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The incompressible limit for compressible MHD equations in L^p type critical spaces

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ABSTRACT

In this paper, the global well-posedness and the low Mach number limit for compressible viscous magnetohydrodynamic equations *in the critical* L^p*-type Besov space* are considered. More precisely, we will show that, in the isentropic case, the solution of compressible magnetohydrodynamic equations will converge to that of incompressible magnetohydrodynamic equations when the Mach number tends to zero in the critical L^p framework. Moreover, the convergence rate will be obtained.

1. Introduction

We consider the incompressible limit of the following multi-dimensional compressible magnetohydrodynamic equations in critical Besov spaces (see [27,28]):

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u\otimes u) - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u + \nabla P = (\nabla\times H)\times H, \\ \partial_t H - \nabla\times(u\times H) = -\nabla\times(\alpha\nabla\times H), \qquad \nabla\cdot H = 0, \end{cases} \tag{1.1}$$

where $\rho\in\mathbb{R}_+$ is the density, $u\in\mathbb{R}^n$ stands for the velocity and $H\in\mathbb{R}^n$ is the magnetic field. The pressure P is an increasing and convex function of ρ for $\rho>0$. The viscosity coefficients μ, λ are two constants satisfying $\mu>0$, $\nu\triangleq\lambda+2\mu>0$. The constant $\alpha>0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field.

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Introducing the viscosity operator

$$\mathcal{A} \triangleq \mu\Delta + (\lambda+\mu)\nabla\mathrm{div},$$

and using the identities

$$(\nabla \times H) \times H = (H\cdot\nabla)H - \frac{1}{2}\nabla(|H|^2),$$

$$\nabla\times(\nabla\times H) = \nabla(\nabla\cdot H) - \Delta H,$$

$$\nabla\times(u\times H) = u(\nabla\cdot H) - H(\nabla\cdot u) + H\cdot\nabla u - u\cdot\nabla H.$$

We can rewrite (1.1) into the following equations.

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u\otimes u) - \mathcal{A}u + \nabla P = H\cdot\nabla H - \frac{1}{2}\nabla(|H|^2), \\ \partial_t H + u\cdot\nabla H - H\cdot\nabla u - \alpha\Delta H = -H(\nabla\cdot u), \qquad \nabla\cdot H = 0. \end{cases} \tag{1.2}$$

It is a common sense that slightly compressible flows should not differ much from incompressible flows. In fact, the incompressible MHD equations read

$$\begin{cases} \partial_t v + v\cdot\nabla v - \mu\Delta v + \nabla\pi = B\cdot\nabla B, \\ \partial_t B + v\cdot\nabla B - B\cdot\nabla v - \alpha\Delta B = 0, \\ \nabla\cdot v = \nabla\cdot B = 0, \end{cases} \tag{1.3}$$

which should be considered to be relevant for describing the compressible MHD fluids in the low Mach number regime.

This may be justified formally by rescaling the time variable by $t^\varepsilon = \varepsilon t$($\varepsilon$ denotes the Mach number) and performing the change of unknown

$$\rho(t,x) = \rho^\varepsilon(x,t^\varepsilon), \quad u(t.x) = \varepsilon u^\varepsilon(t^\varepsilon,x), \quad H(t,x) = \varepsilon H^\varepsilon(t^\varepsilon,x).$$

Let $\lambda = \varepsilon\lambda^\varepsilon,\ \mu = \varepsilon\mu^\varepsilon,\ \alpha = \varepsilon\alpha^\varepsilon$, then the system for $(\rho^\varepsilon, u^\varepsilon, H^\varepsilon)$ satisfies

$$\begin{cases} \partial_t\rho^\varepsilon + \mathrm{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \mathrm{div}(\rho^\varepsilon u^\varepsilon\otimes u^\varepsilon) - \mu^\varepsilon\Delta u^\varepsilon - (\lambda^\varepsilon+\mu^\varepsilon)\nabla\mathrm{div}u^\varepsilon + \frac{\nabla P^\varepsilon}{\varepsilon^2} \\ \qquad\qquad = H^\varepsilon\cdot\nabla H^\varepsilon - \frac{1}{2}\nabla(|H^\varepsilon|^2), \\ \partial_t H^\varepsilon + u^\varepsilon\cdot\nabla H^\varepsilon - H^\varepsilon\cdot\nabla u^\varepsilon - \alpha_\varepsilon\Delta H^\varepsilon = -H^\varepsilon(\nabla\cdot u^\varepsilon), \qquad \nabla\cdot H^\varepsilon = 0, \end{cases} \tag{1.4}$$

where $P^\varepsilon \triangleq P(\rho^\varepsilon)$. For simplicity of presentation, we assume that $\lambda^\varepsilon, \mu^\varepsilon, \alpha^\varepsilon$, are all constants, independent of ε, and still denote them as λ, μ, α with a little abuse of notations.

Formally, it is clear that if $(\rho^\varepsilon, u^\varepsilon, H^\varepsilon)$ tends to some function (ρ, v, B), then we have $\nabla P^\varepsilon \to 0$ when ε goes to 0. Hence, if P' does not vanish, the limit density has to be a constant. Taking the limit in the mass equation $(1.4)_1$ implies the limit v is divergence-free. Returning to Eqs. $(1.4)_2$ and $(1.4)_3$, we can conclude that (v, B) must satisfy (1.3) for some π.

The heuristics has already been justified rigorously in different contexts. Hu–Wang [23] proved convergence of weak solutions of the compressible MHD equations to that of the incompressible MHD equations. Jiang–Ju–Wang obtained convergence of weak solutions for the compressible MHD equations to the strong solutions of the ideal incompressible MHD equations in the whole space in [26] or to the strong solution of the viscous incompressible MHD equations in torus in [25]. Li [29] studied the incompressible limit of the viscous isentropic compressible MHD equations for local smooth solutions with well-prepared data. Readers can also see [18,21] and references therein for more discussions and related results about this topic.

We remark that all the above results were carried out in the framework of Sobolev spaces. Note that system (1.2) is scaling invariant under the following transformation:

$$\begin{aligned}&(\rho_0(x), u_0(x); u_0(x)) \to (\rho_0(\lambda x), \lambda u_0(\lambda x); \lambda H_0(\lambda x)),\\&(\rho(t,x), u(t,x); H(t,x)) \to (\rho(\lambda^2 t, \lambda x), \lambda u(\lambda^2 t, \lambda x); \lambda H(\lambda^2 t, \lambda x)),\end{aligned}$$

up to a change of the Pressure P into $\lambda^2 P$. This indicates the following definition of the critical space.

Definition 1.1. A functional space is called a critical space if the associated norm is invariant under the transformation $(\rho(t,x), u(t,x); H(t,x)) \to (\rho(\lambda^2 t, \lambda x), \lambda u(\lambda^2 t, \lambda x); \lambda H(\lambda^2 t, \lambda x))$ (up to a constant independent of λ).

We strive to study the incompressible limit for critical regularity assumption of system (1.2) consistent with those of the well-posedness for the system (1.3). Mu [36] studied the incompressible limit for small-data global existence and Li–Mu–Wang [30] obtained the incompressible limit for large-data local existence. Both of their works are restricted in L^2 critical Besov spaces. In this paper we focus on the case of ill prepared data of the form $(\rho_0^\varepsilon = 1 + \varepsilon a_0^\varepsilon, u_0^\varepsilon, H_0^\varepsilon)$ so that acoustic waves have to be considered, where $(a_0^\varepsilon, u_0^\varepsilon, H_0^\varepsilon)$ are bounded in L^p critical Besov spaces. Without loss of generality, we assume that $P'(1) = 1$.

Set $\rho^\varepsilon = 1 + \varepsilon a^\varepsilon$, then $(a^\varepsilon, u^\varepsilon, H^\varepsilon)$ satisfies

$$\begin{cases}\partial_t a^\varepsilon + \dfrac{\mathrm{div} u^\varepsilon}{\varepsilon} = -\mathrm{div}(a^\varepsilon u^\varepsilon),\\ \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \mathcal{A} u^\varepsilon + \dfrac{\nabla a^\varepsilon}{\varepsilon}\\ \qquad\qquad = -K(\varepsilon a^\varepsilon)\dfrac{\nabla a^\varepsilon}{\varepsilon} - I(\varepsilon a^\varepsilon)\mathcal{A} u^\varepsilon + \dfrac{1}{1+\varepsilon a^\varepsilon}\big((\nabla \times H^\varepsilon) \times H^\varepsilon\big),\\ \partial_t H^\varepsilon + u^\varepsilon \cdot \nabla H^\varepsilon - H^\varepsilon \cdot \nabla u^\varepsilon - \alpha \Delta H^\varepsilon = -H^\varepsilon(\nabla \cdot u^\varepsilon), \qquad \nabla \cdot H^\varepsilon = 0,\end{cases} \tag{1.5}$$

where $I(\varepsilon a^\varepsilon) = \frac{\varepsilon a^\varepsilon}{1+\varepsilon a^\varepsilon}$ and $K(\varepsilon a^\varepsilon) = \frac{P'(1+\varepsilon a^\varepsilon)}{1+\varepsilon a^\varepsilon} - 1$.

In order to be more specific, let us pause for a while and introduce the notation and function spaces that will be used throughout the paper. We will denote a generic constant by C which may be different from line to line and denote $A \le CB$ by $A \lesssim B$. The notation $A \approx B$ means $A \le CB$ and $B \le CA$.

Littlewood–Paley theory and Besov spaces First, we introduce the Littlewood–Paley decomposition. There exist two radial smooth functions $\varphi(x), \chi(x)$ supported in the annulus $\mathcal{C} = \{\xi \in \mathbb{R}^n : 3/4 \le |\xi| \le 8/3\}$ and the ball $B = \{\xi \in \mathbb{R}^n : |\xi| \le 4/3\}$, respectively such that

$$\chi(\xi) + \sum_{j\ge 0} \varphi(2^{-j}\xi) = 1 \quad \forall \xi \in \mathbb{R}^n.$$

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \forall \xi \in \mathbb{R}^n \diagdown \{0\}.$$

The homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low-frequency cut-off operators $\dot{S}_j$ are defined for all $j \in \mathbb{Z}$ by

$$\dot{\Delta}_j f = \varphi(2^{-j}D)f, \quad \dot{S}_j f = \sum_{k\le j-1} \dot{\Delta}_k f = \chi(2^{-j}D)f.$$

With our choice of φ, it is easy to see that

$$\begin{aligned} &\dot{\Delta}_j \dot{\Delta}_k f = 0 \quad \text{if } |j-k| \geq 2, \\ &\dot{\Delta}_j (\dot{S}_{k-1} f \dot{\Delta}_k f) = 0 \quad \text{if } |j-k| \geq 5. \end{aligned} \tag{1.6}$$

The next Bernstein-Type inequality will be repeatedly used through the paper.

Lemma 1.1 (*Lemma 2.1 of [1]*). *Let $\mathcal{C}$ be an annulus and B a ball. A constant C exists such that for any nonnegative integer k, any couple (p,q) in $[1,\infty]^2$ with $q \geq p \geq 1$, and any function u of L^p, we have*

$$Supp\hat{u} \subset \lambda B \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1} \lambda^{k+n(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p},$$

$$Supp\hat{u} \subset \lambda \mathcal{C} \Rightarrow C^{-k-1} \lambda^k \|u\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^p} \leq C^{k+1} \lambda^k \|u\|_{L^p}.$$

Denote $\mathcal{Z}'(\mathbb{R}^n)$ by the dual space of

$$\mathcal{Z}(\mathbb{R}^n) \triangleq \{f \in \mathcal{S}(\mathbb{R}^n) : \partial^\alpha \hat{f}(0) = 0, \forall \alpha \in (\mathbb{N} \cup 0)^n\}.$$

Then the definition of the homogeneous Besov space is the following.

Definition 1.2. Let s be a real number and (p,r) be in $[1,\infty]^2$. The homogeneous Besov space $\dot{B}^s_{p,r}$ consists of those distributions $u \in \mathcal{Z}'(\mathbb{R}^n)$ such that

$$\|u\|_{\dot{B}^s_{p,r}} \triangleq \Big(\sum_{j \in \mathbb{Z}} 2^{jsr} \|\dot{\Delta}_j u\|^r_{L^p}\Big)^{\frac{1}{r}} < \infty.$$

Since we work with time-dependent functions valued in Besov spaces, we introduce the norms

$$\|u\|_{L^q_T \dot{B}^s_{p,r}} := \big\| \|u(t,\cdot)\|_{\dot{B}^s_{p,r}} \big\|_{L^q(0,T)}.$$

Also, when performing the parabolic estimate, it is natural to use the following quantity

$$\|u\|_{\tilde{L}^q_T \dot{B}^s_{p,r}} \triangleq \Big(\sum_{j \in \mathbb{Z}} 2^{jsr} \|\dot{\Delta}_j u\|^r_{L^q_T(L^p)}\Big)^{\frac{1}{r}}.$$

The index T will be omitted if $T = +\infty$ and we shall denote by $\tilde{\mathcal{C}}_b(\dot{B}^s_{p,r})$ the subset of functions $\tilde{L}^\infty(\dot{B}^s_{p,r})$ which are continuous from $\mathbb{R}_+$ to $\dot{B}^s_{p,r}$.

An important estimate for the heat equation in Besov spaces is expressed in the following

Lemma 1.2. *Let $p,q,r \in [1,\infty]$, $s \in \mathbb{R}$. Assume that $u_0 \in \dot{B}^{s-1}_{p,r}$, $f \in \tilde{L}^1_T \dot{B}^{s-1}_{p,r}$. Let u be a solution of the equation*

$$\partial_t u - \mu \Delta u = f, \quad u|_{t=0} = u_0. \tag{1.7}$$

Then for $t \in [0,T]$, there holds

$$\|u\|_{\tilde{L}^q_T \dot{B}^{s-1+2/q}_{p,r}} \leq C\big(\|u_0\|_{\dot{B}^{s-1}_{p,r}} + \|f\|_{\tilde{L}^1_T \dot{B}^{s-1}_{p,r}}\big). \tag{1.8}$$

Moreover, $u \in \mathcal{C}((0,T]; \dot{B}^{s-1}_{p,r})$ if $r < \infty$. Readers can see [1] and [3] for its proof.

Restricting to the case of small-data global solutions, the corresponding well-posedness result for (1.3) reads as follows:

Theorem 1.1. *Let* $(v_0, B_0) \in \dot{B}^{n/p-1}_{p,1}$ *with* $\nabla \cdot v_0 = \nabla \cdot B_0 = 0$, $p < +\infty$. *Then there exists a* $\eta \sim \eta(\mu, \alpha)$ *such that when*

$$\|(v_0, B_0)\|_{\dot{B}^{n/p-1}_{p,1}} \leq \eta,$$

then system(1.3) has a unique global solution $(v, B) \in \tilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}^{\frac{n}{p}-1}_{p,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{\frac{n}{p}+1}_{p,1})$ *and satisfies*

$$\|(v, B)\|_{\tilde{L}^\infty \dot{B}^{\frac{n}{p}-1}_{p,1} \cap L^1 \dot{B}^{\frac{n}{p}+1}_{p,1}} \leq C\|(v_0, B_0)\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}},$$

where the constant C *depends only on* μ, α, n *and* p.

Remark 1.1. Since we cannot find the proof of Theorem 1.1 directly in the literature, we will give it in Appendix B briefly.

