

A New Algorithm To Evaluate Terminal Heads Of Length K

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ABSTRACT

Finding the terminal heads of length k of a given string in a context-free language has proven to be essential in the design of $LL(k)$ and $LR(k)$ parser generators. Improvement on this technique would greatly enhance of performance of LL/LR parser generation. Aho and Ullman have proposed a method FIRST to toggle this problem. This paper presents an innovative alternative to FIRST called THREAD. We conduct performance evaluation and conclude that our method performs better under some scenarios such as when the string is long.

CCS Concepts

•Software and its engineering → General programming languages; *Translator writing systems and compiler generators*; •Theory of computation → Grammars and context-free languages;

Keywords

$First_k(\alpha)$; $THREAD_k(\alpha)$; Terminal heads; $LR(k)$

1. INTRODUCTION

The algorithm that finds the first k terminal heads of a given string in a context-free grammar is denoted as FIRST in this paper. The construction of both top-down(LL) and bottom-up(LR) parsers is aided by FIRST. During LL/LR parsing, FIRST allows us to choose which production to apply, based on the next input symbol. The most widely used case is the computation of FIRST when $k = 1$, because $LALR(1)$, $LL(1)$, $SLR(1)$ and $LR(1)$ are the most efficient algorithms in parsing programming language grammars. On the other hand, when $k > 1$, algorithms like $LL(k)$ with a lookahead of more than 1 tokens are more complex but they can often handle more complex and certain ambiguous context free grammars and are often used in language translation and natural language processing.

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The most popular real world parser generator solutions are Bison and ANTLR. Bison is a general-purpose parser generator that converts an annotated context-free grammar into a deterministic LR or generalized LR (GLR) parser employing LALR(1) parser tables[7, 6, 2]. ANTLR on the other hand is a $LL(k)$ parser generator[10, 1]. Even though Bison and ANTLR undertake different paths in the parsing strategy they choose, both of them computes the FIRST set directly or indirectly in their parsing processes. The main contribution of this paper is the proposal of a new innovative approach in computing the FIRST sets.

In the rest of the paper, we will first present the notation conventions and the definition of FIRST in section 2. After which, we will survey related works on computing the FIRST set in section 3. In section 4, we will present our innovative approach denoted as THREAD and display a few examples of applying THREAD. A performance study of our method compare to the state of the art solution is presented in Section 5. Lastly, we conclude the whole paper in Section 6.

In this paper we present a new algorithm, which we call $THREAD_k(\alpha)$, to evaluate the terminal heads of length k of a given string in a context-free grammar. It is an alternative to the $FIRST_k(\alpha)$ algorithm of Aho and Ullman [4], and takes a very different approach. In this paper, we will present the algorithm, give examples, compare to the method of Aho and Ullman, and discuss other related issues.

Since $FIRST_k(\alpha)$ is a fundamental algorithm that works as a basic building block in compiler theory and practice, its improvement should have wide impact.

2. FIRST

2.1 Notation Conventions

Alphabet: A set of symbols, where a symbol is a non-divisible basic element of the alphabet.

String: A sequence of symbols concatenated together. We represent the length of a string s as $|s|$. A string is said to *vanish* if it can derive the empty string. We use Greek letters $\alpha, \beta, \gamma, \dots$ to represent strings. An empty string is represented by ϵ .

(G)grammar: A grammar for a language L is defined as a 4-tuple $G = (N, \Sigma, P, S)$.

N : A set of non-terminal symbols. A non-terminal symbol can appear on either the left or right side of productions. We use upper case Roman letters A, B, C, \dots to represent non-terminals.

Σ : A set of terminal symbols disjoint from the set N . A

terminal symbol appears only on the right side of productions. We use lower case Roman letters a, b, c, \dots to represent terminals

P : A set of productions.

S : The start symbol from which the production rules originate from.

k – head: A k – head of a string S is a string which is made of the first k symbols of S , or the first k symbols of any string that can derive from S .

k – terminalhead or k – thead: A k – head of S which is made up of terminal strings only.

EXAMPLE 2.1. Given grammar $G1$:

$X \rightarrow XY \mid a, Y \rightarrow b \mid \epsilon$

Here a and b are terminal symbols because they appear only on the right side of the productions of $G1$. X and Y are non-terminal symbols because they can appear on the left side of the productions of $G1$. Y vanishes because it can derive the empty string. The shortest string X can derive is a , therefore it does not vanish because it cannot derive the empty string.

Given string $\alpha = XY$, its 1-head can be X or a , and its 2-head can be XY , XX , aY , Xb , ab or aa . Its 1-thead is a , and its 2-thead can be aa or ab .

2.2 Definition of FIRST

For a CFG $G = (N, \Sigma, P, S)$,

$FIRST_k(\alpha) = \{x \mid \alpha \xrightarrow{lm} x\beta \text{ and } |x| = k \text{ or } \alpha \xrightarrow{*} x \text{ and } |x| < k\}$

Where \xrightarrow{lm} means 0 or more steps of leftmost derivatives.

In short, $FIRST_k(\alpha)$ consists of all terminal prefixes of length k (or less if α derives a terminal string of length less than k) of the terminal strings that can be derived from α . Following are two examples on the calculation of $FIRST_k(\alpha)$.

