Nonlinear Differ. Equ. Appl. (2021) 28:12
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https://doi.org/10.1007/s00030-021-00674-y

Nonlinear Differential Equations and Applications NoDEA

Liouville theorem of the 3D stationary MHD system: for D-solutions converging to non-zero constant vectors

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Abstract. In this paper, we derive the Liouville theorem of D-solutions to the stationary MHD system under the asymptotic assumption: one of the velocity field and the magnetic field approaches zero and the other approaches a non zero constant vector at infinity. Our result extends the corresponding one of D-solutions to the Navier–Stokes equations when the velocity approaches a non zero constant vector at infinity.

Mathematics Subject Classification. 35Q30, 76N10.

Keywords. Incompressible, MHD system, Liouville theorem.

1. Introduction

The stationary 3D incompressible MHD equations read

$$\begin{cases} u\cdot\nabla u+\nabla p-h\cdot\nabla h-\kappa\Delta u=0,\\ u\cdot\nabla h-h\cdot\nabla u-\nu\Delta h=0,\\ \nabla\cdot u=0,\\ \nabla\cdot h=0,\end{cases}\tag{1.1}$$

where $u(x)$, $h(x)\in\mathbb{R}^3, p(x)\in\mathbb{R}$ represent the velocity vector, the magnetic field and the scalar pressure. $\kappa,\nu>0$ are constants. Physically $(1.1)_1$ represents the conservation of momentum while $(1.1)_2$ is the diffusive *Maxwell-Faraday equation* which describes the Faraday's law of induction. $(1.1)_3$ shows the conservation of mass and $(1.1)_4$ is the Gauss's law for magnetism. The MHD system, which describes the large scale, slow dynamics of plasmas, is a fundamental system of partial differential equations in nature. For more background of the MHD system, we refer readers to [12].

Published online: 01 February 2021

When $h \equiv 0$, the MHD system is reduced to the Navier–Stokes system

$$\begin{cases} u \cdot \nabla u + \nabla p - \kappa \Delta u = 0, \\ \nabla \cdot u = 0. \end{cases} \tag{1.2}$$

In the 1930s, the pioneering work Leray [13] studied the existence of a weak solution to (1.2) by using the variation method. Also, such a weak solution approaches a constant vector at infinity and satisfies the bounded Dirichlet integral $\int_{\mathbb{R}^3} |\nabla u|^2 dx < +\infty$, which is often referred to as "D-solutions". The smoothness of D-solutions is easy to prove by the properties of elliptic partial differential equations. However, the uniqueness of D-solutions has been a long and old open problem. Later Finn [7] and Ladyzheskaya [11] show that any D-solution in a 3D exterior domain converges to a prescribed constant vector u_∞ at infinity. Finn also suggested a class of *physically reasonable solution* in 3D, which satisfies

$$u(x) = O(|x|^{-1}) \quad \text{if } u_\infty = 0;$$
$$u(x) - u_\infty = O(|x|^{-\frac{1}{2}-\varepsilon}) \quad \text{if } u_\infty \neq 0.$$

Babenko [1] proved any D-solution is physically reasonable provided $u_\infty \neq 0$. See also [6] for a short proof.

When the spacial dimension is 2, Gilbarg–Weinberger [9] proved the following asymptotic estimate of vorticity w:

$$|w(x)| = o(|x|^{-\frac{3}{4}} (\log |x|)^{\frac{1}{8}}), \quad \text{as } |x| \to \infty.$$

Therefore, by Biot-Savart law and the incompressibility, the constancy of u immediately follows by the vanishing of w since the vorticity equation in 2D is

$$-\kappa \Delta w + u \cdot \nabla w = 0,$$

which satisfies the maximum principle.

However, the related 3-dimensional problem is totally different due to the lack of the maximum principle. Nevertheless, the Liouville theorem of the case that $u_\infty \neq 0$ still follows due to a fast decay estimate of u, which may not hold for the case $u_\infty = 0$. See Theorem X.7.2 in [8]. Nowadays the related Liouville-type theorem for $u_\infty = 0$ is still open. Under some extra assumptions, positive answers were given by tremendous authors. For example, Galdi [8] proved vanishing result provided $u \in L^{9/2}(\mathbb{R}^3)$, while Chae [3] provided $\Delta u \in L^{6/5}(\mathbb{R}^3)$. See [2–4,8,10,18,20] etc. for some recent progress in this aspect.

In this paper, we consider the related Liouville-type theorem of the MHD system under the following D-condition (1.3) on both u and h. The existence and smoothness of D-solutions to (1.1) with (1.3) can be easily proven by following Leray [13] and Galdi [8]. Here we prove the trivialness of solutions by assuming one of u and h approaches zero and the other approaches a non zero constant vector. There are also some related works to consider the Liouville-type theorem of D-solutions to the MHD system when both u and h approach to zero at infinity. See, for, example, [5,14]. Here goes the main result:

Theorem 1.1. *Let (u,h) be a smooth solution to the MHD equations (1.1) satisfies the following $D-$condition*

$$\int_{\mathbb{R}^3} \left(|\nabla u|^2 + |\nabla h|^2\right) dx < \infty, \tag{1.3}$$

and

$$u(x) \to u_\infty, \quad h(x) \to h_\infty, \quad \textit{uniformly as} \quad |x| \to \infty,$$

where $u_\infty, h_\infty \in \mathbb{R}^3$ are two constant vector. Then $(u,h) \equiv (u_\infty, h_\infty)$ follows if one of the following two conditions holds

(i) $u_\infty \neq 0$, $h_\infty = 0$.

(ii) $u_\infty = 0$, $h_\infty \neq 0$ *and* $\kappa = \nu$.

□

The strategy of proving Theorem 1.1 is motivated by the formal estimates of the linear Oseen flow (see [6,8] e.g.), which is derived by applying a multiplier estimate of Lizorkin [15] and the basic perturbation theory. Due to the appearance of the first order derivative terms of u and h, lower order integrability of the unknowns are achieved. Then the Liouville-type theorem follows by testing the equations with proper functions.

