VV186: Honors Mathematics

Math Foundations

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Pay attention to...

Several things I want you to pay attention to:

1. Be interactive. Feel free to interrupt me at any time if you want to ask something or simply make some comments. You are free to discuss with your friend if you want, as long as your discussion is related to the course contents and your voice won't effect other students.
2. Speak everything in English during the RC. This might be hard at the beginning, but you will soon get used to that.
3. "Question everything." Do not pretend to have understood everything. Maths is about strictness, abstraction and generalization. Understanding every basic concept is essential in our course. I will be quite "push" on checking your conceptual understanding. This process will be **annoying, tedious, but rewarding**. So Get prepared.
4. Open the camera if you can. This is important for me. It is hard to keep talking for more than one hour without seeing any feedback from you. Convince yourself that your small action does make my life easier. So please open your camera.

Outline

What is the seven properties of natural numbers' operations?

Natural Number

What is the seven properties of natural numbers' operations?

1. associativity$\times 2$
2. existence of neutral element$\times 2$
3. commutativity$\times 2$
4. distributivity.

Notation for Addition & Multiplication

$$a_1 + a_2 + \cdots + a_n =: \sum_{j=1}^{n} a_j =: \sum_{1 \le j \le n} a_j$$

$$a_1 \cdot a_2 \cdots a_n =: \prod_{j=1}^{n} a_j =: \prod_{1 \le j \le n} a_j.$$

Often one wants to show that some statement frame $A(n)$ is true for all $n \in \mathbb{N}$ with $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Mathematical induction works by establishing two statements:

(I) $A(n_0)$ is true.

(II) $A(n+1)$ is true whenever $A(n)$ is true for $n \geq n_0$, i.e.,

$$\underset{\substack{n\in\mathbb{N}\\ n\geq n_0}}{\forall} \big(A(n) \Rightarrow A(n+1)\big)$$

Straightforward Exercise:

Prove

For all $n \in \mathbb{N}$, $5^n - 1$ is divisible by 4.

Let's see an interesting problem. What is going wrong in the proof?

> 1.3.5. Example. Let us use mathematical induction to argue that every set of $n \geq 2$ lines in the plane, no two of which are parallel, meet in a common point.
>
> The statement is true for $n = 2$, since two lines are not parallel if and only if they meet at some point. Since these are the only lines under considerations, this is the common meeting point of the lines.
>
> We next assume that the statement is true for n lines, i.e., any n non-parallel lines meet in a common point. Let us now consider $n + 1$ lines, which we number 1 through $n + 1$. Take the set of lines 1 through n; by the induction hypothesis, they meet in a common point. The same is true of the lines $2, \dots, n + 1$. We will now show that these points must be identical.

> Assume that the points are distinct. Then all lines $2, \dots, n$ must be the same line, because any two points determine a line completely. Since we can choose our original lines in such a way that we consider distinct lines, we arrive at a contradiction. Therefore, the points must be identical, so all $n+1$ lines meet in a common point. This completes the induction proof.
>
> Where is the mistake in the above "proof" of our (obviously false) supposition?

Hint: Start by considering the case where $n = 3$, find some contradiction to invalidate the above proof.

Remark: Induction can sometimes be tricky. Before using it, examine the complex level of the problem, because using induction is sometimes much more complicated than using a "direct" method to prove a statement.

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z} \wedge q \neq 0 \right\}$$

What is the 12 properties of $\mathbb{Q}$?

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z} \land q \neq 0 \right\}$$

What is the 12 properties of $\mathbb{Q}$?

- $\mathbb{Q}$ inherits all the 7 properties of $\mathbb{N}$
- inverse element of $+$ and $\cdot$
- trichotomy law
- closed under $+$ and $\cdot$

Simple Exercise: With these properties, prove

1. if $a \neq 0$ and $a \cdot b = a \cdot c$, then $b = c$.
2. Furthermore,if $a \cdot b = 0$, then either$a = 0$ or $b = 0$.
3. $a - b = b - a$ only if $a = b$

Triangle Inequality

For all rational numbers $a, b \in Q$, we have

$$|a + b| \leq |a| + |b|.$$

Furthermore,

$$|a + b| \geq ||a| - |b||$$

Remark: They are quite useful in proof. We will see example(s) later.

Boundness

How do we define these concepts?

1. bounded/unbounded
2. max, min
3. sup, inf

What's the relationship between 1. and 2.? (how to prove?) Does a set in $\mathbb{Q}/\mathbb{R}$ necessarily has a max or sup?

