Network Coding Benefits Age of Information in Broadcast Channels

Xingran Chen, Shaochong Wang, Renpu Liu, Shirin Saeedi Bidokhti

***Abstract*—We investigated the benefits of coding on the age of information and rate in memoryless broadcast channels with M users. A novel framework of network age of information (AoI) was proposed and two classes of coding actions were considered. Within these classes, deterministic policies were devised, which revealed the tradeoff between the AoI and rate. Lower and upper bounds were derived on the AoI of users. Simulations showed that network coding increases the rate and decreases the AoI simultaneously, and the benefits of coding increase with the number of users.**

I. Introduction

A. Motivation

Sending status updates in a timely manner has significant importance in the Internet of Things (IoT) applications. A new concept *Age of Information (AoI)*, which was introduced in [1], measures the timeliness of information at a remote system. In other words, AoI measures, at the receiving side, how much time has passed since the generation time of the latest received packet.

In the context of broadcast packet erasure channels (BPECs) with feedback, the impact of network coding on AoI is not apriori clear when BPECs have multiple users, though coding is beneficial on AoI in BPECs with 2 users [31]. In this work, we aim to seek the efficiency both in AoI and rate. Furthermore, we reveal the relationship between AoI and rate considering network coding. On one hand, a higher rate effectively corresponds to a smaller delay (both in the sense that the queues get emptied faster and in the sense that fewer uses of the network are needed in total to transmit a fixed number of information bits), hence may correspond to a smaller AoI. On the other hand, to achieve high rates with coding, we have incur delay by waiting for the arrival/generation of other packets for the purpose of coding as well as prioritizing their transmission, hence may incur larger AoI.

The following questions will be answered in this paper: 1) Can we devise scheduling policies that schedule different coding actions, as opposed to schemes that schedule users, and show the benefit of coding, in terms of age and rate, over uncoded schemes such as those proposed in [2], [3]? 2) What is the trends of benefits of coding? Will the benefits increase or decrease with the number of users, the erasure probability and the generation rate of packets? 3) What is the relationship between AoI and rate? In a fixed generation rate, can we devise scheduling policies which provide benefit for both AoI and rate?

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B. Related Work

In practice, it is not always effective to update the information as fast as possible for it may cause further delay in the network queues. In [4], a single source and server setup were considered under First-Come First-Serve (FCFS) queue management and it was shown that there is an optimal update rate that minimizes time-average AoI. Further extensions to networks of multiple sources and servers with and without packet management were studied in [5]–[9]. AoI has also been investigated in network management and scheduling. In particular, [10] proposes scheduling policies to optimize the overall age in a wireless network. Maintaining equally up-to-date and synchronized information from multiple sources is studied in [11]. In [12], scheduling algorithms are designed to minimize AoI in wireless broadcast channels. [13] devises scheduling policies to minimize average AoI under throughput constraints in wireless multi-access networks. The minimum age of time-varying wireless channels with interference constraints is obtained in [14]–[16] with and without channel state information.

In coding theory, previous work has mainly studied point to point channels. For example, [17]–[19] consider erasure channels, propose coding schemes, and analyze the resulting average or peak AoI in various setups. More recently, [20] proves that when the source alphabet and channel input alphabet have the same size, a Last-Come First-Serve (LCFS) with no buffer policy is optimal. Considering erasure channels with FCFS M/G/1 queues, [21] finds an optimal block length for channel coding to minimize the average age and average peak age of information. [22] investigates the impact of coding on AoI in a two-user broadcast symbol erasure channel with feedback, and the proposed adaptive coding scheme improves the AoI at the weak user by orders of magnitude without compromising the AoI at the strong user. The AoI optimization problem in MIMO broadcast channels with various numbers of users is considered in [23]. Under various assumptions on the size of each update and the number of antennas at the transmitter and the receivers, [23] identifies the corresponding age-optimal precoding and transmission scheduling strategies. The effects of quantization and coding on the estimation quality is revealed in [24], [25], which optimizes sampling times such that minimizes the long term average minimum mean square error for each coding scheme. [26] analyzes the timeliness of information when a fixed number of packets is encoded using networking coding, and coded packets are sent through a multicast wireless network, with independent erasure channels between the source and multiple destinations.

BPECs and their variants have been considered in previous work such as [27]–[30] and rate-optimal coding algorithms are designed using (network) coding ideas. The key idea is due to [27] where it is shown that the entire capacity region of two-user BPECs with feedback can be attained by XOR-ing overheard packets. More precisely, suppose a packet p, intended for user 1, is broadcasted and only received at user 2. Scheduling algorithms re-transmit this packet because it is not received at its intended receiver. However, one may be able to exploit coding opportunities by tracking such packets (and this is possible through the available feedback). For example, in a similar manner, a packet q, intended for user 2, may get transmitted and received only at receiver 1. Now instead of re-transmitting p and q in two uses of the network, one can transmit the XOR packet $p \oplus q$ which is simultaneously useful for both users.

C. Contributions

In this paper, we study a memoryless broadcast channel with M users. Based on our previous work [31], we seek the impact of coding on AoI and rate. Our fist goal is to minimize the expected weighted sum of AoI (EAoI). This metric looks at the expected time-average AoI, normalized by the number of source nodes M. Our second goal is to both minimize EAoI and maximize the rate, and hence to reveal the tradeoff between AoI and rate.

We first propose a novel framework of network AoI on the broadcast channels under transmission mechanism with coding (Section II and Section III). Based on the new framework, we show that only the latest packet in every (virtual) queue matters when the goal is to seek the minimization of AoI.

Then, we derive two general lower bounds on AoI for any transmission policy (Section IV). The first lower bound is derived by assuming that there is always a fresh packet to be delivered, which is the tighter one when the generation rate is small, and reveals the relationship between AoI and rate. The second lower bound is derived by assuming that all packets are delivered instantaneously upon arrival, which is the tighter one when the generation rate is large, and reveals the relationship between AoI and generation rate.

The policies with coding are classified into two categories, regular-coding policies and coded-coding policies (which will be defined in Section II). In the former class, we propose an age-based Max-Weight (MW) policy, to minimize the AoI (Section V). The upper bound of the proposed policy is given. Comparing to the policies without coding, simulation results show that coding is beneficial, i.e., the AoI under regular-coding policies is smaller than that under policies without coding. In addition, the benefits increase with the number of users as well as the erasure probability and decrease with the generation rate.

In order to find the relationship between AoI and rate, we propose a age-mix-rate MW policy to minimize AoI and maximize rate simultaneously (Section VI), and the upper bound of the proposed MW policy is given. Under regular-coding policies, the AoI decreases with rate. Fixing the AoI, the channel with more users has larger rate; and fixing the rate, the channel with less users has smaller AoI.

Finally, we propose a age-mix-rate MW policy in the latter class (coded-coding policies) with 3 users (Section VII). Simulations show that the age-mix-rate MW policy in the latter class (respectively, policies without coding) provides the largest (respectively, smallest) channel capacity. Approaching the channel capacity, EAoI reach the minimum. Under the age-mix-rate MW policy in the latter class (respectively, policies without coding), the channel has the smallest (respectively, highest) minimum EAoI. So coding can decrease AoI and increase rate simultaneously.

D. Notation

We use the notations $\mathbb{E}(\cdot)$ and $\Pr(\cdot)$ for expectation and probability, respectively. We denote scalars with lower case letters, e.g. s. Denote vectors as lower case letters with underline, e.g., $\underline{s}$. Random variables are denoted by capital letters, e.g. S. Sets are denoted by calligraphy letters, e.g. $\mathcal{S}$. We use M to denote the number of users, K to denote the time horizon. $[n]$ denotes the set $\{1, 2, \cdots, n\}$. For two vectors $\underline{x}$ and $\underline{z}$, $\underline{x} \preceq \underline{z}$ represents $x_i \leq z_i$ for all i. For two sets $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \subset \mathcal{B}$ represents that $\mathcal{A}$ is a subset of $\mathcal{B}$; $\mathcal{A} \subsetneq \mathcal{B}$ represents that $\mathcal{A}$ is a proper subset of $\mathcal{B}$;

II. System Model

We generalize our original model in [31], where transmission occurs on a noisy network which we model by a BPEC with M users and a transmitter. In the beginning of time slot k, a packet intended for user i is generated with probability θ_i. Let $G_i(k) = 1$ represents that a new packet is generated (for user i) in time slot k, i.e., $\Pr\left(G_i(k) = 1\right) = \theta_i$ for $k = 1, 2, \cdots$. Every packet for user i is successfully delivered to user i with probability $1 - \epsilon_i$, $0 \leq \epsilon_i < 1$, and lost with probability ϵ_i. If a packet is not delivered to the intended user, then it can be cached by other user(s) casually. Let $\sigma_{\mathcal{I}}^{(i)}$ represent the probability that a packet for user i is cached by users in $\mathcal{I}$. Denote ϵ_0 as the probability that a packet is erased by all users. Then,

$$\sum_{\mathcal{I} \subset [M] \setminus i} \sigma_{\mathcal{I}}^{(i)} + \epsilon_0 = \epsilon_i, \quad i = 1, 2, \cdots, M. \tag{1}$$

Suppose that the *transmission delay* is one time slot. The encoder can track packets in caches using the available feedback, which act as side information for the users, and can be exploited in the cache design at the encoder. In particular, the encoder can encode the packets in the cache and form more efficient coded packets for transmission. We call a packet *coded packet* if it is encoded by more than one packets, otherwise we call it a *regular packet*. In this work, we consider XOR operation in encoding.

Consider a coded packet x. When x is received by user i, x is a *perfectly-decoded packet (for user i)* if x can be decoded into regular packets (by user i). If x is not a perfectly-decoded packet, we call x is an *imperfectly-decode packet (for user i)*. Policies with coding can be classified into two categories: 1) only perfectly-decoded packets for user i can be cached by user i, 2) user i can cache both perfectly-coded and imperfectly-coded packets. We define the former category as *regular-coding policies*, and the latter one as *coded-coding policies*. In addition, policies without coding actions are defined as *non-coding policies*. We first investigate regular-coding policies in the rest of this section, as well as Section III – Section VI, and then investigate coded-coding policies in Section VII. The non-coding policies are investigated in Appendix A and Appendix B. We can interpret regular-coded policies by the following example.

Example 1. *Consider three users* 1, 2, 3 *and the corresponding packets* $\{a_t\}_{t\geq 1}$, $\{b_t\}_{t\geq 1}$ *and* $\{c_t\}_{t\geq 1}$ *intended for users* 1, 2, 3, *respectively. In some time slot, suppose that user* 1 *have stored* c_t, *user* 2 *have stored* c_{t+1}, *and user* 3 *have stored* a_{t+1}. *If the encoder transmits the packet* $a_{t+1} \oplus c_{t+1}$, *and user* 1 *receives it. Recall that user* 1 *has* c_t, *so* $a_{t+1} \oplus c_{t+1}$ *can not be recovered by user* 1. *Then,* $a_{t+1}\oplus c_{t+1}$ *is discarded.*

Furthermore, user i can *recover* x if he can decode x and obtain the regular packet for him. If the encoder transmits packet x, then the user(s) recovering x can gain information from x. We define the user(s) who can recover x as the *potential receiver(s)* in the current time. Thus, in every time slot, we only consider the potential receivers.

Consider regular-coding policies. The encoder is modeled by a network of virtual queues. Let $Q_{i,i}$ denote the queue of incoming packets for user i. If a packet in $Q_{i,i}$ is not delivered to user i, but received by other user(s), then the packet will be cached in the corresponding cache(s). Define $Q_{i,\mathcal{S}}$ as the set such that packets for user i are received by the users in $\mathcal{S}$, where $\mathcal{S} \subset [M]\backslash i$. An example of the virtual queues of a BPEC with 3 users is given in Figure 1. Suppose that packet $a \in Q_{i,\mathcal{S}}$, then $a \in Q_{i,\mathcal{S}'}$ if $\mathcal{S}' \supset \mathcal{S}$. For notational clearness, we give the following tie-break rule.

Assumption 1. *(Tie-break Rule) If* $a \in Q_{i,\mathcal{S}}$, *for any* $\mathcal{S}' \subset \mathcal{S}$ *and* $\mathcal{S}' \neq \mathcal{S}$, $a \notin Q_{i,\mathcal{S}'}$.

