

# The Interplay of Competition and Cooperation Among Service Providers (Part II)

Xingran Chen, Mohammad Hassan Lotfi, Saswati Sarkar

**Abstract**—This paper investigates the incentives of mobile network operators (MNOs) for acquiring additional spectrum to offer mobile virtual network operators (MVNOs) and thereby inviting competition for a common pool of end users (EUs). We consider interactions between two service providers, an MNO and an MVNO, when the EUs 1) must choose one of them 2) have the option to defect to an outside option should the SP duo offer unsatisfactory access fees or qualities of service. We formulate a multi-stage hybrid of cooperative bargaining and non-cooperative games in which the two SPs jointly determine their spectrum acquisitions, allocations and mutual money flows through the bargaining game, and subsequently individually determine the access fees for the EUs through the non-cooperative game. We identify when the overall equilibrium solutions exist, when it is unique and characterize the equilibrium solutions when they exist. The characterizations are easy to compute, and are in closed form or involve optimizations in only one decision variable. The hybrid framework allows us to determine whether and by how much the different entities benefit due to the cooperation in spectrum acquisition decision.

**Index Terms**—Resource Sharing, Game Theory, Bargaining Game, Nash Bargaining Solution, Nash equilibrium

## I. INTRODUCTION

**T**WO different classes of service providers co-exist in the current wireless service provider (SP) market: Mobile Networks Operators (MNOs) and Mobile Virtual Network Operators (MVNOs). The MNO acquires  $I_L$  spectrum from a regulator, which he offers to a MVNO in exchange of money, and the MVNO uses  $I_F$  amount of this spectrum. Both SPs earn by selling wireless plans to end users (EUs); the MNO earns additionally by leasing her spectrum to the MVNO. Thus, they both cooperate, by sharing spectrum; they also compete, for a common pool of EUs. They clearly make different decisions, which affect their subscriptions; their payoffs have different expressions and their decisions also follow

different constraints, eg,  $I_L$  has no upper limit, while  $0 \leq I_F \leq I_L$ . In a sequence of two papers we investigate the economics of the interplay of the competition and cooperation between an MNO and an MVNO.

In the prequel, Part I, we consider that the SPs arrive at their decisions individually, in the current paper we consider that the SPs arrive at certain decisions together, and then arrive at other decisions individually. Specifically, in this paper, the SPs together decide the spectrum they acquire (i.e.,  $I_L, I_F$ ) to maximize their overall profits, and the marginal reservation fee  $s$  that the MVNO pays to the MNO for using the spectrum the MNO offers. Here  $s$  is decided so as to split the proceeds between the SPs in accordance with the subscription revenue each generates, which in turn depends on the prior preferences of the EUs for them. Subsequently, each SP individually decides the access fees for the EUs. The  $I_L, I_F, s$  are obtained as the solution of a cooperative bargaining game, and the access fees are obtained as solutions of a non-cooperative game. The bargaining and the non-cooperative games together constitute a sequential game. In contrast, in the prequel, each decision variable is selected through a non-cooperative game, each of which constitutes a stage of a sequential game. Also, the marginal reservation fee is considered a fixed parameter, and the MNO and MVNO individually decide the spectrum he acquires, and subsequently individually decide the access fees for the EUs. Note that the marginal reservation fee is indeed a market-driven parameter in a large spectrum market with many MNOs and MVNOs; in such a scenario the marginal reservation fee may be driven by the overall market evolution, and is beyond the control of individual MNOs and MVNOs. This is the case that the prequel considers. In a smaller market, the marginal reservation fee would be chosen as a decision variable through a negotiation between the MNO and the MVNO concerned. This is the case this paper considers.

The economics of the interactions of resource sharing among service providers have been investigated in many works, e.g. [4]–[28]. Non-cooperative games were considered in [5]–[7], [10], [15]–[17]. However, from a global point of view, the resources are utilized inefficiently when self-interested users share resources non-cooperatively [1], [4]. For the cooperative situation,

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coalitional games were investigated in [12], [13], [26], and [28], and bargaining games were studied in [8], [9], [11], [20], [22], and [24]. Optimization and fuzzy logic based frameworks were respectively considered in [14], [18], [19], [21], and in [23]. However, these works do not capture the dynamics of the interplay of competition and cooperation between MNOs and MVNOs, whose roles are fundamentally different from each other. Innovations in the realm of modeling and analysis become necessary to address this dynamics coupled with provider heterogeneity. We have positioned the prequel in context of the limited amount of literature available specifically for MNOs and MVNOs, none of these existing works study joint decisions of these SPs, and therefore do not venture into the multistage hybrid of bargaining and non-cooperative games that we consider in this paper.

We now describe the contributions of this paper. First, we consider a base case in which one MNO and one MVNO compete for EUs in a common pool, and the EUs choose one of the SPs through a hoteling model for subscription (Section II). We formulate the sequential hybrid of bargaining and non-cooperative games that model the dynamics of the SP interactions (Section II-A), and identify the salient properties of its equilibrium solutions when they exist (Section II-B). We obtain conditions for existence and uniqueness of the equilibrium solutions in terms of system parameters, and characterize them when they exist (Section II-C). We prove that the bargaining framework yields a collusive outcome in which the MNO acquires the minimum amount of spectrum that he is mandated to and the MVNO leases either all or nothing of this spectrum from the MNO (though the MVNO is allowed to lease any amount of this spectrum). The equilibrium solutions are easy to compute and reveal several underlying insights: eg, only the SP that is apriori more popular retains the spectrum leased from the regulator in its entirety. This spectrum sharing arrangement is obtained strategically to motivate the EUs to choose the SP that offers higher price so that the overall subscription revenue is maximized (since the proceeds are shared between the SPs anyway). Comparing the payoffs of the SPs and the access fees for the EUs in this paper with those obtained in Part I, we show that joint decision on spectrum acquisition conclusively benefits the SPs by considerably enhancing their payoffs. The joint decision provides only nuanced benefits for the EUs, by securing cheaper access fees for them, while simultaneously guiding more EUs to more expensive service by having the more apriori popular SP retain the acquired spectrum in its entirety, and thereby provide better quality of service to the EUs (Section II-D).

Next, we allow the EUs to defect to an outside option if neither of the two SPs offer a desirable combi-

nation of access fee and quality of service. We also allow the SPs to have exclusive additional customer bases to draw from depending on his spectrum acquisition and the price he offers (Section III). In this scenario we show that there are two equilibrium solutions, both of which yield a milder version of the collusive outcome than in the base case, in that the MNO may acquire higher than the mandated minimum amount of spectrum (Sections III-A, III-B). This happens because the EUs have an outside option to desert to, and the SPs have exclusive customer bases to gain from, depending on the price and the qualities of service they offer. The two equilibrium solutions differ in which of the SPs retain the spectrum leased from the regulator. The SP that retains the entire spectrum gets a higher payoff in each case. Under both equilibrium-type solutions, each SP increases his payoff compared to what he gets when the SPs decide their spectrum acquisitions individually.

## II. BASE CASE

We formulate the dynamics of interaction between the SPs as a sequential hybrid of bargaining and non-cooperative games in Section II-A, we identify some salient properties of its equilibrium-type solutions in Section II-B and characterize the equilibrium-type solutions in Section II-C. Using these solutions, we assess how the SPs and the EUs fare due to the cooperation between the SPs in jointly deciding their spectrum acquisitions, compared to when they decide everything individually, through analysis in Sections II-C and through numerical computations in Section II-D.

### A. Model

We start with by recapitulating notations that are similar to Part I and the current paper. We denote MNO as  $SP_L$  and MVNO as  $SP_F$ .  $SP_L$  offers  $I_L$  amount of spectrum (which it acquires from a central regulator) to  $SP_F$  in exchange of money, and  $SP_F$  uses  $I_F$  amount of this spectrum. Clearly,  $0 \leq I_F \leq I_L$ . We denote the marginal leasing fee (per spectrum unit) that  $SP_L$  pays the central regulator as  $\gamma$ , marginal reservation fee  $SP_F$  pays to  $SP_L$  by  $\tilde{s}$ , an additional remuneration that  $SP_L$  transfers to  $SP_F$  by  $\theta$ , the fraction of EUs that  $SP_F$  and  $SP_L$  attract as  $n_F$  and  $n_L$ , respectively, and the access fee that  $SP_F$  and  $SP_L$  charge the EUs as  $p_F$  and  $p_L$ , respectively. Let  $c$  be the transaction cost incurred by a SP for each subscription. The  $SP_L$  incurs a spectrum acquisition cost of  $\gamma I_L^2$ , and  $SP_F$  pays to  $SP_L$

a leasing fee of  $sI_F^2$ . Thus,  $SP_L$ ,  $SP_F$  receive payoffs  $\pi_F, \pi_L$  respectively, where:

$$\pi_F = n_F(p_F - c) - \tilde{s}I_F^2 + \theta \quad (1)$$

$$\pi_L = n_L(p_L - c) + \tilde{s}I_F^2 - \gamma I_L^2 - \theta. \quad (2)$$

The above equations are similar to (1), (2) of Part I, with the exception of the introduction of  $\theta$  whose significance will be explained later.

