VV186: Honors Mathematics

Functions & Differentiation

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RC Policy

Several things I want you to pay attention to:

1. Be interactive. Feel free to interrupt me at any time if you want to ask something or simply make some comments. You are free to discuss with your friend if you want, as long as your discussion is related to the course contents and your voice won't effect other students.
2. Speak everything in English during the RC. This might be hard at the beginning, but you will soon get used to that.
3. "Question everything." Do not pretend to have understood everything. Maths is about strictness, abstraction and generalization. Understanding every basic concept is essential in our course. I will be quite "push" on checking your conceptual understanding. This process will be **annoying, tedious, but rewarding**. So Get prepared.

Outline

1. Assignment

2. Functions
   - Landau Symbol
   - Continuity
   - Inverse Functions
   - Uniform Continuity

3. Differential Calculus
   - “Linear Approximation”

Assignment 4

1. Feedback is posted on VV186 Piazza - Feedback for Assignment 4.
2. Come to OH if you have further questions.

2.4.12. Definition. Let f, ϕ be real- or complex-valued functions defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of Ω. We say that

$$f(x) = O(\phi(x)) \qquad\qquad \text{as } x \to x_0$$

if and only if

$$\underset{C>0}{\exists}\ \underset{\varepsilon>0}{\exists}\ \underset{x\in\Omega}{\forall} \quad |x - x_0| < \varepsilon \quad \Rightarrow \quad |f(x)| \le C|\phi(x)| \tag{2.4.2}$$

2.4.17. Definition. Let f, ϕ be real- or complex-valued functions defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of Ω. We say that

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if and only if

$$\underset{C>0}{\forall}\ \underset{\varepsilon>0}{\exists}\ \underset{x\in\Omega\setminus\{x_0\}}{\forall} \quad |x - x_0| < \varepsilon \quad \Rightarrow \quad |f(x)| < C|\phi(x)| \tag{2.4.3}$$

Remark:

1. An intuitive interpretation of big- and small-O symbol:
   - $f(x) = O(g(x))$ as $x \to x_0$: $f(x)$ is not significantly greater than $g(x)$ as $x \to x_0$.
   - $f(x) = o(g(x))$ as $x \to x_0$: $f(x)$ is significantly less than $g(x)$ as $x \to x_0$.
2. Big-O symbol is very common in computer science. You will encounter it frequently in VE203, VE280, VE281, VE477 ...
3. Small-O symbol is very common in our course (e.g. derivative).
4. Notice that when $x_0 = \infty$, the definition needs to be adjusted.
5. We cannot know the exact behavior of a function from Landau notation.

2.4.23. Theorem. Let f, ϕ be a real- or complex-valued functions defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of Ω. If $x_0 \in \Omega$, we require $\phi(x_0) > 0$. Suppose that exists some $C \geq 0$ such that

$$\lim_{x \to x_0} \frac{|f(x)|}{|\phi(x)|} = C. \tag{2.4.4}$$

Then $f(x) = O(\phi(x))$ as $x \to x_0$.

Remark: This theorem is useful when (2.4.4) exists. However, one can still have $f(x) = O(\phi(x))$ as $x \to x_0$ even if it does not exist. *The requirement that $\phi(x_0) > 0$ is a convention made by Landau.

Equivalent Condition of Small O

2.4.24. Theorem. Let f, ϕ be a real- or complex-valued functions defined on an interval $I \subset \mathbb{R}$ and let $x_0 \in \overline{I}$. Then

$$\lim_{x \to x_0} \frac{|f(x)|}{|\phi(x)|} = 0 \qquad \Leftrightarrow \qquad f(x) = o(\phi(x)) \text{ as } x \to x_0. \tag{2.4.5}$$

Remark: Notice that this theorem is a sufficient and necessary condition. With $f(x) = o(g(x))$ as $x \to x_0$, we cannot conclude that $f(x) \to 0$ as $x \to x_0$. A counter example is: $f(x) = 1, g(x) = \frac{1}{|x|}, x_0 = 0$.

Common Results

$$O(f(x)) + O(g(x)) = O(|f(x)| + |g(x)|),$$
$$O(f(x))O(g(x)) = O(f(x)g(x)),$$
$$O(f(x))o(g(x)) = o(f(x)g(x)),$$
$$O(O(f(x))) = O(f(x)),$$
$$o(O(f(x)) = o(f(x)).$$

2.5.1. Definition. Let $\Omega \subset \mathbb{R}$ be any set and $f \colon \Omega \to \mathbb{R}$ be a function defined on Ω. Let $x_0 \in \Omega$. We say that f is ***continuous at*** x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

If $U \subset \Omega$, we say that f is ***continuous on*** U if f is continuous at every $x_0 \in U$.

We say that f is ***continuous on its domain***, or simply ***continuous***, if f is continuous at every $x_0 \in \Omega$.

Question: What are the three requirements of continuity?

