Z. Angew. Math. Phys. (2023) 74:9

https://doi.org/10.1007/s00033-022-01909-3

Zeitschrift für angewandte Mathematik und Physik ZAMP

BKM-type blow-up criterion of the inviscid axially symmetric Boussinesq system involving a single component of velocity

Zijin Li and Xinghong Pan

Abstract. In this paper, we give a BKM-type blow-up criterion, which involves only a single component of the velocity, for the inviscid axially symmetric Boussinesq system. More precisely, we will show that if the vorticity of the swirl part of the velocity belongs to $L^1(0, T_*, L^\infty)$, then the solution is regular up to time T_*. At present, our results can not be easily generalized to the classical $L^1_t BMO$ regularity criterion, which will be considered in our further works.

Mathematics Subject Classification. 35Q35, 76D05.

Keywords. Inviscid Boussinesq system, Axially symmetric, Regularity criterion, One component.

Contents

Published online: 03 December 2022

1. Introduction

The 3D inviscid Boussinesq system reads

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla p = \rho \boldsymbol{e_3}, \\ \partial_t \rho + u\cdot\nabla\rho - \kappa\Delta\rho = 0, \\ \nabla\cdot u = 0. \end{cases} \tag{1.1}$$

Here, $u \in \mathbb{R}^3$ is the velocity, while $p \in \mathbb{R}$ and $\rho \in \mathbb{R}$ represent the pressure and the temperature fluctuation, respectively. $\boldsymbol{e_3} = (0,0,1)^T$ is the unit vector in the vertical direction, and $\kappa > 0$ stands for the constant thermal diffusivity, respectively, which is assumed to be one without loss of generality in the following.

Physically, Eq. $(1.1)_1$ describes the conservation law of the momentum with the influence of buoyant effect $\rho\boldsymbol{e_3}$, while $(1.1)_2$ represents the temperature fluctuation with diffusivity, and the third line describes the incompressibility of the fluid. This system, which models the convection of an incompressible flow driven by the buoyant effect of a thermal field, plays a vital role in atmospheric science and geophysical applications. It is closely related to a type of the Rayleigh-Bénard convection, which occurs in a horizontal layer of conductive fluid heated from below. For detailed physical background, we refer readers to [17,18, 20,21].

Our proof in this paper will partially carried out on the framework of cylindrical coordinates $(r,\ \theta,\ z)$. To be more specific, for $x = (x_1,\ x_2,\ x_3) \in \mathbb{R}^3$,

$$r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \arctan\frac{x_2}{x_1}, \quad z = x_3.$$

A solution of (1.1) is called an axially symmetric solution, if and only if

$$\begin{cases} u = u_r(t,r,z)\boldsymbol{e_r} + u_\theta(t,r,z)\boldsymbol{e_\theta} + u_z(t,r,z)\boldsymbol{e_z}, \\ \rho = \rho(t,r,z), \end{cases}$$

satisfy the system (1.1). Here, the basis vectors $\boldsymbol{e_r}, \boldsymbol{e_\theta}, \boldsymbol{e_z}$ are

$$\boldsymbol{e_r} = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \quad \boldsymbol{e_\theta} = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), \quad \boldsymbol{e_z} = (0,0,1).$$

Choosing $\kappa = 1$ for simplicity, (1.1) can be simplified to

$$\begin{cases} \partial_t u_r + (u_r\partial_r + u_z\partial_z)u_r + \partial_r p = 0, \\ \partial_t u_\theta + (u_r\partial_r + u_z\partial_z)u_\theta + \dfrac{u_\theta u_r}{r} = 0, \\ \partial_t u_z + (u_r\partial_r + u_z\partial_z)u_z + \partial_z p = \rho, \\ \partial_t \rho + (u_r\partial_r + u_z\partial_z)\rho - \Delta\rho = 0, \\ \nabla\cdot u = \partial_r u_r + \dfrac{u_r}{r} + \partial_z u_z = 0. \end{cases} \tag{1.2}$$

The vorticity of the axially symmetric velocity field u is given by

$$w(t,r,z) = w_r(t,r,z)\boldsymbol{e_r} + w_\theta(t,r,z)\boldsymbol{e_\theta} + w_z(t,r,z)\boldsymbol{e_z},$$

where

$$w_r = -\partial_z u_\theta, \quad w_\theta = \partial_z u_r - \partial_r u_z, \quad w_z = \partial_r u_\theta + \frac{u_\theta}{r}.$$

By taking spacial derivatives of $(1.2)_{1,2,3}$, one concludes that (w_r, w_θ, w_z) satisfies

$$\begin{cases} \partial_t w_r + (u_r\partial_r + u_z\partial_z)w_r = (w_r\partial_r + w_z\partial_z)u_r, \\ \partial_t w_\theta + (u_r\partial_r + u_z\partial_z)w_\theta = \dfrac{u_r}{r}w_\theta + \dfrac{1}{r}\partial_z(u_\theta)^2 - \partial_r\rho, \\ \partial_t w_z + (u_r\partial_r + u_z\partial_z)w_z = (w_r\partial_r + w_z\partial_z)u_z. \end{cases} \tag{1.3}$$

Now we are ready to present the main result. A Beale–Kato–Majda-type condition on the swirl part of the velocity, stated as the following

$$\int_0^{T_*} \|\nabla \times (u_\theta \boldsymbol{e_\theta})(t,\cdot)\|_{L^\infty}\,\mathrm{d}t \le C_* < \infty, \tag{1.4}$$

indicates the regularity of the solution.

Theorem 1.1. *For any $0 < T_* < \infty$, let $(u,\rho) \in C\big([0,T_*); H^m(\mathbb{R}^3)\big)$ $(3 \le m \in \mathbb{N})$ be the unique solution of (1.2) with the initial data $(u_0, \rho_0) \in H^m(\mathbb{R}^3)$, satisfying $r\rho_0$, $\partial_r\rho_0/r \in L^2$. Then, $(u,\rho)(t,\cdot)$ keeps in $H^m(\mathbb{R}^3)$ beyond $t = T_*$ if and only if (1.4) holds.*

□

Condition (1.4) automatically holds for any $T_* > 0$ provided $u_\theta \equiv 0$. Thus as a corollary of Theorem 1.1, we have the global well-posedness of the solution to the swirl-free incompressible axially symmetric Boussinesq system. This was already shown by authors in [10]. Nevertheless, in the current paper, we will derive an exact temporal growth of the global-in-time solution. To do this, one needs the following:

Definition 1.2. We say $A \le \Phi_{k,\alpha}(t)$, if there exist α, $c > 0$ such that

$$A \le c\underbrace{\exp\Big(c\exp\big(\cdots\exp(ct^\alpha)\big)}_{k \text{ times exponents}}\Big).$$

Now we are ready for our second result of the paper:

Theorem 1.3. *Suppose $m \in \mathbb{N}$ and $m \ge 3$, let (u,ρ) be the unique strong solution of (1.2) with initial data $(u_0,\rho_0) \in H^m$, satisfying $r\rho_0$, $\partial_r\rho_0/r \in L^2$ and $\nabla\cdot u_0 = u_0\cdot\boldsymbol{e_\theta} \equiv 0$. Then, (u,ρ) is globally well-posed and satisfies the following temporal asymptotic property:*

$$\|(u,\rho)(t,\cdot)\|_{H^m}^2 \le \Phi_{4,3}(t), \quad \forall t > 0. \tag{1.5}$$

□

Now we outline the idea of the proof of Theorems 1.1 and 1.3. Generally speaking, the proof, which is totally derived in the framework of Sobolev spaces, could be mainly divided into three parts: the fundamental estimates, the medium estimates, and the higher-order estimates. The fundamental estimates, which is independent of the condition (1.4), are derived by basic energy estimate of system (1.1) and maximum principle for certain parabolic systems. More specifically, we arrive at the following estimates by the end of this part:

$$\|u(t,\cdot)\|_{L^2}^2 \le C_0(1+t^2);$$

$$\|\rho(t,\cdot)\|_{L^p}^2 + \int_0^t \|\nabla\rho(s,\cdot)\|_{L^2}^2\mathrm{d}s \le C_0, \quad \forall p \in [2,\infty].$$

The part of medium estimates is the most skilful part in this proof. With a BKM's criterion for just one component of the velocity field, one can hardly obtain the first order derivative estimates of the solution directly from the part of fundamental estimates. Instead, three special quantities

$$\Omega := \frac{w_\theta}{r}, \quad J = \frac{w_r}{r}, \quad \mathcal{N} := \frac{\partial_r\rho}{r},$$

which are of the same scaling as the second order spatial derivative of (u,ρ), are carefully chosen to construct a self-closed system. See (3.4). Here, part of quantities are motivated by [4] in which the authors introduced Ω and J to study the regularity criteria of axially symmetric Navier–Stokes equations. At the end of this part, we conclude the following self-closed estimate:

$$\|(\Omega,J)(t,\cdot)\|^2_{L^2\cap L^6}+\|\mathcal{N}(t,\cdot)\|^2_{L^2}+\int_0^t\|\nabla\mathcal{N}(s,\cdot)\|^2_{L^2}\mathrm{d}s\le\Phi_{1,3}(t),$$

for any $t\le T_*$.

