_r u+u)\|^2_{X_{\tau,\kappa_0-1/2,1/2}}\|\partial_z g\|^2_{X_{\tau,3}}).\end{aligned}$$

Then by using (7.6) in Corollary 7.1, we have

$$\begin{aligned}&\sum_{j\in\mathbb{N}}(j+1)^{-1}\|R^2_{j,\kappa_0}(t)\|^2_{L^2(\theta_2)}dt\\&\quad\lesssim\langle t\rangle^{1/2}\|g\|^2_{X_{\tau,5}}\|\partial_z g\|^2_{X_{\tau,\kappa_0}}\\&\qquad+\langle t\rangle^{1/2}(\|u\|^2_{X_{\tau,\kappa_0,1/2}}+\|u\|^{2/3}_{X_{\tau,\kappa_0+11/2,1/2}}\|u\|^{4/3}_{X_{\tau,\kappa_0,1/2}})\|\partial_z g\|^2_{X_{\tau,3}}\\&\quad\lesssim\langle t\rangle^{1/2}\|g\|^2_{X_{\tau,5}}\|\partial_z g\|^2_{X_{\tau,\kappa_0}}.\\&\qquad+\langle t\rangle^{1/2}(\|g\|^2_{X_{\tau,\kappa_0}}\|\partial_z g\|^2_{X_{\tau,3}}+\|u\|^{2/3}_{X_{\tau,\kappa_0+11/2,1/2}}\|u\|^{4/3}_{X_{\tau,\kappa_0,1/2}}\|\partial_z g\|^2_{X_{\tau,3}}).\end{aligned}$$

Then using the *a priori* estimates (3.5) in Lemma 3.1, (3.21) in Lemma 3.2 and Young inequality, we can obtain that

$$\begin{aligned}&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{7+\delta}{2}}\sum_{j=0}^{\infty}(j+1)^{-1}\|R_j^2(t)\|^2_{L^2(\theta_2)}\\&\quad\lesssim \lambda^{-1/2}\sqrt{\epsilon}\int_0^T(\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|u\|^2_{X_{\tau,\kappa+5/2}}+\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g\|^2_{X_{\tau,\kappa_0}})dt. \qquad (7.10)\end{aligned}$$

Estimate of term R_j^4

For term R_j^4, using (3.24) in Lemma 3.3, the incompressibility and (3.21), (3.22) in Lemma 3.2, we have

$$\begin{aligned}&\|R^4(t)\|^2_{X_{\tau,\kappa_0-1/2}}\\&\quad=\left\|v\left(\langle t\rangle^{-1}\int_z^{\infty}ud\bar{z}-\frac{z}{\langle t\rangle}u\right)\right\|^2_{X_{\tau,\kappa_0-1/2}}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \langle t\rangle^{1/2}\|\partial_z v\|^2_{X_{\tau,3,1/2}} \left\|\left(\langle t\rangle^{-1}\int_z^\infty u d\bar{z} - \frac{z}{\langle t\rangle}u\right)\right\|^2_{X_{\tau,\kappa_0-1/2,1/2}} \\
&\quad + \langle t\rangle^{1/2}\|\partial_z v\|^2_{X_{\tau,\kappa_0-1/2,1/2}} \left\|\left(\langle t\rangle^{-1}\int_z^\infty u d\bar{z} - \frac{z}{\langle t\rangle}u\right)\right\|^2_{X_{\tau,3,1/2}} \\
&\lesssim \langle t\rangle^{1/2}(\|u\|^2_{X_{\tau,5,1/2}}\|\partial_z u\|^2_{X_{\tau,\kappa_0-1/2,1/2}} \\
&\quad + \|r\partial_r u + 2u\|^2_{X_{\tau,\kappa_0-1/2,1/2}}\|\partial_z u\|^2_{X_{\tau,3,1/2}}).
\end{aligned}
$$

Here, at the last line, we have used (3.21) twice and (3.22) once to bound the term

$$
\langle t\rangle^{-1}\int_z^\infty u d\bar{z} - \frac{z}{\langle t\rangle}u.
$$

Then by using (7.6), we have

$$
\begin{aligned}
&\|R^4(t)\|^2_{X_{\tau,\kappa_0-1/2}} \\
&\qquad\lesssim \langle t\rangle^{1/2}\|u\|^2_{X_{\tau,5,1/2}}\|\partial_z g\|^2_{X_{\tau,\kappa_0}} \\
&\qquad\quad + \langle t\rangle^{1/2}(\|u\|^2_{X_{\tau,\kappa_0,1/2}} + \|u\|^{2/3}_{X_{\tau,\kappa_0+11/2,1/2}}\|u\|^{4/3}_{X_{\tau,\kappa_0,1/2}})\|\partial_z g\|^2_{X_{\tau,3,1/2}} \\
&\qquad\lesssim \langle t\rangle^{1/2}(\|u\|^2_{X_{\tau,\kappa_0,1/2}}\|\partial_z g\|^2_{X_{\tau,\kappa_0}} + \|u\|^{2/3}_{X_{\tau,\kappa+5/2}}\|g\|^{4/3}_{X_{\tau,\kappa_0}}\|\partial_z g\|^2_{X_{\tau,3,1/2}}).
\end{aligned}
$$

Then using the *a priori* estimates (3.5) in Lemma 3.1, (3.21) in Lemma 3.2 and Young inequality, we can obtain that

$$
\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{7+\delta}{2}}\|R^4(t)\|^2_{X_{\tau,\kappa_0-1/2}}dt \\
&\qquad\lesssim \lambda^{-1/2}\sqrt{\epsilon}\int_0^T (\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g\|^2_{X_{\tau,\kappa_0}} + \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|u\|^2_{X_{\tau,\kappa+5/2}})dt. \quad (7.11)
\end{aligned}
$$

Estimate of term R_j^7

For term R_j^7, first using (3.22) in Lemma 3.2, then (3.24) in Lemma 3.3, and then incompressibility, we have

$$
\begin{aligned}
\|R_j^7\|^2_{X_{\tau,\kappa_0-1/2}} &= \left\|\frac{z}{\langle t\rangle}\int_z^\infty v\partial_z u d\bar{z}\right\|^2_{X_{\tau,\kappa_0-1/2}} \\
&\lesssim \|v\partial_z u\|^2_{X_{\tau,\kappa_0-1/2}} \\
&\lesssim \langle t\rangle^{1/2}(\|\partial_z v\|^2_{X_{\tau,3,1/2}}\|\partial_z u\|^2_{X_{\tau,\kappa_0-1/2,1/2}} \\
&\quad + \|\partial_z v\|^2_{X_{\tau,\kappa_0-1/2,1/2}}\|\partial_z u\|^2_{X_{\tau,3,1/2}})
\end{aligned}
$$

$$\lesssim \langle t\rangle^{1/2}(\|u\|^2_{X_{\tau,5,1/2}}\|\partial_z u\|^2_{X_{\tau,\kappa_0-1/2,1/2}} + \|r\partial_r u+2u\|^2_{X_{\tau,\kappa_0-1/2,1/2}}\|\partial_z u\|^2_{X_{\tau,3,1/2}}).$$

The rest is the same as R^4_j. Then we can obtain that

$$\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{7+\delta}{2}}\|R^7_j(t)\|^2_{X_{\tau,\kappa_0-1/2}}dt \lesssim \lambda^{-1/2}\int_0^T (\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g\|^2_{X_{\tau,\kappa_0}} + \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|u\|^2_{X_{\tau,\kappa+5/2}})dt. \quad (7.12)$$

