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Liouville theorem of D-solutions to the stationary magnetohydrodynamics system in a slab

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ABSTRACT

In this paper, we study Liouville theorems of D-solutions to the stationary magnetohydrodynamic system in a slab. We will prove trivialness of the velocity and the magnetic field with various boundary conditions. In some boundary conditions, the additional assumption that the horizontal angular component(s) of the velocity or (and) the magnetic field is (are) axially symmetric is needed. More precisely, five types of boundary conditions will be considered: the vertical periodic boundary condition for the velocity and the magnetic field, the Navier-slip boundary condition for the velocity, the perfectly conducting or insulating boundary condition for the magnetic field, the non-slip boundary condition for the velocity, and the perfectly conducting or insulating boundary condition for the magnetic field. One of our innovations is that we do not impose finite Dirichlet integral assumption on the magnetic field compared with previous works.

I. INTRODUCTION

In this paper, we consider the stationary magnetohydrodynamic (MHD) system

$$\begin{cases} u\cdot\nabla u + \nabla p - b\cdot\nabla b - \Delta u = 0, \\ u\cdot\nabla b - b\cdot\nabla u - \Delta b = 0, \\ \nabla\cdot u = \nabla\cdot b = 0 \end{cases} \tag{1.1}$$

in $\mathbb{R}^2\times I$, where $I=[0,1]$ is a one-dimensional interval. $u(x)$, $b(x)\in\mathbb{R}^3$, and $p(x)\in\mathbb{R}$ represent the velocity vector, the magnetic field, and the scalar pressure, respectively. The MHD system, which describes the state of the fluid flows of plasma, is a fundamental partial differential equation in nature. For the background of the MHD system, we refer the reader to Ref. 1 for more details. We note that if $b\equiv 0$, the MHD system is reduced to the Navier–Stokes system.

The main aim of our paper is to study the Liouville-type theorem of D-solutions of the stationary MHD system (1.1). This study is partly motivated by the related Liouville problem of the stationary Navier–Stokes equation, which has attracted much attention in recent years and is still far from being fully understood. See, for example, Refs. 2–11 and references therein.

For a domain $\Omega\in\mathbb{R}^3$ and $1\le p\le\infty$, $L^p(\Omega)$ denotes the usual Lebesgue integrable space with norm $\|\cdot\|_{L^p(\Omega)}$, and a function $f\in L^p_{\mathrm{loc}}(\Omega)$ means that $f\in L^p(\Omega')$ for any bounded domain Ω' with $\overline{\Omega'}\subset\Omega$. The symbol ∂_i stands for $\frac{\partial}{\partial x_i}$.

We say that (u,b) is a D-solution of (1.1), which means that $(u,b)\in D^{1,2}_{\mathrm{loc}}(\mathbb{R}^2\times I)$ is a weak solution of (1.1), satisfying the following finite Dirichlet integral condition:

$$\int_{\mathbb{R}^2\times I}|Du|^2dx < +\infty,$$

where the functional space $D^{1,2}_{\text{loc}}(\mathbb{R}^2\times I)$ is defined by

$$D^{1,2}_{\text{loc}}(\mathbb{R}^2\times I) := \left\{f\in L^1_{\text{loc}}(\mathbb{R}^2\times I)\middle|\partial_i f\in L^2_{\text{loc}}(\mathbb{R}^2\times I), i=1,2,3\right\}.$$

Remark 1.1. Here, our D-solutions are weaker than those considered in the previous works, such as Refs. 12 and 13, where the finite Dirichlet integral assumption on the magnetic field is imposed.

In the following, sometimes, we will carry out some of the proofs in the cylindrical coordinates (r,θ,x_3). That is, for $x=(x_1,\,x_2,\,x_3)\in\mathbb{R}^3$, $r=\sqrt{x_1^2+x_2^2}$ and $\theta=\arctan\frac{x_2}{x_1}$. The solution of the incompressible stationary magnetohydrodynamic system is given as

$$u = u^r(r,\theta,x_3)e_r + u^\theta(r,\theta,x_3)e_\theta + u^3(r,\theta,x_3)e_3,$$

$$b = b^r(r,\theta,x_3)e_r + b^\theta(r,\theta,x_3)e_\theta + b^3(r,\theta,z)e_3,$$

where the basis vectors $e_r, e_\theta,$ ande_3 are

$$e_r=\left(\frac{x_1}{r},\frac{x_2}{r},0\right),\quad e_\theta=\left(-\frac{x_2}{r},\frac{x_1}{r},0\right),\quad e_3=(0,0,1).$$

Denote the one-dimensional periodic domain $[0,1]$ by $\mathbf{S}$ and $x_h=(x_1,x_2)$. Our first result is given as follows:

Theorem 1.2. *Let (u,h) be a bounded weak solution to the problem*

$$\begin{cases} u\cdot\nabla u+\nabla p-b\cdot\nabla b-\Delta u=0 & in\quad \mathbb{R}^2\times\mathbf{S},\\ u\cdot\nabla b-b\cdot\nabla u-\Delta b=0 & in\quad \mathbb{R}^2\times\mathbf{S},\\ \nabla\cdot u=\nabla\cdot b=0 & in\quad \mathbb{R}^2\times\mathbf{S},\\ (u,b)(x_h,x_3)=(u,b)(x_h,x_3+1) & for\quad x_h\in\mathbb{R}^2, \end{cases} \tag{1.2}$$

with the finite Dirichlet integral $\int_{\mathbb{R}^2\times\mathbf{S}}|\nabla u(x)|^2dx<+\infty$.

If u^θ, b^θ are axially symmetric (independent of θ) and $\lim\limits_{|x_h|\to+\infty} b=0$, then we have $u=(0,0,c)$ and $b=(0,0,0)$ for some constant c.

Remark 1.3. If $b\equiv 0$, our result in Theorem 1.2 is an improvement of Ref. 3, Theorem 1.3 where vanishing of D-solutions for the Navier–Stokes equations is proved with the additional assumption that all the components of the velocity are axially symmetric and vanishing at far field $|x_h|\to\infty$. In addition, our result improves that in Ref. 13, Theorem 1.1 and Corollary 1.1, where many more assumptions on the velocity and the magnetic field are imposed.

Despite u and b being periodic in the vertical direction, our method is also valid for D-solutions of certain boundary value problems of the magnetohydrodynamic system (1.1) in the slab $\mathbb{R}^2\times[0,1]$. Below is a corollary that deals with the Navier-slip boundary condition for the velocity and the perfectly conducting or insulating boundary condition for the magnetic field.