Finally, one notices the full-components BKM-type criterion

$$\int_0^t\|(\nabla\times u,\nabla\rho)(s,\cdot)\|_{L^\infty}\mathrm{d}s\le C_*<\infty \tag{1.6}$$

for Boussinesq system will imply the validity of Theorem 1.1. To achieve (1.6), we will carry out the following bootstrapping approach of the higher-order energy estimates:

I. **$L^\infty_tL^2\cap L^2_t\dot H^1$ estimate of $(r\rho)$:**

$$\|r\rho(t,\cdot)\|^2_{L^2}+\int_0^t\|\nabla(r\rho)(s,\cdot)\|^2_{L^2}\mathrm{d}s\le C_0(1+t^3);$$

II. **$L^\infty_tL^2\cap L^2_t\dot H^1$ estimate of $\nabla\rho$:**

$$\|\nabla\rho(t,\cdot)\|^2_{L^2}+\int_0^t\|\nabla^2\rho(s,\cdot)\|^2_{L^2}\mathrm{d}s\le\Phi_{1,3}(t);$$

III. **$L^\infty_tL^2\cap L^\infty_tL^6$ estimate of ∇b:**

$$\|\nabla b(t,\cdot)\|_{L^2\cap L^6}\le\Phi_{2,3}(t);$$

IV. **$L^\infty_tL^2\cap L^2_t\dot H^1$ estimates of $\nabla^2\rho$:**

$$\|\nabla^2\rho(t,\cdot)\|^2_{L^2}+\int_0^t\|\nabla^3\rho(s,\cdot)\|^2_{L^2}\mathrm{d}s\le\Phi_{2,3}(t);$$

V. **$L^1_tL^\infty$ estimate of $(w_\theta,\nabla\rho)$:**

$$\int_0^t\|(w_\theta,\nabla\rho)(s,\cdot)\|_{L^\infty}\mathrm{d}s\le\Phi_{2,3}(t);$$

VI. **The Full-components BKM-type criterion**

$$\int_0^t\|(\nabla\times u,\nabla\rho)(s,\cdot)\|_{L^\infty}\,\mathrm{d}s\le C_*<\infty$$

and the Estimate (1.5) in Theorem 1.3.

Before ending our introduction, we review some previous results related to the Boussinesq system (with or without diffusion). Many works and efforts have been made to study the well-posedness of the Cauchy problem for the Boussinesq system. In 2D case, Chae [7] and Hou-Li [11] independently proved the global regularity of solutions to the 2D Boussinesq system. And also Chae [7] considers the case of zero viscosity and nonzero diffusion. See [1,12] for related results in critical spaces. For the 3D case, Abidi et al. [2] and Hmidi-Rousset [9,10] proved the global well-posedness of the Cauchy problem for the 3D axisymmetric Boussinesq system without swirl. Readers can see [6,15] and references therein for more regularity results on the Boussinesq system.

Throughout the paper, $C_{a,b,c,...}$ denotes a positive constant depending on a, b, c, ... which may be different from line to line. We also apply $A \lesssim B$ to denote $A \leq CB$. Meanwhile, $A \simeq B$ means both $A \lesssim B$ and $B \lesssim A$. $[\mathcal{A}, \mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$ denotes the communicator of the operator $\mathcal{A}$ and the operator $\mathcal{B}$. L stands for a multi-index such that $L = (l_1, l_2, l_3)$ where $l_1, l_2, l_3 \in \mathbb{N} \cup \{0\}$ and $|L| = l_1 + l_2 + l_3$, $\nabla^L = \partial_{x_1}^{l_1} \partial_{x_2}^{l_2} \partial_{x_3}^{l_3}$.

We use standard notations for Lebesgue and Sobolev functional spaces in $\mathbb{R}^3$: For $1 \leq p \leq \infty$ and $k \in \mathbb{N}$, L^p denotes the Lebesgue space with norm

$$\|f\|_{L^p} := \begin{cases} \left(\int\limits_{\mathbb{R}^3} |f(x)|^p \mathrm{d}x \right)^{1/p}, & 1 \leq p < \infty, \\ \underset{x \in \mathbb{R}^3}{esssup} |f(x)|, & p = \infty, \end{cases}$$

while $W^{k,p}$ denotes the usual Sobolev space and $\dot{W}^{k,p}$ denotes the usual homogeneous Sobolev space with their norm and semi-norm

$$\|f\|_{W^{k,p}} := \sum_{0 \leq |L| \leq k} \|\nabla^L f\|_{L^p},$$

$$|f|_{\dot{W}^{k,p}} := \sum_{|L| = k} \|\nabla^L f\|_{L^p},$$

respectively. We also simply denote H^k and $\dot{H}^k$ instead of $W^{k,p}$ and $\dot{W}^{k,p}$ provided $p = 2$. For a function $f \in L^p \cap L^q$ with $1 \leq p, q \leq \infty$, we denote its Yudovich-type norm as

$$\|f\|_{L^p \cap L^q} = \max \{\|f\|_{L^p}, \|f\|_{L^q}\}.$$

We do not distinguish functional spaces for scalar or vector-valued functions in this paper since it will be clear from the context. For any Banach space X, we say $v : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ belongs to the Bochner space $L^p(0, T; X)$, if

$$\|v(t, \cdot)\|_X \in L^p\big((0, T)\big).$$

And we usually use $L^p_T X$ for short notation of $L^p(0, T; X)$.

The remaining of this paper is devoted to proving Theorems 1.1 and 1.3. Before that, some preliminaries are given in Sect. 2.

2. Preliminaries

At the beginning, let us introduce the well-known *Gagliardo–Nirenberg* interpolation inequality. We list here without proof.

Lemma 2.1. (Gagliardo–Nirenberg) *Fix $q, r \in [1, \infty]$ and $j, m \in \mathbb{N} \cup \{0\}$ with $j \leq m$. Suppose that $f \in L^q \cap \dot{W}^{m,r}$ and there exists a real number $\alpha \in [j/m, 1]$ such that*

$$\frac{1}{p} = \frac{j}{3} + \alpha \left(\frac{1}{r} - \frac{m}{3} \right) + \frac{1 - \alpha}{q}.$$

Then, $f \in \dot{W}^{j,p}(\mathbb{R}^d)$ and there exists a constant $C > 0$ such that

$$\|\nabla^j f\|_{L^p} \leq C \|\nabla^m f\|_{L^r}^{\alpha} \|f\|_{L^q}^{1-\alpha},$$

except the following two cases:

(i) $j = 0$, $mr < d$ *and* $q = \infty$; *(In this case, it is necessary to assume also that either* $u \to 0$ *at infinity, or* $u \in L^s(\mathbb{R}^d)$ *for some* $s < \infty$.*)*
(ii) $1 < r < \infty$ *and* $m - j - 3/r \in \mathbb{N}$. *(In this case, it is necessary to assume also that* $\alpha < 1$.*)*

□

Using the Biot–Savart law and the L^p boundedness of Calderon–Zygmund singular integral operators, we have the following lemma whose detailed proof can be found in [3,5].

Lemma 2.2. *Let* $u = u_r\boldsymbol{e_r} + u_\theta\boldsymbol{e_\theta} + u_z\boldsymbol{e_z}$ *be an axially symmetric vector field,* $w = \nabla \times u = w_r\boldsymbol{e_r} + w_\theta\boldsymbol{e_\theta} + w_z\boldsymbol{e_z}$ *and* $b = u_r\boldsymbol{e_r} + u_z\boldsymbol{e_z}$. *Then, we have*

$$\|\nabla u\|_{L^p} \leq C_p\|w\|_{L^p} \tag{2.1}$$

and

$$\|\nabla b\|_{L^p} \leq C_p\|w_\theta\|_{L^p} \tag{2.2}$$

for all $1 < p < \infty$.

□

The proof of the following lemma can be found in [16] (equation (A.5)) and [19] (Proposition 2.5).

Lemma 2.3. *Define* $\Omega := \frac{w_\theta}{r}$. *For* $1 < p < +\infty$, *there exists an absolute constant* $C_p > 0$ *such that*

$$\left\|\nabla\frac{u_r}{r}(t,\cdot)\right\|_{L^p} \leq C_p\|\Omega(t,\cdot)\|_{L^p}. \tag{2.3}$$

□

Finally, the following 1 dimensional Hardy inequality could be found in [8] (Theorem 330).

Lemma 2.4. *If* $p > 1$, $\sigma \neq 1$, f *is a nonnegative measurable function and* F *is defined by*

$$F(x) = \int_0^x f(t)\mathrm{d}t, \quad (\sigma > 1), \quad F(x) = \int_x^\infty f(t)\mathrm{d}t, \quad (\sigma < 1).$$

Then,

$$\int_0^\infty x^{-\sigma}F^p\mathrm{d}x < \left(\frac{p}{|\sigma - 1|}\right)^p \int_0^\infty x^{-\sigma}(xf)^p\mathrm{d}x,$$

unless $f \equiv 0$.

□

Then, a direct corollary holds automatically:

Corollary 2.5.

$$\left\|\frac{u_\theta}{r}(t,\cdot)\right\|_{L^\infty} \leq \frac{1}{2}\|w_z(t,\cdot)\|_{L^\infty} \tag{2.4}$$

for any $t > 0$.

