The Interplay of Competition and Cooperation Among Service Providers (Part I)

Xingran Chen, Mohammad Hassan Lotfi, Saswati Sarkar

***Abstract*—This paper investigates the incentives of mobile network operators (MNOs) for acquiring additional spectrum to offer mobile virtual network operators (MVNOs) and thereby inviting competition for a common pool of end users (EUs). We consider a base case and two generalizations: (i) one MNO and one MVNO, (ii) one MNO, one MVNO and an outside option, and (iii) two MNOs and one MVNO. In each of these cases, we model the interactions of the service providers (SPs) using a sequential game, identify when the Subgame Perfect Nash Equilibrium (SPNE) exists, when it is unique and characterize the SPNE when it exists. The characterizations are easy to compute, and are in closed form or involve optimizations in only one decision variable. We identify metrics to quantify the interplay between cooperation and competition, and evaluate those as also the SPNEs to show that cooperation between MNO and MVNO can enhance the payoffs of both, while increased competition due to the presence of additional MNOs is beneficial to EUs but reduces the payoffs of the SPs.**

***Index Terms*—Resource Sharing, Game Theory, Sequential Game, Subgame Perfect Nash Equilibrium**

I. Introduction

NOWADAYS wireless service providers (SPs) are divided into (i) mobile network operators (MNOs) that lease spectrum from a regulator like FCC, and (ii) mobile virtual network operators (MVNOs) that obtain spectrum from one or more MNOs. MVNOs can distinguish their plans from MNOs by bundling their service with other products, offering different pricing plans for End-Users (EUs), or building a good reputation through a better customer service. Although traditionally wireless service has been offered only by MNOs, in recent years, the number of MVNOs has been rapidly growing. According to [4], between June 2010 and June 2015, the number of MVNOs increased by 70 percent worldwide, reaching 1,017 as of June 2015. Even some MNOs developed their own MVNOs. An example of which is Cricket wireless which is owned by AT&T and offers a prepaid wireless service to EUs. Another example of MVNOs is the Google's Project Fi in which the customer's service is handled using Wi-Fi hotspots wherever/whenever they exist; elsewhere the service is handled using the spectrum of a number of MNOs, eg, Sprint, T-Mobile or U.S. Cellular networks.

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In this work, we consider the economics of the interaction among MNOs and MVNOs. We seek to understand why and under what conditions the MNOs cooperate with the MVNOs by offering some of their spectrum to the MVNOs, and thereby inviting competition for a common pool of EUs. We consider scenarios where the MNOs decide on acquiring new spectrum, and in exchange for a fee offer those to MVNOs, which decide to acquire some of the spectrum offered. The SPs decide on their pricing strategies for the EUs, and the EUs decide to opt for one of them, or neither, if the access fees and the qualities of service are not satisfactory. The spectrum acquisition and pricing decisions of the SPs determine their respective profits. We characterize their equilibrium choices. We obtain metrics that quantify the cooperation and competition of the SPs in terms of their spectrum investments and subscriptions of EUs, which help quantify the interplay between competition and cooperation under the equilibrium choices.

We consider a hotelling model in which a continuum of undecided EUs decide which of the SPs they want to buy their wireless plan from, if at all. The EUs have different preferences for each SP. These preferences can be because of different services and qualities that SPs offer. For example, the MVNOs may be able to offer a free or cheap international call plan through VoIP, or an SP may have an infamous customer service. The preference for a SP also increases with the spectrum she acquires. If, for example, EUs have high preferences for MVNOs, then the MNOs may prefer to lease some of their spectrum to the MVNOs and receive their share of profit through the MVNOs, instead of competing for EUs by lowering their access fees. On the other hand, if EUs have high preferences for the MNOs, the MNOs may not offer spectrum to the MVNOs and seek to attract the EUs directly. Thus, cooperation is mutually beneficial only in some scenarios, which we seek to identify.

We start with by considering a base case in which one MNO and one MVNO compete for EUs in a common pool, and the EUs have to choose one of the SPs (Section II). We consider a sequential game in which the SPs decide their spectrum investments and access fees for the EUs (Section II-A). We subsequently seek the Subgame Perfect Nash Equilibrium (SPNE) of the game using backward induction (Section II-E), and identify conditions under which the SPNE outcome of the game exists and is unique, and characterize the SPNE whenever it exists. The SPNE is simple to compute, as 1) the amount of spectrum the MNO invests turns out to be the value that maximizes a function involving only one decision variable 2) the amount of spectrum the MVNO leases from the MNO is a simple closed form expression involving the amount that the MNO offers it and the leasing fee 3) the access fees for the EUs constitute simple closed form expressions of the spectrum the SPs acquire. The characterizations provide several insights. The spectrum acquired by the MNO never falls below a threshold which depends only on the leasing fee to the MVNO and preferences for the SPs. When the spectrum equals this threshold, the MVNO reserves the entire spectrum that the MNO offers it. Thus cooperation is high in this case. As the MNO acquires higher amounts of spectrum, the MVNO reserves progressively lower amounts, leading to lower degrees of cooperation. Numerical computations reveal that the MNO acquires minimal amount of spectrum only when the leasing fee to the MVNO is smaller than a threshold (Section II-D). The SPNE characterizations show that higher degrees of cooperation invariably reduces (enhances, respectively) the efficacy of the MNO (MVNO, respectively) in competing for the EUs; yet, higher degrees of cooperation enhance the payoffs of both the SPs as our numerical computations reveal. The MNO's loss in revenue from subscription is more than compensated by the leasing fees obtained from the MVNO.

Next, we generalize the hotelling model for EU subscription in the base case by incorporating an additional demand function (Section III). The effects of the demand function are two-fold. First, the demand function models the attrition in the number of EUs of SPs if the spectrum investment or price of both SPs is not desirable for EUs. Thus, in effect, an EU may opt for neither SP if neither offers a price-quality combo that is to his satisfaction, which is equivalent to opting for outside options. Second, the demand function models an exclusive additional customer base for each of the SPs to draw from depending on her investment and the price she offers. We characterize the unique interior SPNE outcome of the game (Section III-A). Numerical results reveal that the general behavior of the SPNE outcome are as in the base case (Section III-B).

Finally, we generalize the base case to include competition between MNOs. We consider a wireless market with two MNOs and one MVNO, in which EUs choose one of the three SPs (Section IV). We generalize the hotelling model to consider three players instead of two in the classical ones (Section IV-A), and characterize the unique SPNE outcome (Section IV-B). The characterizations show that this enhanced competition 1) increases the degree of cooperation, as the MVNO acquires all the spectrum that the MNOs offer, and 2) is beneficial to EUs, as the amounts of spectrum of SPs acquires are higher, and the SPs charge the EUs less.

While in this work we consider that the SPs arrive at their decisions individually, in the accompanying sequel we consider that the SPs arrive at certain decisions as a group, and then arrive at other decisions individually (Part II). Also, here we assume that the per unit leasing fee the MVNO pays to MNO(s) is a fixed parameter, which is beyond the control of individual MNOs and MVNOs, perhaps determined by the overall market evolution. Note that the overall market may consider several MNOs and MVNOs, whose presence we consider in the generalizations (Section III, IV). We investigate the implications of different values of this fee on the SPNE and the payoffs. In the sequel we consider that the SPs cooperatively characterize this fee as a decision variable in a bargaining framework (Part II).

We position our work in context of the existing literature for interaction among MNOs and MVNOs, and defer to the sequel a more detailed literature review on the interaction among arbitrary SPs (Part II). In [1] MNOs seek to maximize the joint profit of MNO and MVNO. The selection of access fees by a MNO is formulated as a maximization problem in which the sales of the MNO is expressed as a function of only the price he selects. In contrast we consider that each SP seeks to maximize his individual profit and obtain the access fees they select and the spectrum they acquire, which also determine how the EUs choose between the SPs. Thus we need to dwell in the realm of a hierarchical game rather than a single stage optimization. A scenario very different from ours is considered in [2]: the SPs *do not* compete for consumer market shares but for the proportion of resource they are going to use. The interaction between the SPs is a hierarchical game in which the MNO and MVNO choose their access fees, the MVNO also decide investment in content/advertising. The access fees become roots of a fourth order polynomial equation which is computed numerically. The closest to our work is [3], which considers a dynamic three-level sequential game of spectrum sharing between one MNO and one MVNO. The focus is however complementary to ours.

Unlike our work, [3] does not consider decisions of the 1) MNO pertaining to how much spectrum to acquire from a regulatory body 2) MVNO pertaining to how much of the MNO's spectrum offer it ought to accept (it assumes that the MVNO uses the entire spectrum the MNO offers). We also generalize our model to consider multiple MNOs and an MVNO, which [3] does not. [3] however considers a decision of the MVNO that we do not, i.e., how much it would invest in content generation. The customer subscription models are also entirely different. We consider a one-shot game involving a continuum of EUs in which the SP choice of each EU is based on his intrinsic preferences for the SPs and the spectrum investments of the SPs. [3] considers a multi-time slot game in which a discrete number of EUs choose between the SPs based on their experiences in the previous slots and their estimates of the quality of service the SPs they had not chosen apriori offer. The games we consider fundamentally differ in that the SPNE need not exist in ours (we identify necessary and sufficient conditions for its existence), while it always exists in that in [3]. By exploiting the structure of the game, we are able to obtain closed form expressions for the various decisions we consider, in the SPNE, whenever it exists. [3] computes the SPNE only numerically through the solution of a multi-slot stochastic dynamic program (DP). Our SPNE characterization is easy to compute, while DPs usually suffer from the curse of dimensionality.

II. Base case

We present the system model in which we formulate the payoffs and strategies of SPs, and the utilities and decisions of EUs (Section II-A). Next, we formulate the interaction between different entities as a sequential game (Section II-B). Subsequently, we characterize the conditions under for the existence and the uniqueness of the SPNE, obtain closed form expressions for the SPNE when it exists (Section II-C). We present numerical results in Section II-D. We prove the analytical results in Section II-E, Appendix B and Appendix D.

A. Model

SPs: We consider one MNO (SP$_L$, L represents leader) and one MVNO (SP$_F$, F represents follower) which compete for a common pool of undecided EUs. SP$_L$ offers I_L amount of spectrum (which it acquires from a regulator) to SP$_F$ in exchange of money, and SP$_F$ uses I_F amount of this spectrum. Clearly, $0 \leq I_F \leq I_L$. For simplicity of analysis and formulation, we assume that $I_L \geq \delta > 0$, where δ is a parameter of choice. This assumption is not significantly restrictive as δ may be chosen as low a positive quantity as one desires. Both SP$_L$ and SP$_F$ earn by selling wireless plans to EUs; SP$_L$ earns additionally by leasing her spectrum to SP$_F$. We assume that both SP$_L$ and SP$_F$ have access to separate spectrum, which they can use to serve the EUs who join them, above and beyond the I_L, I_F amounts they strategically acquire. For example, a SP$_F$ like Google's Project Fi serves customers using Wi-Fi hotspots and the spectrum of 3 MNOs (Sprint, T-Mobile or U.S. Cellular networks). Also, SP$_L$ may acquire additional spectrum from the regulator which it does not offer SP$_F$.

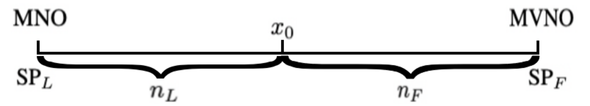

Fig. 1: The hotelling model for the base case. The EUs in $[0, x_0]$ ($[x_0, 1]$, respectively) prefer SP$_L$ (SP$_F$, respectively). The former fraction of EUs is n_L, the latter is n_F. x_0 is farther off from SP$_L$ as t_L becomes lower and $v_L - v_F$ become higher.

We denote the marginal leasing fee (per spectrum unit) that SP$_L$ pays the regulator as γ, marginal reservation fee SP$_F$ pays to SP$_L$ by s, the fraction of EUs that SP$_F$ and SP$_L$ attract as n_F and n_L, respectively, and the access fee that SP$_F$ and SP$_L$ charge the EUs as p_F and p_L, respectively. Since SP$_L$ wants to lease out some of her spectrum to SP$_F$ with profit motive, it is reasonable to assume that $s > \gamma$. We assume that s, γ are pre-determined. The strategies of SPs are to choose the investment levels (I_L, I_F) and the access fees for EUs (p_L, p_F) so as to maximize their overall payoffs, which we formulate next.

SP$_F$ and SP$_L$ respectively earn revenues of $n_F(p_F - c), n_L(p_L - c)$ from EU subscription, where c is the transaction cost SPs incur in subscription. We expect the cost of reserving spectrum to be strictly convex, i.e. the cost of investment per spectrum unit increases with the amount of spectrum[1]. For simplicity in analysis, we consider these costs to be quadratic and discuss generalizations in Remark 3. That is, SP$_L$ incurs a spectrum acquisition cost of γI_L^2, and SP$_F$ pays to SP$_L$ a leasing fee of sI_F^2. Thus, the payoffs of SPs are:

$$\pi_F = n_F(p_F - c) - sI_F^2 \tag{1}$$

$$\pi_L = n_L(p_L - c) + sI_F^2 - \gamma I_L^2. \tag{2}$$

[1]These costs do not satisfy the economy of scale; the regulator may mandate such structures to stop excessive acquisition by big SPs seeking to control the market, which has limited spectrum supply, and drive out smaller SPs or new entrants. Incidentally, several seminal works have considered strictly convex investment costs, e.g. [5] and [6].

EUs: We use a hotelling model to describe how EUs choose between the SPs. We assume that SP$_L$ is located at 0, SP$_F$ is located at 1, and EUs are distributed uniformly along the unit interval $[0, 1]$ (Figure 1). The closer an EU to an SP, the more this EU prefers this SP to the other. Note that the notion of closeness and distance is used to model the preference of EUs, and may not be the same as physical distance. Let t_L (t_F) be the unit transport cost of EUs for SP$_L$ (SP$_F$), the EU located at $x \in [0, 1]$ incurs a cost of $t_L x$ (respectively, $t_F(1-x)$) when joining SP$_L$ (respectively, SP$_F$).

$$\begin{aligned} u_L(x) =& v^L - (p_L + t_L x) \\ u_F(x) =& v^F - (p_F + t_F(1-x)) \,. \end{aligned} \tag{3}$$

The EU at x receives utilities $u_L(x), u_F(x)$ respectively from SP$_L$ and SP$_F$, and joins the SP that gives it the higher utility.

The first component of the utility functions comprises of the "static factors", namely v^L and v^F of SP$_L$ and SP$_F$, respectively. The static factor of an SP is the same for all EUs, which depends on the local presence, its existing spectrum beyond I_L or I_F and its reputation in the region, quality of the customer-service, ease of usage for the online portals, etc. However, the static factors do not depend on strategies of SPs, such as the access fees, the investment levels, etc.

The second component, i.e., $p_L + t_L x$ or $p_F + t_F(1-x)$, is denoted as the "strategy factor". The strategy factors depend on the strategies of the SPs, namely their access fees and the spectrum I_L, I_F they acquire. Clearly, the utilities would decrease with the access fees, we consider the dependence to be linear. As SP$_F$ acquires greater fraction of the additional spectrum SP$_L$ offers him, SP$_F$ becomes more desirable and SP$_L$ less desirable to the EUs. Denote $t_L = I_F/I_L$ and $t_F = (I_L - I_F)/I_L$. Then the impact of quality of service in the decision of EUs is captured through t_L and t_F. For example, when $I_F = I_L$, i.e., SP$_F$ leases the entire I_L spectrum from SP$_L$ and SP$_L$ can use none of it, then $t_F = 0$ and $t_L = 1$. This gives SP$_F$ an advantage over SP$_L$ in attracting EUs. Similarly, even when $I_F = 0$, i.e., SP$_F$ leases no spectrum from SP$_L$, $t_F = 1$ and $t_L = 0$, SP$_F$ has an advantage over SP$_L$. But subscription may still be divided in both the above extreme cases. This happens since both SP$_F$ and SP$_L$ have access to separate spectrum as reflected in the static factors v^F, v^L. Note that the pair of transport cost ($t_L = I_F/I_L, t_F = 1 - t_L$) is one of the many functions that can be considered. We choose this model specifically since it captures the essence of the model, and is analytically tractable.

Finally, the strategy factors incorporate intrinsic preference of the EUs towards the SPs through the coordinate x, which presents the local distance in the utility model. If an EU is for example close to SP$_F$, x is high and $1-x$ is low, and it is deemed to have a higher intrinsic preference for SP$_F$, as compared to SP$_L$. The intrinsic preference may be developed through pre-existing and ongoing relations the EU has with the SPs, e.g., if an EU is already availing of other services from an SP, the EU will have a stronger intrinsic preference for the SP, due to convenience of billing etc. Higher intrinsic preferences enhance utilities of the SP for the EUs. The impact of the strategies of the SPs on the EUs will depend on their intrinsic preferences for the EUs, which is captured in the term $t_L x$ or $t_F(1-x)$ in the utility. Note that the intrinsic preference is different for different EUs unlike the static factor.

As is common for hotelling models, we assume that each EU chooses exactly one SP to subscribe to, i.e., the market is "fully covered". An equivalent assumption is to consider the static factors v^L and v^F to be sufficiently large so that the utility of EUs for buying a wireless plan is positive regardless of the choice of SP. We would in effect relax this assumption in Section III.

SP$_F$'s leasing of spectrum from SP$_L$ constitute an act of cooperation. Thus, we call I_F/I_L *the degree of cooperation*. Since SP$_F$ and SP$_L$ compete to attract EUs, the split of subscription (n_L, n_F) represent the level of competition. Since the amount of spectrum SP$_F$ leases from SP$_L$ determines the split of subscription, there is a natural interplay between cooperation and competition, that these metrics will enable us to quantify.

B. The sequential game framework

The interaction among SPs and EUs can be formulated as a sequential game. As a leader of the game, SP_L makes the first moves. The timing and the stages of the game are as following:

- **Stage 1:** SP$_L$ decides on the amount of spectrum, I_L, to acquire.
- **Stage 2:** SP$_F$ decides on the amount of spectrum to lease from SP$_L$, I_F.
- **Stage 3:** SP$_L$ and SP$_F$ determine the access fees for the EUs, p_L and p_F, respectively.
- **Stage 4:** Each EU subscribes to the SP that gives it the higher utility.

Remark 1. *We assume that the decision of investments (I_L and I_F) happens before the decisions of access fees (p_L and p_F), guided by the fact that spectrum investment decisions are long-term ones, and are therefore expected to be constants over longer time horizons in comparison to subscription pricing decisions.*

Definition 1. *A strategy is a Subgame Perfect Nash Equilibrium (SPNE) if and only if it constitutes a Nash Equilibrium (NE) of every subgame of the game.*

We refer to a SPNE choice of spectrum investments and access fees by the SPs as $(I_L^*, I_F^*, p_L^*, p_F^*)$, and the EU subscriptions for the SPs under the same as n_L^*, n_F^*, should a SPNE exist.

C. The SPNE outcome

We next identify the conditions under which SPNE exists, characterize the SPNE when it exists, and examine its uniqueness.

