VV186: Honors Mathematics

Sequence & Real Functions

Xingjian Zhang

Univerity of Michigan-Shanghai Jiao Tong University Joint Institute

October 13, 2020

Outline

1. Sequence
   - Cauchy Sequence

2.2.36. Definition. Let M be a set. A map $\varrho\colon M \times M \to \mathbb{R}$ is called a ***metric*** if

(i) $\varrho(x, y) \geq 0$ for all $x, y \in M$ and $\varrho(x, y) = 0$ if and only if $x = y$.

(ii) $\varrho(x, y) = \varrho(y, x)$

(iii) $\varrho(x, z) \leq \varrho(x, y) + \varrho(y, z)$.

The pair (M, ϱ) is then called a ***metric space***.

Remark: Metric space is a very important structure in maths. It describes how the "distance" between two elements in a set is measured. In the future, we will discuss a similar structure that has more nice properties: *Normed Vector Space*, which is also endowed with some distance function. Try to compare metric & norm, metric space & normed vector space by finding their similarities & differences (when we learn both of them).

Examples of Metric Space

1. $\mathbb{R}^+$ with $d(x, y) = |\log(y/x)|$

1. $\mathbb{R}^+$ with $d(x, y) = |\log(y/x)|$
2. $\mathrm{Conv}(\mathbb{R})$ with $d(x, y) = \sup|x - y|$. Notice that x, y are real convergent sequences. (We will see this example in details later!)

Examples of Metric Space

1. $\mathbb{R}^+$ with $d(x,y) = |\log(y/x)|$
2. $\text{Conv}(\mathbb{R})$ with $d(x,y) = \sup|x-y|$. Notice that x, y are real convergent sequences. (We will see this example in details later!)
3. * *The Discrete Metric.* Any set M with $d(x,y) = 0$ if $x = y$, $d(x,y) = 1$ otherwise. This shows for any set, there is always a metric space associated with it. Moreover, by this metric, the set of any single point is an open ball (why?), and therefore every subset is open – The space is discrete (has the *discrete topology*.)

Examples of Metric Space

1. $\mathbb{R}^+$ with $d(x, y) = |\log(y/x)|$
2. $\mathrm{Conv}(\mathbb{R})$ with $d(x, y) = \sup |x - y|$. Notice that x, y are real convergent sequences. (We will see this example in details later!)
3. * *The Discrete Metric.* Any set M with $d(x, y) = 0$ if $x = y$, $d(x, y) = 1$ otherwise. This shows for any set, there is always a metric space associated with it. Moreover, by this metric, the set of any single point is an open ball (why?), and therefore every subset is open – The space is discrete (has the *discrete topology*.)
4. * We will come back to metric space when we learn normed vector space. We will then conclude that any normed vector space is a metric space.

Generalizing Convergence

We can hence define convergence of sequences in metric spaces (M, ϱ), where convergence of a sequence $(a_n)\colon \mathbb{N} \to M$ is determined by (2.2.1),

$$\lim_{n\to\infty} a_n = a \qquad :\Leftrightarrow \qquad \underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{n>N}{\forall}\ a_n \in B_\varepsilon(a).$$

where

$$B_\varepsilon(a) = \{y \in M\colon \varrho(y, a) < \varepsilon\}, \qquad \varepsilon > 0, \qquad a \in M.$$

Remark: $B_\varepsilon(a)$ is the (generalized) *open ball*. It describes the neighborhood of some element a in the set M. (Where did we define the open ball in the previous lectures?) The open ball turns out to be an important concept in *VV285*.

We then well-define the boundedness of sequence by it.