Estimate of term R^3_j, R^5_j and R^6_j

For term R^3_j, using (3.24) in Lemma 3.3, and the incompressibility, we have

$$\begin{aligned}\sum_{j\in\mathbb{N}}(j+1)^{-1}\|R^3_j(t)\|^2_{L^2(\theta_2)} &= \|u^2\|^2_{X_{\tau,\kappa_0-1/2}}\\ &\lesssim \langle t\rangle^{1/2}(\|\partial_z u\|^2_{X_{\tau,3,1/2}}\|u\|^2_{X_{\tau,\kappa_0-1/2,1/2}} + \|\partial_z u\|^2_{X_{\tau,\kappa_0-1/2,1/2}}\|u\|^2_{X_{\tau,3,1/2}})\\ &\lesssim \langle t\rangle^{1/2}(\|\partial_z u\|^2_{X_{\tau,3,1/2}}\|g\|^2_{X_{\tau,\kappa_0}} + \|\partial_z g\|^2_{X_{\tau,\kappa_0}}\|u\|^2_{X_{\tau,3,1/2}}). \quad (7.13)\end{aligned}$$

For term R^5_j, using (3.24) in Lemma 3.3 and (3.21) and (3.22) in Lemma 3.2, we have

$$\begin{aligned}\sum_{j\in\mathbb{N}}(j+1)^{-1}\|R^5_j(t)\|^2_{L^2(\theta_2)} &= \left\|u\frac{z}{\langle t\rangle}\int_z^\infty u d\bar z\right\|^2_{X_{\tau,\kappa_0-1/2}}\\ &\lesssim \langle t\rangle^{1/2}\|\partial_z u\|^2_{X_{\tau,3,1/2}}\left\|\frac{z}{\langle t\rangle}\int_z^\infty u d\bar z\right\|^2_{X_{\tau,\kappa_0-1/2,1/2}}\\ &\quad\times\langle t\rangle^{1/2}\|\partial_z u\|^2_{X_{\tau,\kappa_0-1/2,1/2}}\left\|\frac{z}{\langle t\rangle}\int_z^\infty u d\bar z\right\|^2_{X_{\tau,3,1/2}}\\ &\lesssim \langle t\rangle^{1/2}(\|\partial_z u\|^2_{X_{\tau,3,1/2}}\|u\|^2_{X_{\tau,\kappa_0-1/2,1/2}} + \|\partial_z u\|^2_{X_{\tau,\kappa_0-1/2,1/2}}\|u\|^2_{X_{\tau,3,1/2}})\\ &\lesssim \langle t\rangle^{1/2}(\|\partial_z u\|^2_{X_{\tau,3,1/2}}\|g\|^2_{X_{\tau,\kappa_0}} + \|\partial_z g\|^2_{X_{\tau,\kappa_0}}\|u\|^2_{X_{\tau,3,1/2}}). \quad (7.14)\end{aligned}$$

For term R^6_j, using (3.22) in Lemma 3.2, we have

$$\|R^6(t)\|^2_{X_{\tau,\kappa_0-1/2}} = 2\left\|\frac{z}{\langle t\rangle}\int_z^\infty u^2 d\bar z\right\|^2_{X_{\tau,\kappa_0-1/2}} \lesssim \|u^2\|^2_{X_{\tau,\kappa_0-1/2}}.$$

Then the rest is the same as estimates for R^3_j.

Then combining estimates in (7.13), (7.14), using the *a priori* estimates (3.5) in Lemma 3.1 and (3.21) in Lemma 3.2, we can obtain that

$$\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{7+\delta}{2}}\|(R^3,R^5,R^6)(t)\|^2_{X_{\tau,\kappa_0-1/2}}dt \lesssim \lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g\|^2_{X_{\tau,\kappa_0}}dt. \tag{7.15}$$

Now, inserting estimates in (7.9)–(7.12) and (7.15) into (7.4), we can obtain (7.1) in Lemma 7.1.

7.2. *Estimates of the first-order z-derivative*

Lemma 7.3. *Under the assumption of* (3.4), *for sufficiently small* ϵ, *there exits a constant* C *such that for any* $t\in(0,T]$, *we have the following estimate*:

$$\begin{aligned}&\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z g(t)\|^2_{X_{\tau,\kappa_1}}+\delta\int_0^T\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z^2 g(t)\|^2_{X_{\tau,\kappa_1}}dt\\&\quad+\lambda\sqrt{\epsilon}\int_0^T\langle t\rangle^{\frac{7-\delta}{2}}\eta(t)\|\partial_z g(t)\|^2_{X_{\tau,\kappa_1+1/2}}dt\\&\le C\|\partial_z g(0)\|^2_{X_{\tau,\kappa_1}}+\int_0^T\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g(t)\|^2_{X_{\tau,\kappa_1}}dt\\&\quad+C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g(t)\|^2_{X_{\tau,\kappa_0}}dt.\end{aligned}\tag{7.16}$$

Proof. Now, applying $M_{j,\kappa_1}\partial_z[r]^j\partial_r^j$ to the first equation of (7.2), we can obtain that

$$\left[\partial_t+\lambda\sqrt{\epsilon}(j+1)-\partial_z^2+\frac{1}{\langle t\rangle}\right]\partial_z g_{j,\kappa_1}=\sum_{i=1}^{7}\partial_z R^i_{j,\kappa_1}.$$

Performing space variable energy estimates, we can have

$$\begin{aligned}&\frac{d}{dt}\|\partial_z g_{j,\kappa_1}(t)\|^2_{L^2(\theta_2)}+\delta\|\partial_z^2 g_{j,\kappa_1}(t)\|^2_{L^2(\theta_2)}+\frac{5-\delta}{2\langle t\rangle}\|\partial_z g_{j,\kappa_1}(t)\|^2_{L^2(\theta_2)}\\&\quad+2(j+1)\lambda\sqrt{\epsilon}\eta(t)\|\partial_z g_{j,\kappa_1}(t)\|^2_{L^2(\theta_2)}\\&\le 2\left|\left\langle\sum_{i=1}^{7}\partial_z R^i_{j,\kappa_1},\partial_z g_{j,\kappa_1}\right\rangle_{\theta_2}\right|.\end{aligned}$$

Multiplying the above equality by $\langle t\rangle^{\frac{7-\delta}{2}}$ and using integration by parts for the right hand of the above inequality, and then integrating the resulted equation from

0 to t for any $t \in (0, T]$, we can achieve that

$$\begin{aligned}
&\langle t\rangle^{\frac{7-\delta}{2}} \|\partial_z g_{j,\kappa_1}(t)\|^2_{L^2(\theta_2)} + \delta \int_0^T \langle t\rangle^{\frac{7-\delta}{2}} \|\partial_z^2 g_{j,\kappa_1}(t)\|^2_{L^2(\theta_2)} dt \\
&\quad + 2\lambda\sqrt{\epsilon}(j+1) \int_0^T \langle t\rangle^{\frac{7-\delta}{2}} \eta(t) \|\partial_z g_{j,\kappa_1}(t)\|^2_{L^2(\theta_2)} dt \\
&\leq \|\partial_z g_{j,\kappa_1}(0)\|^2_{L^2(\theta_2)} + \int_0^T \langle t\rangle^{\frac{5-\delta}{2}} \|\partial_z g_{j,\kappa_1}(t)\|^2_{L^2(\theta_2)} dt \\
&\quad + \int_0^T \langle t\rangle^{\frac{7-\delta}{2}} \left| \left\langle \sum_{i=1}^7 R^i_{j,\kappa_1}, \partial_z^2 g_{j,\kappa_1} + \frac{z}{2\langle t\rangle} \partial_z g_{j,\kappa_1} \right\rangle_{\theta_2} \right| dt.
\end{aligned}$$

