The Interplay of Competition and Cooperation Among Service Providers (Part II)

Xingran Chen, Saswati Sarkar, Mohammad Hassan Lotfi,

Abstract—This paper investigates the incentives of mobile network operators (MNOs) for acquiring additional spectrum to offer mobile virtual network operators (MVNOs) and thereby inviting competition for a common pool of end users (EUs). We consider interactions between two service providers, a MNO and an MVNO, when the EUs 1) must choose one of them 2) have the option to defect to an outside option should the SP duo offer unsatisfactory access fees or qualities of service. We formulate a multi-stage hybrid of cooperative bargaining and non-cooperative games in which the two SPs jointly determine their spectrum acquisitions, allocations and mutual money flows through the bargaining game, and subsequently individually determine the access fees for the EUs through the non-cooperative game. We identify when the overall equilibrium solutions exist, when it is unique and characterize the equilibrium solutions when they exist. The characterizations are easy to compute, and are in closed form or involve optimizations in only one decision variable. The hybrid framework allows us to determine whether and by how much the different entities benefit due to the cooperation in spectrum acquisition decision.

Index Terms—Resource Sharing, Game Theory, Bargaining Game, Nash Bargaining Solution, Nash equilibrium

1 INTRODUCTION

1.1 Motivation and Overview

TWO different classes of service providers co-exist in the current wireless service provider (SP) market: Mobile Networks Operators (MNOs) and Mobile Virtual Network Operators (MVNOs). The MNO acquires I_L spectrum from a regulator, which he offers to a MVNO in exchange of money, and the MVNO uses I_F amount of this spectrum. Both SPs earn by selling wireless plans to end users (EUs); the MNO earns additionally by leasing her spectrum to the MVNO. Thus, they both cooperate, by sharing spectrum; they also compete, for a common pool of EUs. They clearly make different decisions, which affect their subscriptions; their payoffs have different expressions and their decisions also follow different constraints, eg, $\delta \leq I_L$ and $0 \leq I_F \leq I_L$. In a sequence of two papers we investigate the economics of the interplay of the competition and cooperation between an MNO and an MVNO.

1.2 Relation with the Sequel

In the prequel, Part I, we consider that the SPs arrive at their decisions individually, in the current paper we consider that the SPs arrive at certain decisions together, and then arrive at other decisions individually. Specifically, in this paper, the SPs together decide the spectrum they acquire (i.e., I_L, I_F) to maximize their overall profits, and the marginal reservation fee s that the MVNO pays to the MNO for using the spectrum the MNO offers. Here s is decided so as to split the proceeds between the SPs in accordance with the subscription revenue each generates, which in turn depends on the prior preferences of the EUs for them. Subsequently, each SP individually decides the access fees for the EUs. The I_L, I_F, s are obtained as the solution of a cooperative bargaining game, and the access fees are obtained as solutions of a non-cooperative game. The bargaining and the non-cooperative games together constitute a sequential game. In contrast, in the prequel, each decision variable is selected through a non-cooperative game, each of which constitutes a stage of a sequential game. Also, the marginal reservation fee is considered a fixed parameter, and the MNO and MVNO individually decides the spectrum each acquires, and subsequently individually decides the access fees for the EUs. Note that the marginal reservation fee is indeed a market-driven parameter in a large spectrum market with many MNOs and MVNOs; in such a scenario the marginal reservation fee may be driven by the overall market evolution, and is beyond the control of individual MNOs and MVNOs. This fee may also be beyond the control of individual MNOs and MVNOs, when it is determined by an external regulator to influence the interaction between different providers perhaps to protect the interests of the EUs. These are the cases that the prequel considers. In a smaller market and in absence of regulatory intervention, the marginal reservation fee would be chosen as a decision variable through a negotiation between the MNO and the MVNO concerned. This is the case this paper considers.

1.3 Positioning vis-a-vis the State-of-the-Art

The economics of the interactions of resource sharing among service providers have been investigated in many works. In the prequel, we have distinguished our contributions from those in the genre of non-cooperative interaction between

- *Xingran Chen is with the Electrical and System Engineering Department of the University of Pennsylvania, PA, 19104. E-mail: xingranc@seas.upenn.edu*
- *Saswati Sarkar is with the Electrical and System Engineering Department of the University of Pennsylvania, PA, 19104. E-mail: swati@seas.upenn.edu*
- *Mohammad Hassan Lotfi is at Capital One. E-mail: mohammadhassanlotfi@gmail.com*

the SPs, since there we considered that they arrive at their decisions individually. In this paper, since the SPs together decide the spectrum they acquire, we review the state of the art on cooperative interaction between providers, which have invariably been modeled by coalitional and bargaining games.

Coalitional games were investigated in [10], [11] and [22]. Transferable and nontransferable payoff coalitional games were used in [10] and [11], respectively, to model cooperation among service providers through joint deploying and pooling of resources and serving each others' customers. Both papers concluded that cooperation substantially enhances individual providers' payoffs. In [22], MNOs weighed between building individual networks or entering into network and spectrum sharing agreements. Coalitional games with transferable and nontransferable utility were built to show that a cost division policy guaranteed coalition stability.

Bargaining games were studied in [5], [6], [8], [9], [16], [18], and [20]. The cooperation between selfish nodes was formulated as two-person bargaining games in [5], [8], both nodes were seen to perform better than if they work independently. In [6], nodes in a wireless network seek to agree on a fair and efficient allocation of spectrum. Nash Bargaining Solution (NBS) achieves the best tradeoff between fairness and efficiency. A dynamic incomplete information bargaining was built in [16], where the primary user does not have complete information of the second user energy cost. NBS can lead to a win-win situation, i.e., data rate of both users are improved. [18] investigated the joint uplink subchannel and power allocation problem in cognitive small cells with imperfect channel state information. [20] modeled a situation of dynamic spectrum access by a set of cognitive radio enabled nodes as a bargaining game where the nodes bargain among themselves in a distributed manner to agree upon a sharing rule of the channels. The selfish strategies of the players affect system wide performance. Other optimization models were introduced in [12], [14], [15], [17], and fuzzy logic based frameworks was considered in [19].

However, these works do not consider the dynamics of the interplay of competition and cooperation between MNOs and MVNOs, whose roles are fundamentally different from each other. The principal difference is that while both MNO and MVNO earn by selling wireless plans to the EUs, the MNO earns additionally by leasing spectrum to the MVNO. Thus, they make different decisions, which affect their subscriptions, and their payoffs have different expressions. To our knowledge, [9] is the only work in the domain of cooperative interaction between SPs, that also considers the dynamics of providers whose roles are similar to those of the MNO and MVNO. This paper considers that the spectrum the MNO acquires is exogenously determined, whereas we consider this as a joint decision of the 2 SPs. This leads to an additional stage in our multi-stage formulation. The subscription models for the EUs are also different, though in both cases the EUs choose between the SPs based on the access fees and the spectrum availability (quality of service in [9]). Though our model is more general in that it consists of an additional decision variable and additional stage, we are able to obtain the closed form expressions for SPNE 1) access fees, 2) the amount of spectrum the MVNO leases from the MNO, and 3) the reservation fee the MVNO pays to the MNO. In contrast, [9] only proves that the SPNE access fees exist, and provides the feasibility region of 2) and 3). We also obtain closed form expressions for SPNE spectrum acquisition of the MNO from the central regulator, which [9] considers as a given parameter. We also generalize our model and results to allow for an outside option, which [9] does not. The only other papers that consider the dynamics of MNO and MVNO, namely [4], [7] and [13], have considered only non-cooperative decisions by the SPs. We have therefore distinguished these from our contributions in the prequel, which is closer to them.

1.4 Contribution

We now describe the contributions of this paper. First, we consider a base case in which one MNO and one MVNO compete for EUs in a common pool, and the EUs choose one of the SPs through a hoteling model for subscription (Section 2). We formulate the sequential hybrid of bargaining and non-cooperative games that model the dynamics of the SP interactions (Section 2.1), and identify the salient properties of its equilibrium solutions when they exist (Section 2.2). We obtain conditions for existence and uniqueness of the equilibrium solutions in terms of system parameters, and characterize them when they exist (Section 2.3). We prove that the bargaining framework yields a collusive outcome in which the MNO acquires the minimum amount of spectrum that he is mandated to and the MVNO leases either all or nothing of this spectrum from the MNO (though the MVNO is allowed to lease any amount of this spectrum). The equilibrium solutions are easy to compute and reveal several underlying insights: eg, only the SP that is apriori more popular retains the spectrum leased from the regulator in its entirety. This spectrum sharing arrangement is obtained strategically to motivate the EUs to choose the SP that offers higher price so that the overall subscription revenue is maximized (since the proceeds are shared between the SPs anyway). Comparing the payoffs of the SPs and the access fees for the EUs in this paper with those obtained in Part I, we show that joint decision on spectrum acquisition conclusively benefits the SPs by considerably enhancing their payoffs. The joint decision provides only nuanced benefits for the EUs, by securing cheaper access fees for them, while simultaneously guiding more EUs to more expensive service by having the more apriori popular SP retain the acquired spectrum in its entirety, and thereby provide better quality of service to the EUs. Accordingly, as compared to individual decisions, for some parameter values the EU-resource-cost metric that we define in Part I is higher under the joint decision, and lower for the rest (Section 2.4).

Next, we allow the EUs to defect to an outside option if neither of the two SPs offer a desirable combination of access fee and quality of service. We also allow the SPs to have exclusive additional customer bases to draw from depending on his spectrum acquisition and the price he offers (Section 3). In this scenario we show that there are two equilibrium solutions, both of which yield a milder version of the collusive outcome than in the base case, in that the MNO may acquire higher than the mandated minimum amount of spectrum (Sections 3.1, 3.2). This happens because the EUs have an outside option to desert to, and the SPs have exclusive customer bases to gain from, depending on the price and the qualities of service they offer. The two equilibrium solutions differ in which of the SPs retain the spectrum leased from the regulator. The SP that retains the entire spectrum gets a higher payoff in each case. Under both equilibrium-type solutions, each SP increases his payoff compared to what he gets when the SPs decide their spectrum acquisitions individually. Also, the EU-resource-cost metric is invariably higher than when the SPs decide their spectrum acquisitions individually.

2 Base Case

We formulate the dynamics of interaction between the SPs as a sequential hybrid of bargaining and non-cooperative games in Section 2.1, we identify some salient properties of its equilibrium-type solutions in Section 2.2 and characterize the equilibrium-type solutions in Section 2.3. Using these solutions, we assess how the SPs and the EUs fare due to the cooperation between the SPs in jointly deciding their spectrum acquisitions, compared to when they decide everything individually, through analysis in Sections 2.3 and through numerical computations in Section 2.4.

2.1 Model

We start with by recapitulating notations that are similar to Part I and the current paper. We denote MNO as SP$_L$ and MVNO as SP$_F$. SP$_L$ offers I_L amount of spectrum (which it acquires from a central regulator) to SP$_F$ in exchange of money, and SP$_F$ uses I_F amount of this spectrum. Clearly, $0 \leq I_F \leq I_L$. We denote the marginal leasing fee (per spectrum unit) that SP$_L$ pays the central regulator as γ, marginal reservation fee SP$_F$ pays to SP$_L$ by $\tilde{s}$, an additional remuneration that SP$_L$ transfers to SP$_F$ by θ, the fraction of EUs that SP$_F$ and SP$_L$ attract as n_F and n_L, respectively, and the access fee that SP$_F$ and SP$_L$ charge the EUs as p_F and p_L, respectively. Let c be the transaction cost incurred by a SP for each subscription. The SP$_L$ incurs a spectrum acquisition cost of γI_L^2, and SP$_F$ pays to SP$_L$ a leasing fee of sI_F^2. Thus, SP$_L$, SP$_F$ receive payoffs π_F, π_L respectively, where:

$$\pi_F = n_F(p_F - c) - \tilde{s}I_F^2 + \theta \tag{1}$$

$$\pi_L = n_L(p_L - c) + \tilde{s}I_F^2 - \gamma I_L^2 - \theta. \tag{2}$$

The above equations are similar to (1), (2) of Part I, with the exception of the introduction of θ whose significance will be explained later.