Corollary 1.4. Let (u,h) be a bounded weak solution to the magnetohydrodynamic system

$$\begin{cases} u\cdot\nabla u+\nabla p-b\cdot\nabla b-\Delta u=0 & in\quad \mathbb{R}^2\times[0,1],\\ u\cdot\nabla b-b\cdot\nabla u-\Delta b=0 & in\quad \mathbb{R}^2\times[0,1],\\ \nabla\cdot u=\nabla\cdot b=0 & in\quad \mathbb{R}^2\times[0,1],\\ u\cdot n=0,\ \nabla\times u\times n=0 & on\quad \mathbb{R}^2\times(\{0\}\cup\{1\}),\\ b\cdot n=0,\ \nabla\times b\times n=0,\quad \text{or}\quad b\times n=0 & on\quad \mathbb{R}^2\times(\{0\}\cup\{1\}), \end{cases}$$

with the finite Dirichlet integral $\int_{\mathbb{R}^2\times[0,1]}|\nabla u(x)|^2dx<\infty$. Here, n is the unit outward normal vector on the boundary. Then, the following holds:

(i) *Under the boundary condition that $u\cdot n=0, \nabla\times u\times n=0$ and $b\cdot n=0, \nabla\times b\times n=0$, we have $(u,b)=0$, provided that u^θ, b^θ are axially symmetric and $\lim\limits_{|x_h|\to+\infty} b=0$.*

(ii) *Under the boundary condition that $u\cdot n=0, \nabla\times u\times n=0$ and $b\times n=0$, we have $(u,b)=0$, provided that u^θ are axially symmetric and $\lim\limits_{|x_h|\to+\infty} b=0$.*

Remark 1.5. The well-posedness problems of Corollary 1.4 can be found in Refs. 14 and 15 and references therein. Our results in Corollary 1.4 are an improvement of Ref. 13, Corollary 1.2. Note that under the insulating boundary condition for the magnetic field, the axially symmetric assumption on b^θ is not needed.

Indeed, it is a subtle problem to give a suitable boundary condition for the magnetic field mathematically due to the fact that the magnetic field satisfies a system of elliptic equations up to leading order with the additional divergence-free constraint, i.e.,

$$\begin{cases} -\Delta b = \nabla \times (u \times b) & \text{in} \quad \mathbb{R}^2 \times I, \\ \nabla \cdot b = 0 & \text{in} \quad \mathbb{R}^2 \times I. \end{cases}$$

Hence, the standard elliptic boundary condition, such as the homogeneous Dirichlet boundary condition, may lead to an overdetermined problem. The perfectly conducting and insulating boundary conditions in Corollary 1.4 for the magnetic field are characterized as those with these boundary conditions for b, and the corresponding Stokes and Laplacian operators are identical, which is not true, in general, for the Dirichlet boundary condition; see, for example, Refs. 16 and 17. Hence, usually, we do not impose the Dirichlet boundary condition $b = 0$ on the boundary.

In addition, the well-posedness problem for the stationary MHD system in the case that the velocity satisfies the (non-slip) boundary condition and the magnetic field satisfies the perfectly conducting or insulating boundary condition has been established. See Refs. 18 and 19 and references therein. We have the following theorem concerning the Liouville theorem of the MHD system with the same boundary conditions.

Theorem 1.6. *Let (u, h) be a bounded weak solution to the magnetohydrodynamic system*

$$\begin{cases} u \cdot \nabla u + \nabla p - b \cdot \nabla b - \Delta u = 0 & in \quad \mathbb{R}^2 \times [0,1], \\ u \cdot \nabla b - b \cdot \nabla u - \Delta b = 0 & in \quad \mathbb{R}^2 \times [0,1], \\ \nabla \cdot u = \nabla \cdot b = 0 & in \quad \mathbb{R}^2 \times [0,1], \\ u = 0 & on \quad \mathbb{R}^2 \times (\{0\} \cup \{1\}), \\ b \cdot n = 0, \ \nabla \times b \times n = 0, \quad \text{or} \quad b \times n = 0 & on \quad \mathbb{R}^2 \times (\{0\} \cup \{1\}), \end{cases} \tag{1.3}$$

with the finite Dirichlet integral $\int_{\mathbb{R}^2 \times [0,1]} |\nabla u(x)|^2 dx < \infty$. Here, n is the unit outward normal vector on the boundary. Then, the following holds:

(i) *$u \equiv 0$ and $b \equiv (c_1, c_2, 0)$ when $u \equiv 0$ and $b \cdot n = 0, \nabla \times b \times n = 0$ on $\mathbb{R}^2 \times (\{0\} \cup \{1\})$.*
(ii) *$u \equiv 0$ and $b \equiv (0, 0, c_3)$ when $u \equiv 0$ and $b \times n = 0$ on $\mathbb{R}^2 \times (\{0\} \cup \{1\})$.*

Here, c_1, c_2, and c_3 are three constants.

Remark 1.7. In Theorem 1.6, if $b \equiv 0$, our results go back to the Liouville theorem for the Navier–Stokes equations in Ref. 3, Theorem 1.1.

This paper is organized as follows. In Sec. I, we give a useful proposition concerning gradient estimates of ∇V for the divergence equation $\nabla \cdot V = f$ in a cylinder, which will be used to show that the L^2 mean oscillation of the pressure p is bounded. Section I is devoted to proving Theorem 1.2, Corollary 1.4. Using a similar but different argument as that in the Proof of Theorem 1.2, especially when dealing with the oscillation estimate of the pressure p, Theorem 1.6 will be proven in Sec. III D.

Throughout this paper, we use C to denote a generic constant, which may be different from line to line. We also apply $A \lesssim B$ to denote $A \le CB$. For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we write $x = (x_h, x_3)$ for simplicity. We denote that$B(x_h^0, r) := \{x \in \mathbb{R}^2 : |x_h - x_h^0| < r\}$. We simply denote that $B_r := B(0_h, r)$. For a vector function $f = (f^1, f^2, f^3)$, f^h denotes (f^1, f^2). ∇_h or ∂_h denotes (∂_1, ∂_2) with a little abuse of notation if no confusion is caused.

II. PRELIMINARY

For $\alpha \in [0,1]$, define the domain $\Omega_{R,\alpha} = \{x_h \in \mathbb{R}^2 \| x_h| \le R\} \times [0, R^\alpha]$, where $R \ge 1$. We consider the following problem:
Given

$$f \in L^2(\Omega_{R,\alpha}), \qquad \text{with} \qquad \int_{\Omega_{R,\alpha}} f = 0, \tag{2.1}$$

find a vector field $V : \Omega_{R,\alpha} \to \mathbb{R}^3$ such that

$$\nabla \cdot V = f, \quad V \in W_0^{1,2}(\Omega_{R,\alpha}), \qquad \|\nabla V\|_{L^2} \le c_0 \|f\|_{L^2}, \tag{2.2}$$

with $c_0 = c_0(\Omega_{R,\alpha})$. For our purpose, we need an explicit estimate of the c_0 constant depending on the radius R and α.

The first solution of this problem is given by Bogovskiĭ.[20,21] See also Lemma III.3.1 of Ref. 22.

Lemma 2.8. Let $\Omega = \{\bar{x}_h \in \mathbb{R}^2 \| \bar{x}_h| \le 1\} \times [0,1]$. *Then, for any* $\bar{f} \in L^2(\Omega)$, *satisfying*

$$\bar{f} \in L^2(\Omega), \qquad \text{with} \qquad \int_\Omega \bar{f} = 0,$$

there exists a constant C and a vector function $\bar{V} : \Omega \to \mathbb{R}^3$ *such that*

$$\nabla \cdot \bar{V} = \bar{f}, \quad \bar{V} \in W_0^{1,2}(\Omega), \qquad \|\nabla \bar{V}\|_{L^2} \le C\|\bar{f}\|_{L^2}, \tag{2.3}$$

where C is an absolute constant.

Next, we use the above lemma and a scaling argument to deduce the following proposition. We mention that the constant on the right-hand side is also an absolutely constant c, which is independent of the diameter of the domain $\Omega_{R,\alpha}$.