Cauchy Sequence

The fun part starts! We define the *Cauchy sequences*, whose elements' distance becomes smaller and smaller:

> 2.2.40. Definition. A sequence (a_n) in a metric space (M, ϱ) is called a ***Cauchy sequence*** if
>
> $$\underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{m,n>N}{\forall}\ \varrho(a_m, a_n) < \varepsilon.$$

Remark:

1. Every Cauchy sequence is bounded.

Cauchy Sequence

The fun part starts! We define the *Cauchy sequences*, whose elements' distance becomes smaller and smaller:

> 2.2.40. Definition. A sequence (a_n) in a metric space (M, ϱ) is called a ***Cauchy sequence*** if
>
> $$\underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{m,n>N}{\forall}\ \varrho(a_m, a_n) < \varepsilon.$$

Remark:

1. Every Cauchy sequence is bounded.
2. Every convergent sequence is a Cauchy sequence.

Cauchy Sequence

The fun part starts! We define the *Cauchy sequences*, whose elements' distance becomes smaller and smaller:

> 2.2.40. Definition. A sequence (a_n) in a metric space (M, ϱ) is called a ***Cauchy sequence*** if
>
> $$\underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{m,n>N}{\forall}\ \varrho(a_m, a_n) < \varepsilon.$$

Remark:

1. Every Cauchy sequence is bounded.
2. Every convergent sequence is a Cauchy sequence.
3. However, the reverse is not necessarily true.

Cauchy Sequence

The fun part starts! We define the *Cauchy sequences*, whose elements' distance becomes smaller and smaller:

> 2.2.40. Definition. A sequence (a_n) in a metric space (M, ϱ) is called a ***Cauchy sequence*** if
>
> $$\underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{m,n>N}{\forall}\ \varrho(a_m, a_n) < \varepsilon.$$

Remark:

1. Every Cauchy sequence is bounded.
2. Every convergent sequence is a Cauchy sequence.
3. However, the reverse is not necessarily true.
4. The metric space where all Cauchy sequences converge is called a *complete* metric space.

Cauchy Sequence

The fun part starts! We define the *Cauchy sequences*, whose elements' distance becomes smaller and smaller:

> 2.2.40. Definition. A sequence (a_n) in a metric space (M, ϱ) is called a ***Cauchy sequence*** if
>
> $$\underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{m,n>N}{\forall}\ \varrho(a_m, a_n) < \varepsilon.$$

Remark:

1. Every Cauchy sequence is bounded.
2. Every convergent sequence is a Cauchy sequence.
3. However, the reverse is not necessarily true.
4. The metric space where all Cauchy sequences converge is called a *complete* metric space.
5. A *completion* of an incomplete metric space is obtained by adding all the limits of Cauchy sequences in that space. Moreover, the completion of it can be constructed as a set of *equivalence classes* of Cauchy sequences in it – That explains why Cauchy sequences are important!

A Recap on Construction of $\mathbb{R}$

Given $\mathbb{Q}$, we may consider the set of all sequences in $\mathbb{Q}$ that converge to a limit. Denote this set by $\text{Conv}(\mathbb{Q})$. Each sequence $(a_n) \in \text{Conv}(\mathbb{Q})$ is associated uniquely to a number $a \in \mathbb{Q}$, namely its limit. We can now say that two sequences are equivalent if they have the same limit, i.e.,

$$(a_n) \sim (b_n) \quad :\Leftrightarrow \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n. \tag{2.2.5}$$

Remark:

1. $\sim$ denotes an *equivalence relation* between two elements in some set.

We denote the set of all sequences with the same limit as a sequence (a_n) by $[(a_n)]$. Such a set is called a ***(equivalence) class*** and the set of all classes is denoted $\text{Conv}(\mathbb{Q})/\sim$.

Since each rational number is represented by a class (why?) we see that the rational numbers may be identified with the set of all classes of convergent sequences:

$$\mathbb{Q} \simeq \text{Conv}(\mathbb{Q})/\sim .$$

Remark:

1. The equivalence relation is extremely useful in this situation. Because if R is an equivalence relation on a set M, then for all $a, b \in M$, either $[a] \cap [b] = \varnothing$ or $[a] = [b]$. (Why? And Why is this even useful?)