It seems that the system (1.1) is over-determined, since there are 8 equations but only 7 unknowns in (1.1). However, under the conditions in this paper, we remark here that the last equation in the system (1.1) can be derived by the rest. The reason is: acting the divergence operator on $(1.1)_2$ and noting the divergence free of u, it follows that

$$u \cdot \nabla(\nabla \cdot h) - \Delta(\nabla \cdot h) = 0.$$

Testing this equation by $\varphi_R(\nabla \cdot h)$ (the definition of the cut-off function φ_R can be found in Sect. 3, around (3.10) and (3.11).) and integration by parts, we have

$$\int_{\mathbb{R}^3} \varphi_R |\nabla(\nabla \cdot h)|^2 dx = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \cdot h|^2 \left(\Delta \varphi_R + u \cdot \nabla \varphi_R\right) dx. \tag{1.4}$$

By the D-condition (1.3) and the boundedness of u (Since u is assumed to be smooth and it converges to a constant vector uniformly as $|x| \to \infty$.), the right hand side of (1.4) follows that

$$\int_{\mathbb{R}^3} |\nabla \cdot h|^2 \left(\Delta \varphi_R + u \cdot \nabla \varphi_R\right) dx \le C \left(\frac{1}{R^2} + \frac{\|u\|_{L^\infty(\mathbb{R}^3)}}{R}\right) \int_{\mathbb{R}^3} |\nabla h|^2 dx \to 0 \quad \text{as } R \to \infty.$$

Therefore one derives that $\nabla \cdot h \equiv c$ for some constant c from the left hand side of (1.4). One more time by the D-condition of h, we have $\int_{\mathbb{R}^3} |\nabla \cdot h|^2 dx \le \int_{\mathbb{R}^3} |\nabla h|^2 dx < +\infty$, which indicates that $\nabla \cdot h \equiv 0$.

This paper is organized as follows, some preliminary work, such as the basic estimates and perturbation analysis of the related linear operator, will be derived in Sect. 2. In Sect. 3, we focus on the proof of Theorem 1.1.

Let us introduce some notations at the end of the introduction. Throughout this paper, $C(c_1, c_2, \ldots, c_n)$ denotes a positive constant depending on c_1, c_2, $\ldots$ c_n which may be different from line to line. We also apply $A \lesssim B$ to denote $A \le CB$. We denote by $B_r(x_0) := \{x \in \mathbb{R}^3 : |x - x_0| < r\}$. We simply denote by $B_r := B_r(0)$. The symbol ∂_i stands for $\frac{\partial}{\partial x_i}$, for $i = 1, 2, 3$, while L stands for a multi-index such that $L = (l_1, l_2, l_3)$ where $l_1, l_2, l_3 \in \mathbb{N} \cup \{0\}$ and $|L| = l_1 + l_2 + l_3$, $\nabla^L = \partial_{x_1}^{l_1} \partial_{x_2}^{l_2} \partial_{x_3}^{l_3}$. For a domain $\Omega \subset \mathbb{R}^3$, $1 \le p \le \infty$ and $k \in \mathbb{N}$, $L^p(\Omega)$ denotes the usual Lebesgue space with norm

$$\|f\|_{L^p(\Omega)} := \begin{cases} \left(\int_\Omega |f(x)|^p dx \right)^{1/p}, & 1 \le p < \infty, \\ \underset{x \in \Omega}{esssup} |f(x)|, & p = \infty, \end{cases}$$

while $W^{k,p}(\Omega)$ denotes the usual Sobolev space and $\dot{W}^{k,p}(\Omega)$ denotes the usual homogeneous Sobolev space with their norm and semi-norm

$$\|f\|_{W^{k,p}(\Omega)} := \sum_{0 \le |L| \le k} \|\nabla^L f\|_{L^p(\Omega)},$$

$$|f|_{\dot{W}^{k,p}(\Omega)} := \sum_{|L| = k} \|\nabla^L f\|_{L^p(\Omega)},$$

respectively. $C^\infty(\Omega)$ denotes the space of smooth functions on Ω. $\mathcal{S}(\mathbb{R}^d)$ means the space of rapid decreasing smooth functions on $\mathbb{R}^d$, and the dual space $\mathcal{S}'(\mathbb{R}^d)$ stands for the space of tempered distributions. For any $f \in \mathcal{S}'(\mathbb{R}^d)$, $\mathcal{F} f$ and $\mathcal{F}^{-1} f$ denote its Fourier transform and inverse Fourier transform, respectively. $\mathcal{P}_n$ stands for the space of polynomials in $\mathbb{R}^3$ with their degree no bigger than n.

2. Preliminaries

At first, we introduce the following multiplier theorem by Lizorkin [15], which generalized the related theories from Marcinkiewicz [16] and Mikhlin [17].

Lemma 2.1. (Lizorkin) *Let $d \ge 2$, $H = \{\xi \in \mathbb{R}^d : |\xi_i| > 0, \text{ for } i = 1, 2, \ldots, d\}$ and let $\Phi : H \to \mathbb{C}$ be a function such that the derivatives $\partial_1^{k_1} \cdots \partial_d^{k_d} \Phi$ exist and are continuous for all $k_1, \ldots, k_d \in \{0, 1\}$. Assume that there are real numbers $\beta \in [0, 1)$ and $M > 0$ such that*

$$|\xi_1|^{k_1 + \beta} \cdots |\xi_d|^{k_d + \beta} |\partial_1^{k_1} \cdots \partial_d^{k_d} \Phi| \le M \tag{2.1}$$

for all $\xi \in H$ and $k_1, \ldots, k_d \in \{0, 1\}$. Then the operator $T : \mathcal{S}(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ defined by

$$Tf(x) = \mathcal{F}^{-1}(\Phi(\cdot) \hat{f}(\cdot))(x), \quad x \in \mathbb{R}^d,$$

has a unique extension $T : L^q(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$, where $1 < q < \beta^{-1}$ and $\frac{1}{p} = \frac{1}{q} - \beta$. Further more

$$\|Tf\|_{L^p(\mathbb{R}^d)} \le C(q, r, M) \|f\|_{L^q(\mathbb{R}^d)}, \forall f \in L^q(\mathbb{R}^d).$$

□

We refer readers to [8], Section VII.4 for more details about Lemma 2.1. To prove the main theorem, the following estimates of linear systems (2.2) and (2.4) are key points:

Proposition 2.2. *Let f_1, $f_2 \in L^q(\mathbb{R}^3)$, $g \in W^{1,q}(\mathbb{R}^3)$ with $1 < q < 2$ and $r = \left(\frac{1}{q} - \frac{1}{2}\right)^{-1}$. Then we have the following two results*