Proposition 2.9. Let $\Omega_{R,\alpha}$ *be as mentioned above. Then, for any* $f \in L^2(\Omega_{R,\alpha})$, *satisfying (2.1), problem (2.2) has one solution V such that for the constant* c_0 *in (2.2), we have the following estimate:*

$$\begin{aligned} &\|\nabla_h V^h\|_{L^2(\Omega_{R,\alpha})} + \|\partial_3 V^3\|_{L^2(\Omega_{R,\alpha})} \le C\|f\|_{L^2(\Omega_{R,\alpha})},\\ &\|\partial_3 V^h\|_{L^2(\Omega_{R,\alpha})} \le CR^{1-\alpha}\|f\|_{L^2(\Omega_{R,\alpha})}, \quad \|\nabla_h V^3\|_{L^2(\Omega_{R,\alpha})} \le CR^{\alpha-1}\|f\|_{L^2(\Omega_{R,\alpha})}, \end{aligned} \tag{2.4}$$

where C is independent of $\Omega_{R,\alpha}$.

Proof. For $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \Omega$, define

$$\bar{f}(\bar{x}_1, \bar{x}_2, \bar{x}_3) := f(R\bar{x}_1, R\bar{x}_2, R^\alpha \bar{x}_3) = f(x_1, x_2, x_3).$$

It is easy to see that $\bar{f}$ satisfies the assumption in Lemma 2.8. Therefore, by Lemma 2.8, there exists a vector function $\bar{V} : \Omega \to \mathbb{R}^3$, satisfying (2.3). Then, for $x \in \Omega_{R,\alpha}$, define

$$\begin{aligned} V(x_1,x_2,x_3) &= \big(V^1(x_1,x_2,x_3), V^2(x_1,x_2,x_3), V^3(x_1,x_2,x_3)\big)\\ &= \left(R\bar{V}^1\left(\frac{x_1}{R}, \frac{x_2}{R}, \frac{x_3}{R^\alpha}\right), R\bar{V}^2\left(\frac{x_1}{R}, \frac{x_2}{R}, \frac{x_3}{R^\alpha}\right), R^\alpha \bar{V}^3\left(\frac{x_1}{R}, \frac{x_2}{R}, \frac{x_3}{R^\alpha}\right)\right). \end{aligned}$$

By a direct computation, we have

$$\nabla \cdot V = f, \quad V \in W_0^{1,2}(\Omega_{R,\alpha}) \qquad \text{in} \quad x \quad \text{variables.}$$

Now, we estimate the L^2 norm of ∇V. First, we have

$$\begin{aligned} &\|\nabla_h V^h\|^2_{L^2(\Omega_{R,\alpha})} + \|\partial_3 V^3\|^2_{L^2(\Omega_{R,\alpha})}\\ &= \sum_{i,j=1}^{2} \int_0^{R^\alpha} \int_{|x_h|\le R} \left|\frac{\partial V^j}{\partial x_i}\right|^2 dx_h dx_3 + \int_0^{R^\alpha} \int_{|x_h|\le R} \left|\frac{\partial V^3}{\partial x_3}\right|^2 dx_h dx_3\\ &= \int_0^{R^\alpha} \int_{|x_h|\le R} \left(\sum_{i,j=1}^{2} \left|\frac{\partial \bar{V}^j}{\partial \bar{x}_i}\right|^2 \left(\frac{x_h}{R}, \frac{x_3}{R^\alpha}\right) + \left|\frac{\partial \bar{V}^3}{\partial \bar{x}_3}\right|^2 \left(\frac{x_h}{R}, \frac{x_3}{R^\alpha}\right)\right) dx_h dx_3\\ &= R^{2+\alpha} \int_0^1 \int_{|\bar{x}_h|\le 1} \left(\sum_{i,j=1}^{2} \left|\frac{\partial \bar{V}^j}{\partial \bar{x}^i}\right|^2 (\bar{x}_h, \bar{x}_3) + \left|\frac{\partial \bar{V}^3}{\partial \bar{x}_3}\right|^2 (\bar{x}_h, \bar{x}_3)\right) d\bar{x}_h d\bar{x}_3\\ &\le CR^{2+\alpha}\|\nabla \bar{V}\|^2_{L^2(\Omega)}. \end{aligned}$$

Then,

$$\begin{aligned}\|\nabla_h V^3\|^2_{L^2(\Omega_{r,\alpha})} &= \sum_{i=1}^{2}\int_0^{R^\alpha}\int_{|x_h|\le R}\left|\frac{\partial V^3}{\partial x_i}\right|^2 dx_h dx_3\\ &= R^{2(\alpha-1)}\int_0^{R^\alpha}\int_{|x_h|\le R}\sum_{i=1}^{2}\left|\frac{\partial \bar V^3}{\partial \bar x^i}\right|^2\left(\frac{x_h}{R},\frac{x_3}{R^\alpha}\right)dx_h dx_3\\ &= R^{3\alpha}\int_0^1\int_{|\bar x_h|\le 1}\sum_{i=1}^{2}\left|\frac{\partial \bar V^3}{\partial \bar x^i}\right|^2(\bar x_h,\bar x_3)d\bar x_h d\bar x_3\\ &\le CR^{3\alpha}\|\nabla \bar V\|^2_{L^2(\Omega)}.\end{aligned}$$

At last,

$$\begin{aligned}\|\partial_3 V^h\|^2_{L^2(\Omega_{R,\alpha})} &= \sum_{i=1}^{2}\int_0^{R^\alpha}\int_{|x_h|\le r}\left|\frac{\partial V^i}{\partial x_3}\right|^2 dx_h dx_3\\ &= R^{2(1-\alpha)}\int_0^{R^\alpha}\int_{|x_h|\le R}\sum_{i=1}^{2}\left|\frac{\partial \bar V^i}{\partial \bar x^3}\right|^2\left(\frac{x_h}{R},\frac{x_3}{R^\alpha}\right)dx_h dx_3\\ &= R^{4-\alpha}\int_0^1\int_{|\bar x_h|\le 1}\sum_{i=1}^{2}\left|\frac{\partial \bar V^3}{\partial \bar x^i}\right|^2(\bar x_h,\bar x_3)d\bar x_h d\bar x_3\\ &\le CR^{4-\alpha}\|\nabla \bar V\|^2_{L^2(\Omega)}.\end{aligned}$$

In addition, it is easy to see that

$$\|f\|^2_{L^2(\Omega_{R,\alpha})} = R^{2+\alpha}\|\bar f\|^2_{L^2(\Omega)}.$$

Combining the above estimates and (2.3), we can get (2.4), which finishes the Proof of Proposition 2.9.

III. PROOF OF THEOREM 1.2

The proof is divided into three parts. First, we will show that the horizontal radial components of the velocity and the magnetic field u^r and b^r actually belong to $L^2(\mathbb{R}^2\times\mathbf{S})$. Second, using Proposition 2.9, we give an L^2 mean oscillation estimate for the pressure p in a cylinder. At last, we prove the trivialness of u and b.

A. L^2 estimates for u^r and b^r

Lemma 3.10. Under assumptions of Theorem 1.2, we have

$$\|(u^r,b^r)\|_{L^2(\mathbb{R}^2\times\mathbf{S})} + \|\nabla b^r\|_{L^2(\mathbb{R}^2\times\mathbf{S})} < C^*.$$

Here, C^* is a constant depending on $\|\nabla u\|_{L^2(\mathbb{R}^2\times\mathbf{S})}$ and $\|(u,b)\|_{L^\infty(\mathbb{R}^2\times\mathbf{S})}$.