Assumption 1 shows that if $a \in Q_{i,\mathcal{S}}$, then $\mathcal{S}$ is the largest set of users who cache a. In other words, packet a can not be in two queues, i.e., the map from packets to queues is a surjection. From the definition of $Q_{i,\mathcal{S}}$, in time slot k, the

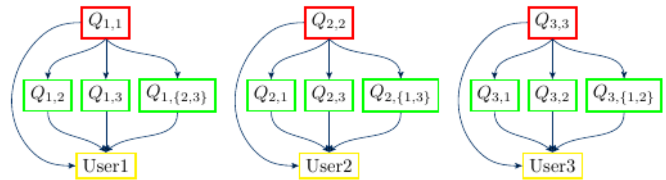

Fig. 1: The virtual queues of 3-user under regular-coding policies.

(regular) packets for user i received by other users are in the set

$$\cup_{\mathcal{S}\subset[M]\backslash i}\{a|a \in Q_{i,\mathcal{S}}\} \tag{2}$$

and packets kept by user i are in the set

$$\cup_{j\neq i,\mathcal{S}\subset[M]\backslash j,i\in\mathcal{S}}\{a|a \in Q_{j,\mathcal{S}}\}. \tag{3}$$

III. Age of Information

To capture the freshness of information, we introduce the concept *Age of Information* [1], which is defined as follows

Definition 1 (Age of Information [1])**.** *Consider a source-destination pair. Let* $\{t_k\}_k$ *be the times at which packets are generated and* $\{t'_k\}_k$ *be the times at which packets are received at the destination. At any time* ξ, *denote* $N(\xi) = \max\{k|t'_k \leq \xi\}$, *and* $u(\xi) = t_{N(\xi)}$. *The Age of Information (AoI) at the destination is* $\Delta(t) = t - u(t)$.

Denote $h_i(k)$ as the AoI of user i. Recall that transmission delay is unit. From Definition 1, $h_i(k)$ increases linearly in time when there is no delivery of packets to user i and drops with every delivery to a value that represents how old the received packet is. However, each user receives either new or old packets. If a packet older than the current received one is received, which can not bring reduction of AoI to user i, then $h_i(k)$ increases as before. We give a variant of AoI for user i.

Definition 2. *Denote the generation time of the packet received by user* i *in time slot* k *as* $v_i(k)$. $h_i(k)$ *evolves as* $h_i(k) = \min\{h_i(k-1)+1, k - v_i(k)\}$ *and* $h_i(0) = 0$.

We aim to find policy π that minimize the following expected weighted sum of age of information (EAoI) at the users:

$$\mathbb{E}\left[\frac{1}{MK}\sum_{k=1}^{K}\sum_{i=1}^{M}\alpha_i h_i^{\pi}(k)\right], \tag{4}$$

where $\alpha_1, \alpha_2, \cdots, \alpha_M$ are weights, hence the minimum AoI is given by the following optimization problem.

$$\min_{\pi\in\Pi}\lim_{K\to\infty}\mathbb{E}[J_K^{\pi}], \text{ where } J_K^{\pi} = \frac{1}{MK}\sum_{k=1}^{K}\sum_{i=1}^{M}\alpha_i h_i^{\pi}(k). \tag{5}$$

In order to formulate $h_i(k)$ precisely, it is sufficient to obtain the AoI of (virtual) queues. We first consider $Q_{i,i}$.

Lemma 1. *If $a_j \in Q_{i,i}$ has the generation time k_j, $j \in \{1,2\}$, and $k_2 > k_1$, then transmitting a_2 can not be worse than transmitting a_1 in terms of AoI.*

Remark 1. *Lemma 1 can reduce the number of transmitting actions. Equivalently, Lemma 1 implies that we can assume that the buffer size of $Q_{i,i}$ is 1.*

Proof. If a_j is delivered to user i in time slot k, then from Definition 1, the AoI of user i is $\min\{k-k_j, h_i(k-1)+1\}$. Note that $k-k_1 > k-k_2$, hence

$$\min\{k-k_1, h_i(k-1)+1\} \geq \min\{k-k_2, h_i(k-1)+1\},$$

which implies a_2 provides a larger AoI reduction (for user i) than that of a_1. Then, transmitting a_2 can not be worse than transmitting a_1 in terms of AoI. □

Denote the AoI of $Q_{i,i}$ as $w_{i,i}(k)$. Note that all packets in $Q_{i,i}$ are regular. Denote the generation time of latest packet as k'. Based on Lemma 1, we define $w_{i,i}(k) = \min\{k-k', h_i(k)\}$, and $w_{i,i}(0) = h_i(0)$. If $k-k' > h_i(k)$, then packets in $Q_{i,i}$ are older than the latest one received by user i, so packets in $Q_{i,i}$ are useless in terms of AoI in time slot k. In addition, $w_{i,i}(k)$ drops to 0 if a new packet is generated, otherwise it increases by 1. Thus, the recursion of $w_{i,i}(k)$ is

$$w_{i,i}(k+1) = \begin{cases} 0 & G_i(k)=1 \\ \min\{w_{i,i}(k)+1, h_i(k)+1\} & G_i(k)=0 \end{cases}. \quad (6)$$

Lemma 2. *If $a_j \in Q_{i,\mathcal{S}}$ has the generation time k_j, $j \in \{1,2\}$, $k_2 > k_1$, then, encoding a_2 can not be worse than encoding a_1 in terms of AoI.*

Remark 2. *Lemma 2 can reduce the number of coding actions. Lemma 2 implies we can assume that $Q_{i,\mathcal{S}}$ has buffer size 1.*

Proof. Without loss of generality, suppose a_j, $j \in \{1,2\}$ (for user i) can be encoded with other packets $b_1, b_2, \cdots, b_l$. Denote the coded packet as c_j, $j \in \{1,2\}$, respectively. Then, c_1 is a perfectly-coded packet for user i if and only if c_2 is a perfectly-coded packet for user i. We only consider the case where c_1, c_2 are perfectly-coded packets for user i, otherwise the AoI of user i has no changes after receiving c_1, c_2. We prove the lemma by two step.

Step 1. Note that a_j can be encoded with $b_1, b_2, \cdots, b_l$. Then, c_1 and c_2 can provide the same AoI reduction to the potential receivers (except user i), respectively. If the coded packet is delivered to user i, then from Definition 1, the AoI of user i is $\min\{k-k_j, h_i(k-1)+1\}$ when received c_j. Note that $k-k_1 > k-k_2$, e.g., c_2 provides smaller AoI for user i. Then, encoding a_2 can not be worse than encoding a_1 in terms of AoI.

Step 2. In the future time slot, suppose that a_2 has been received, and a_1 can be encoded with packets $b'_1, b'_2, \cdots, b'_m$, denoted by c'. If c' is delivered to user i, by Definition 2, c' can not provide AoI reduction for user i. Then, the coded packet encoded by $b'_1, b'_2, \cdots, b'_m$ can provide the same AoI reduction (as c') for the potential receivers except user i.

By **Step 1** and **Step 2**, encoding a_2 can not be worse than encoding a_1. □

Similarly, denote the AoI of $Q_{i,\mathcal{S}}$ as $w_{i,\mathcal{S}}(k)$. Let the generation time of the latest packet in $Q_{i,\mathcal{S}}$ be k'. We define $w_{i,\mathcal{S}}(k) = \min\{k-k', h_i(k)\}$ and $Q_{i,\mathcal{S}}(0) = h_i(0)$. Thus, $w_{i,\mathcal{S}}(k)$ increases by 1 unless $Q_{i,\mathcal{S}}$ receives a newer packet. If packet $a \in Q_{i,\mathcal{S}}$ is received by other users in $\mathcal{I}$, $\mathcal{I} \cap \mathcal{S} = \varnothing$, from Assumption 1, $a \in Q_{i,\mathcal{I}\cup\mathcal{S}}$ and $a \notin Q_{i,\mathcal{S}}$. Let $t_{i,\mathcal{S}}(k) = 1$ (respectively, $t_{i,i}(k) = 1$) represents the latest packet in $Q_{i,\mathcal{S}}$ (respectively, $Q_{i,i}$) is encoded and transmitted in time slot k, otherwise $t_{i,\mathcal{S}}(k) = 0$ (respectively, $t_{i,i}(k) = 0$). Similarly, let $d_i(k) = 1$ (respectively, $d_{\mathcal{S}}(k) = 1$) represents that user i (respectively, users in $\mathcal{S}$) receive(s) a packet, otherwise $d_i(k) = 0$ (respectively, $d_{\mathcal{S}}(k) = 0$). Note that $w_{i,\mathcal{S}}(k) = h_i(k)$ when $Q_{i,\mathcal{S}}$ is empty, then the recursion of $w_{i,\mathcal{S}}(k)$ is

$$w_{i,\mathcal{S}}(k+1) = \begin{cases} \min\{w_{i,i}(k)+1, h_i(k)+1\} & \mathcal{P}^{(1)}_{i,\mathcal{S}}(k) \\ \min\{w_{i,\mathcal{S}'}(k)+1, h_i(k)+1\} & \mathcal{P}^{(2)}_{i,\mathcal{S}}(k) \\ w_{i,\mathcal{S}}(k)+1 & \text{otherwise} \end{cases}. \quad (7)$$

where

$$\mathcal{P}^{(1)}_{i,\mathcal{S}}(k) = \{d_i(k)=0, d_{\mathcal{S}}(k)=1, t_{i,i}(k)=1\}$$

and

$$\mathcal{P}^{(2)}_{i,\mathcal{S}}(k) = \{\mathcal{S}' \subset \mathcal{S}, d_i(k)=0, d_{\mathcal{S}\setminus\mathcal{S}'}(k)=1, t_{i,\mathcal{S}'}(k)=1\}.$$

From (6) and (7), the recursion of $h_i(k)$ is

$$h_i(k+1) = \begin{cases} w_{i,i}(k)+1 & t_{i,i}(k)=1, d_i(k)=1 \\ w_{i,\mathcal{S}}(k)+1 & t_{i,\mathcal{S}}(k)=1, d_i(k)=1 \\ h_i(k)+1 & \text{otherwise} \end{cases}. \quad (8)$$

A. Coding Actions

Lemma 1 and Lemma 2 reduce the number of coding actions, in time slot k only the latest packet in every queue matters in terms of AoI. Recall that we only consider potential receivers in each time slot. Then, the encoder decides among the following actions, denoted by $A(k)$, and defined below:

- $A(k) = Q_{i,i}$: a packet is transmitted from $Q_{i,i}$;
- $A(k) = \oplus_{j=1}^{l} Q_{\tau_j,\mathcal{S}_{\tau_j}}$: a coded packet is transmitted from $Q_{\tau_1,\mathcal{S}_{\tau_1}}, Q_{\tau_2,\mathcal{S}_{\tau_2}}, \cdots, Q_{\tau_l,\mathcal{S}_{\tau_l}}$, and users $\tau_1, \tau_2, \cdots, \tau_l$ are potential receivers.

IV. Lower Bound

In this section, we derive lower bounds of AoI among all policies. Before investigating AoI, we consider the channel capacity. This is because the lower bound of AoI depends on the channel capacity [31], [32].

A. Capacity Outer Bound

To obtain the channel capacity is sophisticated due to these coding actions. Then we start by providing the capacity outer bound [33, Section III]. Denote $\hat{C}$ as the channel capacity outer bound. Let π is a permutation of $[M]$ such that $\pi(M-i+1)=\hat{\pi}(i)$, where $\hat{\pi}$ is the permutation defined in [33, Definition 1]. Let $\epsilon_{\mathcal{I}}$ be the probability that an erasure occurs for all users in $\mathcal{I}$. Denote $\hat{\epsilon}_{\pi(i)}=\epsilon_{\cup_{j=1}^{i}\{\pi(j)\}}$. From [33, Lemma 3, Lemma 4], we can obtain the outer bound $\hat{C}$

$$\hat{C}=\cap_{\pi}\hat{C}_{\pi} \tag{9}$$

where

$$\hat{C}_{\pi}=\Big\{(R_1,R_2,\cdots,R_M)|\sum_{i\in[M]}\frac{R_{\pi(i)}}{1-\hat{\epsilon}_{\pi(i)}}\leq 1\Big\}. \tag{10}$$

Consider a special class of channels, symmetric and independent channels, we have following result.