By using Cauchy inequality and (3.22) to the right hand of the above inequality, and then summing the resulted equations over $j \in \mathbb{N}$, we can obtain that

$$\begin{aligned}
&\langle t\rangle^{\frac{7-\delta}{2}} \|\partial_z g(t)\|^2_{X_{\tau,\kappa_1}} + \delta \int_0^T \langle t\rangle^{\frac{7-\delta}{2}} \|\partial_z^2 g(t)\|^2_{X_{\tau,\kappa_1}} dt \\
&\quad + \lambda\sqrt{\epsilon}(j+1) \int_0^T \langle t\rangle^{\frac{7-\delta}{2}} \eta(t) \|\partial_z g(t)\|^2_{X_{\tau,\kappa_1}} dt \\
&\leq \|\partial_z g(0)\|^2_{X_{\tau,\kappa_1}} + \int_0^T \langle t\rangle^{\frac{5-\delta}{2}} \|\partial_z g(t)\|^2_{X_{\tau,\kappa_1}} dt \\
&\quad + C \int_0^T \langle t\rangle^{\frac{7-\delta}{2}} \sum_{i=1}^7 \|R^i\|^2_{X_{\tau,\kappa_1}} dt.
\end{aligned} \tag{7.17}$$

□

Lemma 7.4. *We have the following estimates*:

$$\sum_{j=1}^\infty \sum_{i=1}^6 \|R^i\|^2_{X_{\tau,\kappa_1}} \lesssim \langle t\rangle^{1/2} (\|g\|^2_{X_{\tau,5}} \|\partial_z g\|^2_{X_{\tau,\kappa_1+2}} + \|\partial_z g\|^2_{X_{\tau,5}} \|g\|^2_{X_{\tau,\kappa_1+2}}). \tag{7.18}$$

Proof. Proof of this lemma is repeatedly use of (3.21) and (3.22) in Lemma 3.2, product estimates in (3.23) to (3.25) in Lemma 3.3 and the relation between u and g. Since it is a routing estimate, we omit the details.

Then by using the *a priori* estimates in (3.5) in Lemma 3.1, (7.18) and (3.21) in Lemma 3.2, we see that

$$\int_0^T \langle t\rangle^{\frac{7-\delta}{2}} \sum_{i=1}^7 \|R^i\|^2_{X_{\tau,\kappa_1}} dt \lesssim \lambda^{-1/2}\sqrt{\epsilon} \int_0^T \langle t\rangle^{\frac{5-\delta}{2}} \|\partial_z g\|^2_{X_{\tau,\kappa_0}} dt.$$

Inserting the above inequality into (7.17), we can obtain (7.16) in Lemma 7.3.

□

7.3. *Estimates of the second-order z-derivative*

Lemma 7.5. *Under the assumption of* (3.4), *for sufficiently small* ϵ, *there exits a constant* C *such that for any* $t \in (0, T]$, *we have the following estimate*:

$$\begin{aligned}
&\langle t\rangle^{\frac{9-\delta}{2}}\|\partial_z^2 g(t)\|^2_{X_{\tau,\kappa_2}} + \delta\int_0^T \langle t\rangle^{\frac{9-\delta}{2}}\|\partial_z^3 g(t)\|^2_{X_{\tau,\kappa_2}}dt\\
&\quad + \lambda\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{9-\delta}{2}}\eta(t)\|\partial_z^2 g(t)\|^2_{X_{\tau,\kappa_2+1/2}}dt\\
&\leq C\|\partial_z^2 g(0)\|^2_{X_{\tau_0,\kappa_2}} + \int_0^T \langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z^2 g(t)\|^2_{X_{\tau,\kappa_2}}dt\\
&\quad + C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z^2 g(t)\|^2_{X_{\tau,\kappa_1}}dt.
\end{aligned} \tag{7.19}$$

Proof. Now, applying $M_{j,\kappa_2}\partial_z^2[r]^j\partial_r^j$ to the first equation of (7.4), we can obtain that

$$\left[\partial_t + \lambda\sqrt{\epsilon}(j+1) - \partial_z^2 + \frac{1}{\langle t\rangle}\right]\partial_z^2 g_{j,\kappa_2} = \sum_{i=1}^{7}\partial_z^2 R^i_{j,\kappa_2}.$$

Performing spacial energy estimates, we can have

$$\begin{aligned}
&\frac{d}{dt}\|\partial_z^2 g_{j,\kappa_2}(t)\|^2_{L^2(\theta_2)} + \delta\|\partial_z^3 g_{j,\kappa_2}(t)\|^2_{L^2(\theta_2)} + \frac{5-\delta}{2\langle t\rangle}\|\partial_z^2 g_{j,\kappa_2}(t)\|^2_{L^2(\theta_2)}\\
&\quad + 2(j+1)\lambda\sqrt{\epsilon}\eta(t)\|\partial_z^2 g_{j,\kappa_2}(t)\|^2_{L^2(\theta_2)}\\
&\leq 2\left|\left\langle \sum_{i=1}^{6}\partial_z^2 R^i_{j,\kappa_2}, \partial_z^2 g_{j,\kappa_2}\right\rangle_{\theta_2}\right|.
\end{aligned}$$

Multiplying the above equality by $\langle t\rangle^{\frac{9-\delta}{2}}$ and using integration by parts for the right hand of the above inequality, and then integrating the resulted equation from 0 to t for any $t \in (0, T]$, we can achieve that

$$\begin{aligned}
&\langle t\rangle^{\frac{9-\delta}{2}}\|\partial_z^2 g_{j,\kappa_2}(t)\|^2_{L^2(\theta_2)} + \delta\int_0^T \langle t\rangle^{\frac{9-\delta}{2}}\|\partial_z^3 g_{j,\kappa_2}(t)\|^2_{L^2(\theta_2)}dt\\
&\quad + 2\lambda\sqrt{\epsilon}(j+1)\int_0^T \langle t\rangle^{\frac{9-\delta}{2}}\eta(t)\|\partial_z^2 g_{j,\kappa_2}(t)\|^2_{L^2(\theta_2)}dt\\
&\leq \|\partial_z^2 g_{j,\kappa_2}(0)\|^2_{L^2(\theta_2)} + \int_0^T \langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z g_{j,\kappa_2}(t)\|^2_{L^2(\theta_2)}dt\\
&\quad + \int_0^T \langle t\rangle^{\frac{9-\delta}{2}}\left|\left\langle \sum_{i=1}^{7}\partial_z R^i_{j,\kappa_2}, \partial_z^3 g_{j,\kappa_2} + \frac{z}{2\langle t\rangle}\partial_z^2 g_{j,\kappa_2}\right\rangle_{\theta_2}\right|dt.
\end{aligned}$$

By using Cauchy inequality and (3.22) to the right hand of above inequality, and then summing the resulted equations over $j \in \mathbb{N}$, we can obtain that

$$\begin{aligned} &\langle t\rangle^{\frac{9-\delta}{2}} \|\partial_z^2 g(t)\|^2_{X_{\tau,\kappa_2}} + \delta \int_0^T \langle t\rangle^{\frac{9-\delta}{2}} \|\partial_z^3 g(t)\|^2_{X_{\tau,\kappa_2}} dt \\ &\quad + \lambda\sqrt{\epsilon}(j+1) \int_0^T \langle t\rangle^{\frac{9-\delta}{2}} \eta(t) \|\partial_z^2 g(t)\|^2_{X_{\tau,\kappa_2}} dt \\ &\le \|\partial_z^2 g(0)\|^2_{X_{\tau,\kappa_2}} + \int_0^T \langle t\rangle^{\frac{7-\delta}{2}} \|\partial_z^2 g(t)\|^2_{X_{\tau,\kappa_2}} dt \\ &\quad + C_\delta \int_0^T \langle t\rangle^{\frac{9-\delta}{2}} \sum_{i=1}^7 \|\partial_z R^i\|^2_{X_{\tau,\kappa_2}} dt. \end{aligned} \tag{7.20}$$

Similar as Lemma 7.4, we have the following lemma. □

Lemma 7.6. *We have the following estimates*:

$$\begin{aligned} \sum_{i=1}^7 \|\partial_z R^i\|^2_{X_{\tau,\kappa_2}} &\lesssim \langle t\rangle^{1/2} (\|g\|^2_{X_{\tau,5}} \|\partial_z^2 g\|^2_{X_{\tau,\kappa_2+2}} + \|\partial_z g\|^2_{X_{\tau,5}} \|\partial_z g\|^2_{X_{\tau,\kappa_2+2}} \\ &\quad + \|\partial_z^2 g\|^2_{X_{\tau,5}} \|g\|^2_{X_{\tau,\kappa_2+2}}). \end{aligned} \tag{7.21}$$

Proof. Proof of this lemma is also repeatedly use of (3.21) and (3.22) in Lemma 3.2, product estimates in (3.23) to (3.25) in Lemma 3.3 and the relation between u and g. Since it is a routing estimate, we omit the details.