Proof. In cylindrical coordinates, the divergence-free condition $(1.2)_3$ is translated as

$$\nabla\cdot u = \frac{1}{r}\partial_r(ru^r) + \frac{1}{r}\partial_\theta u^\theta + \partial_3 u^3 = 0,\quad \nabla\cdot h = \frac{1}{r}\partial_r(rb^r) + \frac{1}{r}\partial_\theta b^\theta + \partial_3 b^3 = 0.$$

Since u^θ, b^θ are independent of θ, we have

$$\nabla\cdot u = \frac{1}{r}\partial_r(ru^r) + \partial_3 u^3 = 0,\quad \nabla\cdot h = \frac{1}{r}\partial_r(rb^r) + \partial_3 b^3 = 0. \tag{3.1}$$

Now, we integrate the first equation of (3.1) in $\mathbf{S}$ about x_3, and thus, we get

$$\partial_r\int_0^1 (ru^r)dx_3 = -r\int_0^1 \partial_3 u^3 dx_3 = -r[u^3(x_h,1) - u^3(x_h,0)] = 0,$$

where at the last line, we have used the periodic boundary condition. Hence, we have

$$r\int_0^1 u^r dx_3 = f(\theta).$$

By setting $r = 0$, we can actually get $f(\theta) \equiv 0$, which means that

$$\int_0^1 u^r dx_3 = 0. \tag{3.2}$$

In addition, we can obtain

$$\int_0^1 b^r dx_3 = 0. \tag{3.3}$$

1. L^2 estimate of u^r

By using (3.2), the one-dimensional Poincaré inequality indicates that

$$\begin{aligned}\int_{\mathbb{R}^2\times\mathbf{S}} |u^r|^2 dx &= \int_{\mathbb{R}^2\times\mathbf{S}} \left| u^r - \frac{1}{|\mathbf{S}|}\int u^r dx_3 \right|^2 dx_3 dx_h \\ &\lesssim \int_{\mathbb{R}^2}\int_0^1 |\partial_3 u^r|^2 dx_3 dx_h \lesssim \int_{\mathbb{R}^2\times\mathbf{S}} |\nabla u|^2 dx < \infty.\end{aligned}$$

2. L^2 estimate of ∇b^r

By a direct computation, we can see that b^r satisfies

$$u\cdot\nabla b^r - b\cdot\nabla u^r = (\Delta - \frac{1}{r^2})b^r. \tag{3.4}$$

Let $\phi(s)$ be a smooth cut-off function satisfying

$$\phi(s) = \begin{cases} 1, & s\in[0,1], \\ 0, & s\geq 2, \end{cases} \tag{3.5}$$

with the usual property that ϕ, ϕ', and ϕ'' are bounded. Set $\phi_R(y_h) = \phi(\frac{|y_h|}{R})$, where R is a large positive number. Now, testing (3.4) with $b^r\phi_R$, we obtain

$$\int_{\mathbb{R}^2\times\mathbf{S}} -(\Delta - \frac{1}{r^2})b^r(b^r\phi_R)dx = \int_{\mathbb{R}^2\times\mathbf{S}} -(u\cdot\nabla b^r - b\cdot\nabla u^r)(b^r\phi_R)dx.$$

Integration by parts indicates that

$$\begin{aligned}&\int_{\mathbb{R}^2\times\mathbf{S}} |\nabla b^r|^2\phi_R dx - \frac{1}{2}\int_{\mathbb{R}^2\times\mathbf{S}} |b^r|^2\Delta\phi_R dx + \int_{\mathbb{R}^2\times\mathbf{S}} \frac{|b^r|^2}{r^2}\phi_R dx \\ &= -\frac{1}{2}\int_{\mathbb{R}^2\times\mathbf{S}} u\cdot\nabla|b^r|^2\phi_R dx - \int_{\mathbb{R}^2\times\mathbf{S}} (u^r b^r) b\cdot\nabla\phi_R dx - \int_{\mathbb{R}^2\times\mathbf{S}} (b\cdot\nabla b^r)u^r\phi_R dx \\ &= \frac{1}{2}\int_{\mathbb{R}^2\times\mathbf{S}} |b^r|^2 u\cdot\nabla\phi_R dx - \int_{\mathbb{R}^2\times\mathbf{S}} (u^r b^r) b\cdot\nabla\phi_R dx - \int_{\mathbb{R}^2\times\mathbf{S}} (b\cdot\nabla b^r)u^r\phi_R dx.\end{aligned}$$

By remembering that ϕ_R is a function of r, independent of θ, we have

$$u\cdot\nabla\phi_R = u^r\partial_r\phi_R, \quad b\cdot\nabla\phi_R = b^r\partial_r\phi_R.$$

Then, using Cauchy inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}^2\times I} |\nabla b^r|^2\phi_R dx + \int_{\mathbb{R}^2\times\mathbf{S}} \frac{|b^r|^2}{r^2}\phi_R dx \\ &\lesssim \frac{1}{R^2}\int_{B_{2R}\times\mathbf{S}} |b^r|^2 dx + \frac{\|b^r\|_{L^\infty}^2}{2R}\int_{B_{2R}\times\mathbf{S}} |u^r| dx + \frac{1}{2}\int_{\mathbb{R}^2\times\mathbf{S}} |\nabla b^r|^2\phi_R dx + \frac{\|b\|_{L^\infty}^2}{2}\int_{B_{2R}\times\mathbf{S}} |u^r|^2\phi_R dx \\ &\leq \frac{\|b^r\|_{L^\infty}^2}{R^2}\int_{B_{2R}\times\mathbf{S}} dx + \frac{\|b^r\|_{L^\infty}^2}{2R}\left(\int_{B_{2R}} |u^r|^2 dx\right)^{1/2}\left(\int_{B_{2R}\times\mathbf{S}} dx\right)^{1/2} \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^2\times\mathbf{S}} |\nabla b^r|^2\phi_R dx + \frac{\|b\|_{L^\infty}^2}{2}\int_{B_{2R}\times\mathbf{S}} |u^r|^2\phi_R dx \\ &\leq C\|b\|_{L^\infty}^2(1 + \|u^r\|_{L^2}^2) + \frac{1}{2}\int_{\mathbb{R}^2\times\mathbf{S}} |\nabla b^r|^2\phi_R dx.\end{aligned}$$

Therefore, we get

$$\int_{\mathbb{R}^2\times\mathbf{S}} |\nabla b^r|^2 \phi_R dx \lesssim \|b\|_{L^\infty}^2 (1+\|u^r\|_{L^2}^2).$$

By letting $R \to +\infty$, we have

$$\|\nabla b^r\|_{L^2(\mathbb{R}^2\times\mathbf{S})} \lesssim \|b\|_{L^\infty}(1+\|u^r\|_{L^2}) < +\infty. \tag{3.6}$$

3. L^2 estimate of b^r

By using (3.3), the one-dimensional Poincaré inequality indicates that

$$\begin{aligned}\int_{\mathbb{R}^2\times\mathbf{S}} |b^r|^2 dx &= \int_{\mathbb{R}^2\times\mathbf{S}} \left| b^r - \frac{1}{|\mathbf{S}|}\int_{\mathbf{S}} b^r dx_3\right|^2 dx_3 dx_h \\ &\lesssim \int_{\mathbb{R}^2}\int_0^1 |\partial_3 b^r|^2 dx_3 dx_h \\ &\lesssim \int_{\mathbb{R}^2\times\mathbf{S}} |\nabla b^r|^2 dx < +\infty.\end{aligned}$$

□

B. L^2 mean oscillation estimate of the pressure p

Lemma 3.11. Let (u,b) be the solution of (1.2), and then, we have, for $n\in\mathbb{N}\backslash\{0\}$,

$$\|p-p_n\|_{L^2(B_n\times\mathbf{S})} \le C_0 n, \tag{3.7}$$

where $C_0 = C_0(\|(u,b)\|_{L^\infty}, \|\nabla u\|_{L^2})$ and $p_n := \frac{1}{|B_n\times\mathbf{S}|}\int_{B_n\times\mathbf{S}} p dx$ is the average of p on $B_n\times\mathbf{S}$.