We use a hotelling model to describe how EUs choose between the SPs. EUs are distributed uniformly along the unit interval  $[0, 1]$ , and  $SP_L$  and  $SP_F$  are respectively located at 0, 1 (Figure 1 of Part I). Let  $t_L$  ( $t_F$ ) be the unit transport cost of EUs for  $SP_L$  ( $SP_F$ ), the EU located at  $x \in [0, 1]$  incurs a cost of  $t_L x$  (respectively,  $t_F(1-x)$ ) when joining  $SP_L$  (respectively,  $SP_F$ ). The transport costs capture the impact of the qualities of services the SPs offer on the subscription of the EUs, which in turn depend on the spectrum they acquire:  $t_L = I_F/I_L$ ,  $t_F = 1 - t_L$ .  $v_L, v_F$  represent prior preferences of the EUs for  $SP_L, SP_F$  respectively, which is the same for all EUs, and do not depend on the strategies of the SPs, i.e.,  $I_L, I_F, p_L, p_F$ . Let  $v^L - v^F = \Delta$ . The EU at  $x$  receives utilities  $u_L(x), u_F(x)$  respectively from  $SP_L$  and  $SP_F$ , and joins the SP that gives it the higher utility, where:

$$\begin{aligned} u_L(x) &= v^L - (p_L + t_L x) \\ u_F(x) &= v^F - (p_F + t_F(1-x)). \end{aligned} \quad (3)$$

We now mention the major differences with Part I. Here, we consider a hybrid of bargaining and non-cooperative games to model the dynamics of the interaction between  $SP_L$  and  $SP_F$ . The two SPs jointly decide on the spectrum acquisitions ( $I_L, I_F$ ), so as to maximize the overall profit, but individually decide on the access fees for EUs,  $p_L, p_F$ . The SPs also split the profit, by selecting the marginal reservation fee  $\tilde{s}$ , and the additional remuneration  $\theta$ . Thus,  $\tilde{s}, \theta$  are new decision variables<sup>1</sup>. The SPs decide  $I_L, I_F, \tilde{s}, \theta$  through a bargaining process. If the SPs,  $SP_L, SP_F$ , are not able to agree on these, they receive their respective *disagreement payoffs*,  $d_L, d_F$ , which we assume to be equal to their payoffs in the sequential non-cooperative game whose outcome was characterized in Part I (Theorems 1, 2).

<sup>1</sup>A question that arises is if the SPs jointly decide the spectrum acquisitions, why would they not jointly select the access fees too. The answer is two-fold. First,  $SP_L$  offers the spectrum he acquires to  $SP_F$ , a part of which  $SP_F$  uses - thus, they share the spectrum anyhow, that is, the spectrum usage is inherently cooperative. On the other hand, they are competing for the same pool of EUs, it is therefore natural that the access fees will be determined competitively, thus such decisions must be individual. Second, in practice, the spectrums are acquired for larger time intervals, while access fees are updated more frequently. Joint decisions between two SPs involves substantial coordination and negotiation, which is infeasible on shorter time scales.

The disagreement payoff is for example higher for a SP who is apriori more popular, i.e., has a larger  $v_L$  or  $v_F$ , (eg, Figure 4 of Part I). The disagreement payoffs also depend on the marginal fee per spectrum unit  $s$  the  $SP_F$  pays the  $SP_L$  in the event of a disagreement. This marginal fee is a parameter determined by the overall spectrum market, as assumed for  $s$  in Part I. We also define a *bargaining power* of the SPs. Let  $0 \leq w \leq 1$  be the relative bargaining power of the  $SP_F$  over  $SP_L$ : the higher the  $w$ , more is  $SP_F$ 's bargaining power.

In the event of agreement, the SPs decide their shares of the overall profit, and thereby  $\tilde{s}, \theta$ , commensurate with their disagreement payoffs and bargaining powers; higher values of the latter two fetch higher shares of the profit. Since  $\tilde{s}$  will have no significance in deciding the shares if  $I_F$  is decided as 0 (refer to (1) and (2)), we have considered the additional remuneration transfer decision variable  $\theta$  (which was not in Part I). Note that  $\theta$  can be positive or negative, and the sign reflects the direction of the money flow.

When the SPs jointly decide the spectrum to acquire, so as to maximize the overall profits, a collusive outcome may occur in which both SPs jointly decrease the amount of spectrum acquisitions while maintaining a specific relative difference that yields the best outcome. The reason is that EUs decide based on the ratio of the investment by SPs and not the absolute values. Thus, regulatory intervention may be desirable. Therefore, we consider that a regulator enforces a minimum spectrum acquisition amount of  $L_0$  on  $SP_L$ , i.e.,  $I_L \geq L_0 > 0$ <sup>2</sup>.

We formulate a bargaining framework and use the *Nash Bargaining Solution* (NBS) to characterize  $I_F, I_L, \tilde{s}, \theta$ :

**Definition 1.** Nash Bargaining Solution (NBS): is the unique solution (in our case the tuple of the payoffs of  $SP_L$  and  $SP_F$ ) that satisfies the four "reasonable" axioms (Invariant to affine transformations, Pareto optimality, Independence of irrelevant alternatives, and Symmetry) characterized in [2].

From standard game theoretic results in [2], the optimal solution of the following maximization,  $(\pi_L^*, \pi_F^*)$ , constitute the Nash Bargaining Solution:

$$\begin{aligned} \max_{\pi_L, \pi_F} \quad & (\pi_F - d_F)^w (\pi_L - d_L)^{1-w} \\ \text{s.t.} \quad & (\pi_L, \pi_F) \in U, \quad (\pi_L, \pi_F) \geq (d_L, d_F) \end{aligned} \quad (4)$$

<sup>2</sup> $L_0$  may not be the same as the minimum required amount for  $I_L, \delta$ , assumed in Part I. This is because collusion does not naturally arise in the non-cooperative selection in part I. Thus, a minimum amount  $\delta$  was mandated merely for convenience of analysis, and  $\delta$  was assumed small everywhere. Here, the minimum amount  $L_0$  is imposed as a regulatory intervention to ensure some minimum quality of service for the EUs in presence of collusion between the SPs.

where

$$U = \left\{ (\pi_F, \pi_L) \mid \begin{array}{l} \pi_F = n_F(p_F - c) - \tilde{s}I_F^2 + \theta \\ \pi_L = n_L(p_L - c) + \tilde{s}I_F^2 - \theta - \gamma I_L^2 \end{array} \right\} \\ \cap \{I_L \geq L_0, 0 \leq I_F \leq I_L\}.$$

**Remark 1.** Thus, the payoffs of the individual SPs after bargaining is no less than their disagreement payoffs.

**Remark 2.** The above optimization is guaranteed to have a feasible solution if  $L_0$  is lower than the spectrum acquisition of  $SP_L$  that corresponds to his disagreement payoff; it need not have a feasible solution otherwise.

The SPs decide  $I_L, I_F, \tilde{s}, \theta$  as per the following sequential hybrid of bargaining and non-cooperative games:

- (1)  $SP_L$  and  $SP_F$  jointly decide  $(I_L, I_F, \tilde{s}, \theta)$  through the bargaining game (4).
- (2)  $SP_L$  and  $SP_F$  determine the  $p_L$  and  $p_F$ , respectively, and individually, to maximize their payoffs  $\pi_L, \pi_F$ , based on  $I_L, I_F, \tilde{s}, \theta$  determined in the previous stage. The process constitutes a non-cooperative game.
- (3) EUs decide to subscribe to one of the SPs based on  $I_L, I_F, p_L, p_F$  determined in the previous stages and prior preferences  $v^L, v^F$ . A EU at location  $x$  chooses the SP that provides it a higher utility as per the expressions in (3).

From the above,  $n_F, n_L, p_L, p_F$  are determined in Stage 2 based on  $I_L, I_F, \tilde{s}, \theta$  determined in Stage 1, as solution of (4). Thus,  $n_F, n_L, p_L, p_F$  are functions of  $I_L, I_F, \tilde{s}, \theta$ ; therefore the latter are the decision variables in optimization (4). Thus optimization (4) is

$$\begin{aligned} \max_{I_L, I_F, \tilde{s}, \theta} \quad & (\pi_F - d_F)^w (\pi_L - d_L)^{1-w} \\ \text{s.t.} \quad & 0 \leq I_F \leq I_L, \quad I_L \geq L_0 \\ & \pi_F = n_F(p_F - c) - \tilde{s}I_F^2 + \theta \\ & \pi_L = n_L(p_L - c) + \tilde{s}I_F^2 - \theta - \gamma I_L^2 \\ & (\pi_L, \pi_F) \geq (d_L, d_F) \end{aligned} \quad (5)$$

**Definition 2.** We define  $(I_L^*, I_F^*, \tilde{s}^*, \theta^*, p_L^*, p_F^*, n_L^*, n_F^*)$  as an equilibrium-type solution, when  $I_L^*, I_F^*, \tilde{s}^*, \theta^*$  constitute the optimum solution of (5),  $p_L^*, p_F^*$  the Nash equilibrium of the non-cooperative game in Stage 2 of the sequential game described above, and  $n_L^*, n_F^*$  the corresponding EU subscriptions. Let  $(\pi_L^*, \pi_F^*)$  be the corresponding payoffs of the SPs,

**Remark 3.** There is for example no equilibrium-type solution if (5) does not have a feasible solution.

Note that the framework presented above is identical to that in Sections II-A, II-B of Part I except that 1)  $I_L^*, I_F^*, \tilde{s}^*, \theta^*$  are determined as solutions of a bargaining game as opposed to  $I_L^*, I_F^*$  being obtained as SPNE of a

non-cooperative game and 2)  $s$  being a fixed parameter in and  $\theta$  not being invoked in Part I. Thus, once we get an optimum  $(I_L^*, I_F^*, \tilde{s}^*, \theta^*)$ , from (5), the access fee for EUs ( $p_L^*$  and  $p_F^*$ ) and the split of EUs ( $n_L^*$  and  $n_F^*$ ) between SPs can be determined from the results in Part I, namely Theorems 1, 2, depending on the value of  $\Delta$ . In fact, Theorems 1, 2 of Part I show that  $p_L^*, p_F^*, n_L^*, n_F^*$  are expressions only of  $I_L^*, I_F^*$ , while  $\tilde{s}^*, \theta^*$  will be needed to determine the payoffs of the individual SPs,  $\pi_L^*, \pi_F^*$ , a fact that we will use in the next Section.

We therefore focus on determining  $(I_L^*, I_F^*, \tilde{s}^*, \theta)$  in the next two sections.

### B. Properties of the equilibrium-type solutions

We now obtain identify some properties of the equilibrium-type solutions.