We use a hotelling model to describe how EUs choose between the SPs. EUs are distributed uniformly along the unit interval $[0,1]$, and SP$_L$ and SP$_F$ are respectively located at $0, 1$ (Figure 1 of Part I). Let t_L (t_F) be the unit transport cost of EUs for SP$_L$ (SP$_F$), the EU located at $x \in [0,1]$ incurs a cost of $t_L x$ (respectively, $t_F(1-x)$) when joining SP$_L$ (respectively, SP$_F$). The transport costs capture the impact of the qualities of services the SPs offer on the subscription of the EUs, which in turn depend on the spectrum they acquire: $t_L = I_F/I_L, t_F = 1 - t_L$. v_L, v_F represent prior preferences of the EUs for SP$_L$, SP$_F$ respectively, which is the same for all EUs, and do not depend on the strategies of the SPs , i.e., I_L, I_F, p_L, p_F. The EU at x receives utilities $u_L(x), u_F(x)$ respectively from SP$_L$ and SP$_F$, and joins the SP that gives it the higher utility, where:

$$\begin{aligned} u_L(x) &= v^L - (p_L + t_L x) \\ u_F(x) &= v^F - (p_F + t_F(1-x)). \end{aligned} \tag{3}$$

As in Part I, we denote $\Delta = v^L - v^F$.

We now mention the major differences with Part I. Here, we consider a hybrid of bargaining and non-cooperative games to model the dynamics of the interaction between SP$_L$ and SP$_F$. The two SPs jointly decide on the spectrum acquisitions (I_L, I_F), so as to maximize the overall profit, but individually decide on the access fees for EUs, p_L, p_F. The SPs also split the profit, by selecting the marginal reservation fee $\tilde{s}$, and the additional remuneration θ. Thus, $\tilde{s}, \theta$ are new decision variables[1]. The SPs decide $I_L, I_F, \tilde{s}, \theta$ through a bargaining process. If the SPs, SP$_L$ SP$_F$, are not able to agree on these, they receive their respective *disagreement payoff*s, d_L, d_F, which we assume to be equal to their payoffs in the sequential non-cooperative game whose outcome was characterized in Part I (Theorems 1, 2). The disagreement payoff is for example higher for a SP who is apriori more popular, i.e., has a larger v_L or v_F, (eg, Figure 4 of Part I). The disagreement payoffs also depend on the marginal fee per spectrum unit s the SP$_F$ pays the SP$_L$ in the event of a disagreement. This marginal fee is a parameter determined by the overall spectrum market, as assumed for s in Part I. We also define a *bargaining power* of the SPs. Let $0 \leq w \leq 1$ be the relative bargaining power of the SP$_F$ over SP$_L$: the higher the w, more is SP$_F$'s bargaining power.

In the event of agreement, the SPs decide their shares of the overall profit, and thereby $\tilde{s}, \theta$, commensurate with their disagreement payoffs and bargaining powers; higher values of the latter two fetch higher shares of the profit. Since $\tilde{s}$ will have no significance in deciding the shares if I_F is decided as 0 $\big($refer to (1) and (2)$\big)$, we have considered the additional remuneration transfer decision variable θ (which was not in Part I). Note that θ can be positive or negative, and the sign reflects the direction of the money flow.

When the SPs jointly decide the spectrum to acquire, so as to maximize the overall profits, a collusive outcome may occur in which both SPs jointly decrease the amount of spectrum acquisitions while maintaining a specific relative

1. A question that arises is if the SPs jointly decide the spectrum acquisitions, why would they not jointly select the access fees too. The answer is two-fold. First, SP$_L$ offers the spectrum he acquires to SP$_F$, a part of which SP$_F$ uses - thus, they share the spectrum anyhow, that is, the spectrum usage is inherently cooperative. On the other hand, they are competing for the same pool of EUs, it is therefore natural that the access fees will be determined competitively, thus such decisions must be individual. Second, in practice, the spectrums are acquired for larger time intervals, while access fees are updated more frequently. Joint decisions between two SPs involves substantial coordination and negotiation, which is infeasible on shorter time scales.

difference that yields the best outcome. The reason is that EUs decide based on the ratio of the investment by SPs and not the absolute values. Thus, regulatory intervention may be desirable. Therefore, we consider that a regulator enforces a minimum spectrum acquisition amount of L_0 on SP$_L$, i.e., $0 < L_0 \leq I_L$. Recall that we have a minimum required amount for I_L, δ, in part I, L_0 may not be the same as the δ. This is because collusion does not naturally arise in the non-cooperative selection in part I. Thus, a minimum amount δ was mandated merely for convenience of analysis, and δ was assumed small everywhere. Here, the minimum amount L_0 is imposed as a regulatory intervention to ensure some minimum quality of service for the EUs in presence of collusion between the SPs.

We formulate a bargaining framework and use the *Nash Bargaining Solution* (NBS) to characterize $I_F, I_L, \tilde{s}, \theta$:

Definition 1. Nash Bargaining Solution (NBS): *is the unique solution (in our case the tuple of the payoffs of SP$_L$ and SP$_F$) that satisfies the four "reasonable" axioms (Invariant to affine transformations, Pareto optimality, Independence of irrelevant alternatives, and Symmetry) characterized in [2].*

From standard game theoretic results in [2], the optimal solution of the following maximization, (π_L^*, π_F^*), constitute the Nash Bargaining Solution:

$$\begin{aligned} \max_{\pi_L, \pi_F} \quad & (\pi_F - d_F)^w (\pi_L - d_L)^{1-w} \\ s.t \quad & (\pi_L, \pi_F) \in U, \;\; (\pi_L, \pi_F) \geq (d_L, d_F) \end{aligned} \tag{4}$$

where

$$U = \left\{ (\pi_F, \pi_L) \Big| \begin{matrix} \pi_F = n_F(p_F - c) - \tilde{s} I_F^2 + \theta \\ \pi_L = n_L(p_L - c) + \tilde{s} I_F^2 - \theta - \gamma I_L^2 \end{matrix} \right\} \\ \cap \left\{ L_0 \leq I_L, 0 \leq I_F \leq I_L \right\}.$$

Remark 1. *Thus, the payoffs of the individual SPs after bargaining is no less than their disagreement payoffs.*

Remark 2. *The above optimization is guaranteed to have a feasible solution if L_0 is lower than the spectrum acquisition of SP$_L$ that corresponds to his disagreement payoff; it need not have a feasible solution otherwise.*

The SPs decide $I_L, I_F, \tilde{s}, \theta$ as per the following sequential hybrid of bargaining and non-cooperative games:

- **Stage 1:** SP$_L$ and SP$_F$ jointly decide $(I_L, I_F, \tilde{s}, \theta)$ through the bargaining game (4).
- **Stage 2:** SP$_L$ and SP$_F$ determine the p_L and p_F, respectively, and individually, to maximize their payoffs π_L, π_F, based on $I_L, I_F, \tilde{s}, \theta$ determined in the previous stage. The process constitutes a non-cooperative game.
- **Stage 3:** EUs decide to subscribe to one of the SPs based on I_L, I_F, p_L, p_F determined in the previous stages and prior preferences v^L, v^F. A EU at location x chooses the SP that provides it a higher utility as per the expressions in (3).

From the above, n_F, n_L, p_L, p_F are determined in Stage 2 based on $I_L, I_F, \tilde{s}, \theta$ determined in Stage 1, as solution of (4). Thus, n_F, n_L, p_L, p_F are functions of $I_L, I_F, \tilde{s}, \theta$; therefore the latter are the decision variables in optimization (4). Thus optimization (4) is

$$\begin{aligned} \max_{I_L, I_F, \tilde{s}, \theta} \quad & (\pi_F - d_F)^w (\pi_L - d_L)^{1-w} \\ s.t \quad & 0 \leq I_F \leq I_L, \;\; L_0 \leq I_L \\ & \pi_F = n_F(p_F - c) - \tilde{s} I_F^2 + \theta \\ & \pi_L = n_L(p_L - c) + \tilde{s} I_F^2 - \theta - \gamma I_L^2 \\ & (\pi_L, \pi_F) \geq (d_L, d_F) \end{aligned} \tag{5}$$

Definition 2. *We define $(I_L^*, I_F^*, \tilde{s}^*, \theta^*, p_L^*, p_F^*, n_L^*, n_F^*)$ as an equilibrium-type solution, when $I_L^*, I_F^*, \tilde{s}^*, \theta^*$ constitute the optimum solution of (5), p_L^*, p_F^* the Nash equilibrium of the non-cooperative game in Stage 2, and n_L^*, n_F^* the corresponding EU subscriptions in Stage 3. Let (π_L^*, π_F^*) be the corresponding payoffs of the SPs,*

If an equilibrium-type solution exists, it may be determined through backward induction, starting from the last stage (stage 3) of the game and proceeding backward.

Remark 3. *There is for example no equilibrium-type solution if* (5) *does not have a feasible solution.*

Note that the framework presented above is identical to that in Sections 2.1, 2.2 of Part I except that 1) $I_L^*, I_F^*, \tilde{s}^*, \theta^*$ are determined as solutions of a bargaining game as opposed to I_L^*, I_F^* being obtained as SPNE of a non-cooperative game and 2) s being a fixed parameter in and θ not being invoked in Part I. Thus, once we get an optimum $(I_L^*, I_F^*, \tilde{s}^*, \theta^*)$, from (5), the access fee for EUs (p_L^* and p_F^*) and the split of EUs (n_L^* and n_F^*) between SPs can be determined from the results in Part I, namely Theorems 1, 2, depending on the value of Δ. In fact, Theorems 1, 2 of Part I show that $p_L^*, p_F^*, n_L^*, n_F^*$ are expressions only of I_L^*, I_F^*:

Theorem 1. *[Theorem 1 of Part I] Let $|\Delta| < 1$. The SPNE $p_L^*, p_F^*, n_L^*, n_F^*$ are:*

(1) $p_L^* = c + \frac{2}{3} - \frac{I_F^*}{3I_L^*} + \frac{\Delta}{3}, \quad p_F^* = c + \frac{1}{3} + \frac{I_F^*}{3I_L^*} - \frac{\Delta}{3},$

(2) $n_L^* = \frac{\Delta}{3} + \frac{2}{3} - \frac{I_F^*}{3I_L^*}$, $n_F^* = \frac{I_F^*}{3I_L^*} + \frac{1}{3} - \frac{\Delta}{3}$.

Theorem 2 (Theorem 2 of Part I)**.** *The SPNE $p_L^*, p_F^*, n_L^*, n_F^*$ are*

(1) $\Delta \geq 1$*:*

$$p_F^* = p_L^* - \Delta, n_L^* = 1, \; n_F^* = 0,$$

and p_L^* can be chosen any value in $[c+1, c+\Delta]$.

(2) $\Delta = 1$ *: The following interior strategy constitute an additional SPNE:*

$$p_L^* - c = n_L^* = 2/3, p_F^* - c = n_F^* = 1/3.$$

(3) $\Delta < -1$:

$$p_L^* = p_F^* + \Delta - 1, n_L^* = 0, \; n_F^* = 1,$$

and p_L^* can be chosen any value in $[c+1, c-\Delta]$.

Using the above, we now proceed to determine $(I_L^*, I_F^*, \tilde{s}^*, \theta)$ in the next two sections. These, together with $\tilde{s}^*, \theta^*$, will provide the payoffs of the individual SPs, π_L^*, π_F^*.

2.2 Properties of the equilibrium-type solutions

We now obtain identify some properties of the equilibrium-type solutions.

