VV186: Honors Mathematics

Math Foundations

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Outline

Introduction

- Xingjian Zhang
- Junior Student
- Major in ECE (JI) & DS (UMich)
- VV285 TA in SU2020
- Key words:
  1. Conceptual Understanding
  2. Interdisciplinary Spirits
  3. Motivation and Essence

Pay attention to...

Several things I want you to pay attention to:

1. Be interactive. Feel free to interrupt me at any time if you want to ask something or simply make some comments. You are free to discuss with your friend if you want, as long as your discussion is related to the course contents and your voice won't effect other students.
2. Speak everything in English during the RC. This might be hard at the beginning, but you will soon get used to that.
3. "Question everything." Do not pretend to have understood everything. Maths is about strictness, abstraction and generalization. Understanding every basic concept is essential in our course. I will be quite "push" on checking your conceptual understanding. This process will be **annoying, tedious, but rewarding**. So Get prepared.

Each RC will contain some of these four parts:

- Recap of lecture
- Exercise
- Sample solutions to assignment problems
- Extension of lecture
  (marked with "*". e.g. "* Propositional Logic and Predicate Logic")

Apart from regular RC, *big RC* will be held days before mid/final in order to help you prepare the exam better.

RC is **optional**. i.e. You are free to attend none of the RC if you think it is not helpful to you. Generally speaking, I will choose exercise whose difficulty is appropriate for most students. Thus, my RC might not be the right place for you if you want to seek for extremely difficult, exciting and interesting math problems.

I am personally excited in interdisciplinary thinking. So I will also try to give some examples in other math-related subjects. e.g. mechanics, electromagnetics, circuits, computer/data science and etc.

Notation

A *notation* is a system of graphics or symbols, characters and abbreviated expressions, used (for example) in scientific disciplines to represent technical facts and quantities by convention.

Please follow Professor's notations. Keeping consistent in using notations is important because it helps you obtain a clear mind when we have a lot of complex concepts. Additionally, it makes your work more readable (for both your teammates and TAs).

Deduction will be given in assignment/exam if there is abuse/misuse of notations.

Logic Foundation

Test your conceptual understanding:

1. statement
2. negation
3. conjunction, disjunction
4. implication, equivalence
5. tautology, contradiction
6. contraposition (contrapositive)
7. logically equivalent

Ask yourself:

- How are they **defined**?
- What are they essentially? (2,3,4)
- What are their notations, if any?
- What is the difference between "equivalence" and "logically equivalent"?

Remark: In reality, however, we will sometimes live with the ambiguity of equivalence and logically equivalent.

Logical Quantifiers

Test your conceptual understanding:

1. universal quantifier
2. existential quantifier
3. vacuously true

Ask yourself:

- How are they defined?
- What are their notations?

* Propositional Logic and Predicate Logic

Propositional logic is the study of propositions, where a proposition is a statement that is either true or false. Propositional logic may be used to encode simple arguments that are expressed in natural language, and to determine their validity. The validity of an argument may be determined from truth tables, or using inference rules such as modus ponens to establish the conclusion via deductive steps. Predicate logic allows complex facts about the world to be represented, and new facts may be determined via deductive reasoning. Predicate calculus includes predicates, variables and quantifiers, and a predicate is a characteristic or property that the subject of a statement can have.[1]

Without quantifiers, we could only define the propositional logic. This topic will be discussed in details in *VE203: Discrete Mathematics*. Here, we only need a basic background knowledge of logics.

[1] Propositional and Predicate Logic - Gerard O'Regan

Two simple exercises:

1. Let A, B, C be statements. Use truth table to prove that

$$(A \vee B) \wedge C \equiv (A \wedge C) \vee (B \wedge C).$$

2. Interpret the following definition using English:

   The real or complex sequence $(a_n) : \Omega \to X, \Omega \subset \mathbb{N}$, $X = \mathbb{R}$ or $\mathbb{C}$, is said to *converge* with limit $a \in X$ if

$$\underset{\varepsilon>0}{\forall} \quad \underset{N\in\mathbb{N}}{\exists} \quad \underset{n>N}{\forall} \quad |a_n - a| < \varepsilon$$

   Find the negation of above predicate. *(If its negation is true, we say that the sequence is not convergent with a.)*

Set Foundation

Test your conceptual understanding:

1. set
2. empty set
3. (proper) subset
4. power set
5. cardinality

Ask yourself:

- How are they defined?
- How to prove two sets A, B are equal? (at least two ways)
- Why empty set $\varnothing$ is a subset of any set?

Set Foundation

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- Why empty set $\varnothing$ is a subset of any set?