(i) *For the 3-D stationary linear system*

$$\begin{cases} -\kappa\Delta v + \partial_{x_1} v + \nabla p = f_1, \\ -\nu\Delta\eta + \partial_{x_1}\eta = f_2, \qquad\qquad in \quad \mathbb{R}^3 \\ \nabla\cdot v = g, \end{cases} \tag{2.2}$$

there exists a unique $(v,\eta,p) \in \left(\dot{W}^{2,q}(\mathbb{R}^3)\cap L^r(\mathbb{R}^3)\right)^6 \times \left(\dot{W}^{1,q}(\mathbb{R}^3)/\mathcal{P}_0(\mathbb{R}^3)\right)$ such that

$$\begin{aligned} &\|(\nabla^2 v, \nabla^2\eta)\|_{L^q(\mathbb{R}^3)} + \|(v,\eta)\|_{L^r(\mathbb{R}^3)} + \|\nabla p\|_{L^q(\mathbb{R}^3)} \\ &\quad \le C(q,\kappa,\nu)\left(\|f_1\|_{L^q(\mathbb{R}^3)} + \|f_2\|_{L^q(\mathbb{R}^3)} + \|g\|_{W^{1,q}(\mathbb{R}^3)}\right). \end{aligned} \tag{2.3}$$

(ii) *For the 3-D stationary linear system*

$$\begin{cases} \partial_{x_1} w_1 + \nabla p - \Delta w_1 = f_1, \\ -\partial_{x_1} w_2 + \nabla p - \Delta w_2 = f_2, \quad in \quad \mathbb{R}^3, \\ \nabla\cdot w_1 = g, \end{cases} \tag{2.4}$$

there exists a unique $(w_1,w_2,p) \in \left(\dot{W}^{2,q}(\mathbb{R}^3)\cap L^r(\mathbb{R}^3)\right)^6 \times \left(\dot{W}^{1,q}(\mathbb{R}^3)/\mathcal{P}_0(\mathbb{R}^3)\right)$ such that

$$\begin{aligned} &\|(\nabla^2 w_1, \nabla^2 w_2)\|_{L^q(\mathbb{R}^3)} + \|(w_1,w_2)\|_{L^r(\mathbb{R}^3)} + \|\nabla p\|_{L^q(\mathbb{R}^3)} \\ &\quad \le C(q)\left(\|f_1\|_{L^q(\mathbb{R}^3)} + \|f_2\|_{L^q(\mathbb{R}^3)} + \|g\|_{W^{1,q}(\mathbb{R}^3)}\right). \end{aligned} \tag{2.5}$$

Proof. For the proof of (i), applying the Fourier transform, we obtain from (2.2) that

$$\begin{cases} \kappa|\xi|^2\hat{v}(\xi) + i\xi_1\hat{v}(\xi) + i\xi\hat{p}(\xi) = \hat{f}_1(\xi), \\ \nu|\xi|^2\hat{\eta}(\xi) + i\xi_1\hat{\eta}(\xi) = \hat{f}_2(\xi), \qquad\qquad \text{in} \quad \mathbb{R}^3. \\ i\xi\cdot\hat{v} = \hat{g}(\xi), \end{cases}$$

Direct calculation shows

$$\begin{aligned} \hat{v}(\xi) &= \left(\kappa|\xi|^2 + i\xi_1\right)^{-1}\left[\left(id - \frac{\xi\otimes\xi}{|\xi|^2}\right)\cdot\hat{f}_1(\xi) + \left(\frac{\xi\xi_1}{|\xi|^2} - i\kappa\xi\right)\hat{g}(\xi)\right], \\ \hat{\eta}(\xi) &= \left(\nu|\xi|^2 + i\xi_1\right)^{-1}\hat{f}_2(\xi), \\ \hat{p}(\xi) &= -\frac{i\xi\cdot\hat{f}_1(\xi)}{|\xi|^2} + \left(\kappa + \frac{i\xi_1}{|\xi|^2}\right)\hat{g}(\xi). \end{aligned}$$

To prove the assertion (2.3), we only need to prove that the Fourier multipliers $\Phi_0(\xi) := \left(|\xi|^2 + i\xi_1\right)^{-1}$ satisfies the conditions in Lemma 2.1 with $\beta = \frac{1}{2}$ and $\Phi_{jk}(\xi) := \xi_j\xi_k\left(|\xi|^2 + i\xi_1\right)^{-1}$ with $\beta = 0$, respectively. In the below we only prove the first one since the second one is more transparent. For all $\xi \in H = \{\xi \in \mathbb{R}^d : \ |\xi_i| > 0, \ \text{for } i = 1, 2, \dots, d\}$, Young's inequality yields

$$|\xi_1|^{1/2}|\xi_2|^{1/2}|\xi_3|^{1/2}|\Phi_0(\xi)| \le \frac{C\left(|\xi_1| + \xi_2^2 + \xi_3^2\right)}{\sqrt{\xi_1^2 + |\xi|^4}} \le C.$$

This proves (2.1) for any $\xi \in H$ when $k_1 = k_2 = k_3 = 0$. The proof for non-zero (k_1, k_2, k_3) is similar.

Now we focus on the proof of Case (ii). First we apply the Fourier transform, the system (2.4) is solved by

$$\begin{aligned}
\hat{w_1}(\xi) &= \left(|\xi|^2 + i\xi_1\right)^{-1}\left[\left(id - \frac{\xi \otimes \xi}{|\xi|^2}\right) \cdot \hat{f}_1(\xi) + \left(\frac{\xi\xi_1}{|\xi|^2} - i\xi\right)\hat{g}(\xi)\right],\\
\hat{w_2}(\xi) &= \left(|\xi|^2 - i\xi_1\right)^{-1}\left[\hat{f}_2(\xi) - \frac{\xi \otimes \xi}{|\xi|^2} \cdot \hat{f}_1(\xi) + \left(\frac{\xi\xi_1}{|\xi|^2} - i\xi\right)\hat{g}(\xi)\right],\\
\hat{p}(\xi) &= -\frac{i\xi \cdot \hat{f}_1(\xi)}{|\xi|^2} + \left(1 + \frac{i\xi_1}{|\xi|^2}\right)\hat{g}(\xi).
\end{aligned}$$