Lemma 3. *Consider a symmetric and independent channel with M users, i.e., $\epsilon_i=\epsilon$ for all $i\in[M]$. The capacity region $\hat{\mathcal{C}}$ is given by*

$$\hat{\mathcal{C}}=\Big\{(R_1,\cdots,R_M)|0\leq R_i\leq\frac{1}{\sum_{j=1}^{M}1/(1-\epsilon^j)}\Big\}. \tag{11}$$

Proof. Note that the channel is symmetric and independent, i.e., $\epsilon_1=\epsilon_2=\cdots=\epsilon_M=\epsilon$. Then, $\epsilon_{\mathcal{I}}=\epsilon^{|\mathcal{I}|}$. By the symmetry, for any two different permutation π_1, π_2, $\hat{C}_{\pi_1}=\hat{C}_{\pi_2}$. Since $\epsilon_{\mathcal{I}}=\epsilon^{|\mathcal{I}|}$, then from (9) and (10),

$$\hat{\mathcal{C}}=\Big\{(R_1,\cdots,R_M)|0\leq R_i\leq\frac{1}{\sum_{j=1}^{M}1/(1-\epsilon^j)}\Big\}.$$

□

B. Lower Bound

From [31], [32], the lower bound of AoI depends on the channel capacity. Based on Section IV-A, we obtain the relationship between AoI and capacity outer bound. Similar to [32, Section III], we derive two lower bounds on the achievable age performance. The first lower bound, denoted by LB_1, is derived by assuming that there is always a fresh (regular) packet to be delivered. The second lower bound, denoted by LB_2, is derived by assuming that all (regular) packets are delivered instantaneously upon their arrivals.

Theorem 1. *For any scheduling policy π,*

(1) *Let R_i be defined in* (9) *and* (10)*,*

$$LB_1=\frac{M}{2\sum_{i=1}^{M}\frac{R_i}{\alpha_i}}+\sum_{i=1}^{M}\frac{\alpha_i}{2M};$$

(2) $LB_2=\frac{1}{M}\sum_{i=1}^{M}\frac{\alpha_i}{\theta_i}$.

Proof. For a large time horizon K and look at the packets intended for user i. Let $N_i(K)$ denote the number of recovered packets up to and including time slot K. Now consider the m^{th} and $(m+1)^{th}$ recovered packets and denote the delivery time of them at user i by $T_i(m)$ and $T_i(m+1)$, respectively. The inter-delivery time

$$I_i(m)=T_i(m+1)-T_i(m) \tag{12}$$

is the time between these two consecutive deliveries. Upon arrival of the m^{th} recovered packet for user i, the age of information of user i drops to the value $D_i(m)$ which represents how much delay the packet has experienced in the system. Let L_i be the number of remaining time slots after the last packet recovery for user i. Now define $\Gamma_i(m)$ as the sum of age functions $h_i(k)$, where k is in the interval $[T_i(m),T_i(m+1))$:

$$\Gamma_i=\sum_{k=T_i(m)}^{T_i(m)+I_i(m)-1}h_i(k) \tag{13}$$

$$=\frac{1}{2}I_i^2(m)-\frac{1}{2}I_i(m)+D_i(m-1)I_i(m). \tag{14}$$

It follows that in the limit of large K, we have

$$J^{\pi}(M)=\lim_{K\to\infty}\Big[\frac{1}{M}\sum_{i=1}^{M}\alpha_i\frac{1}{K}\sum_{m=1}^{N_i(K)}\Gamma_i(m)\Big]. \tag{15}$$

Using this formulation, we next lower bound EAoI. Let R_i denote the capacity outer bound of user i, defined in (9) and (10). Note that in the limit of large K, $\frac{N_i(K)}{K}$ is the rate/throughput of user i and

$$\lim_{K\to\infty}\frac{N_i(K)}{K}\leq R_i. \tag{16}$$

Then, we prove the first part of Theorem 1. Consider any scheduling policy and a large time-horizon K. The EAoI defined in (5) can be re-written in terms of $\Gamma_i(m)$:

$$J_K^{\pi}=\frac{1}{M}\sum_{i=1}^{M}\frac{\alpha_i}{K}\Big(\sum_{m=1}^{N_i(K)}\Gamma_i(m)+\frac{1}{2}L_i^2+D_i(N_i(K))L_i-\frac{1}{2}L_i\Big).$$

Since $D_i(m)\geq 1$ for all $1\leq m\leq N_i(K)$, we can lower bound (15) by substituting $D_i(m-1)=1$. Using similar steps as [2, Eqns. (9) - (14)],

$$\lim_{K\to\infty}\mathbb{E}\big[J_K^{\pi}\big]\geq\lim_{K\to\infty}\mathbb{E}\Big[\frac{1}{2M}\sum_{i=1}^{M}\frac{\alpha_i K}{N_i(K)}+\frac{\sum_{i=1}^{M}\alpha_i}{2M}\Big]. \tag{17}$$

Now note that by the Cauchy-Schwarz inequality, we have

$$\lim_{K\to\infty}\mathbb{E}\Big[\sum_{i=1}^{M}\frac{N_i(K)}{\alpha_i K}\Big]\mathbb{E}\Big[\sum_{i=1}^{M}\frac{\alpha_i K}{N_i(K)}\Big]\geq M^2,$$

and thus

$$\lim_{K\to\infty}\mathbb{E}\Big[\sum_{i=1}^{M}\frac{\alpha_i K}{N_i(K)}\Big]\geq M^2/\sum_{i=1}^{M}\frac{R_i}{\alpha_i}. \tag{18}$$

Inserting this back into (17), we obtain

$$J^{\pi}(M)\geq\frac{M}{2\sum_{i=1}^{M}\frac{R_i}{\alpha_i}}+\frac{\sum_{i=1}^{M}\alpha_i}{2M}.$$

Next, we prove the second part of Theorem 1. Suppose that all packets are recovered instantaneously with one time-unit delay. A lower bound to EAoI in this scenario constitutes a lower bound to EAoI in our setup. Let $X_i(m)$ denote the inter arrival time between m^{th} and $(m+1)^{th}$ packets. $\{X_i(m)\}_m$ is a geometric i.i.d sequence. Under the assumption of instantaneous recovery, $I_i(m) = X_i(m)$. It hence follows from (14) that

$$\Gamma_i(m) = \frac{1}{2}X_i^2(m) + \frac{1}{2}X_i(m).$$

Thus, similar with [2], the time-average AoI of user i is

$$\begin{aligned}\mathbb{E}[h_i] &= \lim_{K\to\infty} \frac{1}{K}\sum_{k=1}^{K} h_i(k)\\ &= \frac{\mathbb{E}[X_i^2(m)]}{2\mathbb{E}[X_i(m)]} + \frac{1}{2} = \frac{2-\theta_i}{2\theta_i} + \frac{1}{2}\end{aligned}$$

and

$$J^\pi(M) \geq \frac{1}{M}\sum_{i=1}^{M}\frac{\alpha_i}{\theta_i}.$$

□

V. Age-based Max-Weight Policy

From Lemma 1 and Lemma 2, only the latest packet in every queue matters in terms of AoI. With the objective of minimizing AoI, it is sufficient to encode or transmit the latest packet in every (virtual) queue. In addition, we can assume that if packet a (for user i) is recovered, then all packets (for user i) in caches older a are discarded. We give the following assumption, which will be released in Section VI and Section VII.

Assumption 2. *The (virtual) queue $Q_{i,i}$ and $Q_{i,\mathcal{S}}$ has buffer size* 1.

Under Assumption 2, the stability region is $0 < \theta_i \leq 1$ for all i. To schedule packets efficiently, similar to [2], [3], [31], we propose age-based Max-Weight (AMW) policies based on (6), (7) and (8). We define *age-gain* of queue $Q_{i,i}$, $Q_{i,\mathcal{S}}$ (for user i), where $\mathcal{S} \subset [M]\backslash i$,

$$\delta_{i,i}(k) = h_i(k) - w_{i,i}(k) \tag{19}$$

$$\delta_{i,\mathcal{S}}(k) = h_i(k) - w_{i,\mathcal{S}}(k). \tag{20}$$

$\delta_{i,i}(k)$, $\delta_{i,\mathcal{S}}(k)$ quantifies how much the instantaneous user's age of information reduces upon the successful delivery from the encoder. If $Q_{i,i}$ or $Q_{i,\mathcal{S}}$ is empty or contains old packets in time slot k, then by (19) and (20), $\delta_{i,i}(k) = 0$ or $\delta_{i,\mathcal{S}}(k) = 0$.

Now we devise deterministic policies using techniques from Lyapunov Optimization. Denote

$$\vec{h}(k) = (h_1(k), h_2(k), \cdots, h_M(k))$$

and

$$\vec{s}(k) = \Big(\{h_i(k)\}_i, \{w_{i,i}(k)\}_i, \{w_{i,\mathcal{S}}(k)\}_{i,\mathcal{S}\subset[M]\backslash i}\Big). \tag{21}$$

Define the Lyapunov function

$$L(\vec{h}(k)) = \sum_{i=1}^{M}\alpha_i h_i(k), \tag{22}$$

and the one-slot Lyapunov Drift

$$\Theta(\vec{h}(k)) = \mathbb{E}[L(\vec{h}(k+1)) - L(\vec{h}(k))|\vec{s}(k)]. \tag{23}$$

We devise the AMW policy such that it minimizes the one-slot Lyapunov drift:

Definition 3. *In each slot k, the AMW policy chooses the action that has the maximum weight as shown in following Table:*

$A(k)$	Weights
$Q_{i,i}$	$\alpha_i\delta_{i,i}(k)(1-\epsilon_i)$
$\oplus_{j\in[l]}Q_{\tau_j,\mathcal{S}_{\tau_j}}$	$\sum_{j=1}^{l}\alpha_\tau\delta_{\tau_j,\mathcal{S}_{\tau_l}}(k)(1-\epsilon_{\tau_j})$

Theorem 2. *The AMW policy defined in Definition 3 minimizes the one-slot Lyapunov Drift in each slot.*

Proof. Note that we only consider potential receivers. Let $r_{i,i}(k)$, $r_{i,\mathcal{S}}(k)$ represent the number of packets delivered to user i in time slot k, $r_{i,i}(k) = t_{i,i}(k)d_i(k)$ and $r_{i,\mathcal{S}}(k) = t_{i,\mathcal{S}}(k)d_i(k)$. Then,

$$r_{i,i}(k), r_{i,\mathcal{S}}(k) \in \{0,1\}$$

and

$$r_{i,i}(k) + \sum_{\mathcal{S}\subset[M]\backslash i} r_{i,\mathcal{S}}(k) \leq 1.$$

From (8),

$$\begin{aligned} h_i(k+1) &= r_{i,i}(k)\big(w_{i,i}(k)+1\big)\\ &+ \sum_{\mathcal{S}\subset[M]\backslash i} r_{i,\mathcal{S}}(k)(w_{i,\mathcal{S}}(k)+1)\\ &+ \big(1 - r_{i,i}(k) - \sum_{\mathcal{S}\subset[M]\backslash i} r_{i,\mathcal{S}}(k)\big)(h_i(k)+1).\end{aligned} \tag{24}$$

Using (24), we can re-write the Lyapunov Drift as follows:

$$\begin{aligned}\Theta(\vec{h}(k)) =& \mathbb{E}\big[L(\vec{h}(k+1)) - L(\vec{h}(k))|\vec{s}(k)\big]\\ =& \sum_{i=1}^{M}\alpha_i\mathbb{E}\Big[\sum_{\mathcal{S}\subset[M]\backslash i} r_{i,\mathcal{S}_i}(k)\big(w_{i,\mathcal{S}}(k) - h_i(k)\big)\\ &+ r_{i,i}(k)\big(w_{i,i}(k) - h_i(k)\big) + 1\Big]\\ =& -\sum_{i=1}^{M}\alpha_i\mathbb{E}\big[\sum_{\mathcal{S}\subset[M]\backslash i} r_{i,\mathcal{S}}(k)\delta_{i,\mathcal{S}}(k) + r_{i,i}(k)\delta_{i,i}(k)\big]\\ &+ \sum\alpha_i.\end{aligned}$$