We define the *aggregate excess profit* to be the additional profit yielded from the cooperation in the bargaining framework:

Definition 3. *Aggregate Excess Profit (u_{excess}): The aggregate excess profit is defined as*

$$\begin{aligned} u_{excess} =& \pi_L - d_L + \pi_F - d_F \\ =& n_F(p_F - c) + n_L(p_L - c) - \gamma I_L^2 - d_L - d_F \end{aligned} \tag{6}$$

We have argued in the last paragraph of Section 2.1 that the equilibrium-type $p_L^*, p_F^*, n_L^*, n_F^*$ are expressions only of I_L^*, I_F^*. Thus, under the equilibrium-type solutions, u_{excess} is only a function of I_F^*, I_L^*, d_F, d_L. We denote $u_{excess}^* = u_{excess}|_{I_L=I_L^* \& I_F=I_F^*}$.

Theorem 3. *The equilibrium-type payoffs of SPs satisfy the following property:*

$$\pi_L^* = (1-w)u_{excess}^* + d_L \tag{7}$$
$$\pi_F^* = w u_{excess}^* + d_F. \tag{8}$$

Remark 4. *The SPs split u_{excess}^* based on their relative bargaining power, SP$_F$ obtains a portion w, and SP$_L$ obtains the rest. Each SP's payoff equals his share of this aggregate excess profit plus his disagreement payoff. Thus, his payoff increases with his bargaining power and his disagreement payoff; the latter depends on $|\Delta|, s, \gamma$.*

Proof. From (2) in [3], the NBS (π_L^*, π_F^*) satisfies:

$$\frac{\pi_F^* - d_F}{w} = \frac{\pi_L^* - d_L}{1-w}. \tag{9}$$

$$\text{From (9),} \quad \pi_L^* - d_L = \frac{1-w}{w}(\pi_F^* - d_F). \tag{10}$$

Substituting (10) into (6), we have

$$u_{excess}^* = \frac{1}{w}(\pi_F^* - d_F).$$

Thus, (7) follows. Next,

$$\pi_L^* - d_L = \frac{1-w}{w}(\pi_F^* - d_F) = (1-w)u_{excess}$$

Thus, (8) follows. □

Since $0 < w < 1$, from (7), (8), $\pi_L^* \geq d_L$ and $\pi_F^* \geq d_F$ if and only if $u_{excess}^* \geq 0$.

Now, we can solve maximization (5) in two steps: 1) obtain the optimum I_L^*, I_F^* by Theorem 4, 2) obtain the optimum $\tilde{s}^*, \theta^*$ by (12) and (13).

Theorem 4. *The optimum (I_L^*, I_F^*) of (5) are also the optimum solutions of*

$$\begin{aligned} \max_{I_L, I_F} \quad & u_{excess} \\ s.t. \quad & L_0 \leq I_L, \; 0 \leq I_F \leq I_L \\ & u_{excess} \geq 0 \end{aligned} \tag{11}$$

Remark 5. *Thus, the equilibrium-type (I_L^*, I_F^*) can be obtained by solving a maximization that seeks to maximize the overall payoffs of the two SPs.*

Proof. From (7) and (8)

$$(\pi_F - d_F)^w (\pi_L - d_L)^{1-w} = w^w (1-w)^{1-w} u_{excess}.$$

Since $0 < w < 1$, maximizing the objective function of Theorem 4, is equivalent to maximizing u_{excess}. Right after defining u_{excess}, we have argued that u_{excess}^* is a function only of I_F^*, I_L^*, d_L, d_F. Thus, u_{excess}^* does not depend on $\tilde{s}^*, \theta^*$. We have already argued that $(\pi_L, \pi_F) \geq (d_L, d_F)$ is equivalent to $u_{excess} \geq 0$. □

Since u_{excess} is a function only of I_L, I_F, d_L, d_F as noted right after its definition, the choice of $\tilde{s}, \theta$ does not affect u_{excess}. But, $\tilde{s}^*, \theta^*$ must be determined so as to split $u_{excess}^* - d_L - d_F$ into π_L^*, π_F^*, as per (2) and (8) $\big($(1), (7) follow from (2) and (8)$\big)$. From (2) and (8),

$$\theta^* - \tilde{s}^*(I_F^*)^2 = w u_{excess}^* + d_F - n_F^*(p_F^* - c).$$

When $I_F^* = 0$, θ^* is unique; otherwise, there may be multiple values of $\tilde{s}^*, \theta^*$ which accomplish the above. When $I_F^* > 0$, we choose $\theta^* = 0$ and $\tilde{s}^*$ to satisfy the above equation. Our solution utilizes additional remuneration transfer only when SP$_F$ does not reserve any spectrum offered by SP$_L$ and thus that route for transfer of money between the SPs to ensure their commensurate shares is closed. Thus,

$$\tilde{s}^* = \begin{cases} \dfrac{1}{(I_F^*)^2}(n_F^*(p_F^* - c) - d_F - w u_{excess}^*) & I_F^* > 0 \\ s^* \text{ has no significance} & I_F^* = 0 \end{cases} \tag{12}$$

$$\theta^* = \begin{cases} 0 & I_F^* > 0 \\ d_F + w u_{excess}^* - n_F^*(p_F^* - c) & I_F^* = 0 \end{cases} \tag{13}$$

Remark 6. *Intuitively, as SP$_F$'s bargaining power (w) increases, he should get a larger share of the overall revenue. Thus, the marginal reservation fee he pays SP$_L$ ought to decrease and the additional remuneration he receives from SP$_L$ ought to increase. The analysis above confirms this intuition. From* (11), *the equilibrium-type $I_L^*, I_F^*, u_{excess}^*$ do not depend on w. Since the equilibrium-type $n_L^*, n_F^*, p_L^*, p_F^*$ depend only on I_L^*, I_F^*, other than parameters such as Δ, $\tilde{s}^*$ (respectively, θ^*) is a linearly decreasing (respectively, increasing) function of w, from* (12) *and* (13).

2.3 Characterizing the equilibrium-type solutions

We now characterize the equilibrium type solutions. Unless otherwise mentioned, the proofs have been relegated to Appendix A.

Theorem 5. *Let $|\Delta| < 1$. The following holds for each equilibrium-type solution that may exist: $I_L^* = L_0$, and*

(1) *If $-1 < \Delta < 0$, $I_F^* = L_0$, and s^* is obtained by* (12), *and $\theta^* = 0$.*
(2) *If $0 < \Delta < 1$, $I_F^* = 0$, s^* has no significance, and θ^* is obtained by* (13).
(3) *If $\Delta = 0$, both the above constitute equilibrium-type solutions if there exists any equilibrium-type solution.*

Assuming that the equilibrium-type solution exists, Theorem 5 gives the following insights. SP$_L$ always acquires

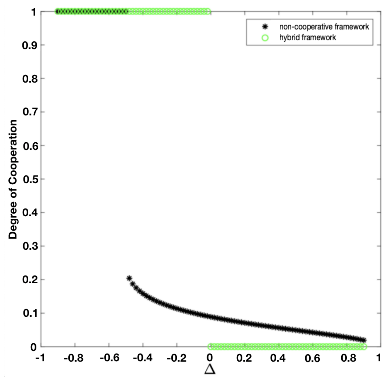

Fig. 1. The degree of cooperation vs. Δ.

minimum amount (L_0) of spectrum from a regulator. This is because the EUs must choose between the SP_L and SP_F, and both determine their spectrum acquisition together so as to maximize the overall profits and subsequently split their profits. The lack of competition leads to a collusive outcome in which they together opt for the minimum overall spectrum acquisition from the regulator. In contrast, when SP_L, SP_F decide their spectrum acquisitions separately, I_L^* exceeds the minimum mandated amount (Theorem 1 of Part I). This happens because each SP seeks to maximize his profit through a sequence of non-cooperative games.

The equilibrium-type solutions differ in how the spectrum acquired from the regulator is split between SP_L and SP_F. This happens because the SPs decide the split of the acquired spectrum jointly to maximize their overall profits, which is accomplished if more EUs choose a SP that charges more. To ensure this, the more apriori popular SP retains the entire leased spectrum; 1) SP_F if $v^L < v^F$, 2) SP_L if $v^L > v^F$. If both are equally popular apriori, i.e., $v^L = v^F$, both the above options constitute equilibrium-type solutions. Then, even if the more apriori popular SP charges a high price, more EUs would choose him because of his greater prior popularity and because he can offer better quality of service through the acquisition of the leased spectrum in its entirety. Thus, the more popular SP gets the lion share of subscription revenue, which he shares with the other. Thus, if SP_F is more popular, he pays SP_L $\tilde{s}^* I_F^{*2}$ amount ($I_F^* = I_L^* = L_0$ here); if SP_L is more popular, he pays SP_F θ^* amount ($I_F^* = 0, I_L^* = L_0$ here). $\tilde{s}^* > 0$ in the first case, and θ^* is 0 and positive respectively in the two cases, as Theorem 6 will show.

We now consider the degree of cooperation, i.e., I_F^*/I_L^*, which clearly equals 0 or 1: these respectively arise if SP_L and SP_F are respectively more apriori popular. If both SPs have equal priors, the degree of cooperation can be either 0 or 1. In contrast, when SP_L, SP_F decide their spectrum acquisitions separately, I_F^* can be between 0 and I_L^* (Theorem 1 of Part I). Figure 1 elucidates this distinction. The curve for the individual spectrum acquisitions has been obtained from Theorem 1 of Part I considering at each Δ, $\tilde{s}$ to be that which maximizes the sum of the disagreement payoffs.

Now, $\tilde{s}^*, \theta^*$ can be obtained from (12) and (13) respectively, $p_L^*, p_F^*, n_L^*, n_F^*$ can be obtained from Theorem 1 leading to the following overall equilibrium-type solutions:

Theorem 6. *Let* $|\Delta| < 1$*:*

(1) *If* $-1 < \Delta < 0$, *if an equilibrium-type solution exists, it is:* $(I_F^*, I_L^*) = (L_0, L_0)$, $\tilde{s}^*$ *is obtained by* (12), $\theta^* = 0$, *and*
$$p_L^* = c + \frac{1}{3} + \frac{\Delta}{3},\ p_F^* = c + \frac{2}{3} - \frac{\Delta}{3}$$
$$n_L^* = \frac{1}{3} + \frac{\Delta}{3},\ n_F^* = \frac{2}{3} - \frac{\Delta}{3}$$

(2) *If* $0 < \Delta < 1$, *if an equilibrium-type solution exists, it is:* $(I_F^*, I_L^*) = (0, L_0)$, $\tilde{s}^*$ *is of no significance,* θ^* *is obtained by* (13), *and*
$$p_L^* = c + \frac{2}{3} + \frac{\Delta}{3},\ p_F^* = c + \frac{1}{3} - \frac{\Delta}{3}$$
$$n_L^* = \frac{2}{3} + \frac{\Delta}{3},\ n_F^* = \frac{1}{3} - \frac{\Delta}{3}$$

(3) *If* $\Delta = 0$, *if an equilibrium-type solution exists, the equilibrium-type solutions are:*
- $(I_F^*, I_L^*) = (0, L_0)$, $\tilde{s}^*$ *is of no significance,* θ^* *is obtained by* (13),
$$p_L^* = c + \frac{2}{3} = n_L^* + c,\ p_F^* = c + \frac{1}{3} = n_F^* + c$$
- $(I_F^*, I_L^*) = (L_0, L_0)$, $\tilde{s}^*$ *is obtained by* (12), $\theta^* = 0$,
$$p_L^* = c + \frac{1}{3} = n_L^* + c,\ p_F^* = c + \frac{2}{3} = n_F^* + c.$$

Thus, considering only the values of $\tilde{s}^*, \theta^*$ given by (12), (13), the equilibrium-type solution is easy to compute and unique when it exists, when $|\Delta| < 1$, with the only exception being at $\Delta = 0$, at which there are either 0 or 2 equilibria. The insights on $p_L^*, p_F^*, n_L^*, n_F^*$ are otherwise similar to those presented after Theorem 1 in Part I.

