VV186: Honors Mathematics

Sequence & Real Functions

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Univerity of Michigan-Shanghai Jiao Tong University Joint Institute

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JOINT INSTITUTE
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RC Policy

Several things I want you to pay attention to:

1. Be interactive. Feel free to interrupt me at any time if you want to ask something or simply make some comments. You are free to discuss with your friend if you want, as long as your discussion is related to the course contents and your voice won't effect other students.

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2. Speak everything in English during the RC. This might be hard at the beginning, but you will soon get used to that.
3. "Question everything." Do not pretend to have understood everything. Maths is about strictness, abstraction and generalization. Understanding every basic concept is essential in our course. I will be quite "push" on checking your conceptual understanding. This process will be **annoying, tedious, but rewarding**. So Get prepared.

Outline

Assignment 2

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2. Come to OH if you have further questions.

2.2.36. Definition. Let M be a set. A map $\varrho\colon M \times M \to \mathbb{R}$ is called a ***metric*** if

(i) $\varrho(x, y) \geq 0$ for all $x, y \in M$ and $\varrho(x, y) = 0$ if and only if $x = y$.
(ii) $\varrho(x, y) = \varrho(y, x)$
(iii) $\varrho(x, z) \leq \varrho(x, y) + \varrho(y, z)$.

The pair (M, ϱ) is then called a ***metric space***.

Remark: Metric space is a very important structure in maths. It describes how the "distance" between two elements in a set is measured. In the future, we will discuss a similar structure that has more nice properties: *Normed Vector Space*, which is also endowed with some distance function. Try to compare metric & norm, metric space & normed vector space by finding their similarities & differences (when we learn both of them).

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2. Conv($\mathbb{R}$) with $d(x, y) = \sup|x - y|$. Notice that x, y are real convergent sequences. (We will see this example in details later!)
3. * *The Discrete Metric.* Any set M with $d(x, y) = 0$ if $x = y$, $d(x, y) = 1$ otherwise. This shows for any set, there is always a metric space associated with it. Moreover, by this metric, the set of any single point is an open ball (why?), and therefore every subset is open – The space is discrete (has the *discrete topology*.)

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4. * We will come back to metric space when we learn normed vector space. We will then conclude that any normed vector space is a metric space if we define the metric according to its norm.

We can hence define convergence of sequences in metric spaces (M, ϱ), where convergence of a sequence $(a_n)\colon \mathbb{N} \to M$ is determined by (2.2.1),

$$\lim_{n\to\infty} a_n = a \qquad :\Leftrightarrow \qquad \underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{n>N}{\forall}\ a_n \in B_\varepsilon(a).$$

where

$$B_\varepsilon(a) = \{y \in M\colon \varrho(y, a) < \varepsilon\}, \qquad \varepsilon > 0, \qquad a \in M.$$

Remark: $B_\varepsilon(a)$ is the (generalized) *open ball*. It describes the neighborhood of some element a in the set M. (Where did we define the open ball in the previous lectures?) The open ball turns out to be an important concept in *VV285*.

We then well-define the boundedness of sequence by it.

The fun part starts! We define the *Cauchy sequences*, whose elements' distance becomes smaller and smaller:

> 2.2.40. Definition. A sequence (a_n) in a metric space (M, ϱ) is called a ***Cauchy sequence*** if
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> $$\underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{m,n>N}{\forall}\ \varrho(a_m, a_n) < \varepsilon.$$

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Remark:

1. Every Cauchy sequence is bounded.
2. Every convergent sequence is a Cauchy sequence.
3. However, the reverse is not necessarily true.
4. The metric space where all Cauchy sequences converge is called a *complete* metric space.
5. A *completion* of an incomplete metric space is obtained by adding all the limits of Cauchy sequences in that space. Moreover, the completion of it can be constructed as a set of *equivalence classes* of Cauchy sequences in it – That explains why Cauchy sequences are important!

A Recap on Construction of $\mathbb{R}$

Given $\mathbb{Q}$, we may consider the set of all sequences in $\mathbb{Q}$ that converge to a limit. Denote this set by $\text{Conv}(\mathbb{Q})$. Each sequence $(a_n) \in \text{Conv}(\mathbb{Q})$ is associated uniquely to a number $a \in \mathbb{Q}$, namely its limit. We can now say that two sequences are equivalent if they have the same limit, i.e.,

$$(a_n) \sim (b_n) \qquad :\Leftrightarrow \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n. \tag{2.2.5}$$

Remark:

1. $\sim$ denotes an *equivalence relation* between two elements in some set.

