VV186: Honors Mathematics

Functions & Differentiation

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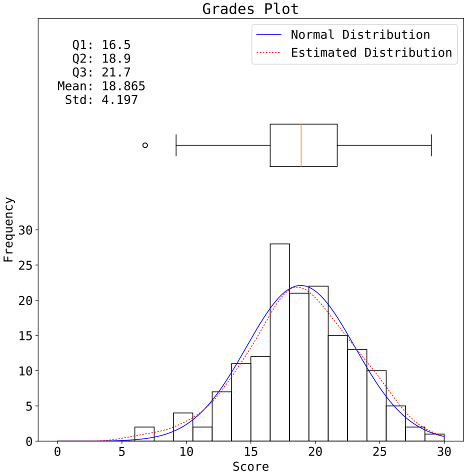
RC Policy

Several things I want you to pay attention to:

1. Be interactive. Feel free to interrupt me at any time if you want to ask something or simply make some comments. You are free to discuss with your friend if you want, as long as your discussion is related to the course contents and your voice won't effect other students.
2. Speak everything in English during the RC. This might be hard at the beginning, but you will soon get used to that.
3. "Question everything." Do not pretend to have understood everything. Maths is about strictness, abstraction and generalization. Understanding every basic concept is essential in our course. I will be quite "push" on checking your conceptual understanding. This process will be **annoying, tedious, but rewarding**. So Get prepared.

Outline

About Midterm 1



For a continuous increasing function f,[1]

- $\lim\limits_{x \to -\infty} f = 0$
- $\lim\limits_{x \to \infty} f = \infty$

How to decompose the proof? (We have discussed this strategy in our first RC.)

1. Prove $\operatorname{ran} f \subset (0, \infty)$
2. Prove $\operatorname{ran} f \supset (0, \infty)$

How to prove 1.?

- Any $f(x_0) \leq 0$ leads to a contradiction.
- Use the definition of *limit* and *strictly increasing* to show how they contradict rather than simply stating a contradiction occurs.

How to prove 2.?

- $\forall y \in (0, \infty)\, \exists x \in \mathbb{R} : f(x) = y$
- Use the *intermediate value theorem* and the definition of *limit* to show how x can be found.

[1] Mean of Ex.6: 1.09'(4')

Comments

- The exam is quite different from high school exam. Do not panic if you did not get a nearly full mark. Instead, compare with *mean*, *quartiles*.
- Take a deep breathe. Reflect why you did well in some exercises but bad in some others.
- Think like a mathematician, write like a mathematician. Especially in proof, you are expected to give a strict proof based on our strict definition of concepts. (I see many people consider ∞ as if it was a number.)
- If you feel it necessary for you to see what happens in grading for your exam paper. Go to Horst's exam inspection session. It will be announced recently.
- If you feel frustrated, definitely come to my OH so we can have a chat. That will help you figure out what you can do to improve.

Leibniz Notation

Leibniz Notation is a powerful, versatile, flexible notation system for calculus (and even differential geometry, etc). We regard dy and dx as infinitesimal elements, therefore have a good intuition of them. We can then rewrite many useful formulas in Leibniz notation:

- Chain rule:
$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$
- Inverse function theorem:
$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

Remark:

- In the coming future, we will see its power in integral calculus, too.
- Despite its flexibility, keep in mind we still need to verify these formulas like above using strict definition.
- You are free to use Leibniz notation in your coursework if you want.

Exercise

We define the *hyperbolic cosine* and *hyperbolic sine* functions by

$$\cosh, \sinh : \mathbb{R} \to \mathbb{R}, \quad \cosh x := \frac{1}{2}\left(e^x + e^{-x}\right), \quad \sinh x := \frac{1}{2}\left(e^x - e^{-x}\right)$$

