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A Regularity Condition of 3d Axisymmetric Navier-Stokes Equations

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Abstract In this paper, we study the regularity of 3d axisymmetric Navier-Stokes equations under a prior point assumption on v^r or v^z. That is, the weak solution of the 3d axisymmetric Navier-Stokes equations v is smooth if

$$rv^r \geq -1; \quad \text{or} \quad r\left|v^r(t,x)\right| \leq Cr^\alpha,\ \alpha \in (0,1]; \quad \text{or} \quad r\left|v^z(t,x)\right| \leq Cr^\beta,\ \beta \in [0,1];$$

where r is the distance from the point x to the symmetric axis.

Keywords Axisymmetric · Navier-Stokes equations · Regularity criteria

Mathematics Subject Classification 35Q30 · 76N10

1 Introduction

In the cylindrical coordinates (r,θ,z), we have $x=(x_1,x_2,x_3)=(r\cos\theta, r\sin\theta, z)$ and the axisymmetric solution of the incompressible Navier-Stokes equations is given as

$$v = v^r(r,z,t)e_r + v^\theta(r,z,t)e_\theta + v^z(r,z,t)e_z,$$

where the basis vectors e_r, e_θ, e_z are

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \qquad e_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), \qquad e_z = (0,0,1).$$

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The components v^r, v^θ, v^z, independent of θ, satisfy

$$\begin{cases} \partial_t v^r + (b \cdot \nabla) v^r - \dfrac{(v^\theta)^2}{r} + \partial_r p = \left(\Delta - \dfrac{1}{r^2}\right) v^r, \\ \partial_t v^\theta + (b \cdot \nabla) v^\theta + \dfrac{v^\theta v^r}{r} = \left(\Delta - \dfrac{1}{r^2}\right) v^\theta, \\ \partial_t v^z + (v \cdot \nabla) v^z + \partial_z p = \Delta v^z, \\ b = v^r e_r + v^z e_z, \qquad \nabla \cdot b = \partial_r v^r + \dfrac{v^r}{r} + \partial_z v^z = 0. \end{cases} \tag{1.1}$$

The finite energy smooth solutions of the Navier-Stokes equations satisfy the following energy identity

$$\left\| v(t) \right\|_{L^2}^2 + 2 \int_0^t \left\| \nabla v(s) \right\|_{L^2}^2 ds = \| v_0 \|_{L^2}^2 < +\infty. \tag{1.2}$$

For the axisymmetric velocity v, we can also compute the vorticity $w = \nabla \times v = w^r e_r + w^\theta e_\theta + w^z e_z$ as follows

$$w^r = -\partial_z v^\theta, \qquad w^\theta = \partial_z v^r - \partial_r v^z, \qquad w^z = \left(\partial_r + \frac{1}{r}\right) v^\theta.$$

And (w^r, w^θ, w^z) satisfies

$$\begin{cases} \partial_t w^r + (b \cdot \nabla) w^r - \left(\Delta - \dfrac{1}{r^2}\right) w^r - \left(w^r \partial_r + w^z \partial_z\right) v^r = 0, \\ \partial_t w^\theta + (b \cdot \nabla) w^\theta - \left(\Delta - \dfrac{1}{r^2}\right) w^\theta - 2 \dfrac{v^\theta}{r} \partial_z v^\theta - w^\theta \dfrac{v^r}{r} = 0, \\ \partial_t w^z + (b \cdot \nabla) w^z - \Delta w^z - \left(w^r \partial_r + w^z \partial_z\right) v^z = 0. \end{cases} \tag{1.3}$$

Let us recall some results on the study of the axisymmetric Navier-Stokes equations. When the swirl v^θ is vanishing, Ladyzhenskaya [12] and Ukhovskii-Yudovich [18] independently proved that finite energy weak solutions are regular for all time, see also Leonardi-Malek-Necas-Pokorny [16]. If the swirl v^θ is non-trivial, global in-time regularity of solutions is still open. But recently, tremendous efforts and progress have been made on the regularity of solutions to the axisymmetric Navier-Stokes equations, see [1, 6–8] etc. In [3] and [4], Chen-Strain-Yau-Tsai proved that the suitable weak solution is regular if it satisfies $r|v| \le C < \infty$. The same result was obtained by Koch-Nadirashvili-Seregin-Sverak in [10] by using a different method. Lei and Zhang in [13] extend their results under a more general critical assumption on the drift term $b \in L^\infty([0,T), BMO^{-1})$.