We define the *aggregate excess profit* to be the additional profit yielded from the cooperation in the bargaining framework:

**Definition 3.** *Aggregate Excess Profit ( $u_{excess}$ ):* The aggregate excess profit is defined as

$$\begin{aligned} u_{excess} &= \pi_L - d_L + \pi_F - d_F \\ &= n_F(p_F - c) + n_L(p_L - c) - \gamma I_L^2 - d_L - d_F \end{aligned} \quad (6)$$

We have argued in the last paragraph of Section II-A that the equilibrium-type  $p_L^*, p_F^*, n_L^*, n_F^*$  are expressions only of  $I_L^*, I_F^*$ . Thus, under the equilibrium-type solutions,  $u_{excess}$  is only a function of  $I_F^*, I_L^*, d_F, d_L$ . We denote  $u_{excess}^* = u_{excess}|_{I_L=I_L^* \& I_F=I_F^*}$ .

**Theorem 1.** The equilibrium-type payoffs of SPs satisfy the following property:

$$\pi_L^* = (1 - w)u_{excess}^* + d_L \quad (7)$$

$$\pi_F^* = wu_{excess}^* + d_F. \quad (8)$$

**Remark 4.** The SPs split  $u_{excess}^*$  based on their relative bargaining power,  $SP_F$  obtains a portion  $w$ , and  $SP_L$  obtains the rest. Each SP's payoff equals his share of this aggregate excess profit plus his disagreement payoff. Thus, his payoff increases with his bargaining power and his disagreement payoff; the latter depends on  $|\Delta|, s, \gamma$ .

*Proof.* From (2) in [3], the NBS  $(\pi_L^*, \pi_F^*)$  satisfies:

$$\frac{\pi_F^* - d_F}{w} = \frac{\pi_L^* - d_L}{1 - w}. \quad (9)$$

$$\text{From (9), } \pi_L^* - d_L = \frac{1 - w}{w}(\pi_F^* - d_F). \quad (10)$$

Substituting (10) into (6), we have

$$u_{excess}^* = \frac{1}{w}(\pi_F^* - d_F).$$

Thus, (7) follows. Next,

$$\pi_L^* - d_L = \frac{1-w}{w}(\pi_F^* - d_F) = (1-w)u_{excess}$$

Thus, (8) follows.  $\square$

Since  $0 < w < 1$ , from (7), (8),  $\pi_L^* \geq d_L$  and  $\pi_F^* \geq d_F$  if and only if  $u_{excess}^* \geq 0$ .

Now, we can solve maximization (5) in two steps: 1) obtain the optimum  $I_L^*, I_F^*$  by Theorem 2, 2) obtain the optimum  $\tilde{s}^*, \theta^*$  by (12) and (13).

**Theorem 2.** *The optimum  $(I_L^*, I_F^*)$  of (5) are also the optimum solutions of*

$$\begin{aligned} \max_{I_L, I_F} \quad & u_{excess} \\ \text{s.t.} \quad & I_L \geq L_0, 0 \leq I_F \leq I_L \\ & u_{excess} \geq 0 \end{aligned} \quad (11)$$

**Remark 5.** *Thus, the equilibrium-type  $(I_L^*, I_F^*)$  can be obtained by solving a maximization that seeks to maximize the overall payoffs of the two SPs.*

*Proof.* From (7) and (8)

$$(\pi_F - d_F)^w (\pi_L - d_L)^{1-w} = w^w (1-w)^{1-w} u_{excess}.$$

Since  $0 < w < 1$ , maximizing the objective function of Theorem 2, is equivalent to maximizing  $u_{excess}$ . Right after defining  $u_{excess}$ , we have argued that  $u_{excess}^*$  is a function only of  $I_F^*, I_L^*, d_L, d_F$ . Thus,  $u_{excess}^*$  does not depend on  $\tilde{s}^*, \theta^*$ . We have already argued that  $(\pi_L, \pi_F) \geq (d_L, d_F)$  is equivalent to  $u_{excess} \geq 0$ .  $\square$

Since  $u_{excess}$  is a function only of  $I_L, I_F, d_L, d_F$  as noted right after its definition, the choice of  $\tilde{s}, \theta$  does not affect  $u_{excess}$ . But,  $\tilde{s}^*, \theta^*$  must be determined so as to split  $u_{excess}^* - d_L - d_F$  into  $\pi_L^*, \pi_F^*$ , as per (2) and (8) ((1), (7) follow from (2) and (8)). From (2) and (8),

$$\theta^* - \tilde{s}^*(I_F^*)^2 = wu_{excess}^* + d_F - n_F^*(p_F^* - c).$$

When  $I_F^* = 0$ ,  $\theta^*$  is unique; otherwise, there may be multiple values of  $\tilde{s}^*, \theta^*$  which accomplish the above. When  $I_F^* > 0$ , we choose  $\theta^* = 0$  and  $\tilde{s}^*$  to satisfy the above equation. Our solution utilizes additional remuneration transfer only when  $SP_F$  does not reserve any spectrum offered by  $SP_L$  and thus that route for transfer of money between the SPs to ensure their commensurate shares is closed. Thus,

$$\tilde{s}^* = \begin{cases} \frac{1}{(I_F^*)^2} (n_F^*(p_F^* - c) - d_F - wu_{excess}^*) & I_F^* > 0 \\ s^* \text{ has no significance} & I_F^* = 0 \end{cases} \quad (12)$$

$$\theta^* = \begin{cases} 0 & I_F^* > 0 \\ d_F + wu_{excess}^* - n_F^*(p_F^* - c) & I_F^* = 0 \end{cases} \quad (13)$$

**Remark 6.** *Intuitively, as  $SP_F$ 's bargaining power ( $w$ ) increases, he should get a larger share of the overall revenue. Thus, the marginal reservation fee he pays  $SP_L$  ought to decrease and the additional remuneration he receives from  $SP_L$  ought to increase. The analysis above confirms this intuition. From (11), the equilibrium-type  $I_L^*, I_F^*, u_{excess}^*$  do not depend on  $w$ . Since the equilibrium-type  $n_L^*, n_F^*, p_L^*, p_F^*$  depend only on  $I_L^*, I_F^*$ , other than parameters such as  $\Delta, \tilde{s}^* (\theta^*, \text{respectively})$  is a linearly decreasing (increasing, respectively) function of  $w$ , from (12) and (13).*

### C. Characterizing the equilibrium-type solutions

We now characterize the equilibrium type solutions. Unless otherwise mentioned, the proofs have been relegated to Appendix A.

**Theorem 3.** *Let  $|\Delta| < 1$ . The following holds for each equilibrium-type solution that may exist:  $I_L^* = L_0$ , and*

- (1) *If  $-1 < \Delta < 0$ ,  $I_F^* = L_0$ , and  $s^*$  is obtained by (12), and  $\theta^* = 0$ .*
- (2) *If  $0 < \Delta < 1$ ,  $I_F^* = 0$ ,  $s^*$  has no significance, and  $\theta^*$  is obtained by (13).*
- (3) *If  $\Delta = 0$ , both the above constitute equilibrium-type solutions if there exists any equilibrium-type solution.*

Assuming that the equilibrium-type solution exists, Theorem 3 gives the following insights.  $SP_L$  always acquires minimum amount ( $L_0$ ) of spectrum from a regulator. This is because the EUs must choose between the  $SP_L$  and  $SP_F$ , and both determine their spectrum acquisition together so as to maximize the overall profits and subsequently split their profits. The lack of competition leads to a collusive outcome in which they together opt for the minimum overall spectrum acquisition from the regulator. In contrast, when  $SP_L, SP_F$  decide their spectrum acquisitions separately,  $I_L^*$  exceeds the minimum mandated amount (Theorem 1 of Part I). This happens because each SP seeks to maximize his profit through a sequence of non-cooperative games.

The equilibrium-type solutions differ in how the spectrum acquired from the regulator is split between  $SP_L$  and  $SP_F$ . This happens because the SPs decide the split of the acquired spectrum jointly to maximize their overall profits, which is accomplished if more EUs choose a SP that charges more. To ensure this, the more apriori popular SP retains the entire leased spectrum; 1)  $SP_F$  if  $v^L < v^F$ , 2)  $SP_L$  if  $v^L > v^F$ . If both are equally popular apriori, i.e.,  $v^L = v^F$ , both the above options constitute equilibrium-type solutions. Then, even if the more apriori popular SP charges a high price,

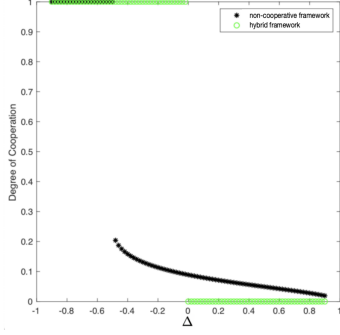


Fig. 1. The degree of cooperation vs.  $\Delta$ .

more EUs would choose him because of his greater prior popularity and because he can offer better quality of service through the acquisition of the leased spectrum in its entirety. Thus, the more popular SP gets the lion share of subscription revenue, which he shares with the other. Thus, if  $SP_F$  is more popular, he pays  $SP_L$   $\tilde{s}^* I_F^{*2}$  amount ( $I_F^* = I_L^* = L_0$  here); if  $SP_L$  is more popular, he pays  $SP_F$   $\theta^*$  amount ( $I_F^* = 0, I_L^* = L_0$  here).  $\tilde{s}^* > 0$  in the first case, and  $\theta^*$  is 0 and positive respectively in the two cases, as Theorem 4 will show.

We now consider the degree of cooperation, i.e.,  $I_F^*/I_L^*$ , which clearly equals 0 or 1: these respectively arise if  $SP_L$  and  $SP_F$  are respectively more apriori popular. If both SPs have equal priors, the degree of cooperation can be either 0 or 1. In contrast, when  $SP_L$ ,  $SP_F$  decide their spectrum acquisitions separately,  $I_F^*$  can be between 0 and  $I_L^*$  (Theorem 1 of Part I). Figure 1 elucidates this distinction. The curve for the individual spectrum acquisitions has been obtained from Theorem 1 of Part I considering at each  $\Delta$ ,  $\tilde{s}$  to be that which maximizes the sum of the disagreement payoffs.