EXAMPLE 2.2. Given grammar $S \rightarrow NM, N \rightarrow st, M \rightarrow bc$. We want to find $FIRST_k(\alpha)$ for string $\alpha = NM$. This is a trivial case, we can just plug N and M into NM to obtain $\alpha = stbc$. Calculation of $FIRST_k(\alpha)$ is easy: $FIRST_1(\alpha) = s$, $FIRST_2(\alpha) = st$, $FIRST_3(\alpha) = stb$, $FIRST_4(\alpha) = stbc$.

EXAMPLE 2.3. Given grammar $S \rightarrow NML$, where $N \rightarrow Ns \mid \epsilon$, $M \rightarrow Mt \mid \epsilon$. Here ϵ is the empty string. We want to find $FIRST_k(\alpha)$ for string $\alpha = NML$.

Then actually $N = s^*$, and $M = t^*$, and $\alpha = s^*t^*bc$.

$FIRST_1(\alpha) = \{s, t, b\}$,

$FIRST_2(\alpha) = \{ss, st, sb, tt, tb, bc\}$,

$FIRST_3(\alpha) = \{sss, sst, ssb, stt, stb, sbc, ttt, ttb, tbc\}$,

$FIRST_4(\alpha) = \{ssss, ssst, sssb, sstt, sstb, ssbc, sttt, sttb, stbc, tttt, ttbt, ttbc\}$.

3. RELATED WORK

We did a survey on the work related to the calculation of FIRST set and found the following literatures.

3.1 Computing Lookahead Sets

Lookahead set can be defined in terms of FIRST and FOLLOW sets. An efficient computation of FIRST set would lead to an efficient way to compute lookahead sets.

The work of DeRemer and Pennello[5] tries to provide an efficient way to compute the LALR(1) lookahead sets in

time linear in the size of the relations. On the other hand, Madsen and Kristensen[8] work on methods for computing LALR(k) lookahead sets.

The work of Kristensen and Madsen [8] discussed computing $FIRST_k$ for finding lookahead strings, which is needed by their LALR(k) algorithm. Their method is based on simulating all steps involved in parsing starting from a relevant state in a LR(0) machine. Given an example of calculating $LALR_k$ for $[A \rightarrow \bullet\alpha]$, their method wants to obtain the sets $\cup\{FIRST_k(\psi_i) \mid i = 1, 2, \dots, n\}$ for all items $[B_i \rightarrow \varphi_i \bullet A\psi_i]$, which "may be computed by simulating all possible steps that the parse algorithm may take starting in the state $GOTO_k(S, A)$ with an empty parse stack." They further pointed out that the set $\cup FIRST_k(\psi_i)$ is not enough, and proceeded to discuss how to cover edgy cases such as when the grammar is circular or contains ϵ -productions, and ended their discussion with cases where the simulated parsing might fail due to circularity.

The PhD thesis of Parr [10] proposed a method to compute $FIRST_k(\alpha)$. This is used in the implementation of LL(k) parser generator ANTLR. Parr's PhD thesis introduces the GLA grammar representation in chapter 3, and explains lookahead computation and representation in chapter 4. Basically, a data structure called GLA (Grammar Lookahead Automata) is used to represent grammars. To calculate LR(k) lookahead, do a constrained walk of a GLA, and the lookahead strings are stored as a lookahead DFA (Deterministic Finite Automata). He also discussed how to solve the cycle issue with cache mechanism. This is similar to the method of Kristensen and Madsen in that it utilizes the parsing machine to do the computation and tightly integrates the calculation of lookahead strings with parsing, and in that none of them is a standalone method to calculate $FIRST_k(\alpha)$.

All the above work redefined LOOK-AHEAD to their favor to avoid the direct computation of FIRST and FOLLOW sets. They leverage the parsing machine for lookahead set computation and tightly integrate the lookahead computation with parsing. In summary, they did not provide a standalone solution to compute FIRST set.

3.2 Method of Aho and Ullman

The only stand alone algorithm we were able to find comes from Aho and Ullman[4].

They first define an operator \oplus_k such that: given an alphabet Σ and two sets $A \subseteq \Sigma^*$, $B \subseteq \Sigma^*$, $S = A \oplus_k B$ is the set of all strings formed from the ordered concatenation of string pairs (a, b) , where $a \in A$, $b \in B$, and the length of strings in S is less than or equal to k .

Now given a context free grammar $G = (N, \Sigma, P, S)$ and a string $\alpha = X_1X_2X_n$ in $(N \cup \Sigma)^*$, $FIRST_k(\alpha) = FIRST_k(X_1) \oplus_k FIRST_k(X_2) \oplus_k \dots \oplus_k FIRST_k(X_n)$, the prove can be found in the book. From this point on, we only need to calculate $FIRST_k(X)$ for any X .