We denote the set of all sequences with the same limit as a sequence (a_n) by $[(a_n)]$. Such a set is called a ***(equivalence) class*** and the set of all classes is denoted $\text{Conv}(\mathbb{Q})/\sim$.

Since each rational number is represented by a class (why?) we see that the rational numbers may be identified with the set of all classes of convergent sequences:

$$\mathbb{Q} \simeq \text{Conv}(\mathbb{Q})/\sim .$$

Remark:

1. The equivalence relation is extremely useful in this situation. Because if R is an equivalence relation on a set M, then for all $a, b \in M$, either $[a] \cap [b] = \varnothing$ or $[a] = [b]$. (Why? And Why is this even useful?)
2. The equivalence classes **partition** the set M!

We denote the set of all sequences with the same limit as a sequence (a_n) by $[(a_n)]$. Such a set is called a ***(equivalence) class*** and the set of all classes is denoted $\text{Conv}(\mathbb{Q})/\sim$.

Since each rational number is represented by a class (why?) we see that the rational numbers may be identified with the set of all classes of convergent sequences:

$$\mathbb{Q} \simeq \text{Conv}(\mathbb{Q})/\sim .$$

Remark:

1. The equivalence relation is extremely useful in this situation. Because if R is an equivalence relation on a set M, then for all $a, b \in M$, either $[a] \cap [b] = \varnothing$ or $[a] = [b]$. (Why? And Why is this even useful?)
2. The equivalence classes **partition** the set M!
3. $\simeq$ denotes a *bijection* exists between two sets.

We denote the set of all sequences with the same limit as a sequence (a_n) by $[(a_n)]$. Such a set is called a ***(equivalence) class*** and the set of all classes is denoted $\mathrm{Conv}(\mathbb{Q})/\sim$.

Since each rational number is represented by a class (why?) we see that the rational numbers may be identified with the set of all classes of convergent sequences:

$$\mathbb{Q} \simeq \mathrm{Conv}(\mathbb{Q})/\sim .$$

Remark:

1. The equivalence relation is extremely useful in this situation. Because if R is an equivalence relation on a set M, then for all $a, b \in M$, either $[a] \cap [b] = \varnothing$ or $[a] = [b]$. (Why? And Why is this even useful?)
2. The equivalence classes **partition** the set M!
3. $\simeq$ denotes a *bijection* exists between two sets.
4. $M/\sim$ denotes the set of all equivalence classes in M **partitioned** by $\sim$.

We can now consider a larger class of sequences, that of Cauchy sequences of rational numbers, denoted by Cauchy($\mathbb{Q}$). Since every convergent sequence is a Cauchy sequence, Conv($\mathbb{Q}$) $\subset$ Cauchy($\mathbb{Q}$).

Furthermore, we say that two Cauchy sequences are equivalent not if they have the same limit (because they might not converge) but rather if their difference converges to zero:

$$(a_n) \sim (b_n) \qquad :\Leftrightarrow \qquad \lim_{n\to\infty}(a_n - b_n) = 0. \tag{2.2.6}$$

Remark:

1. We define a new relationship between two Cauchy sequences, which is more general than the previous equivalence relationship.

We can now consider a larger class of sequences, that of Cauchy sequences of rational numbers, denoted by Cauchy($\mathbb{Q}$). Since every convergent sequence is a Cauchy sequence, Conv($\mathbb{Q}$) $\subset$ Cauchy($\mathbb{Q}$).

Furthermore, we say that two Cauchy sequences are equivalent not if they have the same limit (because they might not converge) but rather if their difference converges to zero:

$$(a_n) \sim (b_n) \qquad :\Leftrightarrow \qquad \lim_{n\to\infty}(a_n - b_n) = 0. \tag{2.2.6}$$

Remark:

1. We define a new relationship between two Cauchy sequences, which is more general than the previous equivalence relationship.
2. We therefore consider this $\sim$ to be a generalization of the previous $\sim$ – So we reuse the same symbol.