Now,  $\tilde{s}^*, \theta^*$  can be obtained from (12) and (13) respectively,  $p_L^*, p_F^*, n_L^*, n_F^*$  can be obtained from parts 3 and 4 of Theorem 1 of Part I leading to the following overall equilibrium-type solutions:

**Theorem 4.** Let  $|\Delta| < 1$ :

- (1) If  $-1 < \Delta < 0$ , if an equilibrium-type solution exists, it is:  $(I_F^*, I_L^*) = (L_0, L_0)$ ,  $\tilde{s}^*$  is obtained by (12),  $\theta^* = 0$ , and

$$p_L^* = c + \frac{1}{3} + \frac{\Delta}{3}, p_F^* = c + \frac{2}{3} - \frac{\Delta}{3}$$

$$n_L^* = \frac{1}{3} + \frac{\Delta}{3}, n_F^* = \frac{2}{3} - \frac{\Delta}{3}$$

- (2) If  $0 < \Delta < 1$ , if an equilibrium-type solution exists, it is:  $(I_F^*, I_L^*) = (0, L_0)$ ,  $\tilde{s}^*$  is of no significance,  $\theta^*$

is obtained by (13), and

$$p_L^* = c + \frac{2}{3} + \frac{\Delta}{3}, p_F^* = c + \frac{1}{3} - \frac{\Delta}{3}$$

$$n_L^* = \frac{2}{3} + \frac{\Delta}{3}, n_F^* = \frac{1}{3} - \frac{\Delta}{3}$$

- (3) If  $\Delta = 0$ , if an equilibrium-type solution exists, the equilibrium-type solutions are:

- $(I_F^*, I_L^*) = (0, L_0)$ ,  $\tilde{s}^*$  is of no significance,  $\theta^*$  is obtained by (13),

$$p_L^* = c + \frac{2}{3} = n_L^* + c, p_F^* = c + \frac{1}{3} = n_F^* + c$$

- $(I_F^*, I_L^*) = (L_0, L_0)$ ,  $\tilde{s}^*$  is obtained by (12),  $\theta^* = 0$ ,

$$p_L^* = c + \frac{1}{3} = n_L^* + c, p_F^* = c + \frac{2}{3} = n_F^* + c.$$

Thus, considering only the values of  $\tilde{s}^*, \theta^*$  given by (12), (13), the equilibrium-type solution is easy to compute and unique when it exists, when  $|\Delta| < 1$ , with the only exception being at  $\Delta = 0$ , at which there are either 0 or 2 equilibria. The equilibrium-type The insights on  $p_L^*, p_F^*, n_L^*, n_F^*$  are otherwise similar to those presented after Theorem 1 in Part I.

**Corollary 1.** The sum of payoffs of each of the possible equilibrium-solutions presented in Theorem 4 is:

$$\pi^* = \pi_L^* + \pi_F^* = (1/3 - |\Delta|/3)^2 + (2/3 + |\Delta|/3)^2 - \gamma L_0^2. \quad (14)$$

*Proof.* First, from (1) and (2), we have

$$\pi_L + \pi_F = n_L(p_L - c) + n_F(p_F - c) - \gamma I_L^2.$$

From Theorem 4,  $n_L^* = p_L^* - c$ ,  $n_F^* = p_F^* - c$ , and  $I_L^* = L_0$ . Then inserting  $n_L^*, n_F^*, p_L^*, p_F^*$ , and  $I_L^*$  into the above equation, we have the desired result.  $\square$

Again, assuming that the equilibrium solution exists in each case, the total payoff of the SPs decreases with the minimum mandated amount of spectrum acquisition  $L_0$ . This is expected as this reduction is in effect equivalent to relaxation of a constraint in a maximization, which increases the maximum value. Intuitively, the SPs increase their overall payoffs if they are allowed to get away with acquiring really small amounts of spectrums; since the EUs must choose one of the SPs, the joint subscription revenues of the SPs is not affected as long as both SPs acquire small amounts of spectrum. The sum also decreases with increase in the marginal reservation fee the central regulator charges. The sum is maximized at  $|\Delta| = 1$ , i.e., when one of the two SPs is apriori substantially more popular than the other, thus, he can attract most of the EUs despite charging a high amount. This enhances the overall subscription revenue.

Note that the sum does not depend on the disagreement payoffs, and therefore does not depend on the marginal reservation fee the  $SP_F$  pays the  $SP_L$  in the event of a disagreement, i.e., the  $s$  the market provides.

We provide a necessary and sufficient condition for the existence of equilibrium-type solutions, in terms of parameters  $\Delta, \gamma, L_0$  and disagreement payoffs  $d_L, d_F$ .

**Theorem 5.** *Let  $|\Delta| < 1$ . At least one equilibrium-type solution exists if and only if*

$$\begin{aligned} \pi^* &= (1/3 - |\Delta|/3)^2 + (2/3 + |\Delta|/3)^2 - \gamma L_0^2 \\ &\geq d_L + d_F = d. \end{aligned}$$

**Remark 7.** *The disagreement payoffs  $d_L, d_F$  depend on the market-dependent marginal reservation fee  $s$  the  $SP_F$  pays the  $SP_L$  in the event of disagreement. Thus, this  $s$  only determines if an equilibrium-type solution exists, but not its values. Clearly, such solutions do not exist for large  $\gamma, L_0$ , which is consistent with the insights developed in Remarks 2, 3. In contrast, for  $|\Delta| < 1$ , the SPNE always exists, and is unique, when the SPs decide everything individually (Theorem 1 of Part I).*

We now consider  $|\Delta| \geq 1$ . As in Part I, this region is not of much interest due to the insurmountable difference between the prior preferences for the SPs. We show that in this case equilibrium-type solutions exist only for very small values of  $L_0$ . Since these solutions provide  $I_L^* = L_0$ , even the solutions are of limited practical utility. We state the results for completeness. Let  $s, \delta$  constitute the parameters that provide the disagreement payoff (from the sequential game of Part I). Let  $\gamma < s$ .

**Theorem 6.** *If  $\Delta \leq \sqrt{\gamma/s} - 2$  or  $\Delta \geq 1$ , the equilibrium-type solutions exist; if  $\Delta \in (\sqrt{\gamma/s} - 2, -1]$ , no equilibrium-type solution exists.*

- (1) *If  $\Delta \leq \sqrt{\gamma/s} - 2$  and  $L_0 \leq \frac{1}{\sqrt{2s}}$ , the equilibrium-type solutions are:  $I_L^* = L_0$ ,  $I_F^* \in [0, L_0]$ ,  $s^*$  is obtained by (12),  $\theta^*$  is obtained by (13), and*

$$\begin{aligned} p_L^* &= p_F^* + \Delta + 1, \quad c + 1 \leq p_F^* \leq c - \Delta - 1, \\ n_L^* &= 0, \quad n_F^* = 1, \end{aligned}$$

*If  $L_0 > \frac{1}{\sqrt{2s}}$ , no equilibrium-type solution exists.*

- (2) *If  $\Delta \geq 1$  and  $L_0 \leq \delta$ , the equilibrium-type solutions are:  $I_L^* = L_0$ ,  $I_F^* \in [0, L_0]$ ,  $s^*$  is obtained by (12),  $\theta^*$  is obtained by (13), and*

$$\begin{aligned} p_F^* &= p_L^* - \Delta, \quad c + 1 \leq p_L^* \leq c + \Delta, \\ n_L^* &= 1, \quad n_F^* = 0. \end{aligned}$$

*If  $L_0 > \delta$ , no equilibrium-type solution exists.*

**Remark 8.** *When  $\Delta \leq \sqrt{\gamma/s} - 2$ , Theorem 2 of Part I shows that the disagreement payoffs are attained when*

*$SP_L$  acquires  $\frac{1}{\sqrt{2s}}$  resource. If  $L_0 > \frac{1}{\sqrt{2s}}$ , equilibrium-type solution does not exist per the intuitions in Remarks 2. More specifically, in this case, the SPs together attain payoffs lower than the total disagreement payoffs, as they are forced to acquire greater amounts of spectrum than what they did for acquiring their disagreement payoffs. This does not increase the total subscription revenue as the EUs must choose one of the SPs, but increases the total cost incurred in spectrum acquisition from the regulator. Thus, the aggregate excess payoff is negative. Hence there is no equilibrium-type solution. If  $\Delta \geq 1$  and  $L_0 > \delta$ , equilibrium-type solutions do not exist for similar reasons which follow from an application of Theorem 2 of Part I.*

#### D. Numerical results

We numerically investigate the payoffs, the degree of cooperation, the investment levels, and the split of EUs to the SPs for  $|\Delta| < 1$  and different values of other parameters. We set  $\gamma = 0.5$ ,  $c = 1$ ,  $w = 0.2$ , and consider two cases: 1)  $\Delta = -0.5$ ; 2)  $\Delta = 0.5$ .  $SP_F$  is apriori more popular in the first, and  $SP_L$  in the second. We refer to the sum of equilibrium-type solution payoffs of the SPs as  $\pi^*$ , and disagreement payoffs of the SPs as  $d$ .

We first examine the condition for existence of equilibrium-type solutions, given in Theorem 5, by varying  $L_0$  between  $[0.1, 1]$ , and different values of  $s$  used to obtain the disagreement payoffs. As expected from Theorem 5, Figure 2 show that  $\pi^*$  decreases with  $L_0$ , and does not depend on  $s$ . As mentioned in Remark 7,  $d$  depends on  $s$ , and from Theorem 5  $d$  does not depend on  $L_0$ . Thus, the plots of  $d$  are parallel to the x-axis in Figure 2. We note that  $d$  initially increases with  $s$ , and then reaches its maximum value, at  $s = s_{best} = 23.5$  and subsequently decreases. We consider  $s = 0.8, 1, 1.2, s_{best}$ . Figure 2 show the region in which  $d \leq \pi^*$ , for different values of  $s$ , it is the region of existence of equilibrium-type solutions as per Theorem 5. For a given  $s$ , we do not plot  $\pi^*$ , once it falls below  $d$ ; thus the curves for  $\pi^*$  corresponding to a specific  $s$  stop whenever they meet the  $d$  for that  $s$ . The region in which equilibrium-type solutions exist is smallest at  $s = s_{best}$  and much larger at  $s = 0.8$ . Referring to Corollary 1 and Theorem 5, in this region  $\pi^* - d$  shows the gain in overall payoffs of the SPs through joint decision on spectrum acquisitions. The gain is naturally the smallest at  $s = s_{best}$ , but significant at other values of  $s$ .