1. Show that $\sinh : \mathbb{R} \to \mathbb{R}$ is bijective (and hence invertible). Verify that

$$(\sinh x)' = \cosh x$$

   then express the derivative in terms of sinh.
2. The inverse function of sinh is called the area hyperbolic sine, written Arsinh: $\mathbb{R} \to \mathbb{R}$. Use the inverse function theorem to prove that

$$(\operatorname{Arsinh} x)' = \frac{1}{\sqrt{x^2+1}}$$

Extrema

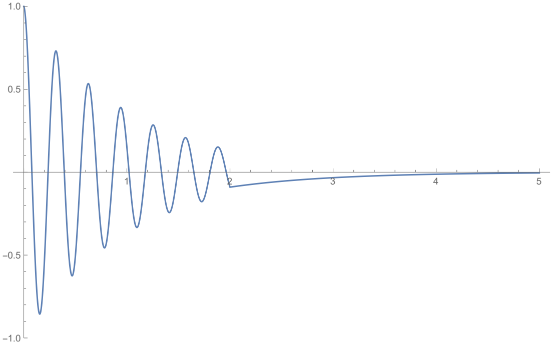
3.2.2. Theorem. Let f be a function and $(a, b) \subset \operatorname{dom} f$ an open interval. If $x \in (a, b)$ is a maximum (or minimum) point for f on (a, b) and if f is differentiable at x, then $f'(x) = 0$.

Remark: Notice the condition: why open interval?

3.2.5. Definition. A function f is said to have a ***critical point*** at $x \in \operatorname{dom} f$ if $f'(x) = 0$. The value $f(x)$ is then called a ***critical value*** of f.

Therefore, when finding the (local and global) maxima and minima of a function on a set $\Omega \subset \operatorname{dom} f$, we need to check

1. the critical points of f,
2. the boundary points of Ω,
3. interior points of Ω where f is not differentiable.



Point out

1. global maxima
2. global minima
3. all local maxima
4. all local minima
5. critical points
6. boundary points

True or False?

If $f'(x) = 0$, then x is a local extrema point for f.

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False, counter-example:

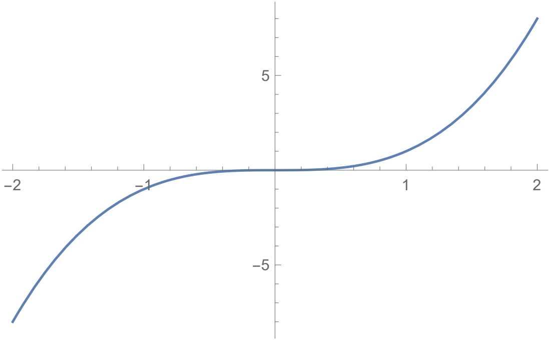


Figure: $f(x) = x^3$

3.2.6. Rolle's Theorem. Let f be a real function and $a < b \in \mathbb{R}$ such that $[a, b] \in \operatorname{dom} f$. Assume that f is continuous on $[a, b]$ and differentiable on (a, b) and that $f(a) = f(b)$. Then there exists a number $x \in (a, b)$ such that $f'(x) = 0$.

3.2.7. Mean Value Theorem. Let f be a real function and $a < b \in \mathbb{R}$ such that $[a, b] \in \operatorname{dom} f$. Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a number $x \in (a, b)$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

3.2.8. Corollary. Let f be a real function and $I \subset \operatorname{dom} f$. Assume that $f' = 0$ on I. Then f is constant on I.

3.2.10. Corollary. Let f be a real function and $I \subset \operatorname{dom} f$. Assume that $f' > 0$ on I. Then f is strictly increasing on I. If $f' < 0$ on I, f is strictly decreasing on I.

Here come two important theorems which are easily confused.

3.2.11. Theorem. Let f be a real function and $x \in \operatorname{dom} f$ such that $f'(x) = 0$. If $f''(x) > 0$, then f has a local minimum at x; if $f''(x) < 0$, then f has a local maximum at x.

3.2.12. Theorem. Let f be a real function and $x \in \operatorname{dom} f$ such that $f''(x)$ exists. If f has a local minimum at x, then $f''(x) \geq 0$; if f has a local maximum at x, then $f''(x) \leq 0$.

3.2.13. Definition. Let $\Omega \subset \mathbb{R}$ be any set and $I \subset \Omega$ an interval. A function $f : \Omega \to \mathbb{R}$ is called ***strictly convex*** on I if for all $a, x, b \in I$ with $a < x < b$,

$$\frac{f(x) - f(a)}{x - a} < \frac{f(b) - f(a)}{b - a}.$$

We say that f is ***strictly concave*** on I if $-f$ is strictly convex. If we replace "$<$" by "$\leq$" above, f is simply called convex and $-f$ concave.