The rest is almost the same as the proof in Case (i). We omit the details. □

Remark 2.3. In this remark, we explain why the Liouville-type theorem of the D-solution to the Navier–Stokes equations with u tends to a nonzero constant vector at infinity was solved, but the case that u approaches 0 is still open. If $u \to u_\infty \ne 0$, by the scaling invariant and the orthogonal transform invariant of the Navier–Stokes equations, we may assume $u_\infty = (1, 0, 0)$ without loss of generality. Denoting $v = u - u_\infty$, we find the linear part of the equations of v are Oseen equations. In proving the Liouville-type theorem, the essential difference of the linear Oseen flow

$$\begin{cases} -\kappa\Delta v + \partial_{x_1} v + \nabla p = f_1, \\ \nabla \cdot v = g. \end{cases} \quad \text{in} \quad \mathbb{R}^3$$

and the linear Stokes flow

$$\begin{cases} -\kappa\Delta v + \nabla p = f_1, \\ \nabla \cdot v = g. \end{cases} \quad \text{in} \quad \mathbb{R}^3$$

happens in the $i\xi_1$ term in $\Phi_0(\xi)$ in the proof of Proposition 2.2. Due to the presence of $i\xi_1$ term, $\beta = \frac{1}{2}$ is guaranteed to be a legal index in the condition (2.1), while we are only allowed to choose $\beta = \frac{2}{3}$ otherwise. This results in a lower order integrability of v (as well as η in the related MHD system in (2.3)). Furthermore it allows a perturbation scheme of the following linear system with an L^2 potential. □

The following lemma is a perturbation of Proposition 2.2, which reads

Lemma 2.4. *Assume* $1 < q < 2$, $f_1, f_2 \in L^q(\mathbb{R}^3)$, $g \in W^{1,q}(\mathbb{R}^3)$ *and* $\mathcal{M} \in \left(L^2(\mathbb{R}^3)\right)^{6\times 6}$. (v, h, p) *satisfies the following stationary linear system*

$$\begin{cases} -\Delta\begin{pmatrix} \kappa v \\ \nu\eta \end{pmatrix} + \partial_{x_1}\begin{pmatrix} v \\ \eta \end{pmatrix} + \mathcal{M}\begin{pmatrix} v \\ \eta \end{pmatrix} + \begin{pmatrix} \nabla p \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \\ \nabla \cdot v = g, \end{cases} \quad \text{in} \quad \mathbb{R}^3. \tag{2.6}$$

There exists a small constant $\varepsilon_0 > 0$ *such that if*

$$\|\mathcal{M}\|_{L^2(\mathbb{R}^3)} < \varepsilon_0,$$

then there exists a unique $(v, \eta, p) \in \left(\dot{W}^{2,q}(\mathbb{R}^3) \cap L^r(\mathbb{R}^3)\right)^6 \times \left(\dot{W}^{1,q}(\mathbb{R}^3)/\mathcal{P}_0(\mathbb{R}^3)\right)$ *solves* (2.6) *such that*

$$\begin{aligned} &\|(\nabla^2 v, \nabla^2 \eta)\|_{L^q(\mathbb{R}^3)} + \|(v, \eta)\|_{L^r(\mathbb{R}^3)} + \|\nabla p\|_{L^q(\mathbb{R}^3)} \\ &\quad \le C(q, \kappa, \nu, \varepsilon_0) \left(\|f_1\|_{L^q(\mathbb{R}^3)} + \|f_2\|_{L^q(\mathbb{R}^3)} + \|g\|_{W^{1,q}(\mathbb{R}^3)}\right), \end{aligned} \tag{2.7}$$

where $r = \left(\frac{1}{q} - \frac{1}{2}\right)^{-1}$.

Proof. We denote the Banach spaces

$$\begin{aligned} \mathcal{X} &:= \left(\dot{W}^{2,q}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)\right)^6 \times \dot{W}^{1,q}(\mathbb{R}^3)/\mathcal{P}_0(\mathbb{R}^3), \\ \mathcal{Y} &:= \left(L^q(\mathbb{R}^3)\right)^6 \times W^{1,q}(\mathbb{R}^3), \end{aligned}$$

with norms

$$\begin{aligned} \|(v, \eta, p)\|_{\mathcal{X}} &:= \|(\nabla^2 v, \nabla^2 \eta)\|_{L^q(\mathbb{R}^3)} + \|(v, \eta)\|_{L^r(\mathbb{R}^3)} + \|\nabla p\|_{L^q(\mathbb{R}^3)}, \\ \|(f_1, f_2, g)\|_{\mathcal{Y}} &:= \|f_1\|_{L^q(\mathbb{R}^3)} + \|f_2\|_{L^q(\mathbb{R}^3)} + \|g\|_{W^{1,q}(\mathbb{R}^3)}, \end{aligned}$$

respectively. By Proposition 2.2, the operator $\mathcal{T}$ which is defied by

$$\begin{aligned} &\mathcal{T} : \ \mathcal{X} \mapsto \mathcal{Y} \\ &(v, \eta, p) \mapsto (-\Delta v + \partial_{x_1} v + \nabla p, \ -\Delta \eta + \partial_{x_1} \eta, \ \nabla \cdot v), \end{aligned}$$

admits a bounded inverse operator $\mathcal{T}^{-1}$ and

$$\|(\nabla^2 v, \nabla^2 \eta)\|_{L^q(\mathbb{R}^3)} + \|(v, \eta)\|_{L^r(\mathbb{R}^3)} + \|\nabla p\|_{L^q(\mathbb{R}^3)} \le C\|\mathcal{T}(v, h, p)\|_{\mathcal{Y}}.$$