2.5.4. Theorem. Let $\Omega \subset \mathbb{R}$ be any set and $f\colon \Omega \to \mathbb{R}$ be a function defined on Ω. Let $x_0 \in \Omega$. Then the following are equivalent:

1. f is continuous at x_0;
2. for any real sequence (a_n) with $a_n \to x_0$, $\lim\limits_{n\to\infty} f(a_n) = f(x_0)$;
3. $\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in\operatorname{dom} f}{\forall} : |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon.$

2.5.10. Theorem. Let f, g be real functions such that $\lim_{x \to x_0} g(x) = L$ exists and f is continuous at $L \in \operatorname{dom} f$. Then

$$\lim_{x \to x_0} f(g(x)) = f(L).$$

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Proof:
Given $\varepsilon > 0$. Since f is continuous at L, $\exists\delta > 0$ such that
$\forall y \in \operatorname{dom} f : |L - y| < \delta \Rightarrow |f(L) - f(y)| < \varepsilon$.

We fix such $\delta > 0$. Since $\lim_{x\to x_0} g(x) = L$, there is some $\tilde{\delta} > 0$ such that
$\forall x \in \operatorname{dom} g : |x_0 - x| < \tilde{\delta} \Rightarrow |L - g(x)| < \delta \Rightarrow |f(L) - f(g(x))| < \varepsilon$.

Useful Results regarding Continuity

1. Locally sign-preserving property (2.5.11. Lemma)
2. The Bolzano Intermediate Value Theorem (2.5.12 & 13. Theorem)
3. Fixed Point Theorem (2.5.14. Theorem) [1]
4. Locally boundedness (2.5.15. Lemma)
5. Globally boundedness on closed intervals (2.5.16. Proposition)
6. Existence of global extrema on closed intervals (2.5.17. Theorem)

Remark: Remember these results. They are useful in proof, and help you get a better sense towards continuity.

[1] * *Fixed point theorem* is a very powerful tool. For example, in VE203, we will prove bi-directional injection leads to a bijection between two sets using it.

Exercise

1. Assume $f \in C([0, 2a])$, $a > 0$, $f(0) = f(2a)$.[2] Prove that there exists some $\zeta \in [0, a]$ such that $f(\zeta) = f(\zeta + a)$.

2. Let $f : [0, +\infty) \to \mathbb{R}$ be a continuous function such that $\lim_{x \to \infty} f(x) < \infty$ exists. Show that f is uniformly continuous on its domain.

[2] $C(I)$ denotes the set of all continuous function on I.

2.5.18. Definition. Let $\Omega, \widetilde{\Omega} \subset \mathbb{R}$ and $f\colon \Omega \to \widetilde{\Omega}$ a function. We say that f is

- ***injective*** if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for all $x_1, x_2 \in \Omega$;
- ***surjective*** if for every $y \in \widetilde{\Omega}$ there exists an $x \in \Omega$ such that $f(x) = y$ (i.e., if $\operatorname{ran} f = \widetilde{\Omega}$);
- ***bijective*** if for every $y \in \widetilde{\Omega}$ there exists a ***unique*** $x \in \Omega$ such that $f(x) = y$ (i.e., f is injective and surjective).

For $f\colon \Omega \to \widetilde{\Omega}$ we would like to define the ***inverse function of f***, denoted by f^{-1} as

$$f^{-1}\colon \widetilde{\Omega} \to \Omega, \qquad\qquad f(x) \mapsto x.$$

If f^{-1} exists, we say that f is ***invertible*** on Ω.

Remark:

- If f is bijective, f^{-1} exists.
- A continuous strictly monotonic function on an interval is bijective.
- A continuous bijective function on an interval is strictly monotonic.

Image and Pre-Image of Sets

Write down the definition of them from memory.

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2.5.21. Definition. Let $\Omega \subset \mathbb{R}$ and $f\colon \Omega \to \mathbb{R}$. Then for any $A \subset \Omega$ and $B \subset \operatorname{ran} f$

$$f(A) := \Big\{ y \in \mathbb{R} \colon \underset{x \in A}{\exists} f(x) = y \Big\}$$

is called the ***image of*** A and

$$f^{-1}(B) := \Big\{ x \in \Omega \colon \underset{y \in B}{\exists} f(x) = y \Big\}$$

is called the ***pre-image of*** B. Note that the symbol $f^{-1}(B)$ makes sense whether or not f is invertible.

Remark:

- $f(\operatorname{dom} f) = \operatorname{ran} f$.
- $f^{-1}(\operatorname{ran} f) = f(\operatorname{dom} f)$.

2.5.23. Definition. Let $I \subset \mathbb{R}$ be an interval and $f : \Omega \to \mathbb{R}$ a function with $I \subset \Omega$. Then f is called ***uniformly continuous on I*** if and only if

$$\underset{\varepsilon>0}{\forall} \underset{\delta>0}{\exists} \underset{x,y \in I}{\forall} |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Clearly, if f is uniformly continuous on I, then f is also continuous on I.