Then by using (7.21), the *a priori* estimates (3.5) in Lemma 3.1 and (3.22) in Lemma 3.2, we see that

$$\int_0^T \langle t\rangle^{\frac{9-\delta}{2}} \sum_{i=1}^7 \|\partial_z R^i\|^2_{X_{\tau,\kappa_2}} dt \lesssim \lambda^{-1/2}\sqrt{\epsilon} \int_0^T \langle t\rangle^{\frac{7-\delta}{2}} \|\partial_z^2 g\|^2_{X_{\tau,\kappa_1}} dt.$$

Inserting the above inequality into (7.23), we can obtain (7.19) in Lemma 7.5. □

7.4. *Estimates of the third-order z-derivative*

Lemma 7.7. *Under the assumption of* (3.4), *for sufficiently small* ϵ, *there exits a constant* C *such that for any* $t \in (0, T]$, *we have the following estimate*:

$$\begin{aligned} &\langle t\rangle^{\frac{11-\delta}{2}} \|\partial_z^3 g(t)\|^2_{X_{\tau,\kappa_3}} + \delta \int_0^T \langle t\rangle^{\frac{11-\delta}{2}} \|\partial_z^4 g(t)\|^2_{X_{\tau,\kappa_3}} dt \\ &\quad + \lambda\sqrt{\epsilon} \int_0^T \langle t\rangle^{\frac{11-\delta}{2}} \eta(t) \|\partial_z^3 g(t)\|^2_{X_{\tau,\kappa_3+1/2}} dt \end{aligned}$$

$$\leq C\|\partial_z^3 g(0)\|^2_{X_{\tau_0,\kappa_3}} + \int_0^T \langle t\rangle^{\frac{9-\delta}{2}} \|\partial_z^3 g(t)\|^2_{X_{\tau,\kappa_3}} dt$$
$$+ C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{9-\delta}{2}} \|\partial_z^3 g(t)\|^2_{X_{\tau,\kappa_2}} dt. \tag{7.22}$$

Proof. Now, applying $M_{j,\kappa_3}\partial_z^3[r]^j\partial_r^j$ to the first equation of (7.4), we can obtain that

$$\left[\partial_t + \lambda\sqrt{\epsilon}(j+1) - \partial_z^2 + \frac{1}{\langle t\rangle}\right]\partial_z^3 g_{j,\kappa_3} = \sum_{i=1}^{7} \partial_z^3 R^i_{j,\kappa_3}.$$

Performing spacial energy estimates, we can have

$$\frac{d}{dt}\|\partial_z^3 g_{j,\kappa_3}(t)\|^2_{L^2(\theta_2)} + \delta\|\partial_z^4 g_{j,\kappa_3}(t)\|^2_{L^2(\theta_2)} + \frac{5-\delta}{2\langle t\rangle}\|\partial_z^3 g_{j,\kappa_3}(t)\|^2_{L^2(\theta_2)}$$
$$+ 2(j+1)\lambda\sqrt{\epsilon}\eta(t)\|\partial_z^3 g_{j,\kappa_3}(t)\|^2_{L^2(\theta_2)}$$
$$\leq 2\left|\left\langle \sum_{i=1}^{7}\partial_z^3 R^i_{j,\kappa_3}, \partial_z^3 g_{j,\kappa_3}\right\rangle_{\theta_2}\right|.$$

Multiplying the above equality by $\langle t\rangle^{\frac{11-\delta}{2}}$ and using integration by parts for the right hand of the above inequality, and then integrating the resulted equation from 0 to t for any $t \in (0,T]$, we can achieve that

$$\langle t\rangle^{\frac{11-\delta}{2}}\|\partial_z^3 g_{j,\kappa_3}(t)\|^2_{L^2(\theta_2)} + \delta\int_0^T \langle t\rangle^{\frac{11-\delta}{2}}\|\partial_z^4 g_{j,\kappa_3}(t)\|^2_{L^2(\theta_2)} dt$$
$$+ 2\lambda\sqrt{\epsilon}(j+1)\int_0^T \langle t\rangle^{\frac{11-\delta}{2}}\eta(t)\|\partial_z^3 g_{j,\kappa_3}(t)\|^2_{L^2(\theta_2)} dt$$
$$\leq \|\partial_z^3 g_{j,\kappa_3}(0)\|^2_{L^2(\theta_2)} + 3\int_0^T \langle t\rangle^{\frac{9-\delta}{2}}\|\partial_z^3 g_{j,\kappa_3}(t)\|^2_{L^2(\theta_2)} dt$$
$$+ \int_0^T \langle t\rangle^{\frac{11-\delta}{2}}\left|\left\langle \sum_{i=1}^{7}\partial_z^2 R^i_{j,\kappa_3}, \partial_z^4 g_{j,\kappa_2} + \frac{z}{2\langle t\rangle}\partial_z^3 g_{j,\kappa_3}\right\rangle_{\theta_2}\right| dt.$$

By using Cauchy inequality and (3.22) to the right hand of above inequality, and then summing the resulted equations over $j \in \mathbb{N}$, we can obtain that

$$\langle t\rangle^{\frac{11-\delta}{2}}\|\partial_z^3 g(t)\|^2_{X_{\tau,\kappa_3}} + \delta\int_0^T \langle t\rangle^{\frac{11-\delta}{2}}\|\partial_z^4 g(t)\|^2_{X_{\tau,\kappa_3}} dt$$
$$+ \lambda\sqrt{\epsilon}(j+1)\int_0^T \langle t\rangle^{\frac{11-\delta}{2}}\eta(t)\|\partial_z^3 g(t)\|^2_{X_{\tau,\kappa_3}} dt$$

$$\leq \|\partial_z^3 g(0)\|^2_{X_{\tau,\kappa_3}} + \int_0^T \langle t\rangle^{\frac{9-\delta}{2}} \|\partial_z^3 g(t)\|^2_{X_{\tau,\kappa_3}} dt + \delta^{-1} \int_0^T \langle t\rangle^{\frac{11-\delta}{2}} \sum_{i=1}^7 \|\partial_z^2 R^i\|^2_{X_{\tau,\kappa_3}} dt. \tag{7.23}$$

Similar as Lemma 7.4, we have the following lemma. □

Lemma 7.8. *We have the following estimates*:

$$\sum_{i=1}^7 \|\partial_z^2 R^i\|^2_{X_{\tau,\kappa_3}} \lesssim \langle t\rangle^{1/2}(\|g\|^2_{X_{\tau,5}} \|\partial_z^3 g\|^2_{X_{\tau,\kappa_3+2}} + \|\partial_z g\|^2_{X_{\tau,5}} \|\partial_z^2 g\|^2_{X_{\tau,\kappa_3+2}}. + \|\partial_z^2 g\|^2_{X_{\tau,5}} \|\partial_z g\|^2_{X_{\tau,\kappa_3+2}} + \|\partial_z^3 g\|^2_{X_{\tau,5}} \|g\|^2_{X_{\tau,\kappa_3+2}}). \tag{7.24}$$

Proof. Proof of this lemma is also repeatedly use of (3.21) and (3.22) in Lemma 3.2, product estimates in (3.23) to (3.25) in Lemma 3.3 and the relation between u and g. Since it is a routing estimate, we omit the details.

