VV186: Honors Mathematics

Vector Space & Sequence of Real Functions

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December 2, 2020

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Outline

3.3.1. Definition. A triple $(V, +, \cdot)$ is called a ***real vector space*** (or ***real linear space***) if

1. V is any set;
2. $+\colon V \times V \to V$ is a map (called addition) with the following properties:
   - $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$ (***associativity***),
   - $u + v = v + u$ for all $u, v \in V$ (***commutativity***),
   - there exists an element $e \in V$ such that $v + e = v$ for all $v \in V$ (***existence of a neutral element***),
   - for every $v \in V$ there exists an element $-v \in V$ such that $v + (-v) = e$;
3. $\cdot\colon \mathbb{R} \times V \to V$ is a map (called scalar multiplication) with the following properties:
   - $1 \cdot u = u$ for all $u \in V$,
   - $\lambda \cdot (u + v) = \lambda \cdot u + \lambda \cdot v$ for all $\lambda \in \mathbb{R}$, $u, v \in V$,
   - $(\lambda + \mu) \cdot u = \lambda \cdot u + \mu \cdot u$ for all $\lambda, \mu \in \mathbb{R}$, $u \in V$,
   - $(\lambda\mu) \cdot u = \lambda \cdot (\mu \cdot u)$ for all $\lambda, \mu \in \mathbb{R}$, $u \in V$.

Remark:

- Remember all of the 9 properties in the definition.
- For both of "$+$" and "$\cdot$", the *codomain* is V. i.e. Both operators should map into original set V.
- Distinguish clearly a *complex* and a *real* vector space. It is determined by the *domain* of *scalar multiplication*. (Is $\mathbb{C}^n$ a complex or a real vector space?)
- Some common notation:
  1. $\mathbb{R}^n$
  2. $\mathbb{C}^n$
  3. $\mathcal{P}_n$
  4. $C(\Omega, \mathbb{R})$
  5. $C^k(\Omega, \mathbb{R})$
  6. $C^\infty(\Omega, \mathbb{R})$
  7. ℓ^∞
  8. c_0[1]

[1]See Slide p.340 if you forget them.

Subspace

We defined the *subspace*:

> 3.3.4. Definition. Let $(V, +, \cdot)$ be a real or complex vector space. If $U \subset V$ and $(U, +, \cdot)$ is also a vector space, then we say that $(U, +, \cdot)$ is a ***subspace*** of $(V, +, \cdot)$.

We have a simple way to verify a candidate $(U, +, \cdot)$ is indeed a subspace of $(V, +, \cdot)$, given that $(V, +, \cdot)$ is a vector space:

> 3.3.6. Lemma. Let $(V, +, \cdot)$ be a real (complex) vector space and $U \subset V$. If $u_1 + u_2 \in U$ for $u_1, u_2 \in U$ and $\lambda u \in U$ for all $u \in U$ and $\lambda \in \mathbb{R}$ ($\mathbb{C}$), then $(U, +, \cdot)$ is a subspace of $(V, +, \cdot)$.

True or False? (Give counter-examples or prove.)

1. Given a vector space V, and its two non-empty subspaces V_1, V_2, then $V_1 \cup V_2$ is a subspace of V
2. Given a vector space V, and its two subspaces V_1, V_2, then $V_1 \cap V_2$ is a subspace of V

Normed Vector Space I

Norm is the "generalized length function" in a vector space. We defined it by three properties:

3.3.8. Definition. Let V be a real (complex) vector space. Then a map $\|\cdot\|: V \to \mathbb{R}$ is said to be a ***norm*** if for all $u, v \in V$ and all $\lambda \in \mathbb{R}$ ($\mathbb{C}$),

1. $\|v\| \geq 0$ for all $v \in V$ and $\|v\| = 0$ if and only if $v = 0$,
2. $\|\lambda \cdot v\| = |\lambda| \cdot \|v\|$,
3. $\|u + v\| \leq \|u\| + \|v\|$.

The pair $(V, \|\cdot\|)$ is called a ***normed vector space*** or a ***normed linear space***.

Remark:

- A norm is a (unary) function. So we can naturally investigate into it as considering a function. (e.g. How to prove a norm is continous?)
- Any normed vector space can also be considered as a metric space. (How should we define the metric $\rho(x, y)$ according to $\|\cdot\|$?)
- Some common notation:
  1. $\|x\|_2$ in $\mathbb{R}^n$ (Euclidean norm)
  2. $\|x\|_p$ in $\mathbb{R}^n$ for $p \in \mathbb{N}^+$
  3. $\|x\|_\infty$ in $\mathbb{R}^n$
  4. $\|(a_n)\|_\infty$ in ℓ^∞ or c_0
  5. $\|f\|_\infty$ in $C([a, b])$

True or False? (Give counter-examples or prove.)