Therefore,

$$\begin{aligned}\Theta(\vec{h}(k)) =& \sum_{i=1}^{M}\alpha_i - \sum_{i=1}^{M}1_{\{A(k)=Q_{i,i}\}}\alpha_i\delta_{i,i}(k)(1-\epsilon_i)\\ &- \sum_{\tau_1,\tau_2,\cdots,\tau_l} 1_{\{A(k)=\oplus_{j\in[l]}Q_{\tau_j,\mathcal{S}_{\tau_j}}\}}\\ &\times\sum_{j=1}^{l}\alpha_{\tau_j}\delta_{i,\mathcal{S}_{\tau_j}}(k)(1-\epsilon_{\tau_j}).\end{aligned} \tag{25}$$

From (25), we have the desired results. $\square$

Theorem 3. *The EAoI achieved by the proposed AMW policy is upper bounded by*

$$UB^{AMW} = \frac{1}{M}\sum_{i=1}^{M}\frac{\alpha_i}{\theta_i} + \frac{\sum_{i=1}^{M}\frac{1}{1-\epsilon_i}}{M}\sum_{i=1}^{M}\alpha_i. \tag{26}$$

Proof. From the proof of Theorem 2,

$$\begin{aligned}\Theta(\vec{h}(k)) =& \sum_{i=1}^{M}\alpha_i\mathbb{E}\Big[\sum_{\mathcal{S}\subset[M]\setminus i} r_{i,\mathcal{S}}(k)\big(w_{i,\mathcal{S}}(k)-h_i(k)\big)\\ &+ r_{i,i}(k)\big(w_{i,i}(k)-h_i(k)\big)+1\Big].\end{aligned}$$

From (7), $w_{i,\mathcal{S}}(k) \le h_i(k)$ for all $\mathcal{S}$, i, thus

$$\begin{aligned}\Theta(\vec{h}(k)) \le& \sum_{i=1}^{M}\alpha_i\mathbb{E}\Big[r_{i,i}(k)\big(w_{i,i}(k)-h_i(k)\big)+1\Big]\\ =& \sum_{i=1}^{M}\alpha_i\mathbb{E}\Big[r_{i,i}(k)\big(w_{i,i}(k)-h_i(k)\big)\Big]+\sum_{i=1}^{M}\alpha_i.\end{aligned}$$

To obtain the AMW policy in Definition 3, we minimize $\Theta(\vec{h}(k))$ in every time slot, then suppose that $Q_{i,i}$ is not empty and we transmit packets in $Q_{i,i}$, i.e., $t_{i,i}=1$, $t_{j,j}=0$ for all $j \neq i$, thus we have

$$\min\Theta(\vec{h}(k)) \le \alpha_i(1-\epsilon_i)\mathbb{E}\big[w_{i,i}(k)-h_i(k)\big]+\sum_{i=1}^{M}\alpha_i$$

which implies

$$\frac{\min\Theta(\vec{h}(k))}{1-\epsilon_i} \le \alpha_i\mathbb{E}\big[w_{i,i}(k)-h_i(k)\big]+\frac{\sum_{i=1}^{M}\alpha_i}{1-\epsilon_i}.$$

Then, let $\gamma = \sum_{i=1}^{M}\frac{1}{1-\epsilon_i}$,

$$\gamma\min\Theta(\vec{h}(k)) \le \sum_{i=1}^{M}\alpha_i\mathbb{E}\big[w_{i,i}(k)-h_i(k)\big]+\gamma\sum_{i=1}^{M}\alpha_i.$$

Note that $L\big(\vec{h}(0)\big)=0$ and

$$\mathbb{E}[L\big(\vec{h}(k+1)\big)] = \mathbb{E}[L\big(\vec{h}(k)\big)]+\min\Theta(\vec{h}(k)).$$

Again, summing over k, we obtain

$$\begin{aligned}&0 \le \gamma\mathbb{E}[L\big(\vec{h}(K+1)\big)]\\ &\le \sum_{k=1}^{K}\sum_{i=1}^{M}\alpha_i\mathbb{E}\big[w_{i,i}(k)-h_i(k)\big]+K\gamma\sum_{i=1}^{M}\alpha_i.\end{aligned} \tag{27}$$

From (6), $w_{i,i}(k) \le W_i(k)$, where $W_i(k)$ is defined in

$$W_i(k+1) = \begin{cases}0 & G_i(k)=1\\ W_i(k)+1 & G_i(k)=0\end{cases}. \tag{28}$$

Substituting (28) into (27),

$$\mathbb{E}[\sum_{k=1}^{K+1}\sum_{i=1}^{M}\alpha_i h_i(k)] \le \mathbb{E}[\sum_{k=1}^{K}\sum_{i=1}^{M}\alpha_i W_i(k)]+K\gamma\sum_{i=1}^{M}\alpha_i. \tag{29}$$

Let $K\to\infty$, note that $W_i(k)$ is a geometric random variable with parameter θ_i, we have

$$\begin{aligned}&\lim_{K\to\infty}\frac{1}{KM}\mathbb{E}[\sum_{k=1}^{K+1}\sum_{i=1}^{M}\alpha_i h_i(k)]\\ &\le \frac{1}{M}\sum_{i=1}^{M}\alpha_i\mathbb{E}[W_i(k)]+\frac{\gamma}{M}\sum_{i=1}^{M}\alpha_i.\end{aligned} \tag{30}$$

Since $\mathbb{E}[W_i(k)] = \frac{1}{\theta_i}$, then

$$\lim_{K\to\infty}\mathbb{E}[J_K^{MW}] \le \frac{1}{M}\sum_{i=1}^{M}\frac{\alpha_i}{\theta_i}+\frac{\gamma}{M}\sum_{i=1}^{M}\alpha_i.$$

$\square$

VI. Tradeoff between AoI and Rate

From [33], coding actions can enhance the rate of broadcast channels. Thus, we investigate the tradeoff between AoI and rate. From Section V, the AMW policy would not encode or transmit packets in queues with age-gain 0. Although old packets have no contribution to AoI reduction, they can enhance the rate, which implies old packets are useful if we consider the rate. Then, we release Assumption 2. Suppose that every queue has *infinite* buffer size.

To capture the impact of old packets, the queue length is introduced into the AMW policy. We propose age-mix-rate Max-Weight (AMR) policies to schedule packets efficiently. Define $Y_i(k)$ as the queue length of user i. Let $Y_i(k)$ equal to the number of packets intended for user i which are kept in the cache at the *beginning* of time slot k. Then,

$$Y_i(k+1) = \sum_{\tau=0}^{k}G_i(\tau)-\sum_{\tau=0}^{k}\big(r_{i,i}(\tau)+\sum_{\mathcal{S}\subset[M]\setminus i}r_{i,\mathcal{S}}(\tau)\big) \tag{31}$$

where $r_{i,i}(k) = t_{i,i}(k)d_i(k)$ and $r_{i,\mathcal{S}}(k) = t_{i,\mathcal{S}}(k)d_i(k)$ represents the number of packets recovered by user i from $Q_{i,i}$ and $Q_{i,\mathcal{S}}$, respectively, in time slot k. The value of $\sum_{\tau=0}^{k}G_i(\tau)$ is the number of packets that user i have been generated before $k+1$ and $\sum_{\tau=0}^{k}\big(r_{i,i}(\tau)+\sum_{\mathcal{S}\in[M]\setminus i}r_{i,\mathcal{S}}(\tau)\big)$ is the total number of packets actually recovered in the same interval. Suppose that the generation rate θ is fixed, then maximizing the rate is equivalent to minimizing the queue lengths. In particular, if $Y_i(k)=0$, then all packets for user i are recovered during time interval $[0,k-1]$. Therefore the minimum AoI and maximum rate can be obtained by the following optimization,

$$\min_{A(k)}\lim_{K\to\infty}\mathbb{E}[\tilde{J}_K] \tag{32}$$

where

$$\tilde{J}_K = \frac{1}{MK}\sum_{k=1}^{K}\sum_{i=1}^{M}\big(\alpha_i h_i(k)+\beta_i Y_i(k)\big). \tag{33}$$

From (31), the recursion of $Y_i(k)$ is

$$Y_i(k+1) = Y_i(k) - r_{i,i}(k) - \sum_{\mathcal{S}\in[M]\setminus i} r_{i,\mathcal{S}}(k) + G_i(k). \tag{34}$$

where $Y_i(0)=0$. Denote $\vec{Y}(k) = (Y_1(k), Y_2(k), \cdots, Y_M(k))$. We define the Lyapunov function $L(\vec{h}(k))$ as

$$L(\vec{h}(k), \vec{Y}(k)) = (1-\lambda)\sum_{i=1}^{M} \alpha_i h_i(k) + \lambda \sum_{i=1}^{M} \beta_i Y_i^2(k), \tag{35}$$

where $0 \le \lambda \le 1$. We define the one-slot Lyapunov Drift as

$$\Theta(k) = \mathbb{E}\Big[L(\vec{h}(k+1), \vec{Y}(k+1)) - L(\vec{h}(k), \vec{Y}(k)) | \vec{s}(k)\Big] \tag{36}$$

where $\vec{s}(k)$ is defined in (21). Given $\{\vec{s}(\tau)\}_{\tau=0}^{k}$, $\{r_{i,i}(\tau)\}_{\tau=0}^{k}$ and $\{r_{i,\mathcal{S}}(\tau)\}_{\tau=0}^{k}$ are known, hence $\{Y_i(\tau)\}_{\tau=0}^{k}$ is known. Define the *rate-gain* of user i in time slot k as

$$\begin{aligned} f_i(k) =& 1_{\{Y_i(k)=0\}}\beta_i\theta_i(1-\epsilon_i) \\ &+ 1_{\{Y_i(k)>0\}}\beta_i(2Y_i(k)+2\theta_i-1)(1-\epsilon_i). \end{aligned} \tag{37}$$

Definition 4. *In each slot k, the AMR policy chooses the action that has the maximum weight as shown in following Table:*

$A(k)$	Weights
$Q_{i,i}$	$(1-\lambda)\alpha_i\delta_{i,i}(k)(1-\epsilon_i) + \lambda f_i(k)$
$\oplus_{j\in[l]} Q_{\tau_j,\mathcal{S}_{\tau_j}}$	$(1-\lambda)\sum_{j=1}^{l} \alpha_{\tau_j}\delta_{\tau_j,\mathcal{S}_{\tau_j}}(k)(1-\epsilon_{\tau_j})$ $+\lambda\sum_{j=1}^{l} f_{\tau_j}(k)$

Remark 3. *$f_i(k)$ can be regarded as the throughput-gain of actions.*

Theorem 4. *The AMR policy defined in Definition 4 minimizes the one-slot Lyapunov Drift in each slot.*

Proof. Based on the definitions of $r_{i,i}(k)$ and $r_{i,\mathcal{S}}(k)$, we have

$$r_{i,i}(k), r_{i,\mathcal{S}}(k) \in \{0,1\}$$

and

$$r_{i,i}(k) + \sum_{\mathcal{S}\subset[M]\setminus i} r_{i,\mathcal{S}}(k) \le 1.$$

Let $\Theta(k) = (1-\lambda)\Theta_1(k) + \lambda\Theta_2(k)$, where

$$\begin{aligned} \Theta_1(k) =& \mathbb{E}\Big[\sum_{i=1}^{M}\alpha_i h_i(k+1) - \sum_{i=1}^{M}\alpha_i h_i(k)|\vec{s}(k)\Big] \\ \Theta_2(k) =& \mathbb{E}\Big[\sum_{i=1}^{M}\beta_i Y_i^2(k+1) - \sum_{i=1}^{M}\beta_i Y_i^2(k)|\vec{s}(k)\Big]. \end{aligned}$$

Similar with the proof of Theorem 2,

$$\begin{aligned} \Theta_1(\vec{h}(k)) =& -\sum_{i=1}^{M}\alpha_i\mathbb{E}\big[\sum_{\mathcal{S}} r_{i,\mathcal{S}}(k)\delta_{i,\mathcal{S}}(k) + r_{i,i}(k)\delta_{i,i}(k)\big] \\ &+ \sum \alpha_i. \end{aligned} \tag{38}$$

Recall that $Y_i(k)$ is known. Then, we consider

$$\tilde{\Theta}_2(k) = \mathbb{E}\Big[\sum_{i=1}^{M}\beta_i Y_i^2(k+1)|\vec{s}(k)\Big]$$

Note that $r_{i,i}^2(k) = r_{i,i}(k)$, $r_{i,\mathcal{S}}^2(k) = r_{i,\mathcal{S}}(k)$ and $G_i^2(k) = G_i(k)$. Note that $\Pr\big(G_i(k)=1\big) = \theta_i$ and $\Pr\big(G_i(k)=0\big) = 1-\theta_i$. We consider the following two cases.