Corollary 1. *The sum of payoffs of each of the possible equilibrium-solutions presented in Theorem 6 is:*
$$\pi^* = \pi_L^* + \pi_F^* = (1/3 - |\Delta|/3)^2 + (2/3 + |\Delta|/3)^2 - \gamma L_0^2. \tag{14}$$

Proof. First, from (1) and (2), we have
$$\pi_L + \pi_F = n_L(p_L - c) + n_F(p_F - c) - \gamma I_L^2.$$
From Theorem 6, $n_L^* = p_L^* - c$, $n_F^* = p_F^* - c$, and $I_L^* = L_0$. Then inserting n_L^*, n_F^*, p_L^*, p_F^*, and I_L^* into the above equation, we have the desired result. □

Again, assuming that the equilibrium solution exists in each case, the total payoff of the SPs decreases with the minimum mandated amount of spectrum acquisition L_0. This is expected as this reduction is in effect equivalent to relaxation of a constraint in a maximization, which increases the maximum value. Intuitively, the SPs increase their overall payoffs if they are allowed to get away with acquiring really small amounts of spectrums; since the EUs must choose one of the SPs, the joint subscription revenues of the SPs is not affected as long as both SPs acquire small amounts of spectrum. The sum also decreases with increase in the marginal reservation fee the central regulator charges. The sum is maximized at $|\Delta| = 1$, i.e., when one of the two SPs is apriori substantially more popular than the other, thus, he can attract most of the EUs despite charging a high amount. This enhances the overall subscription revenue. Note that the sum does not depend on the disagreement payoffs, and therefore does not depend on the marginal reservation fee the SP_F pays the SP_L in the event of a disagreement, i.e., the s the market provides.

We provide a necessary and sufficient condition for the existence of equilibrium-type solutions, in terms of parameters Δ, γ, L_0 and disagreement payoffs d_L, d_F.

Theorem 7. *Let $|\Delta| < 1$. At least one equilibrium-type solution exists if and only if*

$$\begin{aligned}\pi^* =&(1/3 - |\Delta|/3)^2 + (2/3 + |\Delta|/3)^2 - \gamma L_0^2 \\ \geq& d_L + d_F = d.\end{aligned}$$

Remark 7. *The disagreement payoffs d_L, d_F depend on the market-dependent marginal reservation fee s the SP_F pays the SP_L in the event of disagreement. Thus, this s only determines if an equilibrium-type solution exists, but not its values. Clearly, such solutions do not exist for large γ, L_0, which is consistent with the insights developed in Remarks 2, 3. In contrast, for $|\Delta| < 1$, the SPNE always exists, and is unique, when the SPs decide everything individually (Theorem 1 of Part 1).*

We now consider the EU-resource-cost metric introduced in the last paragraph of Section 2.1 of Part I, quantified as $I_F/p_F + (I_L - I_F)/p_L$. From Theorem 6

Theorem 8. *The EU-resource-cost metric in the SPNE is*

$$\begin{cases} L_0/(c + \dfrac{2}{3} - \dfrac{\Delta}{3}) & -1 < \Delta < 0 \\ L_0/(c + \dfrac{2}{3}) & \Delta = 0 \\ L_0/(c + \dfrac{2}{3} + \dfrac{\Delta}{3}) & 0 < \Delta < 1 \end{cases}.$$

Thus, the EU-resource-cost metric is clearly an increasing function of L_0. This is intuitive as in the SPNE the SPs together acquire exactly L_0 amount of spectrum. The Theorem goes beyond this intuition by identifying the exact nature of the dependence. It increases with increase in Δ, when $-1 < \Delta < 0$, reaches its maximum value at $\Delta = 0$, and decreases with increase in Δ, when $0 < \Delta < 1$. Thus, the EUs are best off, when the static factors are equal. It also decreases in increase in c, since the SPs increase the access fee for the EUs with increase in c.

We now consider $|\Delta| \geq 1$. As in Part I, this region is not of much interest due to the insurmountable difference between the prior preferences for the SPs. We show that in this case equilibrium-type solutions exist only for very small values of L_0. Since these solutions provide $I_L^* = L_0$, even the solutions are of limited practical utility. We state the results for completeness. Let s, δ constitute the parameters that provide the disagreement payoff (from the sequential game of Part I). Let $\gamma < s$.

Theorem 9. *If $\Delta \leq -1$ or $\Delta \geq 1$, the equilibrium-type solutions exist.*

(1) *If $\Delta \leq -1$ and $L_0 \leq \frac{1}{\sqrt{2s}}$, the equilibrium-type solutions are: $I_L^* = L_0$, $I_F^* \in [0, L_0]$, s^* is obtained by* (12), *θ^* is obtained by* (13), *and*

$$\begin{aligned}&p_L^* = p_F^* + \Delta + 1,\ c + 1 \leq p_F^* \leq c - \Delta - 1, \\ &n_L^* = 0,\ n_F^* = 1,\end{aligned}$$

If $L_0 > \frac{1}{\sqrt{2s}}$, no equilibrium-type solution exists.

(2) *If $\Delta \geq 1$ and $L_0 \leq \delta$, the equilibrium-type solutions are: $I_L^* = L_0$, $I_F^* \in [0, L_0]$, s^* is obtained by* (12), *θ^* is obtained by* (13), *and*

$$\begin{aligned}&p_F^* = p_L^* - \Delta,\ c + 1 \leq p_L^* \leq c + \Delta. \\ &n_L^* = 1,\ n_F^* = 0.\end{aligned}$$

If $L_0 > \delta$, no equilibrium-type solution exists.

Remark 8. *When $\Delta \leq -1$, Theorem 2 of Part I shows that the disagreement payoffs are attained when SP_L acquires $\frac{1}{\sqrt{2s}}$ resource. If $L_0 > \frac{1}{\sqrt{2s}}$, equilibrium-type solution does not exist per the intuitions in Remarks 2. More specifically, in this case, the SPs together attain payoffs lower than the total disagreement payoffs, as they are forced to acquire greater amounts of spectrum than what they did for acquiring their disagreement payoffs. This does not increase the total subscription revenue as the EUs must choose one of the SPs, but increases the total cost incurred in spectrum acquisition from the regulator. Thus, the aggregate excess payoff is negative. Hence there is no equilibrium-type solution. If $\Delta \geq 1$ and $L_0 > \delta$, equilibrium-type solutions do not exist for similar reasons which follow from an application of Theorem 2 of Part I.*

2.4 Numerical results

We numerically investigate the payoffs, the degree of cooperation, the investment levels, and the split of EUs to the SPs for $|\Delta| < 1$ and different values of other parameters. We set $\gamma = 0.5$, $c = 1$, $w = 0.2$, and consider two cases: 1) $\Delta = -0.5$; 2) $\Delta = 0.5$. SP_F is apriori more popular in the first, and SP_L in the second. We refer to the sum of equilibrium-type solution payoffs of the SPs as π^*, and disagreement payoffs of the SPs as d.

We first examine the condition for existence of equilibrium-type solutions, given in Theorem 7, by varying L_0 between $[0.1, 1]$, and different values of s used to obtain the disagreement payoffs. As expected from Theorem 7, Figure 2 show that π^* decreases with L_0, and does not depend on s. As mentioned in Remark 7, d depends on s, and from Theorem 7 d does not depend on L_0. Thus, the plots of d are parallel to the x-axis in Figure 2. We note that d initially increases with s, and then reaches its maximum value, at $s = s_{best} = 23.9$ and subsequently decreases. We consider $s = 0.8, 1, 1.2, s_{best}$. Figure 2 show the region in which $d \leq \pi^*$, for different values of s, it is the region of existence of equilibrium-type solutions as per Theorem 7. For a given s, we do not plot π^*, once it falls below d; thus the curves for π^* corresponding to a specific s stop whenever they meet the d for that s. The region in which equlibrium-type solutions exist is smallest at $s = s_{best}$ and much larger at $s = 0.8$. Referring to Corollary 1 and Theorem 7, in this region $\pi^* - d$ shows the gain in overall payoffs of the SPs through joint decision on spectrum acquisitions. The gain is naturally the smallest at $s = s_{best}$, but significant at other values of s.

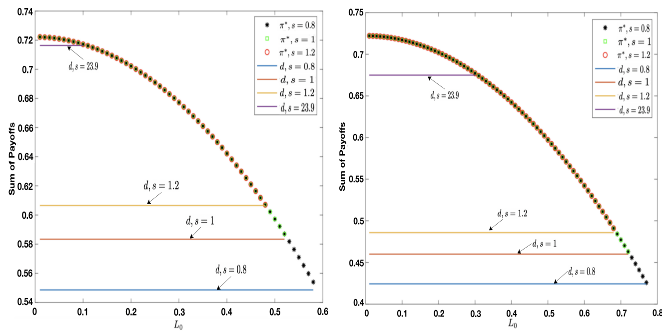

Fig. 2. Sum of equilibrium-type solution payoffs of SPs, π^*, when $\Delta = -0.5$ (left) and when $\Delta = 0.5$ (right) vs. L_0, sum of disagreement payoffs of SPs is d.

Figures 3, 4 demonstrate the payoff gain of each SP due to joint decision on spectrum acquisition (Remark 1). For $\Delta \in \{-0.5, 0.5\}$, the figures plot their disagreement payoffs, and equilibrium-solution payoffs, in the region that equilibrium solutions exist, as given by Theorem 7 and illustrated in Figure 2. The payoffs under the equilibrium solutions now depend on s as the disagreement payoffs depend on s (Equations (7) and (8)). First consider $\Delta = -0.5$. Figure 3 shows that the payoff of SP$_L$ (SP$_F$, respectively) increases (decreases, respectively) with s. Both payoffs decrease with L_0. Also, the gain in payoff for each SP beyond his disagreement payoff, due to joint decision on spectrum, is considerable for low L_0, but decreases as L_0 increases (the SPs are forced to acquire higher amounts of spectrum for large L_0 to deliver the minimum quality of service mandated by the regulator). The payoff of SP$_F$ is higher than that of SP$_L$ in this case because SP$_F$ is more apriori popular (as $\Delta < 0$). When $\Delta = 0.5$, Figure 4 shows that $\pi_L^* > \pi_F^*$, i.e., SP$_L$ has higher payoff in this case, which is intuitive as SP$_L$ is more apriori popular ($\Delta > 0$). The observations are otherwise similar to those for Figure 3.

The above observations for existence of equilibrium-type

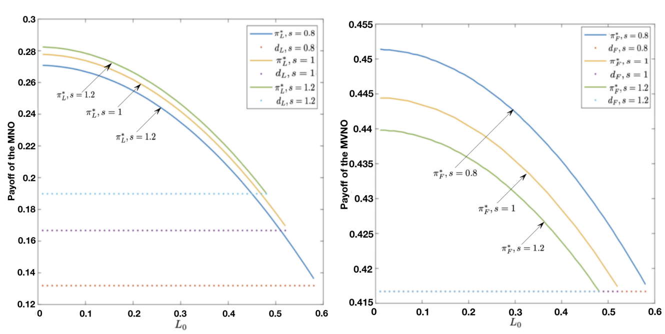

Fig. 3. Equilibrium-type solution and disagreement payoffs of SP$_L$ (left), SP$_F$ (right) when $\Delta = -0.5$ vs. L_0.

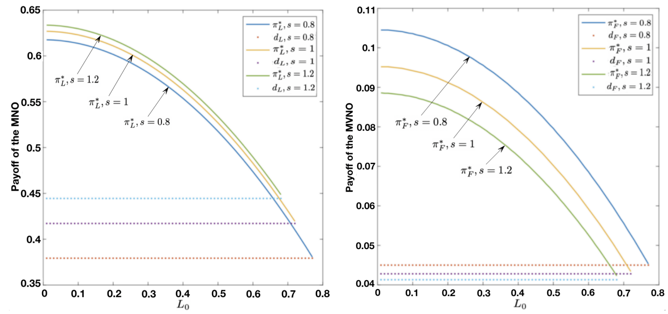

Fig. 4. Equilibrium-type solution and disagreement payoffs of SP$_L$ (left), SP$_F$ (right) when $\Delta = 0.5$ vs. L_0.