We denote $v^L - v^F$ as Δ. Since $0 \le t_L, t_F \le 1$, $0 \le x \le 1$, in the expressions for utilities in (3), $|\Delta| \ge 1$ provides a near insurmountable disadvantage to one of the SPs through the static factors; this SP might have to choose a significantly lower price to recoup. Thus, we focus on the range $|\Delta| < 1$. As stated before, we assume δ is small, and let $\delta < \sqrt{\frac{2-\Delta}{9s}}$, which reduces to $\delta < \sqrt{\frac{2}{9s}}$ in the special case that $v^L = v^F$.

Theorem 1. *Let* $|\Delta| < 1$. *The SPNE is:*

(1) *Any solution of the following maximization is* I_L^**:*

$$\max_{I_L} \pi_L(I_L) = \left(\frac{2+\Delta}{3} - \frac{1-\Delta}{27sI_L^2 - 3}\right)^2 + s\left(\frac{(1-\Delta)I_L}{9sI_L^2 - 1}\right)^2 - \gamma I_L^2$$
$$s.t.\ I_L \ge \sqrt{\frac{2-\Delta}{9s}}$$

(2) I_F^* *is characterized in*

$$I_F^* = \begin{cases} \dfrac{(1-\Delta)I_L}{9I_L^2 s - 1} & \text{if } I_L > \sqrt{\dfrac{2-\Delta}{9s}} \\ I_L & \text{if } I_L = \sqrt{\dfrac{2-\Delta}{9s}} \end{cases}$$

(3) $p_L^* = c + \frac{2}{3} - \frac{I_F^*}{3I_L^*} + \frac{\Delta}{3}$, $\quad p_F^* = c + \frac{1}{3} + \frac{I_F^*}{3I_L^*} - \frac{\Delta}{3}$.

(4) $n_L^* = \frac{\Delta}{3} + \frac{2}{3} - \frac{I_F^*}{3I_L^*}$, $n_F^* = \frac{I_F^*}{3I_L^*} + \frac{1}{3} - \frac{\Delta}{3}$.

Remark 2. *The SPNE is unique provided the maximization in Theorem 1 (1) has a unique solution, which appears to be the case as per our extensive numerical computations. Otherwise every solution of this maximization leads to a distinct SPNE.*

The proof is given in Appendix D-A. The SPNE is easy to compute, despite the expressions being cumbersome. This is because I_L^* can be obtained as a maximizer of an expression that involves only one decision variable, I_L, and fixed parameters s, γ, Δ. I_F^* has been expressed as a closed form function involving I_L^* and the fixed parameters s, Δ. $p_L^*, p_F^*, n_L^*, n_F^*$ have been expressed as closed form functions of I_F^*/I_L^* and the fixed parameters c, Δ.

From Theorem 1 (3) the price the EUs receive from SP$_L$ (SP$_F$, respectively) decrease (increase, respectively) with increase in the degree of cooperation (I_F/I_L). Thus, since at least one of the SPs reduce the price, the EUs benefit from higher degree of cooperation.

From Theorem 1 (3) and (4), $n_L^* = p_L^* - c, n_F^* = p_F^* - c$. Thus, SPNE subscriptions of the SPs increase with increase in the access fees they announce. This counter-intuitive feature arises because the subscriptions also depend on the spectrum acquisitions of the SPs, through the transport costs $t_L = I_F/I_L$ and $t_F = 1 - t_F$ in the utilities specified in (3).

From Theorem 1 (1), in the SPNE, SP$_L$ acquires at least $\sqrt{\frac{2-\Delta}{9s}}$ amount of spectrum. From Theorem 1 (2), when I_L^* equals this minimum, then SP$_F$ reserves all the available spectrum, i.e., $I_L^* = I_F^*$ (note that I_F^* is continuous at $I_L = \sqrt{\frac{2-\Delta}{9s}}$). Thus, SP$_L$ can not use any of I_L^*. However, from Theorem 1 (4), SP$_L$ is still able to attract a positive fraction of EUs: $n_L^* = \frac{\Delta+1}{3} > 0$ since $-1 < \Delta \le 1$. This is because EUs have spectrum other than I_L^*, I_F^* as captured in the absolute values of v^L, v^F.

From Theorem 1 (1) and (2), when I_L^* exceeds its minimum value, then SP$_F$ reserves only a fraction of available spectrum ($I_F^* < I_L^*$). Note that in this case, $\frac{dI_F^*}{dI_L} < 0$. Thus, the higher the amount of available spectrum, the lower would be the amount of spectrum reserved by SP$_F$. Also, I_F^* is decreasing with s.

The SPNE depends on the static factors v^L, v^F only through their difference Δ. As expected, with increase (decrease, respectively) in Δ, SP$_L$ (SP$_F$ respectively) can increase its access fee p_L^* (p_F^*, respectively). The minimum value of his spectrum acquisition I_L^* increases with decrease in Δ, to offset the competitive advantage the static factors provide. Through our numerical computations, we elucidate how I_L^*, I_F^* and the payoffs otherwise vary with Δ.

The results illustrate the interplay between cooperation and competition. From Theorem 1 (4), the subscription n_L^* (n_F^*, respectively) of SP$_L$ (SP$_F$) decreases (increases, respectively) with the degree of cooperation (I_F/I_L). Thus, the higher the degree of cooperation, lesser (greater, respectively) is the competition efficacy of SP$_L$ (SP$_F$, respectively). A natural question arises: why would the SP$_L$ then cooperate with the SP$_F$? From (1) and (2), Theorem 1 (3), (4), $\pi_L = n_L^{*2} + sI_F^{*2} - \gamma I_L^{*2}$, and $\pi_F = n_F^{*2} - sI_F^{*2}$. On the one hand, if the degree of cooperation increases, then the amount of subscribers

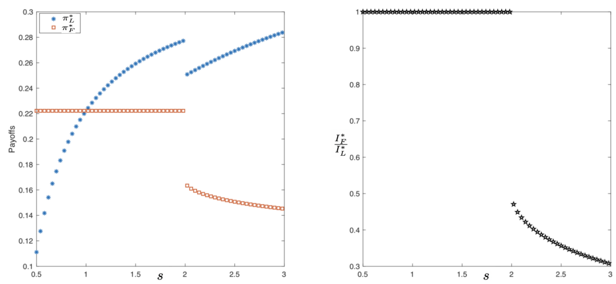

Fig. 2: Payoffs (left) and the degree of cooperation (right) vs. s. Here, $\gamma = 0.5$, $c = 1$, $\Delta = 0$.

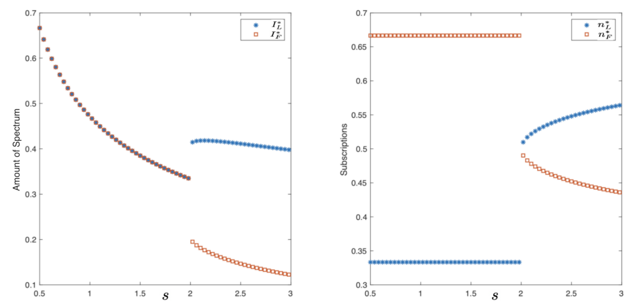

Fig. 3: Investment decisions (left), the split of subscription (right) vs. s. Here, $\gamma = 0.5$, $c = 1$, $\Delta = 0$.

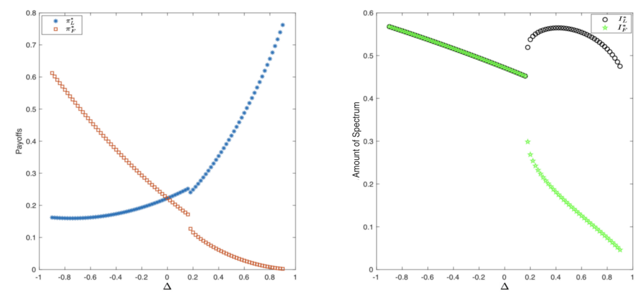

Fig. 4: Payoffs (left), investment decisions (right) vs. Δ. Here, $\gamma = 0.5$, $c = 1$, $s = 1$.

of SP$_L$ decreases, thus the revenue SP$_L$ earn from the subscribers decreases. On the other hand, the payoff of SP$_L$ increases through sI_F^{*2}. Thus the second factor may offset the first, and the payoff of SP$_L$ may increase due to cooperation. Note that it is not a zero sum game, thus, the payoffs of both players may simultaneously increase due to cooperation. We illustrate these phenomena definitively through our numerical computations in the next section.

Finally, in the extreme case that $|\Delta| \geq 1$:

Theorem 2. **(1)** $\Delta \geq 1$*: The SPNE is*

$$I_L^* = \delta, I_F^* = 0, p_F^* = p_L^* - \Delta, n_L^* = 1,\ n_F^* = 0,$$

and p_L^* *can be chosen any value in* $[c+1, c+\Delta]$.

(2) $\Delta = 1$ *: The following interior strategy constitute an additional SPNE*

$$I_L^* = I_F^* = \frac{1}{3\sqrt{s}},\ \ p_L^* - c = n_L^* = 2/3, p_F^* - c = n_F^* = 1/3.$$

(3) $\Delta \leq \sqrt{\frac{\gamma}{s}} - 2$ *: The SPNE strategy is:*

$$I_F^* = I_L^* = \frac{1}{\sqrt{2s}}, p_L^* = p_F^* + \Delta - 1, n_L^* = 0,\ n_F^* = 1,$$

and p_L^* *can be chosen any value in* $[c+1, c-\Delta]$.

(4) $\Delta \in (\sqrt{\frac{\gamma}{s}} - 2, -1)$*: No SPNE strategy exists.*

We prove this theorem in Appendix D-B. As is intuitive, for large Δ, all EUs subscribe to SP$_L$, despite lower access fees selected by SP$_F$; the reverse happens in the other extreme, despite lower access fees selected by SP$_F$. The extremes therefore lead to "corner equilibria", which correspond to $0, 1$ as the degrees of cooperation. The SPNE is non-unique in both these extremes. For a certain range of Δ, the SPNE does not exist.

D. Numerical results

Figure 2 shows the payoffs (left) and the degree of cooperation (right) under different s when $\Delta = 0$. The degree of cooperation reaches the maximum ($= 1$), i.e., $I_F^* = I_L^*$ when s is less than a threshold (≈ 2). In this case, SP$_L$ generates most of its revenue from the reservation fee paid by SP$_F$. As expected, π_L^* increases with s. From Theorem 1 (1), (2) (4), when $I_F^* = I_L^*$, I_L^* equals its minimum value $\sqrt{\frac{2}{9s}}$, and $n_F^* = 1/3 + I_F^*/3I_L^* = 2/3$, thus $\pi_F^* = n_F^{*2} - sI_F^{*2}$ is a constant which is independent of s. When s is larger than this threshold, $I_F^*/I_L^* < 1$, and decreases with s. In this case, I_L^* exceeds its minimum value, and SP$_F$ leases only a portion of the new spectrum invested by SP$_L$, i.e., $I_F^* < I_L^*$. Thus, SP$_L$ generates more of its revenue from EUs. The payoff of SP$_L$ (SP$_F$) first jumps to a lower value at this threshold, and then increases (decreases) with s. At this threshold, the degree of cooperation also jumps to a lower value (< 1). Thus, higher degrees of cooperation can enhance the payoff of both SPs, and the reservation fee s enhances (reduces) the payoff of SP$_L$ (SP$_F$).

Figure 3 shows the SPNE level of investment (left) and subscriptions of SPs (right) when $\Delta = 0$. It reconfirms that when s is smaller than a threshold, SP$_F$ leases the entire spectrum SP$_L$ offers, and after that threshold, SP$_F$ leases only a portion of the new spectrum offered by SP$_L$. Also, I_L^* strictly decreases with s throughout. When s is small, $I_F^* = I_L^*$, n_F^* and n_L^* are constant ($n_L^* = 1/3$, $n_F^* = 2/3$) independent of γ and s, and $n_F^* > n_L^*$. After the threshold, n_F^* decreases and n_L^* increases with s (because I_F^*/I_L^* decreases with s in Figure 2 (right)).

Comparing Figure 2 (right) and Figure 3 (right) we note that higher degrees of cooperations increase (decrease) the competition efficacy of SP_F (SP_L, respectively).

Figure 4 plots the payoffs (left) and I_L, I_F (right) as a function of Δ when $|\Delta| < 1$, the region in which the SPNE exists uniquely. We set $s = 1$. As expected, the payoff of SP_L (SP_F, respectively) increase (decrease, respectively) with increase in Δ. With increase in Δ, I_L, I_F may either increase or decrease, depending on whether additional spectrum provides "bang for the buck" by enticing commensurate number of EUs which depends on the EUs' prior biases (static factors) for or against the SPs. The figure shows which is the case.

E. SPNE Analysis

We use backward induction to characterize SPNE strategies, starting from the last stage of the game and proceeding backward. For simplicity and brevity, we present this analysis only for the important special case of $\Delta = 0$, and defer the general case to Appendix D-A. Thus, we prove Theorem 1 while applying $\Delta = 0$ in the corresponding expressions. Specific Theorems 3, 5, 32 are proven in Appendix B.

Stage 4: We first characterize the equilibrium division of EUs between SPs, i.e., n_L^* and n_F^*, using the knowledge of the strategies chosen by the SPs in Stages 1∼3.

Definition 2. *x_0 is the indifferent location between the two service providers if $u_L(x_0) = u_F(x_0)$ (Figure 1).*

By the full market coverage assumption, if $0 < x_0 < 1$, then EUs in the interval $[0, x_0]$ join SP_L and those in the interval $[x_0, 1]$ join SP_F. If $x_0 \leq 0$, all EUs choose SP_F; and if $x_0 \geq 1$, all EUs choose SP_L (Figure 1).

From Definition 2, $u_F(x_0) = v - t_F(1-x_0) - p_F = v - t_L x_0 - p_L = u_L(x_0)$. Since $t_L + t_F = 1$, then $x_0 = \frac{t_F + p_F - p_L}{t_L + t_F} = t_F + p_F - p_L$. Thus,

$$x_0 = t_F + p_F - p_L \tag{4}$$

Thus, since EUs are distributed uniformly along $[0, 1]$, the fraction of EUs with each SP is:

$$n_L = \begin{cases} 0, & \text{if} \quad x_0 \leq 0 \\ x_0, & \text{if} \quad 0 < x_0 < 1 \\ 1, & \text{if} \quad x_0 \geq 1 \end{cases}, \; n_F = 1 - n_L, \tag{5}$$

where x_0 is defined in (4) and $n_F = 1 - n_L$ (Figure 1).

Only "interior" strategies may be SPNE, as:

Theorem 3. *In the SPNE it must be that $0 < x_0 < 1$.*

Stage 3: SP_L and SP_F determine their access fees for EUs, p_L and p_F, respectively, to maximize their payoffs.

Lemma 1. *The payoffs of SPs are:*

$$\begin{aligned} \pi_L &= (t_F + p_F - p_L)(p_L - c) + sI_F^2 - \gamma I_L^2 \\ \pi_F &= (t_L + p_L - p_F)(p_F - c) - sI_F^2 \end{aligned} \tag{6}$$

Proof. From (5), substitute $(n_L, n_F) = (t_F + p_F - p_L, 1 - n_L)$ into (1) and (2), and get (6). □

We next obtain the SPNE p_F^* and p_L^* which maximize the payoffs π_L and π_F of the SPs respectively.

Theorem 4. *The SPNE pricing strategies are:*

$$p_L^* = c + \frac{2}{3} - \frac{I_F}{3I_L}, \quad p_F^* = c + \frac{1}{3} + \frac{I_F}{3I_L} \tag{7}$$

Proof. p_F^* and p_L^* must satisfy the first order condition, i.e., $\frac{d\pi_F}{dp_F} = 0$ and $\frac{d\pi_L}{dp_L} = 0$. Thus, $p_F^* = c + \frac{I_L + I_F}{3I_L}$ & $p_L^* = c + \frac{2I_L - I_F}{3I_L}$. p_F^* and p_L^* are the unique SPNE strategies if they yield $0 < x_0 < 1$ and no unilateral deviation is profitable for SPs. We establish these respectively in Parts A and B.

Part A. From (7), $x_0 = \frac{I_L^* - I_F^*}{I_L^*} + p_F^* - p_L^* = \frac{2I_L^* - I_F^*}{3I_L^*}$. Since $I_L^* \geq I_F^*$ and $I_L^* > 0$, then $0 < x_0 < 1$.

Part B. Since $\frac{d^2\pi_F}{dp_F^2} < 0, \frac{d^2\pi_L}{dp_L^2} < 0$, a local maxima is also a global maximum, and any solution to the first order conditions maximize the payoffs when $0 < x_0 < 1$, and no unilateral deviation by which $0 < x_0 < 1$ would be profitable for the SPs. Now, we show that unilateral deviations of the SPs leading to $n_L = 0, n_F = 1$ and $n_L = 1, n_F = 0$ is not profitable. Note that the payoffs of the SPs, (1) and (2), are continuous as $n_L \downarrow 0$, and $n_L \uparrow 1$ (which subsequently yields $n_F \uparrow 1$ and $n_F \downarrow 0$, respectively). Thus, the payoffs of both SPs when selecting p_L and p_F as the solutions of the first order conditions are greater than or equal to the payoffs when $n_L = 0$ and $n_L = 1$. Thus, the unilateral deviations under consideration are not profitable for the SPs. □

Remark 3. *The proof shows that x_0, p_L^*, p_F^* do not depend on the specific nature of the costs of leasing spectrum I_F, I_L, neither does n_L^*, n_F^* from* (5). *Thus the SPNE expressions for these would remain the same for any other cost function. But, the SPNE of investment levels (I_L^*, I_F^*) as obtained in the next results depend on the specific nature of these functions.*

Stage 2: SP_F decides on the amount of spectrum to be leased from SP_L, I_F, with the condition that $0 \leq I_F \leq I_L$, to maximize π_F.

Theorem 5. *The SPNE spectrum acquired by SP_F is:*

$$I_F^* = \begin{cases} \dfrac{I_L}{9I_L^2 s - 1} & \text{when} \quad I_L > \sqrt{\dfrac{2}{9s}} \\ I_L & \text{when} \quad \delta \le I_L \le \sqrt{\dfrac{2}{9s}} \end{cases} \tag{8}$$

Stage 1: SP_L chooses the amount of spectrum I_L to lease from the regulator, to maximize π_L.

Theorem 6. *The solution of the following maximization constitutes the SPNE spectrum acquired by SP_L, I_L^* :*

$$\begin{aligned} &\max_{I_L} \pi_L = \frac{1}{9}(2 - \frac{1}{9sI_L^2 - 1})^2 + s(\frac{I_L}{9sI_L^2 - 1})^2 - \gamma I_L^2 \\ &s.t.\ I_L \ge \sqrt{\frac{2}{9s}}. \end{aligned} \tag{9}$$

Let $\Delta = 0$. Theorem 1 follows from Theorems 3, 4 5, 32. Theorem 3 allows us to consider only interior SPNE. Parts (1) and (2) of Theorem 1 follow respectively from Theorems 32 and 5. Part (3) follows from Theorem 4, part (4) from Theorem 4 and (5).