We can see that Theorem 1.1 is not related to energy arguments, but to our knowledge, all the present results about proving the incompressible limit of (1.2) to (1.3) strongly rely on the L^2 type norms estimates in order to get rid of the dependence on ε. This is due to the presence of the singular first skew symmetric terms(which disappear when performing L^2 estimates) in the following linearized equations of (1.5)

$$\begin{cases} \partial_t a^\varepsilon + \dfrac{\mathrm{div} u^\varepsilon}{\varepsilon} = f^\varepsilon, \\ \partial_t u^\varepsilon - \mathcal{A} u^\varepsilon + \dfrac{\nabla a^\varepsilon}{\varepsilon} = g^\varepsilon. \end{cases} \tag{1.9}$$

However, the singular terms do not affect the divergence-free part $\mathcal{P}u^\varepsilon$ of the velocity and the magnetic field H^ε, for which just satisfy the heat equation (1.7). Thus we can expect to deal with $\mathcal{P}u^\varepsilon$ and H^ε by means of a L^p type approach the same as that in Theorem 1.1. At the same time, for low frequencies, the singular terms dominate the evolution of a^ε and of the potential $\mathcal{P}^\perp u^\varepsilon$ of the velocity, which prevent us to use a $L^p(p \neq 2)$ type approach and restrict us to handle it only in L^2 type spaces as the wave equation is ill-posed in the $L^p(p \neq 2)$ type space. For the high frequencies, we will see that a^ε and $\mathcal{P}^\perp u^\varepsilon$ tend to behave like the solution of a damped equation and of a heat equation, respectively, and are tractable in L^p type spaces.

Now let us recall some results related on the magnetohydrodynamics equations and on the Navier–Stokes equations(i.e. $H = 0$ in system (1.1)). For MHD equations, Suen–Hoff [38] established the global weak solutions for small data, which was extended to the case where the initial data may contain large oscillation or vacuum in [32,35]. Hu–Wang [24] obtained the global existence and large-time behavior of general weak solutions with finite energy in the framework of [19,20,34]. Li–Su–Wang [31] obtained the local strong solution with large initial data. Suen [37] established some blow-up criteria for (1.1). Li–Yu [33] obtained the optimal decay rate of smooth solutions near equilibrium. In [14,17], The authors analyze a method for approximation of weak solutions to incompressible MHD equations in unbounded domains and justify the convergence of smooth solutions of the Navier–Stokes–Maxwell equations towards smooth solutions of the classical 2D parabolic MHD equations in the case of vanishing dielectric constant. For the Navier–Stokes equations, Danchin [6] proved the global well-posedness of isentropic Navier–Stokes equations in the critical Besov spaces near equilibrium. Later, this result was extended to more general Besov spaces in [2,5,22]. In a series of papers by Danchin [7,10,11], the local well-posedness of solutions to the Navier–Stokes equations with large data was obtained. Recently an optimal decay rate of the solution in critical Besov space was obtained by Xu–Danchin [13]. In [9,8], the zero Mach number limit of the isentropic Navier–Stokes equation was studied in the whole space or torus in the L^2 critical Besov Space. In [15], the authors study incompressible limits for the Navier–Stokes system on unbounded domains under slip boundary conditions. Recently, Danchin–He [12] extended the result to the full Navier–Stokes–Fourier system in the L^p critical Besov space. Readers can also see [16] for the study of low Mach number limits for the quantum-hydrodynamics system.

Our paper is arranged as follows. In Section 2, we give the main result. In Section 3, the global existence of solutions to system (1.5) is established. While Section 4 is devoted to proving the zero mach limit of system (1.2) to system (1.3). Some useful estimates in Besov spaces will be given in Appendix A. Finally, in Appendix B, we will briefly give the proof of Theorem 1.1.

2. Main results

First, we introduce some notations. For $z \in \mathcal{S}'(\mathbb{R}^n)$, the low frequency and high frequency parts of z with respect to a parameter ε are defined as

$$z^{\ell,\varepsilon} := \sum_{2^j\varepsilon\le R_0} \dot{\Delta}_j z \quad \text{and} \quad z^{h,\varepsilon} := \sum_{2^j\varepsilon> R_0} \dot{\Delta}_j z,$$

where R_0 is a sufficiently large constant depending on $n,\ \lambda,\ \mu$ and α. The corresponding semi-norms are

$$\|z\|^{\ell,\varepsilon}_{\dot{B}^s_{p,r}} := \Big(\sum_{2^j\varepsilon\le R_0} 2^{jsr}\|\dot{\Delta}_j z\|^r_{L^p}\Big)^{\frac{1}{r}} \quad \text{and} \quad \|z\|^{h,\varepsilon}_{\dot{B}^s_{p,r}} := \Big(\sum_{2^j\varepsilon> R_0} \|\dot{\Delta}_j z\|^r_{L^p}\Big)^{\frac{1}{r}}.$$

We consider a family of initial data $(a_0^\varepsilon, u_0^\varepsilon, H_0^\varepsilon)$ so that
- $(a_0^\varepsilon, \mathcal{P}^\perp u_0^\varepsilon)^{\ell,\varepsilon} \in \dot{B}^{n/2-1}_{2,1}$,
- $(a_0^\varepsilon)^{h,\varepsilon} \in \dot{B}^{n/p}_{p,1}, (\mathcal{P}^\perp u_0^\varepsilon)^{h,\varepsilon} \in \dot{B}^{n/p-1}_{p,1}$,
- $(\mathcal{P}u_0^\varepsilon, H_0^\varepsilon) \in \dot{B}^{n/p-1}_{p,1}$.

Our assumptions on the data induce us to look for a solution $(a^\varepsilon, u^\varepsilon, H^\varepsilon)$ of (1.5) in the following space X_p^ε
- $(a^\varepsilon, \mathcal{P}^\perp u^\varepsilon)^{\ell,\varepsilon} \in \widetilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}^{n/2-1}_{2,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{n/2+1}_{2,1})$,
- $(a^\varepsilon)^{h,\varepsilon} \in \widetilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}^{n/p}_{p,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{n/p}_{p,1})$, $(\mathcal{P}^\perp u^\varepsilon)^{h,\varepsilon} \in \widetilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{n/p+1}_{p,1})$,
- $(\mathcal{P}u^\varepsilon, H^\varepsilon) \in \widetilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{n/p+1}_{p,1})$,

which is endowed with the norm

$$\begin{aligned} \|(a^\varepsilon, u^\varepsilon, H^\varepsilon)\|_{X_p^\varepsilon} := {} & \|(a^\varepsilon, \mathcal{P}^\perp u^\varepsilon)\|^{\ell,\varepsilon}_{\tilde{L}^\infty \dot{B}^{n/2-1}_{2,1} \cap L^1 \dot{B}^{n/2+1}_{2,1}} \\ & + \varepsilon\|a^\varepsilon\|^{h,\varepsilon}_{\tilde{L}^\infty \dot{B}^{n/p}_{p,1}} + \varepsilon^{-1}\|a^\varepsilon\|^{h,\varepsilon}_{L^1 \dot{B}^{n/p}_{p,1}} \\ & + \|\mathcal{P}^\perp u^\varepsilon\|^{h,\varepsilon}_{\tilde{L}^\infty \dot{B}^{n/p-1}_{p,1} \cap L^1 \dot{B}^{n/p+1}_{p,1}} \\ & + \|(\mathcal{P}u^\varepsilon, H^\varepsilon)\|_{\tilde{L}^\infty \dot{B}^{n/p-1}_{p,1} \cap L^1 \dot{B}^{n/p+1}_{p,1}}. \end{aligned} \tag{2.1}$$

We state our main result as follows

Theorem 2.1. *Let $n \ge 2$ and p satisfies $2 \le p \le \min\big(4, 2n/(n-2)\big)$ and $p \ne 4$ if $n = 2$. Assume that the initial data $(a_0^\varepsilon, u_0^\varepsilon, H_0^\varepsilon)$ are as above. Then there exist two positive constants, η and M, depending only on $n,\ \lambda,\ \mu,\ \alpha$ and the function K, such that if*

$$C_0^\varepsilon := \|(a_0^\varepsilon, \mathcal{P}^\perp u_0^\varepsilon)\|^{\ell,\varepsilon}_{\dot{B}^{n/2-1}_{2,1}} + \varepsilon\|a_0^\varepsilon\|^{h,\varepsilon}_{\dot{B}^{n/p}_{p,1}} + \|\mathcal{P}^\perp u_0^\varepsilon\|^{h,\varepsilon}_{\dot{B}^{n/p-1}_{p,1}} + \|(\mathcal{P}u_0^\varepsilon, H_0^\varepsilon)\|_{\dot{B}^{n/p-1}_{p,1}} \le \eta,$$

then the system (1.5) has a unique global solution $(a^\varepsilon, u^\varepsilon, H^\varepsilon)$ in X_p^ε such that

$$\|(a^\varepsilon, u^\varepsilon, H^\varepsilon)\|_{X_p^\varepsilon} \le MC_0^\varepsilon. \tag{2.2}$$

In addition, $(a^\varepsilon, \mathcal{P}^\perp u^\varepsilon)$ *converges to* 0 *when* ε *goes to zero and* $(\mathcal{P}u^\varepsilon, H^\varepsilon)$ *converges to the solution of system* (1.3) *in the following function spaces if* $(\mathcal{P}u_0^\varepsilon, H_0^\varepsilon) \to (v_0, B_0)$ *in the corresponding spaces.*

Case $n \geq 3$*:*

$$\begin{aligned}
&\|(a^\varepsilon, \mathcal{P}^\perp u^\varepsilon)\|_{\tilde{L}^2 \dot{B}_{p,1}^{\frac{n+1}{p}-\frac{1}{2}}} \leq M C_0^\varepsilon \varepsilon^{\frac{1}{2}-\frac{1}{p}}\\
&\|(\mathcal{P}u^\varepsilon - v, H^\varepsilon - B)\|_{\tilde{L}^\infty \dot{B}_{p,1}^{\frac{n+1}{p}-\frac{3}{2}}} + \|(\mathcal{P}u^\varepsilon - v, H^\varepsilon - B)\|_{L^1 \dot{B}_{p,1}^{\frac{n+1}{p}+\frac{1}{2}}}\\
\leq{}& M\Big(\|(\mathcal{P}u_0^\varepsilon - v_0, H_0^\varepsilon - B_0)\|_{\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{3}{2}}} + C_0^\varepsilon \varepsilon^{\frac{1}{2}-\frac{1}{p}}\Big).
\end{aligned} \tag{2.3}$$

Case $n = 2$*:*

$$\begin{aligned}
&\|(a^\varepsilon, \mathcal{P}^\perp u^\varepsilon)\|_{\tilde{L}^2 \dot{B}_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}}} \leq M C_0^\varepsilon \varepsilon^{\kappa(\frac{1}{2}-\frac{1}{p})}\\
&\|(\mathcal{P}u^\varepsilon - v, H^\varepsilon - B)\|_{\tilde{L}^\infty \dot{B}_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}-1}} + \|(\mathcal{P}u^\varepsilon - v, H^\varepsilon - B)\|_{L^1 \dot{B}_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}+1}}\\
\leq{}& M\Big(\|(\mathcal{P}u_0^\varepsilon - v_0, H_0^\varepsilon - B_0)\|_{\dot{B}_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}-1}} + C_0^\varepsilon \varepsilon^{\kappa(\frac{1}{2}-\frac{1}{p})}\Big),
\end{aligned} \tag{2.4}$$

where the constant κ *satisfies* $0 \leq \kappa \leq \frac{1}{2}$ *and* $\kappa < \frac{8-2p}{p-2}$.

3. Global existence for fixed ε in system (1.5)

Making change of unknowns

$$(a, u, H)(t, x) := \varepsilon(a^\varepsilon, u^\varepsilon, H^\varepsilon)(\varepsilon^2 t, \varepsilon x) \tag{3.1}$$

and the change of data

$$(a_0, u_0, H_0)(x) := \varepsilon(a_0^\varepsilon, u_0^\varepsilon, H_0^\varepsilon)(\varepsilon x). \tag{3.2}$$

Then we note that $(a^\varepsilon, u^\varepsilon, H^\varepsilon)$ solves (1.5) if and only if (a, u, H) solves the following system

$$\begin{cases}
\partial_t a + \nabla \cdot u = -\nabla \cdot (au),\\
\partial_t u + u \cdot \nabla u - \mathcal{A}u + \nabla a\\
\qquad\qquad = -K(a)\nabla a - I(a)\mathcal{A}u + \dfrac{1}{1+a}\big(((\nabla \times H) \times H)\big),\\
\partial_t H + u \cdot \nabla H - H \cdot \nabla u - \alpha \Delta H = -H(\nabla \cdot u), \qquad \nabla \cdot H = 0,
\end{cases} \tag{3.3}$$

where $I(a) = \frac{a}{1+a}$ and $K(a) = \frac{P'(1+a)}{1+a} - 1$.

For simplicity, we denote

$$z^\ell := z^{\ell,1} \quad \text{and} \quad z^h = z^{h,1},$$

$$\|z\|^\ell_{\dot{B}^s_{p,r}} := \Big(\sum_{2^j \leq R_0} 2^{jsr} \|\dot{\Delta}_j z\|^r_{L^p}\Big)^{\frac{1}{r}} \quad \text{and} \quad \|z\|^h_{\dot{B}^s_{p,r}} := \Big(\sum_{2^j > R_0} \|\dot{\Delta}_j z\|^r_{L^p}\Big)^{\frac{1}{r}}.$$

Up to a harmless constant, we have

$$\|(a_0^\varepsilon, \mathcal{P}^\perp u_0^\varepsilon)\|^{\ell,\varepsilon}_{\dot{B}^{n/2-1}_{2,1}} + \varepsilon\|a_0^\varepsilon\|^{h,\varepsilon}_{\dot{B}^{n/p}_{p,1}} + \|\mathcal{P}^\perp u_0^\varepsilon\|^{h,\varepsilon}_{\dot{B}^{n/p-1}_{p,1}} + \|(\mathcal{P}u_0^\varepsilon, H_0^\varepsilon)\|_{\dot{B}^{n/p-1}_{p,1}},$$

$$
\begin{aligned}
&:= \|(a_0,\mathcal{P}^{\perp}u_0)\|^{\ell}_{\dot B^{n/2-1}_{2,1}} + \|a_0\|^{h}_{\dot B^{n/p}_{p,1}} + \|\mathcal{P}^{\perp}u_0\|^{h}_{\dot B^{n/p-1}_{p,1}} + \|(\mathcal{P}u_0,H_0)\|_{\dot B^{n/p-1}_{p,1}}\\
&:= \|(a_0,u_0,H_0)\|_{X_p(0)}
\end{aligned}
$$

and

$$\|(a,u,H)\|_{X_p} := \|(a,u,H)\|_{X_p^1} = \|(a^{\varepsilon},u^{\varepsilon},H^{\varepsilon})\|_{X_p^{\varepsilon}}.$$

3.1. A priori estimates to the solution of system (3.3)

Step 1: the incompressible part of the velocity and the magnetic field. Applying $\mathcal{P}$ to the momentum equation $(3.3)_2$, we can have

$$
\begin{cases}
\partial_t\mathcal{P}u - \mu\Delta\mathcal{P}u = \mathcal{P}\Big(-u\cdot\nabla u - I(a)\mathcal{A}u + \dfrac{1}{1+a}\big((\nabla\times H)\times H\big)\Big),\\
\partial_t H - \alpha\Delta H = -u\cdot\nabla H + H\cdot\nabla u - H(\nabla\cdot u), \qquad \nabla\cdot H = 0.
\end{cases}
$$