Case 1. If $Y_i(k)=0$ and $G_i(k)=0$, then $r_{i,i}(k) = r_{i,\mathcal{S}}(k) = 0$, hence $Y_i(k+1)=0$. If $G_i(k)=1$, then

$$\begin{aligned} &\mathbb{E}\big[\beta_i Y_i^2(k+1)|\vec{s}(k)\big] \\ =& 1_{\{Y_i(k)=0\}}\beta_i\theta_i\mathbb{E}\Big[r_{i,i}(k) + \sum_{\mathcal{S}} r_{i,\mathcal{S}}(k) + 1 \\ &- 2\big(r_{i,i}(k) + \sum_{\mathcal{S}} r_{i,\mathcal{S}}(k)\big)|\vec{s}(k)\Big] \\ =& 1_{\{Y_i(k)=0\}}\beta_i\theta_i\big(1 - \mathbb{E}[r_{i,i}(k)] - \sum_{\mathcal{S}}\mathbb{E}[r_{i,\mathcal{S}}(k)]\big). \end{aligned}$$

Case 2. If $Y_i(k)>0$, note that $G_i(k)$ is independent of $r_{i,i}(k)$ and $r_{i,\mathcal{S}}(k)$, then

$$\begin{aligned} &\mathbb{E}\big[\beta_i Y_i^2(k+1)|\vec{s}(k)\big] \\ =& \beta_i\mathbb{E}\Big[Y_i^2(k) + r_{i,i}(k) + \sum_{\mathcal{S}} r_{i,\mathcal{S}}(k) + G_i(k) \\ &-2Y_i(k)\big(r_{i,i}(k) + \sum_{\mathcal{S}} r_{i,\mathcal{S}}(k)\big) + 2Y_i(k)G_i(k) \\ &-2\big(r_{i,i}(k) + \sum_{\mathcal{S}} r_{i,\mathcal{S}}(k)\big)G_i(k)|\vec{s}(k)\Big], \end{aligned}$$

which implies

$$\begin{aligned} &\mathbb{E}\big[\beta_i Y_i^2(k+1)|\vec{s}(k)\big] \\ =& 1_{\{Y_i(k)>0\}}\beta_i\big(Y_i^2(k) + \theta_i + 2\theta_i Y_i(k)\big) \\ &- 1_{\{Y_i(k)>0\}}\beta_i\big(\mathbb{E}[r_{i,i}(k)] + \sum_{\mathcal{S}}\mathbb{E}[r_{i,\mathcal{S}}(k)]\big)(2Y_i(k)+2\theta_i-1). \end{aligned}$$

Therefore, if $A(k) = Q_{i,i}$ (when $Q_{i,i}$ is not empty), then $r_{i,i}(k)=1$ with probability $1-\epsilon_i$,

$$\begin{aligned} \tilde{\Theta}_2(k) =& -f_i(k) + \sum_{i=1}^{M} 1_{\{Y_i(k)=0\}}\beta_i\theta_i \\ &+ \sum_{i=1}^{M} 1_{\{Y_i(k)>0\}}\beta_i\big(Y_i^2(k) + \theta_i + 2\theta_i Y_i(k)\big) \\ =& -f_i(k) + \sum_{i=1}^{M}\beta_i\big(Y_i^2(k) + \theta_i + 2\theta_i Y_i(k)\big) \end{aligned}$$

where $f_i(k)$ is defined in (37).

If $A(k) = \oplus_{j\in[l]} Q_{\tau_j,\mathcal{S}_{\tau_j}}$ (when $Q_{\tau_j,\mathcal{S}_{\tau_j}}$ are not empty), then $r_{\tau_j,\mathcal{S}_{\tau_j}}(k) = 1$ with probability $1-\epsilon_{\tau_j}$,

$$\begin{aligned}\tilde{\Theta}_2(k) =& -\sum_{j=1}^{l} f_{\tau_j}(k) + \sum_{i=1}^{M} 1_{\{Y_i(k)=0\}}\beta_i\theta_i \\ &+ \sum_{i=1}^{M} 1_{\{Y_i(k)>0\}}\beta_i\big(Y_i^2(k)+\theta_i+2\theta_i Y_i(k)\big) \\ =& -\sum_{j=1}^{l} f_{\tau_j}(k) + \sum_{i=1}^{M}\beta_i\big(Y_i^2(k)+\theta_i+2\theta_i Y_i(k)\big).\end{aligned}$$

For user i, note that erasure probability is ϵ_i, by (37),

$$\begin{aligned}\tilde{\Theta}_2\big(\vec{h}(k)\big) =& -\sum_{i} 1_{\{A(k)=Q_{i,i}\}} f_i(k) \\ &- \sum_{\tau_j,\mathcal{S}_{\tau_j},j\in[l]} 1_{\{A(k)=\oplus_{j\in[l]}Q_{\tau_j,\mathcal{S}_{\tau_j}}\}} f_{\tau_j}(k) \\ &+ \sum_{i=1}^{M}\beta_i\big(Y_i^2(k)+\theta_i+2\theta_i Y_i(k)\big).\end{aligned} \tag{39}$$

From (38) and (39), we can obtain the desired results. □

Theorem 5. *The EAoI under the proposed AMR policy has upper bounded* (26).

Proof. From the proof of Theorem 4,

$$\begin{aligned}\Theta_1(k) =& \sum_{i=1}^{M}\alpha_i\mathbb{E}\Big[\sum_{\mathcal{S}\subset[M]\setminus i} r_{i,\mathcal{S}_i}(k)\big(w_{i,\mathcal{S}}(k)-h_i(k)\big) \\ &+ r_{i,i}(k)\big(w_{i,i}(k)-h_i(k)\big)+1\Big] \\ \le& \sum_{i=1}^{M}\alpha_i\mathbb{E}\Big[r_{i,i}(k)\big(w_{i,i}(k)-h_i(k)\big)+1\Big].\end{aligned}$$

The last inequality holds since from (7), $w_{i,\mathcal{S}}(k) \le h_i(k)$ for all $\mathcal{S}$, i.

Then, we consider $\Theta_2(k)$. Let $r_{i,i}(k) = 1$, then $r_{j,j}(k) = 0, j \ne i$ and $r_{i,\mathcal{S}}(k) = 0$, note that $\mathbb{E}[G_i(k)] = \theta_i$, then

$$\begin{aligned}\Theta_2(k) =& 1_{\{Y_i(k)=0\}}\theta_i(1-1) \\ &+ 1_{\{Y_i(k)>0\}}\big(1-2Y_i(k)\big)(1-\theta_i) \\ =& 1_{\{Y_i(k)>0\}}\big(1-2Y_i(k)\big)(1-\theta_i).\end{aligned}$$

To obtain the AMR policy in Definition 4, we minimize $\Theta(k)$ in every time slot, then let $r_{i,i}(k) = 1, r_{j,j}(k) = 0, j \ne i$, and $r_{i,\mathcal{S}}(k) = 0$, then we have

$$\begin{aligned}\min\Theta(k) \le& (1-\lambda)\Big(\alpha_i\big(w_{i,i}(k)-h_i(k)\big)+\sum_{i=1}^{M}\alpha_i\Big) \\ &+ \lambda\beta_i 1_{\{Y_i(k)>0\}}\big(1-2Y_i(k)\big)(1-\theta_i).\end{aligned}$$

Note that $Y_i(k)$ is a positive integer, then $\big(1-2Y_i(k)\big)(1-\theta) < 0$, thus

$$\min\Theta(\vec{h}(k)) \le \alpha_i\big(w_{i,i}(k)-h_i(k)\big) + \sum_{i=1}^{M}\alpha_i.$$

The rest proof is exact the same as the proof of Theorem 3. □

VII. Coded-coding Policies with Three Users

Now we consider coded-coding policies, where imperfectly-coded packets can be cached in caches. Caching coded packets can bring more coding opportunity in the system. However, the number of coding actions increases exponentially due to coded packets. Thus, we consider BPECs with 3 users. Denote the packets intended for user 1, user 2 and user 3 as $\{a_t\}_{t\ge1}$, $\{b_t\}_{t\ge1}$ and $\{c_t\}_{t\ge1}$, respectively. For clear representation, we suppose that a_t (respectively, b_t, c_t) is older than a_{t+1} (respectively, b_{t+1}, c_{t+1}). Define the *length* of a coded packet as the number of regular packets encoded within it. Let $\mathcal{X} = \big\{\{1,2\},\{1,3\},\{2,3\}\big\}$. If a coded packet $x = a_{t_1} \oplus b_{t_2}$, then we call that x *contains* a_{t_1} and b_{t_2}. A regular packet in $Q_{i,\mathcal{S}}$ will not be removed even if it is encoded in a coded packet, which is cached by another user. in First of all, we give the following assumption.

Assumption 3. *User* 1 *(respectively, user* 2*, user* 3*) can not store (after decoding) coded packets containing* a_t *(respectively,* b_t*,* c_t*).*

Assumption 3 can be interpreted by the following example. The reasons why we have Assumption 3 are that 1) the assumption can simplify the coding actions, 2) the complementary event of Assumption 3 happens with a very small probability.

Example 2. *Consider a independent and symmetric channel. In a ideal situation,* c_1 *is generated in time slot* 0*,* a_1 *is generated in time slot* 1*,* b_1 *is generated in time slot* 2*, and no packets are generated in time slots* 3 *and* 4*. Suppose that under the AMR policy,* c_1 *is transmitted in time slot* 0*, but only user* 2 *receives it.* a_1 *is transmitted in time slot* 1*, but only user* 2 *receives it.* b_1 *is transmitted in time slot* 2*, but only user* 3 *receives it. Then, in time slot* 3*,* $b_1 \oplus c_1$ *is transmitted because it can bring largest age gain and throughput gain, but only user* 1 *receives it. Consequently, in time slot* 4*, transmitting* $a_1 \oplus b_1 \oplus c_1$ *can bring largest age gain and throughput gain. Suppose user* 3 *receives the coded packet, note that user* 3 *caches* b_1*, so after decoding the coded packet, user* 3 *obtains* b_1 *and* $a_1 \oplus c_1$*. By Assumption 3,* $a_1 \oplus c_1$ *can not be stored by user* 3*.*

In Example 2, the probability of $a_1 \oplus b_1 \oplus c_1$ being received by user 3 equals to $(\theta(1-\theta)^2\epsilon(1-\epsilon)\epsilon)^3 \times (1-\theta)^3\epsilon^2(1-\epsilon) \times (1-\theta)^3(1-\epsilon) < \epsilon^8(1-\theta)^{12}$. So $\epsilon^8(1-\theta)^{12}$ is a very small value for proper ϵ and θ. Therefore, under Assumption 3, we obtain a good approximation of coded-coding policies when BPECs have 3 users.

Then, we need to reduce the number of coding actions.

Lemma 4. *Coding actions which encode* 2 *different regular packets for a same user can be removed.*

Proof. Without loss of generality, suppose that a coded packet, denoted by x, is encoded by a_1, a_2 and other (regular or coded) packets. We consider the following two cases.

Case 1. If 1) one of a_1 and a_2 is a regular packet or 2) both a_1 and a_2 are regular packets, then user 1 is one of the potential receiver. Note that a_1 and a_2 are in caches (in forms of regular packets or coded packets), then a_1 and a_2 are not recovered by user 1. Then, user 1 is not the potential receiver (of x), which implies it is not necessary for coded packet x to exist.

Case 2. If a_1 and a_2 are in two different coded packets, from **Case 1**, user 1 can not recover packet x, so user 1 is not the potential receiver, i.e., coded packet x is transmitted to user 2 and user 3. Note that $x = a_1 \oplus a_2 \oplus y$, where y is either a regular packet or a coded packet. Note that a_1 and a_2 are in two different coded packets, then users $2, 3$ can not recover x, hence users $2, 3$ are not potential receivers, which implies it is not necessary for coded packet x to exist. □

From Lemma 4, the length of coded packets is at most 3. If coded packets have length 3, then they must have forms as $a_{t_1} \oplus b_{t_2} \oplus c_{t_3}$. Let $Q_{i,i}$ and $Q_{i,\mathcal{S}}$ be defined in Section II, where $i \in [3]$ and $\mathcal{S} \subset [3] \backslash i$. If the encoded transmits a coded packet with length 3, then users $1, 2, 3$ are the potential receivers. So coded packets which are cached must have length 2. Let $\mathcal{I} \in \mathcal{X}$ and $l = [3] \backslash \mathcal{I}$. Define $Q_{\mathcal{I},l}$ as the queue such that coded packets encoded by regular packets for users in $\mathcal{I}$ are received by user l.