Then by using (7.24), the *a priori* estimates (3.5) in Lemma 3.1 and (3.22) in Lemma 3.2, we see that

$$\int_0^T \langle t\rangle^{\frac{11-\delta}{2}} \sum_{i=1}^7 \|\partial_z^2 R^i\|^2_{X_{\tau,\kappa_1}} dt \lesssim \lambda^{-1/2}\sqrt{\epsilon} \int_0^T \langle t\rangle^{\frac{9-\delta}{2}} \|\partial_z^3 g\|^2_{X_{\tau,\kappa_2}} dt.$$

Inserting the above inequality into (7.23), we can obtain (7.22) in Lemma 7.7. □

8. Estimates of the Linearly Good Unknown $\mathcal{G}$

After we obtain the faster decay rate for low order Gevrey-2 energy of the unknowns u through the linearly good unknowns g. In this section, we focus on the Gevrey-2 estimates of the linearly good unknowns $\mathcal{G}$ and its z-derivative. It will induce faster decay rate for low order Gevrey-2 energy of the auxiliary functions $\mathcal{A}$ and $\mathcal{B}$ as displayed in (3.5).

8.1. *Estimates of $\mathcal{G}$*

Lemma 8.1. *Under the assumption of* (3.4), *for sufficiently small* ϵ, *there exits a constant* C *such that for any* $t \in (0, T]$, *we have the following estimate*:

$$\langle t\rangle^{\frac{5-\delta}{2}} \|\mathcal{G}(t)\|^2_{X_{\tau,\kappa_2,1-\delta/2}} + \delta \int_0^T \langle t\rangle^{\frac{5-\delta}{2}} \|\partial_z \mathcal{G}(t)\|^2_{X_{\tau,\kappa_2,1-\delta/2}} dt + \lambda\sqrt{\epsilon} \int_0^T \langle t\rangle^{\frac{5-\delta}{2}} \eta(t) \|\mathcal{G}(t)\|^2_{X_{\tau,\kappa_2+1/2,1-\delta/2}} dt$$

$$\begin{aligned}
&\leq C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T (\langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|\mathcal{G}(t)\|^2_{X_{\tau,\kappa_2+1/2,1-\delta/2}} \\
&\quad + \langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z\mathcal{G}(t)\|^2_{X_{\tau,\kappa_2,1-\delta/2}})dt + C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{A}(t)\|^2_{X_{\tau,\kappa}}dt \\
&\quad + C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T (\langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|g(t)\|^2_{X_{\tau,\kappa_0+1/2}} + \langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g(t)\|^2_{X_{\tau,\kappa_0}})dt.
\end{aligned} \tag{8.1}$$

Proof. First, we derive of the equation satisfied by $\mathcal{G}$. By multiplying $\frac{z}{2\langle t\rangle}$ to the first equation of (3.1), we see that

$$\begin{aligned}
&[\partial_t - \partial_z^2]\frac{z}{2\langle t\rangle}\int_z^{+\infty}\mathcal{A}d\bar{z} - \frac{1}{\langle t\rangle}\mathcal{G} \\
&\quad = \sqrt{\epsilon}\frac{z}{2\langle t\rangle}\langle t\rangle^{\delta-1}r\partial_r v - \frac{z}{2\langle t\rangle}(ur\partial_r + v\partial_z)\int_z^{+\infty}\mathcal{A}d\bar{z}.
\end{aligned} \tag{8.2}$$

Then subtracting (8.2) from (4.1), we have

$$\begin{aligned}
&\left[\partial_t - \partial_z^2 + \frac{1}{\langle t\rangle}\right]\mathcal{G} + \frac{1}{\langle t\rangle}\mathcal{G} \\
&\quad = \sqrt{\epsilon}\langle t\rangle^{-\delta-1}r\partial_r(r\partial_r u + 2u) - \sqrt{\epsilon}\frac{z}{2\langle t\rangle}\langle t\rangle^{-\delta-1}r\partial_r v \\
&\qquad - ur\partial_r\mathcal{A} + \frac{z}{2\langle t\rangle}ur\partial_r\int_z^{+\infty}\mathcal{A}d\bar{z} + \partial_z ur\partial_r\int_z^{+\infty}\mathcal{A}d\bar{z} \\
&\qquad - \frac{z}{2\langle t\rangle}v\mathcal{A} - v\partial_z\mathcal{A} + (r\partial_r u + 2u)\mathcal{A} \\
&\quad := Q^1 + Q^2 + Q^3
\end{aligned} \tag{8.3}$$

with

$$\partial_z\mathcal{G}|_{z=0} = 0, \quad \lim_{z\to+\infty}\mathcal{G} = 0.$$

Now, multiplying $M_{j,\kappa_2}[r]^j\partial_r^j$ to (8.3), we can obtain that

$$\partial_t\mathcal{G}_{j,\kappa_2} + \lambda\sqrt{\epsilon}\eta(t)(j+1)\mathcal{G}_{j,\kappa_2} - \partial_z^2\mathcal{G}_{j,\kappa_2} + \frac{1}{\langle t\rangle}\mathcal{G}_{j,\kappa_2} = \sum_{i=1}^3 Q^i_{j,\kappa_2}.$$

Similar as (4.6), we can have

$$\begin{aligned}
&\langle[\partial_t + \lambda\delta\sqrt{\epsilon}\eta(t)(j+1) - \partial_z^2]\mathcal{G}_{j,\kappa_2}, \mathcal{G}_{j,\kappa_2}(t)\rangle_{\theta_{2\nu}} \\
&\quad = \frac{1}{2}\frac{d}{dt}\|\mathcal{G}_{j,\kappa_2}(t)\|^2_{L^2(\theta_{2\nu})} + \|\partial_z\mathcal{G}_{j,\kappa_2}(t)\|^2_{L^2(\theta_{2\nu})}
\end{aligned}$$

$$+\frac{4-\nu}{4\langle t\rangle}\|\mathcal{G}_{j,\kappa_2}(t)\|^2_{L^2(\theta_{2\nu})}+\frac{\nu-\nu^2}{8}\left\|\frac{z}{\langle t\rangle}\mathcal{G}_{j,\kappa_2}(t)\right\|^2_{L^2(\theta_{2\nu})}$$
$$+(j+1)\lambda\sqrt{\epsilon}\eta(t)\|\mathcal{G}_{j,\kappa_2}(t)\|^2_{L^2(\theta_{2\nu})}.$$

Using (3.21) in Lemma 3.2, we have

$$\begin{aligned}
&2\langle[\partial_t+\lambda\sqrt{\epsilon}\eta(t)(j+1)-\partial_z^2]\mathcal{G}_{j,\kappa_2},\mathcal{G}_{j,\kappa_2}(t)\rangle_{\theta_{2\nu}}\\
&\quad\ge\frac{d}{dt}\|\mathcal{G}_{j,\kappa_2}(t)\|^2_{L^2(\theta_{2\nu})}+\frac{\delta}{2-\delta}\|\partial_z\mathcal{G}_{j,\kappa_2}(t)\|^2_{L^2(\theta_{2\nu})}\\
&\qquad+\frac{4+\frac{2-2\delta}{2-\delta}\nu}{2\langle t\rangle}\|\mathcal{G}_{j,\kappa_2}(t)\|^2_{L^2(\theta_{2\nu})}+2(j+1)\lambda\sqrt{\epsilon}\eta(t)\|\mathcal{G}_{j,\kappa_2}(t)\|^2_{L^2(\theta_{2\nu})}.
\end{aligned}$$