We can now consider a larger class of sequences, that of Cauchy sequences of rational numbers, denoted by Cauchy($\mathbb{Q}$). Since every convergent sequence is a Cauchy sequence, $\text{Conv}(\mathbb{Q}) \subset \text{Cauchy}(\mathbb{Q})$.

Furthermore, we say that two Cauchy sequences are equivalent not if they have the same limit (because they might not converge) but rather if their difference converges to zero:

$$(a_n) \sim (b_n) \qquad :\Leftrightarrow \qquad \lim_{n\to\infty}(a_n - b_n) = 0. \tag{2.2.6}$$

Remark:

1. We define a new relationship between two Cauchy sequences, which is more general than the previous equivalence relationship.
2. We therefore consider this $\sim$ to be a generalization of the previous $\sim$ – So we reuse the same symbol.
3. Finally, we have a larger set:

$$\text{Cauchy}(\mathbb{Q})/\sim \supset \text{Conv}(\mathbb{Q})/\sim \simeq \mathbb{Q}$$

A Recap on Construction of $\mathbb{R}$

The set $\text{Cauchy}(\mathbb{Q})/\sim$ incorporates the rational numbers and by its construction every Cauchy sequence (a_n) in $\text{Cauchy}(\mathbb{Q})/\sim$ has a limit, namely precisely the object represented by the class $[(a_n)]$. We write

$$\mathbb{R} := \text{Cauchy}(\mathbb{Q})/\sim$$

and call this set the ***real numbers***.

Remark:

1. We then **define** the set of real numbers to be this larger set.

A Recap on Construction of $\mathbb{R}$

The set $\text{Cauchy}(\mathbb{Q})/\sim$ incorporates the rational numbers and by its construction every Cauchy sequence (a_n) in $\text{Cauchy}(\mathbb{Q})/\sim$ has a limit, namely precisely the object represented by the class $[(a_n)]$. We write

$$\mathbb{R} := \text{Cauchy}(\mathbb{Q})/\sim$$

and call this set the ***real numbers***.

Remark:

1. We then **define** the set of real numbers to be this larger set.
2. Recall how did we construct the natural numbers $\mathbb{N}_{def}$? Sets!

Regarding $\simeq$

A way to think about $\simeq$, a bijection exists between two sets A and B, is that the abstract entities A and B are equivalent and **essentially the same** – they are both sets, so they contain no information about repetition or order, and since there is a bijection between them, we can obtain one from the other simply by relabeling the elements.

To some extent, this also explains why we use other abstraction to construct natural numbers or real numbers: we are looking for a correspondence between a known concept and a new concept.

* An explicit definition will be given in VE203.[1]

[1] *Morphisms and Isomorphisms.

Example of Abstract Incomplete Metric Space

Let's see one abstract metric space that is not complete.

We define the set of all real sequences that vanish from some points to be U. i.e.

$$U = \{(a_n) | \exists N \in \mathbb{N}, \forall n > N : a_n = 0\}.$$

It is clear that $U \subset c_0$, where

$$c_0 = \{(a_n) | \lim a_n = 0\}.$$

We use the metric

$$\rho((a_n), (b_n)) = \sup |a_n - b_n|.$$

Define a sequence of sequences $(a_n^{(N)}) \in U$,

$$(a_n^{(N)}) = \begin{cases} \frac{1}{n}, \ n \leq N \\ 0, \ n > N \end{cases}$$

Is $(a_n^{(N)})$ a Cauchy sequence in (U, ρ)? Does $(a_n^{(N)})$ has a limit in this space? What is the completion of (U, ρ)?

Exercise

Have Fun

And

Learn Well![2]

[2]Special acknowledgement to former TA Zhang Leyang, who offered many exercises and advice to my recitation class.