Using the estimate (1.8) in Lemma 1.2 for the heat equation, we have

$$
\begin{aligned}
&\|\mathcal{P}u\|_{\tilde L^\infty \dot B^{\frac{n}{p}-1}_{p,1}\cap L^1\dot B^{\frac{n}{p}+1}_{p,1}} \lesssim \|\mathcal{P}u_0\|_{\dot B^{\frac{n}{p}-1}_{p,1}}\\
&\quad + \Big\|\mathcal{P}\Big(-u\cdot\nabla u - I(a)\mathcal{A}u + \frac{1}{1+a}\big((\nabla\times H)\times H\big)\Big)\Big\|_{L^1\dot B^{\frac{n}{p}-1}_{p,1}}.
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
&\|H\|_{\tilde L^\infty \dot B^{\frac{n}{p}-1}_{p,1}\cap L^1\dot B^{\frac{n}{p}+1}_{p,1}}\\
&\lesssim \|H_0\|_{\dot B^{\frac{n}{p}-1}_{p,1}} + \|-u\cdot\nabla H + H\cdot\nabla u - H(\nabla\cdot u)\|_{L^1\dot B^{\frac{n}{p}-1}_{p,1}}.
\end{aligned}
\tag{3.5}
$$

Now we come to estimate the right hand of (3.4) and (3.5). Using (A.9) and (A.10), we have

$$
\begin{aligned}
\|I(a)\nabla^2 u\|_{\tilde L^1\dot B^{\frac{n}{p}-1}_{p,1}} &\lesssim \|I(a)\|_{\tilde L^\infty\dot B^{\frac{n}{p}}_{p,1}}\|\nabla^2 u\|_{L^1\dot B^{\frac{n}{p}-1}_{p,1}}.\\
&\lesssim (1+\|a\|_{\tilde L^\infty\dot B^{\frac{n}{p}}_{p,1}})^{n/2+1}\|a\|_{\tilde L^\infty\dot B^{\frac{n}{p}}_{p,1}}\|u\|_{L^1\dot B^{\frac{n}{p}+1}_{p,1}}.
\end{aligned}
$$

Using z to denote u or H, with a little abuse of notation, we have

$$\|z\nabla z\|_{\tilde L^1\dot B^{\frac{n}{p}-1}_{p,1}} \lesssim \|z\|_{\tilde L^\infty\dot B^{\frac{n}{p}-1}_{p,1}}\|z\|_{L^1\dot B^{\frac{n}{p}+1}_{p,1}}.$$

Besides, we have $\frac{1}{1+a}\big((\nabla\times H)\times H\big) = (\nabla\times H)\times H - I(a)\big((\nabla\times H)\times H\big)$. The estimate of the term $(\nabla\times H)\times H$ is the same as $z\nabla z$, so we focus on the term $I(a)\big((\nabla\times H)\times H\big)$. Using (A.9) and (A.10), we have

$$
\begin{aligned}
\|I(a)H\nabla H\|^{h}_{L^1\dot B^{n/p-1}_{p,1}} &\lesssim \|I(a)\|_{\tilde L^\infty\dot B^{n/p}_{p,1}}\|H\nabla H\|_{L^1\dot B^{n/p-1}_{p,1}}.\\
&\lesssim (1+\|a\|_{\tilde L^\infty\dot B^{\frac{n}{p}}_{p,1}})^{n/2+1}\|a\|_{\tilde L^\infty\dot B^{\frac{n}{p}}_{p,1}}\|H\nabla H\|_{L^1\dot B^{\frac{n}{p}+1}_{p,1}}.\\
&\lesssim (1+\|a\|_{\tilde L^\infty\dot B^{\frac{n}{p}}_{p,1}})^{n/2+1}\|a\|_{\tilde L^\infty\dot B^{\frac{n}{p}}_{p,1}}\|H\|_{\tilde L^2\dot B^{n/p}_{p,1}}\|\nabla H\|_{\tilde L^2\dot B^{n/p-1}_{p,1}}.
\end{aligned}
$$

Also, by Bernstein inequality,

$$
\begin{aligned}
\|a\|_{\tilde L^\infty\dot B^{\frac{n}{p}}_{p,1}} &\lesssim \|a\|^{\ell}_{\tilde L^\infty\dot B^{\frac{n}{p}}_{p,1}} + \|a^h\|_{\tilde L^\infty\dot B^{\frac{n}{p}}_{p,1}}\\
&\lesssim R_0\|a\|^{\ell}_{\tilde L^\infty\dot B^{\frac{n}{2}-1}_{2,1}} + \|a^h\|_{\tilde L^\infty\dot B^{\frac{n}{p}}_{p,1}}.
\end{aligned}
$$

We deduce from the above inequalities

$$\|(\mathcal{P}u,H)\|_{\tilde{L}^\infty \dot{B}^{\frac{n}{p}-1}_{p,1}\cap L^1\dot{B}^{\frac{n}{p}+1}_{p,1}} \lesssim \|(\mathcal{P}u_0,H_0)\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + R_0(1+R_0\|(a,u,H)\|_{X_p})^{n/2+2}\|(a,u,H)\|^2_{X_p}. \tag{3.6}$$

Step 2: the low frequency part of $(a,\mathcal{P}^\perp u)$. Now we come to estimate the low frequencies of $(a,\mathcal{P}^\perp u)$ which satisfies

$$\begin{cases} \partial_t a + \operatorname{div}\mathcal{P}^\perp u = -\operatorname{div}(au),\\ \partial_t\mathcal{P}^\perp u - \Delta\mathcal{P}^\perp u + \nabla a\\ \qquad = -\mathcal{P}^\perp\Big(u\cdot\nabla u + K(a)\nabla a + I(a)\mathcal{A}u + \dfrac{1}{1+a}\big((\nabla\times H)\times H\big)\Big). \end{cases} \tag{3.7}$$

The standard energy estimates for the barotropic linearized equations (see [1] or [6]) indicate

$$\begin{aligned} \|(a,\mathcal{P}^\perp u)\|^\ell_{\tilde{L}^\infty \dot{B}^{\frac{n}{2}-1}_{2,1}\cap L^1\dot{B}^{\frac{n}{2}+1}_{2,1}} &\lesssim \|(a_0,\mathcal{P}^\perp u_0)\|^\ell_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \|\operatorname{div}(au)\|^\ell_{L^1\dot{B}^{\frac{n}{2}-1}_{2,1}}\\ &\quad + \|\mathcal{P}^\perp(u\cdot\nabla u)\|^\ell_{L^1\dot{B}^{\frac{n}{2}-1}_{2,1}} + \|\mathcal{P}^\perp\big(K(a)\nabla a\big)\|^\ell_{L^1\dot{B}^{\frac{n}{2}-1}_{2,1}}\\ &\quad + \|\mathcal{P}^\perp\big(I(a)\mathcal{A}u\big)\|^\ell_{L^1\dot{B}^{\frac{n}{2}-1}_{2,1}}\\ &\quad + \|\mathcal{P}^\perp\big(\frac{1}{1+a}\big((\nabla\times H)\times H\big)\big)\|^\ell_{L^1\dot{B}^{\frac{n}{2}-1}_{2,1}}. \end{aligned}$$

Let us bound the right hand of the above inequality term by term. First $\operatorname{div}(au) = a\nabla\cdot u + u\cdot\nabla a$. By using $f=a, g=\nabla u, r_3=\infty, r_4=1, \gamma=0, r_1=r_2=2$ in (A.7) and interpolations, we have

$$\begin{aligned} \|(a\nabla\cdot u)\|^\ell_{L^1\dot{B}^{n/2-1}_{2,1}} &\lesssim \|a\|_{\tilde{L}^\infty\dot{B}^{n/p-1}_{p,1}}\|\nabla u\|_{L^1\dot{B}^{n/p}_{p,1}} + \|a\|_{\tilde{L}^2\dot{B}^{n/p}_{p,1}}\|\nabla u\|_{\tilde{L}^2\dot{B}^{n/p-1}_{p,1}}\\ &\lesssim \|(a,u,H)\|^2_{X_p}. \end{aligned} \tag{3.8}$$

Using $f=u, g=\nabla a, r_1=r_2=2, r_3=\infty, r_4=1, \gamma=-1$ in (A.6) and interpolations, we have

$$\begin{aligned} \|u\cdot\nabla a\|^\ell_{L^1\dot{B}^{n/2-1}_{2,1}} &\lesssim \|u\|_{\tilde{L}^2\dot{B}^{n/p}_{p,1}}\|\nabla a\|_{\tilde{L}^2\dot{B}^{n/p-1}_{p,1}}\\ &\quad + \|u\|_{\tilde{L}^2\dot{B}^{n/p}_{p,1}}\|\nabla a\|^\ell_{\tilde{L}^2\dot{B}^{n/2-1}_{2,1}} + R_0\|u\|_{\tilde{L}^\infty\dot{B}^{n/p-1}_{p,1}}\|\nabla a\|^h_{L^1\dot{B}^{n/p-1}_{p,1}}\\ &\lesssim R_0\|(a,u,H)\|^2_{X_p}. \end{aligned} \tag{3.9}$$

Now let us bound $u\cdot\nabla u$ in $L^1(\dot{B}^{n/2-1}_{2,1})$. Using $f=u, g=\nabla u, r_3=\infty, r_4=1, \gamma=0, r_1=r_2=2$ in (A.7) and interpolations, we have

$$\begin{aligned} \|(u\cdot\nabla u)\|^\ell_{L^1\dot{B}^{n/2-1}_{2,1}} &\lesssim \|u\|_{\tilde{L}^\infty\dot{B}^{n/p-1}_{p,1}}\|\nabla u\|_{L^1\dot{B}^{n/p}_{p,1}} + \|u\|_{\tilde{L}^2\dot{B}^{n/p}_{p,1}}\|\nabla u\|_{\tilde{L}^2\dot{B}^{n/p-1}_{p,1}}\\ &\lesssim \|(a,u,H)\|^2_{X_p}. \end{aligned} \tag{3.10}$$

In order to bound $K(a)\nabla a$, we apply $f=K(a), g=\nabla a, r_1=r_2=2, \gamma=0, r_3=\infty, r_4=1, \gamma=-1$ in (A.6) and interpolations, then we get

$$\|K(a)\cdot\nabla a\|^\ell_{L^1\dot{B}^{n/2-1}_{2,1}} \lesssim \|K(a)\|_{\tilde{L}^2\dot{B}^{n/p}_{p,1}}\|\nabla a\|_{\tilde{L}^2\dot{B}^{n/p-1}_{p,1}}$$

$$
\begin{aligned}
&+\|K(a)\|_{\tilde L^2\dot B^{n/p}_{p,1}}\|\nabla a\|^{\ell}_{\tilde L^2\dot B^{n/2-1}_{2,1}}+R_0\|K(a)\|_{\tilde L^\infty\dot B^{n/p-1}_{p,1}}\|\nabla a\|^{h}_{L^1\dot B^{n/p-1}_{p,1}}\\
&\lesssim R_0\big(1+\|a\|_{L^\infty\dot B^{n/p}_{p,1}}\big)^{n/2+1}\|(a,u,H)\|^2_{X_p}. \qquad (3.11)
\end{aligned}
$$

To handle the term $I(a)\mathcal{A}u$, we apply $f=I(a), g=\nabla^2u, r_1=r_3=\infty, r_2=r_4=1, \gamma=-1$ in (A.7) and interpolations. Then we get

$$
\begin{aligned}
\|I(a)\mathcal{A}u\|^{\ell}_{L^1\dot B^{n/2-1}_{2,1}}&\lesssim\big(R_0\|I(a)\|_{\tilde L^\infty\dot B^{n/p-1}_{p,1}}+\|I(a)\|_{\tilde L^\infty\dot B^{n/p}_{p,1}}\big)\|\nabla^2u\|_{L^1\dot B^{n/p-1}_{p,1}}\\
&\lesssim R_0\big(1+\|a\|_{L^\infty\dot B^{n/p}_{p,1}}\big)^{n/2+1}\|(a,u,H)\|^2_{X_p}. \qquad (3.12)
\end{aligned}
$$

We rewrite $\frac{1}{1+a}(\nabla\times H)\times H$ into $(\nabla\times H)\times H-I(a)(\nabla\times H)\times H$. The estimate of the first term is essentially the same as $u\cdot\nabla u$, so we mainly focus on the bound of the second term. By applying $f=I(a), g=H\nabla H, r_1=r_3=\infty, r_2=r_4=1, \gamma=-1$ in (A.7), we get

$$
\|I(a)H\nabla H\|^{\ell}_{L^1\dot B^{n/2-1}_{2,1}}\lesssim\big(R_0\|I(a)\|_{\tilde L^\infty\dot B^{n/p-1}_{p,1}}+\|I(a)\|_{\tilde L^\infty\dot B^{n/p}_{p,1}}\big)\|H\nabla H\|_{L^1\dot B^{n/p-1}_{p,1}}.
$$

On the other hand, using (A.9), we have

$$
\|H\nabla H\|_{L^1\dot B^{n/p-1}_{p,1}}\lesssim\|H\|_{\tilde L^2\dot B^{n/p}_{p,1}}\|\nabla H\|_{\tilde L^2\dot B^{n/p-1}_{p,1}}.
$$

The above two inequalities indicate that

$$
\|I(a)(\nabla\times H)\times H\|^{\ell}_{L^1\dot B^{n/2-1}_{2,1}}\lesssim R_0\big(1+\|a\|_{L^\infty\dot B^{n/p}_{p,1}}\big)^{n/2+1}\|(a,u,H)\|^3_{X_p}. \qquad (3.13)
$$

Putting all the estimates from (3.8) to (3.13), we can get

$$
\begin{aligned}
\|(a,\mathcal{P}^\perp u)\|^{\ell}_{\tilde L^\infty\dot B^{\frac n2-1}_{2,1}\cap L^1\dot B^{\frac n2+1}_{2,1}}&\lesssim\|(a_0,\mathcal{P}^\perp u_0)\|^{\ell}_{\dot B^{\frac n2-1}_{2,1}}\\
&\quad+R_0\big(1+R_0\|(a,u,H)\|_{X_p}\big)^{n/2+2}\|(a,u,H)\|^2_{X_p}. \qquad (3.14)
\end{aligned}
$$

Step 3: Effective velocity. We follow the approach in [22] to estimate the high frequencies of $\mathcal{P}^\perp u$. Introduce the following "effective velocity":

$$
w:=\mathcal{P}^\perp u+(\Delta)^{-1}\nabla a.
$$

Then from (3.7), we get

$$
\begin{aligned}
\partial_t w-\Delta w&=\mathcal{P}^\perp(au)+w-(-\Delta)^{-1}\nabla a\\
&\quad-\mathcal{P}^\perp\Big(u\cdot\nabla u+K(a)\nabla a+I(a)\mathcal{A}u+\frac{1}{1+a}\big((\nabla\times H)\times H\big)\Big).
\end{aligned}
$$

Applying the high-frequency estimate (1.8) of the heat equation to the above w equation, we obtain

$$
\begin{aligned}
\|w\|^{h}_{\tilde L^\infty\dot B^{n/p-1}_{p,1}\cap L^1\dot B^{n/p+1}_{p,1}}&\lesssim\|w_0\|^{h}_{\dot B^{n/p-1}_{p,1}}+\|w\|^{h}_{L^1\dot B^{n/p-1}_{p,1}}+\|a\|^{h}_{L^1\dot B^{n/p-2}_{p,1}}\\
&\quad+\|\mathcal{P}^\perp(au)\|^{h}_{L^1\dot B^{n/p-1}_{p,1}}+\|\mathcal{P}^\perp(u\cdot\nabla u)\|^{h}_{L^1\dot B^{n/p-1}_{p,1}}+\|\mathcal{P}^\perp(K(a)\nabla a)\|^{h}_{L^1\dot B^{n/p-1}_{p,1}}\\
&\quad+\|\mathcal{P}^\perp(I(a)\mathcal{A}u)\|^{h}_{L^1\dot B^{n/p-1}_{p,1}}+\Big\|\mathcal{P}^\perp\Big(\frac{1}{1+a}\big((\nabla\times H)\times H\big)\Big)\Big\|^{h}_{L^1\dot B^{n/p-1}_{p,1}}. \qquad (3.15)
\end{aligned}
$$

Owning to the high frequency cut-off, when $2^j>R_0$, we have

$$
\|w\|^{h}_{L^1\dot B^{n/p-1}_{p,1}}\lesssim R_0^{-2}\|w\|^{h}_{L^1\dot B^{n/p+1}_{p,1}}\quad\text{and}\quad\|a\|^{h}_{L^1\dot B^{n/p-2}_{p,1}}\lesssim R_0^{-2}\|a\|^{h}_{L^1\dot B^{n/p}_{p,1}}.
$$

If R_0 is sufficient large, the term $\|w\|^h_{L^1\dot B^{n/p-1}_{p,1}}$ can be absorbed by the left hand side of (3.15). The other terms satisfy the quadratic estimates. We proceed as follows.