By taking $\nu=1-\delta/2$, we have

$$\begin{aligned}
&\frac{d}{dt}\|\mathcal{G}_{j,\kappa_2}(t)\|^2_{L^2(\theta_{2-\delta})}+\frac{\delta}{2-\delta}\|\partial_z\mathcal{G}_{j,\kappa_2}(t)\|^2_{L^2(\theta_{2-\delta})}+\frac{5-\delta}{2\langle t\rangle}\|\mathcal{G}_{j,\kappa_2}(t)\|^2_{L^2(\theta_{2-\delta})}\\
&\qquad+2(j+1)\lambda\sqrt{\epsilon}\eta(t)\|\mathcal{G}_{j,\kappa_2}(t)\|^2_{L^2(\theta_{2-\delta})}\\
&\quad\le 2\left\langle\sum_{i=1}^3 Q^i_{j,\kappa_2},\mathcal{G}_{j,\kappa_2}(t)\right\rangle_{\theta_{2-\delta}}.
\end{aligned}\tag{8.4}$$

Performing the energy estimate as before and using Cauchy inequality, we can obtain

$$\begin{aligned}
&\langle t\rangle^{\frac{5-\delta}{2}}\|\mathcal{G}_{j,\kappa_2}(t)\|^2_{L^2(\theta_{2-\delta})}+\frac{\delta}{2-\delta}\int_0^T\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z\mathcal{G}_{j,\kappa_2}(t)\|^2_{L^2(\theta_{2-\delta})}dt\\
&\qquad+\lambda\sqrt{\epsilon}(j+1)\int_0^T\langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|\mathcal{G}_{j,\kappa_2}(t)\|^2_{L^2(\theta_{2-\delta})}dt\\
&\quad\lesssim\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T\langle t\rangle^{\frac{7+\delta}{2}}(j+1)^{-1}\|(Q^1,Q^2,Q^3)_{j,\kappa_2}\|^2_{L^2(\theta_{2-\delta})}dt.
\end{aligned}$$

Summing the above inequality over $j\in\mathbb{N}$ indicates that

$$\begin{aligned}
&\langle t\rangle^{\frac{5-\delta}{2}}\|\mathcal{G}(t)\|^2_{X_{\tau,\kappa_2,1-\delta/2}}+\frac{\delta}{2-\delta}\int_0^T\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z\mathcal{G}(t)\|^2_{X_{\tau,\kappa_2,1-\delta/2}}dt\\
&\qquad+\lambda\sqrt{\epsilon}\int_0^T\langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|\mathcal{G}(t)\|^2_{X_{\tau,\kappa_2,1-\delta/2}}dt\\
&\quad\lesssim\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T\langle t\rangle^{\frac{7+\delta}{2}}\|(Q^1,Q^2,Q^3)\|^2_{X_{\tau,\kappa_2-1/2,1-\delta/2}}dt.
\end{aligned}\tag{8.5}$$

□

Estimates of Q^1

By using (3.22) in Lemma 3.2 and incompressibility, it is easy to see that

$$\begin{aligned}\|Q^1\|^2_{X_{\tau,\kappa_2-1/2,1-\delta/2}} &\lesssim \epsilon\langle t\rangle^{-2-2\delta}\|(r\partial_r u+2u)\|^2_{X_{\tau,\kappa_2+3/2,1-\delta/2}}\\ &\lesssim \epsilon\langle t\rangle^{-2-2\delta}\|u\|^2_{X_{\tau,\kappa_2+7/2,1-\delta/2}}\\ &\lesssim \epsilon\langle t\rangle^{-2-2\delta}\|g\|^2_{X_{\tau,\kappa_0+1/2}}.\end{aligned}$$

From this, we have that

$$\frac{1}{\lambda\sqrt{\epsilon}}\int^T\langle t\rangle^{\frac{7+\delta}{2}}\|Q^1\|^2_{X_{\tau,\kappa_2-1/2,1-\delta/2}}dt \lesssim \frac{\sqrt{\epsilon}}{\lambda}\int_0^T\langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|g\|^2_{X_{\tau,\kappa_0+1/2}}dt. \quad (8.6)$$

Estimates of Q^3

By using (3.22) in Lemma 3.2 and (3.24) in Lemma 3.3, we have

$$\begin{aligned}\|Q^3\|^2_{X_{\tau,\kappa_2-1/2,1-\delta/2}} &\lesssim \|v\partial_z\mathcal{A}\|^2_{X_{\tau,\kappa_2-1/2,1-\delta/2}} + \|(r\partial_r u+2u)\mathcal{A}\|^2_{X_{\tau,\kappa_2-1/2,1-\delta/2}}\\ &\quad + \|\partial_z(v\mathcal{A})\|^2_{X_{\tau,\kappa_2-1/2,1-\delta/2}}\\ &\lesssim \langle t\rangle^{1/2}(\|(r\partial_r u+2u)\|^2_{X_{\tau,3,1/2}}\|\partial_z\mathcal{A}\|^2_{X_{\tau,\kappa_2-1/2,1/2}}\\ &\quad + \|\partial_z v\|^2_{X_{\tau,\kappa_2-1/2,1/2}}\|\partial_z\mathcal{A}\|^2_{X_{\tau,3,1/2}})\\ &\lesssim \langle t\rangle^{1/2}(\|u\|^2_{X_{\tau,5,1/2}}\|\partial_z\mathcal{G}\|^2_{X_{\tau,\kappa_2-1/2,1-\delta/2}}\\ &\quad + \|u\|^2_{X_{\tau,\kappa_2+3/2,1/2}}\|\partial_z\mathcal{A}\|^2_{X_{\tau,3,1/2}}).\end{aligned}$$

Here, at the last line, we have used the fact that

$$\|\partial_z\mathcal{A}\|^2_{X_{\tau,\kappa_2-1/2,1/2}} \lesssim \|\partial_z\mathcal{G}\|^2_{X_{\tau,\kappa_2-1/2,1-\delta/2}}.$$

From this, by using (3.5) in Lemma 3.1, we have

$$\begin{aligned}&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T\langle t\rangle^{\frac{7+\delta}{2}}\|Q^2\|^2_{X_{\tau,\kappa_2-1/2,1-\delta/2}}dt\\ &\quad\lesssim \lambda^{-1/2}\sqrt{\epsilon}\int_0^T(\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z\mathcal{G}\|^2_{X_{\tau,\kappa_2,1-\delta/2}} + \langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|g\|^2_{X_{\tau,\kappa_0+1/2}})dt. \quad (8.7)\end{aligned}$$

Estimates of Q^2

By using (3.23)–(3.25) in Lemma 3.3, (7.6) in Corollary 7.1 and (3.5) in Lemma 3.6, we have

$$\begin{aligned}\|Q^2\|^2_{X_{\tau,\kappa_2-1/2,1-\delta/2}} &\lesssim \|ur\partial_r\mathcal{A}\|^2_{X_{\tau,\kappa_2-1/2,1-\delta/2}} + \left\|\partial_z\left(u\int_z^\infty r\partial_r\mathcal{A}d\bar{z}\right)\right\|^2_{X_{\tau,\kappa_2-1/2,1-\delta/2}}\\ &\quad + \left\|\partial_z u\int_z^\infty r\partial_r\mathcal{A}d\bar{z}\right\|^2_{X_{\tau,\kappa_2-1/2,1-\delta/2}}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \langle t\rangle^{1/2}(\|\partial_z u\|^2_{X_{\tau,3,1/2}}\|r\partial_r\mathcal{A}\|^2_{X_{\tau,\kappa_2-1/2,1/2}}\\
&\quad + \|\partial_z u\|^2_{X_{\tau,\kappa_2-1/2,1/2}}\|r\partial_r\mathcal{A}\|^2_{X_{\tau,3,1/2}})\\
&\lesssim \langle t\rangle^{1/2}(\|\partial_z u\|^2_{X_{\tau,3,1/2}}\|\mathcal{A}\|^{4/3}_{X_{\tau,\kappa_2-1/2,1/2}}\|\mathcal{A}\|^{2/3}_{X_{\tau,\kappa_2-1/2+6,1/2}}.\\
&\quad + \|\partial_z u\|^2_{X_{\tau,3,1/2}}\|\mathcal{A}\|^2_{X_{\tau,\kappa_2-1/2,1/2}} + \|\partial_z g\|^2_{X_{\tau,\kappa_0}}\|\mathcal{A}\|^2_{X_{\tau,5,1/2}})\\
&\lesssim (\langle t\rangle^{-\frac{6-\delta}{2}}\|\mathcal{G}\|^{4/3}_{X_{\tau,\kappa_2,1-\delta/2}}\|\mathcal{A}\|^{2/3}_{X_{\tau,\kappa,1/2}} + \langle t\rangle^{-\frac{6-\delta}{2}}\|\mathcal{G}\|^2_{X_{\tau,\kappa_2}}\\
&\quad + \langle t\rangle^{-\frac{4-\delta}{2}}\|\partial_z g\|^2_{X_{\tau,\kappa_0}}).
\end{aligned}
$$