III. EUs with Outside Options

We now generalize our framework to consider a scenario in which the EUs from the common pool the SPs are competing over, may not choose either of the two SPs if the service quality-price tradeoff they offer is not satisfactory. In effect, there is an outside option for the EUs. Also, each SP has an exclusive additional customer base which can provide customers beyond the common pool depending on the service quality and access fees they offer. We introduce these modifications through demand functions we describe next.

Definition 3. *The fraction[2] of EUs with each SP is*

$$\tilde{n}_L = \alpha n_L + \tilde{\varphi}_L(p_L, I_L), \quad \tilde{n}_F = \alpha n_F + \tilde{\varphi}_F(p_F, I_F),$$

where

$$\begin{aligned} \tilde{\varphi}_L(p_L, I_L) &= k' - \theta' p_L + b'(I_L - I_F), \\ \tilde{\varphi}_F(p_F, I_F) &= k' - \theta' p_F + b' I_F \end{aligned}$$

and $\alpha > 0$, k', θ' and b' are constants.

Here, n_L, n_F represent fractional subscriptions from the common pool as before, and are determined in Stage 4 of the sequential game described in Section II-B, based on the utilities specified in (3), with $v^L = v^F$ for simplicity. The demand functions $\tilde{\varphi}_L(.,.)$ and $\tilde{\varphi}_F(.,.)$ can be positive or negative. A positive value denotes attracting EUs presumably from an exclusive additional customer base beyond the common pool, and a negative value denotes losing some of the EUs in the common pool to an outside option. The size of the common pool may be different from the exclusive additional customer bases of the SPs; to account for this disparity, we multiply the fractional subscriptions from the common pool, n_L, n_F with a constant α.

Considering $\theta' = \alpha$, for analytical tractability:

$$\begin{aligned} \tilde{n}_L &= \alpha\big(n_L + \varphi_L(p_L, I_L)\big), \\ \tilde{n}_F &= \alpha\big(n_F + \varphi_F(p_F, I_F)\big), \end{aligned} \tag{10}$$

with $k = k'/\alpha$, $b = b'/\alpha$, and

$$\begin{aligned} \varphi_L(p_L, I_L) &= k - p_L + b(I_L - I_F), \\ \varphi_F(p_F, I_F) &= k - p_F + bI_F \end{aligned} \tag{11}$$

The formulation is the same as in Sections II-A, II-B, with $\tilde{n}_L, \tilde{n}_F$ replacing n_L, n_F in (1) and (2). We characterize the SPNE strategies in section III-A, and provide numerical results in Section III-B.

A. The SPNE outcome

For simplicity, we consider only interior SPNE strategies, that is, $0 < n_L^*, n_F^* < 1$. We define functions $f(I_L)$, $g(I_L)$, $\pi_L(I_F)$ and sets $\mathbb{L}_1$, $\mathbb{L}_2$ as follows:

$$\begin{aligned} &g(I_L) = \frac{b}{15}I_L + \frac{1}{15} - \frac{c}{3} + \frac{k}{3},\ f(I_L) = \frac{1}{5I_L} + \frac{b}{5}, \\ &\theta(y) = 2\alpha\big(\frac{b}{5}I_L + \frac{1}{5} + g(I_L) - f(I_L)y\big)^2 + sy^2 - \gamma I_L^2, \\ &\mathbb{L}_1 = \{s > 2\alpha f^2(I_L) + 2\alpha f(I_L)g(I_L)/I_L,\ g(I_L) \ge 0, \\ &\qquad \delta \le I_L < \frac{4}{b}\}, \end{aligned}$$

$$\begin{aligned} \mathbb{L}_2 = &\{0 \le I_L < \frac{4}{b}\} \cap \Big(\{g(I_L) \ge 0, \\ &2\alpha f^2(I_L) \le s \le 2\alpha f^2(I_L) + 2\alpha f(I_L)g(I_L)/I_L\} \\ &\cup \{2\alpha f^2(I_L) + 4\alpha f(I_L)g(I_L)/I_L \ge s,\ 2\alpha f^2(I_L) > s\}\Big). \end{aligned}$$

With $\delta < 4/b$, we prove in Appendix E:

Theorem 7. *The interior SPNE strategies are:*

(1) I_L^* *is characterized in*

$$I_L^* = \underset{I_L}{\operatorname{argmax}} \Big(\max_{I_L \in \mathbb{L}_1} \theta(\frac{-2\alpha f(I_L)g(I_L)}{2\alpha f^2(I_L) - s}),\ \max_{I_L \in \mathbb{L}_2} \theta(I_L) \Big)$$

(2) I_F^* *is characterized in*

$$I_F^* = \begin{cases} \dfrac{-2\alpha f(I_L)g(I_L)}{2\alpha f^2(I_L) - s} & \textit{if } I_L \in \mathbb{L}_1 \\ I_L & \textit{if } I_L \in \mathbb{L}_2 \end{cases}$$

(3) $p_L^* = \frac{1}{15} + \frac{2c}{3} + \frac{k}{3} + \frac{I_L^* - I_F^*}{5I_L^*} - \frac{b}{5}I_F^* + \frac{4b}{15}I_L^*$, $p_F^* = \frac{1}{15} + \frac{2c}{3} + \frac{k}{3} + \frac{I_F^*}{5I_L^*} + \frac{b}{15}I_L^* + \frac{b}{5}I_F^*$.

[2] The fraction may be replaced with actual number (of EUs) in this case, by altering scale factors in this expression and in those of the payoffs. Our results hold for both interpretations as we do not use $0 \le \tilde{n}_L, \tilde{n}_F \le 1$ in any derivation. We use $0 \le n_L, n_F \le 1$ though.

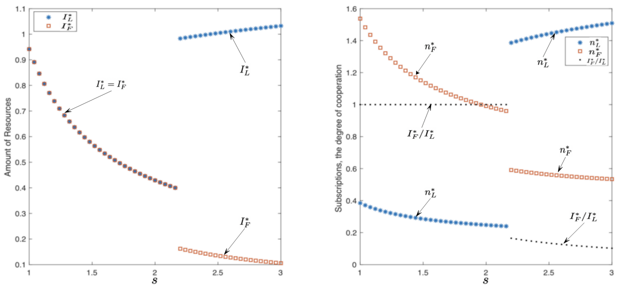

Fig. 5: Spectrum (left), degree of cooperation and subscriptions (right) vs. s Here, $\gamma = 0.8$, $c = k = 1$, $b = 2$.

(4) $\tilde{n}_L^* = \frac{I_L^* - I_F^*}{I_L^*} + p_F^* - 2p_L^* + k + bI_L^* - bI_F^*$, $\tilde{n}_F^* = \frac{I_F^*}{I_L^*} + p_L^* - 2p_F^* + k + bI_F^*$

Despite the expressions being cumbersome, the characterization is easy to compute, as in Theorem 1, and lead to important insights, as enumerated below.

$$\tilde{n}_L^* = \frac{3}{5}(1 - \frac{I_F^*}{I_L^*}) + \varphi_L(p_L, I_L) + \frac{2b}{5}I_F^* - \frac{b}{5}I_L^*$$
$$\tilde{n}_F^* = 1 - \frac{3}{5}(1 - \frac{I_F^*}{I_L^*}) + \varphi_F(p_F, I_F) - \frac{2b}{5}I_F^* + \frac{b}{5}I_L^*$$

In both equations, intuitively, the first term, $\frac{3}{5}(1 - \frac{I_F^*}{I_L^*}), 1 - \frac{3}{5}(1 - \frac{I_F^*}{I_L^*})$, represents the subscription from the common pool, if there had been no attrition to an outside option. The second and third terms represent the impacts of the attritions as also the additions from the exclusive customer bases. The first term depends on the degree of cooperation similar to the the base case specified in part (4) of Theorem 1. In the special case that $b = 0$, i.e., when the demand functions depend only on the access fees, the third term is 0 and the demand functions capture the impact of attrition and additions in the SPNE expression for the subscriptions. For $b > 0$, the second and the third term together become $k - p_L^* + \frac{b}{5}I_L^*(4 - 3I_F^*/I_L^*)$ in the expression for $\tilde{n}_L^*$, and $k - p_F^* + \frac{b}{5}I_L^*(1 + 3I_F^*/I_L^*)$ in that for $\tilde{n}_F^*$. Thus, higher degree of cooperation decreases (increases, respectively) the subscription for SP$_L$ (SP$_F$, respectively) even in these terms, and therefore, overall, like in the base case. Note that the subscriptions represent the efficacy in competition. However, as in the base case, the decrease in subscription does not directly lead to reduction in overall payoffs of SP$_L$, as the deficit may be compensated through income generated by leasing spectrum to SP$_F$.

B. Numerical results

Figure 5 show that now, both n_L^*, n_F^* can decrease (eg, with changes in s) because of attrition to the outside option possibly due to decrease of I_L^*, I_F^*. We note this when s is below a threshold. Otherwise, the trends resemble Figures 2 and 3 (the base case).

IV. The 3-player model

We now generalize our framework to consider competition between MNOs, rather than that only between an MNO and an MVNO. In a 3-player model, we consider two MNOs and one MVNO competing for a common pool of EUs in a covered market (i.e., each EU needs to opt for exactly one SP). We present the model in Section IV-A, and characterize the SPNE in Section IV-B. We show that the competition among multiple SPs reduces their payoffs, but benefits the EUs: the SPs acquire higher amounts of spectrum (hence provide higher service quality), and charge the EUs less. The competition also reduces the payoffs of SPs. We prove the results in Appendix C and Appendix F.

A. Model

We consider a symmetric model and seek a symmetric equilibrium i.e., the strategies of the MNOs are the same, and the MVNO leases the same amount of spectrum from each MNO. Thus, in the SPNE, $I_L = I_{L_1} = I_{L_2}$, $I_F = I_{F_1} = I_{F_2}$, $p_L = p_{L_1} = p_{L_2}$, and $n_L = n_{L_1} = n_{L_2}$. The total amount spectrum of SPs is $2I_L$. Thus, each MNO retains $I_L - I_F$ spectrum. We define the payoffs of MVNO and MNOs as

$$\pi_F = n_F(p_F - c) - 2sI_F^2 \quad (12)$$
$$\pi_L = n_L(p_L - c) + sI_F^2 - \gamma I_L^2 \quad (13)$$

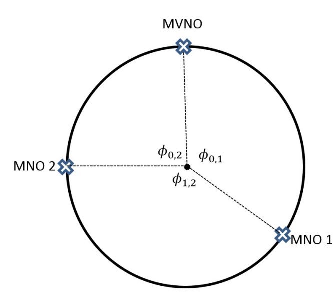

Fig. 6: The hoteling model for the three players case

To accommodate the three SPs, we modify the hotelling model. The EUs are uniformly distributed along a circle of radius 1 on which the SPs are virtually located (Figure 6). Since the radius is 1, each arc length equals the corresponding angle. Thus, the number of EUs located 1) between the MVNO and MNO$_i$ is $\phi_{0,i}$ and 2) between the MNOs is $\phi_{1,2}$.

We consider that $\phi_{0,1}$, $\phi_{0,2}$ and $\phi_{1,2}$ reflect the natural preferences of EUs for SPs (intuitively, for example, those in the arc $\phi_{0,1}$ would have stronger preference for the MVNO and MNO$_1$, and so on). We allow the preferences to depend on spectrum investments by defining these arcs as: $\phi_{0,1} = \phi_{0,2} = h_1(I_L, I_F)$ and $\phi_{1,2} = h_2(I_L, I_F)$ for some functions h_1 and h_2 (considering that the model is symmetric). We can now consider the transport cost as a parameter $t > 0$ rather than a function of I_L, I_F, unlike in Section II. We focus on the special case that $v^L = v^F = v$.

Similar to (3), if an EU is located in the arc of $\phi_{0,1}$, at a distance of x from the MVNO,

$$\begin{aligned} u_{MVNO} &= v - tx - p_F \\ u_{MNO_1} &= v - t(\phi_{0,1} - x) - p_L \\ u_{MNO_2} &= v - t \cdot \min(x + \phi_{0,2}, \phi_{0,1} - x + \phi_{1,2}) - p_L \end{aligned} \tag{14}$$

By calculation, if $x \leq \phi_{0,1}/2$, then $u_{MNO_1} \leq u_{MVNO}$, and $u_{MNO_2} = v - t(x + \phi_{0,2}) - p_L < u_{MVNO}$. Then, EUs choose MVNO. If $x > \phi_{0,1}/2$, then $u_{MVNO} < u_{MNO_1}$, and $u_{MNO_2} = v - t(\phi_{0,1} - x + \phi_{1,2}) - p_L < u_{MNO_1}$. Then, EUs choose MNO$_1$ instead of MNO$_2$.

Similarly, due to symmetry, if an EU is located in the arc of $\phi_{0,2}$, he does not choose MNO$_1$, and suppose the distance from the EU to the MVNO is x, thus

$$\begin{aligned} u_{MVNO} &= v - tx - p_F \\ u_{MNO_2} &= v - t(\phi_{0,2} - x) - p_L \end{aligned} \tag{15}$$

If an EU is located in the arc of $\phi_{1,2}$, at a distance of x to the MNO$_1$, then his utility is;

$$\begin{aligned} u_{MNO_1} &= v - tx - p_L, \\ u_{MNO_2} &= v - t(\phi_{1,2} - x) - p_L \\ u_{MVNO} &= v - t \cdot \min(x + \phi_{0,1}, \phi_{1,2} - x + \phi_{0,2}) - p_F \end{aligned} \tag{16}$$

Now we have the following lemma,

Lemma 2. *If $p_L - p_F \geq t\phi_{0,1}$, then all EUs choose the MVNO; if $p_L - p_F < t\phi_{0,1}$, then EUs located in the arc of $\phi_{1,2}$ do not choose the MVNO.*

Henceforth, we only consider $p_L - p_F < t\phi_{0,1}$, as:

Theorem 8. *No SPNE strategy exists if $p_L - p_F \geq t\phi_{0,1}$.*

Now, from Lemma 2 and the discussion above, the MVNO and MNO$_i$ (MNO$_1$ and MNO$_2$, respectively) compete to attract the EUs located only on the arc of $\phi_{0,i}$ ($\phi_{1,2}$, respectively). Thus, we define the number of EUs of any two SPs depends only on their total investment levels, i.e., for a constant ζ,

$$\begin{aligned} \phi_{01} = \phi_{02} &= \zeta \frac{2I_F + I_L - I_F}{2I_L} = \zeta \frac{I_F + I_L}{2I_L}, \\ \phi_{12} &= \zeta \frac{2(I_L - I_F)}{I_L} = \zeta \frac{I_L - I_F}{I_L}. \end{aligned}$$

B. The SPNE outcome

With $\delta < \frac{\pi}{2}\sqrt{\frac{t}{3s}}$, we prove in Appendix C:

Theorem 9. *The unique symmetric SPNE strategy, with I_L^*, p_L^* representing the choices of, and n_L^* subscription to, each MNO, and I_F^*, p_F^*, n_F^* the corresponding quantities for the MVNO, is:*

$$I_L^* = I_F^* = \frac{\pi}{2}\sqrt{\frac{t}{3s}},\ p_L^* = p_F^* = t\pi + c,\ n_F^* = 2n_L^* = \pi.$$

Remark 4. *The MVNO leases the entire new spectrum from each MNO. The degree of cooperation, I_F^*/I_L^* is 1. The characterization of the SPNE is easy to compute.*

We compare the outcome of the 3-player model with the 2-player model, to understand the impact of the competition between the MNOs. To ensure consistency of comparison, we modify the 2-player model of the base case in Section II as follows: (1) The transport cost is t instead of $t_L = I_F/I_L$ and $t_F = 1 - t_L$. (2) EUs are distributed uniformly along the interval $[0, 2\pi]$ instead of $[0, 1]$, since in the 3-player model, the total amount of EUs is 2π (3) $v^L = v^F = v$. By the same analysis method in Section II, we prove in Appendix F:

Corollary 1. *In the 2-player game formulation, the unique SPNE strategies are:*

$$I_L^* = \delta,\ I_F^* = 0,\ p_L^* = p_F^* = 2t\pi + c,\ n_F^* = n_L^* = \pi.$$

Comparing Theorem 9 and Corollary 1, we note that due to the competition by an additional MNO, SPs acquire higher amounts of spectrum in the 3-player model, i.e., the two MNOs order additional spectrum, and the MVNO leases the entire new spectrum from each MNO. The SPs charge the EUs less too: $t\pi + c$, as opposed to $2t\pi + c$ in the 2-player model. In both models, the MNO(s) and the MVNO divide the EUs equally: in the 2-player model, each SP has half of the EUs (π), while in the 3-player model, the MVNO has half of the EUs (π), and each MNO has a quarter of the EUs ($\pi/2$).

Finally, from (12) and (13), for 3 players, the payoffs are: (1) $\frac{5t\pi^2}{6}$ for each MNO, and (2) $\frac{t\pi^2}{12}(7 - \frac{\gamma}{s})$ for the MVNO. For 2 players, the payoffs are $2t\pi^2 - \delta^2$ and $2t\pi^2$ for the MNO and the MVNO respectively. Thus, the SPs earn more in the 2-player model, since fewer SPs compete for the same number of EUs.

V. Future Research

Future research includes generalization to accommodate 1) arbitrary, potentially non-convex, spectrum reservation fee functions that the SP$_F$ pay the SP$_L$ and the SP$_L$ pay the regulator 2) arbitrary transport cost t_L, t_F functions of the spectrum acquired by the SPs, I_L, I_F 3) arbitrary number of MNOs and MVNOs.

Appendix A
On quadratic function maximization

Lemma 3. *Define a quadratic function $f(x) = ax^2 + bx + c$ with $a \neq 0$. The maximum of $f(x)$ in an interval $[d, e] (d < e)$ can be obtained by the following rules:*

(1) *If $a > 0$, and define the midpoint of the interval $M = \frac{d+e}{2}$, then $f_{\max}(x) = f(d)$ if $M < -\frac{b}{2a}$; $f_{\max}(x) = f(e)$ if $M \geq -\frac{b}{2a}$.*
(2) *If $a < 0$, i.e., $f(x)$ is concave, then $f_{\max}(x) = f(d)$ if $d \geq -\frac{b}{2a}$; $f_{\max}(x) = f(e)$ if $e \leq -\frac{b}{2a}$; $f_{\max}(x) = f(-\frac{b}{2a})$ if $d < -\frac{b}{2a} < e$.*

Proof. **(1)**. Since $a > 0$, then $f(x)$ is convex, thus the maximum point can only be obtained at the boundary points, i.e., $x = d$ or $x = e$. Thus,

$$f(d) - f(e) = (a(d + e) + b)(d - e). \tag{17}$$

Let $M < -\frac{b}{2a}$. Since $a > 0$, $M < -\frac{b}{2a} \Leftrightarrow \frac{d+e}{2} < -\frac{b}{2a} \Leftrightarrow (d + e)a + b < 0$. Note $d - e < 0$, from (17), $f(d) - f(e) = (a(d+e)+b)(d-e) > 0$, which implies $f_{\max}(x) = f(d)$. Similarly, if $M \geq -\frac{b}{2a}$, note $a > 0$, then $M \geq -\frac{b}{2a} \Leftrightarrow \frac{d+e}{2} \geq -\frac{b}{2a} \Leftrightarrow (d + e)a + b \geq 0$. Since $d - e < 0$, then from (17), $f(d) - f(e) = (a(d + e) + b)(d - e) \leq 0$, which implies $f_{\max}(x) = f(e)$.