Lemma 5. *Coding actions which encode packets from $Q_{\mathcal{I}_1,l_1}$, $Q_{\mathcal{I}_2,l_2}$, $l_1 \neq l_2$, can be removed.*

Proof. Consider a coded packet $x = x_1 \oplus x_2$, where $x_1 \in Q_{\mathcal{I}_1,l_1}$ and $x_2 \in Q_{\mathcal{I}_2,l_2}$. Note that $\mathcal{I}_1, \mathcal{I}_2 \in \mathcal{X}$, then, coded packet x must has 2 different regular packets intended for a same user. By Lemma 4, these coding actions can be removed. □

Lemma 6. *Coding actions which encode and transmit packets only from $Q_{\mathcal{I},l}$ can be removed.*

Proof. Consider a packet $x \in Q_{\mathcal{I},l}$. Without loss of generality, let $\mathcal{I} = \{1, 2\}$, $l = 3$. Then, x has the form $x = a_{t_1} \oplus b_{t_2}$. Therefore, transmitting x from $Q_{\mathcal{I},l}$ is equivalent to transmitting a coded packet encoded by a_{t_1} from $Q_{1,\mathcal{S}_1}$ and b_{t_2} from $Q_{2,\mathcal{S}_2}$. So it is not necessary to transmit packets encoded only from $Q_{\mathcal{I},l}$, where $\mathcal{I} \in \mathcal{X}$. These coding actions can be removed. □

From Lemma 6, packets in $Q_{\mathcal{I},l}$ can be encoded with packets in $Q_{l,\mathcal{I}}$, in this case the potential receivers are user 1, user 2 and user 3; or packets in $Q_{\mathcal{I},l}$ can be encoded with packets in $Q_{l,i}$, where $\mathcal{I} = \{i, j\}$, in this case the potential receivers are user l and user i. From Lemma 4, Lemma 5 and Lemma 6, in each time slot k, the encoder decides among the following actions, denoted by $A(k)$, and defined below:

- $A(k) = Q_{i,i}$: a packet is transmitted from $Q_{i,i}$, $i \in [3]$;
- $A(k) = Q_{i,\mathcal{S}_i} \oplus Q_{j,\mathcal{S}_j}$, $Q_{1,\mathcal{S}_1} \oplus Q_{2,\mathcal{S}_2} \oplus Q_{3,\mathcal{S}_3}$, $i, j \in [3], i \neq j$.
- $A(k) = Q_{l,\mathcal{I}} \oplus Q_{\mathcal{I},l}$, $Q_{l,i} \oplus Q_{\mathcal{I},l}$: a coded packet is encoded by a packet from $Q_{l,\mathcal{I}}$ or $Q_{l,i}$ and a packet from $Q_{\mathcal{I},l}$ where $i, l \in [3]$, $i \neq l$.

Now, we consider the AoI of packets in $Q_{\mathcal{I},l}$. Without loss of generality, consider $\mathcal{I} = \{1, 2\}$, $l = 3$. For $x \in Q_{\{1,2\},3}$, $x = a_{t_1} \oplus b_{t_2}$. Define the AoI of x as $(h_{x,1}(k), h_{x,2}(k))$, where the first element, $h_{x,1}(k)$, is the AoI of a_{t_1}, and the second element, $h_{x,2}$, is the AoI of b_{t_2}.

Lemma 7. *Under AMR policies, suppose that $x_1, x_2 \in Q_{\mathcal{I},l}$ and x_1 enters $Q_{\mathcal{I},l}$ before x_2. Then $(h_{x_2,1}, h_{x_2,2}) \preceq (h_{x_1,1}, h_{x_1,2})$.*

Proof. Note that the AMR policy is a one-step optimal algorithm. If a coded packet $x = a_{t_1} \oplus b_{t_2}$ is transmitted in time slot k, then $a_{t_1} \oplus b_{t_2}$ has the largest gain (age-gain and throughput-gain) in the same type. Since $x_1, x_2 \in Q_{\mathcal{I},l}$, then x_2 are in the same type as x_1. Since x_1 enters $Q_{\mathcal{I},l}$ before x_2, then x_2 must have larger gain than that of x_1, otherwise the encoder will always transmit x_1 when packets in the type of x_1 are transmitted.

Note that x_1 and x_2 have the same throughput-gain $\big($from (37)$\big)$, and x_2 has larger age-gain than that of x_1, then at least one of the two regular packets encoded within x_2 are newer than the corresponding ones in x_1, respectively. Hence, $(h_{x_2,1}, h_{x_2,2}) \preceq (h_{x_1,1}, h_{x_1,2})$. □

From Lemma 7, we define the AoI of $Q_{\mathcal{I},l}$ as the AoI of the latest packet in $Q_{\mathcal{I},l}$. Let $\mathcal{I} = \{i, j\}$ and $l = [3] \backslash \mathcal{I}$, denote the generation time of the regular packets (encoded in the latest packet) for user i (respectively, user j) as v_i (respectively, v_j). Then, the AoI of packet x as $w_{\mathcal{I},l}(k)$,

$$w_{\mathcal{I},l}(k) = \big(\min\{k - v_i, h_i(k-1) + 1\}, \min\{k - v_j, h_j(k-1) + 1\}\big) \tag{40}$$

where $w_{\mathcal{I},l}(0) = (h_i(0), h_j(0))$. Similar with (19) and (20), we can define *age-gain of $Q_{\mathcal{I},j}$* as

$$\delta_{\mathcal{I},l}(k) = h_{\mathcal{I}}(k) - w_{\mathcal{I},l}(k) \triangleq (\delta_{\mathcal{I},l}^{(i)}(k), \delta_{\mathcal{I},l}^{(j)}(k)) \tag{41}$$

where $h_{\mathcal{I}}(k) = \big(h_i(k), h_j(k)\big)$.

Let $t_{\mathcal{I},j}(k) = 1$ represents the latest packet in $Q_{\mathcal{I},j}$ is encoded and transmitted in time slot k, otherwise $t_{\mathcal{I},j}(k) = 0$. Similar with Section VI, we define the queue length user i as

$$\begin{aligned} Y_i(k+1) = &\sum_{\tau=0}^{k} G_i(\tau) - \sum_{\tau=0}^{k} r_{i,i}(k) \\ &- \sum_{\tau=0}^{k} \sum_{\mathcal{S}} r_{i,\mathcal{S}}(k) - \sum_{\tau=0}^{k} \sum_{\mathcal{I}: i \in \mathcal{I}} r_{\mathcal{I},l}^{(i)}(k) \end{aligned} \tag{42}$$

where $r_{i,i}(k)$, $r_{i,\mathcal{S}}(k)$ are define in Section VI, and $r_{\mathcal{I},j}(k) = (r_{\mathcal{I},l}^{(i)}(k), r_{\mathcal{I},l}^{(j)}(k))$, $r_{\mathcal{I},l}^{(i)}(k) = t_{\mathcal{I},l}(k) d_i(k)$, and $r_{\mathcal{I},l}^{(j)}(k) = t_{\mathcal{I},l}(k) d_j(k)$ represents the number of packets delivered to user

i and user j from $Q_{\mathcal{I},l}$ in time slot k, respectively. Then, the recursion of $Y_i(k)$ is

$$\begin{aligned} Y_i(k+1) =& Y_i(k) + G_i(k) - r_{i,i}(k) \\ & - \sum_{\mathcal{S}} r_{i,\mathcal{S}}(k) - \sum_{\mathcal{I}:i\in\mathcal{I}} r^{(i)}_{\mathcal{I},l}(k) \end{aligned} \tag{43}$$

where $Y_i(0) = 0$.

Therefore the minimum AoI is obtained by the following optimization,

$$\min_{\pi\in\Pi} \quad \lim_{K\to\infty} \mathbb{E}[\tilde{J}_K] \tag{44}$$

where

$$\tilde{J}_K = \frac{1}{MK} \sum_{k=1}^{K} \sum_{i=1}^{M} \big(\alpha_i h_i(k) + \beta_i Y_i(k)\big). \tag{45}$$

Now we define the Lyapunov function $L\big(\vec{h}(k), \vec{Y}(k)\big)$ as

$$L\big(\vec{h}(k), \vec{Y}(k)\big) = (1-\lambda) \sum_{i=1}^{M} \alpha_i h_i(k) + \lambda \sum_{i=1}^{M} \beta_i Y_i^2(k), \tag{46}$$

where λ is a positive number. We also define the one-slot Lyapunov Drift as

$$\Theta(k) = \mathbb{E}\Big[L(\vec{h}(k+1), \vec{Y}(k+1)) - L(\vec{h}(k), \vec{Y}(k)) | \vec{b}(k)\Big] \tag{47}$$

where

$$\vec{b}(k) = \Big(\{h_i(k)\}_i, \{w_{i,i}(k)\}_i, \{w_{i,\mathcal{S}}(k)\}_i, \{w_{\mathcal{I},l}(k)\}_l\Big).$$

Given $\{\vec{b}(\tau)\}_{\tau=0}^k$, $\{r_{i,i}(\tau)\}_{\tau=0}^k$, $\{r_{i,\mathcal{S}}(\tau)\}_{\tau=0}^k$ and $\{r_{\mathcal{I},l}(\tau)\}_{\tau=0}^k$ are known, hence $\{Y_i(\tau)\}_{\tau=0}^k$ is known.

Definition 5. *In each slot k, the AMR policy chooses the action that has the maximum weight as follows: $A(k) = Q_{i,i}, \oplus_j Q_{j,\mathcal{S}_j}$ have the same weights as that in Definition 4 and*

$A(k)$	Weights
$Q_{l,i} \oplus Q_{l,\mathcal{I}}$	$(1-\lambda)\alpha_l \delta_{l,i}(k)(1-\epsilon_l) + \lambda f_i(k)$ $(1-\lambda)\alpha_i \delta^{(i)}_{\mathcal{I},l}(k)(1-\epsilon_i) + \lambda f_l(k)$
$Q_{l,\mathcal{I}} \oplus Q_{\mathcal{I},l}$	$(1-\lambda)\alpha_l \delta_{l,\mathcal{I}}(k)(1-\epsilon_l)$ $+(1-\lambda)\sum_{\mathcal{I}:i\in\mathcal{I}} \alpha_i \delta^{(i)}_{\mathcal{I},l}(k)(1-\epsilon_i)$ $+\lambda \sum_{i\in[3]} f_i(k)$

Theorem 6. *The AMR policy defined in Definition 5 minimizes the one-slot Lyapunov Drift in each slot.*

Proof. From the definition of $r_{i,i}(k)$, $r_{i,\mathcal{S}}(k)$ and $r_{\mathcal{I},j}(k)$, we have

$$r_{i,i}(k), r_{i,\mathcal{S}}(k), r^{(i)}_{\mathcal{I},l}(k), r^{(j)}_{\mathcal{I},l}(k) \in \{0,1\}$$

and

$$r_{i,i}(k) + \sum_{\mathcal{S}\subset[M]\setminus i} r_{i,\mathcal{S}}(k) + \sum_{\mathcal{I}:i\in\mathcal{I}} r^{(i)}_{\mathcal{I},l}(k) \le 1$$

Let $\Theta(k) = (1-\lambda)\Theta_1(k) + \lambda\Theta_2(k)$, where

$$\begin{aligned} \Theta_1(k) =& \mathbb{E}\Big[\sum_{i=1}^{M} \alpha_i h_i(k+1) - \sum_{i=1}^{M} \alpha_i h_i(k) | \vec{b}(k)\Big] \\ \Theta_2(k) =& \mathbb{E}\Big[\sum_{i=1}^{M} \beta_i Y_i^2(k+1) - \sum_{i=1}^{M} \beta_i Y_i^2(k) | \vec{b}(k)\Big]. \end{aligned}$$

Similar with the proof of Theorem 2,

$$\begin{aligned} \Theta_1(k) =& -\sum_{i=1}^{M} \alpha_i \mathbb{E}\big[\sum_{\mathcal{S}} r_{i,\mathcal{S}}(k)\delta_{i,\mathcal{S}}(k) + r_{i,i}(k)\delta_{i,i}(k) \\ & + \sum_{i\in\mathcal{I}} r^{(i)}_{\mathcal{I},l}(k)\delta^{(i)}_{\mathcal{I},l}(k)\big] + \sum \alpha_i. \end{aligned} \tag{48}$$

Note that $Y_i(k)$ is a known value. Then, we consider

$$\tilde{\Theta}_2(k) = \sum_{i}^{M} \mathbb{E}\big[\beta_i Y_i^2(k+1) | \vec{b}(k)\big].$$

Then, let $r_i(k) = r_{i,i}(k) + \sum_{\mathcal{S}} r_{i,\mathcal{S}}(k) + \sum_{\mathcal{I}:i\in\mathcal{I}} r^{(i)}_{\mathcal{I},l}(k)$, and $r_i(k) \le 1$,

$$\begin{aligned} \tilde{\Theta}_2(k) =& \lambda \sum_{i} \mathbb{E}\Big[Y_i^2(k) + r_i(k) + G_i(k) - 2Y_i(k)r_i(k) \\ & + 2Y_i(k)G_i(k) - 2r_i(k)G_i(k)\Big]. \end{aligned}$$

Note that $r_i^2(k) = r_i(k)$ and $G_i^2(k) = G_i(k)$, $\Pr\big(G_i(k) = 1\big) = \theta$ and $\Pr\big(G_i(k) = 0\big) = 1-\theta$. Similar with the proof of Theorem 4, we consider the following two cases.