Figures 3, 4 demonstrate the payoff gain of each SP due to joint decision on spectrum acquisition (Remark 1).

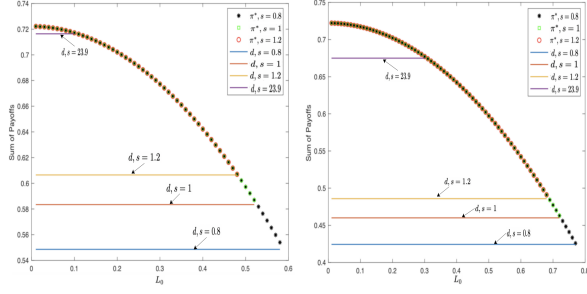


Fig. 2. Sum of equilibrium-type solution payoffs of SPs,  $\pi^*$ , when  $\Delta = -0.5$  (left) and when  $\Delta = 0.5$  (right) vs.  $L_0$ , sum of disagreement payoffs of SPs is  $d$ .

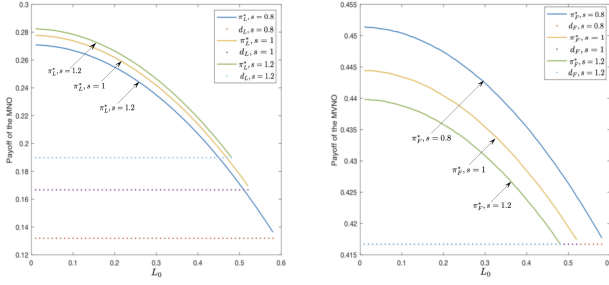


Fig. 3. Equilibrium-type solution and disagreement payoffs of  $SP_L$  (left),  $SP_F$  (right) when  $\Delta = -0.5$  vs.  $L_0$ .

For  $\Delta \in \{-0.5, 0.5\}$ , the figures plot their disagreement payoffs, and equilibrium-solution payoffs, in the region that equilibrium solutions exist, as given by Theorem 5 and illustrated in Figure 2. The payoffs under the equilibrium solutions now depend on  $s$  as the disagreement payoffs depend on  $s$  (Equations (7) and (8)). First consider  $\Delta = -0.5$ . Figure 3 shows that the payoff of  $SP_L$  ( $SP_F$ , respectively) increases (decreases, respectively) with  $s$ . Both payoffs decrease with  $L_0$ . Also, the gain in payoff for each SP beyond his disagreement payoff, due to joint decision on spectrum, is considerable for low  $L_0$ , but decreases as  $L_0$  increases (the SPs are forced to acquire higher amounts of spectrum for large  $L_0$  to deliver the

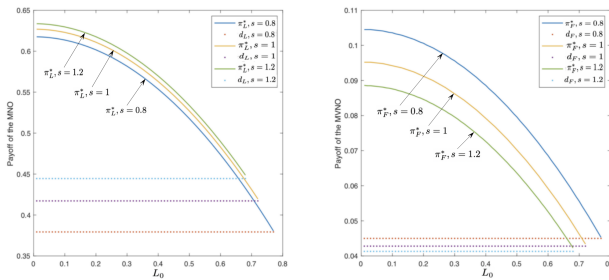


Fig. 4. Equilibrium-type solution and disagreement payoffs of  $SP_L$  (left),  $SP_F$  (right) when  $\Delta = 0.5$  vs.  $L_0$ .

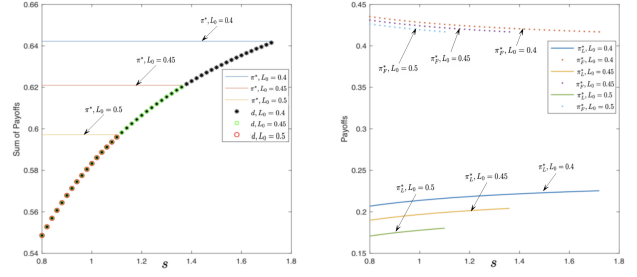


Fig. 5. Sum of equilibrium-type solution payoffs  $\pi^*$ , sum of disagreement payoffs  $d$  (left), individual payoffs (right) vs.  $s$ .

minimum quality of service mandated by the regulator). The payoff of  $SP_F$  is higher than that of  $SP_L$  in this case because  $SP_F$  is more apriori popular (as  $\Delta < 0$ ). When  $\Delta = 0.5$ , Figure 4 shows that  $\pi_L^* > \pi_F^*$ , i.e.,  $SP_L$  has higher payoff in this case, which is intuitive as  $SP_L$  is more apriori popular ( $\Delta > 0$ ). The observations are otherwise similar to those for Figure 3.

The above observations for existence of equilibrium-type solutions and the collective and individual gains in payoffs of the SPs due to joint decision on spectrum acquisition may be reinforced by plotting  $\pi^*, d, \pi_L^*, \pi_F^*$  as functions of  $s$  for few fixed  $L_0$ s (Figure 5). We consider only  $\Delta = -0.5$  here, and  $L_0 = 0.4, 0.45, 0.5$ . Now, in the left figure, plots of  $\pi^*$  are parallel to the  $x$ -axis, while  $d$  increases with increase in  $s$  in the range considered,  $s \in [0.8, 2]$ . We plot  $\pi^*$  only in the region in which the equilibrium-type solutions exist, i.e., where  $\pi^* \geq d$ . This plot also quantifies the gains in collective payoffs by showing how much the flat curves exceed the increasing one, in the region in which they are plotted. The figure in the right show that the payoff of each SP decreases with  $L_0$ , and payoff of  $SP_F$  ( $SP_L$ , respectively) decreases (increases, respectively) with  $s$ . The payoff of  $SP_F$  is higher than that of  $SP_L$ , which is intuitive as  $SP_F$  is apriori more popular in this case.

Our numerical computations thus far reveal that the cooperation in form of joint decisions on spectrum acquisitions benefits the SPs by enhancing their collective and individual payoffs. We now investigate how this enhanced cooperation between the SPs affects the EUs.

Now we investigate the subscriptions and access fees when the reservation fee is  $s_{best}$ , with  $\Delta$  varying in  $(-1, 1)$ . From Theorem 4, subscriptions  $n_L^*, n_F^*$  only depend on  $\Delta$  (and are independent of  $s$  and  $L_0$ ). Figure 6 plots the subscriptions (left). From Theorem 4,  $p_L^* = n_L^* + c$  and  $p_F^* = n_F^* + c$ , so the equilibrium-type subscriptions and access fees exhibit similar behaviors. For  $\Delta < 0$ , i.e., when  $SP_F$  is apriori more popular,



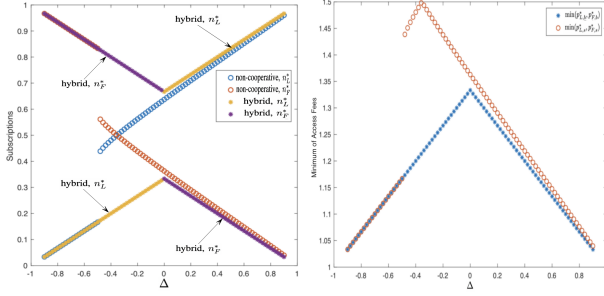


Fig. 6. The subscriptions (left), the access fees (right) vs.  $\Delta$ .

$n_F^* > n_L^*$  for the equilibrium-type solution under joint decision on spectrum acquisition. The difference  $n_F^* - n_L^*$  increases as  $\Delta$  reduces, when  $\Delta < 0$ . The reverse is observed when  $\Delta^* > 0$ . Since  $p_F^* - p_L^* = n_F^* - n_L^*$  throughout, more EUs choose the SP that charges higher; this choice is clearly induced by how the SPs share between them the spectrum  $I_L$  that  $SP_L$  acquires. In some way, this benefits the SPs, enhancing their overall revenue, and harms the EUs by motivating them to pay more. The conclusion is however nuanced as the EUs choose the more expensive option, voluntarily, and only because that option provides better quality of service by retaining the acquired spectrum in its entirety, and was also apriori more popular. The choice is therefore guided, rather than enforced, by having the more apriori popular SP retain the acquired spectrum. When the SPs separately decide their spectrum acquisitions, the trends are similar, through the differences between the subscriptions, and therefore the access fees, is less pronounced. The spectrum is more evenly shared between the SPs (Figure 1), leading to lower access fees and lower qualities of service for more EUs.

In the Figure 6 (right), we plot the minimum of access fees of SPs in both frameworks:  $\min(p_{L,b}^*, p_{F,b}^*)$  ( $\min(p_{L,s}^*, p_{F,s}^*)$ , respectively) represent the minimum access fees when the SPs decide spectrum acquisitions jointly (separately, respectively). The minimum access fee represents the least cost an EU might incur. The minimum is clearly equal or lower for the joint decision case. Thus, joint decisions of the SPs benefits the EUs by providing them cheaper access. But, as we have noted in the previous paragraph, more EUs are induced to select the more expensive option by having it provide the better quality of service and choosing the more popular of the two SPs to do so. Thus, in one perspective, the EUs gain due to enhanced coordination between the SPs, while they lose in another perspective.