3.2.14. Theorem. Let $f : I \to \mathbb{R}$ be be strictly convex on I and differentiable at $a, b \in I$. Then

(i) For any $h > 0$ ($h < 0$) such that $a + h \in I$, the graph of f over the interval $(a, a + h)$ lies below the secant line through the points $(a, f(a))$, $(a + h, f(a + h))$.

(ii) The graph of f over all of I lies above the tangent line through the point $(a, f(a))$.

(iii) If $a < b$, then $f'(a) < f'(b)$.

Convexity and Concavity II

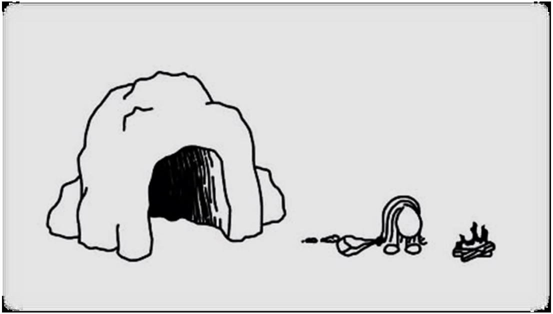
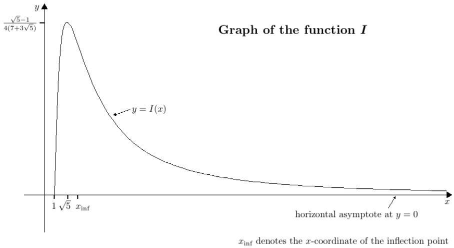


Figure: Concave

Curve Sketching

- 99% possibility to appear in your midterm 2.
- Follow the guidelines provided by Horst. (The rubric will be generally the same as the guidelines.)[2]



Two advice:

1. Do not forget to mark the *asymptote line*.
2. Do not add any **redundant** marks.

[2] Please refer to Slides 312!!!

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \frac{x}{1 + e^{-x}}$$

1. Find all local and global maxima and minima if they exist or explain why they don't exist.
2. Where is f increasing/decreasing?
3. Where is f convex / concave?
4. Find all asymptotes of f.
5. Sketch the graph of f, indicating any asymptotes and marking characteristic points on the axes.

$$\lim_{x \searrow b} \frac{f(x)}{g(x)} = \lim_{x \searrow b} \frac{f'(x)}{g'(x)}$$

if $\lim_{x \searrow b} f(x) = \lim_{x \searrow b} g(x) = 0$ or ∞, and $\lim_{x \searrow b} \frac{f'(x)}{g'(x)}$ exists.

Question:
Find limit of

$$f(x) = \frac{\sqrt{x+3} - \sqrt{3x+1}}{x-1}$$

at $x = 1$.

Vector space is definitely **the most important algebraic structure** in Honors Mathematics. We will exploit many nice properties and useful application of it throughout VV186 to VV286.

Thus, for now, let's remember its definition firmly.

3.3.1. Definition. A triple $(V, +, \cdot)$ is called a ***real vector space*** (or ***real linear space***) if

1. V is any set;
2. $+ \colon V \times V \to V$ is a map (called addition) with the following properties:
   - $(u+v)+w = u+(v+w)$ for all $u, v, w \in V$ (***associativity***),
   - $u+v = v+u$ for all $u, v \in V$ (***commutativity***),
   - there exists an element $e \in V$ such that $v+e = v$ for all $v \in V$ (***existence of a neutral element***),
   - for every $v \in V$ there exists an element $-v \in V$ such that $v+(-v) = e$;
3. $\cdot \colon \mathbb{R} \times V \to V$ is a map (called scalar multiplication) with the following properties:
   - $1 \cdot u = u$ for all $u \in V$,
   - $\lambda \cdot (u+v) = \lambda \cdot u + \lambda \cdot v$ for all $\lambda \in \mathbb{R}$, $u, v \in V$,
   - $(\lambda + \mu) \cdot u = \lambda \cdot u + \mu \cdot u$ for all $\lambda, \mu \in \mathbb{R}$, $u \in V$,
   - $(\lambda\mu) \cdot u = \lambda \cdot (\mu \cdot u)$ for all $\lambda, \mu \in \mathbb{R}$, $u \in V$.