Also, considering on one velocity component, it has been shown that in [11] the axisymmetric solution is smooth in $(0, T) \times \mathbb{R}^3$ when $v^r \in L_t^s L_x^q$ with $3/q + 2/s \le 1$. Neustupa and Pokorny [17] proved the regularity of one component (either v^θ or v^r) implies regularity of the other components of the solutions. Chae-Lee [2] proved regularity assuming a zero-dimensional integral norm on w^θ: $w^\theta \in L_t^s L_x^q$ with $3/q + 2/s \le 2$. Also regularity results come from the work of Jiu-Xin [8] under the assumption that another zero-dimensional scaled norms $\int_{Q_R} (R^{-1}|w^\theta|^2 + R^{-3}|v^\theta|^2) dz$ is sufficiently small for $R > 0$, small enough. See more refined results in [9] and the work of Zhang-Zhang [20].

Most recently, Chen-Fang-Zhang in [5] proved that if rv^θ satisfies $r|v^\theta| \le Cr^\alpha, \alpha > 0$, then v is regular without any other a prior assumptions. Later, Lei-Zhang in [14] improved their result by assuming $r|v^\theta| \le C|\ln r|^{-2}$ for small r. Also Wei in [19] improved the *log* power from -2 to $-\frac{3}{2}$.

Our result is a complement of theirs. We prove the regularity of the solution by assuming Hölder continuity on rv^r and rv^z. Below is the main theorem.

Theorem 1.1 *Let v is an axisymmetric weak solution of the Navier-Stokes equations in $(0,T)\times\mathbb{R}^3$ with the axisymmetric initial data $v_0 \in H^2(\mathbb{R}^3)$ and $\nabla\cdot v_0 = 0$. If rv^r or rv^z satisfies*

$$rv^r \ge -1; \quad or \quad r\left|v^r(t,x)\right| \le Cr^\alpha,\ \alpha\in(0,1]; \quad or \quad r\left|v^z(t,x)\right| \le Cr^\beta,\ \beta\in[0,1];$$

Then v is smooth in $(0,T]\times\mathbb{R}^3$.

Remark 1.1 The case that $r|v^z| \le C$, where $\beta = 0$ implied the regularity of the solution has already been proved in propositon 4.2 of [15].

Remark 1.2 Now we can not prove the regularity of the solution under the assumption $rv^r < -1$.

2 Proof of the Regularity Criteria

First, we state two lemmas which will be very useful in the proof of the theorem.

Lemma 2.1 *Assume v is a smooth axisymmetric solution of (1.1) on $(0,T)$, then v can be stated as follow.*

$$v = v^\theta e_\theta + \nabla\times(\psi e_\theta) = -\partial_z\psi e_r + v^\theta e_\theta + \frac{\partial_r(r\psi)}{r}e_z, \tag{2.1}$$

where ψ is the angular component of the stream function of v.

Next we give a general Sobolev-Hardy inequality whose proof can be found in Lemma 2.4 of [5]. We omit the detail.

Lemma 2.2 *Set $\mathbb{R}^n = \mathbb{R}^k\times\mathbb{R}^{n-k}$ with $2\le k\le n$, and write $x=(x',z)\in\mathbb{R}^k\times\mathbb{R}^{n-k}$. For $1<q<n, 0\le s\le q$ and $s<k$, let $q_*\in[q,\frac{q(n-s)}{n-q}]$. Then there exists a positive constant $C=C(s,q,n,k)$ such that for all $f\in C_0^\infty(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n}\frac{|f|^{q_*}}{|x'|^s}dx \le C\|f\|_q^{\frac{n-s}{q_*}-\frac{n}{q}+1}\|\nabla f\|_q^{\frac{n}{q}-\frac{n-s}{q_*}}.$$

