# VV186: Honors Mathematics

## Series

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# RC Policy

Several things I want you to pay attention to:

1. Be interactive. Feel free to interrupt me at any time if you want to ask something or simply make some comments. You are free to discuss with your friend if you want, as long as your discussion is related to the course contents and your voice won't effect other students.
2. Speak everything in English during the RC. This might be hard at the beginning, but you will soon get used to that.
3. "Question everything." Do not pretend to have understood everything. Maths is about strictness, abstraction and generalization. Understanding every basic concept is essential in our course. I will be quite "push" on checking your conceptual understanding. This process will be **annoying, tedious, but rewarding**. So Get prepared.

# Outline

# Sequences in Vector Space

Recall how we define convergent sequence in a metric space. Since that every normed vector space is also a metric space, we can then define convergent sequence in it.

In the following, we assume that $(V, \|\cdot\|)$ is a normed vector space.

A ***sequence in a vector space*** $V$ is a map $(a_n)\colon \mathbb{N} \to V$. We say that $(a_n)$ converges to $a \in V$ if

$$\underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{n>N}{\forall} : \|a_n - a\| < \varepsilon.$$

# Sequence of Functions

We now consider two kinds of convergence for functions.

1. ***pointwise convergence***: For every $x \in [-1, 1]$,

$$f_n(x) \xrightarrow{n\to\infty} f(x) \qquad :\Leftrightarrow \qquad |f_n(x) - f(x)| \xrightarrow{n\to\infty} 0$$

2. ***uniform convergence***: Each $f_n$ is an element of the normed vector space $C([-1, 1])$, and so is $f$. Then

$$f_n \xrightarrow{n\to\infty} f \qquad :\Leftrightarrow \qquad \|f_n - f\|_\infty \xrightarrow{n\to\infty} 0.$$

**Remark:**

- Notice that the second definition is more natural in the sense that we consider the convergence of function in a normed vector space of **functions**. (What is this vector space?)
- What is the difference between the pointwise and uniform convergence of functions ...
- Uniform convergence $\Rightarrow$ pointwise convergence. (Why?) If both limit exist,

# Recall...

Describe in words the difference between **pointwise convergence of function sequence** and **uniform convergence of function sequence**. How are these two concepts different? If you can, state some properties that they have in common or that serve to differentiate them from each other. Is one of them also always an example of the other? Give examples.

This exercise is left for you!

# How to Calculate

Here is a common procedure to find the limit of a function sequence $(f_n)$

1. Fix each $x \in \Omega$, find the pointwise limit of $(f_n)$, denoted by $f$.
2. Fix each $k \in \mathbb{N}$, find an explicit expression of $\|f_k(x) - f(x)\|$, or an estimate of it.
3. If 2. $\to 0$ as $k \to \infty$, we have $(f_n)$ converges uniformly to $f$. Otherwise, the convergence is pointwise but not uniform.

# Exercise

Find the pointwise limit of the sequence of functions $(f_n)$ on $\mathbb{R}$, where

$$f_n(x) = \frac{x^2 + nx}{n}.$$

Is this a uniform convergence?

# Completeness of Function Space

The metric space $(C([a,b]),\rho)$ is complete, where $\rho(f,g) = \|f-g\|_\infty$.

A uniformly convergent sequence of continuous functions will always converge to a continuous function.

3.4.3. Theorem. Let $[a, b] \subset \mathbb{R}$ be a closed interval. Let $(f_n)$ be a sequence of continuous functions defined on $[a, b]$ such that $f_n(x)$ converges to some $f(x) \in \mathbb{R}$ as $n \to \infty$ for every $x \in [a, b]$. If the sequence $(f_n)$ converges uniformly to the thereby defined function $f : [a, b] \to \mathbb{R}$, then $f$ is continuous.

# Definition of Series

3.5.1. Definition. Let $(a_n)$ be a sequence in a normed vector space $(V, \|\cdot\|)$. Then we say that $(a_n)$ is ***summable*** with sum $s \in V$ if

$$\lim_{n\to\infty} s_n = s, \qquad s_n := \sum_{k=0}^{n} a_k.$$

We call $s_n$ the ***nth partial sum*** of $(a_n)$. We use the notation

$$\sum_{k=0}^{\infty} a_k \qquad \text{or simply} \qquad \sum a_k \tag{3.5.1}$$

**Remark:** The word "series" has two meanings:

1. the limit of $(s_n)$.
2. the procedure of summing the sequence $(a_n)$.