Now, we consider system (1.2) in $B_n\times[0,n]$. In Proposition 2.9, set $\alpha = 1$ and $R = n$, with $n\in\mathbb{N}\backslash\{0\}$, and choose $f = p - p_n$. Then, there exists V in $B_n\times[0,n]$ satisfying

$$\nabla\cdot V = p - p_n, \qquad \|\nabla V\|_{L^2(B_n\times[0,n])} \le C\|p-p_n\|_{L^2(B_n\times[0,n])},$$

where C is an absolute constant, independent of n. Now, multiplying $(1.2)_1$ with V and integrating $B_n\times[0,n]$, we get

$$\int_{B_n\times[0,n]} \nabla(p-p_n)\cdot V dx = \int_{B_n\times[0,n]} (\Delta u - u\cdot\nabla u + b\cdot\nabla b)\cdot V dx.$$

Integration by parts indicates that

$$\begin{aligned}&\int_{B_n\times[0,n]} (p-p_n)^2 dx \\ &= \int_{B_n\times[0,n]} (p-p_n)\nabla\cdot V dx \\ &= -\int_{B_n\times[0,n]} (\Delta u - u\cdot\nabla u + b\cdot\nabla b)\cdot V dx \\ &= \sum_{i,j=1}^3 \int_{B_n\times[0,n]} \partial_i u^j \partial_i V^j + \partial_i(u^i u^j - b^i b^j) V^j dx \\ &= \sum_{i,j=1}^3 \int_{B_n\times[0,n]} (\partial_i u^j - u^i u^j + b^i b^j)\partial_i V^j dx \\ &\le \|\nabla V\|_{L^2(B_n\times[0,n])}\left(\|\nabla u\|_{L^2(B_n\times[0,n])} + \||u|^2\|_{L^2(B_n\times[0,n])} + \||b|^2\|_{L^2(B_n\times[0,n])}\right) \\ &\le \|\nabla V\|_{L^2(B_n\times[0,n])}\left(\|\nabla u\|_{L^2(B_n\times[0,n])} + \|(u,b)\|_{L^\infty}^2 \|1\|_{L^2(B_n\times[0,n])}\right).\end{aligned}$$

Using Cauchy–Schwartz inequality, we have

$$\begin{aligned}
&\int_{B_n\times[0,n]}(p-p_n)^2dx\\
&\le\|\nabla V\|_{L^2(B_n\times[0,n])}\Big(\|\nabla u\|_{L^2(B_n\times[0,n])}+n^{3/2}\|(u,b)\|^2_{L^\infty(B_n\times[0,n])}\Big)\\
&\le\varepsilon\|\nabla V\|^2_{L^2(B_n\times[0,n])}+C_\varepsilon\Big(\|\nabla u\|^2_{L^2(B_n\times[0,n])}+\|(u,b)\|^4_{L^\infty(B_n\times[0,n])}n^3\Big)\\
&\le C\varepsilon\|p-p_n\|^2_{L^2(B_n\times[0,n])}+C_\varepsilon\Big(\|\nabla u\|^2_{L^2(B_n\times[0,n])}+\|(u,b)\|^4_{L^\infty(B_n\times[0,n])}n^3\Big).
\end{aligned}$$

By choosing ε small enough, we can obtain

$$\|p-p_n\|_{L^2(B_n\times[0,n])}\le C\Big(\|\nabla u\|_{L^2(B_n\times[0,n])}+\|(u,b)\|^2_{L^\infty(B_n\times[0,n])}n^{3/2}\Big).$$

Remembering that (u,p) is periodic in the x_3 direction, the above inequality can be rewritten as

$$n^{1/2}\|p-p_n\|_{L^2(B_n\times\mathbf{S})}\le C\Big(n^{1/2}\|\nabla u\|_{L^2(B_n\times\mathbf{S})}+\|(u,b)\|^2_{L^\infty(B_n\times\mathbf{S})}n^{3/2}\Big).$$

This implies (3.7).

C. Trivialness of u and b

Let $\phi(s)$ be the test function in (3.5), and set $\phi_n(y_h)=\phi(\frac{|y_h|}{n})$. Testing the first two equations of (1.2) with $u\phi_n$ and $b\phi_n$, respectively, we achieve that

$$-\int_{\mathbb{R}^2\times\mathbf{S}}u\phi_n\Delta udx=-\int_{\mathbb{R}^2\times\mathbf{S}}u\phi_R(u\cdot\nabla u-b\cdot\nabla b+\nabla(p-p_n))dx,$$

$$-\int_{\mathbb{R}^2\times\mathbf{S}}b\phi_n\Delta bdx=-\int_{\mathbb{R}^2\times\mathbf{S}}b\phi_n(u\cdot\nabla b-b\cdot\nabla u)dx.$$

Direct integration by parts implies that

$$\begin{aligned}
&\int_{\mathbb{R}^2\times\mathbf{S}}|\nabla u|^2\phi_ndx+\sum_{i=1}^3\int_{\mathbb{R}^2\times\mathbf{S}}u^i\nabla u^i\cdot\nabla\phi_ndx\\
&=\frac{1}{2}\int_{\mathbb{R}^2\times\mathbf{S}}|u|^2u\cdot\nabla\phi_ndx+\int_{\mathbb{R}^2\times\mathbf{S}}(p-p_n)u\cdot\nabla\phi_ndx\\
&\quad-\sum_{i,j=1}^3\int_{\mathbb{R}^2\times\mathbf{S}}b_ib_j\partial_iu_j\phi_ndx-\sum_{i,j=1}^3\int_{\mathbb{R}^2\times\mathbf{S}}b_ib_ju_j\partial_i\phi_ndx
\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^2\times\mathbf{S}}|\nabla b|^2\phi_ndx-\frac{1}{2}\int_{\mathbb{R}^2\times\mathbf{S}}|b|^2\Delta\phi_ndx\\
&=\frac{1}{2}\int_{\mathbb{R}^2\times\mathbf{S}}|b|^2u\cdot\nabla\phi_ndx+\sum_{i,j=1}^3\int_{\mathbb{R}^2\times\mathbf{S}}b_ib_j\partial_iu_j\phi_ndx.
\end{aligned}\tag{3.9}$$

Therefore, the following equation is achieved by adding (3.8) and (3.9) together:

$$\begin{aligned}
&\int_{\mathbb{R}^2\times\mathbf{S}}(|\nabla u|^2+|\nabla b|^2)\phi_ndx-\frac{1}{2}\int_{\mathbb{R}^2\times\mathbf{S}}|b|^2\Delta\phi_ndx+\sum_{i=1}^3\int_{\mathbb{R}^2\times\mathbf{S}}u^i\nabla u^i\cdot\nabla\phi_ndx\\
&=\int_{\mathbb{R}^2\times\mathbf{S}}\Big(\frac{1}{2}|u|^2+\frac{1}{2}|b|^2+(p-p_n)\Big)u\cdot\nabla\phi_ndx\\
&\quad-\int_{\mathbb{R}^2\times\mathbf{S}}(b\cdot u)(b\cdot\nabla\phi_n)dx.
\end{aligned}$$