In particular, we pick $n=3$, $k=2$, $q=2$, $q_*\in[2,2(3-s)]$, and assume $0\le s<2$, $r=\sqrt{x_1^2+x_2^2}$. Then there exists a positive constant $C(q_*,s)$ such that for all $f\in C_0^\infty(\mathbb{R}^n)$,

$$\left\|\frac{f}{r^{\frac{s}{q_*}}}\right\|_{q_*} \le C\|f\|_2^{\frac{3-s}{q_*}-\frac{1}{2}}\|\nabla f\|_2^{\frac{3}{2}-\frac{3-s}{q_*}}. \tag{2.2}$$

Proof of Theorem 1.1

Case 1: $rv^r \geq -1$ or $r|v^r(t,x)| \leq Cr^\alpha$, $\alpha \in (0,1]$.

From the equation of v^θ in (1.1), we have

$$\partial_t v^\theta + (b\cdot\nabla)v^\theta - \Delta v^\theta + \frac{rv^r+1}{r^2}v^\theta = 0. \tag{2.3}$$

1. If $rv^r \geq -1$, the coefficient $\frac{rv^r+1}{r^2}$ of v^θ is nonnegative, which has a lower bound 0.
2. If $r|v^r(t,x)| \leq Cr^\alpha$, $\alpha \in (0,1]$, using a simple computation, we can get that

$$\frac{rv^r+1}{r^2} \geq \frac{1-Cr^\alpha}{r^2} \geq c_0,$$

where c_0 is a fixed constant which may be negative. So at last for *Case 1*, we get

$$\frac{rv^r+1}{r^2} \geq \min\{0, c_0\} \tag{2.4}$$

From (2.3) and (2.4), by using Maximum principle and Sobolev embedding inequality, we obtain

$$\sup_t \|v^\theta\|_{L^\infty} \leq \|v_0^\theta\|_{L^\infty} \leq C\|v_0\|_{H^2} < \infty,$$

which implies the regularity of the solution.

Case 2: $r|v^z| < Cr^\beta$, where $\beta = 0$.

Actually this case has already been proven in propositon 4.2 of [15], but for completion of our paper, we state it here again. From (2.1), we have

$$v^r = -\partial_z\psi, \qquad rv^z = \partial_r(r\psi).$$

Then

$$|\psi| = \frac{|\int_0^r sv^z ds|}{r} \leq C,$$

which indicates that ψ is a bounded function which can be embedded in *BMO*. As ψ is the angular component of the stream function of v, we can get that $b \in L^\infty(BMO^{-1})$. The result in [13] implies the regularity of v.

Case 3: $r|v^z| < Cr^\beta$, where $\beta \in (0,1]$.

We will deal with this case by combining a L^4 estimate of v^θ and a L^2 estimate of w^θ. From the equation of v^θ in (1.1), multiplying it with $(v^\theta)^3$ and integrating it over $\mathbb{R}^3$, we can get

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\|v^\theta\|_4^4 &+ \frac{3}{4}\|\nabla(v^\theta)^2\|_2^2 + \int_{\mathbb{R}^3}\frac{(v^\theta)^4}{r^2}dx\\ &= -\int_{\mathbb{R}^3}\frac{v^r}{r}(v^\theta)^4 dx\\ &\triangleq I.\end{aligned} \tag{2.5}$$

From (2.1), we have

$$v^r = -\partial_z \psi, \qquad r v^z = \partial_r (r\psi). \tag{2.6}$$

Then

$$|\psi| = \frac{|\int_0^r s v^z ds|}{r} \le C r^\beta.$$

Using integration by parts and (2.6), we have

$$\begin{aligned} |I| &= \left| -\int_{\mathbb{R}^3} \frac{v^r}{r} \left(v^\theta\right)^4 dx \right| \\ &= \left| \int_{\mathbb{R}^3} \frac{\partial_z \psi}{r} \left(v^\theta\right)^4 dx \right| \\ &= \left| -2 \int_{\mathbb{R}^3} \frac{\psi}{r} \left(v^\theta\right)^2 \partial_z \left(v^\theta\right)^2 dx \right| \\ &\le \frac{1}{4} \left\| \nabla \left(v^\theta\right)^2 \right\|_2^2 + 4 \int_{\mathbb{R}^3} \frac{(v^\theta)^4}{r^{2-\beta}} dx. \end{aligned} \tag{2.7}$$