Notice that we define series in general vector space, so it is natural to investigate into a series of matrices and etc. However, we will focus on real sequences and function sequences in VV186.

# Terminology

For a sequence $(a_n)$ in a normed vector space $(V, \|\cdot\|)$. And define $s_n := \sum_{k=0}^{n} a_k$. Then the following two arguments are equivalent:

1. The sequence $(a_n)$ is summable.
2. The sequence $(s_n)$ is convergent.
3. The series $\sum_{k=0}^{\infty} a_k$ converges.

Think about the difference in description.

# The limit of $(s_n)$

It is easier to deduce whether some sequence is summable than finding the concrete sum. This is analogous to finding whether a sequence is convergent. Why? Because a series is the limit of $(s_n)$, a special sequence.

For example,

- $(a_n) = \dfrac{1}{n^2}$
- $(a_n) = \dfrac{4^n}{n!}$
- ...

Therefore, we develop several test techniques to help us deduce whether a sequence is summable. Before that, let's discuss the *Cauchy criterion* and *absolute convergence* first, which are fundamental tools in the field of series.

3.5.4. Cauchy Criterion. Let $\sum a_k$ be a series in a **complete** vector space $(V, \|\cdot\|)$. Then

$$
\begin{aligned}
\sum a_k \text{ converges} \quad &\Leftrightarrow \quad (s_n)_{n\in\mathbb{N}} \text{ converges, } s_n = \sum_{k=0}^{n} a_k \\
&\Leftrightarrow \quad (s_n) \text{ is Cauchy} \\
&\Leftrightarrow \quad \underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{m>n>N}{\forall}\ \|s_m - s_n\| < \varepsilon \\
&\Leftrightarrow \quad \underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{m>n>N}{\forall}\ \left\| \sum_{k=n+1}^{m} a_k \right\| < \varepsilon
\end{aligned}
$$

**Remark:** Intuitively speaking, cauchy criterion requires a summable series in a complete space to have a sufficiently "flat" tail. It follows several useful corollaries:

1. If summable, $a_k \to 0$ as $k \to \infty$. (Why? What is its contrapositive?)
2. If summable, $A_n := \sum_{k=n}^{\infty} a_k \to 0$ as $n \to \infty$. (Why?)

# Absolute Convergence

**3.5.11. Definition.** A series $\sum a_k$ in a normed vector space $(V, \|\cdot\|)$ is called ***absolutely convergent*** if $\sum \|a_k\|$ converges.

A sequence $(a_k)$ in a normed vector space $(V, \|\cdot\|)$ is said to be ***absolutely summable*** if $\sum a_k$ converges absolutely.

**3.5.12. Theorem.** An absolutely convergent series $\sum a_k$ in a **complete** vector space $(V, \|\cdot\|)$ is convergent.

**Remark:** Absolute convergence of series is a strong requirement of summable. We actually do not need absolute convergence to let a sequence summable. If this is the case, we say the series is *conditionally convergent*.

However, the absolute convergence turns out to be useful because usually a non-negative sequence is easier to deal with.

# Completeness

It is worth noting that the previous two theorems (3.5.4 & 3.5.12) require the vector space $(V, \|\cdot\|)$ to be complete. If you are going to use them in your coursework, the first step is prove the completeness. This is often easy to show but indispensable.

# Infinite Triangle Inequality

3.5.14. Lemma. Let $(V, \|\cdot\|)$ be a complete normed vector space and $\sum a_k$ an absolutely convergent series. Then

$$\left\| \sum_{k=0}^{\infty} a_k \right\| \le \sum_{k=0}^{\infty} \|a_k\|. \tag{3.5.5}$$

Proof.

The triangle inequality yields, for any $n \in \mathbb{N}$,

$$\left\| \sum_{k=0}^{n} a_k \right\| \le \sum_{k=0}^{n} \|a_k\| \le \sum_{k=0}^{\infty} \|a_k\|. \tag{3.5.6}$$

Since the series converges absolutely, it also converges and

$$\left\| \sum_{k=0}^{\infty} a_k \right\| = \left\| \lim_{n\to\infty} \sum_{k=0}^{n} a_k \right\| = \lim_{n\to\infty} \left\| \sum_{k=0}^{n} a_k \right\|$$

since the norm is continuous (i.e., $b_n \to b$ implies $\|b_n\| \to \|b\|$. Why?).
Since the limit exists, the estimate (3.5.6) implies (3.5.5). $\square$

Does the last equality trivially hold? No! How can we show that holds given the norm is continous?