We denote $B_{2n/n} := \{x_h : n \le |x_h| \le 2n\}$ as the dyadic annulus. It follows that

$$
\begin{aligned}
&\int_{\mathbb{R}^2\times\mathbf{S}} \big(|\nabla u|^2 + |\nabla b|^2\big)\phi_n dx \\
&\lesssim \int_{B_{2n/n}\times\mathbf{S}} |b|^2|\Delta\phi_n| dx + \int_{B_{2n/n}\times\mathbf{S}} |u||\nabla u||\nabla\phi_n| dx \\
&\quad + \int_{B_{2n/n}\times\mathbf{S}} |u\cdot\nabla\phi_n|\cdot\big(|u|^2+|b|^2\big) dx \\
&\quad + \int_{B_{2n/n}\times\mathbf{S}} |p-p_n||u\cdot\nabla\phi_n| dx + \int_{B_{2n/n}\times\mathbf{S}} |b||u||b\cdot\nabla\phi_n| dx \\
&:= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
\tag{3.10}
$$

First, I_1 satisfies

$$
I_1 \lesssim \frac{\|b\|^2_{L^\infty(B_{2n/n}\times\mathbf{S})}}{n^2}\int_{B_{2n/n}\times\mathbf{S}} dx \lesssim \|b\|^2_{L^\infty(B_{2n/n}\times\mathbf{S})} \to 0 \qquad \text{as} \quad n\to\infty.
$$

Using Hölder's inequality, I_2 follows that

$$
\begin{aligned}
I_2 &\lesssim \frac{\|u\|_{L^\infty(B_{2n/n}\times\mathbf{S})}}{n}\|\nabla u\|_{L^2((B_{2n/n}\times\mathbf{S})}\|1\|_{L^2((B_{2n/n}\times\mathbf{S})} \\
&\lesssim \|u\|_{L^\infty(B_{2n/n}\times\mathbf{S})}\|\nabla u\|_{L^2((B_{2n/n}\times\mathbf{S})} \to 0 \qquad \text{as} \quad n\to\infty.
\end{aligned}
$$

Next, for I_3, we have

$$
\begin{aligned}
I_3 &\lesssim \|(u,b)\|^2_{L^\infty(B_{2n/n}\times\mathbf{S})}\int_{B_{2n/n}\times\mathbf{S}} |u^r\partial_r\phi_n| dx \lesssim \frac{\|(u,b)\|^2_{L^\infty(B_{2n/n}\times\mathbf{S})}}{n}\|u^r\|_{L^2(B_{2n/n}\times\mathbf{S})}\|1\|_{L^2((B_{2n/n}\times\mathbf{S})} \\
&\lesssim \|(u,b)\|^2_{L^\infty(B_{2n/n}\times\mathbf{S})}\|u^r\|_{L^2(B_{2n/n}\times\mathbf{S})} \to 0 \qquad \text{as} \quad n\to\infty.
\end{aligned}
$$

For I_4, we have

$$
\begin{aligned}
I_4 &\lesssim \int_{B_{2n/n}\times\mathbf{S}} |p-p_n||u^r\partial_r\phi_n| dx \lesssim \frac{1}{n}\|p-p_n\|_{L^2(B_{2n/n}\times\mathbf{S})}\|u^r\|_{L^2((B_{2n/n}\times\mathbf{S})} \\
&\lesssim C_0\|u^r\|_{L^2(B_{2n/n}\times\mathbf{S})} \to 0 \qquad \text{as} \quad n\to\infty.
\end{aligned}
$$

Here, we have applied Hölder's inequality, the boundedness of L^2 mean oscillation of p in dyadic annulus, which is achieved in Subsection III B, and $u^r \in L^2(\mathbb{R}^2\times\mathbf{S})$. Finally, the following estimate is satisfied by I_5:

$$
\begin{aligned}
I_5 &\lesssim \|u\|_{L^\infty(B_{2n/n}\times\mathbf{S})}\|b\|_{L^\infty(B_{2n/n}\times\mathbf{S})}\int_{B_{2n/n}\times\mathbf{S}} |b^r\partial_r\phi_n| dx \\
&\lesssim \frac{\|u\|_{L^\infty(B_{2n/n}\times\mathbf{S})}\|b\|_{L^\infty(B_{2n/n}\times\mathbf{S})}}{n}\|b^r\|_{L^2(B_{2n/n}\times\mathbf{S})}\|1\|_{L^2((B_{2n/n}\times\mathbf{S})} \\
&\lesssim \|u\|_{L^\infty(B_{2n/n}\times\mathbf{S})}\|b\|_{L^\infty(B_{2n/n}\times\mathbf{S})}\|b^r\|_{L^2(B_{2n/n}\times\mathbf{S})} \to 0 \qquad \text{as} \quad n\to\infty.
\end{aligned}
$$

Combining the estimates of I_1, I_2, I_3, I_4, and I_5, (3.10) implies that

$$
\int_{\mathbb{R}^2\times\mathbf{S}} \big(|\nabla u|^2 + |\nabla b|^2\big) dx = 0
$$

by letting $n\to\infty$, which means that u and b are both constant vectors.

Moreover, since $\lim\limits_{|x_h|\to+\infty} b = 0$, we have $b = 0$. In addition, we have

$$
0 = \int_{\mathbf{S}} u^r dx_3 = \int_{\mathbf{S}} (u^1\cos\theta + u^2\sin\theta) dx_3 = u^1\cos\theta + u^2\sin\theta.
$$

By setting $\theta = 0$ and π, we have $u^1 = u^2 = 0$. Hence, we actually have

$$
u = (0,0,c) \quad \text{and} \quad b = (0,0,0)
$$

for some constant c. This finishes the Proof of Theorem 1.2.

D. Proof of corollary 1.4

Proof. For the boundary condition in Corollary 1.4, we translate it as follows:

(i) The perfectly conducting boundary condition for the magnetic field ($b \cdot n = 0, \nabla \times b \times n = 0$) is given by

$$\begin{cases} \partial_3 u^1 = \partial_3 u^2 = u^3 = 0 & \text{on} \quad \mathbb{R}^2 \times (\{0\} \cup \{1\}), \\ \partial_3 b^1 = \partial_3 b^2 = b^3 = 0 & \text{on} \quad \mathbb{R}^2 \times (\{0\} \cup \{1\}). \end{cases}$$

(ii) The insulating boundary condition for the magnetic field ($b \times n = 0$) is given by

$$\begin{cases} \partial_3 u^1 = \partial_3 u^2 = u^3 = 0 & \text{on} \quad \mathbb{R}^2 \times (\{0\} \cup \{1\}), \\ b^1 = b^2 = \partial_3 b^3 = 0 & \text{on} \quad \mathbb{R}^2 \times (\{0\} \cup \{1\}). \end{cases} \tag{3.11}$$

We can extend the solution to be a periodic solution in the x_3 direction.