By Hölder inequality, the operator $\mathcal{E} : \ \mathcal{X} \mapsto \mathcal{Y}$ which is defined by

$$\mathcal{E}(v, \eta, p) = \left(\left[\mathcal{M}\begin{pmatrix} v \\ \eta \end{pmatrix}\right]^T, 0\right)$$

satisfies

$$\begin{aligned} \|\mathcal{E}(v, \eta, p)\|_{\mathcal{Y}} = \left\|\mathcal{M}\begin{pmatrix} v \\ \eta \end{pmatrix}\right\|_{L^q(\mathbb{R}^3)} &\le \|\mathcal{M}\|_{L^2(\mathbb{R}^3)} \|(v, \eta)\|_{L^r(\mathbb{R}^3)} \\ &\le C\|\mathcal{M}\|_{L^2(\mathbb{R}^3)} \|\mathcal{T}(v, h, p)\|_{\mathcal{Y}}, \end{aligned}$$

where $(\cdot)^T$ means the transpose of a matrix. Choosing $\varepsilon_0 = (C(q) + 1)^{-1}$, it follows that

$$C(q)\|\mathcal{M}\|_{L^2(\mathbb{R}^3)} < 1.$$

Then a routine perturbation theory yields if $\|\mathcal{M}\| \le \varepsilon_0$, the operator $\mathcal{T} + \mathcal{E}$ has a bounded inverse which can be represented by

$$(\mathcal{T} + \mathcal{E})^{-1} = \sum_{j=0}^{\infty} (-1)^j \mathcal{E}^j \mathcal{T}^{-j-1},$$

and the estimate (2.9) holds automatically. □

Furthermore, we emphasize here that the result in Lemma 2.4 can also be applied to the following similar system

$$\begin{cases} -\Delta \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \partial_{x_1} \begin{pmatrix} w_1 \\ -w_2 \end{pmatrix} + \mathcal{M} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} \nabla p \\ \nabla p \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \\ \nabla \cdot w_1 = g, \end{cases} \quad \text{in} \quad \mathbb{R}^3. \tag{2.8}$$

We have the following Lemma.

Lemma 2.5. *Under the same conditions of q, f_1, f_2, g and $\mathcal{M}$ in (2.8) as that in Lemma 2.4, there exists a unique $(w_1, w_2, p) \in \left(\dot{W}^{2,q}(\mathbb{R}^3) \cap L^r(\mathbb{R}^3)\right)^6 \times \left(\dot{W}^{1,q}(\mathbb{R}^3)/\mathcal{P}_0(\mathbb{R}^3)\right)$ solving (2.8) such that*

$$\begin{aligned} &\|(\nabla^2 w_1, \nabla^2 w_2)\|_{L^q(\mathbb{R}^3)} + \|(w_1, w_2)\|_{L^r(\mathbb{R}^3)} + \|\nabla p\|_{L^q(\mathbb{R}^3)} \\ &\quad \le C(q, \kappa, \nu, \varepsilon_0) \left(\|f_1\|_{L^q(\mathbb{R}^3)} + \|f_2\|_{L^q(\mathbb{R}^3)} + \|g\|_{W^{1,q}(\mathbb{R}^3)}\right), \end{aligned} \tag{2.9}$$

where $r = \left(\frac{1}{q} - \frac{1}{2}\right)^{-1}$.

The proof of Lemma 2.5 is based on case (ii) of Proposition 2.2, and the method of perturbation in proving Lemma 2.4. We omit the details here. $\square$

3. Proof of Theorem 1.1

Now we are ready for the proof of the main theorem. Since the MHD system (1.1) is invariant under the orthogonal coordinate transform:

$$u_O(x) = Ou(O^T x), \quad h_O(x) = Oh(O^T x), \quad p_O(x) = p(O^T x),$$

where $O \in \mathrm{SO}(\mathbb{R}^3)$ is a 3D rotation matrix. Thus we may assume

$$u_\infty = |u_\infty| \cdot (1, 0, 0)^T.$$

Moreover, by the following scaling invariant of Navier–Stokes-MHD system:

$$u_\lambda(x) = \lambda u(\lambda x), \quad h_\lambda(x) = \lambda h(\lambda x), \quad p_\lambda(x) = \lambda^2 p(\lambda x),$$

we may simplify the constant vector u_∞ by assuming $u_\infty = (1, 0, 0)^T$ without loss of generality. Denoting $v := u - u_\infty$, it follows that

$$\begin{cases} v \cdot \nabla v + \partial_{x_1} v + \nabla p - h \cdot \nabla h - \kappa \Delta v = 0, \\ v \cdot \nabla h + \partial_{x_1} h - h \cdot \nabla v - \nu \Delta h = 0, \\ \nabla \cdot v = 0, \quad \nabla \cdot h = 0, \end{cases} \quad \text{in} \quad \mathbb{R}^3. \tag{3.1}$$

Now we claim a integral property of v and h that

Claim. *Under the same conditions as those in Theorem 1.1, we have*

$$\|v\|_{L^3(\mathbb{R}^3)} + \|h\|_{L^3(\mathbb{R}^3)} < \infty.$$

Proof. According to the D-condition (1.3), there exists an $M > 0$ such that

$$\int_{B_M^c} \left(|\nabla v|^2 + |\nabla h|^2\right) dx < \varepsilon_0^2,$$

where B_M^c is the complement of B_M in $\mathbb{R}^3$. We define $\psi \in C^\infty(\mathbb{R}^3)$ the cut-off function such that

$$\psi(x) = \psi(|x|) = \begin{cases} 1, & \text{if} \quad |x| > 2M; \\ 0, & \text{if} \quad |x| < M, \end{cases} \tag{3.2}$$

and $0 \le \psi(x) \le 1$, $\forall x \in B_{2M} - B_M$. Multiplying ψ on the both sides of (3.1), it follows that

$$\begin{cases} \partial_{x_1} \begin{pmatrix} \psi v \\ \psi h \end{pmatrix} - \Delta \begin{pmatrix} \kappa \psi v \\ \nu \psi h \end{pmatrix} + \begin{pmatrix} (\nabla v)\chi_{B_M^c}, & -(\nabla h)\chi_{B_M^c} \\ (\nabla h)\chi_{B_M^c}, & -(\nabla v)\chi_{B_M^c} \end{pmatrix} \cdot \begin{pmatrix} \psi v \\ \psi h \end{pmatrix} + \begin{pmatrix} \nabla(\psi p) \\ 0 \end{pmatrix} \\ \qquad = \begin{pmatrix} F_1(\psi) \\ F_2(\psi) \end{pmatrix}, \\ \nabla \cdot (\psi v) = \nabla \psi \cdot v, \end{cases} \tag{3.3}$$

where

$$\begin{aligned} F_1(\psi) &= -\, 2\kappa \nabla \psi \cdot \nabla v - \kappa(\Delta \psi) v + v \partial_{x_1} \psi + (\nabla \psi) p; \\ F_2(\psi) &= -\, 2\nu \nabla \psi \cdot \nabla h - \nu(\Delta \psi) h + h \partial_{x_1} \psi. \end{aligned}$$