From this, by using (3.5) in Lemma 3.1 and Young inequality, we have

$$
\begin{aligned}
&\frac{1}{\lambda\sqrt{\epsilon}}\int_0^T \langle t\rangle^{\frac{7+\delta}{2}}\|Q^2\|^2_{X_{\tau,\kappa_2-1/2,1-\delta/2}}dt\\
&\quad\lesssim \lambda^{-1/2}\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|\mathcal{G}\|^2_{X_{\tau,\kappa_2,1-\delta/2}}dt\\
&\qquad\times \lambda^{-1/2}\sqrt{\epsilon}\int_0^T (\langle t\rangle^{\frac{1-\delta}{2}}\eta(t)\|\mathcal{A}\|^2_{X_{\tau,\kappa}} + \langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g\|^2_{X_{\tau,\kappa_0}})dt. \qquad (8.8)
\end{aligned}
$$

Inserting estimates (8.6)–(8.8) into (8.5), we can achieve (8.1) in Lemma 8.1.

8.2. *Estimates of $\partial_z\mathcal{G}$*

Lemma 8.2. *Under the assumption of* (3.4), *for sufficiently small* ϵ, *there exits a constant* C *such that for any* $t \in (0, T]$, *we have the following estimate:*

$$
\begin{aligned}
&\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z\mathcal{G}(t)\|^2_{X_{\tau,\kappa_3,1-\delta/2}} + \delta\int_0^T \langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z^2\mathcal{G}(t)\|^2_{X_{\tau,\kappa_3,1-\delta/2}}dt\\
&\qquad + \lambda\sqrt{\epsilon}\int_0^T \langle t\rangle^{\frac{7-\delta}{2}}\eta(t)\|\partial_z\mathcal{G}(t)\|^2_{X_{\tau,\kappa_3+1/2,1-\delta/2}}dt\\
&\quad\le C\lambda^{-1/2}\int_0^T (\langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|\mathcal{G}(t)\|^2_{X_{\tau,\kappa_2+1/2,1-\delta/2}} + \langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z\mathcal{G}(t)\|^2_{X_{\tau,\kappa_2,1-\delta/2}})dt\\
&\qquad + C\lambda^{-1/2}\sqrt{\epsilon}\int_0^T (\langle t\rangle^{\frac{5-\delta}{2}}\eta(t)\|g(t)\|^2_{X_{\tau,\kappa_0+1/2}} + \langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z g(t)\|^2_{X_{\tau,\kappa_0}})dt.
\end{aligned}
\tag{8.9}
$$

Proof. Now, applying $M_{j,\kappa_3}\partial_z[r]^j\partial_r^j$ to (8.3), we can obtain that

$$
\partial_t\partial_z\mathcal{G}_{j,\kappa_3} + \lambda\sqrt{\epsilon}\eta(t)(j+1)\partial_z\mathcal{G}_{j,\kappa_3} - \partial_z^2\partial_z\mathcal{G}_{j,\kappa_3} + \frac{1}{\langle t\rangle}\partial_z\mathcal{G}_{j,\kappa_3} = \sum_{i=1}^3 \partial_z Q^i_{j,\kappa_3}.
$$

Similar as (8.4), we can have

$$\begin{aligned}
&\frac{d}{dt}\|\partial_z\mathcal{G}_{j,\kappa_3}(t)\|^2_{L^2(\theta_{2-\delta})}+\frac{\delta}{2-\delta}\|\partial_z^2\mathcal{G}_{j,\kappa_3}(t)\|^2_{L^2(\theta_{2-\delta})}+\frac{5-\delta}{2\langle t\rangle}\|\partial_z\mathcal{G}_{j,\kappa_3}(t)\|^2_{L^2(\theta_{2-\delta})}\\
&\quad+2(j+1)\lambda\sqrt{\epsilon}\eta(t)\|\partial_z\mathcal{G}_{j,\kappa_3}(t)\|^2_{L^2(\theta_{2-\delta})}\\
&\quad\le 2\left\langle\sum_{i=1}^3\partial_z Q^i_{j,\kappa_3},\partial_z\mathcal{G}_{j,\kappa_3}(t)\right\rangle_{\theta_{2-\delta}}. \qquad (8.10)
\end{aligned}$$

Performing the energy estimate as before and using Cauchy inequality, we can obtain

$$\begin{aligned}
&\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z\mathcal{G}_{j,\kappa_3}(t)\|^2_{L^2(\theta_{2-\delta})}+\frac{\delta}{2-\delta}\int_0^T\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z^2\mathcal{G}_{j,\kappa_3}(t)\|^2_{L^2(\theta_{2-\delta})}dt\\
&\quad+\lambda\sqrt{\epsilon}(j+1)\int_0^T\langle t\rangle^{\frac{7-\delta}{2}}\eta(t)\|\partial_z\mathcal{G}_{j,\kappa_3}(t)\|^2_{L^2(\theta_{2-\delta})}dt\\
&\quad-\int_0^T\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z\mathcal{G}_{j,\kappa_3}(t)\|^2_{L^2(\theta_{2-\delta})}dt\\
&\quad\lesssim 2\int_0^T\langle t\rangle^{\frac{7-\delta}{2}}\left|\left\langle\sum_{i=1}^3\partial_z Q^i_{j,\kappa_3},\partial_z\mathcal{G}_{j,\kappa_3}(t)\right\rangle_{\theta_{2-\delta}}\right|dt. \qquad (8.11)
\end{aligned}$$

By using integration by parts on z for the right hand of the above inequality, we see that

$$\begin{aligned}
&\left|\left\langle\sum_{i=1}^3\partial_z Q^i_{j,\kappa_3},\partial_z\mathcal{G}_{j,\kappa_3}(t)\right\rangle_{\theta_{2-\delta}}\right|\\
&\quad=\left|\left\langle\sum_{i=1}^3 Q^i_{j,\kappa_3},\partial_z^2\mathcal{G}_{j,\kappa_3}(t)+\frac{(2-\delta)z}{4\langle t\rangle}\partial_z\mathcal{G}_{j,\kappa_3}(t)\right\rangle_{\theta_{2-\delta}}\right|. \qquad (8.12)
\end{aligned}$$