Let $A \in \mathbb{R}$ be any set.

1. We call $x \in \mathbb{R}$ an *interior point* of A if there exists some $\varepsilon > 0$ such that the interval $(x - \varepsilon, x + \varepsilon) \subset A$. The set of interior points of A is denoted by int A.
2. We call $x \in \mathbb{R}$ an *exterior point* of A if there exists some $\varepsilon > 0$ such that the interval $(x - \varepsilon, x + \varepsilon) \cap A = \varnothing$
3. We call $x \in \mathbb{R}$ a *boundary point* of A if for every $\varepsilon > 0$ $(x - \varepsilon, x + \varepsilon) \cap A \neq \varnothing$ and $(x - \varepsilon, x + \varepsilon) \cap A^c \neq \varnothing$. The set of boundary points of A is denoted by ∂A.
4. We call $x \in \mathbb{R}$ an *accumulation point* of A if for every $\varepsilon > 0$ the interval $(x - \varepsilon, x + \varepsilon) \cap A \backslash \{x\} \neq \varnothing$

Remark: We will discuss more on sets and classification of points in $\mathbb{R}^n$ in VV285.

We define *complex numbers* in order to resolve the algebraic incompleteness of real numbers. The formal definition makes use of the ordered pairs, so that $\mathbb{R}^2 \cong \mathbb{C}$. But in practice, we usually use the notation $a + bi$ rather than (a, b) to indicate a complex number z.

We define the modulus or absolute value of a complex number $z = a + bi$ by

$$|z| = \sqrt{a^2 + b^2} = \sqrt{z \cdot \bar{z}}$$

Here $\bar{z} = a - bi$ is called the complex conjugate of z

Let $z_0 \in \mathbb{C}$. Then we define the open ball of radius $R > 0$ centered at z_0 by

$$B_R(z_0) := \{z \in \mathbb{C} : |z - z_0| < R\}$$

Remark: * We will generalize the concept of open ball to $\mathbb{R}^n$ or $\mathbb{C}^n$ in VV285. Additionally, complex analysis is an important topic of VV286, which turns out to be extremely useful in several fields: Fourier Analysis, Improper Integral, and Ordinary Differential Equations.

We use a *static* method to define the functions formally. To be more precise, we use predicates and sets:

2.1.1. Definition. Let X and Y be sets and let P be a predicate with domain $X \times Y$. Let

$$f := \{(x, y) \in X \times Y : P(x, y)\}$$

and assume that P has the property that

$$\underset{(x_1,y_1)\in f}{\forall} \; \underset{(x_2,y_2)\in f}{\forall} \quad x_1 = x_2 \;\Rightarrow\; y_1 = y_2. \tag{2.1.1}$$

Let

$$\operatorname{dom} f := \left\{x \in X : \underset{y\in Y}{\exists} : (x, y) \in f\right\}$$

Then we say that f is a ***function from*** $\operatorname{dom} f$ ***to*** Y. The set $\operatorname{dom} f \subset X$ is called the ***domain*** of f and Y is called the ***codomain*** of f. We also define the ***range*** of f by

$$\operatorname{ran} f := \left\{y \in Y : \underset{x\in X}{\exists} : (x, y) \in f\right\}.$$

Clarification

- $\times$ Co-domain of $f = \operatorname{ran} f$ (Counter-example: $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$.)
- $\surd$ Domain of f contains exactly all the elements that have assignment with an element in f's co-domain.

Notation of Function

Notation of functions is important because you will see them frequently in the coursework, and you are going to declare functions in the same way.

$$f : \Omega \to Y, \qquad x \mapsto f(x)$$

* Notice that the two arrows are different. The first is called "\to" while the second is called "\mapsto" in LaTeX.

If X, Y, Z are sets, $\Omega \subset X$, $\Sigma \subset Y$, and two maps $f\colon \Omega \to Y$, $g\colon \Sigma \to Z$ are given such that $\operatorname{ran} f \subset \Sigma = \operatorname{dom} g$, then we define the ***composition***

$$g \circ f\colon \Omega \to Z, \qquad\qquad x \mapsto g(f(x)).$$

A sequence is essentially a map (function) from subset of $\mathbb{N}$ to $\mathbb{R}$ or $\mathbb{C}$. We allow three notations for sequence:

$$(a_n)_{n\in\Omega} = (a_n) = a_0, a_1, a_2, \ldots$$

Notice that (a_n) denotes a sequence while a_n, the nth term in the sequence, is a value/number.