Figure 7 shows that the bargaining reservation fee  $\tilde{s}^*$  (left, here  $\Delta = -0.5$ ) and the additional remuneration

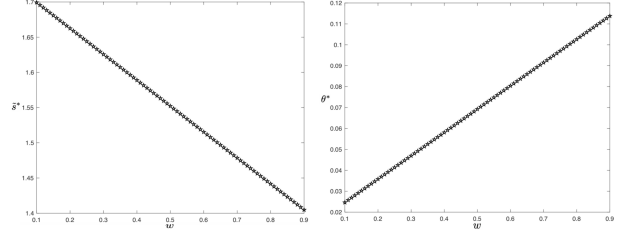


Fig. 7. The bargaining reservation fee  $\tilde{s}^*$  (left), the additional remuneration  $\theta^*$  (right) vs.  $w$ .

$\theta^*$  (right, here  $\Delta = 0.5$ ) linearly respectively decreases and increases with increase in  $w$ , as has been argued both intuitively and analytically in Remark 6. The disagreement payoffs are now obtained at  $s = 1$ .  $\tilde{s}^* > \gamma = 0.5$  even without the constraint that  $\tilde{s} > \gamma$ . Thus,  $SP_L$  always charges the  $SP_F$  a greater marginal reservation fee than he pays the regulator.

### III. EUS WITH OUTSIDE OPTION

We now generalize our framework to provide an outside option to the EUs and exclusive customer bases to draw from to the SPs. Thus, the EUs from the common pool the SPs are competing over, may not choose either of the two SPs, leading to overall attrition, if the service quality-access fee tradeoff they offer is not satisfactory. The exclusive additional customer bases can provide customers beyond the common pool depending on the service quality and access fees the SPs offer. We introduce these modifications through demand functions we describe next.

Similar to equations (10), (11) in Part I, we define the fraction<sup>3</sup> of EUs with each SP as

$$\begin{aligned}\tilde{n}_L &= \alpha(n_L + \varphi_L(p_L, I_L)), \\ \tilde{n}_F &= \alpha(n_F + \varphi_F(p_F, I_F)),\end{aligned}\quad (15)$$

where

$$\begin{aligned}\varphi_L(p_L, I_L) &= k - p_L + b(I_L - I_F), \\ \varphi_F(p_F, I_F) &= k - p_F + bI_F\end{aligned}\quad (16)$$

and  $\alpha > 0$ ,  $k$  and  $b$  are constants.

We also define  $g(I_L) = \frac{b}{15}I_L + \frac{1}{15} - \frac{c}{3} + \frac{k}{3}$ ,  $f(I_L) = \frac{1}{5I_L} + \frac{b}{5} > 0$ .

We characterize the equilibrium-type solutions in Section III-A, and examine its salient properties through numerical computations in Section III-B.

<sup>3</sup>Recall that as in Part I  $n_L, n_F$  are the fraction of EUs from the common pool who subscribe to the EUs, while  $\tilde{n}_L, \tilde{n}_F$  may be the fractions or actual numbers of subscriptions, considering the attrition to the outside option and the additions from the exclusive customer bases. Only scale factors would change in the expressions for  $\tilde{n}_L, \tilde{n}_F$  and the payoffs depending on if  $n_L, n_F$  are fractions or actual numbers.

### A. The equilibrium-type solution

Our goal here is to examine if the availability of the outside option deters the collusive outcome by which the SPs acquire the minimum mandated amount of spectrum from the central regulator. We focus on the region in which at least one interior equilibrium-type solution, i.e.,  $0 < n_L, n_F < 1$  exists, and show that this is indeed the case. The proofs are given in Appendix B.

**Theorem 7.** *Let  $\Delta = 0$ . Either there is no interior equilibrium-type solution, or there are two interior equilibrium-type solutions. They are:*

- (1)  $I_{L,1}^*$  is a solution of

$$\max_{I_L} 2\alpha g^2(I_L) + 2\alpha(f(I_L)I_L + g(I_L))^2 - \gamma I_L^2$$

$$s.t. I_L \geq L_0$$

$I_{F,1}^* = I_{L,1}^*$ ,  $\tilde{s}^*$  is obtained by (12), and  $\theta^* = 0$ .

- (2)  $p_{L,1}^* = \frac{1}{15} + \frac{2c}{3} + \frac{k}{3} + \frac{bI_L^*}{15}$ ,  $p_{F,1}^* = \frac{4}{15} + \frac{2c}{3} + \frac{k}{3} + \frac{4bI_L^*}{15}$ .  
 (3)  $\tilde{n}_{L,1}^* = \frac{2}{15} + \frac{2k}{3} + \frac{2bI_L^*}{15} - \frac{2c}{3}$ ,  $\tilde{n}_{F,1}^* = \frac{8}{15} + \frac{2k}{3} - \frac{2c}{3} + \frac{8bI_L^*}{15}$ .

and

- (1)  $I_{L,2}^* = I_{L,1}^*$ ,  $I_{F,2}^* = 0$ ,  $\tilde{s}^*$  has no significance, and  $\theta^*$  is obtained by (13).  
 (2)  $p_{L,2}^* = p_{F,1}^*$ ,  $p_{F,2}^* = p_{L,1}^*$ .  
 (3)  $n_{L,2}^* = n_{F,1}^*$ ,  $n_{F,2}^* = n_{L,1}^*$ .

We provide a necessary and sufficient condition for the existence of equilibrium-type solutions, in terms of parameters  $\alpha, \gamma, I_{L,1}^*$  and disagreement payoffs  $d_L, d_F$ .

**Theorem 8.** *Interior equilibrium-type solutions exist if and only if*

$$\pi^* = 2\alpha g^2(I_{L,1}^*) + 2\alpha(f(I_{L,1}^*)I_{L,1}^* + g(I_{L,1}^*))^2 - \gamma I_{L,1}^{*2}$$

$$\geq d_L + d_F = d, \quad \text{and} \quad I_L^* < \frac{4}{b}.$$

**Remark 9.** *Solutions do not exist for large  $\gamma$  or small  $\alpha$ , following the insights developed in Remarks 2, 3.*

The equilibrium-type solutions are easy to compute as they involve optimization in one decision variable and closed-form expressions. They are not unique, unlike in Part I (Theorem 7).

Our numerical computations would reveal that  $I_L^*$  exceeds  $L_0$  in some cases. Thus, the deterrent of overall attrition and the incentive of increasing subscription from the exclusive additional bases, induce the SPs to acquire more spectrum than the minimum mandated amount, even when they are jointly deciding the acquisition amounts. Note that  $p_L^*, p_F^*, n_L^*, n_F^*$  are linear increasing function of  $I_L^*$ . Thus, the SPs can increase both their

subscriptions and access fees by acquiring greater overall spectrum  $I_L^*$  from the regulator. Like in the base case,  $I_F^* \in \{0, I_L^*\}$ , and thus the degree of cooperation is either 0 or 1. This is in contrast to the equivalent case in Section III Part I (eg, Figure 5) which show that the degree of cooperation can assume values between 0 and 1. Then, we consider the competition between SPs, i.e., the subscription  $n_L^*$  and  $n_F^*$ . The subscriptions  $n_L^*$  and  $n_F^*$  are constant if there exists no outside option (Theorem 4 (3)); but  $\tilde{n}_L^*$  and  $\tilde{n}_F^*$  change with the spectrum acquisition level of  $SP_L, I_L^*$ , if there exists an outside option.

We can write the first equilibrium-type solution as

$$\tilde{n}_L^* = \frac{1}{5} + \varphi_L(p_L^*, I_L^*) + \frac{bI_L^*}{5},$$

$$\tilde{n}_F^* = \frac{4}{5} + \varphi_F(p_L^*, I_L^*) - \frac{bI_L^*}{5}$$

In both equations, intuitively, the first term,  $\frac{1}{5}, \frac{4}{5}$ , represents the subscription from the common pool, if there had been no attrition to an outside option. The second and third terms represent the impacts of the attrition as also the additions from the exclusive customer bases. In the special case that  $b = 0$ , i.e., when the demand functions depend only on the access fees, the third term is 0 and the demand functions capture the impact of attrition and additions in the expression for the subscriptions. For  $b > 0$ , the second and the third term together become  $k - p_L^* + \frac{b}{5}I_L^*$  in the expression for  $\tilde{n}_L^*$ , and  $k - p_F^* + \frac{4b}{5}I_L^*$  in that for  $\tilde{n}_F^*$ . Thus, higher overall spectrum acquisition increases the subscription for both SPs even in these terms. The intuitions remain same for the second equilibrium-type solution, as the subscriptions are merely swapped.

Finally, when  $L_0 \geq 4/b, \Delta = 0$ , there does not exist an “interior” equilibrium-type solution, that is, in which  $0 < n_L, n_F < 1$ . Future research includes determining (1) whether there exists corner equilibrium-type solutions, or (2) generalization to the case that  $\Delta \neq 0$ .

### B. Numerical results

We set  $b = 2, k = c = 1, w = 0.2$  and  $s = 2$  throughout. For  $s = 2$  the condition for existence of interior equilibrium-type solutions is satisfied for all cases below. Also  $I_L^* < 4/b$  in all cases below.

With  $L_0 = 0.3$ , Figure 8 (left) shows that the spectrum acquisitions for the two equilibrium-type solutions ( $I_L^*$  is the same in both) exceeds  $L_0$  until  $\gamma$  crosses a threshold, and subsequently remains at  $L_0$ . Thus,  $SP_L$  acquires more spectrum when it is cheaper to do so;

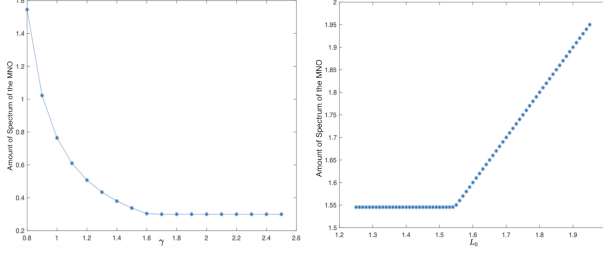


Fig. 8. The spectrum acquisition levels vs.  $\gamma$  (left),  $L_0$  (right)

otherwise settles at the minimum mandated amount. Now, with  $\gamma = 0.8$ , Figure 8 (right) shows that if  $L_0$  is smaller than a threshold ( $= 1.54$ ),  $I_L^*$  exceeds  $L_0$  and equals the threshold value, and subsequently  $I_L^* = L_0$ . Thus,  $I_L^*$  is initially constant and subsequently increases linearly with  $L_0$ .

