VV186: Honors Mathematics

Vector Space & Sequence of Real Functions

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Outline

1. Vector Space

3.3.1. Definition. A triple $(V, +, \cdot)$ is called a ***real vector space*** (or ***real linear space***) if

1. V is any set;
2. $+\colon V \times V \to V$ is a map (called addition) with the following properties:
   - $(u+v)+w = u+(v+w)$ for all $u, v, w \in V$ (***associativity***),
   - $u+v = v+u$ for all $u, v \in V$ (***commutativity***),
   - there exists an element $e \in V$ such that $v+e = v$ for all $v \in V$ (***existence of a neutral element***),
   - for every $v \in V$ there exists an element $-v \in V$ such that $v+(-v) = e$;
3. $\cdot\colon \mathbb{R} \times V \to V$ is a map (called scalar multiplication) with the following properties:
   - $1 \cdot u = u$ for all $u \in V$,
   - $\lambda \cdot (u+v) = \lambda \cdot u + \lambda \cdot v$ for all $\lambda \in \mathbb{R}$, $u, v \in V$,
   - $(\lambda + \mu) \cdot u = \lambda \cdot u + \mu \cdot u$ for all $\lambda, \mu \in \mathbb{R}$, $u \in V$,
   - $(\lambda\mu) \cdot u = \lambda \cdot (\mu \cdot u)$ for all $\lambda, \mu \in \mathbb{R}$, $u \in V$.

Remark:

- Remember all of the 9 properties in the definition.
- For both of "$+$" and "$\cdot$", the *codomain* is V. i.e. Both operators should map into original set V.
- Distinguish clearly a *complex* and a *real* vector space. It is determined by the *domain* of *scalar multiplication*. (Is $\mathbb{C}^n$ a complex or a real vector space?)
- Some common notation:
  1. $\mathbb{R}^n$
  2. $\mathbb{C}^n$
  3. $\mathcal{P}_n$
  4. $C(\Omega, \mathbb{R})$
  5. $C^k(\Omega, \mathbb{R})$
  6. $C^\infty(\Omega, \mathbb{R})$
  7. ℓ^∞
  8. c_0[1]

[1] See Slide p.340 if you forget them.

We defined the *subspace*:

> 3.3.4. Definition. Let $(V, +, \cdot)$ be a real or complex vector space. If $U \subset V$ and $(U, +, \cdot)$ is also a vector space, then we say that $(U, +, \cdot)$ is a ***subspace*** of $(V, +, \cdot)$.

We have a simple way to verify a candidate $(U, +, \cdot)$ is indeed a subspace of $(V, +, \cdot)$, given that $(V, +, \cdot)$ is a vector space:

> 3.3.6. Lemma. Let $(V, +, \cdot)$ be a real (complex) vector space and $U \subset V$. If $u_1 + u_2 \in U$ for $u_1, u_2 \in U$ and $\lambda u \in U$ for all $u \in U$ and $\lambda \in \mathbb{R}$ $(\mathbb{C})$, then $(U, +, \cdot)$ is a subspace of $(V, +, \cdot)$.

Normed Vector Space I

Norm is the "generalized length function" in a vector space. We defined it by three properties:

3.3.8. Definition. Let V be a real (complex) vector space. Then a map $\|\cdot\|\colon V \to \mathbb{R}$ is said to be a ***norm*** if for all $u, v \in V$ and all $\lambda \in \mathbb{R}$ ($\mathbb{C}$),

1. $\|v\| \geq 0$ for all $v \in V$ and $\|v\| = 0$ if and only if $v = 0$,
2. $\|\lambda \cdot v\| = |\lambda| \cdot \|v\|$,
3. $\|u + v\| \leq \|u\| + \|v\|$.

The pair $(V, \|\cdot\|)$ is called a ***normed vector space*** or a ***normed linear space***.

Remark:

- A norm is a (unary) function. So we can naturally investigate into it as considering a function. (e.g. How to prove a norm is continous?)
- Any normed vector space can also be considered as a metric space. (How should we define the metric $\rho(x, y)$ according to $\|\cdot\|$?)
- Some common notation:

1. $\|x\|_2$ in $\mathbb{R}^n$ (Euclidean norm)
2. $\|x\|_p$ in $\mathbb{R}^n$ for $p \in \mathbb{N}^+$
3. $\|x\|_\infty$ in $\mathbb{R}^n$
4. $\|(a_n)\|_\infty$ in ℓ^∞ or c_0
5. $\|f\|_\infty$ in $C([a, b])$

End

Have Fun

And

Learn Well!