To bound $\|\mathcal{P}^\perp(au)\|^h_{L^1\dot B^{n/p-1}_{p,1}}$, applying $f=a, g=u, r_1=r_2=2, \gamma=1$ in (A.8), we obtain

$$\begin{aligned}\|au\|^h_{L^1\dot B^{n/p-1}_{p,1}} &\lesssim R_0^{-1}\|a\|_{\tilde L^2\dot B^{n/p}_{p,1}}\|u\|_{\tilde L^2\dot B^{n/p}_{p,1}}\\ &\lesssim R_0^{-1}\|(a,u,H)\|^2_{X_p}.\end{aligned} \tag{3.16}$$

For the term $\|\mathcal{P}^\perp(u\cdot\nabla u)\|^h_{L^1\dot B^{n/p-1}_{p,1}}$, applying $f=\nabla u, g=u, \gamma=0, r_1=1, r_2=\infty$ in (A.8), we get

$$\begin{aligned}\|u\nabla u\|^h_{L^1\dot B^{n/p-1}_{p,1}} &\lesssim \|u\|_{\tilde L^\infty\dot B^{n/p-1}_{p,1}}\|\nabla u\|_{\tilde L^2\dot B^{n/p}_{p,1}}\\ &\lesssim \|(a,u,H)\|^2_{X_p}.\end{aligned} \tag{3.17}$$

For the term $\|\mathcal{P}^\perp(K(a)\nabla a)\|^h_{L^1\dot B^{n/p-1}_{p,1}}$, applying $f=K(a), g=\nabla a, \gamma=0, r_1=r_2=2$ in (A.8), we get

$$\begin{aligned}\|K(a)\nabla a\|^h_{L^1\dot B^{n/p-1}_{p,1}} &\lesssim \|K(a)\|_{\tilde L^2\dot B^{n/p}_{p,1}}\|\nabla a\|_{\tilde L^2\dot B^{n/p-1}_{p,1}}\\ &\lesssim \big(1+\|a\|_{L^\infty\dot B^{n/p}_{p,1}}\big)^{n/2+1}\|(a,u,H)\|^2_{X_p}.\end{aligned} \tag{3.18}$$

For the term $\|\mathcal{P}^\perp(I(a)\mathcal{A}u)\|^h_{L^1\dot B^{n/p-1}_{p,1}}$, applying $f=I(a), g=\nabla^2 a, \gamma=0, r_1=\infty, r_2=1$ in (A.8), we get

$$\begin{aligned}\|I(a)\mathcal{A}u\|^h_{L^1\dot B^{n/p-1}_{p,1}} &\lesssim \|I(a)\|_{\tilde L^\infty\dot B^{n/p}_{p,1}}\|\nabla^2 u\|_{\tilde L^1\dot B^{n/p-1}_{p,1}}\\ &\lesssim R_0\big(1+\|a\|_{L^\infty\dot B^{n/p}_{p,1}}\big)^{n/2+1}\|(a,u,H)\|^2_{X_p}.\end{aligned} \tag{3.19}$$

In order to bound the cubic term, we decompose it as above with $\frac{1}{1+a}(\nabla\times H)\times H$ into $(\nabla\times H)\times H - I(a)(\nabla\times H)\times H$. The estimate of the first term is essentially the same as $u\cdot\nabla u$, so we mainly focus on the bound of the second term. Applying $f=I(a), g=H\nabla H, \gamma=0, r_1=\infty, r_2=1$ in (A.8), we get

$$\|I(a)H\nabla H\|^h_{L^1\dot B^{n/p-1}_{p,1}} \lesssim \|I(a)\|_{\tilde L^\infty\dot B^{n/p}_{p,1}}\|H\nabla H\|_{L^1\dot B^{n/p-1}_{p,1}}.$$

On the other hand, using (A.9), we have

$$\|H\nabla H\|_{L^1\dot B^{n/p-1}_{p,1}} \lesssim \|H\|_{\tilde L^2\dot B^{n/p}_{p,1}}\|\nabla H\|_{\tilde L^2\dot B^{n/p-1}_{p,1}}.$$

The above two inequalities indicate that

$$\|I(a)(\nabla\times H)\times H\|^h_{L^1\dot B^{n/2-1}_{2,1}} \lesssim R_0\big(1+\|a\|_{L^\infty\dot B^{n/p}_{p,1}}\big)^{n/2+1}\|(a,u,H)\|^3_{X_p}. \tag{3.20}$$

Combining the estimates from (3.15) to (3.20), we conclude that

$$\begin{aligned}\|w\|^h_{\tilde L^\infty\dot B^{n/p-1}_{p,1}\cap L^1\dot B^{n/p+1}_{p,1}} &\lesssim \|w_0\|^h_{\dot B^{n/p-1}_{p,1}} + R_0^{-2}\|a\|^h_{L^1\dot B^{n/p}_{p,1}}\\ &\quad + R_0\big(1+R_0\|(a,u,H)\|_{X_p}\big)^{n/2+2}\|(a,u,H)\|^2_{X_p}.\end{aligned} \tag{3.21}$$

Step 4: The high frequency of the density. We find that a satisfies

$$\partial_t a + u\cdot\nabla a + a = -a\mathrm{div}u - \mathrm{div}w.$$

To bound the high frequency of a, for $2^j > R_0$,

$$\partial_t \dot{\Delta}_j a + u \cdot \nabla \dot{\Delta}_j a + \dot{\Delta}_j a = -\dot{\Delta}_j(a\mathrm{div}u + \mathrm{div}w) + R_j, \tag{3.22}$$

where $R_j := [u \cdot \nabla, \dot{\Delta}_j]a$.

Multiplying (3.22) by $\dot{\Delta}_j a|\dot{\Delta}_j a|^{p-2}$, then integrating on $\mathbb{R}^n \times [0,t]$, we can get

$$\begin{aligned}\|\dot{\Delta}_j a(t)\|_{L^p} + \int_0^t \|\dot{\Delta}_j a\|_{L^p} ds &\lesssim \|\dot{\Delta}_j a_0\|_{L^p} + \int_0^t \|\mathrm{div}u\|_{L^\infty}\|\dot{\Delta}_j a\|_{L^p} ds \\ &+ \int_0^t \|\dot{\Delta}_j(a\mathrm{div}u + \mathrm{div}w)\|_{L^p} ds + \int_0^t \|R_j\|_{L^p} ds.\end{aligned} \tag{3.23}$$

Using (A.9), we have

$$\|a\mathrm{div}u\|_{\dot{B}^{n/p}_{p,1}} \lesssim \|a\|_{\dot{B}^{n/p}_{p,1}} \|\mathrm{div}u\|_{\dot{B}^{n/p}_{p,1}}.$$

Commutator estimates in [1] give that

$$\sum_{j\in\mathbb{Z}} \|R_j\|_{L^p} \lesssim \|\nabla u\|_{\dot{B}^{n/p}_{p,1}} \|a\|_{\dot{B}^{n/p}_{p,1}}.$$

Now multiplying (3.23) by $2^{jn/p}$, using the above two estimates and summing over $2^j > R_0$, then we get

$$\begin{aligned}\|a\|^h_{\tilde{L}^\infty_t \dot{B}^{n/p}_{p,1}} + \int_0^t \|a\|^h_{\dot{B}^{n/p}_{p,1}} ds &\lesssim \|a_0\|^h_{\dot{B}^{n/p}_{p,1}} \\ &+ \int_0^t \|\nabla u\|_{\dot{B}^{n/p}_{p,1}} \|a\|_{\dot{B}^{n/p}_{p,1}} ds + \|w\|^h_{L^1 \dot{B}^{n/p+1}_{p,1}}.\end{aligned}$$

Therefore, we get

$$\|a\|^h_{\tilde{L}^\infty_t \dot{B}^{n/p}_{p,1} \cap L^1_t \dot{B}^{n/p}_{p,1}} \lesssim \|a_0\|^h_{\dot{B}^{n/p}_{p,1}} + R_0\|(a,u,H)\|^2_{X_p} + \|w\|^h_{L^1 \dot{B}^{n/p+1}_{p,1}}. \tag{3.24}$$

Step 5: Close of the a priori estimate. For a suitable small δ, multiply (3.24) by δ and then add it to (3.21). By choosing R_0 sufficiently large, we can get

$$\begin{aligned}&\|a\|^h_{\tilde{L}^\infty_t \dot{B}^{n/p}_{p,1} \cap L^1_t \dot{B}^{n/p}_{p,1}} + \|w\|^h_{\tilde{L}^\infty_t \dot{B}^{n/p-1}_{p,1} \cap L^1_t \dot{B}^{n/p+1}_{p,1}} \\ &\lesssim \|w_0\|^h_{\dot{B}^{n/p-1}_{p,1}} + \|a_0\|^h_{\dot{B}^{n/p}_{p,1}} + R_0\big(1 + R_0\|(a,u,H)\|_{X_p}\big)^{n/2+2}\|(a,u,H)\|^2_{X_p}.\end{aligned}$$

Since $\mathcal{P}^\perp u^h = w^h - (-\Delta)^{-1}\nabla a^h$, the above estimate still holds for a^h and $\mathcal{P}^\perp u^h$. So we have

$$\begin{aligned}&\|a\|^h_{\tilde{L}^\infty_t \dot{B}^{n/p}_{p,1} \cap L^1_t \dot{B}^{n/p}_{p,1}} + \|\mathcal{P}^\perp u\|^h_{\tilde{L}^\infty_t \dot{B}^{n/p-1}_{p,1} \cap L^1_t \dot{B}^{n/p+1}_{p,1}} \\ &\lesssim \|\mathcal{P}^\perp u_0\|^h_{\dot{B}^{n/p-1}_{p,1}} + \|a_0\|^h_{\dot{B}^{n/p}_{p,1}} + R_0\big(1 + R_0\|(a,u,H)\|_{X_p}\big)^{n/2+2}\|(a,u,H)\|^2_{X_p}.\end{aligned} \tag{3.25}$$

Finally, combining the estimates (3.6), (3.14) and (3.25), we have

$$\begin{aligned}\|(a,u,H)\|_{X_p} &\lesssim \|(a_0, \mathcal{P}^\perp u_0)\|^\ell_{\dot{B}^{n/2-1}_{2,1}} + \|a_0\|^h_{\dot{B}^{n/p}_{p,1}} + \|\mathcal{P}^\perp u_0\|^h_{\dot{B}^{n/p-1}_{p,1}} \\ &+ \|(\mathcal{P}u_0, H_0)\|_{\dot{B}^{n/p-1}_{p,1}} + R_0\big(1 + R_0\|(a,u,H)\|_{X_p}\big)^{n/2+2}\|(a,u,H)\|^2_{X_p}.\end{aligned} \tag{3.26}$$

3.2. Global existence of the solution to system (3.3)

Now we come to give the global existence of the solution to system (3.3). Define

$$X_2(T) = \big\{(a,u,H) : a^\ell \in \tilde{L}^\infty(0,T;\dot{B}^{n/2-1}_{2,1}) \cap L^1(0,T;\dot{B}^{n/2+1}_{2,1}); \\ a^h \in \tilde{L}^\infty(0,T;\dot{B}^{n/2}_{2,1}) \cap L^1(0,T;\dot{B}^{n/2}_{2,1}); \\ (u,H) \in \tilde{L}^\infty(0,T;\dot{B}^{n/2-1}_{2,1}) \cap L^1(0,T;\dot{B}^{n/2+1}_{2,1})\big\}, \tag{3.27}$$

with norm

$$\|(a,u,H)\|_{X_2} \\ = \|a\|^\ell_{\tilde{L}^\infty \dot{B}^{n/2-1}_{2,1} \cap L^1 \dot{B}^{n/2+1}_{2,1}} + \|a\|^h_{\tilde{L}^\infty \dot{B}^{n/2}_{2,1} \cap L^1 \dot{B}^{n/2}_{2,1}} + \|(u,H)\|_{\tilde{L}^\infty \dot{B}^{n/2-1}_{2,1} \cap L^1 \dot{B}^{n/2+1}_{2,1}}.$$

Set

$$\|(a_0,u_0,H_0)\|_{X_2(0)} := \|a_0\|^\ell_{\dot{B}^{n/2-1}_{2,1}} + \|a_0\|^h_{\dot{B}^{n/2}_{2,1}} + \|(u_0,H_0)\|_{\dot{B}^{n/2-1}_{2,1}}.$$

Duplicating the proof of the a prior estimate (3.26), we can prove that

$$\|(a,u,H)\|_{X_2} \lesssim \|(a_0,u_0,H_0)\|_{X_2(0)} \\ + R_0\big(1 + R_0\|(a,u,H)\|_{X_p}\big)^{n/2+2}\|(a,u,H)\|_{X_p}\|(a,u,H)\|_{X_2}. \tag{3.28}$$

Now by using the a priori estimates (3.26) and (3.28), we sketch the proof of the global existence of system (3.3). The simplest way is to smooth out the initial data (a_0,u_0,H_0) into a sequence of initial data $(a_{0,k},u_{0,k},H_{0,k})_{k\in\mathbb{N}}$ with

$$a^\ell_{0,k} \in \dot{B}^{n/2-1}_{2,1}, a^h_{0,k} \in \dot{B}^{n/2}_{2,1}, (u_{0,k},H_{0,k}) \in \dot{B}^{n/2-1}_{2,1}$$

and

$$\big\|\big(a_{0,k}-a_0, \mathcal{P}^\perp u_{0,k} - \mathcal{P}^\perp u_0\big)\big\|^\ell_{\dot{B}^{n/2-1}_{2,1}} + \|a_{0,k}-a_0\|^h_{\dot{B}^{n/p}_{p,1}} + \|\mathcal{P}^\perp u_{0,k} - \mathcal{P}^\perp u_0\|^h_{\dot{B}^{n/p-1}_{p,1}} \\ + \|(\mathcal{P}u_{0,k} - \mathcal{P}u_0, H_{0,k}-H_0)\|_{\dot{B}^{n/p-1}_{p,1}} \longrightarrow 0 \quad \text{as} \quad k \to \infty. \tag{3.29}$$

Estimate (3.29) implies that there exists a constant C_0 such that

$$\|(a_{0,k},u_{0,k},H_{0,k})\|_{X_p(0)} \le C\|(a_0,u_0,H_0)\|_{X_p(0)} \le C_0\eta. \tag{3.30}$$

Applying the local existence result in [30], there exists a maximal existence time $T_k > 0$ such that system (3.3) has a unique-in-time solution $(a_k,u_k,H_k) \in X_2(T_k)$ with initial data $(a_{0,k},u_{0,k},H_{0,k})$. Using the definition of Besov space, it is easy to see that $(a_k,u_k,H_k) \in X_p(T_k)$. The a prior estimates (3.26) and (3.30) imply that there exists a M such that

$$\|(a_k,u_k,H_k)\|_{X_p(T_k)} \le M\eta. \tag{3.31}$$

Actually, by choosing η small and using the a prior estimate (3.28) and (3.31), we can get

$$\|(a_k,u_k,H_k)\|_{X_2(T_k)} \lesssim \|(a_k,u_k,H_k)\|_{X_2(0)},$$

which implies that $T_k = \infty$.