(2). If $a < 0$, then $f(x)$ is concave. Since $f'(x) = 2ax + b$, then 1) $f'(x) < 0$ and $f(x)$ is decreasing if $x > -\frac{b}{2a}$, 2) $f'(x) \geq 0$ and $f(x)$ is increasing if $x \leq -\frac{b}{2a}$. **(i)** If $d \geq -\frac{b}{2a}$, then $f(x)$ is decreasing if $x \in [d, e]$, hence $f_{\max}(x) = f(d)$. **(ii)** If $e \leq -\frac{b}{2a}$, then $f(x)$ is increasing if $x \in [d, e]$, hence $f_{\max}(x) = f(e)$. **(iii)** Let $d < -\frac{b}{2a} < e$. Since $f(x)$ is concave, thus $f(x)$ has a unique maximum point (stationary point) $x = -\frac{b}{2a}$, i.e., $f(-\frac{b}{2a}) \geq f(x)$ for all $x \in \mathbb{R}$. If $[d, f]$ contains $-\frac{b}{2a}$, i.e., $d \leq -\frac{b}{2a} \leq f$, then $f(-\frac{b}{2a}) \geq f(x)$ for all $x \in [d, f]$, hence $f_{\max}(x) = f(-\frac{b}{2a})$. ☐

Appendix B
Proofs in the Base case when $v^L = v^F$

Proof of Theorem 3 when $v^L = v^F$.

Proof. Let $(p_L^*, p_F^*, I_L^*, I_F^*)$ be a corner SPNE strategy. Thus, 1) $x_0 \geq 1$, or 2) $x_0 \leq 0$. We arrive at a contradiction for 1) **Step 1** and 2) in **Step 2** respectively.

Lemma 4. $\pi_F^* \geq 0$. *If* $n_F^* > 0$, $p_F^* \geq c$.

Proof. Let $\pi_F^* < 0$. Consider a unilateral deviation in which $I_F = 0, p_F \geq c$. From (12), $\pi_F \geq 0$, leading to a contradiction. Now, let $n_F^* > 0$ and $p_F^* < c$. Thus, $\pi_F^* < 0$ which is a contradiction. ☐

Step 1. Let $x_0^* \geq 1$. Clearly, $n_F^* = 0$ and $n_L^* = 1$. From (2), $\pi_F^* = -sI_F^{*2}$. From Lemma 4, $I_F^* = 0$. Thus, $\pi_F^* = 0, t_F^* = 1$. From (4), $1 \leq x_0^* = t_F^* + p_F^* - p_L^* = 1 + p_F^* - p_L^*$. Thus, $p_F^* \geq p_L^*$.

From (1), $\pi_L^* = p_L^* - c - \gamma I_L^{*2}$. If $p_L^* < c$, then $\pi_L^* < -\gamma\delta^2 < 0$ since $I_L^* \geq \delta$. Consider a unilateral deviation by which $I_L = \delta, p_L = c$, then $\pi_L = -\gamma\delta^2$, which is beneficial for SP$_L$. Thus, $p_L^* \geq c$.

Now, let $p_L^* > c$. Thus, $p_F^* \geq p_L^* > c$. Recall that $x_0^* = 1 + p_F^* - p_L^*$. Consider a unilateral deviation by which $p_F = p_L^* - \epsilon > c$. Now, by (4), $x_0 < 1$, and hence $n_F > 0$. Now, from (2), $\pi_F > 0 = \pi_F^*$. Thus, (I_F^*, p_F^*) is not SP$_F$'s best response to SP$_L$'s choices (I_L^*, p_L^*), which is a contradiction. Hence, $p_L^* = c$.

Now consider another unilateral deviation of SP$_L$, $p_L' = p_F^* + \epsilon$, where $0 < \epsilon < 1$, with all the rest the same. Since $p_L^* \leq p_F^*$, $p_L' > p_L^* = c$.

$$n_L' = x_0' = t_F^* + p_F^* - p_L' = 1 - \epsilon.$$

Then

$$\pi_L' - \pi_L^* = n_L'(p_L' - c) - (p_L^* - c) = (1 - \epsilon)(p_L' - c) > 0.$$

The last inequality follows because $p_L' > c$ and $\epsilon < 1$. Thus, we again arrive at a contradiction.

Step 2. Let $x_0^* \leq 0$. Clearly, $n_F^* = 1, n_L^* = 0$. Since $n_F^* > 0$, by Lemma 4, $p_F^* \geq c$. From (4), $x_0^* = t_F^* + p_F^* - p_L^* \leq 0$. Thus, $p_L^* \geq p_F^* + t_F^*$. Now, from (1),

$$\pi_L^* = sI_F^{*2} - \gamma I_L^{*2}. \tag{18}$$

Consider a unilateral deviation by SP$_L$, by which $p_L' = t_F^* + p_F^* - \epsilon$, $0 < \epsilon < 1$. Then

$$n_L' = x_0' = t_F^* + p_F^* - p_L' = \epsilon > 0$$

Therefore, by (64),

$$\pi_L' - \pi_L^* = n_L'(p_L' - c) = \epsilon(p_F^* - \epsilon + t_F^* - c)$$

Since $p_F^* \geq c$, either $p_F^* = c$ or $p_F^* > c$. If $p_F^* > c$, then let $\epsilon < p_F^* - c$. Then, $\pi_L' - \pi_L^* > 0$. If $p_F^* = c$, then $I_F^* = 0$ (otherwise $\pi_F^* < 0$, which by Lemma 4 implies that p_F^* is not a NE), then $t_F^* = 1$. Thus, $\pi_L' - \pi_L^* > 0$. We again arrive at a contradiction. ☐

By Theorem 3 proved above henceforth we only consider interior SPNE in which $0 < x_0^* < 1$.

Proof of Theorem 5 when $v^L = v^F$.

Substituting p_F and p_L from (7) into (6), using $t_L = I_F/I_L$ and $t_F = 1 - t_L$, SP$_F$'s payoff becomes,

$$\pi_F(I_F; I_L) = (\frac{1}{9I_L^2} - s)I_F^2 + \frac{2}{9I_L}I_F + \frac{1}{9} \tag{19}$$

Thus, the following maximization yields I_F^*:

$$\begin{aligned} \max\ \pi_F(I_F; I_L) = (\frac{1}{9I_L^2} - s)I_F^2 + \frac{2}{9I_L}I_F + \frac{1}{9} \\ s.t\ 0 \leq I_F \leq I_L. \end{aligned} \tag{20}$$

(A). If $I_L = \frac{1}{\sqrt{9s}}$, i.e., $\frac{1}{9I_L^2} - s = 0$, $\pi_F(I_F; I_L)$ is increasing in I_F. Thus, $I_F^* = I_L$.

(B). Let $I_L \neq \frac{1}{\sqrt{9s}}$. Referring to the terminology of Lemma 3, $-b/2a = \frac{I_L}{9I_L^2 s - 1}$, which we denote as F_1.

(B-1). Let $I_L < \frac{1}{\sqrt{9s}}$, i.e., $1 - 9I_L^2 s > 0$. Then π_F is a convex function. Note that $I_F \in [0, I_L]$, and the midpoint of the interval is $I_L/2$. From Lemma 3, since $1 - 9I_L^2 s > 0$, then $F_1 < 0 < I_L/2$, $\Rightarrow$ the maximum is obtained at $I_F^* = I_L$.

(B-2). Let $I_L > \frac{1}{\sqrt{9s}}$, i.e., $1 - 9I_L^2 s < 0$. Then π_F is a concave function. Note that $F_1 = \frac{I_L}{9I_L^2 s - 1} > 0$. From Lemma 3, $0 < F_1 < I_L \Leftrightarrow \sqrt{\frac{2}{9s}} < I_L$ and $F_1 \geq I_L \Leftrightarrow \frac{1}{\sqrt{9s}} < I_L \leq \sqrt{\frac{2}{9s}}$, thus

$$I_F^* = \begin{cases} F_1 & \text{if} \quad \sqrt{\dfrac{2}{9s}} < I_L \\ I_L & \text{if} \quad \dfrac{1}{\sqrt{9s}} < I_L \leq \sqrt{\dfrac{2}{9s}} \end{cases}.$$

Combining **(A)** and **(B)**, we obtain (8). □

Proof of Theorem 32.

Proof. Substituting p_L and p_F from (7) into π_L from (6), using $t_L = I_F/I_L$ and $t_F = \frac{I_L - I_F}{I_L}$, SP$_L$'s payoff becomes:

$$\pi_L(I_L; I_F^*) = (\frac{2}{3} - \frac{I_F^*}{3I_L})^2 + s(I_F)^{*2} - \gamma I_L^2. \tag{21}$$

Now, the following optimization yields I_L^*:

$$\begin{aligned} \max_{I_L} \quad & \pi_L(I_L; I_F^*) = (\frac{2}{3} - \frac{I_F^*}{3I_L})^2 + s(I_F^*)^2 - \gamma I_L^2 \\ s.t \quad & I_L \geq \delta. \end{aligned}$$

(A). From (8), if $\delta \leq I_L \leq \sqrt{\frac{2}{9s}}$, then $I_F^* = I_L$, thus for I_L in this range, the objective function of the optimization is $\frac{1}{9} + (s - \gamma)I_L^2$. This is an increasing function of I_L, since $s > \gamma$. Thus the optimum solution for $I_L \in [\delta, \sqrt{\frac{2}{9s}}]$ is $\sqrt{\frac{2}{9s}}$.

(B). Next, if $I_L > \sqrt{\frac{2}{9s}}$, then $I_F^* = \frac{I_L}{9I_L^2 s - 1}$. Since $I_L = \frac{I_L}{(9I_L^2 s - 1)}$ when $I_L = \sqrt{\frac{2}{9s}}$, then I_F^* is continuous at $I_L = \sqrt{\frac{2}{9s}}$. So $\pi_L(I_L; I_F^*) \to \pi_L|_{I_L = I_F^* = \sqrt{\frac{2}{9s}}}$ as $I_L \downarrow \sqrt{\frac{2}{9s}}$. Therefore, this case also includes the optimum solution of previous case. Thus substituting $I_F^* = \frac{I_L}{9I_L^2 s - 1}$ to (68), (9) is obtained. □

Appendix C
The proofs in the 3-player model

Proof of Lemma 2.

Proof. First, let $p_L - p_F \geq t\phi_{0,1}$. Consider EUs in the arc of $\phi_{1,2}$. Consider an EU at distance x from MNO$_1$. From the symmetry of MNO$_1$ and MNO$_2$, 1) if $x \leq \frac{\phi_{1,2}}{2}$, $u_{MNO_1} \geq u_{MNO_2}$, and 2) if $x > \frac{\phi_{1,2}}{2}$, $u_{MNO_2} \geq u_{MNO_1}$. Since $p_L - p_F \geq t\phi_{0,1}$, 1) if $x < \frac{\phi_{1,2}}{2}$, then $u_{MNO_1} = v - tx - p_L < v - t(x + \phi_{0,1}) - p_F = u_{MVNO}$, and 2) if $x > \frac{\phi_{1,2}}{2}$, then $u_{MNO_2} = v - tx - p_L < v - t(x + \phi_{0,1}) - p_F = u_{MVNO}$. Thus, all the EUs in arc $\phi_{1,2}$ will choose the MVNO.

Note that $\phi_{0,1} = \phi_{0,2}$. Now consider the EUs in arc $\phi_{0,1}$ ($\phi_{0,2}$), at a distance of x from MNO$_1$ (MNO$_2$, respectively). From (14) and (15), $u_{MNO_i} - u_{MVNO} = t\phi_{0,i} - p_L + p_F - 2tx < 0$ since $p_L - p_F \geq t\phi_{0,1}, x > 0$. Thus all these EUs opt for the MVNO.

Let $p_L - p_F < t\phi_{0,1}$. One can similarly show that the EUs in arc $\phi_{1,2}$ choose either MNO$_1$ or MNO$_2$. □

Proof of Theorem 8.

Proof. Since $I_L^* \geq \delta > 0$, $\phi_{0,1}^* = \phi_{0,2}^* > 0$. From Lemma 2, $n_F^* = 2\pi$, and $n_L^* = 0$. Thus,

$$\pi_F^* = 2\pi(p_F^* - c) - 2s(I_F^*)^2, \pi_L^* = sI_F^{*2} - \gamma I_L^{*2}.$$

Let $p_F^* < c$, then $\pi_F^* < 0$. Consider a unilateral deviation of the MVNO, by which $p_F = c, I_F = 0$. Thus, $\pi_F = 0$, and the unilateral deviation is profitable, which is a contradiction. Thus, $p_F^* = c$.

Thus, since $\phi_{0,1}^* > 0$, and from the condition of the theorem, $p_L^* \geq p_F^* + t\phi_{0,1}^* > c$. Consider a unilateral deviation of MNO$_1$, by which $p_L' = p_F^* + t\phi_{0,1} - \epsilon > c$, with $\epsilon > 0$. Now consider the utilities of the EUs in arc $\phi_{0,1}$, at a distance of x from MNO$_1$. From (14),

$$u_{MNO_1}' - u_{MVNO} = t\phi_{0,1}^* - p_L' + p_F^* - 2tx = \epsilon - 2tx.$$

So for $x \in (0, \epsilon/2t)$, $u_{MNO_1} > u_{MVNO}$. Thus $n'_{MNO_1} > 0$.

Since I_F^* and I_L^* are the same as before, then $\pi'_{MNO_1} = n'_{MNO_1}(p'_L - c) + sI_F^{*2} - \gamma I_L^{*2}$. Thus,

$$\pi'_{MNO_1} - \pi^*_{MNO_1} = n'_{MNO_1}(p'_L - c) > 0.$$

The last inequality follows since $p'_L > c$ and $n'_{MNO_1} > 0$. Thus, the unilateral deviation is profitable which leads to a contradiction. □

Proof of Theorem 9.

Due to Theorem 8, we consider that $p_L - p_F < t\phi_{0,1}$ henceforth. We sequentially progress from Stage 4 to Stage 1.

Stage 4: First, we determine the constant ζ.

Lemma 5. *$\zeta = \pi$, and $\phi_{0,1} = \phi_{0,2} = \pi \frac{I_F + I_L}{2I_L}$, $\phi_{1,2} = \pi \frac{I_L - I_F}{I_L}$.*

Proof. $\phi_{01} + \phi_{02} + \phi_{12} = 2\pi$, then $\zeta = \pi$. The rest follows from the definition of ϕ_{01}, ϕ_{02}, and ϕ_{12}. □

By symmetry, we only consider the split of the EUs between the MNO$_1$ and the MVNO.

Theorem 10.

$$n_{MVNO} = \begin{cases} 0 & x_0 \le 0 \\ \pi \dfrac{I_F + I_L}{2I_L} + \dfrac{p_L - p_F}{t} & 0 < x_0 < \phi_{0,1} \\ \pi \dfrac{I_L + I_F}{I_L} & x_0 \ge \phi_{0,1} \end{cases} \tag{22}$$

$$n_{MNO_1} = \begin{cases} \pi & x_0 \le 0 \\ \pi \dfrac{3I_L - I_F}{4I_L} + \dfrac{p_F - p_L}{2t} & 0 < x_0 < \phi_{0,1} \\ \pi \dfrac{I_L - I_F}{2I_L} & x_0 \ge \phi_{0,1} \end{cases} \tag{23}$$

where $x_0 = \frac{\phi_{0,1}}{2} + \frac{p_L - p_F}{2t}$.

Proof. Suppose x_0 is the indifferent location of joining MVNO and MNO1, then:

$$\begin{aligned} & v - tx_0 - p_F = v - t(\phi_{0,1} - x_0) - p_L \\ \Rightarrow & x_0 = \frac{\phi_{0,1}}{2} + \frac{p_L - p_F}{2t}. \end{aligned} \tag{24}$$

Let $x_{MVNO,MNO2}, x_{MNO1,MNO2}$ be the indifferent locations between 1) MVNO and MNO$_2$, and 2) MNO$_1$ and MNO$_2$ respectively. Then, $x_{MVNO,MNO2} = \frac{\phi_{0,2}}{2} + \frac{p_L - p_F}{2t}$, and $x_{MNO1,MNO2} = \frac{\phi_{1,2}}{2}$. Assuming the number of EUs per unit length to be normalized to one, n_{MVNO} equals $x_0 + x_{MVNO,MNO_2}$ if $0 < x_0 < \phi_{0,1}$, 0 if $x_0 \le 0$, and $\phi_{0,1} + \phi_{0,2}$ if $x_0 \ge \phi_{0,1}$. From the symmetry of the game, $x_{MVNO,MNO_2} = x_0$. Now, (22) follows from Lemma 5.

Next, n_{MNO_1} and n_{MNO_2} equal $(\phi_{0,1} - x_0) + x_{MNO_1,MNO_2}$ if $0 < x_0 < \phi_{0,1}$, $\phi_{0,1} + x_{MNO_1,MNO_2}$ if $x_0 \le 0$, and x_{MNO_1,MNO_2} if $x_0 \ge \phi_{0,1}$. Similarly, (23) follows. □

Stage 3: Now we characterize the SPNE access fees.

Theorem 11. *The SPNE access fees of EUs of SPs, (p_F^*, p_L^*) by which $0 < x_0 < \phi_{0,1}$, is:*

$$p_F^* = \frac{t\pi}{3}\frac{I_F + 5I_L}{2I_L} + c,\ p_L^* = \frac{t\pi}{3}\frac{7I_L - I_F}{2I_L} + c. \tag{25}$$

Proof. Substituting (22) and (23) into (12) and (13),

$$\pi_F = (\pi \frac{I_F + I_L}{2I_L} + \frac{p_L - p_F}{t})(p_F - c) - 2sI_F^2 \tag{26}$$

$$\pi_L = (\pi \frac{3I_L - I_F}{4I_L} + \frac{p_F - p_L}{2t})(p_L - c) + sI_F^2 - \gamma I_L^2 \tag{27}$$

p_F^* and p_L^* should be determined to satisfy the first order condition, i.e., $\frac{\pi_F}{dp_F}|_{p_F^*} = 0$ and $\frac{\pi_L}{dp_L}|_{p_L^*} = 0$, thus $p_F^* = \frac{t\pi}{3}\frac{I_F + 5I_L}{2I_L} + c$, $p_L^* = \frac{t\pi}{3}\frac{7I_L - I_F}{2I_L} + c$. Therefore, p_F^* and p_L^* are the unique interior SPNE strategies if 1) they yield $0 < x_0 < \phi_{0,1}$ and $p_L - p_F \le t\phi_{0,1}$, and 2) no unilateral deviation is profitable for SPs. We establish these in Parts A and B respectively.