We denote the set of all sequences with the same limit as a sequence (a_n) by $[(a_n)]$. Such a set is called a ***(equivalence) class*** and the set of all classes is denoted $\mathrm{Conv}(\mathbb{Q})/\sim$.

Since each rational number is represented by a class (why?) we see that the rational numbers may be identified with the set of all classes of convergent sequences:

$$\mathbb{Q} \simeq \mathrm{Conv}(\mathbb{Q})/\sim .$$

Remark:

1. The equivalence relation is extremely useful in this situation. Because if R is an equivalence relation on a set M, then for all $a, b \in M$, either $[a] \cap [b] = \varnothing$ or $[a] = [b]$. (Why? And Why is this even useful?)

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3. $\simeq$ denotes a *bijection* exists between two sets.
4. $M/\sim$ denotes the set of all equivalence classes in M **partitioned** by $\sim$.

We can now consider a larger class of sequences, that of Cauchy sequences of rational numbers, denoted by Cauchy($\mathbb{Q}$). Since every convergent sequence is a Cauchy sequence, Conv($\mathbb{Q}$) $\subset$ Cauchy($\mathbb{Q}$).

Furthermore, we say that two Cauchy sequences are equivalent not if they have the same limit (because they might not converge) but rather if their difference converges to zero:

$$(a_n) \sim (b_n) \qquad :\Leftrightarrow \qquad \lim_{n\to\infty}(a_n - b_n) = 0. \tag{2.2.6}$$

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Remark:

1. We define a new relationship between two Cauchy sequences, which is more general than the previous equivalence relationship.
2. We therefore consider this $\sim$ to be a generalization of the previous $\sim$ – So we reuse the same symbol.
3. Finally, we have a larger set:

$$\text{Cauchy}(\mathbb{Q})/\sim \supset \text{Conv}(\mathbb{Q})/\sim \simeq \mathbb{Q}$$

The set Cauchy($\mathbb{Q}$)$/\sim$ incorporates the rational numbers and by its construction every Cauchy sequence (a_n) in Cauchy($\mathbb{Q}$)$/\sim$ has a limit, namely precisely the object represented by the class $[(a_n)]$. We write

$$\mathbb{R} := \text{Cauchy}(\mathbb{Q})/\sim$$

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1. We then **define** the set of real numbers to be this larger set.
2. Recall how did we construct the natural numbers $\mathbb{N}_{def}$? Sets!

Regarding $\simeq$

A way to think about $\simeq$, a bijection exists between two sets A and B, is that the abstract entities A and B are equivalent and **essentially the same** – they are both sets, so they contain no information about repetition or order, and since there is a bijection between them, we can obtain one from the other simply by relabeling the elements.

To some extent, this also explains why we use other abstraction to construct natural numbers or real numbers: we are looking for a correspondence between a known concept and a new concept.

* An explicit definition will be given in VE203.[1]

[1] *Morphisms and Isomorphisms.

Example of Abstract Incomplete Metric Space

Let's see one abstract metric space that is not complete.

We define the set of all real sequences that vanish from some points to be U. i.e.

$$U = \{(a_n) | \exists N \in \mathbb{N}, \forall n > N : a_n = 0\}.$$

It is clear that $U \subset c_0$, where

$$c_0 = \{(a_n) | \lim a_n = 0\}.$$

We use the metric

$$\rho((a_n), (b_n)) = \sup |a_n - b_n|.$$

Define a sequence of sequences $(a_n)^{(N)}$, where

$$(a_n)^{(N)} = \begin{cases} \frac{1}{n}, \ n \leq N \\ 0, \ n > N \end{cases}$$

Is $(a_n)^{(N)}$ a Cauchy sequence in (U, ρ)? Does $(a_n)^{(N)}$ has a limit in this space?
* What is the completion of (U, ρ)?

Exercise

Let (a_n) be a real sequence satisfies

$$\forall n \in \mathbb{N}: \ |a_{n+2} - a_{n+1}| < c|a_{n+1} - a_n|, \qquad 0 < c < 1.$$

where c is fixed. Is (a_n) convergent or not? Prove.

Termiology

Let's say we have a function f such that

$$f : \Omega \to \mathbb{R}, \qquad x \mapsto f(x)$$

where $\Omega \subset \mathbb{R}$.

