VV186: Honors Mathematics

Integral

Xingjian Zhang

Univerity of Michigan-Shanghai Jiao Tong University Joint Institute

December 5, 2020

Outline

Hierarchy of Integrals

1. Integral of step functions
2. Regulated integral
3. Darboux integral & Riemann Integral
4. Lebesgue integral
5. Gauge integral (Henstock-Kurzweil integral)

Remark: We only study the first three hierarchies. Make sure you remember their definitions and properties.

4.1.13. Definition. A function $f \in L^\infty([a,b])$ is said to be ***regulated*** if for any $\varepsilon > 0$ there exists a step function φ such that

$$\sup_{x\in[a,b]} |f(x) - \varphi(x)| < \varepsilon. \qquad (4.1.6)$$

Equivalently, a regulated function f is a function such that there exists a sequence of step functions (φ_n) which converges uniformly to f. (To see this, we simply define φ_n as the step function satisfying (4.1.6) with $\varepsilon = 1/n$.)

Remark:

- The *equivalent description* of 4.1.13. Definition is more intuitive: We seek for a sequence of step functions that converge **uniformly** to the target function.
- The continous a.e.[1] functions on a closed interval are regulated. i.e. $\mathrm{PC}([a,b]) \subset \mathrm{Reg}([a,b])$

[1] "a.e." is the abbreviate of "almost everywhere".

4.1.18. Definition and Theorem. Let $f \in \operatorname{Reg}([a,b])$ and (φ_n) a sequence in $\operatorname{Step}([a,b])$ converging uniformly to f. Then the ***regulated integral*** of f, defined by

$$\int_a^b f := \lim_{n\to\infty} \int_a^b \varphi_n \tag{4.1.8}$$

exists and does not depend on the choice of (φ_n).

4.1.20. Theorem. Let (f_n) be a sequence of regulated functions such that $f_n \to f$ uniformly for some $f \in L^\infty$, i.e., $\|f - f_n\|_\infty \to 0$. Then f is regulated and

$$\int_a^b f_n \xrightarrow{n\to\infty} \int_a^b f. \tag{4.1.11}$$

Consider the function

$$f\colon [0,1] \to \mathbb{R}, \qquad f(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for some } n \in \mathbb{N} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Is the function regulated?

Exercise

True or False?

1. $f(x) = x^3 + e^x$ on $[-1, 1]$ is regulated.
2. Since $f(x) = x^2$ is regulated on each $[-n, n]$, where $n \in \mathbb{N}$, f is regulated on $\mathbb{R}$ because we can let $n \to \infty$
3. A continuous function is piecewise continuous
4. Let f, g be two real-valued function defined on $[0, 1]$. Furthermore, assume $f - g = x$, then the equation $\int_0^1 (f - g) = \int_0^1 f - \int_0^1 g = \frac{1}{2}$ holds.
5. Let $f \in \text{Reg}([0, 1])$, let g be a real-valued function. Then $f \circ g$ is regulated.

Darboux Integral

4.1.22. Definition. Let $[a, b] \subset \mathbb{R}$ be a closed interval and f a bounded real function on $[a, b]$. Let $\mathscr{U}_f$ denote the set of all step functions u on $[a, b]$ such that $u \geq f$ and $\mathscr{L}_f$ the set of all step functions v on $[a, b]$ such that $v \leq f$. The function f is then said to be ***Darboux-integrable*** if

$$\underline{I}(f) = \sup_{v \in \mathscr{L}_f} \int_a^b v = \inf_{u \in \mathscr{U}_f} \int_a^b u = \overline{I}(f).$$

In this case, the ***Darboux integral of*** f, $\int_a^b f$, is defined to be this common value.

Question: True or false?
If $\int_a^b |f|$ is Darboux integrable, $\int_a^b f$ is Darboux integrable.

Exercise

Consider the function

$$f : [0,1] \to \mathbb{R}, \qquad f(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for some } n \in \mathbb{N} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Is the function Darboux integrable?

4.1.28. Definition. Let $[a, b] \subset \mathbb{R}$ be a closed interval and f a bounded real function on $[a, b]$. Then f is Riemann-integrable with integral

$$\int_a^b f \in \mathbb{R}$$

if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any tagged partition (P, Ξ) on $[a, b]$ with mesh size $m(P) < \delta$

$$\left| \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) - \int_a^b f \right| < \varepsilon.$$

Definition 4.1.28 is quite difficult to work with in practice, so it is fortunate that the following result holds:

4.1.29. Theorem. A bounded real function is Riemann-integrable on $[a, b]$ if and only if it is Darboux-integrable. Moreover, the values of the integrals coincide.

Darboux Integral vs. Riemann Integral

Describe in words the difference between **Darboux Integral** and **Riemann Integral**. How are these two concepts different? If you can, state some properties that they have in common or that serve to differentiate them from each other. Is one of them also always an example of the other? Give examples.

This exercise is left for you!