Part A. Substituting p_L^* and p_F^* into (24), $x_0 = \frac{\phi_{0,1}}{2} + \frac{p_L - p_F}{2t} = \pi(\frac{5}{12} + \frac{I_F}{12I_L}) \in (0, \phi_{0,1})$, since $0 \le I_F \le I_L$ $I_L > 0$. Also, $p_L - p_F = \frac{t\pi}{3}\frac{I_L - I_F}{I_L} < \frac{t\pi}{2}\frac{I_L + I_F}{I_L} = t\phi_{0,1}$.

Part B. Since $\frac{d^2\pi_F}{d(p_F^*)^2} = -\frac{2}{t} < 0$, $\frac{d^2\pi_L}{d(p_L^*)^2} = -\frac{1}{t} < 0$, then p_L^* and p_F^* are the unique maximal solutions of π_L and π_F, respectively for $0 < x_0 < \phi_{0,1}$. Similar to the proof of Theorem 4, any deviation by SPs such that $x_0 \le 0$ or $x_0 \ge \phi_{0,1}$ (which yields $n_L = 1, n_F = 0$ and $n_L = 0, n_F = 1$, respectively) is not profitable. □

Stage 2: We characterize the spectrum SP$_F$ acquires from SP$_L$ in the SPNE.

Theorem 12. *I_F^* is given by:*

$$I_F^* = \begin{cases} \dfrac{5t\pi^2 I_L}{72I_L^2 s - t\pi^2} & \text{if} \quad I_L \ge \dfrac{\pi}{2}\sqrt{\dfrac{t}{3s}} \\ I_L & \text{if} \quad \delta \le I_L < \dfrac{\pi}{2}\sqrt{\dfrac{t}{3s}} \end{cases} \tag{28}$$

Proof. I_F^* is obtained as the optimum solution of

$$\begin{aligned} \max_{I_F} \pi_F = & (\frac{t\pi^2}{36I_L^2} - 2s)I_F^2 + \frac{5t\pi^2}{18I_L}I_F + \frac{25t\pi^2}{36} \\ s.t \quad & 0 \le I_F \le I_L \end{aligned} \tag{29}$$

The objective function follows from substituting (25) into (26). The constraints come from the model assumptions directly.

(A). Let $I_L = \frac{\pi}{6}\sqrt{\frac{t}{2s}}$. Then π_F is increasing in I_F, as $\pi_F = \frac{5t\pi^2}{18I_L}I_F + \frac{25t\pi^2}{36}$. Thus $I_F^* = I_L$.

(B). Let $I_L \neq \frac{\pi}{6}\sqrt{\frac{t}{2s}}$.Referring to the terminology of Lemma 3, $(-b/2a) = -\frac{\frac{5t\pi^2}{18I_L}}{2(\frac{t\pi^2}{36I_L^2}-2s)} = \frac{5t\pi^2 I_L}{72I_L^2 s - t\pi^2}$. We denote this quantity as F_1.

(B-1). Let $I_L < \frac{\pi}{6}\sqrt{\frac{t}{2s}}$. Then π_F is convex. $I_F \in [0, I_L]$. Since $\frac{t\pi^2}{36I_L^2} - 2s > 0$, then $72sI_L^2 - t\pi^2 < 0$, thus $F_1 < 0 < \frac{I_L}{2}$. From Lemma 3, $I_F^* = I_L$.

(B-2). Let $I_L > \frac{\pi}{6}\sqrt{\frac{t}{2s}}$, i.e., $\frac{t\pi^2}{36I_L^2} - 2s < 0$, then π_F is concave, and $F_1 = \frac{5t\pi^2 I_L}{72I_L^2 s - t\pi^2} > 0$. From Lemma 3,

$$I_F^* = \begin{cases} \dfrac{5t\pi^2 I_L}{72I_L^2 s - t\pi^2} & \text{if} \quad I_L \geq \dfrac{\pi}{2}\sqrt{\dfrac{t}{3s}} \\ I_L & \text{if} \quad \dfrac{\pi}{6}\sqrt{\dfrac{t}{2s}} < I_L < \dfrac{\pi}{2}\sqrt{\dfrac{t}{3s}} \end{cases}$$

The desired results come from **(A)**, **(B)** and **(C)**. □

Stage 1: We characterize the spectrum SP$_L$ acquires from the regulator in the SPNE.

Theorem 13. *Any solution to the following maximization problem constitutes* I_L^*.

$$\max_{I_L} \pi_L = \frac{t\pi^2}{18}\left(\frac{7I_L - \frac{5t\pi^2 I_L}{72I_L^2 s - t\pi^2}}{2I_L}\right)^2 + s\left(\frac{5t\pi^2 I_L}{72I_L^2 s - t\pi^2}\right)^2 - \gamma I_L^2$$
$$s.t\; I_L \geq \frac{\pi}{2}\sqrt{\frac{t}{3s}}. \tag{30}$$

Proof. Each MNO chooses its I_L as the solution of the following maximization:

$$\max_{I_L} \quad \pi_L(I_L; I_F^*) = \frac{t\pi^2}{18}\left(\frac{7I_L - I_F^*}{2I_L}\right)^2 + sI_F^{*2} - \gamma I_L^2$$
$$s.t \quad I_L \geq \delta. \tag{31}$$

The objective function follows by substituting (25) into (27). The constraint follows from the modeling assumption. Next, we consider two cases separately: A) $\delta \leq I_L \leq \frac{\pi}{2}\sqrt{\frac{t}{3s}}$ and B) $I_L > \frac{\pi}{2}\sqrt{\frac{t}{3s}}$.

(A). From (28), if $\delta \leq I_L \leq \frac{\pi}{2}\sqrt{\frac{t}{3s}}$, then $I_F^* = I_L$, thus the objective function of (31) is $\frac{t\pi^2}{2} + (s-\gamma)I_L^2$. This is an increasing function of I_L since $s > \gamma$. Thus the optimum solution in this range is $\frac{\pi}{2}\sqrt{\frac{t}{3s}}$.

(B). Next, if $I_L > \frac{\pi}{2}\sqrt{\frac{t}{3s}}$, then $I_F^* = \frac{5t\pi^2 I_L}{72I_L^2 s - t\pi^2}$, thus $\pi_L(I_L, I_F^*) = \pi_L(I_L, \frac{5t\pi^2 I_L}{72I_L^2 s - t\pi^2})$. Note that $I_L = \frac{5t\pi^2 I_L}{72I_L^2 s - t\pi^2}$ when $I_L = \frac{\pi}{2}\sqrt{\frac{t}{3s}}$, then I_F^* is continuous at $I_L = \frac{\pi}{2}\sqrt{\frac{t}{3s}}$. So $\pi_L(I_L; I_F^*) \to \pi_L|_{I_F^* = \frac{\pi}{2}\sqrt{\frac{t}{3s}}}$ as $I_L \to \frac{\pi}{2}\sqrt{\frac{t}{3s}}$. Therefore, this case also includes the optimum solution of previous case. Substituting $I_F^* = \frac{5t\pi^2 I_L}{72I_L^2 s - t\pi^2}$ into (31), we get (30). □

Theorem 14. $I_L^* = I_F^* = \frac{\pi}{2}\sqrt{\frac{t}{3s}}$.

Proof. From (30), we have $\pi_L(I_L) = \frac{t\pi^2}{18}(\frac{7I_L - \frac{5t\pi^2 I_L}{72I_L^2 s - t\pi^2}}{2I_L})^2 + s(\frac{5t\pi^2 I_L}{72I_L^2 s - t\pi^2})^2 - \gamma I_L^2 \triangleq f_1(I_L) + f_2(I_L) + f_3(I_L)$, where $f_1(I_L) = \frac{t\pi^2}{18}(\frac{7}{2} - \frac{5t\pi^2}{144I_L^2 s - 2t\pi^2})^2$, $f_2(I_L) = s(\frac{5t\pi^2 I_L}{72I_L^2 s - t\pi^2})^2$, and $f_3(I_L) = -\gamma I_L^2$. Now we take the derivatives of f_1, f_2, and f_3 with respect to I_L, $\frac{d\pi_L}{dI_L} = f_1'(I_L) + f_2'(I_L) + f_3'(I_L) = \frac{10t^2\pi^4 sI_L^2}{(72I_L^2 s - t\pi^2)^3} \times 19 \cdot (t\pi^2 - 144I_L^2 s) - 2\gamma I_L$. Since $I_L \geq \frac{\pi}{2}\sqrt{\frac{t}{3s}}$, then $t\pi^2 \leq 12I_L^2 s$, thus $72I_L^2 s - t\pi^2 \geq 0$ and $t\pi^2 - 144I_L^2 s \leq 0$, which implies $\frac{df_1}{dI_L} + \frac{df_2}{dI_L} \leq 0$. $\frac{df_3}{dI_L} = -2\gamma I_L < 0$, therefore $\frac{d\pi_L}{dI_L} < 0$ so π_L is a decreasing functions of I_L, so $I_L^* = \frac{\pi}{2}\sqrt{\frac{t}{3s}}$. In addition, $\pi_L^* = \frac{t\pi^2}{2} + (s-\gamma)I_L^* > 0$, and $I_F^* = \frac{5t\pi^2 I_L^*}{72I_L^{*2} s - t\pi^2} = \frac{\pi}{2}\sqrt{\frac{t}{3s}} = I_L^*$. □

Theorem 9 follows from Theorems 10, 11, 14.

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Supplementary Proofs

APPENDIX D
SPNE ANALYSIS OF BASIC CASE

If SP_L invests in the minimum new spectrum, i.e., $I_L = \delta$, and set $p_L = c$, then

$$\pi_L = sI_F^2 - \gamma\delta^2.$$

Thus for any Nash equilibrium (NE) strategy (I_L^*, p_L^*), we have

$$\pi_L^*|_{p_L^*, I_L^*} \geq -\gamma\delta^2.$$

If SP_F leases no new spectrum from SP_L, then $\pi_F = 0$. So for any NE strategy (I_F^*, p_F^*), we have

$$\pi_F^*|_{p_F^*, I_F^*} \geq 0.$$

Stage 4: We first characterize the equilibrium division of EUs between SPs, i.e., n_L^* and n_F^*, using the knowledge of the strategies chosen by the SPs in Stages 1∼3.

Theorem 15. *The indifferent location between the two service providers is*

$$x_0 = v^L - v^F + t_F + p_F - p_L. \tag{32}$$

Proof. From Definition 2,

$$\begin{aligned} u_F(x_0) &= v^F - t_F(1 - x_0) - p_F \\ &= v^L - t_L x_0 - p_L = u_L(x_0). \end{aligned}$$

Note $t_L + t_F = 1$, then

$$\begin{aligned} x_0 =& \frac{v^L - v^F + t_F + p_F - p_L}{t_L + t_F} \\ =& v^L - v^F + t_F + p_F - p_L. \end{aligned}$$

□

The fraction of EUs with each SP (n_L and n_F) is:

$$\begin{aligned} n_L &= \begin{cases} 0, & \text{if} \quad x_0 \leq 0 \\ x_0, & \text{if} \quad 0 < x_0 < 1 \\ 1, & \text{if} \quad x_0 \geq 1 \end{cases} \\ n_F &= 1 - n_L, \end{aligned} \tag{33}$$

where x_0 is defined in (32).

A. The interior SPNE

In this section, we consider the interior SPNE, and the corner SPNE are considered in Appendix D-B.

Stage 3: SP_L and SP_F determine their prices for EUs, p_L and p_F, respectively, to maximize their payoffs.

Lemma 6. *The utility functions of SPs are*

$$\begin{aligned} \pi_L =& (v^L - v^F + t_F + p_F - p_L)(p_L - c) \\ &+ sI_F^2 - \gamma I_L^2 \\ \pi_F =& (v^F - v^L + t_L + p_L - p_F)(p_F - c) \\ &- sI_F^2. \end{aligned} \tag{34}$$

Proof. From (33), substituting $(n_L, n_F) = (v^L - v^F + t_F + p_F - p_L, 1 - n_L)$ into (1) and (2), we get (34). □

In the following theorem, we characterize the SPNE access fees of SPs.

Theorem 16. *The interior SPNE access fees p_L^*, p_F^* are*

$$\begin{aligned} p_L^* =& c + \frac{1}{3} + \frac{I_L - I_F}{3I_L} + \frac{v^L - v^F}{3} \\ p_F^* =& c + \frac{1}{3} + \frac{I_F}{3I_L} + \frac{v^F - v^L}{3}, \end{aligned} \tag{35}$$

and (p_L^*, p_F^*) are unique if and only if

$$v^L - v^F - 1 < \frac{I_F}{I_L} < v^L - v^F + 2. \tag{36}$$

Proof. We complete the proof in two steps: we first obtain interior equilibrium access fees (p_L^*, p_F^*) (**Step 1**); then we get the condition (36) and prove that p_L^* and $p_F^*)$ are the unique interior Nash equilibrium access fees of SP_L and SP_F, respectively (**Step 2**).

Step 1. Consider an interior SPNE, every Nash equilibrium (p_L^*, p_F^*) should satisfy the first order condition. Get π_F and π_L from (34), then p_L^* and p_F^* should be solved by

$$\frac{d\pi_L}{dp_L}|_{p_L^*} = 0, \ \frac{d\pi_F}{dp_F}|_{p_F^*} = 0.$$

Note that $t_L + t_F = 1$, then

$$\begin{aligned} p_L^* =& c + \frac{1}{3} + \frac{I_L - I_F}{3I_L} + \frac{v^L - v^F}{3} \\ p_F^* =& c + \frac{1}{3} + \frac{I_F}{3I_L} + \frac{v^F - v^L}{3}. \end{aligned}$$

Step 2. In this step, we prove that the p_F^* and p_L^* are the unique maximum solutions (in (**A**)). Then, we prove that the condition (36) is sufficient and necessary (in (**B**)). Finally, we show that p_F^* and p_L^* are Nash equilibrium by proving that no unilateral is profitable for SPs (in (**C**)).

(**A**). Taking the second derivative of π_L (π_F) with respect to p_L^* (p_F^*),

$$\frac{d^2\pi_L}{d(p_L^*)^2} = \frac{d^2\pi_F}{d(p_F^*)^2} = -2 < 0,$$

then p_L^* and p_F^* are the unique maximal solutions of π_L and π_F, respectively.

(B). Substituting (35) into (33), we have

$$\begin{aligned}x_0 =&\frac{v^L - v^F}{3} + \frac{2I_L - I_F}{3I_L}\\ =&\frac{v^L - v^F}{3} + \frac{2}{3} - \frac{I_F}{3I_L},\end{aligned}$$

thus

$$\begin{aligned}&0 < x_0 = \frac{v^L - v^F}{3} + \frac{2}{3} - \frac{I_F}{3I_L} < 1\\ \Leftrightarrow & v^L - v^F - 1 < \frac{I_F}{I_L} < v^L - v^F + 2.\end{aligned} \tag{37}$$

From (37), $0 < x_0 < 1$ if and only if (36) holds. Therefore if (36) does not hold, then $x_0 \le 0$ or $x_0 \ge 1$, which implies $n_L = 0, n_F = 1$ or $n_L = 1, n_F = 0$.

(C). Since $\frac{d^2\pi_F}{dp_F^2} < 0, \frac{d^2\pi_L}{dp_L^2} < 0$, a local maxima is also a global maximum, and any solution to the first order conditions maximize the payoffs when $0 < x_0 < 1$, and no unilateral deviation by which $0 < x_0 < 1$ would be profitable for the SPs. Now, we show that unilateral deviations of the SPs leading to $n_L = 0, n_F = 1$ and $n_L = 1, n_F = 0$ is not profitable. Note that the payoffs of the SPs, (1) and (2), are continuous as $n_L \downarrow 0$, and $n_L \uparrow 1$ (which subsequently yields $n_F \uparrow 1$ and $n_F \downarrow 0$, respectively). Thus, the payoffs of both SPs when selecting p_L and p_F as the solutions of the first order conditions are greater than or equal to the payoffs when $n_L = 0$ and $n_L = 1$. Thus, the unilateral deviations under consideration are not profitable for the SPs.

□

Based on the results in Theorem 16, we can obtain the payoffs of SPs as follows,

Lemma 7. *The payoff of SP_F is*

$$\begin{aligned}&\pi_F(I_F; I_L) = (\frac{1}{9I_L^2} - s)I_F^2\\ &+\frac{2(v^F - v^L + 1)}{9I_L}I_F + \frac{(v^F - v^L + 1)^2}{9};\end{aligned} \tag{38}$$

Proof. First, we consider interior equilibrium strategies, from (34) in Theorem 16 , we have

$$\pi_F = (v^F - v^L + t_L + p_L - p_F)(p_F - c) - sI_F^2.$$

Note that $t_L = I_F/I_L$ and $t_F = 1 - t_L$.

(i). Calculate $v^F - v^L + t_L + p_L - p_F$. Substituting p_F and p_L in (35) into $v^F - v^L + t_L + p_L - p_F$, we have

$$\begin{aligned}&v^F - v^L + t_L + p_L - p_F\\ =&v^F - v^L + t_L + \frac{I_L - I_F}{3I_L}\\ &+\frac{v^L - v^F}{3} - \frac{I_F}{3I_L} - \frac{v^F - v^L}{3}\\ =&\frac{v^F - v^L + 1}{3} + \frac{I_F}{3I_L}.\end{aligned}$$

(ii). Calculate $p_F - c$. Substituting p_F in (35) into $p_F - c$, we have

$$\begin{aligned}p_F - c =&c + \frac{1}{3} + \frac{I_F}{3I_L} + \frac{v^F - v^L}{3} - c\\ =&\frac{v^F - v^L + 1}{3} + \frac{I_F}{3I_L}.\end{aligned}$$

From **(i)** and **(ii)**, we can obtain (38).

□

Lemma 8. *The payoff of SP_L is*

$$\begin{aligned}&\pi_L(I_L; I_F) = (\frac{v^L - v^F + 2}{3} - \frac{I_F}{3I_L})^2\\ &+ s(I_F)^2 - \gamma I_L^2;\end{aligned} \tag{39}$$

Proof. First, we consider interior equilibrium strategies, from (34), we have

$$\begin{aligned}\pi_L(I_L) =&(v^L - v^F + t_F + p_F - p_L)(p_L - c)\\ &+sI_F^2 - \gamma I_L^2.\end{aligned}$$

(i). Calculate $v^L - v^F + t_F + p_F - p_L$. Note that $t_L = I_F/I_L$ and $t_F = \frac{I_L - I_F}{I_L}$. From (35), then

$$\begin{aligned}&v^L - v^F + t_F + p_F - p_L\\ =&v^L - v^F + t_F + (c + \frac{1}{3} + \frac{I_F}{3I_L} + \frac{v^F - v^L}{3})\\ &-(c + \frac{1}{3} + \frac{I_L - I_F}{3I_L} + \frac{v^L - v^F}{3})\\ =&\frac{v^L - v^F}{3} + t_F + \frac{2I_F - I_L}{3I_L}\\ =&\frac{v^L - v^F + 2}{3} - \frac{I_F}{3I_L}.\end{aligned}$$

(ii). Calculate $p_L - c$. From (35),

$$\begin{aligned}p_L - c =& c + \frac{1}{3} + \frac{I_L - I_F}{3I_L} + \frac{v^L - v^F}{3} - c\\ =&\frac{1}{3} + \frac{I_L - I_F}{3I_L} + \frac{v^L - v^F}{3}\\ =&\frac{v^L - v^F + 2}{3} - \frac{I_F}{3I_L}.\end{aligned}$$

From **(i)** and **(ii)**, we get (39).