1. Given a vector space V, given two norms $\|\cdot\|_1 : V \to \mathbb{R}; \|\cdot\|_2 : \mathbb{R} \to \mathbb{R}$, then the $\|\cdot\| := \|\cdot\|_2 \circ \|\cdot\|_1$ is a norm of V

Pointwise vs. Uniform Convergence

3.4.1. Definition. Let $\Omega \subset \mathbb{R}$ and (f_n) be a sequence of functions $f_n \colon \Omega \to \mathbb{C}$. We say that the sequence (f_n) converges pointwise to the function $f \colon \Omega \to \mathbb{C}$ if

$$\underset{x\in\Omega}{\forall} |f_n(x) - f(x)| \xrightarrow{n\to\infty} 0.$$

If f is the pointwise limit of (f_n), we say that (f_n) converges ***uniformly*** to f on Ω if

$$\sup_{x\in\Omega} |f_n(x) - f(x)| \xrightarrow{n\to\infty} 0.$$

Remark:

- What are the differences between them?
- What are the relations between them?
- Can you come up with some examples to illustrate them?

How to Calculate

Here is a general procedure to find the limit of a function sequence (f_n)

1. Fix each $x \in \Omega$, find the pointwise limit of (f_n), denoted by f.
2. Fix each $k \in \mathbb{N}$, find an explicit expression of $\|f_k(x) - f(x)\|$, or an estimate of it.
3. If 2. $\to 0$ as $k \to \infty$, we have (f_n) converges uniformly to f. Otherwise, the convergence is pointwise but not uniform.

Results That You Should Know

1. A uniform convergent sequence of continous functions converges to a continous function.[2]
2. The metric space $(C([a, b]), \rho)$ is complete, where $\rho(f, g) = \|f - g\|_{\infty}$. [3]

[2] 3.4.3. Theorem. Slides p.353
[3] 3.4.4. Theorem. Slides p.355

Exercise 9.
Define the functions

$$f_n \colon \mathbb{R} \to \mathbb{R}, \qquad f_n(x) = \frac{|x|^n}{1+|x|^n}$$

for $n \in \mathbb{N}$.

i) Find the pointwise limit of (f_n).

ii) Show that the convergence is not uniform on $\mathbb{R}$.

iii) Show that for any $q > 1$ the convergence on $I_q = \{x \in \mathbb{R} \colon |x| \le 1/q\} \cup \{x \in \mathbb{R} \colon |x| \ge q\}$ is uniform.

(1 + 1 + 2 Marks)

i) Fix $x = x_0$,

If $|x_0| < 1$, $\lim\limits_{n\to\infty} |x_0|^n = 0$, $\lim\limits_{n\to\infty} f_n(x_0) = \dfrac{\lim\limits_{n\to\infty} |x_0|^n}{1 + \lim\limits_{n\to\infty} |x_0|^n} = 0.$

If $|x_0| = 1$, $\lim\limits_{n\to\infty} |x_0|^n = 1$, $\lim\limits_{n\to\infty} f_n(x_0) = \dfrac{\lim\limits_{n\to\infty} |x_0|^n}{1 + \lim\limits_{n\to\infty} |x_0|^n} = 1/2.$

If $|x_0| > 1$, $\lim\limits_{n\to\infty} \dfrac{1}{|x_0|^n} = 0$, $\lim\limits_{n\to\infty} f_n(x_0) = \dfrac{1}{1 + \lim\limits_{n\to\infty} \dfrac{1}{|x_0|^n}} = 1.$

In conclusion, the pointwise limit is $f(x) = \begin{cases} 0, & |x| < 1 \\ 1/2, & |x| = 1 \\ 1, & |x| > 1 \end{cases}.$

ii) We then show the convergence is not uniform.

$$\begin{aligned}\|f - f_n\|_\infty &= \sup_{x\in\mathbb{R}} |f(x) - f_n(x)| \\ &\geq |f(3^{-1/n}) - f_n(3^{-1/n})| \\ &= |f_n(3^{-1/n})| \\ &= 1/4 \nrightarrow 0.\end{aligned}$$

Thus, the convergence is not uniform.

iii) Notice that $f_n(x) = 1 - \dfrac{1}{1+|x|^n}$ increases as $|x|$ increases. Similarly, $1 - f_n(x) = \dfrac{1}{1+|x|^n}$ decreases as $|x|$ increases. Since

$$\begin{aligned}
\|f - f_n\|_\infty &= \sup_{x\in I_q} |f(x) - f_n(x)| \\
&= \max\left\{ \sup_{|x|\le 1/q} |f(x) - f_n(x)|, \sup_{|x|\ge q} |f(x) - f_n(x)| \right\} \\
&= \max\left\{ \sup_{|x|\le 1/q} |f_n(x)|, \sup_{|x|\ge q} |1 - f_n(x)| \right\} \\
&= \max\left\{ \frac{|1/q|^n}{1+|1/q|^n}, \frac{1}{1+|q|^n} \right\} \\
&= \frac{1}{1+|q|^n} \xrightarrow{n\to\infty} 0,
\end{aligned}$$

the sequence is uniformly convergent on I_q.

Tips

Some general tips for the exams:

- Do **not** stay up too late tonight. You want to have a clear mind at 8 am. tomorrow morning.
- The sample exams do not contain exercises regarding vector spaces. However, this does not mean you will not encounter this concept in the exam.
- Allocate your time (100mins) wisely in the exam. Finish the easy problems first, focus on tough ones then. The problems are possibly not arranged in the order of difficulty.

KEEP CALM

AND

TRUST MATHEMATICIANS

(and you, of course!)