Meanwhile, the Sobolev imbedding leads to

$$\|v\|_{L^6(\mathbb{R}^3)} + \|h\|_{L^6(\mathbb{R}^3)} \le C\left(\|\nabla v\|_{L^2(\mathbb{R}^3)} + \|\nabla h\|_{L^2(\mathbb{R}^3)}\right) < \infty,$$

and by acting the divergence operator on $(3.1)_1$, we have

$$-\,\Delta p = \sum_{i,j=1}^{3} \partial_i \partial_j (v^i v^j - h^i h^j), \tag{3.4}$$

which yields

$$p = \sum_{i,j=1}^{3} R_i R_j (v^i v^j - h^i h^j) \in L^3(\mathbb{R}^3), \tag{3.5}$$

with the Riesz transform R_j, $j = 1, 2, 3$, in $\mathbb{R}^3$ [19], under the condition that

$$p(x) \to 0, \quad \text{as } |x| \to \infty.$$

Therefore, by Hölder inequality,

$$\begin{aligned} \|F_1(\psi)\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \le{}& C(\kappa, \nu)\Big(\|\nabla \psi\|_{L^3(\mathbb{R}^3)} \|\nabla v\|_{L^2(\mathbb{R}^3)} + \|\Delta \psi\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \|v\|_{L^6(\mathbb{R}^3)} \\ &+ \|\partial_{x_1} \psi\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \|v\|_{L^6(\mathbb{R}^3)} + \|\nabla \psi\|_{L^2(\mathbb{R}^3)} \|p\|_{L^3(\mathbb{R}^3)}\Big) \\ \le{}& C(M, \kappa, \nu) < \infty. \end{aligned} \tag{3.6}$$

Similarly as (3.6), one has

$$\|F_2(\psi)\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \le C(M, \kappa, \nu) < \infty. \tag{3.7}$$

Finally, for the right hand side of (3.3)$_3$, it follows that

$$\begin{aligned}\|\nabla\psi\cdot v\|_{W^{1,\frac{6}{5}}(\mathbb{R}^3)} &\le \|\nabla\psi\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}\|v\|_{L^6(\mathbb{R}^3)} + \|\nabla^2\psi\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}\|v\|_{L^6(\mathbb{R}^3)} \\ &\quad + \|\nabla\psi\|_{L^3(\mathbb{R}^3)}\|\nabla v\|_{L^2(\mathbb{R}^3)} \\ &\le C(M) < \infty.\end{aligned} \tag{3.8}$$

Applying Lemma 2.4 to the system (3.3), using (3.6), (3.7), (3.8) and by choosing $q = 6/5$, one derives ψv, $\psi h \in L^3(\mathbb{R}^3)$. Thus the claim follows by noting

$$\begin{aligned}\int_{\mathbb{R}^3} |v|^3 dx &\le \int_{\mathbb{R}^3} |\psi v|^3 dx + \int_{B_{2M}} |v|^3 dx \\ &\le \int_{\mathbb{R}^3} |\psi v|^3 dx + \left(\int_{\mathbb{R}^3} |v|^6 dx\right)^{1/2} \left(\int_{B_{2M}} dx\right)^{1/2} < \infty,\end{aligned} \tag{3.9}$$

and the same process for h. Meanwhile, we have $p \in L^{\frac{3}{2}}(\mathbb{R}^3)$ by a similar argument of Riesz transform as (3.4) and (3.5). □

Now we introduce the cut-off function $\varphi \in C_c^\infty(\mathbb{R}^3)$ s.t.

$$\varphi(x) = \varphi(|x|) = \begin{cases} 1, & \text{if} \quad |x| < 1; \\ 0, & \text{if} \quad |x| > 2, \end{cases} \tag{3.10}$$

and $0 \le \varphi(x) \le 1$ for any $1 \le |x| \le 2$. Then for each $R > 0$, we denote

$$\varphi_R(x) := \varphi\left(\frac{|x|}{R}\right). \tag{3.11}$$

We multiply (3.1)$_1$ by $v\varphi_R$ and integrate over $\mathbb{R}^3$ to have

$$\int_{\mathbb{R}^3} v\cdot\nabla v\cdot v\varphi_R dx - \int_{\mathbb{R}^3} h\cdot\nabla h\cdot v\varphi_R dx + \int_{\mathbb{R}^3} \partial_{x_1} v\cdot v\varphi_R dx + \int_{\mathbb{R}^3} \nabla p\cdot v\varphi_R dx = \kappa\int_{\mathbb{R}^3} \Delta v\cdot v\varphi_R dx.$$

Integration by parts over $\mathbb{R}^3$ yields

$$\begin{aligned}&\kappa\int_{\mathbb{R}^3} \varphi_R|\nabla v|^2 dx \\ &\quad = \frac{\kappa}{2}\int_{\mathbb{R}^3} \Delta\varphi_R|v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} |v|^2 v\cdot\nabla\varphi_R dx + \int_{\mathbb{R}^3} pv\cdot\nabla\varphi_R dx \\ &\qquad + \frac{1}{2}\int_{\mathbb{R}^3} |v|^2\partial_{x_1}\varphi_R dx - \sum_{i,j=1}^3 \int_{\mathbb{R}^3} h_i h_j \partial_{x_i} v_j \varphi_R dx - \sum_{i,j=1}^3 \int_{\mathbb{R}^3} h_i h_j v_j \partial_{x_i}\varphi_R dx.\end{aligned} \tag{3.12}$$

Meanwhile, multiplying (3.1)$_2$ by $h\varphi_R$ and integrate over $\mathbb{R}^3$, one derive

$$\int_{\mathbb{R}^3} v\cdot\nabla h\cdot h\varphi_R dx - \int_{\mathbb{R}^3} h\cdot\nabla v\cdot h\varphi_R dx + \int_{\mathbb{R}^3} \partial_{x_1} h\cdot h\varphi_R dx = \nu\int_{\mathbb{R}^3} \Delta h\cdot h\varphi_R dx.$$