Case 1. If $Y_i(k) = 0$, then

$$\begin{aligned} & \mathbb{E}\big[\beta_i Y_i^2(k+1) | \vec{b}(k)\big] \\ =& 1_{\{Y_i(k)=0\}} \beta_i \theta \mathbb{E}\Big[1 - r_i(k) | \vec{b}(k)\Big]. \end{aligned}$$

Case 2. If $Y_i(k) > 0$, then

$$\begin{aligned} & \mathbb{E}\big[\beta_i Y_i^2(k+1) | \vec{b}(k)\big] \\ =& \beta_i \mathbb{E}\Big[Y_i^2(k) + r_i(k) + G_i(k) - 2Y_i(k)r_i(k) \\ & + 2Y_i(k)G_i(k) - 2r_i(k)G_i(k) | \vec{b}(k)\Big], \end{aligned}$$

which implies

$$\begin{aligned} & \mathbb{E}\big[\beta_i Y_i^2(k+1) | \vec{s}(k)\big] \\ =& 1_{\{Y_i(k)>0\}} \beta_i \big(Y_i^2(k) + \theta + 2\theta Y_i(k)\big) \\ & - 1_{\{Y_i(k)>0\}} \beta_i \mathbb{E}\big[r_i(k)\big](2Y_i(k) + 2\theta - 1). \end{aligned}$$

Therefore, if $A(k) = Q_{i,i}$ (when $Q_{i,i}$ is not empty), then $r_{i,i}(k) = 1$ with probability $1-\epsilon_i$,

$$\begin{aligned} \tilde{\Theta}_2(k) =& -f_i(k) + \sum_{i=1}^{M} 1_{\{Y_i(k)=0\}} \beta_i \theta \\ & + \sum_{i=1}^{M} 1_{\{Y_i(k)>0\}} \beta_i \big(Y_i^2(k) + \theta + 2\theta Y_i(k)\big) \\ =& -f_i(k) + \sum_{i=1}^{3} \beta_i \big(Y_i^2(k) + \theta + 2\theta Y_i(k)\big) \end{aligned}$$

where $f_i(k)$ is defined in (37).

If $A(k) = \oplus_{j\in[l]} Q_{\tau_j,\mathcal{S}_{\tau_j}}$ (when $Q_{\tau_j,\mathcal{S}_{\tau_j}}$ are not empty), then $r_{\tau_j,\mathcal{S}_{\tau_j}}(k) = 1$ with probability $1-\epsilon_{\tau_j}$,

$$\begin{aligned}\tilde{\Theta}_2(k) =& -\sum_{j=1}^{l} f_{\tau_j}(k) + \sum_{i=1}^{3} 1_{\{Y_i(k)=0\}}\beta_i\theta \\ &+ \sum_{i=1}^{3} 1_{\{Y_i(k)>0\}}\beta_i\big(Y_i^2(k)+\theta+2\theta Y_i(k)\big). \\ =& -\sum_{j=1}^{l} f_{\tau_j}(k) + \sum_{i=1}^{3}\beta_i\big(Y_i^2(k)+\theta+2\theta Y_i(k)\big).\end{aligned}$$

If $A(k) = Q_{l,\mathcal{I}} \oplus Q_{\mathcal{I},l}$ (when $Q_{l,\mathcal{I}}$ and $Q_{\mathcal{I},l}$ are not empty), then,

$$\begin{aligned}\tilde{\Theta}_2(k) =& -\sum_{i\in[3]} f_i(k) + \sum_{i\in[3]} 1_{\{Y_i(k)=0\}}\beta_i\theta \\ &+ \sum_{i\in[3]} 1_{\{Y_i(k)>0\}}\beta_i\big(Y_i^2(k)+\theta+2\theta Y_i(k)\big). \\ =& -\sum_{i\in[3]} f_i(k) + \sum_{i\in[3]}\beta_i\big(Y_i^2(k)+\theta+2\theta Y_i(k)\big).\end{aligned}$$

If $A(k) = Q_{l,i} \oplus Q_{\mathcal{I},l}$ (when $Q_{l,i}$ and $Q_{\mathcal{I},l}$ are not empty), then,

$$\begin{aligned}\tilde{\Theta}_2(k) =& -f_i(k) - f_j(k) + \sum_{i\in[3]} 1_{\{Y_i(k)=0\}}\beta_i\theta \\ &+ \sum_{i\in[3]} 1_{\{Y_i(k)>0\}}\beta_i\big(Y_i^2(k)+\theta+2\theta Y_i(k)\big) \\ =& -f_i(k) - f_j(k) + \sum_{i\in[3]}\beta_i\big(Y_i^2(k)+\theta+2\theta Y_i(k)\big).\end{aligned}$$

For user i, note that erasure probability is ϵ_i,

$$\begin{aligned}\tilde{\Theta}_2(k) =& -\sum_{i} 1_{\{A(k)=Q_{i,i}\}} f_i(k) \\ &- \sum_{j\in[l]} 1_{\{A(k)=\oplus_{j\in[l]}Q_{\tau_j,\mathcal{S}_{\tau_j}}\}} f_{\tau_j}(k) \\ &- 1_{\{A(k)=Q_{l,\mathcal{I}}\oplus Q_{\mathcal{I},l}\}} \sum_{i\in[3]} f_i(k) \\ &- 1_{\{Q_{l,i}\oplus Q_{\mathcal{I},l}\}}\big(f_l(k)+f_i(k)\big) \\ &+ \sum_{i\in[3]}\beta_i\big(Y_i^2(k)+\theta+2\theta Y_i(k)\big).\end{aligned} \tag{49}$$

From (48) and (49), we can obtain the desired results. □

Theorem 7. *The EAoI under the proposed AMR policy has upper bounded* (26).

Proof. The proof is similar to that of Theorem 5. □

VIII. Numerical Results and Discussion

In this section, we answer the questions proposed in Section I-A. We consider an symmetric and independent channel. Let $\epsilon_i = \epsilon$ and $\theta_i = \theta$ for $i \in [M]$. Consider $M = 3k$ where $k \in \mathbb{N}^+$ and $\alpha_i = i \bmod 3$, $i \in [M]$.

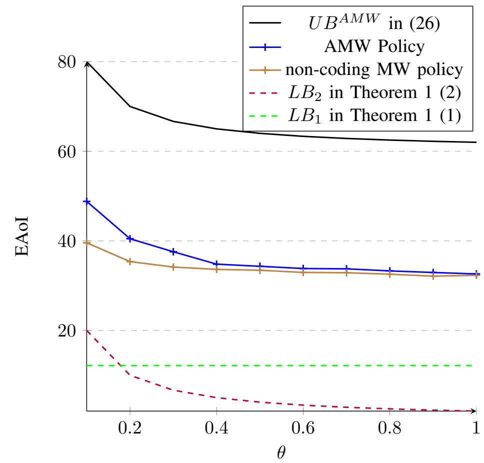

Fig. 2: EAoI as a function of θ with $M = 9$ when $\epsilon = 0.7$.

We first consider the AMW policy in Definition 3 and the non-coding MW policy in Definition 6 in Appendix A. From Assumption 2, the stability region is $0 < \theta \le 1$. Figure 2 plots the EAoI under AMW policy and non-coding MW policy when $M = 9$, as well as the two lower bounds in Theorem 1 and the upper bound in (26), which shows coding is indeed beneficial, and the benefits of coding are significant under small θ.

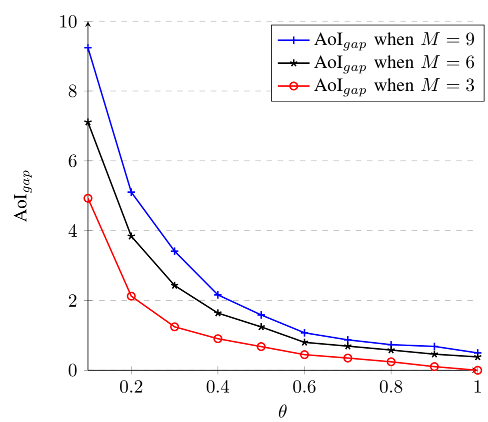

Fig. 3: AoI$_{gap}$ as a function of θ when $\epsilon = 0.7$.

We then consider the gap between the EAoI under the AMW policy in Definition 3 and the EAoI under the non-coding MW policy in Definition 6 in Appendix A. Denote the gap as AoI$_{gap}$. The relationship between AoI$_{gap}$ and θ and M is provided in Figure 3. For fixed M and ϵ, AoI$_{gap}$ decreases with θ, i.e., which implies the benefits of coding decrease with θ. Note that the buffer sizes of (virtual) queues are 1 (Assumption 2). The (expected) number of newly incoming packets increases when θ gets large, which can weaken the impact of coding actions.

Figure 4 plots AoI_{gap} as a function of ϵ when θ is fixed. AoI_{gap} increases with ϵ. This is because lost packets (due to erasure probability) can be cached and provide more coding opportunities in future slots. AoI_{gap} increases slowly when is small, and sharply when is large. In addition, from Figure 2 and Figure 3, for fixed ϵ and θ, the benefits of coding increase as M gets larger.

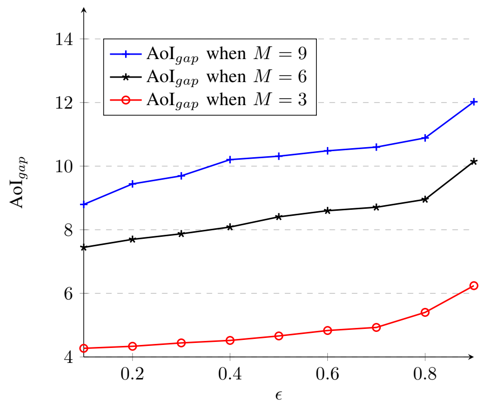

Fig. 4: AoI_{gap} as a function of ϵ when $\theta = 0.1$.

Since every packet has the same contribution in terms of throughput, then let $\beta_i = 1$ for all $i \in [M]$ and $\lambda = 0.9$. The tradeoff between the AoI and rate is illustrated in Figure 5. Under the AMW policy defined in Definition 4, the channel capacity can not be obtained analytically. Then, we obtain the channel capacity numerically. The channel capacity are around $0.32, 0.35, 0.36$ when $M = 3, 6, 9$, respectively. In the stability region, EAoI decreases with rate. When the EAoI is fixed, a BPEC with more users can provide larger rate; when the rate is fixed, a BPEC with less users can provide smaller EAoI.