With  $\gamma = 0.8$ , Figure 9 shows that considering the first equilibrium-type solution,  $\tilde{n}_{L,1}^*$  and  $\tilde{n}_{F,1}^*$  increase with  $L_0$  if  $L_0$  is larger than the threshold. Thus, both can simultaneously increase, or decrease, because of the presence of exclusive additional customer bases and outside options. For this equilibrium-type solution,  $SP_F$  leases the entire  $I_L^*$  spectrum from  $SP_L$ ; thus  $SP_F$  can attract more EUs. Thus,  $\tilde{n}_{F,1}^* > \tilde{n}_{L,1}^*$ . For the second equilibrium type solution, the reverse would happen.

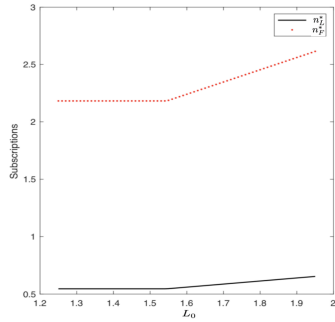


Fig. 9. The split of subscriptions vs.  $L_0$

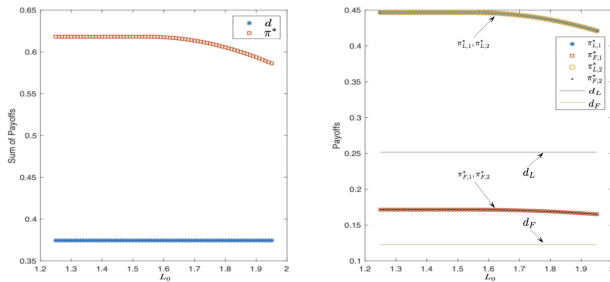


Fig. 10. Sum of equilibrium-type solution payoff,  $\pi^*$ , disagreement payoff,  $d$  (left), payoffs of individual SPs (right) vs.  $L_0$

Figure 10 shows that the total payoff of the two SPs, as also their individual payoffs exceed the corresponding disagreement values, under both equilibrium-type solutions. As in the base case (eg, Figures 2, 3, 4) the total payoff and the individual payoffs decrease with increase in  $L_0$ , for the same reason as described in the paragraph after Corollary 1. In the first equilibrium-type solution,  $SP_F$  leases the entire spectrum  $SP_L$  acquires, while in the second,  $SP_L$  retains this entire spectrum. We observe  $\pi_{L,1}^* < \pi_{F,1}^*$  and  $\pi_{L,2}^* > \pi_{F,2}^*$ . Thus, under Nash bargaining solution, the SP that retains the entire spectrum gets a higher share of the payoff.

#### IV. FUTURE RESEARCH

Extending the sequential hybrid of the cooperative and non-cooperative games to more than two SPs remains a topic of future research. Considering 3 SPs as in Section IV of Part I constitute a starting point towards that end.

#### APPENDIX A

##### PROOFS FOR THEOREMS IN SECTION II-C

We prove Theorem 3 and Theorem 5 in two steps.

Let  $|\Delta| < 1$ . Consider  $(I_L^*, I_F^*, p_L^*, p_F^*, n_L^*, n_F^*)$  that constitute the optimum solution of

$$\begin{aligned} \max_{I_L, I_F} \quad & u_{excess} \\ \text{s.t.} \quad & I_L \geq L_0, \quad 0 \leq I_F \leq I_L \end{aligned} \quad (17)$$

Here  $p_L^*, p_F^*, n_L^*, n_F^*$  are obtained from  $I_L^*, I_F^*$  per parts 3 and 4 of Theorem 1 of Part I:

$$p_L^* = c + \frac{2}{3} - \frac{I_F}{3I_L} + \frac{\Delta}{3}, \quad p_F^* = c + \frac{1}{3} + \frac{I_F}{3I_L} - \frac{\Delta}{3}. \quad (18)$$

$$n_L^* = p_L^* - c, \quad n_F^* = p_F^* - c \quad (19)$$

In **Step 1** we show that any such  $(I_L^*, I_F^*)$  must be of the form given in Theorem 3. Next, note that an optimum solution of (11), should it exist, is also an optimum solution of (17). Since equilibrium-type solutions constitute the optimum solutions of (11), Theorem 3 follows.

In **Step 2** we observe that given the  $I_L^*, I_F^*$  of the possible equilibrium-type solutions mentioned in Theorem 3, 1)  $\bar{s}^*, \theta^*$  of these can be obtained from (12) and (13) respectively, and 2)  $p_L^*, p_F^*, n_L^*, n_F^*$  of these can be obtained from parts 3 and 4 of Theorem 1 of Part I. Accordingly, Theorem 4 follows from Theorem 3, as mentioned before Theorem 4. The total payoff of the two SPs under each of the possible equilibrium-type solutions in Theorem 4 is the same, and is given in Corollary 1. If any possible equilibrium-type solution

listed in Theorem 3 is an equilibrium-type solution, then this total payoff must not exceed the sum of the disagreement payoffs. Next, if this total payoff is not less than the disagreement payoffs, then  $u_{excess} \geq 0$  under the possible equilibrium-type solutions listed in Theorem 3. Thus, these solutions satisfy the additional constraint in (11) (beyond (17)), and therefore constitute its optimum solution too. Thus, these are equilibrium-type solutions. Theorem 5 follows.

### Step 1.

*Proof.* Consider  $(I_L^*, I_F^*, p_L^*, p_F^*, n_L^*, n_F^*)$  that constitute the optimum solution of (17).

Substituting (18) and (19) into (1) and (2), we can get the payoffs of  $SP_F, SP_L$ , for some  $\tilde{s}, \theta$  as:

$$\pi_F^* = \left(\frac{1-\Delta}{3} + \frac{I_F^*}{3I_L^*}\right)^2 - \tilde{s}(I_F^*)^2 + \theta, \quad (20)$$

$$\pi_L^* = \left(\frac{\Delta+2}{3} - \frac{I_F^*}{3I_L^*}\right)^2 + \tilde{s}(I_F^*)^2 - \gamma(I_L^*)^2 - \theta. \quad (21)$$

By Definition 3, substituting (20) and (21) into (6), we can get  $u_{excess}^*$ ,

$$u_{excess}^* = \left(\frac{\Delta+2}{3} - \frac{I_F^*}{3I_L^*}\right)^2 - \gamma(I_L^*)^2 - d_L + \left(\frac{1-\Delta}{3} + \frac{I_F^*}{3I_L^*}\right)^2 - d_F. \quad (22)$$

Denote  $t^* = I_F^*/I_L^*$ , (22) is equivalent to

$$u_{excess}^* = \left(\frac{\Delta+2-t^*}{3}\right)^2 - \gamma(I_L^*)^2 + \left(\frac{1-\Delta+t^*}{3}\right)^2 - d_L - d_F. \quad (23)$$

Now we prove that  $I_L^* = L_0$  by contradiction. Suppose  $I_L^* > L_0$ , then take  $\hat{I}_L = L_0$  and  $\hat{I}_F = I_F^* \frac{L_0}{I_L^*}$ . Thus  $t^* = I_F^*/I_L^* = \hat{I}_F/\hat{I}_L$ . Since  $t^*$  is constant and  $\hat{I}_L < I_L^*$ , then  $u_{excess}$  is higher with  $\hat{I}_F$  and  $\hat{I}_L$  than with  $I_F^*$  and  $I_L^*$ . This contradicts the optimality of  $I_F^*$  and  $I_L^*$ . Therefore,  $I_L^* = L_0$ .

Take the second derivative of  $u_{excess}$  with respect to  $I_F$ ,  $\frac{d^2 u_{excess}}{dI_F^2} = \frac{4}{9I_L^2} > 0$ , then  $u_{excess}$  is convex with respect to  $I_F$ , and the maximum of  $u_{excess}$  must be obtained at the boundaries of  $I_F$ .

Then, we obtain the optimal solution  $I_F^*$ . Note  $0 \leq I_F \leq I_L$ . Substitute  $I_F^* = 0$  and  $I_F^* = I_L^* = L_0$  into (23), we have

$$u_{excess}(0, L_0) - u_{excess}(L_0, L_0) = \frac{4}{9}\Delta.$$

Therefore

$$\begin{aligned} u_{excess}(0, L_0) &> u_{excess}(L_0, L_0) & \text{if } \Delta > 0 \\ u_{excess}(0, L_0) &= u_{excess}(L_0, L_0) & \text{if } \Delta = 0 \\ u_{excess}(0, L_0) &< u_{excess}(L_0, L_0) & \text{if } \Delta < 0 \end{aligned}$$

$$\Rightarrow \begin{cases} (I_F^*, I_L^*) = (0, L_0) & \text{if } 0 < \Delta \leq 1 \\ (I_F^*, I_L^*) = (0 \text{ or } L_0, L_0) & \text{if } \Delta = 0 \\ (I_F^*, I_L^*) = (L_0, L_0) & \text{if } -1 < \Delta < 0 \end{cases}.$$

□

### Proof of Theorem 6.

*Proof.* Once  $I_L^*, I_F^*$  are determined,  $\tilde{s}^*$  is obtained by (12) and  $\theta^*$  is obtained by (13). We obtain  $I_L^*, I_F^*, p_L^*, p_F^*$  in two steps:  $\Delta \leq \sqrt{\frac{\gamma}{s}} - 2$  (Step 1),  $\Delta \geq 1$  (Step 2).