# Tests of Summable Sequences

For all of the following tests, if the domain of discourse is real sequences, we require the real sequence to be (strictly) positive. Recall that in complete space, absolute convergence implies convergence of series.

1. 3.5.9. Convergence of p-Series
2. 3.5.15. Comparison test & Contrapositive
3. 3.5.19. WeierstraßM-test (proof: show pointwise convergence by comparison test, and prove the convergence is uniform)
4. 3.5.22. Root test (proof: comparison test)
5. 3.5.26. Root test using limits (proof: root test)
6. 3.5.28. Ratio test (proof: comparison test)
7. 3.5.30. Ratio test using limits (proof: similar with 3.5.26)
8. 3.5.31. Ratio comparison test (proof: comparison test)
9. 3.5.32. Raabe's test [finer version of ration test] (proof: Bernoulli's inequality & ration comparison test)
10. 3.5.38. Leibniz Theorem.

**Remark:** For 4. and 6., we use upper/lower limits in the tests. This can be quite useful when the limit of the target sequence exists. Why?

# Exercise - Straightforward

Please determine whether the following series converge or not.

1. $\sum \dfrac{4n(n+2)!}{(2n)!}$
2. $\sum \dfrac{\sin(\alpha n)}{n^2}, \quad \alpha \neq 0$
3. $\sum \dfrac{(2n)!}{4^n (n+1)! n!}$
4. $\sum \dfrac{1}{(\ln n)^{\ln n}}$

Prove that if $(a_n)$ is a non-negative non-summable sequence. Show that

$$\sum \frac{a_n}{1+a_n}$$

diverges.[1]

[1] Spivak *Calculus* p.463

**3.5.40. Theorem.** Let $\sum a_k$ and $\sum b_k$ be absolutely convergent series. Then the ***Cauchy product*** $\sum c_k$ given by

$$c_k := \sum_{i+j=k} a_i b_j$$

converges absolutely and $\sum c_k = \left(\sum a_k\right)\left(\sum b_k\right)$.

**3.5.41. Remark.** If $a = (a_k)$ and $b = (b_k)$ are two absolutely summable sequences, the sequence

$$a * b := (c_k), \qquad c_k := \sum_{i+j=k} a_i b_j,$$

is called the ***convolution*** of $a$ and $b$.

**Remark:** This kind of convolution is called the *discrete* convolution. We will learn *continous* convolution in VV286.

# * Discrete Convolution in Engineering Field I

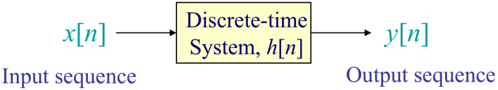


Figure: Discrete Time System

In the field of *Signal Processing*, we can fully characterize a *discrete linear time-invariant system* using a discrete function (essentially a sequence) $h[n]$, called the *impulse response* of this system. To be more precise, the output signal $y$ and the input signal $x$ has the following relation:

$$y[n] = h[n] * x[n]$$

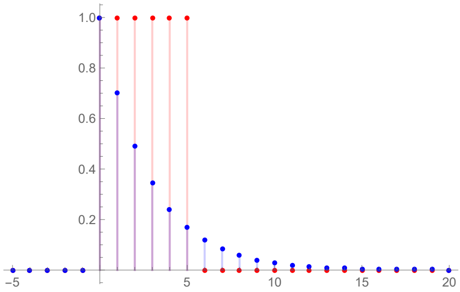


Figure: $h[n]$ and $x[n]$

In this example, our impulse function $h[n]$ is simply a rectangular function. Our input signal $x[n]$ is a decaying signal. Can you imagine the output signal?

# * Discrete Convolution in Engineering Field III

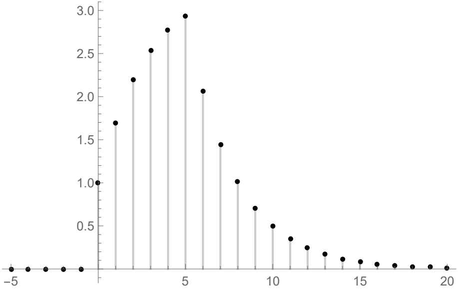


Figure: $y[n]$

Think about how the system modifies the input signal. This process will be discussed in detail in *VE216: Signal and System*, a very fundamental engineering discipline.

Have Fun

And

Learn Well![2]

[2] Special acknowledgement to former TA **Zhang Leyang**, who offered plenty of exercises and advice to my recitation class.