The following steps are used to calculate $FIRST_k(X)$: Define a sets $F_i(X)$ for X in $N \cup \Sigma$ and for increasing values of i , $i \geq 0$, as follow:

1. $F_i(X) = \{a\}$ for all a in Σ and $i \geq 0$;
2. $F_0(X) = \{x \in \Sigma^{*k} \mid A \rightarrow x\alpha \text{ is in } P, \text{ where either } |x|=k \text{ or } |x| < k \text{ and } \alpha = \epsilon\}$
3. Suppose that F_0, F_1, \dots, F_{i-1} have been defined for all

A in N. Then $F_i(A) = \{x | A \rightarrow Y_1, \dots, Y_n \text{ is in } P \text{ and } x \text{ is in } F_i(Y_1) \oplus_k F_i(Y_2) \oplus_k \dots \oplus_k F_i(Y_n)\} \cup F_{i-1}(A)$

4. As $F_{i-1}(A) \subseteq F_i(A) \subseteq \Sigma^{*k}$ for all A and i, eventually we must reach an i for which $F_{i-1}(A) = F_i(A)$ for all A in N. Let $FIRST_k(A) = F_i(A)$ for that value of i.

In summary, the method of Aho and Ullman breaks down the task of evaluating the terminal heads of length k of a string α into applying the \oplus_k operation on the component symbols of α . It solves the second problem by building a table from bottom up like in dynamic programming.

4. THE NEW ALGORITHM

In this section we propose the new algorithm [9] with a simple example, prove the correctness and complexity of the new algorithm.

4.1 The $THEAD_k(\alpha)$ Algorithm

We use $THEAD_k(\alpha)$ as the name of the new algorithm, and also use it to represent the set of terminal heads of string α , where the length of each terminal head string is k. $THEAD_k(\alpha)$ and $FIRST_k(\alpha)$ are equivalent.

To illustrate the algorithm, we define the following notations:

For a string $\alpha = X_1X_2 \dots X_n$, $|\alpha|$ is the length of α ($|\alpha| = n$); $\alpha[i]$ is the i th symbol of string α ; $h(\alpha, k)$ denotes the first k symbols of α , i.e., prefix string of α of length k; $h_v(\alpha, k)$ is a substring of α that consists of the prefix string of α up to the k-th symbol that does not vanish, or the entire α string if it contains less than k symbols that do not vanish; $prod(\alpha, i)$ is the set of strings obtained by applying all possible productions to the i th symbol X_i of α .

We also let T stand for the set of *Terminals*, and NT stand for the set of *Non-Terminals*. T_k stands for the set of strings made of Terminals and whose length is k. \emptyset stands for the empty set.

Algorithm 1, $THEAD_k(\alpha)$, is shown in Figure 1.

In Algorithm 1, **H** and **S** are sets of strings initially empty. **H** is used to hold all k-THEADs and **S** contains all m-THEADS where $m < k$. **L** is an auxiliary ordered list of strings which initially consists just of $h_v(\alpha, k)$.

The following steps are repeated to get i-THEADS where goes from 1 to k:

1. Expansion of Strings in L using *Prod*(Lines 5-10): add to the end of L the result of applying all possible productions to the i^{th} symbol in the current member β of L, omitting strings that are already in L, and truncating all members added which have k or more symbols that do not vanish, by deleting the part of the string following the k-th symbol that does not vanish.
2. Filter Invalid Strings in L (Lines 11-13): remove from L all strings whose i th symbol is a non-terminal. After this step, L should only contain strings whose first i symbols are non-terminals.
3. Extract k-HEADS from L into H (Lines 14-19): remove from L all strings whose prefix of length k consisting entirely of terminals, and add the prefixes of length k involved to the set H.

ALGORITHM 1. $THEAD_k(\alpha)$	
INPUT: STRING $\alpha = X_1X_2 \dots X_n$; Integer k : length of theads.	
OUTPUT: SET H – CONTAINS K-THEADS OF α , AND (OPTIONALLY) SET S – CONTAINS M-THEADS OF α , $M < K$.	
1	$H \leftarrow \emptyset$
2	$S \leftarrow \emptyset$
3	$L \leftarrow \{h_v(\alpha, k)\}$
4	for $i = 1$ to k do
5	foreach string β in L do
6	$\phi = prod(\beta, i)$
7	foreach string γ in ϕ do
8	$L \leftarrow L \cup \{h_v(\gamma, k)\}$
9	end foreach
10	end foreach
11	foreach string β in L do
12	if $\beta[i] \in NT$ then $L \leftarrow L - \{\beta\}$
13	end foreach
14	foreach string β in L do
15	if $h(\beta, k) \in T^k$ then
16	$L \leftarrow L - \{\beta\}$
17	$H \leftarrow H \cup \{h(\beta, k)\}$
18	end if
19	end foreach
20	foreach string β in L do
21	if $ \beta < k$ AND $\beta \in T^{ \beta }$ then
22	$L \leftarrow L - \{\beta\}$
23	$S \leftarrow S \cup \{\beta\}$
24	end if
25	end foreach
26	if $L == \emptyset$ then stop
27	end for

Figure 1: Algorithm $THEAD_k(\alpha)$

4. Extract m -HEADS ($m < k$) from L into S (Lines 20-25): remove from L all strings of length less than k which consist entirely of terminals, and add these to the set S .
5. Early termination (line 26), if L is empty, the algorithm terminates.

After the terminal of the outermost for loop, H will contain the required set of terminal strings of length k of α , i.e., the k -head set of α ; and S will contain the set of terminal strings of length less than k which are derived from α . Obviously, H gives the result of $THEAD_k(\alpha)$.