Remark:
f is uniformly continuous on $I \Leftrightarrow f$ is continuous on I
where $I = [a, b] \subset \operatorname{dom} f$. However, If I is not closed, they are not equivalent any more.

Are the following functions continuous? uniformly continuous?

- $f(x) = 1/x$ on $(0,1)$
- $f(x) = 1/x$ on $(1,2)$
- $f(x) = 1/(1+x^2)$ on $\mathbb{R}$
- $f(x) = x$ on $\mathbb{R}$
- $f(x) = x^2$ on any bounded interval $I \subset \mathbb{R}$
- $f(x) = x^2$ on $\mathbb{R}$

What is the similarity and difference between continuity and uniform continuity?

We all learned *differential calculus* in our high school but we usually called it *“derivative”*. It was often defined in a sloppy way, and lacking of mathematical interpretation. At this moment, we want to formulate a strict definition of *derivative* and *differentiation*. With this solid, scalable definition, we can easily extend the derivative to higher-dimension space[3].

We have continuous functions now but they cannot satisfy us – a continous function may still behave wildly. What we want is even nicer properties: we need a class of functions that can be approximated linearly at any point. They turned out to be extremely useful in various fields.

In this section, you should always keep this sentence in your mind:

The derivative of a function at some point x is just the linear approximation near x.

[3]We will cover them in VV285, where you will know matrices are derivatives of multi-variable functions. At then, you may not be very surprised by this result if you understand the essence of derivate – it is simply linear approximation.

3.1.1. Definition. Let $\Omega \subset \mathbb{R}$ be a set, $x \in \Omega$ an interior point of Ω and $f : \Omega \to \mathbb{R}$ a real function. Then we say that f is ***differentiable*** at x if there exists a linear map L_x such that for all sufficiently small $h \in \mathbb{R}$

$$f(x+h) = f(x) + L_x(h) + o(h) \qquad \text{as } h \to 0. \tag{3.1.5}$$

We say that f is differentiable on some open set $U \subset \Omega$ if f is differentiable at every point of U.

We say that f is differentiable if its domain is an open set and f is differentiable at every point of the domain.

Question: How to prove L_x is unique if it exists?

Remark: Due to the uniqueness of L_x, we define the derivative of f at x:

$$f'(x) := L_x$$

3.1.5. Theorem. Let Ω be a set, $x \in \Omega$ an interior point and $f : \Omega \to \mathbb{R}$ a function that is differentiable at x with derivative $L_x = f'(x)$. Then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{3.1.7}$$

Furthermore, if the limit in (3.1.7) exists for an interior point $x \in \Omega$, then f is differentiable at x and the derivative is given by (3.1.7).

Remark: This theorem makes use od the geometric meaning of derivative (slope). However, I do **not** recommend you to remember the concept of derivate using slope. Because the concept of slope in high-dimension space is quite vague and non-intuitive while "linear approximation" remains to be meaningful in such a case.

Check Your Understanding

True of False?

- L_x is essentially a number for a fixed $x \in \Omega$, because $L_x = \alpha$.
- The derivate of f at x is a line passing through $(x, f(x))$.
- For $f(x) = x^4, f'(x) = 4x^3$, so L_x may not be linear.

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- The derivate of f at x is a line passing through $(x, f(x))$.
- For $f(x) = x^4, f'(x) = 4x^3$, so L_x may not be linear.

Explanation:

- False. L_X is essentially a linear map (function) rather than a real number. We abuse the notation of L_x to represent α. In fact, the notation $L_x = \alpha$ sometimes does not make sense. To remember that L_x is a function, you can write $L_x : x \mapsto \alpha x$.
- False. The derivative of f at x is a linear map. Its graph does not necessarily pass $(x, f(x))$.
- False. This question is very tricky. Do not confuse *derivative at some point* with *function that gives derivative*. Fix $x = x_0$, $4x_0^3$ is just a real number. Therefore, the derivative at x is a linear map $L_x : x \mapsto 4x_0^3 \cdot x$. Essentially, f' is a function that maps some point x to the derivative of f at x. i.e. $f' : x \mapsto L_x$.

Useful Result regarding Differentiation

1. Linearity of differentiation
2. Chain rule
3. Product rule
4. Quotient rule

Exercise

1. Find $f'(x)$, given $f(x) = g(x + g(x)) + \dfrac{1}{g(x)}$. Assume $g(x) > 0, g'(x) \neq 0$ exists.

2. Give an example of a function f such that $\lim_{x\to\infty} f(x)$ exists, but $\lim_{x\to\infty} f'(x)$ does not exist.

3. Prove that if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f'(x)$ both exist, then $\lim_{x\to\infty} f'(x) = 0$.

4. Suppose that f is differentiable and $|f(x) - y(x)| \leq |x - y|^n$ for $n > 1$. Prove that f is constant.

Have Fun

And

Learn Well![4]

[4]Special acknowledgement to former TA **Zhang Leyang**, who offered plenty of exercises and advice to my recitation class.