In case (i), we make even extensions for u^h, b^h, p and odd extensions for u^3, b^3 in the x_3 direction. More precisely, for $x \in \mathbb{R}^2 \times [-1,1]$, set

$$\tilde{p}(x_h, x_3) := \begin{cases} p(x_h, x_3) & (x_h, x_3) \in \mathbb{R}^2 \times [0,1], \\ p(x_h, -x_3) & (x_h, x_3) \in \mathbb{R}^2 \times [-1,0], \end{cases}$$

$$\tilde{u}(x_h, x_3) := \begin{cases} (u^h(x_h, x_3), \quad u^3(x_h, x_3)) & (x_h, x_3) \in \mathbb{R}^2 \times [0,1], \\ (u^h(x_h, -x_3), -u^3(x_h, -x_3)) & (x_h, x_3) \in \mathbb{R}^2 \times [-1,0], \end{cases}$$

$$\tilde{b}(x_h, x_3) := \begin{cases} (b^h(x_h, x_3), \quad b^3(x_h, x_3)) & (x_h, x_3) \in \mathbb{R}^2 \times [0,1], \\ (b^h(x_h, -x_3), -b^3(x_h, -x_3)) & (x_h, x_3) \in \mathbb{R}^2 \times [-1,0]. \end{cases}$$

It is not hard to see that $(\tilde{u}, \tilde{b}, \tilde{p})$ is a weak solution of (1.1) in $\mathbb{R}^2 \times [-1,1]$ and on the boundary,

$$\tilde{u}|_{x_3=-1} = \tilde{u}|_{x_3=1}, \quad \tilde{b}|_{x_3=-1} = \tilde{b}|_{x_3=1}, \quad \tilde{p}|_{x_3=-1} = \tilde{p}|_{x_3=1}.$$

Then, we extend the solution $(\tilde{u}, \tilde{b}, \tilde{p})$ to be a periodic solution in the x_3 direction. By applying Theorem 1.2, we can get that

$$u = (0,0,c) \quad \text{and} \quad b = (0,0,0).$$

Since $u^3|_{x_3=0,1} = 0$, we have $c = 0$.

In case (ii), we make even extensions for u^h, b^3, p and odd extensions for u^3, b^h in the x_3 direction. The same as case (i), we can show that $(u,b) \equiv 0$. Here, we want to emphasize that why in case (ii) the assumption b^θ is axially symmetric is not needed. In this situation, Eq. (3.3) is not guaranteed any more. However, we still have $\nabla b^r \in L^2(\mathbb{R}^2 \times I)$. Then, the boundary condition (3.11) for b implies that $b^r|_{x_3=0,1} = 0$, which, by using one-dimensional Poincaré's inequality, can also validate that $b^r \in L^2(\mathbb{R}^2 \times I)$. □

IV. PROOF OF THEOREM 1.6

The same as Corollary 1.4, for the boundary condition in Theorem 1.6, we have the following:

(i) The perfectly conducting boundary condition for the magnetic field ($b \cdot n = 0, \nabla \times b \times n = 0$) is given by

$$\begin{cases} u = 0 & \text{on} \quad \mathbb{R}^2 \times (\{0\} \cup \{1\}), \\ \partial_3 b^1 = \partial_3 b^2 = b^3 = 0 & \text{on} \quad \mathbb{R}^2 \times (\{0\} \cup \{1\}). \end{cases} \tag{4.1}$$

(ii) The insulating boundary condition for the magnetic field ($b \times n = 0$) is given by

$$\begin{cases} u = 0 & \text{on} \quad \mathbb{R}^2 \times (\{0\} \cup \{1\}), \\ b^1 = b^2 = \partial_3 b^3 = 0 & \text{on} \quad \mathbb{R}^2 \times (\{0\} \cup \{1\}). \end{cases} \tag{4.2}$$

Remark 4.12. Since now u satisfies the non-slip boundary condition, there is no any extension as that in Corollary 1.4.

The idea of proving Theorem 1.6 is similar to but different from that in the Proof of Theorem 1.2, especially when we use Proposition 2.9 to estimate the oscillation of the pressure p.

Lemma 4.13. Under assumptions of Theorem 1.6, we have the following:

(i) *The perfectly conducting boundary condition for the magnetic field is given by*

$$\|(u,b^3)\|_{L^2(\mathbb{R}^2\times[0,1])}+\|\nabla b\|_{L^2(\mathbb{R}^2\times[0,1])}<C^*.$$

(ii) *The insulating boundary condition for the magnetic field is given by*

$$\|(u,b^h)\|_{L^2(\mathbb{R}^2\times[0,1])}+\|\nabla b\|_{L^2(\mathbb{R}^2\times[0,1])}<C^*.$$

Here, C^* is a constant depending on $\|\nabla u\|_{L^2(\mathbb{R}^2\times[0,1])}$ and $\|(u,b)\|_{L^\infty(\mathbb{R}^2\times[0,1])}$.

Proof. The L^2 estimate for u is a simple combination of $\|\nabla u\|_{L^2(\mathbb{R}^2\times[0,1])}<+\infty$, $u|_{x_3=0,1}=0$, and one-dimensional Poincaré inequality. Here, we omit the details.

Considering the equation for b,

$$u\cdot\nabla b-b\cdot\nabla u=\Delta b.$$

Using the same procedure as obtaining (3.6), we can get that

$$\|\nabla b\|_{L^2(\mathbb{R}^2\times[0,1])}\lesssim\|b\|_{L^\infty}(1+\|u\|_{L^2})<+\infty.$$

In case (i) with $b^3|_{x_3=0,1}=0$ and case (ii) with $b^h|_{x_3=0,1}=0$, the one-dimensional Poincaré inequality and the above boundedness of $\|\nabla b\|_{L^2(\mathbb{R}^2\times[0,1])}$ guarantee that $b^3\in L^2(\mathbb{R}^2\times[0,1])$ and $b^h\in L^2(\mathbb{R}^2\times[0,1])$, respectively. □

Lemma 4.14. Let (u,b) be the solution of (1.3); then, we have, for $R\geq 1$,

$$\|p-p_R\|_{L^2(B_R\times[0,1])}\leq C_0R,$$

where $C_0=C_0(\|(u,b)\|_{L^\infty},\|\nabla u\|_{L^2})$ and $p_R:=\frac{1}{|B_R\times[0,1]|}\int_{B_R\times[0,1]}p\,dx$ is the average of p in $B_R\times[0,1]$.

Proof. Since now our solution is not a periodic solution as that in Theorem 1.2, we cannot proceed with our proof as the same as that in Lemma 3.11 in the cylinder $B_n\times[0,n]$. Conversely, we can only consider it in a thin cylinder $B_R\times[0,1]$ for large R.

In Proposition 2.9, set $\alpha=0$ and choose $f=p-p_R$. Then, there exists V in $B_R\times[0,1]$, satisfying $\nabla\cdot V=p-p_R$, and

$$\begin{aligned}&\|\nabla_h V^h\|_{L^2(B_R\times[0,1])}+\|\partial_3V^3\|_{L^2(B_R\times[0,1])}\leq C\|p-p_R\|_{L^2(B_R\times[0,1])},\\&\|\partial_3V^h\|_{L^2(B_R\times[0,1])}\leq CR\|f\|_{L^2(B_R\times[0,1])},\\&\|\nabla_hV^3\|_{L^2(B_R\times[0,1])}\leq CR^{-1}\|p-p_R\|_{L^2(B_R\times[0,1])},\end{aligned}\tag{4.3}$$

where C is an absolute constant, independent of R. Now, multiplying $(1.3)_1$ with V and integrating $B_R\times[0,1]$, we get

$$\begin{aligned}\int_{B_R\times[0,1]}\nabla(p-p_R)\cdot V\,dx&=\int_{B_R\times[0,1]}(\Delta u-u\cdot\nabla u+b\cdot\nabla b)\cdot V\,dx\\&=\int_{B_R\times[0,1]}(\Delta u-u\cdot\nabla u+b\cdot\nabla b)\cdot V\,dx.\end{aligned}$$