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2. $f(x)$ is a real number.
3. What is f essentially?
4. f is a real function.

1. *Polynomial functions* are fundamental functions in real/complex analysis. Later, we use the series of polynomial functions to define *(complex) exponential functions*. Moreover, it turns out to be a major ingredient of Taylor's Theorem – a theorem with huge amounts of engineering applications! We will see some of them in this semester, hopefully.

Elementary Real Functions

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4. *Piecewise functions* are functions defined piecewise. Usually, they are not smooth in shape.
5. *Periodic functions* are functions defined periodically. With Fourier analysis, they turn out to have GREAT significance in several engineering fields, such as *Signal and Systems*,*Fluid mechanics*, *Solid Mechanics*, *Thermodynamics*, *Electromagnetics*, *Optimal Control*...

Floor and Ceiling

We defined two useful functions that round a real number to an integer: *floor* and *ceiling*.

$$\lfloor x \rfloor := \max\{n \in \mathbb{Z} : n \le x\}, \qquad \lceil x \rceil := \min\{n \in \mathbb{Z} : n \ge x\},$$

Both part are trivial. For manipulating functions, essentially we are composing the original function f with some other function g - not any new concept.

* Weird/Crazy Functions

We can construct some really weird and crazy functions.

Go to specialfunctionswiki.org and explore some!

Write the definition of limits of functions, from memory if possible.

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2.4.1. Definition. Let f be a real- or complex-valued function defined on a subset of $\mathbb{R}$ that includes some interval (a, ∞), $a \in \mathbb{R}$. Then f converges to $L \in \mathbb{C}$ as $x \to \infty$, written

$$\lim_{x\to\infty} f(x) = L \qquad :\Leftrightarrow \qquad \underset{\varepsilon>0}{\forall}\ \underset{C>0}{\exists}\ \underset{x>C}{\forall}\ |f(x) - L| < \varepsilon. \tag{2.4.1}$$

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2.4.3. Definition. Let f be a real- or complex-valued function defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of Ω. Then the limit of f as $x \to x_0$ is equal to $L \in \mathbb{C}$, written

$$\lim_{x\to x_0} f(x) = L \quad :\Leftrightarrow \quad \underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in\Omega\setminus\{x_0\}}{\forall}\ |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Remark: These two definitions are essential when proving the convergence of functions. We will see an important theorem later(2.4.9), which is equally important in the proof.

2.4.5. Theorem. Let f and g be real- or complex-valued functions and x_0 an accumulation point of $\operatorname{dom} f \cap \operatorname{dom} g$ such that $\lim\limits_{x \to x_0} f(x)$ and $\lim\limits_{x \to x_0} g(x)$ exist. Then

1. $\lim\limits_{x \to x_0} (f(x) + g(x)) = \lim\limits_{x \to x_0} f(x) + \lim\limits_{x \to x_0} g(x)$,
2. $\lim\limits_{x \to x_0} (f(x) \cdot g(x)) = \big(\lim\limits_{x \to x_0} f(x)\big)\big(\lim\limits_{x \to x_0} g(x)\big)$,
3. $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to x_0} f(x)}{\lim\limits_{x \to x_0} g(x)}$ if $\lim_{x \to x_0} g(x) \neq 0$.

These statements remain true if $x_0 = \pm\infty$.

Remark: Useful, easy to memorize.

Exercise

Find

$$\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}.$$

What's your observation?

Here comes a theorem that is quite important but may not be intuitive.

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A similar result holds for $x_0 = \pm\infty$.

Remark:

1. How to prove?

Limits of Functions & Sequences

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A similar result holds for $x_0 = \pm\infty$.

Remark:

1. How to prove?
2. The limits of functions can be described entirely through sequences.
3. The negation of R.H.S. is useful if we want to prove some sequence does not have a limit at some point/infinity.

Exercise

Prove

$$\sin\frac{1}{x} \text{ does not have a limit at } 0.$$

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  1. Go through the lecture slides. Make sure you remember the definition and theorems. (Time permitting, re-prove every lemma/theorem on your own.)
  2. Go through your assignments. Make sure you remember some concepts introduced in assignments. e.g. $\overline{\lim}$, $\underline{\lim}$, ...
  3. Do the sample exams.
  4. Come to the big RC and OH.
- Good luck!

Have Fun

And

Learn Well![2]

[2] Special acknowledgement to former TA **Zhang Leyang**, who offered plenty of exercises and advice to my recitation class.