Remark: Roughly speaking, we can define a sequence in two ways:

- Explicit definition. e.g. $(a_n) = (2n)$
- Recursive definition. e.g. $a_0 = 0, a_n = a_{n-1} + 2$.

In general, explicit definition contains more information than a recursive definition. For example, if we know the explicit definition of *Fibonacci sequence*, we can calculate its 1000^{th} term in a very short time without recourse to a recursive calculation.[1] Finding the explicit definition of a sequence given its recursive definition is an important topic in *VE203: Discrete Mathematics*.

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[1] $*F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi} = \frac{\varphi^n - \psi^n}{\sqrt{5}}$ where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2} = 1 - \varphi = -\frac{1}{\varphi}$.

Convergence of Sequence I

The definition of convergence (of sequence) is very important. Memorize it!

2.2.4. Definition. The real or complex sequence $(a_n)\colon \Omega \to X$, $\Omega \subset \mathbb{N}$, $X = \mathbb{R}$ or $\mathbb{C}$, is said to ***converge with limit*** $a \in X$ if

$$\underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{n>N}{\forall}\quad |a_n - a| < \varepsilon.$$

We then write "$\lim\limits_{n\to\infty} a_n = a$" or "$a_n \to a$ as $n \to \infty$."

A sequence that does not converge (to any limit) is called ***divergent***.

2.2.5. Remark. Definition 2.2.4 can alternatively be formulated as

$$\lim_{n\to\infty} a_n = a \qquad :\Leftrightarrow \qquad \underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{n>N}{\forall}\ a_n \in B_\varepsilon(a). \tag{2.2.1}$$

where

$$B_\varepsilon(a) = \{z \in X\colon |a - z| < \varepsilon\}, \qquad \varepsilon > 0, \qquad a \in X,$$

for $X = \mathbb{R}$ or $\mathbb{C}$ is called an ***ε-neighbourhood*** of a. We hence say, "For sufficiently large n, a_n is arbitrarily close to a."

2.2.8. Definition. A real sequence (a_n) is called ***divergent to infinity*** if

$$\underset{C>0}{\forall} \quad \underset{N\in\mathbb{N}}{\exists} \quad \underset{n>N}{\forall} \quad a_n > C.$$

It is called ***divergent to minus infinity*** if

$$\underset{C<0}{\forall} \quad \underset{N\in\mathbb{N}}{\exists} \quad \underset{n>N}{\forall} \quad a_n < C.$$

Convergence of Sequence IV

2.2.13. Lemma.

1. The sequence (a_n) converges to a if and only if the sequence $(b_n) := (a_n - a)$ converges to zero, i.e.,

$$a_n \to a \qquad \Leftrightarrow \qquad a_n - a \to 0$$

2. The sequence (a_n) converges to 0 if and only if the sequence $(b_n) = (|a_n|)$ converges to zero, i.e.,

$$a_n \to 0 \qquad \Leftrightarrow \qquad |a_n| \to 0$$

Remark: The above conclusion is useful in proof.

Clarification

- × A sequence may not contain infinitely many terms.
  When we say "sequence" we usually assume that it is infinite. If it is finite in number of terms, we usually say it is a "tuple". Similarly, a subsequence of a sequence is infinite in size, too.
- × A sequence is either convergent or divergent to infinity. (Counter-example: $(a_n) = (-1)^n$)
  iI a sequence is not convergent, we say it is "divergent". However, it doesn't mean it diverge to ∞.

Let (a_n) be a real sequence that converges to some element $L \in \mathbb{R}$. Prove that the sequence $\left(\frac{\sum_{i=1}^{n} a_i}{n}\right)$ is convergent to L.

Exercise

Let (a_n) be a real sequence that converges to some element $L \in \mathbb{R}$. Prove that the sequence $\left(\frac{\sum_{i=1}^{n} a_i}{n}\right)$ is convergent to L.

How do we solve this kind of problem? (use the definition of convergence)

1. Fix ε
2. Find $N \in \mathbb{N}$, so that $\forall n > N : \ldots$

Solution

2.2.22. Definition. A real sequence (a_n) is called

- ***increasing*** if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.
- ***decreasing*** if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.
- ***strictly increasing*** if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$.
- ***strictly decreasing*** if $a_n > a_{n+1}$ for all $n \in \mathbb{N}$.
- ***monotonic*** if it is either increasing or decreasing.