Integration by parts indicates that

$$\begin{aligned}
&\int_{B_R\times[0,1]}(p-p_R)^2dx\\
&=\int_{B_R\times[0,1]}(p-p_R)\nabla\cdot Vdx\\
&=-\int_{B_R\times[0,1]}(\Delta u-u\cdot\nabla u+b\cdot\nabla b)\cdot Vdx\\
&=\int_{B_R\times[0,1]}\sum_{i,j=1}^{3}\partial_i u^j\partial_i V^j+\partial_i(u^iu^j-b^ib^j)V^jdx\\
&=\int_{B_R\times[0,1]}\sum_{i,j=1}^{3}\left(\partial_i u^j-u^iu^j+b^ib^j\right)\partial_i V^jdx\\
&\le\|\nabla V\|_{L^2(B_R\times[0,1])}\left(\|\nabla u\|_{L^2(B_R\times[0,1])}+\|u\|_{L^\infty(B_R\times[0,1])}\|u\|_{L^2(B_R\times[0,1])}\right)\\
&\quad+\underbrace{\int_{B_R\times[0,1]}\sum_{i,j=1}^{3}\left(b^ib^j\right)\partial_i V^jdx}_{\mathcal{L}}.
\end{aligned}\tag{4.4}$$

We need to deal with the term $\mathcal{L}$ more carefully.

Case (i). The perfectly conducting boundary condition for the magnetic field:

By using (4.3) and Cauchy–Schwartz inequality,

$$\begin{aligned}
\mathcal{L}&=\int_{B_R\times[0,1]}\sum_{i,j=1}^{2}\left(b^ib^j\right)\partial_i V^jdx+\int_{B_R\times[0,1]}\sum_{i=3\text{ or }j=3}\left(b^ib^j\right)\partial_i V^jdx\\
&\le\||b|^2\|_{L^\infty(B_R\times[0,1])}\|\nabla_h V^h\|_{L^2(B_R\times[0,1])}\|1\|_{L^2(B_R\times[0,1])}\\
&\quad+\|b\|_{L^\infty(B_R\times[0,1])}\|b^3\|_{L^2(B_R\times[0,1])}\|\nabla V\|_{L^2(B_R\times[0,1])}\\
&\le\varepsilon\|p-p_R\|^2_{L^2(B_R\times[0,1])}+C_\varepsilon R^2,
\end{aligned}$$

where C_ε depends on $\|(u,b)\|_{L^\infty}$, $\|\nabla u\|_{L^2}$, and ε.

Case (ii). The insulating boundary condition for the magnetic field:

By using (4.3) and Cauchy–Schwartz inequality,

$$\begin{aligned}
\mathcal{L}&=\int_{B_R\times[0,1]}\sum_{i,j=1}^{2}(b^3)^2\partial_3 V^3dx+\int_{B_R\times[0,1]}\sum_{i\neq3\text{ or }j\neq3}\left(b^ib^j\right)\partial_i V^jdx\\
&\le\||b|^2\|_{L^\infty(B_R\times[0,1])}\|\partial_3 V^3\|_{L^2(B_R\times[0,1])}\|1\|_{L^2(B_R\times[0,1])}\\
&\quad+\|b\|_{L^\infty(B_R\times[0,1])}\|b^h\|_{L^2(B_R\times[0,1])}\|\nabla V\|_{L^2(B_R\times[0,1])}\\
&\le\varepsilon\|p-p_R\|^2_{L^2(B_R\times[0,1])}+C_\varepsilon R^2.
\end{aligned}$$

Inserting the above two inequalities into (4.4) and using Cauchy–Schwartz inequality, we have

$$\begin{aligned}
&\int_{B_R\times[0,1]}(p-p_n)^2dx\\
&\le\varepsilon R^{-2}\|\nabla V\|^2_{L^2(B_R\times[0,1])}+C_\varepsilon R^2\left(\|\nabla u\|^2_{L^2(B_R\times[0,1])}+\|u\|^2_{L^\infty(B_R\times[0,1])}\|u\|^2_{L^2(B_R\times[0,1])}\right)\\
&\quad+\varepsilon\|p-p_R\|^2_{L^2(B_R\times[0,1])}+C_\varepsilon R^2\\
&\le C\varepsilon\|p-p_n\|^2_{L^2(B_R\times[0,1])}+C_\varepsilon R^2,
\end{aligned}$$

where C_ε depends on $\|(u,b)\|_{L^\infty}$, $\|\nabla u\|_{L^2}$, and ε. By choosing ε small enough, we can obtain

$$\|p-p_R\|_{L^2(B_R\times[0,1])}\le C_0R.$$

This finishes the Proof of Lemma 4.14.

A. Trivialness of u and b

Proof. The proof of trivialness of u and b will be almost the same as that in Subsection III C. Let $\phi(s)$ be the test function in (3.5), and set $\phi_R(y_h) = \phi(\frac{|y_h|}{R})$. Testing the first two equations of (1.3) with $u\phi_R$ and $b\phi_R$, we can get the same estimate as (3.10),

$$\begin{aligned}
&\int_{\mathbb{R}^2\times[0,1]}\big(|\nabla u|^2+|\nabla b|^2\big)\phi_R dx\\
&\lesssim \int_{B_{2R/R}\times[0,1]}|b\|\nabla b\|\nabla\phi_R|dx+\int_{B_{2R/R}\times[0,1]}|u\|\nabla u\|\nabla\phi_R|dx\\
&\quad+\int_{B_{2R/R}\times[0,1]}|u\cdot\nabla\phi_R|\cdot\big(|u|^2+|b|^2\big)dx\\
&\quad+\int_{B_{2R/R}\times[0,1]}|p-p_R\|u\cdot\nabla\phi_R|dx+\int_{B_{2R/R}\times[0,1]}|b\|u\|b\cdot\nabla\phi_R|dx\\
&:=J_1+J_2+J_3+J_4+J_5.
\end{aligned} \tag{4.5}$$

The proof of the fact that $\sum_{i=1}^4 J_i \to 0$ as $R\to+\infty$ is completely the same as that of $I_i(2\le i\le 4)$ in Subsection III C. The only difference comes from J_5 since now we do not have $b^r\in L^2(\mathbb{R}^2\times[0,1])$ in the case that the magnetic field satisfies the perfectly conducting boundary condition. However, since in both boundary conditions we have $u\in L^2(\mathbb{R}^2\times[0,1])$, we can estimate it as follows:

$$J_5 \lesssim \frac{\||b|^2\|_{L^\infty}}{R}\|u\|_{L^2(B_{2R/R}\times[0,1])}\|1\|_{L^2(B_{2R/R}\times[0,1])} \lesssim C\|u\|_{L^2(B_{2R/R}\times[0,1])} \to 0 \quad \text{as} \quad R\to+\infty.$$

Hence, at last, from (4.5), by letting $R\to+\infty$, we can get

$$\int_{\mathbb{R}^2\times[0,1]}\big(|\nabla u|^2+|\nabla b|^2\big)dx = 0,$$

which implies that u and b are constant vectors. In boundary condition (4.1), we have $u=0$ and $b=(c_1,c_2,0)$ for constants c_1 and c_2. In boundary condition (4.2), we have $u=0$ and $b=(0,0,c_3)$ for a constant c_3. This finishes the Proof of Theorem 1.6. □

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DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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