Remark: The sets are defined through predicates. Correspondingly, the operations on sets are defined through operations on predicates.

Set Operation

The *union*, *intersection*, and *difference* of sets is defined by

$$A \cup B := \{x : P_1(x) \vee P_2(x)\}$$
$$A \cap B := \{x : P_1(x) \wedge P_2(x)\}$$
$$A \backslash B := \{x : P_1(x) \wedge (\neg P_2(x))\}$$

We can then further define the *complement* and *disjoint*.

Remark: From the definition above, we can find an analogy between set operation and logic operation. See the following exercise.

Exercise

Prove

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

$$(A \cup B) \backslash C = (A \backslash C) \cup (B \backslash C)$$

in two ways:

- by definition
- by "subset"

Other Notation about Set Operations

$$\bigcup_{k=0}^{n} A_k := A_0 \cup A_1 \cup A_2 \cup \cdots \cup A_n$$

$$\bigcap_{k=0}^{n} A_k := A_0 \cap A_1 \cap A_2 \cap \cdots \cap A_n$$

How to properly generalize the notation to $n = \infty$?

Ordered Pairs

A set does not contain the information about the order of its elements. However, sometimes it is convenient or necessary to have such an ordering. This is achieved by defining an *ordered pair*, denoted by

$$(a, b)$$

If A, B are sets and $a \in A, b \in B$, then we denote the set of all ordered pairs by

$$A \times B := \{(a, b) : a \in A, b \in B\}$$

$A \times B$ is called the *cartesian product* of A and B. we may abbreviate cartesian product using exponents. (What does an element in $\mathbb{R}^3$ represents?)

Remark: * Using ordered pairs, one can then define *ordered triples* and, more generally, *ordered n-tuples*. For elements $x_0, ..., x_n$, define

$$(x_0, ..., x_n) = (x_0, (x_1, ..., x_n))$$

This is an example of a *recursive definition*. Correspondingly, one can define the *n-fold cartesian product* of sets.

Some Insight I

Remark: * Moreover, the natural numbers can be constructed rigorously from set theory using empty set $\varnothing$ and successor operation $S(x) = x \cup \{x\}$. In *Zermelo–Fraenkel (ZF) set theory*, the natural numbers are defined recursively like this:

$$\begin{aligned} &0 := \varnothing \\ &1 := S(\varnothing) = \{\varnothing\} \\ &2 := S(S(\varnothing)) = \{\varnothing, \{\varnothing\}\} \end{aligned}$$

Then we use set $\mathbb{N}_{def}$ to interprets the natural numbers where

$$\mathbb{N}_{def} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \dots\}$$

The above is just a brief introduction of what have been mentioned in the lecture. Do not worry if you cannot get the points because it will be discussed in details in VE203.

Some Insight II

The result comes from a strong historical desire to find a list of axioms from which all mathematical truths could be proved in a first-order system. With the advent of calculus, people began to prove all types of completely incorrect statements by their cavalier manipulation of the infinite. This fact, coupled with the discovery of Russell's Paradox, led to people to believe that there should be a more systematic way of theorem-proving. It was later proved impossible by Godel to find such a comprehensive system, even if infinitely many axioms were allowed. Still, most results of interest can be proved from the set-theoretic construction of mathematics known as ZFC set theory. One of the axioms is the existence of the natural numbers. **It is generally accepted that one should take as few axioms as possible. Also, it actually turns out proof-wise to be easier to construct the integers than assuming their existence, as it would be more difficult to define the addition and multiplication operations on them.** *These are the two largest reasons why. We actually construct the natural numbers in a sense as well.*[2]

The order of introduction to various concepts in vv186-vv285-vv286 is not arranged in a way that from easy to hard, simple to complex. Instead, the first priority is always **strictness**. So it is a very common situation where we use a quite complex structure to represent some seemingly naive concepts. This might not be intuitive, but it is going to be useful and rigorous.

Some examples:

- Ordered pairs
- Natural numbers
- Exponential functions
- Triangular functions
- Matrices
- ...

[2] Why we construct natural numbers by set theory? - math.stackexchange

Tips on Assignments

- Please read the introduction on P.1 of Assignment 1 carefully.
- Discuss with your teammates. If you haven't figure out a solution, learn from your teammate. If you know how to do it, share your ideas with your teammates. Both will help you learn better. (The later way helps you consolidate your knowledge. See Learning From the Feynman Technique)
- The knowledge we learn in the first two weeks are very fundamental. And the real interesting part (*Real Functions, Convergence and Continuity*) is just coming up!

Have Fun

And

Learn Well![3]

[3]Special acknowledgement to former TA Zhang Leyang, who offered many exercises and advice to my recitation class.