The entire algorithm derives a closure of the initial string in L , where each derived string in the closure satisfies the requirements on the length (should be equal to k) of the strings, and on the type of symbols (should be terminal symbol) in the strings.

4.2 A Complete Example of $THEAD_k(\alpha)$

4.3 Proof of Correctness

The prove of the correctness of Algorithm 1 is as follow.

LEMMA 1. *In Algorithm 1, at the end of the i^{th} outer loop cycle (lines 4-27), for each string s in list L , where $s = X_1X_2X_n$, the first i symbols X_1, X_2, \dots, X_i of s (or all the symbols of s if $|s| < i$) are terminals.*

PROOF. Prove by induction. For outer loop cycle $i = 1$, the step of lines 11-13 removes from L all strings whose 1^{st} symbol is a non-terminal. Thus for all the strings remained in L , the 1^{st} symbol is terminal. Now assume at cycle $i = n - 1$, for all the strings in L , the first i symbols are terminals. At cycle $i = n$, the inner loop (lines 5-10) only makes derivations on the n th symbol, and does not introduce any nonterminal symbols to the first $n - 1$ symbols; next, Algorithm 1 removes from L those strings whose n th symbol is a nonterminal (lines 11-13), thus for all the symbols in L , now their first n symbols are terminals. The remaining steps (lines 14-26) do not alter this fact. Therefore Lemma 1 holds. \square

LEMMA 2. *In Algorithm 1, at the end of the i^{th} outer loop cycle, all the possible combinations of i -thead derivations are generated by the inner loop (lines 5-10).*

PROOF. This also can be proved by induction. When $i = 1$, this is obvious from the inner loop. Assume this holds for $i = n - 1$. When $i = n$, for each string s in L , the first $n - 1$ symbols of s are all terminals. In the inner loop, for each string s in L , all the possible productions are applied to the n^{th} symbol of s , thus all the possible terminal and non-terminal symbols at the n th position are generated by string s and included in L . These form new derived strings, appended to the end of L , and processed by the next cycle. Thus Lemma 2 holds. \square

LEMMA 3. *Algorithm 1 ends in k or less outer loop cycles (lines 4-27) when L becomes empty.*

PROOF. From Lemma 1, for all the strings generated in the k^{th} outer loop cycle, their first k symbols are all terminals, these are then removed from L (lines 14-25). In the cycles, all members added to L that have k or more symbols that do not vanish will be truncated (lines 3, 8 and 12). Thus L will be empty at the end of at most the k th loop cycle, and Algorithm 1 ends. \square

THEOREM 1. *When Algorithm 1 ends, all the possible k -thead derivations are included in H , and all m -thead derivations are included in S , where $m < k$.*

PROOF. This follows from Lemma 1, Lemma 2 and Lemma 3. \square

4.4 Complexity Analysis

In Algorithm 1, the complexity of the step of lines 6-9 is $O(|P_{ij}|)$, where $|P_{ij}|$ is the number of possible productions to the i^{th} symbol in the j th member of L . For the loop of lines 5-10, the complexity is $O(|P_{ij}||L|)$.

The complexity of the entire algorithm is hard to analyze directly, but it is easy to see that, since the primary output is set H , the theoretical lower boundary of the number of steps needed is equal to the number of elements in the output set: $\Omega(|H|)$. H is the set of terminal strings of length k of α , so $\Omega(|H|) = \Omega(|T|_k)$, where $|T|$ is the number of terminals in the alphabet. This is the theoretical lower boundary of both time and space requirements. Obviously, it is exponential in nature as expected.

4.5 Comparison with other algorithms

Aho and Ullman's method and our method are both standalone algorithms to compute $FIRST_k(\alpha)$, where the computation rely on a set of production rules of the grammar only, and the parsing machine is not needed. Thus these two methods are better than the other methods in literature research.

Aho and Ullman's method takes a bottom up approach by first calculating $FIRST_i(X)$ for each symbol X , $i = 1, 2, \dots, k$, then combining these building blocks to obtain $FIRST_k(\alpha)$. This is a systematic approach, which is also demonstrated in their handling of $FIRST_1(\alpha)$, which is discussed in [3]. Once the preparation phase is done, for whatever input string, the task boils down to applying the \oplus_k operation on the consisting symbols of the input string, which concatenates elements from each set. However, the systematic nature also means that the overhead must always be taken to achieve good efficiency. From a practical point of view, since input strings are unknown, the entire preparation step must be done and its result be cached for later use.

In comparison, our method takes a top down approach. No previous computation is needed. The algorithm computes $FIRST_k(\alpha)$ on the fly based on symbols included in the input string. No cache is needed. It removes unnecessary overhead strings on the way of computation. In nature, both methods are equivalent. Our method can also be used for the preparation process of Aho and Ullman's method.

Another difference is that the $FIRST_k(\alpha)$ method of Aho and Ullman gives a set of terminal heads whose length $L \leq k$, and this set must be kept during the entire calculation process, only at the very end can we remove those $L < k$. In comparison, our method separates terminal heads into two sets, for one set the length of terminal heads $L = k$, and for the other set $L < k$. The second set where $L < k$ can be ignored from the calculation process.