At last, we get that

$$\|(a_k,u_k,H_k)\|_{X_p(\infty)} \lesssim \|(a_0,u_0,H_0)\|_{X_p(0)} \le M\eta. \tag{3.32}$$

Next, compactness arguments similar to those of [1] or [7] allow us to conclude (a_k,u_k,H_k) weakly converges (up to extraction) to some function (a,u,H) which is a solution of (3.3) with the desired regularity properties and satisfy (2.2) with $\varepsilon = 1$. Scaling back to the original unknowns $(a^\varepsilon,u^\varepsilon,H^\varepsilon)$ completes the proof the global existence part of Theorem 2.1.

4. The incompressible limit: strong convergence of the solution

In this section, we combine Strichartz estimates for the following linear system of acoustics:

$$\begin{cases} \partial_t b + \varepsilon^{-1} \Lambda v = F \\ \partial_t v - \varepsilon^{-1} \Lambda b = G \\ (b, v)|_{t=0} = (b_0, v_0) \end{cases} \tag{4.1}$$

and the uniform estimates in (2.2) for the global solution $(a^\varepsilon, u^\varepsilon, H^\varepsilon)$ so as to establish the strong convergence for $(a^\varepsilon, \mathcal{P}^\perp u^\varepsilon)$ to zero and for $(\mathcal{P} u^\varepsilon, H^\varepsilon)$ to the solution (v, B) of system (1.3) in proper function spaces.

4.1. Convergence to zero of the compressible part

First, we give an estimate for the solution of the linear system (4.1) whose proof can be founded in Proposition 10.30 of [1].

Proposition 4.1. *Let (b, v) be a solution of (4.1). Then, for any $s \in \mathbb{R}$ and positive T (possibly infinite), the following estimate holds:*

$$\|(b, v)\|_{\tilde{L}^r_T \dot{B}^{s+n(\frac{1}{q}-\frac{1}{2})+\frac{1}{r}}_{q,1}} \lesssim \varepsilon^{\frac{1}{r}} \|(b_0, v_0)\|_{\dot{B}^s_{2,1}} + \varepsilon^{1+\frac{1}{r}-\frac{1}{\bar{r}'}} \|(F, G)\|_{\tilde{L}^{\bar{r}'}_T \dot{B}^{s+n(\frac{1}{\bar{q}'}-\frac{1}{2})+\frac{1}{\bar{r}'}}_{\bar{q}',1}} \tag{4.2}$$

with

$$q \geq 2, \frac{2}{r} \leq \min\{1, \gamma(q)\}, \quad (r, q, n) \neq (2, \infty, 3),$$
$$\bar{q} \geq 2, \frac{2}{\bar{r}} \leq \min\{1, \gamma(\bar{q})\}, \quad (\bar{r}, \bar{q}, n) \neq (2, \infty, 3),$$

where $\gamma(q) := (n-1)(\frac{1}{2} - \frac{1}{q})$, $\quad \frac{1}{\bar{q}} + \frac{1}{\bar{q}'} = 1$, *and* $\frac{1}{\bar{r}} + \frac{1}{\bar{r}'} = 1$.

Actually, the above inequality (4.2) still holds for the low frequency, which means that

$$\|(b, v)\|^\ell_{\tilde{L}^r_T \dot{B}^{s+n(\frac{1}{q}-\frac{1}{2})+\frac{1}{r}}_{q,1}} \lesssim \varepsilon^{\frac{1}{r}} \|(b_0, v_0)\|^\ell_{\dot{B}^s_{2,1}} + \varepsilon^{1+\frac{1}{r}-\frac{1}{\bar{r}'}} \|(F, G)\|^\ell_{\tilde{L}^{\bar{r}'}_T \dot{B}^{s+n(\frac{1}{\bar{p}'}-\frac{1}{2})+\frac{1}{\bar{r}'}}_{\bar{p}',1}}. \tag{4.3}$$

In order to prove the convergence to 0 for the $(a^\varepsilon, \mathcal{P}^\perp u^\varepsilon)$, we use the fact that

$$\begin{cases} \partial_t a^\varepsilon + \varepsilon^{-1} \mathrm{div} \mathcal{P}^\perp u^\varepsilon = F^\varepsilon, \\ \partial_t \mathcal{P}^\perp u^\varepsilon + \varepsilon^{-1} \nabla a^\varepsilon = G^\varepsilon, \end{cases} \tag{4.4}$$

where

$$F^\varepsilon := -\mathrm{div}(a^\varepsilon u^\varepsilon)$$

$$G^\varepsilon := \mathcal{P}^\perp \Big(-u^\varepsilon \cdot \nabla u^\varepsilon - K(\varepsilon a^\varepsilon) \frac{\nabla a^\varepsilon}{\varepsilon} + \frac{1}{1+\varepsilon a^\varepsilon} \mathcal{A} u^\varepsilon + \frac{1}{1+\varepsilon a^\varepsilon} \big((\nabla \times H^\varepsilon) \times H^\varepsilon \big) \Big).$$

Obviously, the estimate (4.3) stated in Proposition 4.1 also holds for the system (4.4) since a^ε and $\Lambda^{-1} \mathrm{div} \mathcal{P}^\perp u^\varepsilon$ satisfy (4.1) with source terms F^ε, $\Lambda^{-1} \mathrm{div} G^\varepsilon$ and $\Lambda^{-1} \mathrm{div}$ is a homogeneous multiplier of degree 0. Let us pause for a while and set $\varepsilon = 1$ and

$$F := -\mathrm{div}(au)$$

$$G := \mathcal{P}^{\perp}\Big(-u\cdot\nabla u - K(a)\nabla a + \frac{1}{1+a}\mathcal{A}u + \frac{1}{1+a}\big((\nabla\times H)\times H\big)\Big).$$

Hence, by taking $\bar{q}=2, \bar{r}=\infty, s=n/2-1$ and

— $\frac{1}{r}=\frac{1}{2}-\frac{1}{q}$ for $q\in[2,\frac{2(n-1)}{n-3}]$ if $n\geq 3$,

— $\frac{1}{r}=\kappa(\frac{1}{2}-\frac{1}{q})$ for $q\in[2,\infty]$ if $n=2$, where $\kappa\in[0,\frac{1}{2}]$ will be determined later on, we have

Case $n\geq 3$:

$$\|(a,\mathcal{P}^{\perp}u)\|^{\ell}_{\tilde{L}^{\frac{2q}{q-2}}\dot{B}^{\frac{n-1}{q}-\frac{1}{2}}_{q,1}} \lesssim \|(a_0,\mathcal{P}^{\perp}u_0)\|^{\ell}_{\dot{B}^{n/2-1}_{2,1}} + \|(F,G)\|^{\ell}_{L^1\dot{B}^{n/2-1}_{2,1}}.$$

Case $n=2$:

$$\|(a,\mathcal{P}^{\perp}u)\|^{\ell}_{\tilde{L}^{\frac{2q}{\kappa(q-2)}}\dot{B}^{\frac{n-\kappa}{q}-1+\frac{\kappa}{2}}_{q,1}} \lesssim \|(a_0,\mathcal{P}^{\perp}u_0)\|^{\ell}_{\dot{B}^{n/2-1}_{2,1}} + \|(F,G)\|^{\ell}_{L^1\dot{B}^{n/2-1}_{2,1}}.$$

Duplicating the proof of (3.14), we can get for $n\geq 2$,

$$\|(F,G)\|^{\ell}_{L^1\dot{B}^{n/2-1}_{2,1}} \lesssim (1+\|(a,u,H)\|_{X_p})^{n/2+2}\|(a,u,H)\|_{X_p} \lesssim C_0^1.$$

Then we have the following estimates

$$\begin{aligned}
&\|(a,\mathcal{P}^{\perp}u)\|^{\ell}_{\tilde{L}^{\frac{2q}{q-2}}\dot{B}^{\frac{n-1}{q}-\frac{1}{2}}_{q,1}} \lesssim C_0^1,\ \text{for } n\geq 3;\\
&\|(a,\mathcal{P}^{\perp}u)\|^{\ell}_{\tilde{L}^{\frac{2q}{\kappa(q-2)}}\dot{B}^{\frac{n-\kappa}{q}-1+\frac{\kappa}{2}}_{q,1}} \lesssim C_0^1 \text{ for } n=2.
\end{aligned} \tag{4.5}$$

We also have the estimate

$$\|(a,\mathcal{P}^{\perp}u)\|^{\ell}_{L^1\dot{B}^{n/2+1}_{2,1}} \lesssim C_0^1,\ \text{for } n\geq 2. \tag{4.6}$$

Recall the complex interpolation inequality

$$\|f\|_{\tilde{L}^{r_\theta}\dot{B}^{s_\theta}_{p_\theta,1}} \lesssim \|f\|^{1-\theta}_{\tilde{L}^{r_0}\dot{B}^{s_0}_{p_0,1}}\|f\|^{\theta}_{\tilde{L}^{r_1}\dot{B}^{s_1}_{p_1,1}}, \tag{4.7}$$

with $\frac{1}{r_\theta}=\frac{1-\theta}{r_0}+\frac{\theta}{r_1}, \frac{1}{p_\theta}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}, s_\theta=(1-\theta)s_0+\theta s_1$, and $0<\theta<1, s_0<s_1, p_0<p_1, r_0<r_1$.

When $n\geq 3$, set $q=2p-2$, which satisfies the condition $q\in[2,\frac{2(n-1)}{n-3}]$ if $p\in\min\{4,\frac{2n}{n-2}\}$ and when $n=2$, set $q=2\frac{(p-2)\kappa+p}{(p-2)\kappa+4-p}$, which satisfies $q\geq 2$ if $2\leq p<4$.

Using the interpolation (4.7) between (4.5) and (4.6) with q chosen as above, we can get

$$\begin{aligned}
&\|(a,\mathcal{P}^{\perp}u)\|^{\ell}_{\tilde{L}^2\dot{B}^{\frac{n+1}{p}-\frac{1}{2}}_{p,1}} \lesssim C_0^1 \text{ for } n\geq 3,\\
&\|(a,\mathcal{P}^{\perp}u)\|^{\ell}_{\tilde{L}^2\dot{B}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}}_{p,1}} \lesssim C_0^1 \text{ for } n=2.
\end{aligned} \tag{4.8}$$

Back to the original variables in (4.4), we can have for any $\varepsilon>0$,

$$\begin{aligned}
&\|(a^\varepsilon,\mathcal{P}^{\perp}u^\varepsilon)\|^{\ell,\varepsilon}_{\tilde{L}^2\dot{B}^{\frac{n+1}{p}-\frac{1}{2}}_{p,1}} \lesssim \varepsilon^{\frac{1}{2}-\frac{1}{p}}C_0^\varepsilon, \quad \text{for} \quad n\geq 3,\\
&\|(a^\varepsilon,\mathcal{P}^{\perp}u^\varepsilon)\|^{\ell,\varepsilon}_{\tilde{L}^2\dot{B}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}}_{p,1}} \lesssim \varepsilon^{\kappa(\frac{1}{2}-\frac{1}{p})}C_0^\varepsilon, \quad \text{for} \quad n=2, \kappa\in[0,1/2].
\end{aligned}$$

Now combining the high frequency cut-off in (2.2), we obtain

Case $n\geq 3$:

$$\begin{aligned}\|(a^\varepsilon,\mathcal{P}^\perp u^\varepsilon)\|_{\tilde L^2\dot B^{\frac{n+1}{p}-\frac12}_{p,1}}&\lesssim \|(a^\varepsilon,\mathcal{P}^\perp u^\varepsilon)\|^{\ell,\varepsilon}_{\tilde L^2\dot B^{\frac{n+1}{p}-\frac12}_{p,1}}+\|(a^\varepsilon,\mathcal{P}^\perp u^\varepsilon)\|^{h,\varepsilon}_{\tilde L^2\dot B^{\frac{n+1}{p}-\frac12}_{p,1}}\\&\lesssim \|(a^\varepsilon,\mathcal{P}^\perp u^\varepsilon)\|^{\ell,\varepsilon}_{\tilde L^2\dot B^{\frac{n+1}{p}-\frac12}_{p,1}}+\varepsilon^{\frac12-\frac1p}\|(a^\varepsilon,\mathcal{P}^\perp u^\varepsilon)\|^{h,\varepsilon}_{\tilde L^2\dot B^{\frac{n}{p}}_{p,1}}\\&\lesssim \varepsilon^{\frac12-\frac1p}C_0^\varepsilon.\end{aligned}$$

Case $n=2$: for $\kappa\in[0,\frac12]$,

$$\begin{aligned}\|(a^\varepsilon,\mathcal{P}^\perp u^\varepsilon)\|_{\tilde L^2\dot B^{\frac{n+\kappa}{p}-\frac\kappa2}_{p,1}}&\lesssim \|(a^\varepsilon,\mathcal{P}^\perp u^\varepsilon)\|^{\ell,\varepsilon}_{\tilde L^2\dot B^{\frac{n+\kappa}{p}-\frac\kappa2}_{p,1}}+\|(a^\varepsilon,\mathcal{P}^\perp u^\varepsilon)\|^{h,\varepsilon}_{\tilde L^2\dot B^{\frac{n+\kappa}{p}-\frac\kappa2}_{p,1}}\\&\lesssim \|(a^\varepsilon,\mathcal{P}^\perp u^\varepsilon)\|^{\ell,\varepsilon}_{\tilde L^2\dot B^{\frac{n+\kappa}{p}-\frac\kappa2}_{p,1}}+\varepsilon^{\kappa(\frac12-\frac1p)}\|(a^\varepsilon,\mathcal{P}^\perp u^\varepsilon)\|^{h,\varepsilon}_{\tilde L^2\dot B^{\frac{n}{p}}_{p,1}}\\&\lesssim \varepsilon^{\kappa(\frac12-\frac1p)}C_0^\varepsilon.\end{aligned}$$

This completes the strong convergence of $(a^\varepsilon,\mathcal{P}^\perp u^\varepsilon)$ to 0 in $\tilde L^2\dot B^{\frac{n+1}{p}-\frac12}_{p,1}$ with an explicit rate $\varepsilon^{\frac12-\frac1p}$ for $n\geq 3$ and in $\tilde L^2\dot B^{\frac{n+\kappa}{p}-\frac\kappa2}_{p,1}$ with an explicit rate $\varepsilon^{\kappa(\frac12-\frac1p)}$ for $n=2$.