□

Based on the proof of Theorem 16, the existence of equilibria are showed in the following statement:

Corollary 2. *No corner SPNE access fees exist if* $(I_F, I_L) \in R$*, where*

$$\begin{aligned} R =& \{I_L \geq \delta, 0 \leq I_F \leq I_L\} \\ & \cap \{v^L - v^F - 1 < I_F/I_L < v^L - v^F + 2\} \end{aligned} \tag{40}$$

Proof. From Theorem 16, if (36) holds, then no corner SPNE access fees (p_L^*, p_F^*) exist. Note that $I_L \geq \delta$ and $0 \leq I_F \leq I_L$, thus

$$R \stackrel{def}{=} \{I_L \geq \delta, 0 \leq I_F \leq I_L\} \cap \left\{v^L - v^F - 1 < \frac{I_F}{I_L} < v^L - v^F + 2\right\}.$$

□

In Stage 2 and Stage 1, we characterize the optimum investment levels I_L^* and I_F^* of SPs. To analyze easily, we consider 4 sections: $-1 < v^L - v^F < 1$ (**Section A**), $1 \leq v^L - v^F < 2$ (**Section B**), $-2 < v^L - v^F \leq -1$ (**Section C**), and $|v^L - v^F| \geq 2$ (**Section D**).

Section A: $-1 < v^L - v^F < 1$

In this section, we consider $-1 < v^L - v^F < 1$. First, we show that if an SPNE exists when $-1 < v^L - v^F < 1$, then it must be an interior SPNE (in Proposition 1). Then, we characterize the unique optimum I_F^* (in Theorem 17) and an optimum I_L^* (in Theorem 18), respectively. Finally, we collect the optimum strategies in **Stages 1**∼**4**, and prove that this strategiy $(p_L^*, p_F^*, I_L^*, I_F^*)$ is an interior Nash equilibrium strategy.

Proposition 1. *If an SPNE exists when* $-1 < v^L - v^F < 1$*, then it is an interior SPNE.*

Proof. From Corollary 2, no corner SPNE access fees exist if $(I_L, I_F) \in R$. Note that $-1 < v^L - v^F < 1$, then

$$v^L - v^F - 1 < 0 \leq \frac{I_F}{I_L} \leq 1 < v^L - v^F + 2.$$

Thus from (40),

$$R = \{I_L \geq \delta, 0 \leq I_F \leq I_L\}.$$

So (36) holds for any $I_L \geq \delta$ and $0 \leq I_F \leq I_L$ when $-1 < v^L - v^F < 1$. □

Stage 2: SP_F decides on the amount of spectrum to be leased from SP_L (I_F), with the condition that $0 \leq I_F \leq I_L$, to maximize π_F. Since we assume δ is small, then let $\delta < \min(\sqrt{\frac{v^F - v^L + 2}{9s}}, \frac{1}{\sqrt{9s}})$.

Theorem 17. *If* $-1 < v^L - v^F < 1$*, then the optimum investment level of* SP_F*,* I_F^**, is*

$$I_F^* = \begin{cases} \dfrac{(v^F - v^L + 1)I_L}{9I_L^2 s - 1} & I_L > \sqrt{\dfrac{v^F - v^L + 2}{9s}} \\ I_L & \delta \leq I_L \leq \sqrt{\dfrac{v^F - v^L + 2}{9s}} \end{cases}. \tag{41}$$

Proof. From (38) and Proposition 1, the optimal investment level of SP_F, I_F^*, is a solution of the following optimization problem,

$$\begin{aligned} \max \quad & \pi_F(I_F; I_L) = (\frac{1}{9I_L^2} - s)I_F^2 \\ & + \frac{2(v^F - v^L + 1)}{9I_L} I_F + \frac{(v^F - v^L + 1)^2}{9} \\ s.t \quad & 0 \leq I_F \leq I_L \end{aligned} \tag{42}$$

(A). If $I_L = \frac{1}{\sqrt{9s}}$, then $\pi_F(I_F; I_L)$ is a linear function of I_F, i.e.,

$$\pi_F(I_F; I_L) = \frac{2(v^F - v^L + 1)}{9I_L} I_F + \frac{(v^F - v^L + 1)^2}{9}.$$

Since $-1 < v^L - v^F < 1$, then

$$\frac{2(v^F - v^L + 1)}{9I_L} > 0,$$

$\pi_F(I_F; I_L)$ is an increasing function of I_F, so $I_F^* = I_L$.

(B). If $I_L \neq \frac{1}{\sqrt{9s}}$ and π_F is a quadratic function. We discuss the optimal solutions in two cases: (i) $\delta \leq I_L < \frac{1}{\sqrt{9s}}$, and (ii) $I_L > \frac{1}{\sqrt{9s}}$. We denote F_1 as

$$\frac{d\pi_F}{dI_F}|_{I_F = F_1} = 0 \Rightarrow F_1 = \frac{(v^F - v^L + 1)I_L}{9I_L^2 s - 1}. \tag{43}$$

(B-1). If $\delta \leq I_L < \frac{1}{\sqrt{9s}}$, then π_F is a convex function. Since $I_F \in [0, I_L]$, then the midpoint is $I_L/2$. Note that $-1 < v^L - v^F < 1$ and $1 - 9I_L^2 s > 0$, thus

$$F_1 = \frac{(v^F - v^L + 1)I_L}{9I_L^2 s - 1} < 0 < I_L/2.$$

From Lemma 3, the maximum is obtained at $I_F^* = I_L$.

(B-2). If $I_L > \frac{1}{\sqrt{9s}}$, then π_F is a concave function. Note that $-1 < v^L - v^F < 1$ and $1 - 9I_L^2 s < 0$, then

$$F_1 = \frac{(v^F - v^L + 1)I_L}{9I_L^2 s - 1} > 0.$$

From Lemma 3,

$$\begin{cases} I_F^* = F_1 & \text{if} \quad 0 < F_1 < I_L \\ I_F^* = I_L & \text{if} \quad F_1 \geq I_L \end{cases}.$$

Since

$$0 < F_1 < I_L \Leftrightarrow \sqrt{\frac{v^F - v^L + 2}{9s}} < I_L$$

$$F_1 \geq I_L \Leftrightarrow \frac{1}{\sqrt{9s}} < I_L \leq \sqrt{\frac{v^F - v^L + 2}{9s}},$$

thus

$$\begin{cases} I_F^* = F_1 & \text{if} \quad \sqrt{\frac{v^F - v^L + 2}{9s}} < I_L \\ I_F^* = I_L & \text{if} \quad \frac{1}{\sqrt{9s}} < I_L \leq \sqrt{\frac{v^F - v^L + 2}{9s}} \end{cases}.$$

From **(A)** and **(B)**, we obtain (41). Given v^L, v^F, s and I_L, I_F^* is the unique maximum of π_F, so no unilateral deviation is beneficial for SP$_F$. □

Stage 1: SP$_L$ decides on the amount of spectrum I_L to lease from the central regulator, to maximize her payoff, π_L.

Theorem 18. *If $-1 < v^F - v^L < 1$, then the optimal investment of SP$_L$, I_L^* is, a solution of the following optimization problem:*

$$\begin{aligned} \max_{I_L} \quad & \pi_L(I_L) = (\frac{2 + v^L - v^F}{3} - \frac{v^F - v^L + 1}{27sI_L^2 - 3})^2 \\ & + s(\frac{(v^F - v^L + 1)I_L}{9sI_L^2 - 1})^2 - \gamma I_L^2 \\ s.t \quad & I_L \geq \sqrt{\frac{v^F - v^L + 2}{9s}} \end{aligned} \tag{44}$$

Proof. Substituting I_F^* in (41) into (39), the optimal investment level of SP$_L$, I_L^*, is a solution of the following optimization problem,

$$\begin{aligned} \max_{I_L} \quad & \pi_L(I_L; I_F^*) = (\frac{2 + v^L - v^F}{3} - \frac{I_F^*}{3I_L})^2 \\ & + s(I_F^*)^2 - \gamma I_L^2 \\ s.t \quad & I_L \geq \delta \end{aligned} \tag{45}$$

(A). From (41), if $\delta \leq I_L \leq \sqrt{\frac{v^F - v^L + 2}{9s}}$, then $I_F^* = I_L$, thus (45) is equivalent to

$$\begin{aligned} \max_{I_L} \quad & \pi_L(I_L) = \frac{(1 + v^L - v^F)^2}{9} + (s - \gamma)I_L^2 \\ & \delta \leq I_L \leq \sqrt{\frac{v^F - v^L + 2}{9s}} \end{aligned}$$

Since $s > \gamma$, then $\pi_L(I_L)$ is an increasing function of I_L, thus $I_L^* = \sqrt{\frac{v^F - v^L + 2}{9s}}$. This case can be considered as part of the next part.

(B). Next, if $I_L > \sqrt{\frac{v^F - v^L + 2}{9s}}$, then $I_F^* = \frac{(v^F - v^L + 1)I_L}{9I_L^2 s - 1}$, thus

$$\pi_L(I_L, I_F^*) = \pi_L(I_L, \frac{(v^F - v^L + 1)I_L}{9I_L^2 s - 1}).$$

Note that $I_F^* = I_L$ when

$$I_L = \sqrt{\frac{v^F - v^L + 2}{9s}},$$

then I_F^* is continuous at $I_L = \sqrt{\frac{v^F - v^L + 2}{9s}}$. So

$$\pi_L(I_L; I_F^*) \to \pi_L|_{I_L = I_F^* = \sqrt{\frac{v^F - v^L + 2}{9s}}}$$

as

$$I_L \downarrow \sqrt{\frac{v^F - v^L + 2}{9s}}.$$

Therefore, this case also includes the optimum solution of previous case. Thus in this case (45) is equivalent to

$$\begin{aligned} \max_{I_L} \quad & \pi_L(I_L) = (\frac{2 + v^L - v^F}{3} - \frac{v^F - v^L + 1}{27sI_L^2 - 3})^2 \\ & + s(\frac{(v^F - v^L + 1)I_L}{9sI_L^2 - 1})^2 - \gamma I_L^2 \\ s.t \quad & I_L \geq \sqrt{\frac{v^F - v^L + 2}{9s}} \end{aligned}$$

Given v^L, v^F and s, I_L^* is a maximum of π_L, then no unilateral deviation is beneficial for SP$_L$.

□

Collect all interior equilibria of p_F^*, p_L^*, and I_F^*, I_L^*, we have

Corollary 3. *If $-1 < v^L - v^F < 1$, then the unique SPNE strategy is:*

(1) *I_L^* is characterized by*

$$\begin{aligned} \max_{I_L} \, & \pi_L(I_L) = (\frac{2 + v^L - v^F}{3} - \frac{v^F - v^L + 1}{27sI_L^2 - 3})^2 \\ & + s(\frac{(v^F - v^L + 1)I_L}{9sI_L^2 - 1})^2 - \gamma I_L^2 \\ s.t. \, & I_L \geq \sqrt{\frac{v^F - v^L + 2}{9s}} \end{aligned}$$

(2) *I_F^* is characterized in*

$$I_F^* = \begin{cases} \frac{(v^F - v^L + 1)I_L}{9I_L^2 s - 1} \text{ if } I_L > \sqrt{\frac{v^F - v^L + 2}{9s}} \\ I_L \text{ if } I_L = \sqrt{\frac{v^F - v^L + 2}{9s}} \end{cases}$$

(3) $p_L^* = c + \frac{2}{3} - \frac{I_F^*}{3I_L^*} + \frac{v^L - v^F}{3}$, $\quad p_F^* = c + \frac{1}{3} + \frac{I_F^*}{3I_L^*} + \frac{v^F - v^L}{3}$.

(4) $n_L^* = \frac{v^L - v^F}{3} + \frac{2}{3} - \frac{I_F^*}{3I_L^*}$, $n_F^* = \frac{I_F^*}{3I_L^*} + \frac{1}{3} + \frac{v^F - v^L}{3}$.

Section B: $1 \leq v^L - v^F < 2$

In this section, we consider $1 \leq v^L - v^F < 2$. First, give the conditions under which the interior SPNE may exist (Proposition 2). Then, We obtain an optimum I_F^* (in Theorem 19) and I_L^* (in Theorem 20), respectively. Finally, we find the SPNE I_F^* and I_L^*. Since we assume δ is small, so let

$$\delta < \min\left(\sqrt{\frac{2}{9s(v^L - v^F)} - \frac{1}{9s}}, \frac{1}{\sqrt{2s(v^L - v^F - 1)}}\right).$$

Proposition 2. *If $1 \leq v^L - v^F < 2$, then no corner SPNE strategies exist when*

$$(I_F, I_L) \in \left\{I_L \geq \delta, (v^L - v^F - 1)I_L < I_F \leq I_L\right\}.$$

Proof. From Corollary 2, no corner equilibrium strategies exist if $(I_L, I_F) \in R$. Since

$$0 \leq v^L - v^F - 1 < 1 \tag{46}$$

$$3 \leq v^F - v^L + 2 < 4, \tag{47}$$

then from (40), (46) and (47),

$$R = \left\{I_L \geq \delta, (v^L - v^F - 1)I_L < I_F \leq I_L\right\}.$$

$\square$

Stage 2: SP_F decides on the amount of spectrum to be leased from SP_L (I_F), with the condition that $0 \leq I_F \leq I_L$, to maximize π_F.

Theorem 19. *If $1 \leq v^L - v^F < 2$, then the optimum investment level of SP_F, I_F^* can be obtained by the following rules:*

(1) *if $v^L - v^F = 1$, then $I_F^* \in [0, \frac{1}{\sqrt{9s}}]$ when $I_L = \frac{1}{\sqrt{9s}}$, $I_F^* = I_L$ when $0 \leq I_L < \frac{1}{\sqrt{9s}}$;*

(2) *if $1 < v^L - v^F < 2$ and $\delta \leq I_L \leq \sqrt{\frac{2}{9s(v^L-v^F)} - \frac{1}{9s}}$, then $I_F^* = I_L$;*

(3) *if $1 < v^L - v^F < 2$ and $I_L > \sqrt{\frac{2}{9s(v^L-v^F)} - \frac{1}{9s}}$, then no interior equilibria I_F^* exist.*

Proof. From (38) and Proposition 2, I_F^* is obtained by the following optimization problems,

$$\begin{aligned} \max_{I_F} \quad & \pi_F(I_F; I_L) = \left(\frac{1}{9I_L^2} - s\right)I_F^2 + \frac{2(v^F - v^L + 1)}{9I_L} I_F \\ & + \frac{(v^F - v^L + 1)^2}{9} \\ s.t \quad & (v^L - v^F - 1)I_L < I_F \leq I_L \end{aligned} \tag{48}$$

(A). First, we consider $I_L = \frac{1}{\sqrt{9s}}$, then

$$\pi_F(I_F; I_L) = \frac{2(v^F - v^L + 1)}{9I_L} I_F + \frac{(v^F - v^L + 1)^2}{9}$$

is a linear function of I_F. Since $1 \leq v^L - v^F < 2$, then $\frac{2(v^F - v^L + 1)}{9I_L} \leq 0$. Now we have two sub-cases.

(A-1). If $1 < v^L - v^F < 2$, then $\pi_F(I_F; I_L)$ is a decreasing function of I_F, then

$$I_F^* \downarrow (v^L - v^F - 1)I_L,$$

which means

$$\pi_F(I_F^*; I_L) \to \pi_F((v^L - v^F - 1)I_L; I_L),$$

which means SP_F always wants to make a deviation to get a higher payoff by decreasing the investment level ($I_F \downarrow (v^L - v^F - 1)I_L$). There exists no optimum I_F^* in this case.

(A-2). If $v^L - v^F = 1$, then $\pi_F(I_F; I_L)$ is a constant, and I_F^* can be any number in the interval $(0, \frac{1}{\sqrt{9s}}]$ since $I_L = \frac{1}{\sqrt{9s}}$.

(B). Then, we consider $I_L \neq \frac{1}{\sqrt{9s}}$. π_F is a quadratic function.The symmetric axis

$$F_1 = \frac{(v^F - v^L + 1)I_L}{9I_L^2 s - 1}.$$

(B-1). If $\delta \leq I_L < \frac{1}{\sqrt{9s}}$, then π_F is a convex function. Since $I_L \in \left((v^L - v^F - 1)I_L, I_L\right]$, the midpoint of the interval is $(v^L - v^F)I_L/2$. Note that $1 \leq v^L - v^F < 2$ and $1 - 9sI_L^2 > 0$, then

$$F_1 = \frac{(v^F - v^L + 1)I_L}{9I_L^2 s - 1} \geq 0,$$

From Lemma 3,

$$\begin{cases} I_F^* \to (v^L - v^F - 1)I_L & \frac{v^L - v^F}{2} I_L < F_1 \\ I_F^* = I_L & \frac{v^L - v^F}{2} I_L \geq F_1 \end{cases}.$$

In addition,

$$\frac{v^L - v^F}{2} I_L < F_1 \Leftrightarrow \frac{1}{\sqrt{9s}} > I_L > \sqrt{\frac{2}{9s(v^L - v^F)} - \frac{1}{9s}}$$

$$\frac{v^L - v^F}{2} I_L \geq F_1 \Leftrightarrow \delta \leq I_L \leq \sqrt{\frac{2}{9s(v^L - v^F)} - \frac{1}{9s}}.$$

thus

(i) $I_F^* \downarrow (v^L - v^F - 1)I_L$ when $\frac{1}{\sqrt{9s}} > I_L > \sqrt{\frac{2}{9s(v^L-v^F)} - \frac{1}{9s}}$;

(ii) $I_F^* = I_L$ when $\delta \leq I_L \leq \sqrt{\frac{2}{9s(v^L-v^F)} - \frac{1}{9s}}$.

If $\frac{1}{\sqrt{9s}} > I_L > \sqrt{\frac{2}{9s(v^L-v^F)} - \frac{1}{9s}}$,

$$I_F^* \downarrow (v^L - v^F - 1)I_L,$$

which means

$$\pi_F(I_F^*; I_L) \to \pi_F((v^L - v^F - 1)I_L; I_L),$$

which means SP_F always wants to make a deviation to get a higher payoff by decreasing the investment level ($I_F \downarrow (v^L - v^F - 1)I_L$). There exists no optimum I_F^* in this case.

So the optimum investment level, I_F^*, is

$$I_F^* = I_L$$

when

$$\delta \leq I_L \leq \sqrt{\frac{2}{9s(v^L - v^F)} - \frac{1}{9s}}.$$

(B-2). If $I_L > \frac{1}{\sqrt{9s}}$, then π_F is a concave function. Since $1 \leq v^L - v^F < 2$ and $1 - 9I_L^2 s < 0$, then

$$F_1 = \frac{(v^F - v^L + 1)I_L}{9I_L^2 s - 1} \leq 0 \leq (v^L - v^F - 1)I_L.$$

From Lemma 3, we have

$$I_F^* \downarrow (v^L - v^F - 1)I_L,$$

which means

$$\pi_F(I_F; I_L) \to \pi_F((v^L - v^F - 1)I_L; I_L),$$

which means SP_F always wants to make a deviation to get a higher payoff by decreasing the investment level ($I_F \downarrow (v^L - v^F - 1)I_L$). There exists no optimum I_F^* in this case.