What is a *complex vector space*? What is the difference between real and complex vector spaces in definition?

Here is a list of examples:

1. $\mathbb{R}^n$
2. $\mathbb{C}^n$
3. $C(\Omega)$
4. $C^k(\Omega)$
5. $C^\infty(\Omega)$
6. $\ell^p := \{(x_n) : \sum_n |x_n|^p < \infty\}$
7. $\mathcal{L}(U, V)$: The set of all linear maps from U to V, where U, V are both vector space.
8. * $\mathcal{L}(U, \mathcal{L}(U, V))$
9. $\mathrm{Mat}(m \times n; \mathbb{R})$: The set of all m-by-n matrices.

A vector space may contain a smaller vector space.

3.3.4. Definition. Let $(V, +, \cdot)$ be a real or complex vector space. If $U \subset V$ and $(U, +, \cdot)$ is also a vector space, then we say that $(U, +, \cdot)$ is a ***subspace*** of $(V, +, \cdot)$.

It turns out that we do not bother to prove all 9 properties of $(U, +, \cdot)$ to deduce it is also a vector space:

3.3.6. Lemma. Let $(V, +, \cdot)$ be a real (complex) vector space and $U \subset V$. If $u_1 + u_2 \in U$ for $u_1, u_2 \in U$ and $\lambda u \in U$ for all $u \in U$ and $\lambda \in \mathbb{R}$ $(\mathbb{C})$, then $(U, +, \cdot)$ is a subspace of $(V, +, \cdot)$.

Remark: Every vector space has at least one subspace which contains only its zero element. i.e. $(\{0\}, +, \cdot)$. We call this space the *trivial space*.

True of False?

A real vector space is a subspace of $\mathbb{R}^n$, a complex vector space is a subset of $\mathbb{C}^n$.

True of False?

A real vector space is a subspace of $\mathbb{R}^n$, a complex vector space is a subset of $\mathbb{C}^n$.

False. Real/Complex describes how can we perform scalar multiplication.

Normed Vector Space

With a nice math structure, we still want even more. We want to define a *measure of length* for every element in the vector space.

Normed Vector Space

With a nice math structure, we still want even more. We want to define a *measure of length* for every element in the vector space.

> 3.3.8. Definition. Let V be a real (complex) vector space. Then a map $\|\cdot\|: V \to \mathbb{R}$ is said to be a ***norm*** if for all $u, v \in V$ and all $\lambda \in \mathbb{R}$ $(\mathbb{C})$,
>
> 1. $\|v\| \geq 0$ for all $v \in V$ and $\|v\| = 0$ if and only if $v = 0$,
> 2. $\|\lambda \cdot v\| = |\lambda| \cdot \|v\|$,
> 3. $\|u + v\| \leq \|u\| + \|v\|$.
>
> The pair $(V, \|\cdot\|)$ is called a ***normed vector space*** or a ***normed linear space***.

Remark:

- A normed vector space can be considered as a metric space.
- What's the difference between a metric space and a normed vector space?

True of False?

1. Given a vector space V, and its two non-empty subspaces V_1, V_2, then $V_1 \cup V_2$ is a subspace of V
2. Given a vector space V, and its two subspaces V_1, V_2, then $V_1 \cap V_2$ is a subspace of V
3. The set of all linear maps on $\mathbb{R}$ is a subspace of $\mathrm{C}(\mathbb{R})$
4. Given a vector space $\mathbb{R}^n$, for any two distinct norms $\|\cdot\|_1, \|\cdot\|_2$ of $\mathbb{R}^n$, $\|\cdot\| := \sqrt{\|\cdot\|_1 \|\cdot\|_2}$ is also a norm of $\mathbb{R}^n$
5. Given a vector space V, given two norms $\|\cdot\|_1 : V \to \mathbb{R}; \|\cdot\|_2 : \mathbb{R} \to \mathbb{R}$, then the $\|\cdot\| := \|\cdot\|_2 \circ \|\cdot\|_1$ is a norm of V

Have Fun

And

Learn Well![3]

[3]Special acknowledgement to former TA **Zhang Leyang**, who offered plenty of exercises and advice to my recitation class.