Inserting (8.12) into (8.11), and then summing the above inequality over $j\in\mathbb{N}$ indicates that

$$\begin{aligned}
&\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z\mathcal{G}(t)\|^2_{X_{\tau,\kappa_3,1-\delta/2}}+\frac{\delta}{2-\delta}\int_0^T\langle t\rangle^{\frac{7-\delta}{2}}\|\partial_z^2\mathcal{G}(t)\|^2_{X_{\tau,\kappa_3,1-\delta/2}}dt\\
&\quad+\lambda\sqrt{\epsilon}\int_0^T\langle t\rangle^{\frac{7-\delta}{2}}\eta(t)\|\mathcal{G}(t)\|^2_{X_{\tau,\kappa_3,1-\delta/2}}dt-\int_0^T\langle t\rangle^{\frac{5-\delta}{2}}\|\partial_z\mathcal{G}(t)\|^2_{X_{\tau,\kappa_3,1-\delta/2}}dt\\
&\quad\lesssim\frac{1}{\delta}\int_0^T\langle t\rangle^{\frac{7-\delta}{2}}\|(Q^1,Q^2,Q^3)\|^2_{X_{\tau,\kappa_3,1-\delta/2}}dt. \qquad (8.13)
\end{aligned}$$

□

Estimates of Q^1

This is the same as (8.6), we can obtain that

$$\int_0^T \langle t\rangle^{\frac{7-\delta}{2}} \|Q^1\|^2_{X_{\tau,\kappa_3,1-\delta/2}} dt \lesssim \frac{\sqrt{\epsilon}}{\lambda} \int_0^T \langle t\rangle^{\frac{5-\delta}{2}} \eta(t) \|g\|^2_{X_{\tau,\kappa_0+1/2}} dt. \tag{8.14}$$

Estimates of Q^3

This is the same as (8.7), we can obtain that

$$\begin{aligned} &\frac{1}{\delta} \int_0^T \langle t\rangle^{\frac{7-\delta}{2}} \|Q^3\|^2_{X_{\tau,\kappa_3,1-\delta/2}} dt \\ &\quad \lesssim \lambda^{-1/2}\sqrt{\epsilon} \int_0^T (\langle t\rangle^{\frac{5-\delta}{2}} \|\partial_z \mathcal{G}\|^2_{X_{\tau,\kappa_2,1-\delta/2}} + \langle t\rangle^{\frac{5-\delta}{2}} \eta(t) \|g\|^2_{X_{\tau,\kappa_0+1/2}}) dt. \end{aligned} \tag{8.15}$$

Estimates of Q^2

By using (3.23)–(3.25) in Lemma 3.3, (7.6) in Corollary 7.1 and (3.5) in Lemma 3.6, we have

$$\begin{aligned} \|Q^2\|^2_{X_{\tau,\kappa_3,1-\delta/2}} &\lesssim \|ur\partial_r \mathcal{A}\|^2_{X_{\tau,\kappa_3,1-\delta/2}} + \left\|\partial_z (u \int_z^\infty r\partial_r \mathcal{A} d\bar{z})\right\|^2_{X_{\tau,\kappa_3,1-\delta/2}} \\ &\quad + \left\|\partial_z u \int_z^\infty r\partial_r \mathcal{A} d\bar{z}\right\|^2_{X_{\tau,\kappa_3,1-\delta/2}} \\ &\lesssim \langle t\rangle^{1/2} (\|\partial_z u\|^2_{X_{\tau,3,1/2}} \|r\partial_r \mathcal{A}\|^2_{X_{\tau,\kappa_3,1/2}} \\ &\quad + \|\partial_z u\|^2_{X_{\tau,\kappa_3,1/2}} \|r\partial_r \mathcal{A}\|^2_{X_{\tau,3,1/2}}) \\ &\lesssim \langle t\rangle^{1/2} (\|\partial_z u\|^2_{X_{\tau,3,1/2}} \|\mathcal{G}\|^2_{X_{\tau,\kappa_3+2,1-\delta/2}} + \|\partial_z g\|^2_{X_{\tau,\kappa_0}} \|\mathcal{A}\|^2_{X_{\tau,5,1/2}}). \end{aligned}$$

From this, by using (3.5) in Lemma 3.1, we have

$$\begin{aligned} &\frac{1}{\delta} \int_0^T \langle t\rangle^{\frac{7-\delta}{2}} \|Q^2\|^2_{X_{\tau,\kappa_3,1-\delta/2}} dt \\ &\quad \lesssim \lambda^{-1/2}\sqrt{\epsilon} \int_0^T (\langle t\rangle^{\frac{5-\delta}{2}} \eta(t) \|\mathcal{G}\|^2_{X_{\tau,\kappa_2,1-\delta/2}} + \langle t\rangle^{\frac{5-\delta}{2}} \|\partial_z g\|^2_{X_{\tau,\kappa_0}}) dt. \end{aligned} \tag{8.16}$$

Inserting (8.14)–(8.16) into (8.13), we can obtain (8.9) in Lemma 8.2.

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Appendix A. Proof of Lemma 7.2

Proof. First, we show that

$$\|r\partial_r f\|_{L^2_r} \lesssim \|r^2\partial_r^2 f\|_{L^2_r}^{1/2}\|f\|_{L^2_r}^{1/2} + \|f\|_{L^2_r}. \tag{A.1}$$

Actually, by using integration by parts on r variable and Cauchy–Schwarz inequality, we have

$$\begin{aligned}\int_0^\infty r^2(\partial_r f)^2 dr &= \int_0^\infty r^2\partial_r f df = -\int_0^\infty f d(r^2\partial_r f)\\ &= -\int_0^\infty f(r^2\partial_r^2 f + 2r\partial_r f)dr = -\int_0^\infty f r^2\partial_r^2 f dr + \int_0^\infty f^2 dr\\ &\lesssim \|r^2\partial_r^2 f\|_{L^2_r}\|f\|_{L^2_r} + \|f\|_{L^2_r}^2,\end{aligned}$$

which indicates (A.1). By repeatly using of (A.1), we see that

$$\begin{aligned}\|r^2\partial_r^2 f\|_{L^2_r} &= \|r\partial_r(r\partial_r f) - r\partial_r f\|_{L^2_r} \lesssim \|r\partial_r(r\partial_r f)\| + \|r\partial_r f\|_{L^2_r}\\ &\lesssim \|r^2\partial_r^2(r\partial_r f)\|_{L^2_r}^{1/2}\|r\partial_r f\|_{L^2_r}^{1/2} + \|r\partial_r f\|_{L^2_r}\\ &\lesssim (\|r^3\partial_r^3 f\|_{L^2_r}^{1/2} + \|r^2\partial_r^2 f\|_{L^2_r}^{1/2})\|r\partial_r f\|_{L^2_r}^{1/2} + \|r\partial_r f\|_{L^2_r}\\ &\lesssim (\|r^3\partial_r^3 f\|_{L^2_r}^{1/2}\|r^2\partial_r^2 f\|_{L^2_r}^{1/4}\|f\|_{L^2_r}^{1/4} + \|r^2\partial_r^2 f\|_{L^2_r}^{3/4}\|f\|_{L^2_r}^{1/4})\\ &\quad + (\|r^3\partial_r^3 f\|_{L^2_r}^{1/2}\|f\|_{L^2_r}^{1/2} + \|r^2\partial_r^2 f\|_{L^2_r}^{1/2}\|f\|_{L^2_r}^{1/2})\\ &\quad + \|r^2\partial_r^2 f\|_{L^2_r}^{1/2}\|f\|_{L^2_r}^{1/2} + \|f\|_{L^2_r}.\end{aligned}$$

Then using Young inequality, we can obtain that

$$\|r^2\partial_r^2 f\|_{L^2_r} \lesssim \|r^3\partial_r^3 f\|_{L^2_r}^{2/3}\|f\|_{L^2_r}^{1/3} + \|f\|_{L^2_r}.$$

Inserting this back to (A.1), we can obtain that

$$\|r\partial_r f\|_{L^2_r} \lesssim \|r^3\partial_r^3 f\|_{L^2_r}^{1/3}\|f\|_{L^2_r}^{2/3} + \|f\|_{L^2_r}.$$

Then by taking square of the above inequality, integrating on z with weight $\omega(z)$ and using Höler inequality, we can have (7.5). □

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