4.2. Convergence of the incompressible part

Let us now give the convergence of $(\mathcal{P}u^\varepsilon,H^\varepsilon)$ to the solution (v,B) of (1.3). Set $v^\varepsilon=\mathcal{P}u^\varepsilon-v$, $B^\varepsilon=H^\varepsilon-B$. Applying $\mathcal{P}$ to the second equation of (1.5) and subtracting (1.3) from it yields the following equations

$$\begin{cases}\partial_t v^\varepsilon-\mu\Delta v^\varepsilon=J_1^\varepsilon+J_2^\varepsilon,\\ \partial_t B^\varepsilon-\alpha\Delta B^\varepsilon=J_3^\varepsilon+J_4^\varepsilon,\\ (v^\varepsilon,B^\varepsilon)|_{t=0}=(\mathcal{P}u_0^\varepsilon-v_0,H_0^\varepsilon-B_0),\end{cases}\tag{4.9}$$

with

$$\begin{aligned}J_1^\varepsilon&:=-\mathcal{P}\big(u^\varepsilon\cdot\nabla v^\varepsilon+\mathcal{P}^\perp u^\varepsilon\cdot\nabla v+v^\varepsilon\cdot\nabla v\\&\qquad+u^\varepsilon\cdot\nabla\mathcal{P}^\perp u^\varepsilon-H^\varepsilon\cdot\nabla B^\varepsilon-B^\varepsilon\cdot\nabla B\big),\\ J_2^\varepsilon&=-\mathcal{P}\big(I(\varepsilon a^\varepsilon)\mathcal{A}u^\varepsilon\big)-\mathcal{P}\Big(I(\varepsilon a^\varepsilon)((\nabla\times H^\varepsilon)\times H^\varepsilon)\Big),\\ J_3^\varepsilon&=B^\varepsilon\cdot\nabla u^\varepsilon-u^\varepsilon\cdot\nabla B^\varepsilon+B\cdot\nabla v^\varepsilon\\&\quad-v^\varepsilon\cdot\nabla B+B\cdot\nabla\mathcal{P}^\perp u^\varepsilon-\mathcal{P}^\perp u^\varepsilon\cdot\nabla B,\\ J_4^\varepsilon&=-H^\varepsilon(\nabla\cdot\mathcal{P}^\perp u^\varepsilon).\end{aligned}$$

In what follows, we aim at estimating $(v^\varepsilon,B^\varepsilon)$ in the space $\tilde L^\infty(\dot B^{\frac{n+1}{p}-\frac32}_{p,1})\cap L^1(\dot B^{\frac{n+1}{p}+\frac12}_{p,1})(n\geq 3)$ and $\tilde L^\infty(\dot B^{\frac{n+\kappa}{p}-\frac\kappa2-1}_{p,1})\cap L^1(\dot B^{\frac{n+\kappa}{p}-\frac\kappa2+1}_{p,1})(n=2)$. Set

$$Y_{p,n}:=\|(v^\varepsilon,B^\varepsilon)\|_{\tilde L^\infty\dot B^{\frac{n+1}{p}-\frac32}_{p,1}\cap L^1\dot B^{\frac{n+1}{p}+\frac12}_{p,1}}\quad\text{if } n\geq 3,$$

and

$$Y_{p,n}:=\|(v^\varepsilon,B^\varepsilon)\|_{\tilde L^\infty\dot B^{\frac{n+\kappa}{p}-\frac\kappa2-1}_{p,1}\cap L^1\dot B^{\frac{n+\kappa}{p}-\frac\kappa2+1}_{p,1}}\quad\text{if } n=2.$$

We claim that for any p, satisfying the assumption stated in Theorem 2.1, we have

$$Y_{p,n} \lesssim \varepsilon^{\frac{1}{2}-\frac{1}{p}} C_0^\varepsilon + \|(\mathcal{P}u_0^\varepsilon - v_0, H_0^\varepsilon - v_0)\|_{\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{3}{2}}} \quad \text{if} \quad n \geq 3. \tag{4.10}$$

And

$$Y_{p,n} \lesssim \varepsilon^{\kappa(\frac{1}{2}-\frac{1}{p})} C_0^\varepsilon + \|(\mathcal{P}u_0^\varepsilon - v_0, H_0^\varepsilon - v_0)\|_{\dot{B}_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}-1}} \quad \text{if} \quad n = 2, \tag{4.11}$$

with $\kappa \in [0, 1/2]$ and $\kappa < \frac{8-2p}{p-2}$.

Actually, by virtue of the inequality (1.8), we have

$$Y_{p,n} \lesssim \|(\mathcal{P}u_0^\varepsilon - v_0, H_0^\varepsilon - v_0)\|_{\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{3}{2}}} + \sum_{i=1}^{4} \|J_i^\varepsilon\|_{L^1\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{3}{2}}}, \quad \text{if } n \geq 3, \tag{4.12}$$

and

$$Y_{p,n} \lesssim \|(\mathcal{P}u_0^\varepsilon - v_0, H_0^\varepsilon - v_0)\|_{\dot{B}_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}-1}} + \sum_{i=1}^{4} \|J_i^\varepsilon\|_{L^1\dot{B}_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}-1}} \quad \text{if } n = 2. \tag{4.13}$$

Next we deal with $J_i^\varepsilon (i = 1, 2, 3, 4)$ term by using product and composition estimates in the spirit of the previous sections.

Case $n \geq 3$:

It is easy to see that $\frac{n+1}{p} - \frac{1}{2} \leq \frac{n}{p}$ and $\frac{n+1}{p} - \frac{3}{2} + \frac{n}{p} > 0$, we will repeatedly use Proposition A.2.

For terms in J_1^ε:

$$\begin{aligned} \|u^\varepsilon \cdot \nabla v^\varepsilon\|_{L^1\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{3}{2}}} &\lesssim \|u^\varepsilon\|_{\tilde{L}^\infty\dot{B}_{p,1}^{\frac{n}{p}-1}} \|\nabla v^\varepsilon\|_{L^1\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{1}{2}}} \\ &\lesssim C_0^\varepsilon \|v^\varepsilon\|_{L^1\dot{B}_{p,1}^{\frac{n+1}{p}+\frac{1}{2}}} \lesssim \eta Y_{n,p}, \end{aligned}$$

$$\begin{aligned} \|\mathcal{P}^\perp u^\varepsilon \cdot \nabla v\|_{L^1\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{3}{2}}} &\lesssim \|\mathcal{P}^\perp u^\varepsilon\|_{\tilde{L}^2\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{1}{2}}} \|\nabla v\|_{\tilde{L}^2\dot{B}_{p,1}^{\frac{n}{p}-1}} \\ &\lesssim \varepsilon^{\frac{1}{2}-\frac{1}{p}} C_0^\varepsilon \|v\|_{\tilde{L}^2\dot{B}_{p,1}^{\frac{n}{p}}} \lesssim \eta^2 \varepsilon^{\frac{1}{2}-\frac{1}{p}}, \end{aligned}$$

$$\begin{aligned} \|v^\varepsilon \cdot \nabla v\|_{L^1\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{3}{2}}} &\lesssim \|v^\varepsilon\|_{\tilde{L}^2\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{1}{2}}} \|\nabla v\|_{\tilde{L}^2\dot{B}_{p,1}^{\frac{n}{p}-1}} \\ &\lesssim Y_{p,n} \|v\|_{\tilde{L}^2\dot{B}_{p,1}^{\frac{n}{p}}} \lesssim \eta Y_{p,n}, \end{aligned}$$

$$\begin{aligned} \|u^\varepsilon \cdot \nabla \mathcal{P}^\perp u^\varepsilon\|_{L^1\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{3}{2}}} &\lesssim \|u^\varepsilon\|_{\tilde{L}^2\dot{B}_{p,1}^{\frac{n}{p}}} \|\nabla \mathcal{P}^\perp u^\varepsilon\|_{\tilde{L}^2\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{3}{2}}} \\ &\lesssim \varepsilon^{\frac{1}{2}-\frac{1}{p}} (C_0^\varepsilon)^2 \lesssim \eta^2 \varepsilon^{\frac{1}{2}-\frac{1}{p}}, \end{aligned}$$

$$\begin{aligned} \|H^\varepsilon \cdot \nabla B^\varepsilon\|_{L^1\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{3}{2}}} &\lesssim \|H^\varepsilon\|_{\tilde{L}^2\dot{B}_{p,1}^{\frac{n}{p}}} \|\nabla B^\varepsilon\|_{\tilde{L}^2\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{3}{2}}} \\ &\lesssim C_0^\varepsilon \|B^\varepsilon\|_{\tilde{L}^2\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{1}{2}}} \lesssim \eta Y_{p,n}, \end{aligned}$$

$$\begin{aligned}\|B^\varepsilon\cdot\nabla B\|_{L^1\dot B_{p,1}^{\frac{n+1}{p}-\frac32}} &\lesssim \|B^\varepsilon\|_{\tilde L^\infty \dot B_{p,1}^{\frac{n+1}{p}-\frac32}}\|\nabla B\|_{L^1\dot B_{p,1}^{\frac np}}\\ &\lesssim \eta Y_{p,n}.\end{aligned}$$

Then we get

$$\|J_1^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n+1}{p}-\frac32}} \lesssim \eta Y_{p,n}+\varepsilon^{\frac12-\frac1p}\eta^2. \tag{4.14}$$

For terms in J_2^ε:

$$\begin{aligned}\|I(\varepsilon a^\varepsilon)\mathcal A u^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n+1}{p}-\frac32}} &\lesssim \|I(\varepsilon a^\varepsilon)\|_{\tilde L^\infty \dot B_{p,1}^{\frac{n+1}{p}-\frac12}}\|\nabla^2 u^\varepsilon\|_{L^1\dot B_{p,1}^{\frac np-1}}\\ &\lesssim \big(1+\|\varepsilon a^\varepsilon\|_{\tilde L^\infty \dot B_{p,1}^{n/p}}\big)^{n/2+1}\|\varepsilon a^\varepsilon\|_{\tilde L^\infty \dot B_{p,1}^{\frac{n+1}{p}-\frac12}}\|u^\varepsilon\|_{L^1\dot B_{p,1}^{\frac np+1}},\end{aligned}$$

and

$$\begin{aligned}&\|I(\varepsilon a^\varepsilon)\big((\nabla\times H^\varepsilon)\times H^\varepsilon\big)\|_{L^1\dot B_{p,1}^{\frac{n+1}{p}-\frac32}}\\ &\lesssim \|I(\varepsilon a^\varepsilon)\|_{\tilde L^\infty \dot B_{p,1}^{\frac{n+1}{p}-\frac12}}\|\big((\nabla\times H^\varepsilon)\times H^\varepsilon\big)\|_{L^1\dot B_{p,1}^{\frac np-1}}\\ &\lesssim \big(1+\|\varepsilon a^\varepsilon\|_{L^\infty \dot B_{p,1}^{n/p}}\big)^{n/2+1}\|\varepsilon a^\varepsilon\|_{\tilde L^\infty \dot B_{p,1}^{\frac{n+1}{p}-\frac12}}\|H^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac np}}\|\nabla H^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac np-1}}.\end{aligned}$$

Besides, we have

$$\begin{aligned}\|\varepsilon a^\varepsilon\|_{L^\infty \dot B_{p,1}^{\frac{n+1}{p}-\frac12}} &\lesssim \|\varepsilon a^\varepsilon\|^{\ell,\varepsilon}_{\tilde L^\infty \dot B_{p,1}^{\frac{n+1}{p}-\frac12}}+\|\varepsilon a^\varepsilon\|^{h,\varepsilon}_{\tilde L^\infty \dot B_{p,1}^{\frac{n+1}{p}-\frac12}}\\ &\lesssim \varepsilon^{1/2-1/p}\|a^\varepsilon\|^{\ell,\varepsilon}_{\tilde L^\infty \dot B_{p,1}^{\frac n2-1}}+\varepsilon^{1/2-1/p}\|\varepsilon a^\varepsilon\|^{h,\varepsilon}_{\tilde L^\infty \dot B_{p,1}^{\frac np}}\\ &\lesssim \varepsilon^{1/2-1/p}C_0^\varepsilon.\end{aligned}$$

So the above estimates give

$$\|J_2^\varepsilon\|_{\tilde L^\infty \dot B_{p,1}^{\frac{n+1}{p}-\frac32}} \lesssim \varepsilon^{1/2-1/p}\eta(1+\eta)^{n/2+1}(\eta+\eta^2). \tag{4.15}$$

For terms in J_3^ε:

$$\begin{aligned}\|B^\varepsilon\cdot\nabla u^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n+1}{p}-\frac32}} &\lesssim \|B^\varepsilon\|_{\tilde L^2 \dot B_{p,1}^{\frac{n+1}{p}-\frac12}}\|\nabla u^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac np-1}}\\ &\lesssim \eta Y_{p,n},\end{aligned}$$

$$\begin{aligned}\|u^\varepsilon\cdot\nabla B^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n+1}{p}-\frac32}} &\lesssim \|u^\varepsilon\|_{\tilde L^2 \dot B_{p,1}^{\frac np}}\|\nabla B^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n+1}{p}-\frac32}}\\ &\lesssim C_0^\varepsilon\|B^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n+1}{p}-\frac12}} \lesssim \eta Y_{p,n},\end{aligned}$$

$$\begin{aligned}\|B\cdot\nabla v^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n+1}{p}-\frac32}} &\lesssim \|B\|_{\tilde L^2 \dot B_{p,1}^{\frac np}}\|\nabla v^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n+1}{p}-\frac32}}\\ &\lesssim \eta\|v^\varepsilon\|_{\tilde L^1\dot B_{p,1}^{\frac{n+1}{p}-\frac12}} \lesssim \eta Y_{n,p},\end{aligned}$$

$$\begin{aligned}\|v^\varepsilon\cdot\nabla B\|_{L^1\dot B_{p,1}^{\frac{n+1}{p}-\frac32}}&\lesssim\|v^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n+1}{p}-\frac12}}\|\nabla B\|_{L^1\dot B_{p,1}^{\frac np-1}}\\&\lesssim\eta Y_{p,n},\end{aligned}$$

$$\begin{aligned}\|B\cdot\nabla\mathcal P^\perp u^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n+1}{p}-\frac32}}&\lesssim\|B\|_{\tilde L^2\dot B_{p,1}^{\frac np}}\|\nabla\mathcal P^\perp u^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n+1}{p}-\frac32}}\\&\lesssim\varepsilon^{\frac12-\frac1p}\eta^2,\end{aligned}$$

$$\begin{aligned}\|\mathcal P^\perp u^\varepsilon\cdot\nabla B\|_{L^1\dot B_{p,1}^{\frac{n+1}{p}-\frac32}}&\lesssim\|\mathcal P^\perp u^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n+1}{p}-\frac12}}\|\nabla B\|_{\tilde L^2\dot B_{p,1}^{\frac np-1}}\\&\lesssim\varepsilon^{\frac12-\frac1p}\eta^2.\end{aligned}$$

Then we get

$$\|J_3^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n+1}{p}-\frac32}}\lesssim\eta Y_{p,n}+\varepsilon^{\frac12-\frac1p}\eta^2. \tag{4.16}$$

For J_4^ε:

$$\begin{aligned}\|J_4^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n+1}{p}-\frac32}}&\lesssim\|H^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac np-\frac12}}\|\mathcal P^\perp u^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n+1}{p}-\frac32}}\\&\lesssim\varepsilon^{\frac12-\frac1p}\eta^2.\end{aligned} \tag{4.17}$$

Plugging all the estimates (4.14)–(4.17) into (4.12) and remembering that η is sufficiently small, then we obtain (4.10).