1. A convergent sequence is bounded.
2. A convergent sequence has precisely one limit.
3. *Squeeze theorem.*
4. Every monotonic and bounded (real) sequence is convergent.
5. Let (a_n) be a convergent sequence with limit a. Then any subsequence of (a_n) is convergent with the same limit.
6. Every real sequence has a monotonic subsequence.
7. A number a is an accumulation point of a sequence (a_n) if and only if there exists a subsequence of (a_n) that converges to a.
8. *Theorem of Bolzano-Weierstraß.* Every bounded real sequence has an accumulation point.

Remark: The squeeze theorem is useful in finding the limit of a sequence. (Try to use it to prove $\lim \sqrt[n]{n} = 1$.) The Theorem of Bolzano-Weierstraß turns out to be useful in proving the metric space $(\mathbb{R}, \rho)$ is complete where ρ is the normal Euclidean metric in $\mathbb{R}$. We will further generalize this theorem in VV285, where we consider more general sequence (sequence of vector).

Let (a_n) and (b_n) be convergent real or complex sequences with $a_n \to a$ and $b_n \to b$ for some $a, b \in \mathbb{C}$. Then

(1) $\lim (a_n + b_n) = a + b$

(2) $\lim (a_n \cdot b_n) = ab$

(3) $\lim \frac{a_n}{b_n} = a/b$ if $b \neq 0$

Proof of (3):

We want to prove that $\left|\frac{a_n}{b_n} - \frac{a}{b}\right| \to 0$ as $n \to \infty$. We can first fix some $M \in \mathbb{N}$ such that $\forall n > M : |b_n - b| < |b|/2$. Now, we have $\forall n > M$,

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| = \left|\frac{a_n b - b_n a}{b_n b}\right| = \left|\frac{a_n b - ab + ab - b_n a}{b_n b}\right|$$

$$\leq \frac{|a_n - a|\,|b|}{|b_n b|} + \frac{|a|\,|b - b_n|}{|b_n b|} < \frac{2\,|a_n - a|\,|b|}{b^2} + \frac{2|a|\,|b - b_n|}{b^2}$$

Given $\varepsilon > 0$, choose $N > M$ such that $\forall n > N : |a_n - a| < \frac{|b|\varepsilon}{4}, |b - b_n| < \frac{b^2 \varepsilon}{4|a|}$. Eventually we have

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N : \left|\frac{a_n}{b_n} - \frac{a}{b}\right| < \frac{|b|\varepsilon}{4} \cdot \frac{2|b|}{b^2} + \frac{2|a|}{b^2} \cdot \frac{b^2}{|a|} \cdot \frac{\varepsilon}{4} < \varepsilon$$

Results of Limit Operations II

Extension: Here are some other results:

(1) $\lim \sqrt{a_n} = \sqrt{\lim a_n}$ if $a_n > 0$

(2) $\lim a_n^2 = (\lim a_n)^2$

(3) $\lim \sqrt[n]{n} = 1$

(4) ...

Remark: (1) and (2) hold for higher power, too.

Exercise

Let's see a conceptually interesting exercise:

Let $(p_n)_{n\in\mathbb{N}}$ be a sequence, and

$$p_0 = 1, p_n = \sqrt{2p_{n-1}}$$

Please find its limit, if exists.

Let's see a conceptually interesting exercise:

Let $(p_n)_{n\in\mathbb{N}}$ be a sequence, and

$$p_0 = 1, p_n = \sqrt{2p_{n-1}}$$

Please find its limit, if exists.

What do we need to do?

1. Write down the first several terms on the draft paper, and see if there's some clues. (Result: We guess that the limit exists.)
2. Prove the existence of the limit. (How?)
3. Calculate the limit. (How?)

Solution

About Course Contents:
As you can see, **sequence** is a very important concept in the first part of VV186. So I encourage you to review the definitions, lemmas, and theorems of them carefully before the exam. In particular, proof in the slides are valuable, because it does not only validate every theorem but also show a strict logic that you need to follow in the coursework.

About Assignment:
On the one hand, you might feel Assignment 2 is much harder than Assignment 1. On the other hand, it becomes more and more exciting and interesting. Always try your best to finish the assignments independently, this will greatly helps you prepare the exams. Please feel free to ask questions on the Piazza, including questions regarding assignments. And, feel free to come to TA's OH and have some chat with us. We are glad to help you.

Have Fun

And

Learn Well![2]

[2] Special acknowledgement to former TA Zhang Leyang, who offered many exercises and advice to my recitation class.