Proof. Choosing $\sigma = 2p - 1 > 1$, and $f(r) = r|w_z|$, Lemma 2.4 indicates

$$\int_0^\infty r^{-2p+1}\left(\int_0^r s|w_z(t,s,z)|\mathrm{d}s\right)^p \mathrm{d}r \leq \left(\frac{1}{2}\right)^p \int_0^\infty |w_z(t,r,z)|^p r\mathrm{d}r. \tag{2.5}$$

Noting that

$$|ru_\theta(t,r,z)| = \left|\int_0^r sw_z(t,s,z)\mathrm{d}s\right| \le \int_0^r s|w_z(t,s,z)|\mathrm{d}s,$$

(2.5) implies

$$\int_0^\infty \left|\frac{u_\theta}{r}(t,r,z)\right|^p r\mathrm{d}r \le \left(\frac{1}{2}\right)^p \int_0^\infty |w_z(t,r,z)|^p r\mathrm{d}r. \tag{2.6}$$

Integrating (2.6) with z on $\mathbb{R}$, we know that

$$\left\|\frac{u_\theta}{r}(t,\cdot)\right\|_{L^p} \le \frac{1}{2}\|w_z(t,\cdot)\|_{L^p}.$$

Therefore, (2.4) follows by letting $p \to \infty$.

□

3. Proof of Theorem 1.1

3.1. Fundamental estimates

At the beginning, the following Lemma states fundamental estimates of the system (1.2):

Lemma 3.1. (Fundamental Energy Estimates) *Let (u,ρ) be a smooth solution of (1.2), then we have:*

(i) *For $p \in [2,\infty)$ and $t \in \mathbb{R}_+$,*

$$\|\rho(t,\cdot)\|_{L^p}^p + \int_0^t \int_{\mathbb{R}^3} |\nabla\rho(s,x)|^2|\rho(s,x)|^{p-2}\mathrm{d}x\mathrm{d}s \lesssim_p \|\rho_0\|_{L^p}^p; \tag{3.1}$$

$$\|\rho(t,\cdot)\|_{L^\infty} \le \|\rho_0\|_{L^\infty}. \tag{3.2}$$

(ii) *For $u_0, \rho_0 \in L^2$ and $t \in \mathbb{R}_+$,*

$$\|u(t,\cdot)\|_{L^2}^2 \le C_0(1+t)^2, \tag{3.3}$$

where C_0 depends only on $\|(u_0,\rho_0)\|_{L^2}$.

Proof. Testing $(1.2)_4$ with $p\rho^{p-1}$, it is easy to deduce (3.1). Using the particle trajectory mapping of the velocity $u_r\boldsymbol{e_r} + u_z\boldsymbol{e_z}$, estimate (3.2) are classical for the transport equation. Finally, (3.3) follows from the standard L^2 estimate of the system (1.1). See also [22, Proposition 2.1]. □

3.2. Medium estimates

Starting from fundamental estimates in Lemma 3.1, our first step is to derive a self-closed estimate of the following group of quantities

$$\Omega := \frac{w_\theta}{r}, \quad J := \frac{w_r}{r}, \quad \text{and} \quad \mathcal{N} := \frac{\partial_r\rho}{r}.$$

Using (1.2), direct calculation shows that $(\Omega, J, \mathcal{N})$ satisfies the following reformulated system:

$$\begin{cases} \partial_t \Omega + b\cdot\nabla\Omega = -2\dfrac{u_\theta}{r}J - \mathcal{N}, \\ \partial_t J + b\cdot\nabla J = (w_r\partial_r + w_z\partial_z)\dfrac{u_r}{r}, \\ \partial_t\mathcal{N} + b\cdot\nabla\mathcal{N} - \left(\Delta + \dfrac{2}{r}\partial_r\right)\mathcal{N} = \partial_z u_z\mathcal{N} - \partial_r u_z\dfrac{\partial_z\rho}{r}. \end{cases} \tag{3.4}$$

Here,

$$b = u_r\boldsymbol{e_r} + u_z\boldsymbol{e_z}.$$

In the below, our proof is carried out under the one-component BKM-type condition

$$\int_0^{T_*} \|\nabla\times(u_\theta\boldsymbol{e_\theta})(t,\cdot)\|\,\mathrm{d}t < C_* < \infty,$$

thus all the temporal notations in the following lie in the closed interval $[0, T_*]$ unless we specially emphasize. In particular, the swirl-free case which is considered in Corollary 1.3 satisfies $T_* = \infty$, and so that the temporal asymptotic estimate (1.5) holds for any $t > 0$.

Here is the main result in this subsection, in which a self-closed $L^\infty_T\left(L^2\cap L^6\right)$ estimate of (Ω, J) combining with $L^\infty_T L^2 \cap L^2_T H^1$ estimate of $\mathcal{N}$ is given.

Proposition 3.2. *Let $(\Omega, J, \mathcal{N})$ be defined as above, which solves (3.4) with initial data*

$$(\Omega_0, J_0, \mathcal{N}_0) \in (L^2\cap L^6)\times(L^2\cap L^6)\times L^2.$$

Then, the following space-time estimate holds for any $T\in(0,T_*]$ that

$$\|(\Omega, J)(T,\cdot)\|^2_{L^2\cap L^6} + \|\mathcal{N}(T,\cdot)\|^2_{L^2} + \int_0^T \|\nabla\mathcal{N}(s,\cdot)\|^2_{L^2}\mathrm{d}s \le \Phi_{1,3}(T). \tag{3.5}$$

Proof. The proof of (3.5) is carried out in the following steps: First we deduce the estimate of $\mathcal{N}$ in terms of Ω. Then, we find the Yudovich-type norm of (Ω, J) in (3.5) can be controlled by $L^1_T\left(L^2\cap L^6\right)$ norm of $\mathcal{N}$. Finally we conclude (3.5) by combining those assertions above.

Estimate of $\mathcal{N}$. Now performing L^2 inner product of $\mathcal{N}$, $(3.4)_3$ indicates that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{N}(t,\cdot)\|^2_{L^2} + \|\nabla\mathcal{N}(t,\cdot)\|^2_{L^2} + \int_0^\infty |\mathcal{N}(t,0,z)|^2\mathrm{d}z = \underbrace{\int_{\mathbb{R}^3}\partial_z u_z\mathcal{N}^2\mathrm{d}x}_{N_1} - \underbrace{\int_{\mathbb{R}^3}\partial_r u_z\frac{\partial_z\rho}{r}\mathcal{N}\mathrm{d}x}_{N_2}. \tag{3.6}$$

Using the divergence-free property of u $(1.2)_5$, one obtains that

$$N_1 = -\int_{\mathbb{R}^3}\left(\partial_r u_r + \frac{u_r}{r}\right)\mathcal{N}^2\mathrm{d}x = 2\pi\int_{-\infty}^\infty\int_0^\infty \partial_r(ru_r)\mathcal{N}^2\mathrm{d}r\mathrm{d}z.$$

It is deduced that by applying integration by parts

$$N_1 = -4\pi\int_{-\infty}^\infty\int_0^\infty u_r\mathcal{N}\partial_r\mathcal{N}r\mathrm{d}r\mathrm{d}z = -2\int_{\mathbb{R}^2}\frac{u_r}{r}\partial_r\rho\partial_r\mathcal{N}\mathrm{d}x.$$

Using Hölder inequality, Lemma 2.1, Young inequality and (2.3), N_1 could be bounded in the following way

$$
\begin{aligned}
N_1 &\le 2\left\|\frac{u_r}{r}(t,\cdot)\right\|_{L^\infty}\|\partial_r\rho(t,\cdot)\|_{L^2}\|\partial_r\mathcal{N}(t,\cdot)\|_{L^2}\\
&\le C\left\|\frac{u_r}{r}(t,\cdot)\right\|_{L^6}^{1/2}\left\|\nabla\frac{u_r}{r}(t,\cdot)\right\|_{L^6}^{1/2}\|\partial_r\rho(t,\cdot)\|_{L^2}\|\partial_r\mathcal{N}(t,\cdot)\|_{L^2}\\
&\le C\left\|\nabla\frac{u_r}{r}(t,\cdot)\right\|_{L^2}^{1/2}\left\|\nabla\frac{u_r}{r}(t,\cdot)\right\|_{L^6}^{1/2}\|\partial_r\rho(t,\cdot)\|_{L^2}\|\partial_r\mathcal{N}(t,\cdot)\|_{L^2}\\
&\le \frac14\|\nabla\mathcal{N}(t,\cdot)\|_{L^2}^2+C\left\|\Omega(t,\cdot)\right\|_{L^2}\left\|\Omega(t,\cdot)\right\|_{L^6}\left\|\nabla\rho(t,\cdot)\right\|_{L^2}^2.
\end{aligned}
\tag{3.7}
$$

Meanwhile, noting that $w_\theta=\partial_z u_r-\partial_r u_z$, we find N_2 satisfies

$$
N_2=\int_{\mathbb{R}^3}(\partial_z u_r-w_\theta)\frac{\partial_z\rho}{r}\mathcal{N}\mathrm{d}x=\int_{\mathbb{R}^3}\left(\partial_z\frac{u_r}{r}-\Omega\right)\partial_z\rho\mathcal{N}\mathrm{d}x.
$$