From **(A)** and **(B)**, we obtain the desired results. Given v^L, v^F, s and I_L, if I_F^* exists, then I_F^* is the unique maximum of π_F, so no unilateral deviation is beneficial for SP$_F$.

Stage 1: In this stage, MNO decides on the level of investment I_L with the condition that $I_L \geq \delta$, to maximize his payoff π_L.

Theorem 20. *If* $1 \leq v^L - v^F < 2$*, then the unique optimum investment level of SP*$_L$*,* I_L^**, is* $I_L^* = I_F^* = \frac{1}{\sqrt{9s}}$ *if* $v^L - v^F = 1$*, otherwise no interior SPNE* I_L^* *exists.*

Proof. Substituting I_F^* in Theorem 19 into (39), then the optimal investment level of SP$_L$, I_L^*, is a solution of the following optimization problem,

$$\begin{aligned} \max_{I_L} \quad & \pi_L(I_L; I_F^*) \\ & = (\frac{v^L - v^F + 2}{3} - \frac{I_F^*}{3I_L})^2 + s(I_F^*)^2 - \gamma I_L^2 \\ s.t. \quad & I_L \geq \delta \end{aligned} \tag{49}$$

We have two subcases $1 < v^L - v^F < 2$ (in **(A)**) and $v^L - v^F = 1$ (in **(B)**).

(A). Consider $1 < v^L - v^F < 2$, from Theorem 19 (2), if

$$\delta \leq I_L \leq \sqrt{\frac{2}{9s(v^L - v^F)} - \frac{1}{9s}},$$

then $I_F^* = I_L$, thus the optimization (49) is equivalent to

$$\begin{aligned} \max_{I_L} \quad & \pi_L(I_L) = \frac{(1 - v^F + v^L)^2}{9} + (s - \gamma)I_L^2 \\ s.t \quad & \delta \leq I_L \leq \sqrt{\frac{2}{9s(v^L - v^F)} - \frac{1}{9s}} \end{aligned}$$

Since $s > \gamma$, then $\pi_L(I_L) > 0$ for all $\delta \leq I_L \leq \sqrt{\frac{2}{9s(v^L - v^F)} - \frac{1}{9s}}$, and π_L is an increasing function of I_L, thus $I_L^* = \sqrt{\frac{2}{9s(v^L - v^F)} - \frac{1}{9s}}$.

(B). Consider $v^L - v^F = 1$, from Theorem 19 (1), if $0 \leq I_L < \frac{1}{\sqrt{9s}}$, (49) is equivalent to $\pi_L(I_L; I_F^*) = (s - \gamma)I_L^2$. If $I_L = \frac{1}{\sqrt{9s}}$, then (49) is equivalent to

$$\begin{aligned} \pi_L(I_L; I_F^*) &= \frac{1}{9}(1 - \sqrt{9s}I_F^*)^2 + s(I_F^*)^2 - \frac{\gamma}{9s} \\ &= 2sI_F^{*2} - 2\sqrt{\frac{s}{9}}I_F^* + \frac{1}{9}(1 - \frac{\gamma}{s}). \end{aligned}$$

Since $I_F^* \leq \frac{1}{\sqrt{9s}}$, then $\sqrt{9s}I_F^* \leq 1$, thus π_L is an increasing function of I_F^*, and $\pi_L(I_L; I_F^*) \leq \frac{s-\gamma}{9s}$.

If $I_F^* < \frac{1}{\sqrt{9s}}$, then

$$2sI_F^{*2} - 2\sqrt{\frac{s}{9}}I_F^* + \frac{1}{9}(1 - \frac{\gamma}{s}) < \lim_{I_L \to \frac{1}{\sqrt{9s}}} (s - \gamma)I_L^2,$$

therefore $I_L \uparrow \frac{1}{\sqrt{9s}}$, which implies SP$_L$ always wants to make a deviation, then no SPNE exists.

If $I_F^* = \frac{1}{\sqrt{9s}}$, then $\pi_L(I_L; I_F^*) = \frac{s-\gamma}{9s}$, i.e., $I_L^* = I_F^* = \frac{1}{\sqrt{9s}}$. Thus SPNE exists only when $I_F^* = I_L^* = \frac{1}{\sqrt{9s}}$.

If $v^L - v^F = 1$, then

$$\sqrt{\frac{2}{9s(v^L - v^F)} - \frac{1}{9s}} = \frac{1}{\sqrt{9s}},$$

thus this case can be considered as part of the above part. Therefore $I_L^* = \sqrt{\frac{2}{9s(v^L - v^F)} - \frac{1}{9s}}$ for any $1 \leq v^L - v^F < 2$.

(C). Now we compute π_F, from Theorem 19, denote $t = v^L - v^F$, then

$$\begin{aligned} \pi_F &= n_F(p_F - c) - sI_F^* = (\frac{2-t}{3})^2 - \frac{2}{9t} + \frac{1}{9} \\ &= \frac{1}{9}(t^2 - 4t - \frac{2}{t} + 5) \triangleq f(t) \end{aligned}$$

Taking the derivative with respect to t,

$$\begin{aligned} f'(t) &= \frac{1}{9}(2t - 4 + \frac{2}{t^2}) = \frac{2}{9t^2}(t^3 - 2t^2 + 1) \\ &= \frac{2}{9t^2}(t-1)(t^2 - t - 1) \end{aligned}$$

Therefore, $f'(t) > 0$ when $t \in [\frac{1+\sqrt{5}}{2}, 2)$, and $f'(t) \le 0$ when $t \in [1, \frac{1+\sqrt{5}}{2})$. Thus, $f_{\max}(t) = f(1) = 0$, which implies the possible interior equilibrium is $t = 1$, i.e.,

$$I_F^* = I_L^* = \sqrt{\frac{1}{9s}}.$$

Then, we can calculate

$$p_L^* = c + \frac{2}{3}, \quad p_F^* = c + \frac{1}{3}$$
$$n_L^* = \frac{2}{3}, \quad n_F^* = \frac{1}{3}.$$

It is easy to check that if $v^L - v^F = 1$, then $(I_L^*, I_F^*, p_L^*, p_F^*, n_L^*, n_F^*)$ satisfies Corollary 3.

□

Corollary 4. *If $v^L - v^F = 1$, then the unique SPNE strategy is:*

(1) $I_L^* = \sqrt{\frac{1}{9s}}$.
(2) $I_F^* = \sqrt{\frac{1}{9s}}$.
(3) $p_L^* = c + \frac{2}{3}, \quad p_F^* = c + \frac{1}{3}$.
(4) $n_L^* = \frac{2}{3}, \quad n_F^* = \frac{1}{3}$.

It is easy to check that if $v^L - v^F = 1$, then $(I_L^*, I_F^*, p_L^*, p_F^*, n_L^*, n_F^*)$ satisfies Corollary 3.

Section C: $-2 < v^L - v^F \le -1$

In this section, we consider $-2 < v^L - v^F \le -1$. First, give the conditions under which the interior SPNE may exist (Proposition 3). Then, we prove that no interior SPNE exists (Theorem 22). Since we assume δ is small, then let $\delta < \frac{1}{\sqrt{9s}}$.

Proposition 3. *If $-2 < v^L - v^F \le -1$, then no corner SPNE exist when $I_L \ge \delta$ and $0 \le I_F < (v^L - v^F + 2)I_L$.*

Proof. From Corollary 2, no negative-corner and positive-corner equilibria exist if $(I_L, I_F) \in R$. If $-2 < v^L - v^F \le -1$, then

$$\begin{aligned} 0 <& v^L - v^F + 2 \le 1 \\ -3 <& v^L - v^F - 1 \le -2. \end{aligned}$$

Thus from (40),

$$R = \left\{ I_L \ge \delta, 0 \le I_F < (v^L - v^F + 2)I_L \right\}.$$

□

Stage 2: SP_F decides on the amount of spectrum to be leased from SP_L (I_F), with the condition that $0 \le I_F \le I_L$, to maximize π_F.

Theorem 21. *If $-2 < v^L - v^F \le -1$ and $\pi_F(I_F^*; I_L) \ge 0$, the optimum investment level of SP_F, I_F^*, is obtained by the following rules.*

(1) $I_F^* = \frac{(v^F - v^L + 1)I_L}{9I_L^2 s - 1}$ *when* $I_L > \frac{1}{\sqrt{3s(v^L - v^F + 2)}}$,
(2) *no interior SPNE* I_F^* *exist when* $\delta \le I_L \le \frac{1}{\sqrt{3s(v^L - v^F + 2)}}$.

Proof. From (38), the optimal investment level of SP_F, I_F^*, is the solution of the following optimization problem,

$$\begin{aligned} \max \quad & \pi_F(I_F; I_L) = (\frac{1}{9I_L^2} - s)I_F^2 + \frac{2(v^F - v^L + 1)}{9I_L} I_F \\ & + \frac{(v^F - v^L + 1)^2}{9} \\ s.t \quad & 0 \le I_F < (v^L - v^F + 2)I_L \end{aligned}$$

(A). If $I_L = \frac{1}{\sqrt{9s}}$, then

$$\pi_F(I_F; I_L) = \frac{2(v^F - v^L + 1)}{9I_L} I_F + \frac{(v^F - v^L + 1)^2}{9}$$

is a linear function of I_F. Since $-2 < v^L - v^F \le -1$, then $\frac{2(v^F - v^L + 1)}{9I_L} > 0$, thus $\pi_F(I_F; I_L)$ is an increasing function of I_F. Therefore the optimum investment I_F^*,

$$I_F^* \uparrow (v^L - v^F + 2)I_L,$$

which implies

$$\pi_F(I_F; I_L) \to \pi_F((v^L - v^F + 2)I_L; I_L),$$

which means SP_F always wants to make a deviation to get a higher payoff by increasing the investment level ($I_F \uparrow (v^L - v^F + 2)I_L$). no interior equilibria I_F^* exists in this case.

(B). If $I_L \ne \frac{1}{\sqrt{9s}}$, then π_F is a quadratic function, and the symmetric axis $F_1 = \frac{(v^F - v^L + 1)I_L}{9I_L^2 s - 1}$.

(B-1). If $\delta \le I_L < \frac{1}{\sqrt{9s}}$, then π_F is a convex function. Since $I_F \in [0, (v^L - v^F + 2)I_L)$, the midpoint of the interval is $(v^L - v^F + 2)I_L/2$. Since $-2 < v^L - v^F \le -1$ and $1 - 9I_L^2 s > 0$, then

$$F_1 = \frac{(v^F - v^L + 1)I_L}{9I_L^2 s - 1} < 0 < (v^L - v^F + 2)I_L/2,$$

from Lemma 3 (1),

$$I_F^* \uparrow (v^L - v^F + 2)I_L,$$

which means

$$\pi_F(I_F; I_L) \to \pi_F((v^L - v^F + 2)I_L; I_L),$$

which means SP_F always wants to make a deviation to get a higher payoff by increasing the investment level. Therefore, no interior equilibria I_F^* exists in this case.

(B-2). If $I_L > \frac{1}{\sqrt{9s}}$, then π_F is a concave function. Since $-2 < v^L - v^F \leq -1$ and $1 - 9I_L^2 s < 0$, then

$$F_1 = \frac{(v^F - v^L + 1)I_L}{9I_L^2 s - 1} > 0.$$

From Lemma 3 (2),

- $I_F^* = F_1$ when $0 < F_1 < (v^L - v^F + 2)I_L$;
- $I_F^* \to (v^L - v^F + 1)I_L$ when $F_1 \geq (v^L - v^F + 2)I_L$.

Since

$$0 < F_1 < (v^L - v^F + 2)I_L \Leftrightarrow I_L > \frac{1}{\sqrt{3s(v^L - v^F + 2)}}$$
$$F_1 \geq (v^L - v^F + 2)I_L \Leftrightarrow \frac{1}{\sqrt{9s}} < I_L \leq \frac{1}{\sqrt{3s(v^L - v^F + 2)}},$$

then

- $I_F^* = \frac{(v^F - v^L + 1)I_L}{9I_L^2 s - 1}$ when $I_L > \frac{1}{\sqrt{3s(v^L - v^F + 2)}}$;
- $I_F^* \to (v^L - v^F + 2)I_L$ when $\frac{1}{\sqrt{9s}} < I_L \leq \frac{1}{\sqrt{3s(v^L - v^F + 2)}}$.

Note that

$$I_F^* \uparrow (v^L - v^F + 2)I_L,$$

which means

$$\pi_F(I_F; I_L) \to \pi_F((v^L - v^F + 2)I_L; I_L),$$

which means SP_F always wants to make a deviation to get a higher payoff by increasing the investment level ($I_F \uparrow (v^L - v^F +)I_L$). Thus

$$I_F^* = \frac{(v^F - v^L + 1)I_L}{9I_L^2 s - 1} \text{ when } I_L > \frac{1}{\sqrt{3s(v^L - v^F + 2)}}$$

Since I_F^* is the unique maximum of π_F, so no unilateral deviation is beneficial for SP$_F$. From **(A)** and **(B)**, we obtain the desired results.

□

Stage 1: In this stage, MNO decides on the level of investment I_L with the condition that $I_L \geq \delta$, to maximize his payoff π_L.

Theorem 22. *If $-2 < v^L - v^F \leq -1$, then no interior SPNE I_L^* exists.*

Proof. Substituting I_F^* in Theorem 21 into (39), the optimum investment level of SP$_L$, I_L^*, is a solution of the following optimization problem,

$$\begin{aligned}\max_{I_L} \quad \pi_L = &(\frac{2 + v^L - v^F}{3} - \frac{v^F - v^L + 1}{27I_L^2 s - 3})^2 \\ &+ s(\frac{3(v^F - v^L + 1)I_L}{9I_L^2 s - 1})^2 - \gamma I_L^2 \\ s.t \quad I_L > &\frac{1}{\sqrt{3s(v^L - v^F + 2)}}.\end{aligned}$$

Denote

$$\begin{aligned}f(I_L) =&(\frac{2 + v^L - v^F}{3} - \frac{v^F - v^L + 1}{27I_L^2 s - 3})^2 \\ &+ s(\frac{3(v^F - v^L + 1)I_L}{9I_L^2 s - 1})^2.\end{aligned}$$

Now we prove that $f(I_L)$ is a decreasing function of I_L. Denote

$$f_1(I_L) = (\frac{v^F - v^L + 1}{27I_L^2 s - 3} - \frac{2 + v^L - v^F}{3})^2$$

and

$$f_2(I_L) = s(\frac{3(v^F - v^L + 1)I_L}{9I_L^2 s - 1})^2,$$

then $f(I_L) = f_1(I_L) + f_2(I_L)$. In fact,

$$\begin{aligned}f_1'(I_L) =&2(\frac{v^F - v^L + 1}{27I_L^2 s - 3} - \frac{2 + v^L - v^F}{3}) \\ &\cdot \frac{-(v^F - v^L + 1)}{(27I_L^2 s - 3)^2} 54 I_L s \\ =&\frac{-4(v^F - v^L + 1)I_L s}{(9I_L^2 s - 1)^2} \\ &\cdot (\frac{v^F - v^L + 1}{9I_L^2 s - 1} - (2 + v^L - v^F)),\end{aligned}$$

and

$$\begin{aligned}f_2'(I_L) =&2s\frac{3(v^F - v^L + 1)I_L}{(9I_L^2 s - 1)^3}[3(v^F - v^L + 1)\cdot \\ &(9I_L^2 s - 1) - (3(v^F - v^L + 1)I_L)(18 I_L s)] \\ =&\frac{-18 I_L s(v^F - v^L + 1)^2(9I_L^2 s + 1)}{(9I_L^2 s - 1)^3}\end{aligned}$$

Therefore

$$\begin{aligned}f'(I_L) =&f_1'(I_L) + f_2'(I_L) \\ =&\frac{2(v^F - v^L + 1)I_L s}{(9I_L s - 1)^2}[\frac{-2(v^F - v^L + 1)}{9I_L^2 s - 1} \\ &+ 2(2 + v^L - v^F) \\ &- \frac{9(v^F - v^L + 1)(9I_L^2 s + 1)}{9I_L^2 s - 1}] \\ =&\frac{2(v^F - v^L + 1)I_L s}{(9I_L s - 1)^2}[\frac{-20(v^F - v^L + 1)}{9I_L^2 s - 1} \\ &+ 2(2 + v^L - v^F) - 9(v^F - v^L + 1)] \\ =&\frac{2(v^F - v^L + 1)I_L s}{(9I_L s - 1)^2}[\frac{-20(v^F - v^L + 1)}{9I_L^2 s - 1} \\ &+ 11(v^L - v^F) - 5]\end{aligned}$$

Note that $-2 < v^L - v^F \leq -1$, then $2 \leq 1 - v^L + v^F < 3$ and $0 < v^L - v^F + 2 \leq 1$. Since

$$I_L > \frac{1}{\sqrt{3s(v^L - v^F + 2)}},$$

then

$$9I_L^2 s - 1 > \frac{1 - v^L + v^F}{v^L - v^F + 2} > 0.$$

Thus

$$\frac{-20(v^F - v^L + 1)}{9I_L^2 s - 1} < 0, \quad 11(v^L - v^F) - 5 < 0,$$

so $f'(I_L) < 0$, and $f(I_L)$ is an decreasing function. Therefore $\pi_L(I_L) = f(I_L) - \gamma I_L^2$ is a decreasing function of I_L when $I_L > \frac{1}{\sqrt{3s(v^L - v^F + 2)}}$. Hence

$$I_L^* \downarrow \frac{1}{\sqrt{3s(v^L - v^F + 2)}},$$

which implies

$$\pi_L(I_L) \uparrow \pi_L(\frac{1}{\sqrt{3s(v^L - v^F + 2)}}),$$

which implies SP_L always wants to make a deviation to get a higher payoff by decreasing the investment level ($I_L \downarrow \frac{1}{\sqrt{3s(v^L - v^F + 2)}}$). Therefore there exists no interior equilibria I_L^* in this case. □

Section D: $|v^L - v^F| \geq 2$

Theorem 23. *If $|v^L - v^F| \geq 2$, then no interior Nash equilibrium strategies exist.*

Proof. We calculate R in Corollary 2: If $v^L - v^F \geq 2$, then

$$\frac{I_F}{I_L} > v^L - v^F - 1 \geq 1 \Rightarrow I_F > I_L,$$

which is contradicted by $0 \leq I_F \leq I_L$, thus $R = \varnothing$. Similarly, if $v^L - v^F \leq -2$, then

$$\frac{I_F}{I_L} < v^F - v^L + 2 \leq 0 \Rightarrow I_F < 0,$$

which is contradicted by $0 \leq I_F \leq I_L$, thus $R = \varnothing$. Therefore, (36) does not hold for any $I_L \geq \delta$ and $0 \leq I_F \leq I_L$ when $|v^L - v^F| \geq 2$.