Integrating by parts indicate

$$\nu\int_{\mathbb{R}^3}\varphi_R|\nabla h|^2dx=\frac{\nu}{2}\int_{\mathbb{R}^3}|h|^2\Delta\varphi_R dx+\frac{1}{2}\int_{\mathbb{R}^3}|h|^2v\cdot\nabla\varphi_R dx+\frac{1}{2}\int_{\mathbb{R}^3}|h|^2\partial_{x_1}\varphi_R dx$$
$$+\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}h_ih_j\partial_{x_i}v_j\varphi_R dx. \tag{3.13}$$

Thus we achieve the following equation by adding (3.12) and (3.13) together

$$\begin{aligned}&\int_{\mathbb{R}^3}\varphi_R\left(\kappa|\nabla v|^2+\nu|\nabla h|^2\right)dx\\&=\underbrace{\frac{1}{2}\int_{\mathbb{R}^3}\Delta\varphi_R\left(\kappa|v|^2+\nu|h|^2\right)dx}_{I_1}+\underbrace{\frac{1}{2}\int_{\mathbb{R}^3}\left(|v|^2+|h|^2\right)\partial_{x_1}\varphi_R dx}_{I_2}\\&\quad+\underbrace{\int_{\mathbb{R}^3}pv\cdot\nabla\varphi_R dx}_{I_3}+\underbrace{\frac{1}{2}\int_{\mathbb{R}^3}\left[\left(|v|^2+|h|^2\right)v-2(h\cdot v)h\right]\cdot\nabla\varphi_R dx.}_{I_4}\end{aligned}$$

By Hölder inequality and $|\nabla^L\varphi_R|\lesssim R^{-|L|}$, we have

$$\begin{aligned}|I_1|&\le C\|\Delta\varphi_R\|_{L^\infty(\mathbb{R}^3)}\left(\int_{B_{2R}-B_R}\left(|v|^3+|h|^3\right)dx\right)^{2/3}\left(\int_{B_{2R}-B_R}dx\right)^{1/3}\\&\lesssim R^{-1}\left(\|v\|^2_{L^3(B_{2R}-B_R)}+\|h\|^2_{L^3(B_{2R}-B_R)}\right);\\|I_2|&\le C\|\nabla\varphi_R\|_{L^\infty(\mathbb{R}^3)}\left(\int_{B_{2R}-B_R}\left(|v|^3+|h|^3\right)dx\right)^{2/3}\left(\int_{B_{2R}-B_R}dx\right)^{1/3}\\&\lesssim\|v\|^2_{L^3(B_{2R}-B_R)}+\|h\|^2_{L^3(B_{2R}-B_R)};\\|I_3|&\le C\|\nabla\varphi_R\|_{L^\infty(\mathbb{R}^3)}\left(\int_{B_{2R}-B_R}|p|^{\frac{3}{2}}dx\right)^{2/3}\left(\int_{B_{2R}-B_R}|v|^3dx\right)^{1/3}\\&\lesssim R^{-1}\|p\|_{L^{\frac{3}{2}}(B_{2R}-B_R)}\|v\|_{L^3(B_{2R}-B_R)};\\|I_4|&\le C\|\nabla\varphi_R\|_{L^\infty(\mathbb{R}^3)}\int_{B_{2R}-B_R}\left(|v|^3+|h|^2|v|\right)dx\\&\lesssim R^{-1}\left(\|v\|^3_{L^3(B_{2R}-B_R)}+\|h\|^3_{L^3(B_{2R}-B_R)}\right).\end{aligned}$$

Therefore we find

$$\lim_{R\to\infty}I_j=0,\quad\forall j=1,2,3,4.$$

This results in $(v,h)\equiv0$ and so that $(u,h)\equiv(u_\infty,0)$. This completes the proof of Case (i). To prove Case (ii), we first remark that:

Remark 3.1. Let us explain here that why our method in the proof of Case (i) fails when trying to solve Case (ii) without the condition $\kappa=\nu$, which seems to be not physical. By applying the orthogonal transform and scaling invariant properties of the MHD system, we may assume

$$h_\infty=(-1,0,0)^T$$

without loss of generality. Denoting $\eta := h - h_\infty$, one derive

$$\begin{cases} u\cdot\nabla u + \partial_{x_1}\eta + \nabla p - \eta\cdot\nabla\eta - \kappa\Delta u = 0, \\ u\cdot\nabla\eta + \partial_{x_1}u - \eta\cdot\nabla u - \nu\Delta\eta = 0, \\ \nabla\cdot u = 0, \quad \nabla\cdot\eta = 0, \end{cases} \quad \text{in} \quad \mathbb{R}^3. \tag{3.14}$$

Compared to (3.1), the only difference happens on the "$\partial_{x_1}-$terms" on the first 2 lines of (3.3). However, this only difference results in the linear part of the equations of u and η totally mixed each other, which ends up with the failure of the argument in Proposition 2.2 in solving the related linear system of (3.3). □

Nevertheless, under the condition $\kappa = \nu$ (in the following we suppose $\kappa = \nu = 1$ without loss of generality), we still have the Liouville-type theorem. We give the sketch of the proof in the following since the main idea is identical with that in Case (i).

Sketch of the proof: We denote

$$w_1 = u + \eta; \quad w_2 = u - \eta,$$

we see w_1 and w_2 satisfy the following system

$$\begin{cases} w_2\cdot\nabla w_1 + \partial_{x_1}w_1 + \nabla p - \Delta w_1 = 0, \\ w_1\cdot\nabla w_2 - \partial_{x_1}w_2 + \nabla p - \Delta w_2 = 0, \\ \nabla\cdot w_1 = 0, \quad \nabla\cdot w_2 = 0, \\ \int_{\mathbb{R}^3}|\nabla w_1|^2dx < +\infty, \quad \int_{\mathbb{R}^3}|\nabla w_2|^2dx < +\infty, \end{cases} \quad \text{in} \quad \mathbb{R}^3. \tag{3.15}$$