Using Lemma 2.3, we handle N_2 similarly as N_1 before

$$
\begin{aligned}
N_2 &\le \left(\left\|\nabla\frac{u_r}{r}(t,\cdot)\right\|_{L^3}+\|\Omega(t,\cdot)\|_{L^3}\right)\|\partial_z\rho(t,\cdot)\|_{L^2}\|\mathcal{N}(t,\cdot)\|_{L^6}\\
&\le C\|\Omega(t,\cdot)\|_{L^3}\|\nabla\rho(t,\cdot)\|_{L^2}\|\nabla\mathcal{N}(t,\cdot)\|_{L^2}\\
&\le \frac14\|\nabla\mathcal{N}(t,\cdot)\|_{L^2}^2+\|\Omega(t,\cdot)\|_{L^2}\|\Omega(t,\cdot)\|_{L^6}\|\nabla\rho(t,\cdot)\|_{L^2}^2.
\end{aligned}
\tag{3.8}
$$

Inserting (3.7) and (3.8) into (3.6), one knows that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{N}(t,\cdot)\|_{L^2}^2+\|\nabla\mathcal{N}(t,\cdot)\|_{L^2}^2\lesssim\left(\|\Omega(t,\cdot)\|_{L^2}^2+\|\Omega(t,\cdot)\|_{L^6}^2\right)\|\nabla\rho(t,\cdot)\|_{L^2}^2.
$$

This indicates that, by integrating with t

$$
\|\mathcal{N}(t,\cdot)\|_{L^2}^2+\int_0^t\|\nabla\mathcal{N}(s,\cdot)\|_{L^2}^2\mathrm{d}s\lesssim\|\mathcal{N}_0\|_{L^2}^2+\int_0^t\|\Omega(s,\cdot)\|_{L^2\cap L^6}^2\|\nabla\rho(s,\cdot)\|_{L^2}^2\mathrm{d}s.
\tag{3.9}
$$

Estimate of $(\boldsymbol{\Omega},\boldsymbol{J})$. To close the aforementioned estimates (3.9), we need to derive the L^p estimate $(2\le p\le 6)$ of Ω. Performing the L^p energy estimates $(2\le p\le 6)$ for $(3.4)_1$ and $(3.4)_2$, one arrives at

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\Omega(t,\cdot)\|_{L^p}\lesssim\underbrace{\left\|\frac{u_\theta}{r}(t,\cdot)\right\|_{L^\infty}}_{T_1}\|J(t,\cdot)\|_{L^p}+\|\mathcal{N}(t,\cdot)\|_{L^p},\\
&\frac{\mathrm{d}}{\mathrm{d}t}\|J(t,\cdot)\|_{L^p}\lesssim\|(w_r,w_z)(t,\cdot)\|_{L^\infty}\ \left\|\nabla\frac{u_r}{r}(t,\cdot)\right\|_{L^p}.
\end{aligned}
$$

By Corollary 2.5, the term T_1 could be replaced by $\|w_z(t,\cdot)\|_{L^\infty}$. Recalling the identity $\nabla\times(u_\theta\boldsymbol{e_\theta})=w_r\boldsymbol{e_r}+w_z\boldsymbol{e_z}$, one derives

$$
\|\Omega(t,\cdot)\|_{L^p}\lesssim\|\Omega_0\|_{L^p}+\int_0^t\|\nabla\times(u_\theta\boldsymbol{e_\theta})(s,\cdot)\|_{L^\infty}\|J(s,\cdot)\|_{L^p}\mathrm{d}s+\int_0^t\|\mathcal{N}(s,\cdot)\|_{L^p}\mathrm{d}s,
\tag{3.10}
$$

$$
\begin{aligned}
\|J(t,\cdot)\|_{L^p}&\lesssim\|J_0\|_{L^p}+\int_0^t\|(w_r,w_z)(s,\cdot)\|_{L^\infty}\left\|\nabla\frac{u_r}{r}(s,\cdot)\right\|_{L^p}\mathrm{d}s\\
&\lesssim\|J_0\|_{L^p}+\int_0^t\|\nabla\times(u_\theta\boldsymbol{e_\theta})(s,\cdot)\|_{L^\infty}\|\Omega(s,\cdot)\|_{L^p}\mathrm{d}s.
\end{aligned}
\tag{3.11}
$$

Here, the second line of (3.11) follows from Lemma 2.3. Combining (3.10) and (3.11), using Gronwall inequality, one knows that

$$\|(\Omega, J)(t,\cdot)\|_{L^p} \lesssim \left(\|(\Omega_0, J_0)\|_{L^p} + \int_0^t \|\mathcal{N}(s,\cdot)\|_{L^p} \mathrm{d}s \right) \exp\left(\int_0^t \|\nabla \times (u_\theta \boldsymbol{e_\theta})(s,\cdot)\|_{L^\infty} \mathrm{d}s \right).$$

By the condition (1.4), noting that $t \le T_*$, we conclude for $2 \le p \le 6$ that :

$$\|(\Omega, J)(t,\cdot)\|_{L^p} \le C_{C_*} \left(\|(\Omega_0, J_0)\|_{L^p} + \int_0^t \|\mathcal{N}(s,\cdot)\|_{L^p} \mathrm{d}s \right). \tag{3.12}$$

End proof of (3.5). Choosing $p = 2$ and $p = 6$ in (3.12), one derives

$$\|\Omega(t,\cdot)\|^2_{L^2\cap L^6} \lesssim \|\Omega_0\|^2_{L^2\cap L^6} + \left(\int_0^t \|\mathcal{N}(s,\cdot)\|_{L^2\cap L^6} \mathrm{d}s \right)^2.$$

Using Sobolev inequality and Hölder inequality, it follows that

$$\|\Omega(t,\cdot)\|^2_{L^2\cap L^6} \lesssim \|\Omega_0\|^2_{L^2\cap L^6} + \left(t^2 \sup_{s\in[0,t]} \|\mathcal{N}(s,\cdot)\|^2_{L^2} + t \int_0^t \|\nabla\mathcal{N}(s,\cdot)\|^2_{L^2} \mathrm{d}s \right).$$

Substituting (3.9) into the right hand side of the above inequality, we arrive at

$$\|\Omega(t,\cdot)\|^2_{L^2\cap L^6} \lesssim \|\Omega_0\|^2_{L^2\cap L^6} + (1+t^2) \left(\|\mathcal{N}_0\|^2_{L^2} + \int_0^t \|\Omega(s,\cdot)\|^2_{L^2\cap L^6} \|\nabla\rho(s,\cdot)\|^2_{L^2} \mathrm{d}s \right).$$

This implies, for any $T \le T_*$ and $t \in (0, T]$

$$\|\Omega(t,\cdot)\|^2_{L^2\cap L^6} \lesssim \|\Omega_0\|^2_{L^2\cap L^6} + (1+t^2)\|\mathcal{N}_0\|^2_{L^2} + (1+T^2) \int_0^t \|\Omega(s,\cdot)\|^2_{L^2\cap L^6} \|\nabla\rho(s,\cdot)\|^2_{L^2} \mathrm{d}s.$$

Thus by Gronwall inequality, one has:

$$\|\Omega(t,\cdot)\|^2_{L^2\cap L^6} \lesssim \left(\|\Omega_0\|^2_{L^2\cap L^6} + (1+t^2)\|\mathcal{N}_0\|^2_{L^2} \right) \exp\left((1+T^2) \int_0^t \|\nabla\rho(s,\cdot)\|^2_{L^2} \mathrm{d}s \right).$$

Recalling the fundamental energy estimates (3.1) and (3.3), this indicates

$$\|\Omega(T,\cdot)\|^2_{L^2\cap L^6} \le C_{0,C_*} \left(1+T^2\right) \exp\left(C_0(1+T^3)\right) \le \Phi_{1,3}(T), \quad \forall T \in (0, T_*]. \tag{3.13}$$

Substituting (3.13) into (3.9), using (3.1) and (3.3), one finds

$$\begin{aligned} \|\mathcal{N}(T,\cdot)\|^2_{L^2} + \int_0^T \|\nabla\mathcal{N}(s,\cdot)\|^2_{L^2} \mathrm{d}s \le& \Phi_{1,3}(T) \int_0^T \|\nabla\rho(s,\cdot)\|^2_{L^2} \mathrm{d}s \\ \le& \Phi_{1,3}(T), \quad \forall T \in (0, T_*]. \end{aligned} \tag{3.14}$$

Thus, the proposition is proved by combining (3.12), (3.13) and (3.14).

□

3.3. Higher-order estimates

In this part, as we stated in the introduction, our proof will be presented with the following route:

$$\|r\rho\|_{L_t^\infty L^2\cap L_t^\infty \dot H^1} \to \|\nabla\rho\|_{L_t^\infty L^2\cap L_t^2 \dot H^1} \to \|\nabla b\|_{L_t^\infty (L^2\cap L^6)} \to \|\nabla^2\rho\|_{L_t^\infty L^2\cap L_t^2 \dot H^1} \to \|w_\theta\|_{L_{T_*}^\infty L^\infty}.$$

3.3.1. $L_t^\infty L^2 \cap L_t^2 \dot H^1$ estimate of $(r\rho)$. Based on the medium estimate of the solution in Proposition 3.2, our next step aims at deriving a higher-order derivatives estimate of ρ. Before that, a weighted low order L^2 estimate is necessary. Here is the result:

Proposition 3.3. *Under the same conditions as Theorem 1.1, $r\rho$, which starts from initial data $r\rho_0 \in L^2$, satisfies the following space-time estimate*

$$\|r\rho(t,\cdot)\|_{L^2}^2 + \int_0^t \|\nabla(r\rho)(s,\cdot)\|_{L^2}^2 \mathrm{d}s \le C_0(1+t^3),$$

where $C_0 > 0$ is a constant depending only on the initial data u_0, and ρ_0.