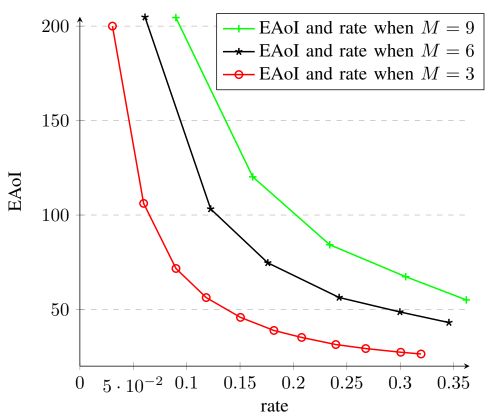

Fig. 5: The tradeoff between AoI and rate when $\epsilon = 0.7$.

When M = 3, the tradeoff between EAoI and rate under the non-coding MW policy, the AMW policy and the AMR policy (defined in Definition 5) is revealed in Figure 6. Among these policies, the AMR policy (respectively, non-coding MW policy) provides the largest (respectively, smallest) channel capacity. Approaching the channel capacity, EAoI reach the minimum. Under the AMR policy (respectively, non-coding MW policy), the channel has the smallest (respectively, largest) minimum EAoI. Thus, the coding actions can decrease AoI and increase rate simultaneously.

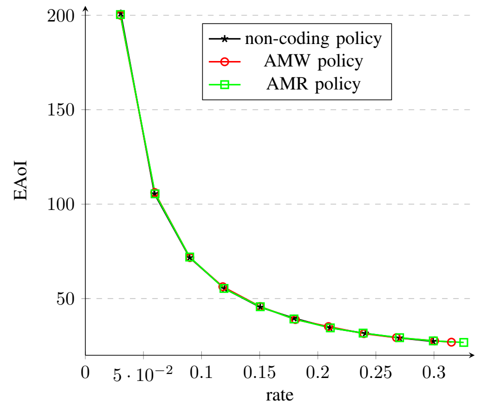

Fig. 6: The tradeoff between AoI and rate (when $M = 3$) under non-coding, AMW and AMR polices when $\epsilon = 0.7$.

IX. Conclusion and Future Directions

We investigated the benefit of coding in memoryless broadcast channel with M users. A novel framework of network AoI was proposed. We proposed two kinds of MW policies, one is AMW policy, which minimizes AoI; the other is AMR policy, which minimizes AoI and maximize rate simultaneously. Two general lower bounds and the upper bound for the proposed MW policies are obtained. The tradeoff of AoI and rate under the AMW and AMR policies was considered. Simulation showed that coding is beneficial, and the benefit increases with the number of users. In addition, when the AoI is fixed, channels with more users have larger rate; when the rate is fixed, channels with less users have smaller AoI.

Future research includes generalizations to accommodate the two scenarios. 1) AMR policies with M users: In this case, coding situation is extremely complex. It is important to propose a simple and efficient framework. 2) Global optimal policies: How to obtain the global optimal policy (rather than a one-step optimal policy) is another important direction.

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Appendix A
Max-Weight Policies in Non-coding Channel

We devise deterministic policies without coding using techniques from Lyapunov Optimization. Denote the ESAoI for Max-Weight policies (in the long run) as $\mathbb{E}[J]$. Denote

$$\vec{h}(k) = (h_1(k), h_2(k), \cdots, h_M(k)).$$

Define the Lyapunov function

$$L(\vec{h}(k)) = \frac{1}{M}\sum_{i=1}^{M} \alpha_i h_i(k), \tag{50}$$

and the one-slot Lyapunov Drift

$$\Theta(\vec{h}(k)) = \mathbb{E}[L(\vec{h}(k+1)) - L(\vec{h}(k))|\vec{h}(k)]. \tag{51}$$

We devise the Max-Weight (MW) policy such that it minimizes the one-slot Lyapunov drift:

Definition 6. *In each slot k, the MW policy chooses the action that has the maximum weight as shown in following Table:*

$A(k)$	Weights
$Q_{i,i}$	$\alpha_i\delta_{i,i}(k)(1-\epsilon_i)$

Theorem 8. *The MW policy defined in Definition 6 minimizes the one-slot Lyapunov Drift in each slot.*

Proof. From the definitions of $t_{i,i}(k)$ and $d_i(k)$, we have $\sum_{i=1}^{M} t_{i,i}(k) = 1$. Let $r_{i,i}(k) = t_{i,i}(k)d_i(k)$ represents the number of packets delivered to user i. Similar to (8), we can write the recursion of $h_i(k)$ as

$$h_i(k+1) = \begin{cases} w_{i,i}(k)+1 & r_{i,i}(k)=1 \\ h_i(k)+1 & \text{otherwise} \end{cases}. \tag{52}$$

From (52),

$$h_i(k+1) = r_{i,i}(k)\big(w_{i,i}(k)+1\big) + \big(1-r_{i,i}(k)\big)\big(h_i(k)+1\big). \tag{53}$$

Using (24), we can re-write the Lyapunov Drift as follows:

$$\begin{aligned}\Theta(\vec{h}(k)) =& \mathbb{E}\big[L(\vec{h}(k+1)) - L(\vec{h}(k))|\vec{s}(k)\big] \\ =& \sum_{i=1}^{M}\alpha_i\mathbb{E}\Big[-r_{i,i}(k)\big(h_i(k)-w_{i,i}(k)\big)-1\Big] \\ =& \sum_{i=1}^{M}\alpha_i\mathbb{E}\big[-r_{i,i}(k)\delta_{i,i}(k)-1\big].\end{aligned}$$

In every time slot, we have

$$\mathbb{E}[r_{i,i}(k)] = 1_{\{A(k)=Q_{i,i}\}}(1-\epsilon_i). \tag{54}$$

From (54), we obtain the desired results. □

Appendix B
Tradeoff between AoI and Rate in Non-coding Channels

The queue length $Y_i(k)$ can be defined as

$$Y_i(k+1) = \sum_{\tau=0}^{k} G_i(\tau) - \sum_{\tau=0}^{k} r_{i,i}(k) \tag{55}$$

where $r_{i,i}(k) = t_{i,i}(k)d_i(k)$ represents the number of packets delivered to user i from $Q_{i,i}$ in time slot k. Then, define the objective function as $\lim_{T\to\infty}\mathbb{E}[\tilde{J}_K^\pi]$, where

$$\tilde{J}_K^\pi = \frac{1}{MK}\sum_{k=1}^{K}\sum_{i=1}^{M}\alpha_i h_i^\pi(k) + \lambda\frac{1}{MK}\sum_{k=1}^{K}\sum_{i=1}^{M}\beta_i Y_i^\pi(k). \tag{56}$$

Therefore the minimum AoI is obtained by the following optimization,

$$\min_{\pi\in\Pi} \quad \lim_{K\to\infty}\mathbb{E}[\tilde{J}_K^\pi] \tag{57}$$

where $\tilde{J}_K^\pi$ is defined in (56).

From (55), the recursion of $Y_i(k)$ is

$$Y_i(k+1) = Y_i(k) - r_{i,i}(k) + G_i(k). \tag{58}$$

where $Y_i(0)=0$.

We define the Lyapunov function $L(\vec{h}(k))$ as

$$L(\vec{h}(k)) = (1-\lambda)\sum_{i=1}^{M}\alpha_i h_i(k) + \lambda\sum_{i=1}^{M}\beta_i Y_i^2(k), \tag{59}$$

where λ is a positive number. We also define the one-slot Lyapunov Drift as

$$\Theta\big(\vec{h}(k)\big) = \mathbb{E}\Big[L(\vec{h}(k+1)) - L(\vec{h}(k))|\vec{h}(k)\Big]. \tag{60}$$

Definition 7. *In each slot k, the MW policy chooses the action that has the maximum weight as shown in following Table:*

$A(k)$	Weights
$Q_{i,i}$	$(1-\lambda)\alpha_i\delta_{i,i}(k)(1-\epsilon_i)+\lambda f_i(k)$

Theorem 9. *The MW policy defined in Definition 7 minimizes the one-slot Lyapunov Drift in each slot.*

Proof. Based on the definitions of $r_{i,i}(k)$, we have

$$r_{i,i}(k)\in\{0,1\}, \quad \sum_{i=1}^{M} r_{i,i}(k) \le 1.$$

Let $\Theta\big(\vec{h}(k)\big) = (1-\lambda)\Theta_1\big(\vec{h}(k)\big) + \lambda\Theta_2\big(\vec{h}(k)\big)$, where

$$\begin{aligned}\Theta_1\big(\vec{h}(k)\big) =& \mathbb{E}\Big[\sum_{i=1}^{M}\alpha_i h_i(k+1) - \sum_{i=1}^{M}\alpha_i h_i(k)|\vec{s}(k)\Big] \\ \Theta_2\big(\vec{h}(k)\big) =& \mathbb{E}\Big[\sum_{i=1}^{M}\beta_i Y_i^2(k+1) - \sum_{i=1}^{M}\beta_i Y_i^2(k)|\vec{s}(k)\Big].\end{aligned}$$

Similar with the proof of Theorem 2,

$$\begin{aligned}\Theta_1(\vec{h}(k)) =& -\sum_{i=1}^{M}\alpha_i\mathbb{E}\big[\sum_{\mathcal{S}} r_{i,\mathcal{S}}(k)\delta_{i,\mathcal{S}}(k) + r_{i,i}(k)\delta_{i,i}(k)\big] \\ &+ \sum\alpha_i.\end{aligned} \tag{61}$$

Recall that given $\big\{\vec{h}(\tau)\big\}_{\tau=1}^{k}$, $r_{i,i}(\tau)$ is known. Thus, $Y_i(k)$ is known values. Then, we consider

$$\tilde{\Theta}_2\big(\vec{h}(k)\big) = \mathbb{E}\Big[\sum_{i=1}^{M}\beta_i Y_i^2(k+1)|\vec{s}(k)\Big]$$

Note that $r_{i,i}^2(k) = r_{i,i}(k)$, $G_i^2(k) = G_i(k)$. Note that $\Pr\big(G_i(k)=1\big) = \theta$ and $\Pr\big(G_i(k)=0\big) = 1-\theta$. We consider the following two cases.

Case 1. If $Y_i(k)=0$ and $G_i(k)=0$, then $r_{i,i}(k)=0$, hence $Y_i(k+1)=0$. If $G_i(k)=1$, then

$$\begin{aligned}&\mathbb{E}\big[\beta_i Y_i^2(k+1)|\vec{h}(k)\big] \\ =& 1_{\{Y_i(k)=0\}}\beta_i\theta\mathbb{E}\Big[1-r_i(k)|\vec{h}(k)\Big].\end{aligned}$$

Case 2. If $Y_i(k) > 0$, note that $G_i(k)$ is independent of $r_{i,i}(k)$, then

$$\begin{aligned}&\mathbb{E}\big[\beta_i Y_i^2(k+1)|\vec{s}(k)\big]\\=&\beta_i\mathbb{E}\Big[Y_i^2(k)+r_{i,i}(k)+G_i(k)-2Y_i(k)r_{i,i}(k)\\&+2Y_i(k)G_i(k)-2r_{i,i}(k)G_i(k)|\vec{h}(k)\Big],\end{aligned}$$

which implies

$$\begin{aligned}&\mathbb{E}\big[\beta_i Y_i^2(k+1)|\vec{h}(k)\big]\\=&1_{\{Y_i(k)>0\}}\beta_i\big(Y_i^2(k)+\theta+2\theta Y_i(k)\big)\\&-1_{\{Y_i(k)>0\}}\beta_i\mathbb{E}\big[r_{i,i}(k)\big](2Y_i(k)+2\theta-1).\end{aligned}$$

Therefore, if $A(k) = Q_{i,i}$ (when $Q_{i,i}$ is not empty), then $r_{i,i}(k) = 1$ with probability $1-\epsilon_i$,

$$\begin{aligned}\tilde{\Theta}_2\big(\vec{h}(k)\big)=&-f_i(k)+\sum_{i=1}^{M}1_{\{Y_i(k)=0\}}\beta_i\theta\\&+\sum_{i=1}^{M}1_{\{Y_i(k)>0\}}\beta_i\big(Y_i^2(k)+\theta+2\theta Y_i(k)\big)\\=&-f_i(k)+\sum_{i=1}^{M}\beta_i\big(Y_i^2(k)+\theta+2\theta Y_i(k)\big).\end{aligned}$$

where $f_i(k)$ is defined in (37). From (61), we can obtain the desired results. $\square$