5. EXAMPLES

In this section we show how $THEAD_k(\alpha)$ and $FIRST_k(\alpha)$ work on the same input string.

EXAMPLE 5.1. *Given grammar G_2 (ϵ is the empty string):*

Table 1: Example 4, round 1 (i=1)

i	j	String added to L	String Sequence Number
1	1	XYZU	1
		YYZU	2
		xYZU	3
		YZU	4
	2	ZYZU	5
		yYZU	6
	3	ZZU	7
	4	yZU	8
		ZU	9
		zYZU	10
	5	XZU	11
	6	zZU	12
	7	XU	13
	8	zU	14
	9	U	15
	10	xZU	16
	11	YU	17
	12	xU	18
	13	u	19
	14	yU	20

Table 2: Example 4, round 2 (i = 2)

i	j	String added to L	String Sequence Number
2	1	xYZU	1
		yYZU	2
		yZU	3
		zYZU	4
		zZU	5
		zU	6
		xZU	7
		xU	8
		yU	9
	2	xZZU	10
		xy	11
	3	yZZU	12
		yy	13
	4	yXU	14
		yz	15
	5	zZZU	16
		zy	17
	6	zXU	18
		zz	19
	7	zu	20
		xXU	21
	8	xz	22
		xu	23
	9	yu	24
		xXZU	25
	10	yXZU	25
	11	yYu	27
	12	yx	28
	13	zXZU	29
	14	zYU	30
	15	zx	31
	16	xYU	32
	17	xx	33
	18		
	19		
	20		
	21		
	22		

$X \rightarrow XY|a|\epsilon$
 $Y \rightarrow Z|y|\epsilon$
 $Z \rightarrow X|z|\epsilon$
 $U \rightarrow u$

Find the set of 2-theads of XYZU using

Algorithm 1: $THEAD_k(\alpha)$.

Since symbols X, Y and Z can all vanish, and U does not vanish, the string XYZU contains less than 2 symbols (i.e., only 1) that do not vanish, therefore we need to include the entire string XYZU as the initial element in the list L. Thus, at the beginning, $L = XYZU$.

First round of operation for $i = 1$ is shown in Table 1.

At this time, the step of lines 5-10 finishes. Next we follow lines 11-25. Remove from L all strings with nonterminals in the i th (first) position; remove from L all strings whose prefixes of length 2 consisting entirely of terminals, and add these prefixes to H; and remove from L all strings of length less than 2 and contains only terminal strings. At this time, we have $H = \{\}$, $S = \{u\}$, $L = \{xYZU, yYZU, yZU, zYZU, zZU, zU, xZU, xU, yU\}$.

The second round where $i = 2$ is shown in Table 2.

Remove all strings with non-terminals in the i th (second) position, remove all strings whose prefixes of length 2 are made up of terminals, and remove all strings of length less than 2 and contains only terminal strings, we have $H = \{xy, yy, zy, zz, zu, xu, xz, yz, yx, yu, zx, xx\}$, $S = \{u\}$, $L = \{\}$.

EXAMPLE 5.2. Given grammar $G2$ as in Example 3.1, find the set of 2-theads of XYZU, this time use the $FIRST_k(\alpha)$ algorithm of Aho and Ullman.

Following the steps in Aho and Ullman's algorithm, we need $FIRST_k(\alpha)$, where $\alpha = XYZU$, and $k = 2$.

$F_i(p) = \{p\}$, for all $p \in x, y, z, u, \epsilon$, and $i \geq 0$.
 $F_0(X) = \{x, \epsilon\}$
 $F_0(Y) = \{y, \epsilon\}$
 $F_0(Z) = \{z, \epsilon\}$
 $F_0(U) = \{u\}$
 $F_1(X) = \{x, y, \epsilon\}$
 $F_1(Y) = \{y, z, \epsilon\}$
 $F_1(Z) = \{z, x, \epsilon\}$
 $F_1(U) = \{u\}$
 $F_2(X) = \{x, y, z, \epsilon\}$
 $F_2(Y) = \{x, y, z, \epsilon\}$
 $F_2(Z) = \{x, y, z, \epsilon\}$
 $F_2(U) = \{u\}$

From this point on $F_i(S) = F_2(S)$ for $i \geq 3$, $S = X, Y, Z, U$. It converges here. Therefore:

$FIRST_2(X) = F_2(X) = \{x, y, z, \epsilon\}$
 $FIRST_2(Y) = F_2(Y) = \{x, y, z, \epsilon\}$
 $FIRST_2(Z) = F_2(Z) = \{x, y, z, \epsilon\}$
 $FIRST_2(U) = F_2(U) = \{u\}$

Note that here $FIRST_2(X)$ contains strings of length less than 2, because we need to keep them in the intermediate steps, as discussed at the end of section 2.4.