Case $n=2$:

In order to use Proposition A.2, we need $\frac{2+\kappa}{p}-\frac\kappa2-1+\frac2p>0$ which indicates that $\kappa<\frac{8-2p}{p-2}$.

For terms in J_1^ε:

$$\begin{aligned}\|u^\varepsilon\cdot\nabla v^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac\kappa2-1}}&\lesssim\|u^\varepsilon\|_{\tilde L^\infty\dot B_{p,1}^{\frac np-1}}\|\nabla v^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac\kappa2}}\\&\lesssim C_0^\varepsilon\|v^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac\kappa2+1}}\lesssim\eta Y_{n,p},\end{aligned}$$

$$\begin{aligned}\|\mathcal P^\perp u^\varepsilon\cdot\nabla v\|_{L^1\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac\kappa2-1}}&\lesssim\|\mathcal P^\perp u^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac\kappa2}}\|\nabla v\|_{L^1\dot B_{p,1}^{\frac np-1}}\\&\lesssim\varepsilon^{\kappa(\frac12-\frac1p)}C_0^\varepsilon\|v^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac np}}\lesssim\varepsilon^{\kappa(\frac12-\frac1p)}\eta^2,\end{aligned}$$

$$\begin{aligned}\|v^\varepsilon\cdot\nabla v\|_{L^1\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac\kappa2-1}}&\lesssim\|v^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac\kappa2}}\|\nabla v\|_{\tilde L^2\dot B_{p,1}^{\frac np-1}}\\&\lesssim Y_{p,n}\|v\|_{\tilde L^2\dot B_{p,1}^{\frac np}}\lesssim\eta Y_{p,n},\end{aligned}$$

$$\begin{aligned}\|u^\varepsilon\cdot\nabla\mathcal P^\perp u^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac\kappa2-1}}&\lesssim\|u^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac np}}\|\nabla\mathcal P^\perp u^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac\kappa2-1}}\\&\lesssim\varepsilon^{\kappa(\frac12-\frac1p)}(C_0^\varepsilon)^2\lesssim\varepsilon^{\kappa(\frac12-\frac1p)}\eta^2,\end{aligned}$$

$$\begin{aligned}\|H^\varepsilon\cdot\nabla B^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac\kappa2-1}}&\lesssim\|H^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac np}}\|\nabla B^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac\kappa2-1}}\\&\lesssim C_0^\varepsilon\|B^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n+1}{p}-\frac12}}\lesssim\eta Y_{p,n},\end{aligned}$$

$$\begin{aligned}\|B^\varepsilon\cdot\nabla B\|_{L^1\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}-1}} &\lesssim \|B^\varepsilon\|_{\tilde L^\infty\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}-1}}\|\nabla B\|_{L^1\dot B_{p,1}^{\frac{n}{p}}}\\ &\lesssim \eta Y_{p,n}.\end{aligned}$$

Then we get

$$\|J_1^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}-1}} \lesssim \eta Y_{p,n}+\varepsilon^{\kappa(\frac12-\frac1p)}\eta^2. \tag{4.18}$$

For terms in J_2^ε:

$$\begin{aligned}\|I(\varepsilon a^\varepsilon)\mathcal{A}u^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}-1}} &\lesssim \|I(\varepsilon a^\varepsilon)\|_{\tilde L^\infty\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}}}\|\nabla^2u^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n}{p}-1}}\\ &\lesssim \big(1+\|\varepsilon a^\varepsilon\|_{\tilde L^\infty\dot B_{p,1}^{n/p}}\big)^{n/2+1}\|\varepsilon a^\varepsilon\|_{\tilde L^\infty\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}}}\|u^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n}{p}+1}},\end{aligned}$$

and

$$\begin{aligned}&\|I(\varepsilon a^\varepsilon)\big((\nabla\times H^\varepsilon)\times H^\varepsilon\big)\|_{L^1\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}-1}}\\ &\lesssim \|I(\varepsilon a^\varepsilon)\|_{\tilde L^\infty\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}}}\|(\nabla\times H^\varepsilon)\times H^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n}{p}-1}}\\ &\lesssim \big(1+\|\varepsilon a^\varepsilon\|_{L^\infty\dot B_{p,1}^{n/p}}\big)^{n/2+1}\|\varepsilon a^\varepsilon\|_{\tilde L^\infty\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}}}\|H^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n}{p}}}\|\nabla H^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n}{p}-1}}.\end{aligned}$$

Besides, we have

$$\begin{aligned}\|\varepsilon a^\varepsilon\|_{L^\infty\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}}} &\lesssim \|\varepsilon a^\varepsilon\|^{\ell,\varepsilon}_{L^\infty\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}}}+\|\varepsilon a^\varepsilon\|^{h,\varepsilon}_{L^\infty\dot B_{p,1}^{\frac{n+1}{p}-\frac12}}\\ &\lesssim \varepsilon^{\kappa(1/2-1/p)}\|a^\varepsilon\|^{\ell,\varepsilon}_{\tilde L^\infty\dot B_{p,1}^{\frac{n}{2}-1}}+\varepsilon^{\kappa(1/2-1/p)}\|\varepsilon a^\varepsilon\|^{h,\varepsilon}_{\tilde L^\infty\dot B_{p,1}^{\frac{n}{p}}}\\ &\lesssim \varepsilon^{\kappa(1/2-1/p)}C_0^\varepsilon.\end{aligned}$$

So the above estimates give

$$\|J_2^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}-1}} \lesssim \varepsilon^{\kappa(1/2-1/p)}\eta(1+\eta)^{n/2+1}(\eta+\eta^2). \tag{4.19}$$

For terms in J_3^ε:

$$\begin{aligned}\|B^\varepsilon\cdot\nabla u^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}-1}} &\lesssim \|B^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}}}\|\nabla u^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n}{p}-1}}\\ &\lesssim \eta Y_{p,n},\end{aligned}$$

$$\begin{aligned}\|u^\varepsilon\cdot\nabla B^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}-1}} &\lesssim \|u^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n}{p}}}\|\nabla B^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}-1}}\\ &\lesssim C_0^\varepsilon\|B^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}}}\lesssim \eta Y_{p,n},\end{aligned}$$

$$\begin{aligned}\|B\cdot\nabla v^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}-1}} &\lesssim \|B\|_{\tilde L^2\dot B_{p,1}^{\frac{n}{p}}}\|\nabla v^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}-1}}\\ &\lesssim \eta\|v^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}}}\lesssim \eta Y_{n,p},\end{aligned}$$

$$\begin{aligned}\|v^\varepsilon\cdot\nabla B\|_{L^1\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}-1}} &\lesssim \|v^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}}}\|\nabla B\|_{\tilde L^2\dot B_{p,1}^{\frac{n}{p}-1}}\\ &\lesssim \eta Y_{p,n},\end{aligned}$$

$$\begin{aligned}\|B\cdot\nabla\mathcal{P}^\perp u^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}-1}} &\lesssim \|B\|_{\tilde L^2\dot B_{p,1}^{\frac{n}{p}}}\|\nabla\mathcal{P}^\perp u^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}-1}}\\ &\lesssim \varepsilon^{\kappa(\frac12-\frac1p)}\eta^2,\end{aligned}$$

$$\begin{aligned}\|\mathcal{P}^\perp u^\varepsilon\cdot\nabla B\|_{L^1\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}-1}} &\lesssim \|\mathcal{P}^\perp u^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}}}\|\nabla B\|_{\tilde L^2\dot B_{p,1}^{\frac{n}{p}-1}}\\ &\lesssim \varepsilon^{\kappa(\frac12-\frac1p)}\eta^2.\end{aligned}$$

Then we get

$$\|J_3^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}-1}} \lesssim \eta Y_{p,n} + \varepsilon^{\kappa(\frac12-\frac1p)}\eta^2. \tag{4.20}$$

For J_4^ε:

$$\begin{aligned}\|J_4^\varepsilon\|_{L^1\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}-1}} &\lesssim \|H^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n}{p}}}\|\mathcal{P}^\perp u^\varepsilon\|_{\tilde L^2\dot B_{p,1}^{\frac{n+\kappa}{p}-\frac{\kappa}{2}-1}}\\ &\lesssim \varepsilon^{\kappa(\frac12-\frac1p)}\eta^2.\end{aligned} \tag{4.21}$$

Plugging all the estimates (4.18)–(4.21) into (4.13) and remembering that η is sufficiently small, then we obtain (4.11). $\square$

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Appendix A. Some estimates in Besov spaces

In what follows, we denote the characteristic function defined in $\mathbb{Z}$ by $\chi\{\cdot\}$ and by $\{c(j)\}_{j\in\mathbb{Z}}$ a sequence on ℓ^1 with the norm $\|\{c(j)\}\|_{\ell^1}=1$.

Lemma A.1. *Let $s,t,\sigma,\tau\in\mathbb{R}$, $2\le p\le 4$ and $1\le r,r_1,r_2,r_3,r_4\le\infty$ with $\frac1r=\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{r_3}+\frac{1}{r_4}$. Then we have the following:*

1. For $2^j\le R_0$, if $s\le n/p$ and $\sigma\le 2n/p-n/2$, then

$$\begin{aligned}&\|\dot\Delta_j(T_fg)\|_{L_T^rL^2}\\ &\le Cc(j)2^{j(n/p-s-t)}\|f\|_{\tilde L_T^{r_1}\dot B_{p,1}^s}\|g\|^\ell_{\tilde L_T^{r_2}\dot B_{2,1}^t}\\ &\quad + C\chi_{\{2^j\sim R_0\}}2^{j(2n/p-n/2-\sigma-\tau)}\|f\|_{\tilde L_T^{r_3}\dot B_{p,1}^\sigma}\|g\|^h_{\tilde L_T^{r_4}\dot B_{p,1}^\tau};\end{aligned} \tag{A.1}$$

or if $\sigma\le 2n/p-n/2$, then

$$\|\dot\Delta_j(T_fg)\|_{L_T^rL^2}$$

$$\leq Cc(j)2^{j(2n/p-n/2-\sigma-\tau)}\|f\|_{\tilde{L}^{r_1}_T\dot{B}^{\sigma}_{p,1}}\|g\|_{\tilde{L}^{r_2}_T\dot{B}^{\tau}_{p,1}}. \tag{A.2}$$

2. For $2^j > R_0$, if $\sigma \leq n/p$, then

$$\begin{aligned}&\|\dot{\Delta}_j(T_fg)\|_{L^r_TL^p}\\ &\leq Cc(j)2^{j(n/p-\sigma-\tau)}\|f\|_{\tilde{L}^{r_1}_T\dot{B}^{\sigma}_{p,1}}\|g\|_{\tilde{L}^{r_2}_T\dot{B}^{\tau}_{p,1}}.\end{aligned} \tag{A.3}$$

Proof. First we decompose T_fg into $T_fg^\ell + T_fg^h$. Thanks to (1.6), we have

$$\begin{aligned}\dot{\Delta}_j(T_fg^\ell) &= \sum_{|k-j|\leq 4}\dot{\Delta}_j(\dot{S}_{k-1}f\dot{\Delta}_kg^\ell)\\ &= \sum_{|k-j|\leq 4}\sum_{k'\leq k-2}\dot{\Delta}_j(\dot{\Delta}_{k'}f\dot{\Delta}_kg^\ell).\end{aligned}$$

Denote $J := \{(k,k') : |k-j| \leq 4, k' \leq k-2\}$, then for $2^j \leq R_0$,

$$\begin{aligned}\|\dot{\Delta}_j(T_fg^\ell)\|_{L^r_TL^2} &\leq \sum_J\|\dot{\Delta}_j(\dot{\Delta}_{k'}f\dot{\Delta}_kg^\ell)\|_{L^r_tL^2}\\ &\lesssim \sum_J 2^{k's}\|\dot{\Delta}_{k'}f\|_{L^{r_1}_tL^p}2^{k'(n/p-s)}2^{kt}\|\dot{\Delta}_kg^\ell\|_{L^{r_2}_tL^2}2^{-kt}\\ &\lesssim c(j)2^{j(n/p-s-t)}\|f\|_{\tilde{L}^{r_1}\dot{B}^s_{p,1}}\|g\|^\ell_{\tilde{L}^{r_2}\dot{B}^t_{2,1}}.\end{aligned}$$

And

$$\begin{aligned}\|\dot{\Delta}_j(T_fg^h)\|_{L^r_TL^2} &\leq \sum_J\|\dot{\Delta}_j(\dot{\Delta}_{k'}f\dot{\Delta}_kg^h)\|_{L^r_tL^2}\\ &\lesssim \sum_{2^k\sim 2^j\sim R_0} 2^{k's}\|\dot{\Delta}_{k'}f\|_{L^{r_3}_tL^p}2^{k'(2n/p-n/2-\sigma)}2^{k\tau}\|\dot{\Delta}_kg^h\|_{L^{r_4}_tL^p}2^{-k\tau}\\ &\lesssim c(j)\chi_{\{2^j\sim R_0\}}2^{j(2n/p-n/2-\sigma-\tau)}\|f\|_{\tilde{L}^{r_3}\dot{B}^s_{p,1}}\|g\|^h_{\tilde{L}^{r_4}\dot{B}^\tau_{p,1}}.\end{aligned}$$

The above two estimates for $\dot{\Delta}_j(T_fg^\ell)$ and $\dot{\Delta}_j(T_fg^h)$ indicate (A.1). While the proof of (A.2) is essentially the same with the estimate of $\dot{\Delta}_j(T_fg^h)$, we omit the details. Now we come to prove (A.3). For $2^j > R_0$ and $\sigma \leq n/p$

$$\begin{aligned}\|\dot{\Delta}_j(T_fg)\|_{L^r_TL^p} &\leq \sum_J\|\dot{\Delta}_j(\dot{\Delta}_{k'}f\dot{\Delta}_kg)\|_{L^r_tL^p}\\ &\lesssim \sum_J\|\dot{\Delta}_{k'}f\|_{L^{r_1}_tL^\infty}\|\dot{\Delta}_kg\|_{L^{r_2}_tL^p}\\ &\lesssim \sum_J 2^{k'\sigma}\|\dot{\Delta}_{k'}f\|_{L^{r_1}_tL^p}2^{k'(n/p-\sigma)}2^{k\tau}\|\dot{\Delta}_kg\|_{L^{r_2}_tL^p}2^{-k\tau}\\ &\lesssim c(j)2^{j(n/p-\sigma-\tau)}\|f\|_{\tilde{L}^{r_1}\dot{B}^\sigma_{p,1}}\|g\|_{\tilde{L}^{r_2}\dot{B}^\tau_{p,1}}. \quad\square\end{aligned}$$