Multiplying the cut-off function ψ in (3.2) on the both sides of (3.15), it follows that

$$\begin{cases} \partial_{x_1}\begin{pmatrix}\psi w_1 \\ -\psi w_2\end{pmatrix} - \Delta\begin{pmatrix}\psi w_1 \\ \psi w_2\end{pmatrix} + \begin{pmatrix} 0 & (\nabla w_1)\chi_{B_M^c} \\ (\nabla w_2)\chi_{B_M^c} & 0\end{pmatrix}\cdot\begin{pmatrix}\psi w_1 \\ \psi w_2\end{pmatrix} + \begin{pmatrix}\nabla(\psi p) \\ \nabla(\psi p)\end{pmatrix} \\ \qquad = \begin{pmatrix} G_1(\psi) \\ G_2(\psi)\end{pmatrix}, \\ \nabla\cdot(\psi w_1) = \nabla\psi\cdot w_1, \end{cases} \tag{3.16}$$

where

$$G_1(\psi) = -\,2\nabla\psi\cdot\nabla w_1 - (\Delta\psi)w_1 + w_1\partial_{x_1}\psi + (\nabla\psi)p;$$
$$G_2(\psi) = -\,2\nabla\psi\cdot\nabla w_2 - (\Delta\psi)w_2 - w_2\partial_{x_1}\psi + (\nabla\psi)p.$$

The same as that shown in (3.6), (3.7) and (3.8), we have

$$\|G_1(\psi)\|_{L^{6/5}(\mathbb{R}^3)} + \|G_1(\psi)\|_{L^{6/5}(\mathbb{R}^3)} + \|\nabla\psi\cdot w_1\|_{W^{1,6/5}(\mathbb{R}^3)} < +\infty.$$

Applying the conclusion in Lemma 2.5 to the system (3.16) and choosing $q = 6/5$, we derive that ψw_1, $\psi w_2 \in L^3(\mathbb{R}^3)$. Then it is easy to see

$$\|w_1\|_{L^3(\mathbb{R}^3)} + \|w_2\|_{L^3(\mathbb{R}^3)} < +\infty$$

by the process in (3.9). Then testing the first equation of system (3.15) with $\varphi_R w_1$ and the second with $\varphi_R w_2$, where the cut-off function φ_R is defined around (3.10) and (3.11). The same as before for proving trivialness of v, h

in system (3.1), we can achieve that $w_1 = w_2 = 0$, which indicates that $(u, h) \equiv (0, h_\infty)$. □

Acknowledgements

Z. Li is supported by Natural Science Foundation of Jiangsu Province (No. BK20200803), National Natural Science Foundation of China (No. 12001285) and the Startup Foundation for Introducing Talent of NUIST (No. 2019r033). X. Pan is supported by Natural Science Foundation of Jiangsu Province (No. BK20180414) and National Natural Science Foundation of China (No. 11801268).

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References

[1] Babenko, K.I.: On the stationary solutions of the problem of flow past a body of a viscous incompressible fluid. Mat. Sbornik **91**, 3–26 (1973); English Transl.: Math. USSR Sbornik **20**(1), 1–25 (1973)

[2] Carrillo, B., Pan, X., Zhang, Q.S., Zhao, N.: Decay and vanishing of some D-solutions of the Navier–Stokes equations. Arch. Ration. Mech. Anal. **237**(3), 1383–1419 (2020)

[3] Chae, D.: Liouville-type theorem for the forced Euler equations and the Navier–Stokes equations. Commun. Math. Phys. **326**, 37–48 (2014)

[4] Chae, D., Wolf, J.: On Liouville type theorem for the stationary Navier–Stokes equations. Calc. Var. Partial Differ. Equ. **58**(3), Art. 111, 11 pp (2019)

[5] Chae, D., Weng, S.: Liouville type theorems for the steady axially symmetric Navier–Stokes and magnetohydrodynamic equations. Discrete Contin. Dyn. Syst. **36**(10), 5267–5285 (2016)

[6] Farwig, R., Sohr, H.: Weighted estimates for the Oseen equations and the Navier–Stokes equations in exterior domains. In: Theory of the Navier–Stokes equations. Ser. Adv. Math. Appl. Sci., vol. 47, pp. 11–30. World Sci. Publ., River Edge (1998)

[7] Finn, R.: On the steady-state solutions of the Navier–Stokes equations, III. Acta Math. **105**(3–4), 197–244 (1961)

[8] Galdi, G.P.: An Introducion to the Mathematical Theory of the Navier–Stokes Equations. Springer, Berlin (2011)

[9] Gilbarg, D., Weinberger, H.F.: Asymptotic properties of steady plane solutions of the Navier–Stokes equations with bounded Dirichlet integral. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **5**(2), 381–404 (1978)

[10] Kozono, H., Terasawa, Y., Wakasugi, Y.: A remark on Liouville-type theorems for the stationary Navier–Stokes equations in three space dimensions. J. Funct. Anal. **272**(2), 804–818 (2017)

[11] Ladyzhenskaya, O.A.: Matematicheskie problemy dinamiki vzyakoi neszhimaemoi zhidkosti, Fizmatgiz, Moscow, 1961, 2nd edn, Nauka, Moscow, 1970. Translation: The Mathematical Theory of Viscous Imcompressible Flow. Gordon and Nreach, New York (1969)

[12] Laudau, L.D., Lifshitz, E.M.: Electrodynamics of Continuous Media, 2nd edn. Pergamon, New York (1984)

[13] Leray, J.: Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique. J. Math. Pures Appl. **12**, 1–82 (1933)

[14] Li, Z., Pan, X.: On the vanishing of some D-solutions to the stationary magnetohydrodynamics system. J. Math. Fluid Mech. **21**(52), 1–13 (2019)

[15] Lizorkin, P.I.: (L_p, L_q)-multipliers of Fourier integrals. Soviet Math. Dokl. **4**, 1420–1424 (1963)

[16] Marcinkiewicz, J.: Sur les multiplicateurs des séries de Fourier. Stud. Math. **8**, 78–91 (1939)

[17] Mikhlin, S.G.: Fourier integrals and multiple singular integrals. Vestn. Leningrad. Univ. Ser. Mat. Meh. Astron. **12**, 143–155 (1957)

[18] Pan, X., Li, Z.: Liouville theorem of axially symmetric Navier–Stokes equations with growing velocity at infinity. Nonlinear Anal. Real World Appl. **56**, 103159 (2020)

[19] Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton. ISBN 0-691-08079-8 (1970)

[20] Seregin, G.: Liouville type theorem for stationary Navier–Stokes equations. Nonlinearity **29**(8), 2191–2195 (2016)

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Received: 28 November 2019.
Accepted: 13 January 2021.