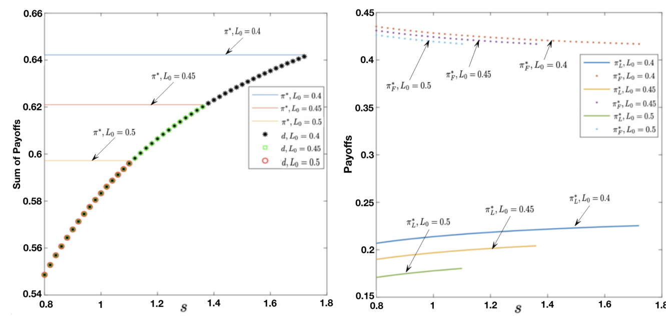

Fig. 5. Sum of equilibrium-type solution payoffs π^*, sum of disagreement payoffs d (left), individual payoffs (right) vs. s.

solutions and the collective and individual gains in payoffs of the SPs due to joint decision on spectrum acquisition may be reinforced by plotting $\pi^*, d, \pi_L^*, \pi_F^*$ as functions of s for few fixed L_0s (Figure 5). We consider only $\Delta = -0.5$ here, and $L_0 = 0.4, 0.45, 0.5$. Now, in the left figure, plots of π^* are parallel to the x-axis, while d increases with increase in s in the range considered, $s \in [0.8, 2]$. We plot π^* only in the region in which the equilibrium-type solutions exist, i.e., where $\pi^* \geq d$. This plot also quantifies the gains in collective payoffs by showing how much the flat curves exceed the increasing one, in the region in which they are plotted. The figure in the right show that the payoff of each SP decreases with L_0, and payoff of SP$_F$ (SP$_L$, respectively) decreases (increases, respectively) with s. The payoff of SP$_F$ is higher than that of SP$_L$, which is intuitive as SP$_F$ is apriori more popular in this case.

Our numerical computations thus far reveal that the cooperation in form of joint decisions on spectrum acquisitions benefits the SPs by enhancing their collective and individual payoffs. We now investigate how this enhanced cooperation between the SPs affects the EUs.

Now we investigate the subscriptions and access fees when the reservation fee is s_{best}, with Δ varying in $(-1, 1)$. From Theorem 6, subscriptions n_L^*, n_F^* only depend on Δ (and are independent of s and L_0). Figure 6 plots the subscriptions (left). From Theorem 6, $p_L^* = n_L^* + c$ and $p_F^* = n_F^* + c$, so the equilibrium-type subscriptions and access fees exhibit similar behaviors. For $\Delta < 0$, i.e., when SP$_F$ is apriori more popular, $n_F^* > n_L^*$ for the equilibrium-type solution under joint decision on spectrum acquisition. The difference

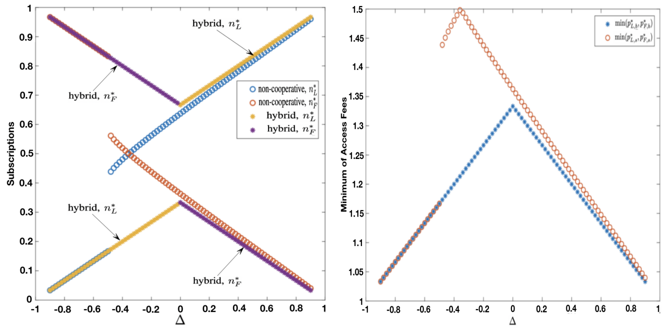

Fig. 6. The subscriptions (left), the access fees (right) vs. Δ.

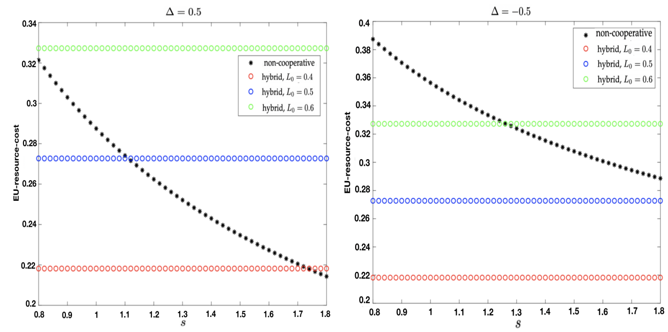

Fig. 7. The EU-resource-cost vs. s under $\Delta = 0.5$ (left) and $\Delta = -0.5$ (right).

$n_F^* - n_L^*$ increases as Δ reduces, when $\Delta < 0$. The reverse is observed when $\Delta > 0$. Since $p_F^* - p_L^* = n_F^* - n_L^*$ throughout, more EUs choose the SP that charges higher; this choice is clearly induced by how the SPs share between them the spectrum I_L that SP$_L$ acquires. In some way, this benefits the SPs, enhancing their overall revenue, and harms the EUs by motivating them to pay more. The conclusion is however nuanced as the EUs choose the more expensive option, voluntarily, and only because that option provides better quality of service by retaining the acquired spectrum in its entirety, and was also apriori more popular. The choice is therefore guided, rather than enforced, by having the more apriori popular SP retain the acquired spectrum. When the SPs separately decide their spectrum acquisitions, the trends are similar, through the differences between the subscriptions, and therefore the access fees, is less pronounced. The spectrum is more evenly shared between the SPs (Figure 1), leading to lower access fees and lower qualities of service for more EUs.

In Figure 6 (right), we plot the minimum of access fees of SPs in both frameworks: $\min(p^*_{L,b}, p^*_{F,b})$ ($\min(p^*_{L,s}, p^*_{F,s})$, respectively) represent the minimum access fees when the SPs decide spectrum acquisitions jointly (separately, respectively). The minimum access fee represents the least cost an EU might incur. The minimum is clearly equal or lower for the joint decision case. Thus, joint decisions of the SPs benefits the EUs by providing them cheaper access. But, as we have noted in the previous paragraph, more EUs are induced to select the more expensive option by having it provide the better quality of service and choosing the more popular of the two SPs to do so. Thus, in one perspective, the EUs gain due to enhanced coordination between the SPs, while they lose in another perspective.

We now plot the EU-resource-cost metric quantified in Theorem 8. Figure 7 shows that for both $\Delta > 0$ or $\Delta < 0$, for some values of L_0, this metric is higher under the joint decision and lower for the rest. As Theorem 8 shows, the EU-resource-cost metric is a linear function of L_0 under the joint decision, and therefore is higher or lower than that for individual decisions (as in Part I) depending on the value of L_0.

3 EUs WITH OUTSIDE OPTION

We now generalize our framework to provide an outside option to the EUs and exclusive customer bases to draw from to the SPs. Thus, the EUs from the common pool the SPs are competing over, may not choose either of the two SPs, leading to overall attrition, if the service quality-access fee tradeoff they offer is not satisfactory. The exclusive additional customer bases can provide customers beyond the common pool depending on the service quality and access fees the SPs offer. We introduce these modifications through demand functions we describe next.

Similar to equations (10), (11) in Part I, we define the fraction[2] of EUs with each SP as

$$\begin{aligned} \tilde{n}_L &= \alpha\big(n_L + \varphi_L(p_L, I_L)\big), \\ \tilde{n}_F &= \alpha\big(n_F + \varphi_F(p_F, I_F)\big), \end{aligned} \tag{15}$$

where

$$\begin{aligned} \varphi_L(p_L, I_L) &= k - p_L + b(I_L - I_F), \\ \varphi_F(p_F, I_F) &= k - p_F + bI_F \end{aligned} \tag{16}$$

and $\alpha > 0$, k and b are constants.

We also define $g(I_L) = \frac{b}{15}I_L + \frac{1}{15} - \frac{c}{3} + \frac{k}{3}$, $f(I_L) = \frac{1}{5I_L} + \frac{b}{5} > 0$.

We characterize the equilibrium-type solutions in Section 3.1, and examine its salient properties through numerical computations in Section 3.2.

3.1 The equilibrium-type solution

Our goal here is to examine if the availability of the outside option deters the collusive outcome by which the SPs acquire the minimum mandated amount of spectrum from the central regulator. We focus on the region in which at least one interior equilibrium-type solution, i.e., $0 < n_L, n_F < 1$ exists, and show that this is indeed the case. The proofs are given in Appendix B.

2. Recall that as in Part I n_L, n_F are the fraction of EUs from the common pool who subscribe to the EUs, while $\tilde{n}_L, \tilde{n}_F$ may be the fractions or actual numbers of subscriptions, considering the attrition to the outside option and the additions from the exclusive customer bases. Only scale factors would change in the expressions for $\tilde{n}_L, \tilde{n}_F$ and the payoffs depending on if n_L, n_F are fractions or actual numbers.

Theorem 10. *Let $\Delta = 0$. Either there is no interior equilibrium-type solution, or there are two interior equilibrium-type solutions. They are:*

(1) $I_{L,1}^*$ *is a solution of*

$$\max_{I_L} 2\alpha g^2(I_L) + 2\alpha(f(I_L)I_L + g(I_L))^2 - \gamma I_L^2$$
$$s.t\ L_0 \leq I_L$$

$I_{F,1}^* = I_{L,1}^*$, $\tilde{s}^*$ *is obtained by* (12), *and* $\theta^* = 0$.

(2) $p_{L,1}^* = \frac{1}{15} + \frac{2c}{3} + \frac{k}{3} + \frac{bI_L^*}{15}$, $p_{F,1}^* = \frac{4}{15} + \frac{2c}{3} + \frac{k}{3} + \frac{4bI_L^*}{15}$.

(3) $\tilde{n}_{L,1}^* = \frac{2}{15} + \frac{2k}{3} + \frac{2bI_L^*}{15} - \frac{2c}{3}$, $\tilde{n}_{F,1}^* = \frac{8}{15} + \frac{2k}{3} - \frac{2c}{3} + \frac{8bI_L^*}{15}$.

and

(1) $I_{L,2}^* = I_{L,1}^*, I_{F,2}^* = 0$, $\tilde{s}^*$ *has no significance, and* θ^* *is obtained by* (13).

(2) $p_{L,2}^* = p_{F,1}^*$, $p_{F,2}^* = p_{L,1}^*$.

(3) $\tilde{n}_{L,2}^* = \tilde{n}_{F,1}^*$, $\tilde{n}_{F,2}^* = \tilde{n}_{L,1}^*$.

We provide a necessary and sufficient condition for the existence of equilibrium-type solutions, in terms of parameters $\alpha, \gamma, I_{L,1}^*$ and disagreement payoffs d_L, d_F.

Theorem 11. *Let $\Delta = 0$. Interior equilibrium-type solutions exist if and only if*

$$\begin{aligned}\pi^* =& 2\alpha g^2(I_{L,1}^*) + 2\alpha(f(I_{L,1}^*)I_{L,1}^* + g(I_{L,1}^*))^2 - \gamma I_{L,1}^{*2} \\ \geq& d_L + d_F = d, \quad \textit{and} \quad I_L^* < \frac{4}{b}.\end{aligned}$$

Remark 9. *Solutions do not exist for large γ or small α, following the insights developed in Remarks 2, 3.*

The equilibrium-type solutions are easy to compute as they involve optimization in one decision variable and closed-form expressions. They are not unique, unlike in Part I (Theorem 7).