Lemma A.2. *Let $\sigma,\tau \in \mathbb{R}$, $2 \leq p \leq 4$ and $1 \leq r, r_1, r_2 \leq \infty$ with $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. Assume that $\sigma + \tau > 0$. Then we have*

$$\begin{aligned}&\|\dot{\Delta}_jR(f,g)\|_{L^r_TL^2}\\ &\leq Cc(j)2^{j(2n/p-n/2-\sigma-\tau)}\|f\|_{\tilde{L}^{r_1}_T\dot{B}^{\sigma}_{p,1}}\|g\|_{\tilde{L}^{r_2}_T\dot{B}^{\tau}_{p,1}},\end{aligned} \tag{A.4}$$

and

$$\begin{aligned}&\|\dot{\Delta}_jR(f,g)\|_{L^r_TL^p}\\ &\leq Cc(j)2^{j(n/p-\sigma-\tau)}\|f\|_{\tilde{L}^{r_1}_T\dot{B}^{\sigma}_{p,1}}\|g\|_{\tilde{L}^{r_2}_T\dot{B}^{\tau}_{p,1}}.\end{aligned} \tag{A.5}$$

Proof. Thanks to (1.6), we have

$$\dot{\Delta}_j R(f,g) = \sum_{k\geq j-3}\sum_{|k-k'|\leq 1} \dot{\Delta}_j(\dot{\Delta}_k f \dot{\Delta}_{k'} g).$$

Denote $J := \{(k,k') : k \geq j-3, |k-k'| \leq 1\}$. Then when $\sigma+\tau > 0$ and $2 \leq p \leq 4$, we have

$$\begin{aligned}
\|\dot{\Delta}_j R(f,g)\|_{L^r_t L^2} &\lesssim 2^{j(2n/p-n/2)} \sum_{(k,k')\in J} \|\dot{\Delta}_k f \dot{\Delta}_{k'} g\|_{L^r_t L^{p/2}} \\
&\lesssim 2^{j(2n/p-n/2)} \sum_{(k,k')\in J} \|\dot{\Delta}_k f\|_{L^{r_1}_t L^p} \|\dot{\Delta}_{k'} g\|_{L^{r_2}_t L^p} \\
&\lesssim 2^{j(2n/p-n/2)} \sum_{(k,k')\in J} 2^{k\sigma}\|\dot{\Delta}_k f\|_{L^{r_1}_t L^p} 2^{-k\sigma} 2^{k\tau}\|\dot{\Delta}_{k'} g\|_{L^{r_2}_t L^p} 2^{-k'\tau} \\
&\lesssim c(j) 2^{j(2n/p-n/2-\sigma-\tau)} \|f\|_{\tilde{L}^{r_1}_t \dot{B}^{\sigma}_{p,1}} \|g\|_{\tilde{L}^{r_2}_t \dot{B}^{\tau}_{p,1}},
\end{aligned}$$

and

$$\begin{aligned}
\|\dot{\Delta}_j R(f,g)\|_{L^r_t L^p} &\lesssim 2^{jn/p} \sum_{(k,k')\in J} \|\dot{\Delta}_k f \dot{\Delta}_{k'} g\|_{L^r_t L^{p/2}} \\
&\lesssim 2^{jn/p} \sum_{(k,k')\in J} \|\dot{\Delta}_k f\|_{L^{r_1}_t L^p} \|\dot{\Delta}_{k'} g\|_{L^{r_2}_t L^p} \\
&\lesssim 2^{jn/p} \sum_{(k,k')\in J} 2^{k\sigma}\|\dot{\Delta}_k f\|_{L^{r_1}_t L^p} 2^{-k\sigma} 2^{k\tau}\|\dot{\Delta}_{k'} g\|_{L^{r_2}_t L^p} 2^{-k'\tau} \\
&\lesssim c(j) 2^{j(n/p-\sigma-\tau)} \|f\|_{\tilde{L}^{r_1}_t \dot{B}^{\sigma}_{p,1}} \|g\|_{\tilde{L}^{r_2}_t \dot{B}^{\tau}_{p,1}}.
\end{aligned}$$

This finishes the proof of Lemma A.2. □

Proposition A.1. *Let $2 \leq p \leq \min\{4, \frac{2n}{n-2}\}$ and $p \neq 4$ if $n = 2$. $1 \leq r, r_1, r_2, r_3, r_4 \leq \infty$ with $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3} + \frac{1}{r_4}$. Then we have*

For $2^j \leq R_0$, $\gamma \in \mathbb{R}$

$$\begin{aligned}
\sum_{2^j\leq R_0} 2^{j(n/2-1)} \|\dot{\Delta}_j(fg)\|_{L^r_t L^2} &\lesssim \|f\|_{\tilde{L}^{r_1}_t \dot{B}^{n/p}_{p,1}} \|g\|_{\tilde{L}^{r_2}_t \dot{B}^{n/p-1}_{p,1}} \\
&\quad + \|f\|_{\tilde{L}^{r_1}_t \dot{B}^{n/p}_{p,1}} \|g\|^{\ell}_{\tilde{L}^{r_2}_t \dot{B}^{n/2-1}_{2,1}} + R_0^{-\gamma} \|f\|_{\tilde{L}^{r_3}_t \dot{B}^{n/p-1}_{p,1}} \|g\|^{h}_{\tilde{L}^{r_4}_t \dot{B}^{n/p+\gamma}_{p,1}};
\end{aligned} \tag{A.6}$$

or $\gamma \leq 0$,

$$\begin{aligned}
\sum_{2^j\leq R_0} 2^{j(n/2-1)} \|\dot{\Delta}_j(fg)\|_{L^r_t L^2} &\lesssim \|f\|_{\tilde{L}^{r_1}_t \dot{B}^{n/p}_{p,1}} \|g\|_{\tilde{L}^{r_2}_t \dot{B}^{n/p-1}_{p,1}} \\
&\quad + R_0^{-\gamma} \|f\|_{\tilde{L}^{r_3}_t \dot{B}^{n/p-1}_{p,1}} \|g\|_{\tilde{L}^{r_4}_t \dot{B}^{n/p+\gamma}_{p,1}}.
\end{aligned} \tag{A.7}$$

For $2^j > R_0$, $0 \leq \gamma \leq 1$

$$\sum_{2^j> R_0} 2^{j(n/p-1)} \|\dot{\Delta}_j(fg)\|_{L^r_t L_p} \lesssim R_0^{-\gamma} \|f\|_{\tilde{L}^{r_1}_t \dot{B}^{n/p}_{p,1}} \|g\|_{\tilde{L}^{r_2}_t \dot{B}^{n/p-1+\gamma}_{p,1}}. \tag{A.8}$$

Proof. Using Bony decomposition $fg = T_f g + T_g f + R(f,g)$. For the low frequency, we choose $s = n/p, t = n/2-1, \sigma = n/p-1, \tau = n/p+\gamma(\gamma \in \mathbb{R})$ in (A.1) for $T_f g$; $\sigma = n/p-1, \tau = n/p$ in (A.2) for $T_g f$ and $\sigma = n/p, \tau = n/p-1$ in (A.4) for $R(f,g)$. Then summing over j for $2^j \leq R_0$ indicates (A.6).

Also we can choose $\sigma = n/p - 1, \tau = n/p + \gamma(\gamma \le 0)$ in (A.2) for $T_f g$; $\sigma = n/p - 1, \tau = n/p$ in (A.2) for $T_g f$ and $\sigma = n/p, \tau = n/p - 1$ in (A.4) for $R(f, g)$. Summing over j indicates (A.7).

For the high frequency. Applying $\sigma = n/p, \tau = n/p - 1 + \gamma$ in (A.3) for $T_f g$ and in (A.5) $R(f, g)$; applying $\sigma = n/p - 1 - \gamma, \tau = n/p$ in (A.3) for $T_g f$ and summing over j for $2^j > R_0$ indicates (A.8).

All these finish the proof of the proposition. □

Proposition A.2 ([4]). *Let $1 \le p, r, r_1, r_2 \le \infty$ with $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, and $s_1, s_2 \in \mathbb{R}$ satisfying $s_1, s_2 \le \frac{n}{p}, s_1 + s_2 > n\max(0, \frac{2}{p} - 1)$. If $f \in \tilde{L}^{r_1}_T(\dot{B}^{s_1}_{p,1})$ and $g \in \tilde{L}^{r_2}_T(\dot{B}^{s_2}_{p,1})$,then we have*

$$\|fg\|_{\tilde{L}^r_T(\dot{B}^{s_1+s_2-n/p}_{p,1})} \lesssim \|f\|_{\tilde{L}^{r_1}_T\dot{B}^{s_1}_{p,1}}\|g\|_{\tilde{L}^{r_2}_T\dot{B}^{s_2}_{p,1}}. \tag{A.9}$$

Proposition A.3 ([5]). *Assume that $F \in W^{[s]+2,\infty}_{loc}$ with $F(0) = 0$. Then for any $s > 0, p, r \in [1, \infty]$, there holds*

$$\begin{aligned}\|F(f)\|_{\tilde{L}^r_T\dot{B}^s_{p,1}} &\le C(1 + \|f\|_{L^\infty_T L^\infty})^{[s]+1}\|f\|_{\tilde{L}^r_T\dot{B}^s_{p,1}}\\ &\le C(1 + \|f\|_{\tilde{L}^\infty_T\dot{B}^{n/p}_{p,1}})^{[s]+1}\|f\|_{\tilde{L}^r_T\dot{B}^s_{p,1}}.\end{aligned} \tag{A.10}$$

Appendix B. Proof of Theorem 1.1

We will prove Theorem 1.1 by applying the Banach contraction mapping principle. We rewrite (1.3) as follows

$$\begin{cases}\partial_t v - \mu\Delta v = \mathcal{P}(B\cdot\nabla B - v\cdot\nabla v),\\ \partial_t B - \alpha\Delta B = B\cdot v - v\cdot\nabla v,\\ \nabla\cdot v = \nabla\cdot B = 0,\\ (v, B)|_{t=0} = (v_0, B_0).\end{cases} \tag{B.1}$$

Let

$$E := \left\{(v, B) : \|(v, B)\|_{\tilde{L}^\infty\dot{B}^{n/p-1}_{p,1}\cap L^1\dot{B}^{n/p+1}_{p,1}} \le K_0, \nabla\cdot v = \nabla\cdot B = 0\right\}.$$

For simplicity, we denote $\|(v, B)\|_E := \|(v, B)\|_{\tilde{L}^\infty\dot{B}^{n/p-1}_{p,1}\cap L^1\dot{B}^{n/p+1}_{p,1}}$.

Define the map $\Phi(\tilde{v}, \tilde{B}) = (v, B)$ be the solution of the following linear equations

$$\begin{cases}\partial_t v - \mu\Delta v = \mathcal{P}(\tilde{B}\cdot\nabla\tilde{B} - \tilde{v}\cdot\nabla\tilde{v}),\\ \partial_t B - \alpha\Delta B = \tilde{B}\cdot\tilde{v} - \tilde{v}\cdot\nabla\tilde{v},\\ \nabla\cdot v = \nabla\cdot B = 0,\\ (v, B)|_{t=0} = (v_0, B_0).\end{cases}$$

First we will show that Φ is a map from E to E by choosing suitably small η and K_0. Then we will prove that Φ is a contraction. The Banach contraction mapping principle indicates Theorem 1.1.

For any $(\tilde{v}, \tilde{B}) \in E$, using the heat estimate (1.8), we have

$$\begin{aligned}\|v\|_E &\le C\|v_0\|_{\dot{B}^{n/p-1}_{p,1}} + C\|\mathcal{P}(\tilde{B}\cdot\nabla\tilde{B} - \tilde{v}\cdot\nabla\tilde{v})\|_{L^1\dot{B}^{n/p-1}_{p,1}}\\ &\le C\|v_0\|_{\dot{B}^{n/p-1}_{p,1}} + \|(\tilde{B}, \tilde{v})\|_{\tilde{L}^\infty\dot{B}^{n/p-1}_{p,1}}\|(\nabla\tilde{B}, \nabla\tilde{v})\|_{L^1\dot{B}^{n/p}_{p,1}}\\ &\le C\eta + C\|(\tilde{B}, \tilde{v})\|^2_E\\ &\le C\eta + CK_0^2,\end{aligned} \tag{B.2}$$

and

$$\begin{aligned} \|B\|_E &\le C\|B_0\|_{\dot B^{n/p-1}_{p,1}} + C\|(\tilde B\cdot\nabla\tilde v - \tilde v\cdot\nabla\tilde B)\|_{L^1\dot B^{n/p-1}_{p,1}} \\ &\le C\|B_0\|_{\dot B^{n/p-1}_{p,1}} + \|(\tilde B,\tilde v)\|_{\tilde L^\infty\dot B^{n/p-1}_{p,1}}\|(\nabla\tilde B,\nabla\tilde v)\|_{L^1\dot B^{n/p}_{p,1}} \\ &\le C\eta + C\|(\tilde B,\tilde v)\|_E^2 \\ &\le C\eta + CK_0^2. \end{aligned} \tag{B.3}$$

Combining (B.2) and (B.3), by choosing η, K_0 suitable small, we get

$$\|(v,B)\|_E \le C\eta + CK_0^2 \le K_0,$$

which indicates that $(v,B)\in E$.

Besides, let $(v_i,B_i) = \Phi(\tilde v_i,\tilde B_i)(i=1,2)$. Then $(V,U) := (v_1-v_2, B_1-B_2)$ is a solution of the following equations

$$\begin{cases} \partial_t V - \mu\Delta V = \mathcal{P}\big((\tilde B_1-\tilde B_2)\cdot\nabla\tilde B_1 + \tilde B_2\cdot\nabla(\tilde B_1-\tilde B_2) \\ \qquad\qquad - (\tilde v_1-\tilde v_2)\cdot\nabla\tilde v_2 - \tilde v_1\cdot\nabla(\tilde v_1-\tilde v_2)\big), \\ \partial_t U - \alpha\Delta U = \big((\tilde B_1-\tilde B_2)\cdot\nabla\tilde v_1 + \tilde B_2\cdot\nabla(\tilde v_1-\tilde v_2) \\ \qquad\qquad - (\tilde v_1-\tilde v_2)\cdot\nabla\tilde B_2 - \tilde v_2\cdot\nabla(\tilde B_1-\tilde B_2)\big) \\ (V,U)|_{t=0} = (0,0). \end{cases} \tag{B.4}$$

Again using the heat estimate (1.8) and Proposition A.2, we have

$$\begin{aligned} \|(V,U)\|_E &\le C\|(\tilde B_1-\tilde B_2,\tilde v_1-\tilde v_2)\|_E(\|(\tilde v_1,\tilde B_1)\|_E + \|(\tilde v_2,\tilde B_2)\|_E) \\ &\le CK_0\|(\tilde B_1-\tilde B_2,\tilde v_1-\tilde v_2)\|_E \\ &\le c_0\|(\tilde B_1-\tilde B_2,\tilde v_1-\tilde v_2)\|_E, \end{aligned}$$

where $c_0 < 1$ if we choose K_0 is sufficiently small. This implies the contraction of Φ. At the same time, Lemma 1.2 implies the continuity of (v,B) about time. $\square$

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