Finally, we can calculate $FIRST_k(\alpha) = FIRST_2(XYZU) = FIRST_2(X) \oplus_2 FIRST_2(Y) \oplus_2 FIRST_2(Z) \oplus_2 FIRST_2(U) = x, y, z, \oplus_2 x, y, z, \oplus_2 x, y, z, \oplus_2 u = \{xx, xy, xz, xu, yx, yy, yz, yu, zx, zy, zu, u\}$

As a last step as discussed at the end of section 1.3.3, we remove strings whose length are less than 2, which is 'u'

k	# of k-threads	Time (sec) By HEAD	Time (sec) By FIRST
1	1	0.000022	0.000108
2	1	0.000009	0.000014
3	1	0.000012	0.00002
4	1	0.000018	0.000017
5	1	0.00002	0.00002
6	1	0.000027	0.000026
7	1	0.000032	0.000021
8	1	0.000072	0.000022
9	1	0.000047	0.000026
10	1	0.000053	0.000055

Table 3: Number of generated k-threads and time spent on input string UUUUUUUUUU, for $k = 1$ to 10

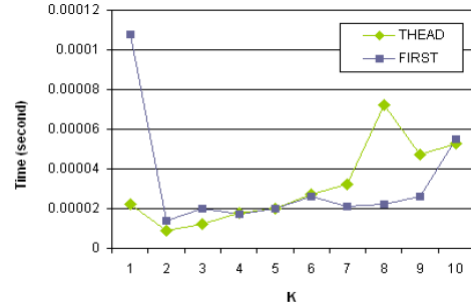


Figure 2: Time cost of $THEAD_k(\alpha)$ versus $FIRST_k(\alpha)$ for $\alpha = UUUUUUUUUU$, $k = 1$ to 10

here, and obtain $\{xx, xy, xz, xu, yx, yy, yz, yu, zx, zy, zu\}$. This is the same result as using our algorithm.

6. PERFORMANCE STUDY

We implemented both the $THEAD_k(\alpha)$ algorithm and the $FIRST_k(\alpha)$ algorithm, and compared their performance. In each experiment, the start time and end time are measured multiple times, and then average start time is subtracted from average end time to obtain the running time. The study was conducted on a Sun Microsystems sun4u Netra 440 server running Solaris. CPU is 1.6GHz, memory is 12 GB. For all the experiments below, test case 2 uses the most memory (hundreds of MB), so memory is not an issue. In the figure legends, $THEAD$ represents $THEAD_k(\alpha)$, and $FIRST$ represents $FIRST_k(\alpha)$.

Grammar $G2$ is used as the testing grammar.

6.1 Test case 1: $\alpha = UUUUUUUUUU$, $k = 1$ to 10

Result is shown in Table 3 and Figure 2. When $\hat{\alpha} = UUUUUUUUUU$, there is only one terminal head, which is u for $k = 1$ to 10. The speed is very fast, at the level of microsecond. The relatively long delay when $k = 1$ for the $FIRST_k(\alpha)$ algorithm should be caused by the initial construction of the $F_i(X)$ table.

6.2 Test case 2: $\alpha = XXXXXXXXXXXX$, $k = 1$ to 8

Result is shown in Table 4 and Figure 3. This is the worst case scenario where the theoretical bound of exponential be-

k	# of k-threads	Time (sec) By THEAD	Time (sec) By FIRST
1	3	0.000242	0.000221
2	9	0.001302	0.00145
3	27	0.00599	0.009041
4	81	0.032146	0.065045
5	243	0.213318	0.425997
6	729	1.463382	3.282263
7	2187	12.21782	26.23495
8	6561	135.462	297.5679

Table 4: Number of generated k-threads and time spent on input string XXXXXXXXXX, for k = 1 to 8

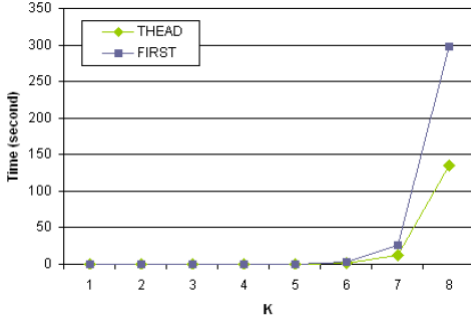


Figure 3: Time cost of $THEAD_k(\alpha)$ versus $FIRST_k(\alpha)$ for $\hat{I}\hat{s} = XXXXXXXXXX$, k = 1 to 8.

havior is observed. This is because each symbol of the input string is a non-terminal (X), which can derive 3 terminals x, y and z. The number of k-threads that can be generated is 3^k . When k is as small as 10, this will take hours to finish. The result is similar when $\hat{I}\hat{s} = YYYYYYYYYY$ or $\hat{I}\hat{s} = ZZZZZZZZZZ$.

6.3 Test case 3: $\alpha = XYZUXYZUYX$, k = 1 to 10

Result is shown in Table 5 and Figure 4. Here α is a randomly generated string. We can see that $THEAD_k(\hat{I}\hat{s})$ performs better than $FIRST_k(\hat{I}\hat{s})$ for k = 1 to 9, but for k = 10, $FIRST_k(\hat{I}\hat{s})$ runs faster. This possibly has to do with the way of implementation: in the implementation of $FIRST_k(\hat{I}\hat{s})$, an ordered list is used to store the strings generated intermediately; for $THEAD_k(\hat{I}\hat{s})$, the list used cannot be ordered, since new inserted strings will need to be processed and have to be attached to the end. When inserting a new generated string to the end of list L, $THEAD_k(\hat{I}\hat{s})$ will search through the entire list to make sure it does not exist yet. To overcome this issue an auxiliary ordered list is used in the implementation. This slows it down when the list is long. Of course, better implementation using more efficient data structure can improve this scenario.