Our numerical computations would reveal that I_L^* exceeds L_0 in some cases. Thus, the deterrent of overall attrition and the incentive of increasing subscription from the exclusive additional bases, induce the SPs to acquire more spectrum than the minimum mandated amount, even when they are jointly deciding the acquisition amounts. Note that $p_L^*, p_F^*, n_L^*, n_F^*$ are linear increasing function of I_L^*. Thus, the SPs can increase both their subscriptions and access fees by acquiring greater overall spectrum I_L^* from the regulator. Like in the base case, $I_F^* \in \{0, I_L^*\}$, and thus the degree of cooperation is either 0 or 1. This is in contrast to the equivalent case in Section 3 Part I (eg, Figure 5) which show that the degree of cooperation can assume values between 0 and 1. Then, we consider the competition between SPs, i.e., the subscription n_L^* and n_F^*. The subscriptions n_L^* and n_F^* are constant if there exists no outside option (Theorem 6 (3)); but $\tilde{n}_L^*$ and $\tilde{n}_F^*$ change with the spectrum acquisition level of SP$_L$, I_L^*, if there exists an outside option.

We can write the first equilibrium-type solution as

$$\begin{aligned}\tilde{n}_L^* =& \frac{1}{5} + \varphi_L(p_L^*, I_L^*) + \frac{bI_L^*}{5}, \\ \tilde{n}_F^* =& \frac{4}{5} + \varphi_F(p_L^*, I_L^*) - \frac{bI_L^*}{5}\end{aligned}$$

In both equations, intuitively, the first term, $\frac{1}{5}$, $\frac{4}{5}$, represents the subscription from the common pool, if there had been no attrition to an outside option. The second and third terms represent the impacts of the attrition as also the additions from the exclusive customer bases. In the special case that $b = 0$, i.e., when the demand functions depend only on the access fees, the third term is 0 and the demand functions capture the impact of attrition and additions in the expression for the subscriptions. For $b > 0$, the second and the third term together become $k - p_L^* + \frac{b}{5} I_L^*$ in the expression for $\tilde{n}_L^*$, and $k - p_F^* + \frac{4b}{5} I_L^*$ in that for $\tilde{n}_F^*$. Thus, higher overall spectrum acquisition increases the subscription for both SPs even in these terms. The intuitions remain same for the second equilibrium-type solution, as the subscriptions are merely swapped.

Finally, when $L_0 \geq 4/b, \Delta = 0$, there does not exist an "interior" equilibrium-type solution, that is, in which $0 < n_L, n_F < 1$. Future research includes determining (1) whether there exists corner equilibrium-type solutions, or (2) generalization to the case that $\Delta \neq 0$.

3.2 Numerical results

We set $b = 2$, $k = c = 1$, $w = 0.2$ and $s = 2$ throughout. For $s = 2$ the condition for existence of interior equilibrium-type solutions is satisfied for all cases below. Also $I_L^* < 4/b$ in all cases below.

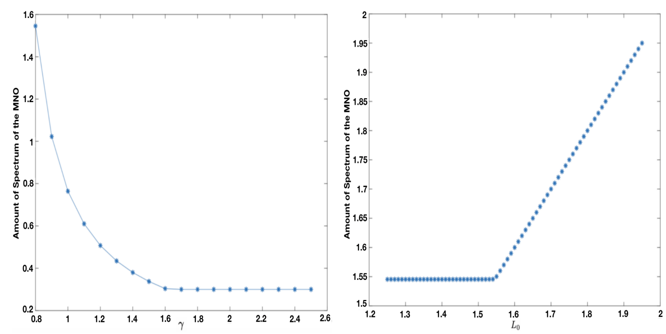

Fig. 8. The spectrum acquisition levels vs. γ (left), L_0 (right)

With $L_0 = 0.3$, Figure 8 (left) shows that the spectrum acquisitions for the two equilibrium-type solutions (I_L^* is the same in both) exceeds L_0 until γ crosses a threshold, and subsequently remains at L_0. Thus, SP$_L$ acquires more spectrum when it is cheaper to do so; otherwise settles at the minimum mandated amount. Now, with $\gamma = 0.8$, Figure 8 (right) shows that if L_0 is smaller than a threshold ($= 1.54$), I_L^* exceeds L_0 and equals the threshold value, and subsequently $I_L^* = L_0$. Thus, I_L^* is initially constant and subsequently increases linearly with L_0.

Figure 9 shows that the total payoff of the two SPs, as also their individual payoffs exceed the corresponding disagreement values, under both equilibrium-type solutions. As in the base case (eg, Figures 2, 3, 4) the total payoff and the individual payoffs decrease with increase in L_0, for the same reason as described in the paragraph after Corollary 1. In the first equilibrium-type solution, SP$_F$ leases the entire

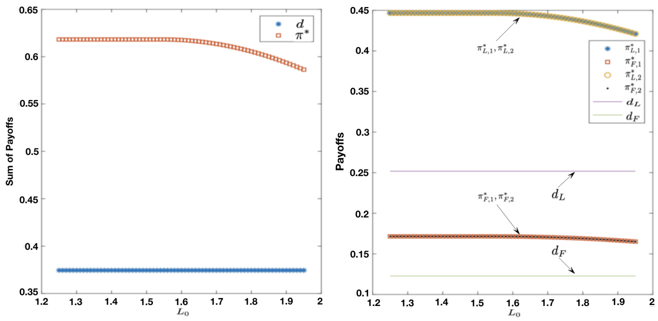

Fig. 9. Sum of equilibrium-type solution payoff, π^*, disagreement payoff, d (left), payoffs of individual SPs (right) vs. L_0

spectrum SP$_L$ acquires, while in the second, SP$_L$ retains this entire spectrum. We observe $\pi^*_{L,1} < \pi^*_{F,1}$ and $\pi^*_{L,2} > \pi^*_{F,2}$. Thus, under Nash bargaining solution, the SP that retains the entire spectrum gets a higher share of the payoff.

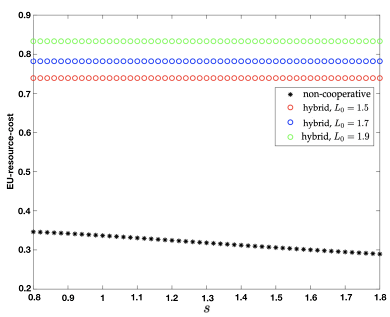

Fig. 10. The EU-resource-benefit vs s

From Theorem 10, by simple calculation, the EU-resource-cost metric is $I^*_{L,1}/p^*_{F,1}$ in either SPNE. Figure 10 shows that this metric is higher when the SPs jointly decide their spectrum acquisitions than when they decide separately (as in Part I). Naturally, for the joint decision case, this metric is constant with respect to s, the reservation fee SP$_F$ pays to SP$_L$. Similar to the base case, this metric increases with L_0.

4 Generalization: Limited spectrum from the central regulator

We now consider that SP$_L$ can lease at most M spectrum units from the central regulator due to paucity at the latter's end, and generalize the results in Sections 2, 3. We naturally assume that $M \geq L_0$ (recall that L_0 is the minimum amount of spectrum that SP$_L$ is required to obtain).

4.1 The base case

Theorems 5, 6, 7, 9 in Section 2.3, reveal that the SPNE strategies in the base case only depend on the lower bound L_0 of I_L. Thus, when I_L is additionally required to be less than or equal to M, SPNE strategies remain the same.

4.2 EUs with outside option

If an outside option exists in the system, then from Theorem 10, the SPNE strategies do not only depend on L_0. Then, Theorem 10 holds with the constraint $L_0 \leq I_L$ in (1) replaced by $L_0 \leq I_L \leq M$. We prove this in Appendix C. The replacement is also intuitive.

The numerical results in Section 3.2 reveal that without this upper bound I^*_L is an interior point. Thus, if M is relatively small, $I^*_L = M$. If M is large, then the values of I^*_L are as given by Theorem 10, and as computed in Section 3.2.

5 Conclusions and Future Research

This paper investigates the incentives of mobile network operators (MNOs) for acquiring additional spectrum to offer mobile virtual network operators (MVNOs) and thereby inviting competition for a common pool of end users (EUs). We consider interactions between two service providers, a MNO and an MVNO, when the EUs 1) must choose one of them 2) have the option to defect to an outside option should the SP duo offer unsatisfactory access fees or qualities of service. The 2 SPs jointly decide their spectrum acquisitions and the money flow between them, and separately decide the access fees for the EUs. We propose a multi-stage hybrid of cooperative bargaining and noncooperative games for modeling the interactions between the SPs, and identify when the overall equilibrium solutions exist, when it is unique and characterize the equilibrium solutions when they exist.

Analytical and numerical results show that the payoffs of both SPs in this hybrid framework are higher than those in noncooperative framework (in Part I). In a market without outside option, EUs in this hybrid framework can attain higher or lower value of resource-cost tradeoff than that in noncooperative framework, while in a market with outside option, EUs typically attain a strictly higher value of this tradeoff than that in noncooperative framework.

Future research includes generalization to accommodate: 1) arbitrary number of SPs 2) non-uniform distribution of EUs between the two SPs in the hotelling model, 3) distinct transaction costs c_L and c_F, 4) potentially non-convex spectrum reservation fee functions that the SP$_F$ pay the SP$_L$ and the SP$_L$ pay the regulator, 5) arbitrary transport cost t_L, t_F functions of the spectrum acquired by the SPs, I_L, I_F. Considering 3 SPs as in Section 4 of Part I constitute a starting point towards 1). Possible starting points towards the others, as also that for moving from 3 SPs to arbitrary number of SPs, have been provided in Section 5 of Part I.

References

[1] G. Hardin, "The tragedy of the commons", *Science*, vol. 162, no. 3859, pp. 1243-1248, December, 1968.

[2] M.J. Osborne and A. Rubinstein, *Bargaining and markets*. Academic press San Diego, 1990, vol. 34.

[3] A. Muthoo, "The economics of bargaining", *Fundamental Economics*, vol. 1, 2002.

[4] A. Banerjee and C. Dippon, "Voluntary relationships among mobile network operators and mobile virtual network operators: An economic explanation", *Information Economics and Policy*, vol. 21, pp. 72 - 84, 2009.

[5] G. Zhang, H. Zhang, and L. Zhao, "Fair resource sharing for cooperative relay networks using nash bargaining solutions", *IEEE Communications Letters*, vol. 13, no. 6, pp. 381-383, 2009.

[6] J. Suris, L. Dasilva, and Z. Han, "Asymptotic optimality for distributed spectrum sharing using bargaining solutions", *IEEE Transactions on Wireless Communications*, vol. 8, no. 10, pp. 5225-5237, 2009.

[7] H. Le Cadre, M. Bouhtou and B. Tuffin, "A pricing model for a mobile network operator sharing limited resource with a mobile virtual network operator", *International Workshop on Internet Charging and QoS Technologies*, pp. 24 - 35, 2009.

[8] G. Zhang, L. Cong, and E. Ding, "Fair and efficient resource sharing for selfish cooperative communication networks using cooperative game theory", *Communications (ICC), 2011 IEEE International Conference on Communications (ICC)*, 2011.

[9] L. Guijarro, V. Pla, B. Tuffin, P. Maille, J. R. Vidal, "Competition and Bargaining in Wireless Networks with Spectrum Leasing," in *2011 IEEE Global Telecommunications Conference (GLOBECOM)*, 2011

[10] C. Singh, S. Sarkar, A. Aram, and A. Kumar, "Cooperative profit sharing in coalition-based resource allocation in wireless networks," *IEEE/ACM Transactions on Networking (TON)*, vol. 20, no. 1, pp. 69-83, 2012.

[11] C. Singh, S. Sarkar, A. Aram, "Provider-Customer Coalitional Games", *IEEE/ACM Transactions on Networking*, vol. 19, no. 5, pp. 1528-1542, 2012.

[12] D. Venmani, Y. Gourhant, and D. Zeghlache, "Divide and share: A new approach for optimizing backup resource allocation in LTE mobile networks backhaul", *Network and service management (cnsm), 2012 8th International Conference and 2012 Workshop on Systems Virtualiztion Management (svm)*, 2012.