Proof. Starting from $(1.2)_4$, one derives $(r\rho)$ satisfies

$$\partial_t(r\rho) + (u_r\partial_r + u_z\partial_z)(r\rho) - \left(\Delta - \frac{1}{r^2}\right)(r\rho) = u_r\rho - 2\partial_r\rho.$$

Taking the L^2 inner product, one finds

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|r\rho(t,\cdot)\|_{L^2}^2 + \|\nabla(r\rho)(t,\cdot)\|_{L^2}^2 + \|\rho(t,\cdot)\|_{L^2}^2 = \underbrace{\int_{\mathbb{R}^3} u_r\rho^2 r\mathrm{d}x}_{R_1} \underbrace{-2\int_{\mathbb{R}^3}\partial_r\rho(r\rho)\mathrm{d}x}_{R_2}. \tag{3.15}$$

Using Hölder inequality, Young inequality and Sobolev inequality, R_1 follows that

$$R_1 \le \|u_r(t,\cdot)\|_{L^2}\|\rho(t,\cdot)\|_{L^3}\|r\rho(t,\cdot)\|_{L^6} \le \frac{1}{4}\|\nabla(r\rho)(t,\cdot)\|_{L^2}^2 + C\|u_r(t,\cdot)\|_{L^2}^2\|\rho(t,\cdot)\|_{L^3}^2. \tag{3.16}$$

Integration by parts, R_2 satisfies

$$R_2 = 2\pi\int_{\mathbb{R}}\int_0^\infty \partial_r\rho(r^2\rho)\mathrm{d}r\mathrm{d}z = -\int_{\mathbb{R}^3}\rho\partial_r(r\rho)\mathrm{d}x - \int_{\mathbb{R}^3}\rho^2\mathrm{d}x.$$

Thus, Hölder inequality and Young inequality indicate that

$$R_2 \le \frac{1}{4}\|\nabla(r\rho)(t,\cdot)\|_{L^2}^2 + C\|\rho(t,\cdot)\|_{L^2}^2. \tag{3.17}$$

Inserting (3.16) and (3.17) into (3.15), one knows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|r\rho(t,\cdot)\|_{L^2}^2 + \|\nabla(r\rho)(t,\cdot)\|_{L^2}^2 \lesssim \|u(t,\cdot)\|_{L^2}^2\|\rho(t,\cdot)\|_{L^3}^2 + \|\rho(t,\cdot)\|_{L^2}^2.$$

Integrating with t, using Lemma 3.1, one arrives

$$\begin{aligned}\|r\rho(t,\cdot)\|_{L^2}^2 + \int_0^t \|\nabla(r\rho)(s,\cdot)\|_{L^2}^2\mathrm{d}s &\lesssim \|r\rho_0\|_{L^2}^2 + t(1+t^2)\|\rho_0\|_{L^3}^2 + t\|\rho_0\|_{L^2}^2 \\ &\le C_0(1+t^3), \quad \forall t > 0.\end{aligned} \tag{3.18}$$

This completes the proof of Proposition 3.3. ☐

3.3.2. $L_t^\infty L^2 \cap L_t^2 \dot{H}^1$ estimate of $\nabla\rho$. Based on the weighted estimate of $r\rho$ in Proposition 3.3, our next step is to derive the $L_t^\infty L^2 \cap L_t^2 \dot{H}^1$ estimate of $\nabla\rho$. The conclusion is as follows:

Proposition 3.4. *Under the same conditions as Theorem 1.1, $\nabla\rho$ satisfies the following space-time estimate*

$$\|\nabla\rho(t,\cdot)\|_{L^2}^2 + \int_0^t \|\nabla^2\rho(s,\cdot)\|_{L^2}^2 \mathrm{d}s \leq \Phi_{1,3}(t).$$

Proof. Applying ∂_r, ∂_z to $(1.2)_4$ respectively, and performing the L^2 inner product for the resulted equations, the following identity is derived

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\nabla\rho(t,\cdot)\|_{L^2}^2 + \|\nabla^2\rho(t,\cdot)\|_{L^2}^2 + \left\|\frac{\partial_r\rho}{r}(t,\cdot)\right\|_{L^2}^2 \\
&= \underbrace{-\int_{\mathbb{R}^3}\partial_r u_r(\partial_r\rho)^2\mathrm{d}x}_{NR_1} \underbrace{-\int_{\mathbb{R}^3}\partial_r u_z\partial_z\rho\partial_r\rho\mathrm{d}x}_{NR_2} \underbrace{-\int_{\mathbb{R}^3}\partial_z u_r\partial_r\rho\partial_z\rho\mathrm{d}x}_{NR_3} \underbrace{-\int_{\mathbb{R}^3}\partial_z u_z(\partial_z\rho)^2\mathrm{d}x}_{NR_4}.
\end{aligned} \tag{3.19}$$

For NR_1, using integrating by parts, it follows that

$$\begin{aligned}
NR_1 &= -4\pi\int_{\mathbb{R}}\int_0^\infty u_r\partial_r\rho\partial_r^2\rho r\mathrm{d}r\mathrm{d}z - 2\pi\int_{\mathbb{R}}\int_0^\infty u_r(\partial_r\rho)^2\mathrm{d}r\mathrm{d}z \\
&= -2\int_{\mathbb{R}^3}\frac{u_r}{r}\partial_r(r\rho)\partial_r^2\rho\mathrm{d}x + 2\int_{\mathbb{R}^3}\frac{u_r}{r}\rho\partial_r^2\rho\mathrm{d}x - \int_{\mathbb{R}^3}\frac{u_r}{r}(\partial_r\rho)^2\mathrm{d}x.
\end{aligned}$$

Thus, Hölder inequality and Young inequality indicate that

$$NR_1 \leq \frac{1}{4}\|\nabla^2\rho(t,\cdot)\|_{L^2}^2 + C\left\|\frac{u_r}{r}(t,\cdot)\right\|_{L^\infty}^2\left(\|\nabla(r\rho)(t,\cdot)\|_{L^2}^2 + \|\rho(t,\cdot)\|_{L^2}^2\right) + \left\|\frac{u_r}{r}(t,\cdot)\right\|_{L^\infty}\|\nabla\rho(t,\cdot)\|_{L^2}^2.$$

Meanwhile, using Lemma 2.3, we have

$$NR_2 + NR_3 = \int_{\mathbb{R}^3}(2\partial_z u_r - w_\theta)\,\partial_z\rho\partial_r\rho\mathrm{d}x \lesssim \|\Omega(t,\cdot)\|_{L^3}\|\nabla(r\rho)(t,\cdot)\|_{L^2}\|\nabla\rho(t,\cdot)\|_{L^6},$$

which indicates the following estimate by interpolation and Young inequality

$$NR_2 + NR_3 \leq \frac{1}{4}\|\nabla^2\rho(t,\cdot)\|_{L^2}^2 + \|\Omega(t,\cdot)\|_{L^2\cap L^6}^2\|\nabla(r\rho)(t,\cdot)\|_{L^2}^2.$$

Similarly, applying the divergence-free property, one can similarly estimate NR_4 in the following way

$$\begin{aligned}
NR_4 =& 2\pi\int_{\mathbb{R}}\int_0^\infty \partial_z(ru_z)(\partial_z\rho)^2\mathrm{d}r\mathrm{d}z = -2\pi\int_{\mathbb{R}}\int_0^\infty \partial_r(ru_r)(\partial_z\rho)^2\mathrm{d}r\mathrm{d}z \\
=& 4\pi\int_{\mathbb{R}}\int_0^\infty ru_r\partial_z\rho\partial_{zr}^2\rho\mathrm{d}r\mathrm{d}z = 2\int_{\mathbb{R}^3}\frac{u_r}{r}\partial_z(r\rho)\partial_{zr}^2\rho\mathrm{d}x \\
\lesssim& \left\|\frac{u_r}{r}(t,\cdot)\right\|_{L^\infty}\|\nabla(r\rho)(t,\cdot)\|_{L^2}\|\nabla^2\rho(t,\cdot)\|_{L^2} \\
\leq& \frac{1}{4}\|\nabla^2\rho(t,\cdot)\|_{L^2}^2 + C\left\|\frac{u_r}{r}(t,\cdot)\right\|_{L^\infty}^2\|\nabla(r\rho)(t,\cdot)\|_{L^2}^2.
\end{aligned}$$

Recall that

$$\left\|\frac{u_r}{r}(t,\cdot)\right\|_{L^\infty} \lesssim \|\Omega(t,\cdot)\|_{L^2}^{1/2}\|\Omega(t,\cdot)\|_{L^6}^{1/2} \le \Phi_{1,3}(t), \quad \forall t \le T_* \tag{3.20}$$

which follows by (3.13) and Lemma 2.3. We insert the above estimates for NR_1–NR_4 into (3.19) to derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho(t,\cdot)\|_{L^2}^2 + \|\nabla^2\rho(t,\cdot)\|_{L^2}^2 \le \Phi_{1,3}(t)\left(\|\nabla(r\rho)(t,\cdot)\|_{L^2}^2 + \|\rho(t,\cdot)\|_{L^2}^2 + \|\nabla\rho(t,\cdot)\|_{L^2}^2\right),$$

which indicates, by using (3.18) and Lemma 3.1, that

$$\|\nabla\rho(t,\cdot)\|_{L^2}^2 + \int_0^t \|\nabla^2\rho(s,\cdot)\|_{L^2}^2 \mathrm{d}s \le \Phi_{1,3}(t). \tag{3.21}$$

This completes the proof of the proposition. □

3.3.3. $L_t^\infty\left(L^2 \cap L^6\right)$ estimate of ∇u. Now let us focus on the velocity field. Using conclusions in Proposition 3.2 and Proposition 3.4, we can derive:

Proposition 3.5. *Under the same conditions as Theorem 1.1, the following $L^2 \cap L^6$ estimate of ∇u*

$$\|\nabla u(t,\cdot)\|_{L^2\cap L^6} \le \Phi_{2,3}(t)$$

holds uniformly for $0 \le t \le T_*$.