4.2.1. Theorem. Let $f\colon [a,b] \to \mathbb{C}$ be continuous and set

$$F\colon [a,b] \to \mathbb{C}, \qquad F(x) := \int_a^x f.$$

Then F is differentiable on (a,b) and

$$F'(x) = f(x), \qquad x \in (a,b).$$

The Fundamental Theorem of Calculus

4.2.2. Definition. Let $\Omega \subset \mathbb{R}$ be an open set and $f : \Omega \to \mathbb{C}$. We say that $F : \Omega \to \mathbb{C}$ is a ***primitive***, ***indefinite integral*** or ***anti-derivative*** of f if $F' = f$. We write

$$F = \int f = \int f(t)\,dt.$$

4.2.3. Lemma. Let $f \in C([a,b], \mathbb{C})$ and F be a primitive of f. Then

$$\int_a^b f = F(b) - F(a).$$

Remark: This result shows that we can regard integration as "inverse differentiation". We can then calculate various integrals simply by guessing their primitives.

Question: Since $\int_{-1}^{1} \frac{1}{x^2+1} = \arctan x\big|_{-1}^{1} = \frac{\pi}{2}$ but $\int_{-1}^{1} \frac{1}{x^2+1} = -\arctan \frac{1}{x}\Big|_{-1}^{1} = -\frac{\pi}{2}$. Where is wrong?

Exercise

Calculate the derivative of $g : [a, b] \to \mathbb{R}$,

$$g(x) = \int_{p(x)}^{q(x)} f,$$

where f, p, q are differentiable on (a, b).

4.2.4. Substitution Rule. Let $f \in \text{Reg}([\alpha, \beta])$ and $g\colon [a, b] \to [\alpha, \beta]$ continuously differentiable. Then

$$\int_a^b (f \circ g)(x) g'(x)\, dx = \int_{g(a)}^{g(b)} f(y)\, dy.$$

4.2.5. Remark. Using the same proof, we see that the substitution rule can be applied to indefinite integrals (primitives):

$$\int (f \circ g)(x) g'(x)\, dx = \int f(y)\, dy \Big|_{y=g(x)}.$$

Remark: We can, to some extent, abuse the Leibniz notation to memorize substitution rule. You are free to use it in the exam.

Exercise

Let $f, g \in C([-a, a])$ such that g is even function and $f(x) + f(-x) = A$. Prove

$$\int_{-a}^{a} fg = A \int_0^a g.$$

Calculate

$$\int \frac{x}{3x^2 + 6x + 10}$$

Calculate

$$\int \frac{1}{\sin(x)\cos^3(x)}$$

4.2.8. Theorem. Let $f, g \in C^1([a,b], \mathbb{C})$. Then

$$\int_a^b f'(x)g(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)\,dx.$$

Exercise

Calculate

$$\int \left(3t + t^2\right) \sin(2t) dt$$

Improper Integrals

An integral

$$\int_a^b f(t)\,dt$$

is called "improper" if

- the domain of integration is unbounded i.e., $a = -\infty$ or $b = \infty$ and/or
- the integrand f is unbounded on (a, b) or otherwise not regulated.

Remark: See detailed definitions on Slides 562. We well defined it using limit.

Exercise

Determine if the following integral converges or diverges. If the integral converges determine its value.

$$\int_0^4 \frac{x}{x^2-9}\,dx$$

$$\int_{-\infty}^0 \frac{e^{\frac{1}{x}}}{x^2}\,dx$$

Cauchy Criterion for Functions

4.2.13. Cauchy Criterion. Let $a \in \mathbb{R}$ and $f : [a, \infty) \to \mathbb{R}$ be integrable on every interval $[a, x]$, $x \in \mathbb{R}$. The improper integral

$$\int_a^\infty f(x)\,dx$$

converges if and only if

$$\underset{\varepsilon>0}{\forall}\ \underset{R>0}{\exists}\ \underset{x,y>R}{\forall} \quad \left|\int_x^y f(t)\,dt\right| < \varepsilon.$$

Results

- 4.2.16. Comparison Test.
- 4.3.1. Integral Test.

Exercise

Use the Comparison Test to determine if the following integral converges or diverges.

$$\int_1^\infty \frac{z-1}{z^4+2z^2}\,dz$$

4.3.5. Taylor's Theorem. Let $I \subset \mathbb{R}$ an open interval and $f \in C^k(I)$. Let $x \in I$ and $y \in \mathbb{R}$ such that $x + y \in I$. Then for all $p \leq k$,

$$f(x+y) = f(x) + \frac{1}{1!}f'(x)y + \cdots + \frac{1}{(p-1)!}f^{(p-1)}(x)y^{p-1} + R_p \quad (4.3.5)$$

with the remainder term

$$R_p := \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} f^{(p)}(x+ty)y^p \, dt.$$

Remark: Memorize the remainder term. (so-called Lagrange form)

We will call

$$T_{f;p,x_0}(x) := f(x) - R_{p+1} = \sum_{n=0}^{p} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \tag{4.3.9}$$

the ***Taylor polynomial of degree*** p ***of*** f ***at*** x_0. The series

$$\lim_{p\to\infty} T_{f;p,x_0}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \tag{4.3.10}$$

is called the ***Taylor series of*** f ***at*** x_0.