**Step 1:**  $\Delta \leq \sqrt{\frac{\gamma}{s}} - 2$  Suppose the reservation fee is  $s$  in the sequential framework with  $s > \gamma$ . From Theorem 2 (3) in Part I,  $n_L^* = 0, n_F^* = 1$ ,

$$\begin{aligned} p_L^* &= p_F^* + \Delta - 1 \\ p_F^* &\in [c + 1, c - \Delta - 1]. \end{aligned} \quad (24)$$

These also constitute the SPNE, together with,

$$I_L' = I_F' = \frac{1}{\sqrt{2s}}, \quad (25)$$

that provides the disagreement payoffs,  $d_L, d_F$ . From (1), (2) in Part I and (25),  $d_L + d_F = p_F^* - c - \frac{\gamma}{2s}$ .

Again, from (1) and (2), under equilibrium-type solution, the payoffs of the SPs are

$$\pi_F = p_F^* - c - \tilde{s}^*(I_F^*)^2 + \theta^*, \quad (26)$$

$$\pi_L = \tilde{s}^* I_F^2 - \gamma(I_L^*)^2 - \theta^*. \quad (27)$$

By Definition 3, substituting (26) and (27) into (6):

$$u_{excess}^* = p_F^* - c - \gamma(I_L^*)^2 - d_L - d_F. \quad (28)$$

Note that  $u_{excess}^*$  is independent of  $I_F$ , then  $I_F^*$  can be any number between  $[0, I_L^*]$ . Therefore,  $I_L^*$  is a solution of the following optimization problem,

$$\begin{aligned} \max_{I_L, I_F} \quad & u_{excess} = p_F^* - c - \gamma I_L^2 - d_L - d_F \\ \text{s.t.} \quad & I_L \geq L_0 \\ & u_{excess} \geq 0 \end{aligned} \quad (29)$$

From (24),  $p_F^*$  is independent of  $I_L$ , so the objective function is a decreasing function of  $I_L$ . Thus,  $I_L^* = L_0$ . Since  $d_L + d_F = p_F^* - c - \frac{\gamma}{2s}$ , then  $u_{excess}^* \geq 0$  is equivalent to  $L_0 \leq \frac{1}{\sqrt{2s}}$ . The result follows.

**Step 2:**  $\Delta \geq 1$ : We first consider the corner SPNE for  $(p_L, p_F, n_L, n_F)$  in Theorem 2 (1) in Part I:  $n_L^* = 1, n_F^* = 0$ , and

$$\begin{aligned} p_F^* &= p_L^* + v^F - v^L \\ p_L^* &\in [c + 1, c + v^L - v^F]. \end{aligned} \quad (30)$$

Along with  $I_L' = \delta, I_F' = 0$ , these also constitute the SPNE that provide the disagreement payoffs. Therefore, from (1) in Part I,  $d_F = 0$  and  $d_L = p_L^* - c - \gamma\delta^2$ . From (1), (2), under an equilibrium-type solution,

$$\begin{aligned} \pi_F^* &= \tilde{s}^*(I_F^*)^2 + \theta^* \\ \pi_L^* &= p_L^* - c + \tilde{s}^*(I_L^*)^2 - \gamma(I_L^*)^2 - \theta^*, \end{aligned} \quad (31)$$

then substituting (31) into (6), we can get  $u_{excess}^*$ :

$$u_{excess}^* = p_L^* - c - \gamma(I_L^*)^2 - d_L - d_F. \quad (32)$$

Note that  $u_{excess}^*$  is independent of  $I_F$ , then  $I_F^*$  can be any number between  $[0, I_L^*]$ . Therefore, the optimum  $I_L^*$  is a solution of the following optimization problem,

$$\begin{aligned} \max_{I_L, I_F} \quad & u_{excess} = p_L^* - c - \gamma I_L^2 - d_L - d_F \\ \text{s.t.} \quad & I_L \geq L_0 \quad u_{excess} \geq 0 \end{aligned} \quad (33)$$

From (30),  $p_L^*$  is independent of  $I_L$ , so the objective function is a decreasing function of  $I_L$ , then  $I_L^* = L_0$ . Note that  $d_L + d_F = p_L^* - c - \gamma\delta^2$ , then  $u_{excess}^* \geq 0$  is equivalent to  $L_0 \leq \delta$ .

Next, we consider  $\Delta = 1$  and the interior SPNE in Theorem 2 (2) in Part I, i.e.,  $0 < n_F^*, n_L^* < 1$ . By similar analysis in Theorem 3, we have  $I_L^* = L_0$  and  $I_F^* = 0$ . Therefore from (18) and (19),  $p_L^* = c + 1, p_F^* = c, n_L^* = 1$ , and  $n_F^* = 0$ , which is contradicted to  $0 < n_F^*, n_L^* < 1$ . Thus no equilibrium-type solution exists in this case.  $\square$

## APPENDIX B

### PROOF OF THEOREMS IN SECTION III-A

Once  $I_L^*, I_F^*$  are determined,  $\tilde{s}^*$  is obtained by (12) and  $\theta^*$  is obtained by (13). We therefore focus on obtaining  $(I_L^*, I_F^*, p_L^*, p_F^*, n_L^*, n_F^*)$  corresponding to the equilibrium-type solutions. Let  $\Delta = 0$ . Consider  $(I_L^*, I_F^*, p_L^*, p_F^*, n_L^*, n_F^*)$  that constitute the optimum solution of (17) (with only the expressions for  $u_{excess}$  differing from Appendix A). Per Theorem 7 (3), (4), Part I, for an interior SPNE,  $I_L^* < 4/b$ , and :

$$\begin{aligned} \tilde{n}_L^* &= \frac{I_L^* - I_F^*}{I_L^*} + p_F^* - 2p_L^* + k + bI_L^* - bI_F^* \\ \tilde{n}_F^* &= \frac{I_F^*}{I_L^*} + p_L^* - 2p_F^* + k + bI_F^*, \end{aligned} \quad (34)$$

$$\begin{aligned} p_L^* &= \frac{1}{15} + \frac{2c}{3} + \frac{k}{3} + \frac{t_F}{5} - \frac{b}{5}I_F + \frac{4b}{15}I_L, \\ p_F^* &= \frac{1}{15} + \frac{2c}{3} + \frac{k}{3} + \frac{t_L}{5} + \frac{b}{15}I_L + \frac{b}{5}I_F. \end{aligned} \quad (35)$$

First, we show that any such  $(I_L^*, I_F^*, p_L^*, p_F^*, n_L^*, n_F^*)$  must be of the form given in Theorem 7. Next, note that an optimum solution of (11), should it exist, is also an optimum solution of (17). Since equilibrium-type solutions constitute the optimum solutions of (11), Theorem 7 follows.

In fact, substituting (15) and (16) in Part I into (1) and (2),

$$\begin{aligned} \pi_F &= \alpha(t_L + k + p_L - 2p_F + bI_F)(p_F - c) - \tilde{s}I_F^2 + \theta \\ \pi_L &= \alpha(t_F + k + p_F - 2p_L + bI_L - bI_F)(p_L - c) \\ &\quad + \tilde{s}I_F^2 - \gamma I_L^2 - \theta. \end{aligned} \quad (36)$$

**Lemma 1.** In any solution of (17),  $I_F^* = I_L^*$  or  $I_F^* = 0$ .

*Proof.* By substituting (36) into  $u_{excess} = \pi_L - d_L + \pi_F - d_F$ , and using  $t_L = I_F/I_L, t_F = 1 - t_L$ ,

$$\begin{aligned} u_{excess} &= 4\alpha f^2(I_L)I_F^2 - 4\alpha f^2(I_L)I_L I_F + 2\alpha g^2(I_L) \\ &\quad + 2\alpha(f(I_L)I_L + g(I_L))^2 - \gamma I_L^2 - d_F - d_L. \end{aligned}$$

$$\text{Next } \frac{d^2 u_{excess}}{dI_F^2} = 8\alpha f^2(I_L) > 0.$$

Thus,  $u_{excess}$  is convex wrt  $I_F$ , and the maximum of  $u_{excess}$  is obtained at the boundary of  $I_F$ :

$$\begin{aligned} u_{excess}|_{I_F=I_L} &= u_{excess}|_{I_F=0} \\ &= 2\alpha g^2(I_L) + 2\alpha(f(I_L)I_L + g(I_L))^2 - \gamma I_L^2 - d_F - d_L. \end{aligned}$$

Thus  $I_F^* = I_L$  or  $I_F^* = 0$ .  $\square$

Also, for any solution of (17),  $I_L^*$  is given by

$$\begin{aligned} \max_{I_L} \quad & 2\alpha g^2(I_L) + 2\alpha(f(I_L)I_L + g(I_L))^2 - \gamma I_L^2 \\ \text{s.t.} \quad & I_L \geq L_0. \end{aligned} \quad (37)$$

Substituting  $I_F^* = I_L^*$  and  $I_F^* = 0$  into (34) and (35), combining with Lemma 1 and (37), it follows that any solution  $(I_L^*, I_F^*, p_L^*, p_F^*, n_L^*, n_F^*)$  of (17) must be of the form given in Theorem 7. Thus, Theorem 7 follows.

From (4) in Part I,  $x_0^* = t_F^* + p_F^* - p_L^*$ , substituting (35),  $t_F^* = (I_L^* - I_F^*)/I_L^*$  into  $x_0^*$ , then we have  $0 < x_0^* < 1$  if and only if  $I_L^* < 4/b$ . The total payoff of the two SPs under each of the possible interior equilibrium-type solutions listed in Theorem 7 is the same, and is given in Theorem 8. If any possible equilibrium-type solution listed in Theorem 7 is an equilibrium-type solution, then this total payoff must not exceed the sum of the disagreement payoffs. Thus, the necessity in Theorem 8

follows. Next, if  $I_L^* < 4/b$ , the  $p_L^*, p_F^*$  in Theorem 8 constitute an interior Nash equilibrium in Stage 2 of the sequential hybrid game. If the total payoff of the possible equilibrium-type solutions in Theorem 8 is not less than the disagreement payoffs, then  $u_{excess} \geq 0$  under them. Thus, these solutions satisfy the additional constraint in (11) (beyond (17)), and therefore constitute its optimum solution too. Thus, the sufficiency in Theorem 8 follows.  $\square$

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