VV186: Honors Mathematics

Integral

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Outline

Hierarchy of Integrals

1. Integral of step functions
2. Regulated integral
3. Darboux integral & Riemann Integral
4. Lebesgue integral
5. Gauge integral (Henstock-Kurzweil integral)

Remark: We only study the first three hierarchies. Make sure you remember their definitions and properties.

Step Functions

4.1.2. Definition. A function $\varphi\colon [a,b] \to \mathbb{R}$ is called a ***step function with respect to a partition*** $P = (a_0, \dots, a_n)$ if there exist numbers $y_i \in \mathbb{R}$, $i = 1, \dots, n$, such that

$$\varphi(t) = y_i \qquad \text{whenever } a_{i-1} < t < a_i \tag{4.1.1}$$

for $i = 1, \dots, n$. We denote the set of all step functions by $\operatorname{Step}([a,b])$

Remark: We can arbitrarily define φ at points a_i and φ is still a step function. For example,

$$\varphi\colon [0,1] \to \mathbb{R}, \qquad \varphi(x) = \begin{cases} 0 & 0 \le x < \frac{1}{2}, \\ \frac{1}{2} & x = \frac{1}{2}, \\ 1 & \frac{1}{2} < x \le 1, \end{cases}$$

Remark: $\operatorname{Step}([a,b])$ is a vector space. $\int$ is linear, and does not depend on values on finite sets (To be more precise, on 0-measure set).

4.1.13. Definition. A function $f \in L^\infty([a,b])$ is said to be ***regulated*** if for any $\varepsilon > 0$ there exists a step function φ such that

$$\sup_{x \in [a,b]} |f(x) - \varphi(x)| < \varepsilon. \qquad (4.1.6)$$

Equivalently, a regulated function f is a function such that there exists a sequence of step functions (φ_n) which converges uniformly to f. (To see this, we simply define φ_n as the step function satisfying (4.1.6) with $\varepsilon = 1/n$.)

Remark:

- The *equivalent description* of 4.1.13. Definition is more intuitive: We seek for a sequence of step functions that converge **uniformly** to the target function.
- The continous a.e.[1] functions on a closed interval are regulated. i.e. $\mathrm{PC}([a,b]) \subset \mathrm{Reg}([a,b])$

[1] "a.e." is the abbreviate of "almost everywhere".

4.1.18. Definition and Theorem. Let $f \in \text{Reg}([a, b])$ and (φ_n) a sequence in $\text{Step}([a, b])$ converging uniformly to f. Then the ***regulated integral*** of f, defined by

$$\int_a^b f := \lim_{n\to\infty} \int_a^b \varphi_n \tag{4.1.8}$$

exists and does not depend on the choice of (φ_n).

4.1.20. Theorem. Let (f_n) be a sequence of regulated functions such that $f_n \to f$ uniformly for some $f \in L^\infty$, i.e., $\|f - f_n\|_\infty \to 0$. Then f is regulated and

$$\int_a^b f_n \xrightarrow{n\to\infty} \int_a^b f. \tag{4.1.11}$$

Consider the function

$$f : [0,1] \to \mathbb{R}, \qquad f(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for some } n \in \mathbb{N} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Is the function regulated?

Exercise

True or False?

1. $f(x) = x^3 + e^x$ on $[-1, 1]$ is regulated.
2. Since $f(x) = x^2$ is regulated on each $[-n, n]$, where $n \in \mathbb{N}, f$ is regulated on $\mathbb{R}$ because we can let $n \to \infty$
3. A continuous function is piecewise continuous
4. Let f, g be two real-valued function defined on $[0, 1]$. Furthermore, assume $f - g = x$, then the equation $\int_0^1 (f - g) = \int_0^1 f - \int_0^1 g = \frac{1}{2}$ holds.
5. Let $f \in \text{Reg}([0, 1])$, let g be a real-valued function. Then $f \circ g$ is regulated.

Darboux Integral

4.1.22. Definition. Let $[a, b] \subset \mathbb{R}$ be a closed interval and f a bounded real function on $[a, b]$. Let $\mathscr{U}_f$ denote the set of all step functions u on $[a, b]$ such that $u \geq f$ and $\mathscr{L}_f$ the set of all step functions v on $[a, b]$ such that $v \leq f$. The function f is then said to be ***Darboux-integrable*** if

$$\underline{I}(f) = \sup_{v \in \mathscr{L}_f} \int_a^b v = \inf_{u \in \mathscr{U}_f} \int_a^b u = \overline{I}(f).$$

In this case, the ***Darboux integral of*** f, $\int_a^b f$, is defined to be this common value.

Question: True or false?
If $\int_a^b |f|$ is Darboux integrable, $\int_a^b f$ is Darboux integrable.

Exercise

Consider the function

$$f : [0,1] \to \mathbb{R}, \qquad f(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for some } n \in \mathbb{N} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Is the function Darboux integrable?

Consider the function

$$f\colon [0,1] \to \mathbb{R}, \qquad f(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for some } n \in \mathbb{N} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Is the function Darboux integrable?

General Strategy

- construct a sequence of step functions $(u_n) > f$,
- construct a sequence of step functions $(l_n) < f$,
- find $\int u_n$ and $\int l_n$,
- verify $\lim\limits_{n\to\infty} \int u_n = \lim\limits_{n\to\infty} \int l_n$.

4.1.28. Definition. Let $[a, b] \subset \mathbb{R}$ be a closed interval and f a bounded real function on $[a, b]$. Then f is Riemann-integrable with integral

$$\int_a^b f \in \mathbb{R}$$

if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any tagged partition (P, Ξ) on $[a, b]$ with mesh size $m(P) < \delta$

$$\left| \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) - \int_a^b f \right| < \varepsilon.$$

Definition 4.1.28 is quite difficult to work with in practice, so it is fortunate that the following result holds:

4.1.29. Theorem. A bounded real function is Riemann-integrable on $[a, b]$ if and only if it is Darboux-integrable. Moreover, the values of the integrals coincide.

Darboux Integral vs. Riemann Integral

Describe in words the difference between **Darboux Integral** and **Riemann Integral**. How are these two concepts different? If you can, state some properties that they have in common or that serve to differentiate them from each other. Is one of them also always an example of the other? Give examples.

This exercise is left for you!

Mean Value Theorem of Integral Calculus

From Assignment

Let $[a, b] \subset \mathbb{R}$ be a closed interval and $f : [a, b] \to \mathbb{R}$ a continuous real function. Then there exists a $\xi \in (a, b)$ such that

$$\int_a^b f(x)dx = (b - a)f(\xi).$$

Forever

Have Fun

And

Learn Well![2]

[2]Special acknowledgement to former TA **Zhang Leyang**, who offered plenty of exercises and advice to my recitation class.