6.4 Test case 4: Average on 100 strings of length 10, k = 1 to 8

100 input strings, each of length 10, are generated from the alphabet of X, Y, Z, U using a random number generator, and then fed to the algorithms to compare their performance. This means the input strings may be like:

k	# of k-threads	Time (sec) By THEAD	Time (sec) By FIRST
1	4	0.000079	0.000315
2	16	0.00038	0.001807
3	63	0.002083	0.016498
4	162	0.011877	0.063377
5	486	0.100032	0.460147
6	1296	0.624867	2.756787
7	2916	3.3104	11.8662
8	4374	14.64284	26.12881
9	6561	62.89018	71.2255
10	6561	94.49193	81.37379

Table 5: Number of generated k-threads and time spent on input string XYZUXYZUYX, for k = 1 to 10

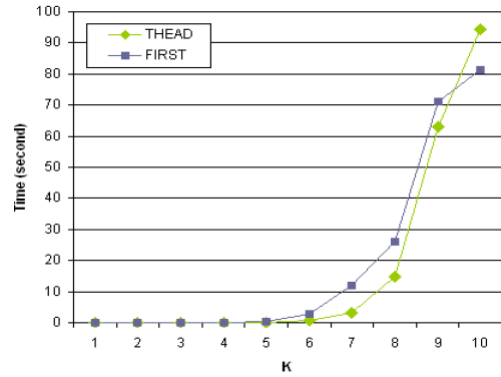


Figure 4: Time cost of $THEAD_k(\alpha)$ versus $FIRST_k(\alpha)$ for $\alpha = XYZUXYZUYX$, k = 1 to 10.

k	Avg # of k-threads	Time (sec) By THEAD	Time (sec) By FIRST
1	3.07	0.000068	0.000177
2	10.37	0.000381	0.000969
3	32.73	0.001999	0.006003
4	95.43	0.011761	0.03663
5	270.25	0.078635	0.246849
6	697.89	0.505454	1.496519
7	1662.39	3.484229	8.207717
8	3669.3	27.723004	55.275918

Table 6: Average number of generated k-threads and time spent on 100 random strings of length 10, for $k = 1$ to 8

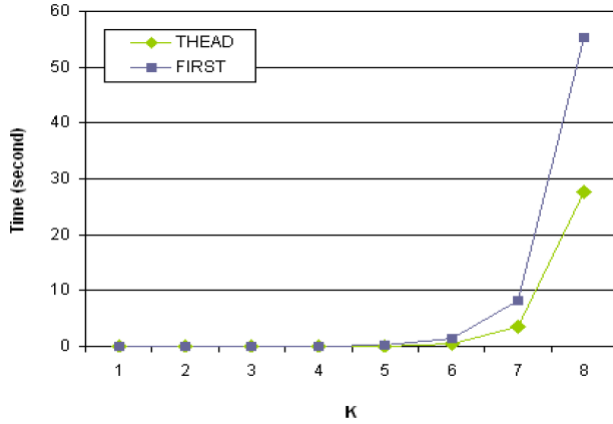


Figure 5: Time cost of $THEAD_k(\alpha)$ versus $FIRST_k(\alpha)$ when k increases. Averaged over 100 strings of length 10

```

1  YXXXYUUUUU
2  UZZUUUZXXY
3  YZZUYZZYZU
4  ZZUZUYZUYZ

```

```

....
100 UUYXXUUXUY

```

Result is shown in Table 6 and Figure 5. Table 6 shows the average number of k-threads generated and average time used by the $THEAD_k(\alpha)$ and $FIRST_k(\alpha)$ algorithms over 100 input strings of length 10, and $k = 1$ to 8. Figure 5 shows the graphical version of the average time used when k increases. It can be seen that the $THEAD_k(\alpha)$ algorithm uses less time.

6.5 Test case 5: 100 strings of length 1 to 100, $k = 2$

In this test case, k is fixed, while the input string is a kprefix of the following randomly generated string, where input string length $|\alpha| = 1$ to 100, i.e., the input strings may be like:

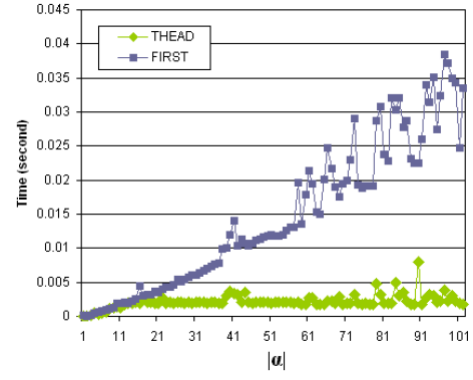


Figure 6: Time cost of $THEAD_k(\alpha)$ versus $FIRST_k(\alpha)$ when $k = 2$, and string length $|\alpha|$ increases

```

1  Y
2  YZ
3  YZZ
4  YZZY
....
100 YZZYXXZYXYXZUXYYUYXZUYYUZXXUYZZ
    YYZXXXXXXUUUYXZZYZYZUXXZZYZZX
    UZUXYZYYYUYZZZZZUZXZYYYYZYUXXZ
    ZUYZUZXUY

```

6.6 Summary

We can draw several conclusions from the experiments.