4.3.6. Definition. Let $I \subset \mathbb{R}$ be an open interval and $f \in C^\infty(I)$. We say that f is ***real-analytic*** or just ***analytic*** at $x_0 \in I$ if there exists a neighborhood $B_\varepsilon(x_0) = (x_0 - \varepsilon, x_0 + \varepsilon) \subset I$ such that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

for all $x \in B_\varepsilon(x_0)$.

Remarks

- *Taylor's series* is extremely useful in engineering fields because it offers a fine approximation of the function's behavior near some point. For example, oscillation analysis.
- However, in reality, since that we can only use polynomials of finite degrees, the taylor expansion degenerates rapidly when approaching to the radius of convergence.
- Some functions are not analytic. i.e. cannot be approximated by Taylor's theorem. But these weird functions never appear in engineering applications.

Euler Gamma Function

The ***Euler gamma function*** is defined through an improper integral,

$$\Gamma : \mathbb{R}_+ \to \mathbb{R}, \qquad \Gamma(t) := \int_0^\infty z^{t-1} e^{-z}\, dz, \qquad t > 0. \tag{4.2.8}$$

Euler Gamma Function is the unique *analytic continuation* of factorial function $(\cdot - 1)!$. It is log-convex on $\mathbb{R}^+$ (increases faster than exponential function). Moreover, it is differentiable.

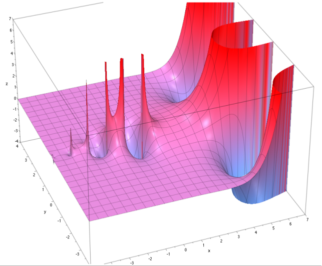


Figure: Real Part Plot

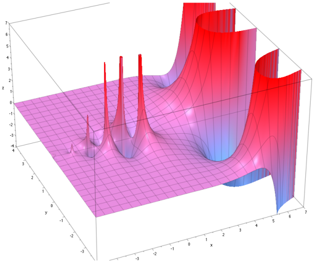
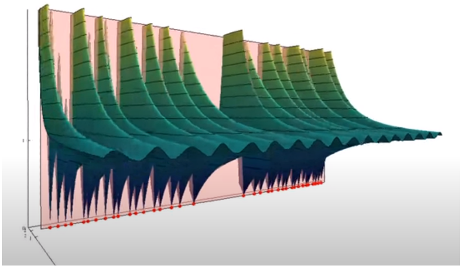


Figure: Imaginary Part Plot

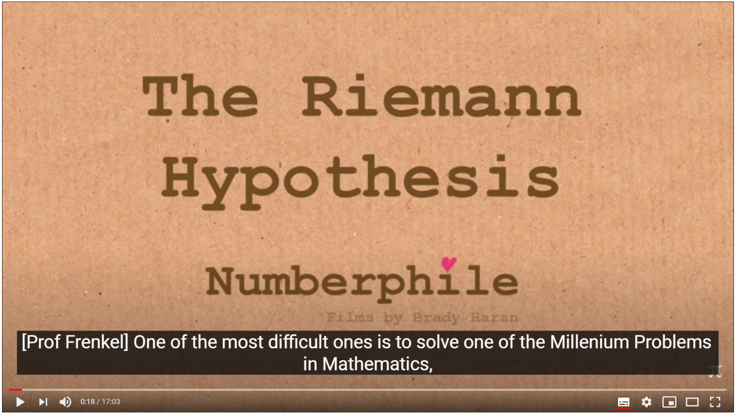
The ***Riemann zeta function*** is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Note that the right-hand side is just the p-series, so we know that $\zeta(s)$ exists for $s > 1$. However, ζ can be extended to a function defined on $\mathbb{R} \setminus \{1\}$. This extension is analogous to, but more complicated than, the extension of the Euler Gamma function (see Slide 579).



Here is a very good video in simple terms illustrating the Riemann Zeta functions, as well as the famous *Riemann Hypothesis*:



[Link] *Riemann Hypothesis - Numberphile*

Go to watch that when you are free!

Abel Sum

We can hence generalize the concept of summability of sequences, by defining the sum to be the limit of the power series; we define the ***Abel sum***

$$\mathrm{A} - \sum_{n=0}^{\infty} a_n := \lim_{x \nearrow 1} \sum_{n=0}^{\infty} a_n x^n$$

for any complex sequence (a_n) for which the right-hand side exists.

Tips

- Practice makes perfect. Here is a website with lots of typical integral problems as well as detailed solutions. You can pick up some as exercises if you want. Paul's Online Notes
- Having taken two vv186 exams, (I believe) you have enough experience and confidence to take the final. The final exam is similar to the two midterm exams in term of duration, compositions.
- You have made your first step into the fascinating and beautiful math world. Congratulations! Helping you guys see the wonderfulness of math is my motivation to be a TA, and I am happy to realize that many of you start to do so rather than bearing its tediousness. I appreciate all of your hard work in this tough semester, and am honored to being your TA. Wish you enjoy this journey and have a good grade!

Forever

Have Fun

And

Learn Well![2]

[2]Special acknowledgement to former TA **Zhang Leyang**, who offered plenty of exercises and advice to my recitation class.