Proof. Due to $(1.2)_2$, $\frac{u_\theta}{r}$ satisfies

$$\partial_t\frac{u_\theta}{r} + (b\cdot\nabla)\frac{u_\theta}{r} + 2\frac{u_r}{r}\cdot\frac{u_\theta}{r} = 0.$$

Performing the L^q estimate and using Gronwall inequality, one derives the following inequality:

$$\left\|\frac{u_\theta}{r}(t,\cdot)\right\|_{L^q} \leqslant \left\|\frac{(u_0)_\theta}{r}\right\|_{L^q} \exp\left(2\int_0^t \left\|\frac{u_r}{r}(s,\cdot)\right\|_{L^\infty} ds\right) \le \Phi_{2,3}(t) \quad \text{for any} \quad q \in [2,\infty].$$

Thus, the L^p estimate of $(1.3)_2$ indicates that

$$\|w_\theta(t,\cdot)\|_{L^p} \lesssim \|(w_0)_\theta\|_{L^p} + \|w_r\|_{L_t^1L^\infty}\left\|\frac{u_\theta}{r}\right\|_{L_t^\infty L^p} + \int_0^t \|w_\theta(s,\cdot)\|_{L^p}\left\|\frac{u_r}{r}(s,\cdot)\right\|_{L^\infty}\mathrm{d}s.$$

Using Gronwall inequality and Sobolev inequality, together with Lemma 3.1, (3.14) and (3.21), it follows that

$$\begin{aligned}\|w_\theta(t,\cdot)\|_{L^2} \le& \left(\|(w_\theta)_0\|_{L^2} + \|w_r\|_{L_t^1L^\infty}\left\|\frac{u_\theta}{r}\right\|_{L_t^\infty L^2} + \|\nabla\rho\|_{L_t^1L^2}\right)\exp\left(\int_0^t\left\|\frac{u_r}{r}(s,\cdot)\right\|_{L^\infty}\mathrm{d}s\right)\\ \le& \Big[1+\Phi_{2,3}(t) + \sqrt{t}\Phi_{2,3}(t) + \sqrt{t}\Big]\Phi_{2,3}(t) \le \Phi_{2,3}(t);\\ \|w_\theta(t,\cdot)\|_{L^6} \le& \left(\|(w_0)_\theta\|_{L^6} + \|w_r\|_{L_t^1L^\infty}\left\|\frac{u_\theta}{r}\right\|_{L_t^\infty L^6} + \|\nabla^2\rho\|_{L_t^1L^2}\right)\exp\left(\int_0^t\left\|\frac{u_r}{r}(s,\cdot)\right\|_{L^\infty}\mathrm{d}s\right)\\ \le& \Big[1+\Phi_{2,3}(t) + \Phi_{1,3}(t)\Phi_{2,3}(t) + \Phi_{1,3}(t)\Big]\Phi_{2,3}(t) \le \Phi_{2,3}(t)\end{aligned}$$

for any $t \le T_*$. Then from (2.2) in Lemma 2.2, we have

$$\|\nabla b(t,\cdot)\|_{L^2\cap L^6} \le \Phi_{2,3}(t).$$

Now it remains to deduce the a similar estimate for $\nabla(u_\theta \boldsymbol{e_\theta})$. By an argument using Calderon–Zygmund singular integral operator, one only needs to prove the same estimate for (w_r, w_z), since $\nabla \times (u_\theta \boldsymbol{e_\theta}) = w_r \boldsymbol{e_r} + w_z \boldsymbol{e_z}$ and div $(u_\theta \boldsymbol{e_\theta}) = 0$. Therefore, preforming the L^p estimates for $(1.3)_{1,3}$ together, one derives

$$\max\left\{\frac{\mathrm{d}}{\mathrm{d}t}\|w_r(t,\cdot)\|_{L^p}^p\,,\ \frac{\mathrm{d}}{\mathrm{d}t}\|w_z(t,\cdot)\|_{L^p}^p\right\} \lesssim \int_{\mathbb{R}^3} (|w_r|^p + |w_z|^p)\,|\nabla b|\mathrm{d}x$$
$$\lesssim \|\nabla \times (u_\theta \boldsymbol{e_\theta})(t,\cdot)\|_{L^\infty} \int_{\mathbb{R}^3} \left(|w_r|^{p-1} + |w_z|^{p-1}\right)|\nabla b|\mathrm{d}x.$$

Using Hölder inequality, it follows that

$$\max\left\{\frac{\mathrm{d}}{\mathrm{d}t}\|w_r(t,\cdot)\|_{L^p}^p\,,\ \frac{\mathrm{d}}{\mathrm{d}t}\|w_z(t,\cdot)\|_{L^p}^p\right\} \lesssim \|\nabla \times (u_\theta \boldsymbol{e_\theta})(t,\cdot)\|_{L^\infty}\left(\|w_r(t,\cdot)\|_{L^p}^{p-1} + \|w_z(t,\cdot)\|_{L^p}^{p-1}\right)\|\nabla b(t,\cdot)\|_{L^p}.$$

This indicates that, by dividing $(\|w_r(t,\cdot)\|_{L^p}^{p-1} + \|w_z(t,\cdot)\|_{L^p}^{p-1})$ on both sides

$$\max\left\{\frac{\mathrm{d}}{\mathrm{d}t}\|w_r(t,\cdot)\|_{L^p}\,,\ \frac{\mathrm{d}}{\mathrm{d}t}\|w_z(t,\cdot)\|_{L^p}\right\} \lesssim \|\nabla \times (u_\theta \boldsymbol{e_\theta})(t,\cdot)\|_{L^\infty}\|\nabla b(t,\cdot)\|_{L^p}.$$

Integrating with t, one concludes for any $p \in [2,6]$ that:

$$\|(w_r, w_z)(t,\cdot)\|_{L^p} \lesssim \|((w_0)_r, (w_0)_z)\|_{L^p} + \|\nabla b\|_{L_t^\infty L^p}\int_0^t \|\nabla \times (u_\theta \boldsymbol{e_\theta})(s,\cdot)\|_{L^\infty}\mathrm{d}s \le \Phi_{2,3}(t).$$

This completes the proof of Proposition 3.5. □

3.3.4. $L_t^\infty L^2 \cap L_t^2 \dot{H}^1$ estimate of $\nabla^2\rho$. To close a higher-order estimate of the solution, a key step in our approach is to derive the $L_t^1 L^\infty$ estimate for $(\nabla \times u, \nabla\rho)$[1]. Before that, the $L_t^\infty L^2 \cap L_t^2 \dot{H}^1$ estimate of $\nabla^2\rho$ should be prepared here:

Proposition 3.6. *Under the same conditions as Theorem 1.1, the following space-time estimate of $\nabla^2\rho$ holds:*

$$\|\nabla^2\rho(t,\cdot)\|_{L^2}^2 + \int_0^t \|\nabla^3\rho(s,\cdot)\|_{L^2}^2\mathrm{d}s \le \Phi_{2,3}(t).$$

Proof. Applying ∇^2 on $(1.2)_4$ and performing the L^2 inner product, one may derive that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2\rho(t,\cdot)\|_{L^2}^2 + \|\nabla^3\rho(t,\cdot)\|_{L^2}^2 =& -2\int_{\mathbb{R}^3}\partial_i u_k \partial_j\partial_k\rho\partial_i\partial_j\rho\mathrm{d}x - \int_{\mathbb{R}^3}\partial_i\partial_j u_k\partial_k\rho\partial_i\partial_j\rho\mathrm{d}x\\ =& -\underbrace{\int_{\mathbb{R}^3}\partial_i u_k\partial_j\partial_k\rho\partial_i\partial_j\rho\mathrm{d}x}_{N^2R_1} + \underbrace{\int_{\mathbb{R}^3}\partial_i u_k\partial_k\rho\partial_i\Delta\rho\mathrm{d}x}_{N^2R_2}.\end{aligned} \tag{3.22}$$

Using Hölder inequality, Young inequality and Sobolev inequality, it follows that

$$\begin{aligned}|N^2R_1| \le& \|\nabla u(t,\cdot)\|_{L^3}\|\nabla^2\rho(t,\cdot)\|_{L^2}\|\nabla^2\rho(t,\cdot)\|_{L^6}\\ \le& \frac{1}{4}\|\nabla^3\rho(t,\cdot)\|_{L^2}^2 + C\|\nabla u(t,\cdot)\|_{L^2}\|\nabla u(t,\cdot)\|_{L^6}\|\nabla^2\rho(t,\cdot)\|_{L^2}^2.\end{aligned} \tag{3.23}$$

[1] This part will be deduced in the next subsection.