First of all, when the input string contains terminal symbols only, the speed is the fastest. When the input string contains non-terminal symbols only, the speed is the slowest, and may lead to the worst case scenario: exponential increase in computation time. For a grammar as simple as G2, when $k = 10$, it will take hours to finish using both algorithms.

In general the $THEAD_k(\alpha)$ algorithm performs better than the $FIRST_k(\alpha)$ algorithm, as shown by test case 4, which is averaged over 100 randomly generated strings of length 10 for $k = 1$ to 10.

However, it is also possible that $FIRST_k(\alpha)$ runs faster than $THEAD_k(\alpha)$, as shown in test case 3 when $k = 10$. Finally, when k is small, but input string is long, $THEAD_k(\alpha)$ will perform better than $FIRST_k(\alpha)$, as shown by test case 5. Actually, for this scenario, the time $THEAD_k(\alpha)$ takes will not increase when the size of the input string increase. However, the time used by $FIRST_k(\alpha)$ will increase linearly according to the length of the input string.

7. IMPLEMENTATION

We briefly discuss the implementation of the two approaches here, which is done in ANSI C from scratch.

To make the comparison of the two algorithms reasonable, it is necessary to implement them with similar data structures.

The major operations involved in both algorithms are set operations. In the current implementation, a set is implemented as a linked list. Search in the set is done by going through the list in linear order. That a linked list is chosen for the implementation is because of the nature of the $THEAD_k(\alpha)$ algorithm: a new generated string has to be

appended to the end of the current set, which makes queue a natural and necessary choice. A queue of unknown size as in the current scenario is in turn naturally implemented as a linked list.

To guarantee similar search experience for both algorithms, an ordered list is used. For the method of Aho and Ullman, this is no problem. But for the THEADk($\hat{I}\hat{s}$) method, the queue (implemented as a list) to be appended to can not be ordered, so an auxiliary list is provided which stores the same strings as the queue but is in sorted order, such that when a search in the auxiliary ordered list does not return a hit, the new string is appended to the end of the queue. The maintenance of two lists in the THEADk($\hat{I}\hat{s}$) algorithm implementation obviously will slow it down to some degree.

This implementation can be improved by providing an auxiliary binary search tree or a hash table to both methods, which works much more efficient when decide if a string exists in a set. This improvement should be of more significance to the performance of the THEADk($\hat{I}\hat{s}$) algorithm implementation according to the above discussion.

Finally, a linked list suffices for all the operations of the THEADk($\hat{I}\hat{s}$) algorithm. For the algorithm of Aho and Ullman, an array is also used to store the pre-computed FIRSTk(X_i) values of all the symbols X_i , such that given a random string $\hat{I}\hat{s} = X_1X_2\cdots X_n$, FIRSTk(X_i) can be retrieved in constant time using index of X_i in the symbol table for the calculation of FIRSTk($\hat{I}\hat{s}$) = $FIRST_k(X_1) \oplus_k FIRST_k(X_2) \oplus_k \cdots \oplus_k FIRST_k(X_n)$.

8. APPLICATION

One application of the algorithm presented here is to be used in a LR(k) parser generation algorithm.

Our study of the LR(k) algorithm shows that the calculation of LR(k) lookahead is one of the major steps involved. We designed and implemented the Edge-Pushing LR(k) algorithm, which depends on the THEADk($\hat{I}\hat{s}$) function to calculate k-lookahead. The Edge-Pushing algorithm is shown below in Figure 7, which is taken from a previous paper [11]. The THEADk($\hat{I}\hat{s}$) algorithm is used on line 13.

The Edge-Pushing algorithm is implemented in the HYACC parser generator, which is available as an open source parser generator [12][13].

9. CONCLUSION

In this paper we have presented a new algorithm to calculate the terminal heads of length k, which is called THEADk($\hat{I}\hat{s}$).

We reviewed relevant literature, of which Aho and Ullman's method is the only previously available standalone algorithm for this purpose. We showed the new algorithm, discussed its correctness and complexity, and compared to previous work. Examples are given to evaluate terminal heads of length k of a given string by using the THEADk($\hat{I}\hat{s}$) algorithm and the FIRSTk($\hat{I}\hat{s}$) method of Aho and Ullman.

An empirical study was conducted to compare the THEADk($\hat{I}\hat{s}$) algorithm and the FIRSTk($\hat{I}\hat{s}$) algorithm. In general, when averaged over a large number of randomly generated input strings, THEADk($\hat{I}\hat{s}$) performs faster than FIRSTk($\hat{I}\hat{s}$). When the input string $\hat{I}\hat{s}$ is long but k is small, THEADk($\hat{I}\hat{s}$) always performs better than FIRSTk($\hat{I}\hat{s}$).

Finally, we discussed the application of the new THEADk($\hat{I}\hat{s}$) algorithm, and pointed out that it has been used to im-

plement the edge-pushing algorithm in the HYACC parser generator.

Due to the fact that FIRSTk($\hat{I}\hat{s}$) is a fundamental algorithm that works as a basic building block in compiler theory and practice, its improvement should have wide impact.

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