[13] H. Le Cadre and M. Bouhtou, "Modeling MNO and MVNO's dynamic interconnection relations: is cooperative content investment profitable for both providers?", *Telecommunication Systems*, vol. 51, pp. 193-217, 2012.

[14] R. Kokku, R. Mahindra, H. Zhang, and S. Rangarajan, "NVS: A Substrate for Virtualizing Wireless Resources in Cellular Networks", *IEEE/ACM Transactions on Networking*, vol. 20, no. 5, pp. 1333-1346, 2012.

[15] M. Tao and Y. Liu, "Spectrum leasing and cooperative resource allocation in cognitive ofdma networks" *Journal of Communications and Networks*, vol. 15, no. 1, pp. 102-110, 2013.

[16] Y. Yan, J. Huang, and J. Wang, "Dynamic bargaining for relay-based cooperative spectrum sharing", *IEEE journal on Selected Areas in Communications*, vol. 31, no. 8, pp. 1480-1493, 2013.

[17] M. Kalil, A. Shami, Y. Ye, "Wireless resources virtualization in LTE systems", *Computer Communications Workshops (INFOCOM WKSHPS), 2014 IEEE Conference on Computer Communications Workshops (INFOCOM WKSHPS)*, 2014.

[18] H. Zhang, C. Jiang, and N. Beaulieu, "Resource allocation for cognitive small cell networks: A cooperative bargaining game theoretic approach", *IEEE Transactions on Wireless Communications*, vol. 14, no. 6, pp. 3481-3493, 2015.

[19] K. Chatzikokolakis, P. Spapis, A. Kaloxylos, G. Beinas, and N. Alonistioti, "Spectrum sharing: A coordination framework enabled by fuzzy logic", *Computer, Information and Telecommunication Systems (CITS), 2015 International Conference on Computer, Information and Telecommunication Systems (CITS)*, IEEE, 2015.

[20] S. Brahma, M. Chatterjee, "Spectrum bargaining: A model for competitive sharing of unlicensed radio spectrum", *IEEE Transactions on Cognitive Communications and Networking*, vol. 1, no. 3, pp. 257-272, 2015.

[21] W. Guo, F. Lai, and D. Zeng, "A Hoteling model with production" *Mathematical Social Sciences*, vol. 73, pp. 40-49, 2015.

[22] L. Cano, A. Capone, and G. Carello, M. Cesana, and M. Passacantando, "Cooperative infrastructure and spectrum sharing in heterogeneous mobile networks", *IEEE Journal on Selected Areas in Communications*, vol. 34, no. 10, pp. 2617-2629, 2016.

[23] M. H. Lotfi and S. Sarkar, "The economics of competition and cooperation between MNOs and MVNOs", *Annual Conference on Information Sciences and Systems (CISS)* IEEE, 2017, pp. 1-6.

Xingran Chen received the BS degree in Statistics from the Central South University, Changsha, China, in 2015. He received the M.A. degree in Applied Mathematics and Computational Science in 2018 from University of Pennsylvania. He is currently working toward the Ph.D. degree in electrical and system engineering at the University of Pennsylvania. His research interests are Internet of Things (IoT), optimization, information and coding theory, and game theory.

Saswati Sarkar received the M.E. degree from the Electrical Communication Engineering Department, Indian Institute of Science, Bangalore, India, in 1996 and the Ph.D. degree from the Electrical and Computer Engineering Department, University of Maryland, College Park, MD, USA, in 2000. She joined the Electrical and Systems Engineering Department, University of Pennsylvania, Philadelphia, PA, USA, as an Assistant Professor in 2000 where she is currently a Professor. Her research interests are in stochastic control, resource allocation, dynamic games, and economics of networks. Dr. Sarkar received the Motorola Gold Medal for the best masters student in the division of electrical sciences at the Indian Institute of Science, and a National Science Foundation (NSF) Faculty Early Career Development Award in 2003. She was an Associate Editor of the IEEE TRANSACTIONS ON WIRELESS COMMUNICATIONS from 2001 to 2006, and of the IEEE/ACM TRANSACTIONS ON NETWORKS from 2008 to 2013.

Mohammad Hassan Lotfi received the B.Sc. degree in electrical engineering from the Sharif University of Technology, Tehran, Iran, in 2011. He received the M.S.c. degree in electrical and system engineering in 2013, the M.A. degree in Statistics in 2017, and the Ph.D. degree in electrical and system engineering in 2017 all from the University of Pennsylvania, Philadelphia, PA, USA. He is currently a Data Scientist at Capital One. His research interests are in economics of networks, game theory, pricing in communications and power networks, and net-neutrality.

Appendix A
Proofs for Theorems in Section 2.3

We prove Theorem 5 and Theorem 7 in two steps.

Let $|\Delta| < 1$. Consider $(I_L^*, I_F^*, p_L^*, p_F^*, n_L^*, n_F^*)$ that constitute the optimum solution of

$$\begin{aligned} \max_{I_L, I_F} \quad & u_{excess} \\ s.t. \quad & L_0 \le I_L,\ 0 \le I_F \le I_L. \end{aligned} \tag{17}$$

Here $p_L^*, p_F^*, n_L^*, n_F^*$ are obtained from I_L^*, I_F^* Theorem 1:

$$p_L^* = c + \frac{2}{3} - \frac{I_F}{3I_L} + \frac{\Delta}{3},\ \ p_F^* = c + \frac{1}{3} + \frac{I_F}{3I_L} - \frac{\Delta}{3}. \tag{18}$$

$$n_L^* = p_L^* - c,\ \ n_F^* = p_F^* - c \tag{19}$$

In **Step 1** we show that any such (I_L^*, I_F^*) must be of the form given in Theorem 5. Next, note that an optimum solution of (11), should it exist, is also an optimum solution of (17). Since equilibrium-type solutions constitute the optimum solutions of (11), Theorem 5 follows.

In **Step 2** we observe that given the I_L^*, I_F^* of the possible equilibrium-type solutions mentioned in Theorem 5, 1) $\tilde{s}^*, \theta^*$ of these can be obtained from (12) and (13) respectively, and 2) $p_L^*, p_F^*, n_L^*, n_F^*$ of these can be obtained from Theorem 1. Accordingly, Theorem 6 follows from Theorem 5, as mentioned before Theorem 6. The total payoff of the two SPs under each of the possible equilibrium-type solutions in Theorem 6 is the same, and is given in Corollary 1. If any possible equilibrium-type solution listed in Theorem 5 is an equilibrium-type solution, then this total payoff must not exceed the sum of the disagreement payoffs. Next, if this total payoff is not less than the disagreement payoffs, then $u_{excess} \ge 0$ under the possible equilibrium-type solutions listed in Theorem 5. Thus, these solutions satisfy the additional constraint in (11) (beyond (17)), and therefore constitute its optimum solution too. Thus, these are equilibrium-type solutions . Theorem 7 follows.

Step 1.

Proof. Consider $(I_L^*, I_F^*, p_L^*, p_F^*, n_L^*, n_F^*)$ that constitute the optimum solution of (17).

Substituting (18) and (19) into (1) and (2), we can get the payoffs of $\mathrm{SP}_F, \mathrm{SP}_L$, for some $\tilde{s}, \theta$ as:

$$\pi_F^* = (\frac{1-\Delta}{3} + \frac{I_F^*}{3I_L^*})^2 - \tilde{s}(I_F^*)^2 + \theta, \tag{20}$$

$$\pi_L^* = (\frac{\Delta+2}{3} - \frac{I_F^*}{3I_L^*})^2 + \tilde{s}(I_F^*)^2 - \gamma(I_L^*)2 - \theta. \tag{21}$$

By Definition 3, substituting (20) and (21) into (6), we can get u_{excess}^*,

$$\begin{aligned} u_{excess}^* =& (\frac{\Delta+2}{3} - \frac{I_F^*}{3I_L^*})^2 - \gamma(I_L^*)2 - d_L \\ &+ (\frac{1-\Delta}{3} + \frac{I_F^*}{3I_L^*})^2 - d_F. \end{aligned} \tag{22}$$

Denote $t^* = I_F^*/I_L^*$, (22) is equivalent to

$$\begin{aligned} u_{excess}^* =& (\frac{\Delta+2-t^*}{3})^2 - \gamma(I_L^*)^2 \\ &+ (\frac{1-\Delta+t^*}{3})^2 - d_L - d_F. \end{aligned} \tag{23}$$

Now we prove that $I_L^* = L_0$ by contradiction. Suppose $I_L^* > L_0$, then take $\hat{I}_L = L_0$ and $\hat{I}_F = I_F^* \frac{L_0}{I_L^*}$. Thus $t^* = I_F^*/I_L^* = \hat{I}_F/\hat{I}_L$. Since t^* is constant and $\hat{I}_L < I_L^*$, then u_{excess} is higher with $\hat{I}_F$ and $\hat{I}_L$ than with I_F^* and I_L^*. This contradicts the optimality of I_F^* and I_L^*. Therefore, $I_L^* = L_0$.

Take the second derivative of u_{excess} with respect to I_F, $\frac{d^2 u_{excess}}{dI_F^2} = \frac{4}{9I_L^2} > 0$, then u_{excess} is convex with respect to I_F, and the maximum of u_{excess} must be obtained at the boundaries of I_F.

Then, we obtain the optimal solution I_F^*. Note $0 \le I_F \le I_L$. Substitute $I_F^* = 0$ and $I_F^* = I_L^* = L_0$ into (23), we have

$$u_{excess}(0, L_0) - u_{excess}(L_0, L_0) = \frac{4}{9}\Delta.$$

Therefore

$$\begin{aligned} &u_{excess}(0, L_0) > u_{excess}(L_0, L_0) \quad \text{if} \quad \Delta > 0 \\ &u_{excess}(0, L_0) = u_{excess}(L_0, L_0) \quad \text{if} \quad \Delta = 0 \\ &u_{excess}(0, L_0) < u_{excess}(L_0, L_0) \quad \text{if} \quad \Delta < 0 \end{aligned}$$

$$\Rightarrow \begin{cases} (I_F^*, I_L^*) = (0, L_0) \quad \text{if} \quad 0 < \Delta \le 1 \\ (I_F^*, I_L^*) = (0 \text{ or } L_0, L_0) \quad \text{if} \quad \Delta = 0 \\ (I_F^*, I_L^*) = (L_0, L_0) \quad \text{if} \quad -1 < \Delta < 0 \end{cases}.$$

□

Proof of Theorem 9.

Proof. Once I_L^*, I_F^* are determined, $\tilde{s}^*$ is obtained by (12) and θ^* is obtained by (13). We obtain $I_L^*, I_F^*, p_L^*, p_F^*$ in two steps: $\Delta \le -1$ (**Step 1**), $\Delta \ge 1$ (**Step 2**).

Step 1: $\Delta \le -1$ Suppose the reservation fee is s in the sequential framework with $s > \gamma$. From Theorem 2 (3) in Part I, $n_L^* = 0, n_F^* = 1$,

$$\begin{aligned} &p_L^* = p_F^* + \Delta - 1 \\ &p_F^* \in [c+1, c-\Delta-1]. \end{aligned} \tag{24}$$

These also constitute the SPNE, together with,

$$I_L' = I_F' = \frac{1}{\sqrt{2s}}, \tag{25}$$

that provides the disagreement payoffs, d_L, d_F. From (1), (2) in Part I and (25), $d_L + d_F = p_F^* - c - \frac{\gamma}{2s}$.