Similarly, one derives

$$\begin{aligned}|N^2R_2| \leq & \|\nabla u(t,\cdot)\|_{L^3}\|\nabla\rho(t,\cdot)\|_{L^6}\|\nabla^3\rho(t,\cdot)\|_{L^2}\\ \leq & \frac{1}{4}\|\nabla^3\rho(t,\cdot)\|_{L^2}^2 + C\|\nabla u(t,\cdot)\|_{L^2}\|\nabla u(t,\cdot)\|_{L^6}\|\nabla^2\rho(t,\cdot)\|_{L^2}^2.\end{aligned} \tag{3.24}$$

Inserting (3.23) and (3.24) in the right hand side of (3.22), one deduces the following by integrating with t:

$$\|\nabla^2\rho(t,\cdot)\|_{L^2}^2 + \int_0^t \|\nabla^3\rho(s,\cdot)\|_{L^2}^2 \mathrm{d}s \lesssim \|\nabla^2\rho_0\|_{L^2}^2 + \|\nabla u\|_{L^\infty(0,t,L^2\cap L^6)}^2 \int_0^t \|\nabla^2\rho(s,\cdot)\|_{L^2}^2 \mathrm{d}s.$$

Using Propositions 3.4, 3.5, together with (3.3), one concludes that

$$\|\nabla^2\rho(t,)\|_{L^2}^2 + \int_0^t \left\|\nabla^3\rho(s,\cdot)\right\|_{L^2}^2 \mathrm{d}s \leq C_0\left(1+\Phi_{2,3}^2(t)\Phi_{1,3}(t)\right) \leq \Phi_{2,3}(t). \tag{3.25}$$

Thus, we complete the proof of Proposition 3.6. □

3.3.5. $L_T^1 L^\infty$ estimate of w_θ and $\nabla\rho$. Now we arrive the $L_t^1L^\infty$ estimates of $\nabla\times u$ and $\nabla\rho$, which is a corner stone in deriving the global-in-time a priori estimates of the solution. Noting that a part of the $\|\nabla\times u\|_{L_t^1L^\infty}$ is already given in Condition (1.4), our task now is to derive the rest. Here goes the proposition:

Proposition 3.7. *Under the same conditions as Theorem 1.1, the following $L_t^1L^\infty$ estimates of $\nabla\times u$ and $\nabla\rho$ follows*

$$\int_0^t \|(w_\theta,\nabla\rho)(s,\cdot)\|_{L^\infty}\mathrm{d}s \leq \Phi_{2,3}(t).$$

Proof. Recall the equation of w_θ:

$$\partial_t w_\theta + (u_r\partial_r + u_z\partial_z)w_\theta = \frac{u_r}{r}w_\theta + \frac{1}{r}\partial_z(u_\theta)^2 - \partial_r\rho. \tag{3.26}$$

Let us introduce the particle trajectory mapping of the velocity $b := u_r\boldsymbol{e_r} + u_z\boldsymbol{e_z}$:

$$X(t,\cdot):\ \mathbb{R}^3\to\mathbb{R}^3$$

which solves the initial value problem:

$$\frac{\partial X(t,x)}{\partial t} = b(t,X(t,x)), \quad X(0,x)=x.$$

Integrating (3.26) along the particle trajectory starting at $x\in\mathbb{R}^3$, one deduces

$$w_\theta(t,X(t,x)) = (w_0)_\theta(x) + \int_0^t \left(\frac{u_r}{r}w_\theta + \frac{1}{r}\partial_z(u_\theta)^2 - \partial_r\rho\right)(s,X(s,x))\mathrm{d}s.$$

Taking the L^∞ norm with $x\in\mathbb{R}^3$, one derives from the previous estimates:

$$\begin{aligned}\|w_\theta(t,\cdot)\|_{L^\infty} \lesssim & \|(w_0)_\theta\|_{L^\infty} + \int_0^t \left\|\frac{u_r}{r}(s,\cdot)\right\|_{L^\infty}\|w_\theta(s,\cdot)\|_{L^\infty}\mathrm{d}s + \left\|\frac{u_\theta}{r}\right\|_{L_t^\infty L^\infty}\|w_r\|_{L_t^1L^\infty} + \|\nabla\rho\|_{L^1(0,t,\,H^2)}\\ \leq & \Phi_{2,3}(t) + \int_0^t \left\|\frac{u_r}{r}(s,\cdot)\right\|_{L^\infty}\|w_\theta(s,\cdot)\|_{L^\infty}\mathrm{d}s.\end{aligned}$$

Recalling the estimate (3.20), Gronwall inequality indicates that

$$\|w_\theta(t,\cdot)\|_{L^\infty} \le \Phi_{2,3}(t) \exp\left(\int_0^t \left\|\frac{u_r}{r}(s,\cdot)\right\|_{L^\infty} \mathrm{d}s\right) < \Phi_{2,3}(t),$$

which follows that

$$\int_0^t \|w_\theta(s,\cdot)\|_{L^\infty} \mathrm{d}s \le \Phi_{2,3}(t). \tag{3.27}$$

Moreover, using Gagliardo–Nirenberg inequality and Hölder inequality, one derives

$$\begin{aligned}\int_0^t \|\nabla\rho(s,\cdot)\|_{L^\infty} \mathrm{d}s &\lesssim \int_0^t \|\nabla\rho(s,\cdot)\|_{L^2}^{\frac{1}{4}} \|\nabla^3\rho(s,\cdot)\|_{L^2}^{\frac{3}{4}} \mathrm{d}s \\ &\lesssim \|\nabla\rho\|_{L^\infty(0,t,L^2)} \left(\int_0^t \|\nabla^3\rho(s,\cdot)\|_{L^2}^2 \mathrm{d}s\right)^{\frac{3}{8}} t^{\frac{5}{8}} \\ &\le \Phi_{1,3}(t)(\Phi_{2,3}(t))^{\frac{3}{8}} t^{\frac{5}{8}} \le \Phi_{2,3}(t).\end{aligned} \tag{3.28}$$

Here, the third inequality follows from estimates (3.21) and (3.25). Thus, Proposition 3.7 is proved by combining (3.27) and (3.28). □

3.3.6. End proof of Theorem 1.1. First, we focus on the following estimates of a triple product form:

Lemma 3.8. *Let $m \in \mathbb{N}$ and $m \ge 2$, $f, g, k \in C_0^\infty(\mathbb{R}^3)$. The following estimate holds:*

$$\left|\int_{\mathbb{R}^3} [\nabla^m,\, f\cdot\nabla] g \nabla^m k \mathrm{d}x\right| \le C\, \|\nabla^m (f,g,k)\|_{L^2}^2\, \|\nabla(f,\, g)\|_{L^\infty}; \tag{3.29}$$

Proof. By applying Hölder inequality, one derives

$$\left|\int_{\mathbb{R}^3} [\nabla^m,\, f\cdot\nabla] g \nabla^m k \mathrm{d}x\right| \le \|[\nabla^m,\, f\cdot\nabla]g\|_{L^2} \|\nabla^m k\|_{L^2}. \tag{3.30}$$

Due to the commutator estimate by Kato–Ponce [13], it follows that

$$\|[\nabla^m,\, f\cdot\nabla]g\|_{L^2} \le C \left(\|\nabla f\|_{L^\infty} \|\nabla^m g\|_{L^2} + \|\nabla g\|_{L^\infty} \|\nabla^m f\|_{L^2}\right). \tag{3.31}$$

Then, (3.29) follows from substituting (3.31) into (3.30). □

Applying ∇^m ($m \in \mathbb{N}$, $m \ge 3$) to the first two equations of (1.1) and performing the L^2 inner product of the resulting equations with $\nabla^m u$ and $\nabla^m \rho$ respectively, we can obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla^m (u,\rho)(t,\cdot)\|_{L^2}^2 + \|\nabla^{m+1}\rho(t,\cdot)\|_{L^2}^2 \\ &= \underbrace{-\int_{\mathbb{R}^3} [\nabla^m,\, u\cdot\nabla] u \nabla^m u \mathrm{d}x}_{I_1} \underbrace{- \int_{\mathbb{R}^3} [\nabla^m,\, u\cdot\nabla]\rho \nabla^m \rho \mathrm{d}x}_{I_2} + \underbrace{\int_{\mathbb{R}^3} \nabla^m \rho \nabla^m u \mathrm{d}x}_{I_3}.\end{aligned} \tag{3.32}$$