In (2.2), let $f = (v^\theta)^2$, $q_* = 2$ and $s = 2 - \beta$, then we have

$$\begin{aligned} 4 \int_{\mathbb{R}^3} \frac{(v^\theta)^4}{r^{2-\beta}} dx &\le C_\beta \left\| \left(v^\theta\right)^2 \right\|_2^\beta \left\| \nabla \left(v^\theta\right)^2 \right\|_2^{2-\beta} \\ &\le \frac{1}{4} \left\| \nabla \left(v^\theta\right)^2 \right\|_2^2 + C_\beta \left\| \left(v^\theta\right)^2 \right\|_2^2. \end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8), we have

$$|I| \le \frac{1}{2} \left\| \nabla \left(v^\theta\right)^2 \right\|_2^2 + C_\beta \left\| \left(v^\theta\right)^2 \right\|_2^2. \tag{2.9}$$

At last, combining (2.5) and (2.9), we have

$$\frac{d}{dt} \|v^\theta\|_4^4 + \left\| \nabla \left(v^\theta\right)^2 \right\|_2^2 + \int_{\mathbb{R}^3} \frac{(v^\theta)^4}{r^2} dx \le C \|v^\theta\|_4^4.$$

Applying Gronwall's inequality, we have

$$\sup_{t \in [0,T)} \|v^\theta\|_4^4 + \int_0^T \left\| \frac{(v^\theta)^2}{r} \right\|_2^2 dt \le \|v_0^\theta\|_4^4 e^{CT} < \infty. \tag{2.10}$$

Next we perform a L^2 estimate of w^θ. From the w^θ equation of (1.3), we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|w^\theta\|_{L^2}^2 + \|\nabla w^\theta\|_{L^2}^2 + \int_{\mathbb{R}^3} \frac{(w^\theta)^2}{r^2} dx \\ &\quad = \int_{\mathbb{R}^3} \frac{v^r}{r} \left(w^\theta\right)^2 dx + \int_{\mathbb{R}^3} w^\theta \frac{\partial_z (v^\theta)^2}{r} dx. \end{aligned} \tag{2.11}$$

We can deal with the first term on the right hand of (2.11) the same as I in (2.5). Thus we can get

$$\begin{aligned}
&\frac{d}{dt}\|w^\theta\|_{L^2}^2 + \|\nabla w^\theta\|_{L^2}^2 + \int_{\mathbb{R}^3} \frac{(w^\theta)^2}{r^2} dx \\
&\le C\|w^\theta\|_{L^2}^2 + C\int_{\mathbb{R}^3} w^\theta \frac{\partial_z (v^\theta)^2}{r} dx \\
&= C\|w^\theta\|_{L^2}^2 - C\int_{\mathbb{R}^3} \partial_z w^\theta \frac{(v^\theta)^2}{r} dx \\
&\le C\|w^\theta\|_{L^2}^2 + \frac{1}{2}\|\nabla w^\theta\|_{L^2}^2 + C\left\|\frac{(v^\theta)^2}{r}\right\|_{L^2}^2 . \qquad (2.12)
\end{aligned}$$

Again, using Gronwall's inequality, we have

$$\sup_{t\in[0,T)} \|w^\theta\|_{L^2}^2 \le e^{CT}\left(\|w_0^\theta\|_{L^2}^2 + \int_0^T \left\|\frac{(v_\theta)^2}{r}\right\|_{L^2}^2 dt\right) < \infty.$$

This implies $w^\theta \in L_t^\infty L_x^2$. From the regularity criteria in [2] (Theorem 1), we proved the regularity of the solution. □

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