Again, from (1) and (2), under equilibrium-type solution, the payoffs of the SPs are

$$\pi_F = p_F^* - c - \tilde{s}^*(I_F^*)^2 + \theta^*, \tag{26}$$

$$\pi_L = \tilde{s}^* I_F^2 - \gamma(I_L^*)^2 - \theta^*. \tag{27}$$

By Definition 3, substituting (26) and (27) into (6):

$$u_{excess}^* = p_F^* - c - \gamma(I_L^*)^2 - d_L - d_F. \tag{28}$$

Note that u^*_{excess} is independent of I_F, then I^*_F can be any number between $[0, I^*_L]$. Therefore, I^*_L is a solution of the following optimization problem,

$$\begin{aligned} \max_{I_L, I_F} \quad & u_{excess} = p^*_F - c - \gamma I_L^2 - d_L - d_F \\ s.t \quad & L_0 \le I_L \\ & u_{excess} \ge 0 \end{aligned} \tag{29}$$

From (24), p^*_F is independent of I_L, so the objective function is a decreasing function of I_L. Thus, $I^*_L = L_0$. Since $d_L + d_F = p^*_F - c - \frac{\gamma}{2s}$, then $u^*_{excess} \ge 0$ is equivalent to $L_0 \le \frac{1}{\sqrt{2s}}$. The result follows.

Step 2: $\Delta \ge 1$**:** We first consider the corner SPNE for (p_L, p_F, n_L, n_F) in Theorem 2 (1) in Part I: $n^*_L = 1, n^*_F = 0$, and

$$\begin{aligned} p^*_F =& p^*_L + v^F - v^L \\ p^*_L \in& [c+1, c + v^L - v^F]. \end{aligned} \tag{30}$$

Along with $I'_L = \delta$, $I'_F = 0$, these also constitute the SPNE that provide the disagreement payoffs. Therefore, from (1) in Part I, $d_F = 0$ and $d_L = p^*_L - c - \gamma\delta^2$. From (1), (2), under an equilibrium-type solution,

$$\begin{aligned} \pi^*_F &= \tilde{s}^*(I^*_F)^2 + \theta^* \\ \pi^*_L &= p^*_L - c + \tilde{s}^*(I^*_F)^2 - \gamma(I^*_L)^2 - \theta^*, \end{aligned} \tag{31}$$

then substituting (31) into (6), we can get u^*_{excess}:

$$u^*_{excess} = p^*_L - c - \gamma(I^*_L)^2 - d_L - d_F. \tag{32}$$

Note that u^*_{excess} is independent of I_F, then I^*_F can be any number between $[0, I^*_L]$. Therefore, the optimum I^*_L is a solution of the following optimization problem,

$$\begin{aligned} \max_{I_L, I_F} \quad & u_{excess} = p^*_L - c - \gamma I_L^2 - d_L - d_F \\ s.t \quad & L_0 \le I_L \quad u_{excess} \ge 0 \end{aligned} \tag{33}$$

From (30), p^*_L is independent of I_L, so the objective function is a decreasing function of I_L, then $I^*_L = L_0$. Note that $d_L + d_F = p^*_L - c - \gamma\delta^2$, then $u^*_{excess} \ge 0$ is equivalent to $L_0 \le \delta$.

Next, we consider $\Delta = 1$ and the interior SPNE in Theorem 2 (2) in Part I, i.e., $0 < n^*_F, n^*_L < 1$. By similar analysis in Theorem 5, we have $I^*_L = L_0$ and $I^*_F = 0$. Therefore from (18) and (19), $p^*_L = c+1$, $p^*_F = c$, $n^*_L = 1$, and $n^*_F = 0$, which is contradicted to $0 < n^*_F, n^*_L < 1$. Thus no equilibrium-type solution exists in this case.

□

Proof of Theorem 8.

Proof. We calculate m^* in 5 cases: $-1 < \Delta < 0$, $\Delta = 0$, $0 < \Delta < 1$, $\Delta \le -1$ and $\Delta \ge 1$.

Case 1. When $-1 < \Delta < 0$. Note that $I^*_L - I^*_F = 0$, then $m^* = I^*_F / p^*_F = L_0/(c + 2/3 - \Delta/3)$.

Case 2. When $\Delta = 0$. If $I^*_F = 0$, then then $m^* = (I^*_L - I^*_F)/p^*_L = L_0/(c + 2/3)$. If $I^*_F = I^*_L$, then $m^* = I^*_F/p^*_F = L_0/(c + 2/3)$.

Case 3. When $0 < \Delta < 1$. Note that $I^*_F = 0$, then $m^* = (I^*_L - I^*_F)/p^*_L = L_0/(c + 2/3 + \Delta/3)$. □

Appendix B
Proof of Theorems in Section 3.1

Once I^*_L, I^*_F are determined, $\tilde{s}^*$ is obtained by (12) and θ^* is obtained by (13). We therefore focus on obtaining $(I^*_L, I^*_F, p^*_L, p^*_F, n^*_L, n^*_F)$ corresponding to the equilibrium-type solutions. Let $\Delta = 0$. Consider $(I^*_L, I^*_F, p^*_L, p^*_F, n^*_L, n^*_F)$ that constitute the optimum solution of (17) (with only the expressions for u_{excess} differing from Appendix A). Per Theorem 7 (3), (4), Part I, for an interior SPNE, $I^*_L < 4/b$, and :

$$\begin{aligned} \tilde{n}^*_L =& \frac{I^*_L - I^*_F}{I^*_L} + p^*_F - 2p^*_L + k + bI^*_L - bI^*_F \\ \tilde{n}^*_F =& \frac{I^*_F}{I^*_L} + p^*_L - 2p^*_F + k + bI^*_F, \end{aligned} \tag{34}$$

$$\begin{aligned} p^*_L =& \frac{1}{15} + \frac{2c}{3} + \frac{k}{3} + \frac{t_F}{5} - \frac{b}{5}I_F + \frac{4b}{15}I_L, \\ p^*_F =& \frac{1}{15} + \frac{2c}{3} + \frac{k}{3} + \frac{t_L}{5} + \frac{b}{15}I_L + \frac{b}{5}I_F. \end{aligned} \tag{35}$$

First, we show that any such $(I^*_L, I^*_F, p^*_F, n^*_L, n^*_F)$ must be of the form given in Theorem 10. Next, note that an optimum solution of (11), should it exist, is also an optimum solution of (17). Since equilibrium-type solutions constitute the optimum solutions of (11), Theorem 10 follows.

In fact, substituting (15) and (16) in Part I into (1) and (2),

$$\begin{aligned} \pi_F =& \alpha(t_L + k + p_L - 2p_F + bI_F)(p_F - c) - \tilde{s}I_F^2 + \theta \\ \pi_L =& \alpha(t_F + k + p_F - 2p_L + bI_L - bI_F)(p_L - c) \\ &+ \tilde{s}I_F^2 - \gamma I_L^2 - \theta. \end{aligned} \tag{36}$$

Lemma 1. *In any solution of* (17), $I^*_F = I^*_L$ *or* $I^*_F = 0$.

Proof. By substituting (36) into $u_{excess} = \pi_L - d_L + \pi_F - d_F$, and using $t_L = I_F/I_L$, $t_F = 1 - t_L$,

$$\begin{aligned} u_{excess} =& 4\alpha f^2(I_L) I_F^2 - 4\alpha f^2(I_L) I_L I_F + 2\alpha g^2(I_L) \\ &+ 2\alpha(f(I_L)I_L + g(I_L))^2 - \gamma I_L^2 - d_F - d_L. \end{aligned}$$

Next $\frac{d^2 u_{excess}}{dI_F^2} = 8\alpha f^2(I_L) > 0$.

Thus, u_{excess} is convex wrt I_F, and the maximum of u_{excess} is obtained at the boundary of I_F:

$$\begin{aligned} & u_{excess}|_{I_F = I_L} = u_{excess}|_{I_F = 0} \\ =& 2\alpha g^2(I_L) + 2\alpha(f(I_L)I_L + g(I_L))^2 - \gamma I_L^2 - d_F - d_L. \end{aligned}$$

Thus $I^*_F = I_L$ or $I^*_F = 0$. □

Also, for any solution of (17), I^*_L is given by

$$\begin{aligned} \max_{I_L} \quad & 2\alpha g^2(I_L) + 2\alpha(f(I_L)I_L + g(I_L))^2 - \gamma I_L^2 \\ s.t \quad & L_0 \le I_L. \end{aligned} \tag{37}$$

Substituting $I^*_F = I^*_L$ and $I^*_F = 0$ into (34) and (35), combining with Lemma 1 and (37), it follows that any solution $(I^*_L, I^*_F, p^*_F, n^*_L, n^*_F)$ of (17) must be of the form given in Theorem 10. Thus, Theorem 10 follows.

From (4) in Part I, $x^*_0 = t^*_F + p^*_F - p^*_L$, substituting (35), $t^*_F = (I^*_L - I^*_F)/I^*_L$ into x^*_0, then we have $0 < x^*_0 < 1$ if and only if $I_L^* < 4/b$. The total payoff of the two SPs under each of the possible interior equilibrium-type solutions listed in Theorem 10 is the same, and is given in Theorem 11. If any possible equilibrium-type solution listed in Theorem 10 is an equilibrium-type solution, then this total payoff must not exceed the sum of the disagreement payoffs. Thus, the necessity in Theorem 11 follows. Next, if $I_L^* < 4/b$, the p_L^*, p_F^* in Theorem 11 constitute an interior Nash equilibrium in Stage 2 of the sequential hybrid game. If the total payoff of the possible equilibrium-type solutions in Theorem 11 is not less than the disagreement payoffs, then $u_{excess} \geq 0$ under them. Thus, these solutions satisfy the additional constraint in (11) (beyond (17)), and therefore constitute its optimum solution too. Thus, the sufficiency in Theorem 11 follows.

□

APPENDIX C
PROOF OF RESULTS IN SECTION 4.2.

In Section 4.2, we had claimed the following Theorem:

Theorem 12. *Let $\Delta = 0$. Either there is no interior equilibrium-type solution, or there are two interior equilibrium-type solutions. They are:*

(1) $I_{L,1}^*$ *is a solution of*

$$\max_{I_L} 2\alpha g^2(I_L) + 2\alpha(f(I_L)I_L + g(I_L))^2 - \gamma I_L^2$$
$$s.t\ L_0 \leq I_L \leq M$$

$I_{F,1}^* = I_{L,1}^*$, $\tilde{s}^*$ *is obtained by* (12), *and* $\theta^* = 0$.

(2) $p_{L,1}^* = \frac{1}{15} + \frac{2c}{3} + \frac{k}{3} + \frac{bI_L^*}{15}$, $p_{F,1}^* = \frac{4}{15} + \frac{2c}{3} + \frac{k}{3} + \frac{4bI_L^*}{15}$.

(3) $\tilde{n}_{L,1}^* = \frac{2}{15} + \frac{2k}{3} + \frac{2bI_L^*}{15} - \frac{2c}{3}$, $\tilde{n}_{F,1}^* = \frac{8}{15} + \frac{2k}{3} - \frac{2c}{3} + \frac{8bI_L^*}{15}$.

and

(1) $I_{L,2}^* = I_{L,1}^*, I_{F,2}^* = 0$, $\tilde{s}^*$ *has no significance, and* θ^* *is obtained by* (13).

(2) $p_{L,2}^* = p_{F,1}^*$, $p_{F,2}^* = p_{L,1}^*$.

(3) $\tilde{n}_{L,2}^* = \tilde{n}_{F,1}^*$, $\tilde{n}_{F,2}^* = \tilde{n}_{L,1}^*$.

The proof of this Theorem is identical to that for Theorem 10 in Appendix B, with the following modification: the optimization problem (37) becomes

$$\begin{aligned} \max_{I_L} \quad & 2\alpha g^2(I_L) + 2\alpha(f(I_L)I_L + g(I_L))^2 - \gamma I_L^2 \\ s.t \quad & L_0 \leq I_L \leq M. \end{aligned}$$

This is because $(I_L^*, I_F^*, p_L^*, p_F^*, n_L^*, n_F^*)$ constitute the optimum solution of

$$\begin{aligned} \max_{I_L, I_F} \quad & u_{excess} \\ s.t. \quad & L_0 \leq I_L \leq M,\ 0 \leq I_F \leq I_L. \end{aligned}$$

Theorem 12 now follows using arguments that are otherwise identical to that for the proof of Theorem 10.