Applying (3.29) in Lemma 3.8, we have I_1–I_2 satisfy

$$I_j \lesssim \|\nabla^m (u,\rho)(t,\cdot)\|_{L^2}^2 \|\nabla(u,\rho)(t,\cdot)\|_{L^\infty}, \quad \forall j = 1,2. \tag{3.33}$$

Meanwhile, I_3 satisfies

$$I_3 \leq \|\nabla^m \rho(t,\cdot)\|_{L^2} \|\nabla^m u(t,\cdot)\|_{L^2} \leq \|\nabla^m (u,\rho)(t,\cdot)\|_{L^2}^2 . \tag{3.34}$$

Substituting (3.33) and (3.34) in (3.32), one derives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(u,\rho)(t,\cdot)\|_{\dot{H}^m}^2 \leq C(1+\|\nabla(u,\rho)(t,\cdot)\|_{L^\infty})\|(u,\rho)(t,\cdot)\|_{\dot{H}^m}^2.$$

Combining this with the fundamental energy estimates (3.1) and (3.3), one conclude the following estimate for full Sobolev norm by interpolation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(u,\rho)(t,\cdot)\|_{H^m}^2 \leq C(1+\underbrace{\|\nabla(u,\rho)(t,\cdot)\|_{L^\infty}}_{L_1})\|(u,\rho)(t,\cdot)\|_{H^m}^2. \tag{3.35}$$

In order to refine L_1 in (3.35) to $\|(\nabla\times u,\nabla\rho)(t,\cdot)\|_{L^\infty}$ so that the estimate in Proposition 3.7 is valid, we introduce the following logarithmic Sobolev inequality:

Lemma 3.9. *For any divergence-free vector field g such that*

$$g: \quad \mathbb{R}^3 \to \mathbb{R}^3$$

and $g \in H^3(\mathbb{R}^3)$, the following estimate holds:

$$\|\nabla g\|_{L^\infty(\mathbb{R}^3)} \lesssim 1+\|\nabla\times g\|_{L^\infty}\log\left(e+\|g\|_{H^3(\mathbb{R}^3)}\right). \tag{3.36}$$

Proof. In Theorem 1 of [14], the authors showed that: For any $1<p<\infty$ and $s>d/p$, there exists a constant $C = C_{d,p,s}$ such that the estimate

$$\|f\|_{L^\infty(\mathbb{R}^d)} \leq C\left(1+\|f\|_{BMO}\log(e+\|f\|_{W^{s,p}(\mathbb{R}^d)})\right) \tag{3.37}$$

holds for all $f \in W^{s,p}(\mathbb{R}^d)$. Noting that

$$\xi\otimes\hat{u} = \frac{\xi}{|\xi|}\otimes\left(\frac{\xi}{|\xi|}\times(\xi\times\hat{u})\right)$$

since $\xi\cdot\hat{u}\equiv 0$[2], and the fact that the Riesz operator is bounded in the BMO space, one derives

$$\|\nabla g\|_{BMO} \lesssim \|\nabla\times g\|_{BMO}.$$

Combining the above relationship and (3.37) (choosing $f:=\nabla g$), noting that $\|\nabla\times g\|_{BMO}\lesssim\|\nabla\times g\|_{L^\infty}$, one concludes the validity of (3.36).

Denoting

$$E_m(t) := \|(u,\rho)(s,\cdot)\|_{H^m}^2, \quad \forall t\leq T_*,$$

using (3.35) and Lemma 3.9, one deduces that

$$E_m'(t) \lesssim (1+\|(\nabla\times u,\nabla\rho)(t,\cdot)\|_{L^\infty}\log(e+E_m(t)))\,(e+E_m(t)).$$

Using Gronwall inequality twice, it follows that

$$e+E_m(t) \leq C\,(e+E_m(0))^{\exp\left(C\int_0^t(1+\|(\nabla\times u,\nabla\rho)(s,\cdot)\|_{L^\infty})\mathrm{d}s\right)}, \quad \forall t\leq T_*, \tag{3.38}$$

Using Proposition 3.7, one deduces

$$\int_0^t (1+\|(\nabla\times u,\,\nabla\rho)(s,\cdot)\|_{L^\infty})\,\mathrm{d}s \leq \Phi_{2,3}(t).$$

[2] Here, $\hat{u}=\hat{u}(\xi)$ is the Fourier transform of u.

Substituting the above estimate into (3.38), one arrives that

$$\sup_{0\le s\le t} \|(u,\rho)(s,\cdot)\|_{H^m}^2 \le \Phi_{4,3}(t),$$

for all $m \in \mathbb{N}$. This completes the proof of Theorem 1.1 and Theorem 1.3. □

Acknowledgements

Z. Li is supported by Natural Science Foundation of Jiangsu Province (No. BK20200803) and National Natural Science Foundation of China (No. 12001285). X. Pan is supported by National Natural Science Foundation of China (Nos. 11801268, 12031006).

References

[1] Abidi, H., Hmidi, T.: On the global well-posedness for Boussinesq system. J. Differ. Equ. **233**(1), 199–220 (2007)
[2] Abidi, H., Hmidi, T., Keraani, S.: On the global regularity of axisymmetric Navier–Stokes–Boussinesq system. Discrete Contin. Dyn. Syst. **29**(3), 737–756 (2011)
[3] Chae, D., Lee, J.: On the regularity of the axisymmetric solutions of the Navier–Stokes equations. Math. Z. **239**(4), 645–671 (2002)
[4] Chen, H., Fang, D., Zhang, T.: Regularity of 3D axisymmetric Navier–Stokes equations. Discrete Cont. Dyn. A **37**(4), 1923–1939 (2017)
[5] Chen, Q., Zhang, Z.: Regularity criterion of axisymmetric weak solutions to the 3D Navier–Stokes equations. J. Math. Anal. Appl. **331**, 1384–1395 (2007)
[6] Cao, C., Wu, J.: Global regularity for the two-dimensional anisotropic Boussinesq equations with vertical dissipation. Arch. Ration. Mech. Anal. **208**(3), 985–1004 (2013)
[7] Chae, D.: Global regularity for the 2D Boussinesq equations with partial viscosity terms. Adv. Math. **203**(2), 497–513 (2006)
[8] Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities, 2nd edn. Cambridge University Press, Cambridge (1952)
[9] Hmidi, T., Rousset, F.: Global well-posedness for the Navier–Stokes–Boussinesq system with axisymmetric data. Ann. l'IHP Anal. Non Linèaire **27**(5), 1227–1246 (2010)
[10] Hmidi, T., Rousset, F.: Global well-posedness for the Euler–Boussinesq system with axisymmetric data. J. Funct. Anal. **260**(3), 745–796 (2011)
[11] Hou, T.Y., Li, C.: Global well-posedness of the viscous Boussinesq equations. Discrete Contin. Dyn. Syst. **12**(1), 1112 (2005)
[12] Hmidi, T., Keraani, S.: On the global well-posedness of the Boussinesq system with zero viscosity. Indiana Univ. Math. J. **58**(4), 1591–1618 (2009)
[13] Kato, T., Ponce, G.: Commutator estimates and the Euler and Navier–Stokes equations. Commun. Pure Appl. Math. **41**(7), 891–907 (1988)
[14] Kozono, H., Taniuchi, Y.: Limiting case of the Sobolev inequality in BMO, with application to the Euler equations. Commun. Math. Phys. **214**, 191–200 (2000)
[15] Larios, A., Lunasin, E., Titi, E.S.: Global well-posedness for the 2D Boussinesq system with anisotropic viscosity and without heat diffusion. J. Differ. Equ. **255**(9), 2636–2654 (2013)
[16] Lei, Z.: On axially symmetric incompressible magnetohydrodynamics in three dimensions. J. Differ. Equ. **259**(7), 3202–3215 (2015)
[17] Majda, A.: Introduction to PDEs and Waves for the Atmosphere and Ocean. AMS/CIMS, Providence (2003)
[18] Marchioro, C., Pulvirenti, M.: Mathematical Theory of Incompressible Nonviscous Fluids. Springer, New York (1994)

[19] Miao, C., Zheng, X.: On the global well-posedness for the Boussinesq system with horizontal dissipation. Commun. Math. Phys. **321**(1), 33–67 (2013)
[20] Mulone, G., Rionero, S.: Necessary and sufficient conditions for nonlinear stability in the magnetic Bénard problem. Arch. Ration. Mech. Anal. **166**(3), 197–218 (2003)
[21] Pedlosky, J.: Geophysical Fluid Dynamics. Springer, New York (1987)
[22] Pan, X.: Global regularity for the 3D non-diffusive MHD-Boussinesq system with axisymmetric data. arXiv:1911.01550

Zijin Li
School of Mathematics and Statistics
Nanjing University of Information Science and Technology
Nanjing 210044
China
e-mail: zijinli@nuist.edu.cn

Zijin Li
Center for Applied Mathematics of Jiangsu Province
Nanjing University of Information Science and Technology
Nanjing 210044
China

Xinghong Pan
School of Mathematics
Nanjing University of Aeronautics and Astronautics
Nanjing 210016
China
e-mail: xinghong_87@nuaa.edu.cn

Xinghong Pan
Key Laboratory of Mathematical Modelling and High Performance Computing of Air Vehicles (NUAA)
MIIT
Nanjing 210016
China

(Received: April 17, 2022; revised: September 24, 2022; accepted: November 20, 2022)