Thus no interior SPNE access fees exist, hence no interior Nash equiliberium strategies exist. □

B. Corner SPNE

We assume δ is small, so let $\delta < \frac{1}{\sqrt{2s}}$ in this section.

Lemma 9. *Consider $x_0 \leq 0$, no corner SPNE strategies exist when $v^L - v^F > -1$.*

Proof. Let $x_0^* \leq 0$. Clearly, $n_F^* = 1$ and $n_L^* = 0$. From (32),

$$p_F^* - p_L^* - v^F + v^L + t_F^* \leq 0. \tag{50}$$

Step 1. We prove $p_F^* - p_L^* - v^F + v^L + t_F^* = 0$.

Assume not, suppose $p_F^* - p_L^* - v^F + v^L + t_F^* < 0$. Consider a unilateral deviation by which $p_F' = p_F^* + \epsilon$, such that $p_F' - p_L^* - v^F + v^L + t_F^* < 0$. From (32), $x_0' = 1$. Now, from (2), $\pi_F' - \pi_F^* = \epsilon > 0$. Thus, (I_F^*, p_F^*) is not SP_F's best response to SP_L's choices (I_L^*, p_L^*), which is a contradiction. Hence, $p_F^* - p_L^* + v^L - v^F + t_F^* = 0$.

Step 2. We prove $p_F^* \geq c$.

From (2), $\pi_F^* = p_F^* - c - sI_F^{*2}$. If $p_F^* < c$, then $\pi_F^* < -sI_F^{*2} < 0$. Consider a unilateral deviation by which $I_F = 0, p_F = c$, then $\pi_F = 0$, which is beneficial for SP_F. Thus, $p_F^* \geq c$.

Step 3. If $v^L - v^F > -1$, then $p_F^* < c + 1$.

If $v^L - v^F > -1$, then let $p_F^* \geq c + 1$. Consider a unilateral deviation by which $p_L = p_L^* - \epsilon$, then $x_0 = p_F^* - p_L - v^F + v^L + t_F^* = \epsilon$. In addition, $p_L = p_L^* - \epsilon = p_F^* + v^L - v^F + t_F^* \geq c + 1 + v^L - v^F$, thus

$$\pi_L - \pi_L^* \geq \epsilon(1 + v^L - v^F - \epsilon).$$

We can choose some $0 < \epsilon < 1$ such that $\pi_L - \pi_L^* > 0$. Hence, $p_F < c + 1$.

Now consider another unilateral deviation of SP_F, $p_F' = p_F^* + \epsilon$, where $0 < \epsilon < 1$, with all the rest the same, then

$$\begin{aligned} n_L' &= x_0' = t_F^* + v^L - v^F + p_F^* - p_L' = \epsilon \\ n_F' &= 1 - n_L' = 1 - \epsilon. \end{aligned}$$

Thus,

$$\begin{aligned} \pi_F' - \pi_F^* &= n_F'(p_F' - c) - (p_F^* - c) \\ &= -\epsilon(p_F^* - c) + (1 - \epsilon)\epsilon \\ &= \epsilon(-p_F^* + c + 1 - \epsilon) > 0. \end{aligned}$$

The last inequality follows because we can choose $0 < \epsilon < 1$ such that $p_F' = p_F^* + \epsilon < c + 1$. Thus, we arrive at a contradiction. □

Lemma 10. *Consider $x_0 \geq 1$, no corner SPNE strategies exist when $v^L - v^F < 1$.*

Proof. Let $x_0^* \geq 1$. Clearly, $n_F^* = 0$ and $n_L^* = 1$. From (32), $1 \leq x_0^* = v^L - v^F + t_F^* + p_F^* - p_L^*$. Thus,

$$p_F^* - p_L^* - v^F + v^L + t_F^* - 1 \geq 0. \tag{51}$$

Step 1. We prove $p_F^* - p_L^* - v^F + v^L = 0$.

Assume not, suppose $p_F^* - p_L^* - v^F + v^L > 0$. Consider a unilateral deviation by which $p'_L = p_L^* + \epsilon$, such that $p_F^* - p'_L - v^F + v^L > 0$. From (32), $x'_0 = 1$. Now, from (1), $\pi'_L - \pi_L^* = \epsilon > 0$. Thus, (I_L^*, p_L^*) is not SP$_L$'s best response to SP$_F$'s choices (I_F^*, p_F^*), which is a contradiction. Hence, $p_F^* - p_L^* + v^L - v^F = 0$.

Step 2. We prove $p_L^* \geq c$.

From (1), $\pi_L^* = p_L^* - c + sI_F^{*2} - \gamma I_L^{*2}$. If $p_L^* < c$, then $\pi_L^* < sI_F^{*2} - \gamma I_L^{*2} \leq sI_F^{*2} - \gamma\delta^2$. Consider a unilateral deviation by which $I_L = \delta, p_L = c$, then $\pi_L = sI_F^{*2} - \gamma\delta^2$, which is beneficial for SP$_L$. Thus, $p_L^* \geq c$.

Step 3. We prove $I_F^* = 0$ and $\pi_F^* = 0$.

For any SPNE (I_F^*, p_F^*), we have $\pi_F^* \geq 0$. Otherwise, assume $\pi_F < 0$, we consider a unilateral deviation $I_F = 0$ and $p_F = c$, then $\pi_F = 0$, which is beneficial for SP$_F$. If $n_F^* = 0$, then $\pi_F^* = -sI_F^{*2} \geq 0 \Rightarrow I_F^* = 0, \pi_F^* = 0$.

Based on Step 3, since $I_F^* = 0$, then

$$t_F^* = \frac{I_L^* - I_F^*}{I_L^*} = 1.$$

Step 4. If $v^L - v^F < 1$,then $p_L < c + 1$.

If $v^L - v^F < 1$, let $p_L^* \geq c + 1$. Thus,

$$p_F^* = p_L^* + v^F - v^L \geq c + v^F - v^L. \tag{52}$$

Recall that $x_0^* = 1 + v^L - v^F + p_F^* - p_L^*$, then consider a unilateral deviation by which $p_F = p_L^* + v^F - v^L - \epsilon > c + 1 + v^F - v^L$. Now, by (32), $x_0 < 1$, and hence $n_F > 0$. Now, from (2), $\pi_F > 0 = \pi_F^*$. Thus, (I_F^*, p_F^*) is not SP$_F$'s best response to SP$_L$'s choices (I_L^*, p_L^*), which is a contradiction. Hence, $p_L^* < c + 1$.

Now consider another unilateral deviation of SP$_L$, $p'_L = p_L^* + \epsilon$, where $0 < \epsilon < 1$, with all the rest the same, then

$$n'_L = x'_0 = t_F^* + v^L - v^F + p_F^* - p'_L = 1 - \epsilon.$$

Then

$$\begin{aligned}\pi'_L - \pi_L^* &= n'_L(p'_L - c) - (p_L^* - c)\\ &= -\epsilon(p_L^* - c) + (1 - \epsilon)\epsilon\\ &= \epsilon(-p_L^* + c + 1 - \epsilon).\end{aligned}$$

The last inequality follows because we can choose $0 < \epsilon < 1$ such that $p'_L = p_L^* + \epsilon < c + 1$. Thus, we arrive at a contradiction. □

Theorem 24. *If $v^L - v^F \leq \sqrt{\frac{\gamma}{s}} - 2$, then the unique corner SPNE strategy is:*

(1) $I_L^* = \frac{1}{\sqrt{2s}}$.
(2) $I_F^* = I_L^*$.
(3) $p_L^* = p_F^* - v^F + v^L - 1$, $c + 1 \leq p_F^* \leq c + v^F - v^L - 1$.
(4) $n_L^* = 0$, $n_F^* = 1$.

Proof. **Step 1.** We prove $p_F^* \leq c + v^F - v^L - t_F^*$.

Suppose $p_F^* > c + v^F - v^L - t_F^*$, then from Step 1 in Lemma 9, $p_L^* = p_F^* - v^F + v^L + t_F^* - t_F^* > c$. Now consider a unilateral deviation of SP$_L$, $p_L = p_L^* - \epsilon$, where $0 < \epsilon < 1$, with all the rest the same, then

$$n_L = x_0 = t_F^* + v^L - v^F + p_F^* - p_L = \epsilon.$$

Thus,

$$\pi_L - \pi_L^* = n_L(p_L - c) = \epsilon(p_L - c) > 0.$$

The last inequality follows because we can choose $0 < \epsilon < 1$ such that $p_L = p_L^* - \epsilon > c$. Thus, $p_F^* > c + v^F - v^L$ can not be an SPNE.

Step 2. We prove $p_F^* \geq c + 1$.

Suppose $p_F^* < c + 1$, consider a unilateral deviation of SP$_F$, $p_F = p_F^* + \epsilon$, where $0 < \epsilon < 1$, with all the rest the same, then

$$\begin{aligned}n_L &= x_0 = t_F^* + v^L - v^F + p_F^* - p_L = \epsilon\\ n_F &= 1 - n_L = 1 - \epsilon.\end{aligned}$$

Thus,

$$\begin{aligned}\pi_F - \pi_F^* &= n_F(p_F - c) - p_F^* + c\\ &= \epsilon(1 - \epsilon - p_F^* + c) > 0.\end{aligned}$$

The last inequality follows because we can choose $0 < \epsilon < 1$ such that $p_F^* + \epsilon < 1 + c$. Thus, $p_F^* < c + 1$ can not be an SPNE.

Therefore from Steps 1, 2, if $\frac{I_F^*}{I_L^*} < 2 + v^L - v^F$, then $c + 1 > c + v^F - v^L - t_F^*$, thus no corner SPNE exists. Now we consider $\frac{I_F^*}{I_L^*} \geq 2 + v^L - v^F$.

Step 3. We prove that no unilateral deviation is beneficial for both SPs when $c + 1 \leq p_F^* \leq c + v^F - v^L - t_F^*$.

Consider another unilateral deviation of SP$_L$, $p'_L = p_L^* - \epsilon$, where $0 < \epsilon < 1$, with all the rest the same, then

$$n'_L = x'_0 = t_F^* + v^L - v^F + p_F^* - p'_L = \epsilon$$

Since $p_L^* = p_F^* - v^F + v^L + t_F^*$, then $p_L^* \in [c + 1 + v^L - v^F + t_F^*, c]$, then

$$\pi'_L - \pi_L^* = n'_L(p'_L - c) < 0,$$

which implies no unilateral deviation is beneficial for SP$_L$.

Consider another unilateral deviation of SP$_F$, $p'_F = p^*_F + \epsilon$, where $0 < \epsilon < 1$, with all the rest the same, then

$$n'_L = x'_0 = t^*_F + v^L - v^F + p'_F - p^*_L = \epsilon$$
$$n'_F = 1 - n'_L = 1 - \epsilon.$$

Thus, note that $c + 1 \leq p^*_F \leq c + v^F - v^L - t^*_F$,

$$\pi'_F - \pi^*_F = n'_F(p'_F - c) - p^*_F + c$$
$$= \epsilon(-p^*_F + c + 1 - \epsilon) \leq -\epsilon^2 < 0.$$

which implies no unilateral deviation is beneficial for SP$_L$.

Step 4. Find I^*_F.

Since p^*_L is independent of I^*_F, then substituting $p^*_F = p^*_L - v^L + v^F - t^*_F$ into (2), thus the optimal investment level of SP$_F$, I^*_F, is the solution of the following optimization problem,

$$\begin{aligned} \max \quad & \pi_F(I_F; I_L) = -sI_F^2 + \frac{I_F}{I_L} + v^F - v^L + p^*_L - c - 1 \\ s.t \quad & (v^L - v^F + 2)I_L \leq I_F \leq I_L \\ & \pi_F(I_F; I_L) \geq 0 \end{aligned}$$

$\pi_F(I_F; I_L)$ is a concave function, and the symmetric axis is $F_2 = \frac{1}{2sI_L} > 0$. From Lemma 3 (2),

- $I^*_F = (v^L - v^F + 2)I_L$ when $F_2 \leq (v^L - v^F + 2)I_L$;
- $I^*_F = F_2$ when $(v^L - v^F + 2)I_L < F_2 < I_L$;
- $I^*_F = I_L$ when $F_2 \geq (v^L - v^F - 1)I_L$.

which is equivalent to

- $I^*_F = (v^L - v^F + 2)I_L$ when $I_L \geq \frac{1}{\sqrt{2s(v^L - v^F + 2)}}$;
- $I^*_F = \frac{1}{2sI_L}$ when $\frac{1}{\sqrt{2s}} < I_L < \frac{1}{\sqrt{2s(v^L - v^F + 2)}}$;
- $I^*_F = I_L$ when $I_L \leq \frac{1}{\sqrt{2s}}$.

Since I^*_F is the unique maximum of π_F, thus no unilateral deviation is beneficial to SP$_F$.

Step 5. Find I^*_L.

Substituting I^*_F from Step 4 into (1), the optimum investment level of SP$_L$, I^*_L, is a solution of the following optimization problem,

$$\begin{aligned} \max_{I_L} \quad & \pi_L(I_L; I^*_F) = sI^{*2}_F - \gamma I_L^2 \\ s.t \quad & I_L \geq \delta \\ & \pi_L(I_L; I^*_F) \geq 0 \end{aligned} \tag{53}$$

Then, we consider three cases $\frac{1}{\sqrt{2s(v^L - v^F - 1)}} \leq I_L$, $\frac{1}{\sqrt{2s}} < I_L < \frac{1}{\sqrt{2s(v^L - v^F - 1)}}$ and $\frac{1}{\sqrt{2s}} \geq I_L \geq \delta$.

(A). If $\frac{1}{\sqrt{2s(v^L - v^F - 1)}} \leq I_L$, then

$$I^*_F = (v^L - v^F + 2)I_L,$$

thus the optimization (53) can be written as

$$\begin{aligned} \max_{I_L} \quad & \pi_{L,1} = (s(v^L - v^F + 2)^2 - \gamma)I_L^2 \\ s.t \quad & \frac{1}{\sqrt{2s(v^L - v^F + 2)}} \leq I_L. \end{aligned}$$

(i). If $s(v^L - v^F + 2)^2 - \gamma < 0$, i.e., $v^L - v^F < \sqrt{\frac{\gamma}{s}} - 2$, then $\pi_{L,1}$ is a decreasing function of I_L, so $I^*_L = \frac{1}{\sqrt{2s(v^L - v^F + 2)}}$, and $\pi^*_{L,1} < 0$.

(ii). If $s(v^L - v^F + 2)^2 - \gamma = 0$, i.e., $v^L - v^F = \sqrt{\frac{\gamma}{s}} - 2$, then $\pi_{L,1} = 0$, thus I^*_L can be any number such that $I_L > \frac{1}{\sqrt{2s(v^L - v^F + 2)}}$.

(iii). If $s(v^L - v^F + 2)^2 - \gamma > 0$, then $\pi_{L,1}$ is an increasing function of I_L, thus $\pi_{L,1} \uparrow \infty$ as $I_L \uparrow \infty$. Therefore, SP$_L$ always wants to make a deviation to get a higher payoff by increasing the investment level ($I_L \uparrow \infty$), so there exist no negative-corner equilibria I^*_L in this case.

(B). If $\frac{1}{\sqrt{2s}} < I_L < \frac{1}{\sqrt{2s(v^L - v^F - 1)}}$, then $I^*_F = \frac{1}{2sI_L}$, thus the optimization (53) is equivalent to

$$\begin{aligned} \max_{I_L} \quad & \pi_{L,2} = \frac{1}{4sI_L^2} - \gamma I_L^2 \\ s.t \quad & \frac{1}{\sqrt{2s}} < I_L < \frac{1}{\sqrt{2s(v^L - v^F + 2)}} \\ & \pi_{L,2}(I_L) \geq 0 \end{aligned}$$

$\pi_{L,2}$ is a decreasing function of I_L, note that $\gamma < s$, denote

$$\pi^*_{L,2} = \pi_{L,2}(\frac{1}{\sqrt{2s}}) = \frac{1}{2}(1 - \frac{\gamma}{s}) > 0,$$

thus $\pi_{L,2} \uparrow \pi^*_{L,2}$ as $I_L \downarrow \frac{1}{\sqrt{2s}}$, which means SP$_L$ always wants to make a deviation to get a higher payoff by decreasing the investment level ($I_L \downarrow \frac{1}{\sqrt{2s}}$). Therefore there exist no negative-corner equilibria I^*_L in this case.

(C). If $\frac{1}{\sqrt{2s}} \geq I_L \geq \delta$, then $I^*_F = I_L$, thus the optimization (53) is equivalent to

$$\begin{aligned} \max_{I_L} \quad & \pi_{L,3} = (s - \gamma)I_L^2 \\ s.t \quad & \frac{1}{\sqrt{2s}} \geq I_L \geq \delta \\ & \pi_{L,3}(I_L) \geq 0 \end{aligned}$$

Note $s > \gamma$, then $\pi_{L,3}$ is an increasing function of I_L, thus $I^*_L = I^*_F = \frac{1}{\sqrt{2s}}$. Denote

$$\pi^*_{L,3} = \pi_{L,3}(\frac{1}{\sqrt{2s}}) = \frac{s - \gamma}{2s}.$$

Note that $\pi^*_{L,3} = \pi^*_{L,2}$.

(D). In this step, we prove that $I_L^* = \frac{1}{\sqrt{2s}}$ is the unique optimum when $v^L - v^F \le \sqrt{\frac{\gamma}{s}} - 2$, and there is no optimum I_L^* when $\sqrt{\frac{\gamma}{s}} - 2 < v^L - v^F \le -1$.

(i). If $v^L - v^F < \sqrt{\frac{\gamma}{s}} - 2$, then

$$\pi_{L,1}^* < 0, \quad \pi_{L,2} < \pi_{L,2}^* = \pi_{L,3}^*,$$

thus

$$\max_{I_L} \pi_L(I_L) = \pi_{L,3}^*,$$

therefore

$$I_L^* = \frac{1}{\sqrt{2s}}.$$

(ii). If $v^L - v^F = \sqrt{\frac{\gamma}{s}} - 2$, then

$$\pi_{L,1} = 0 < \pi_{L,2}^* = \pi_{L,3}^*,$$

and

$$\pi_{L,2} < \pi_{L,2}^* = \pi_{L,3}^*,$$

thus

$$\max_{I_L} \pi_L(I_L) = \pi_{L,3}^*,$$

therefore

$$I_L^* = \frac{1}{\sqrt{2s}}.$$

(iii). If $\sqrt{\frac{\gamma}{s}} - 2 < v^L - v^F \le -1$, then from **(A)**, $\pi_{L,1} \to \infty$, so

$$\pi_{L,1} > \pi_{L,3}^* = \pi_{L,2}^*,$$

thus

$$\max_{I_L} \pi_L(I_L) = \pi_{L,1} \to \infty,$$

which implies SP$_L$ always wants to make a deviation to get a higher payoff by increasing the investment level ($I_L \uparrow \infty$), so there exist no negative-corner equilibria I_L^* in this case.

Therefore, $I_L^* = \frac{1}{\sqrt{2s}}$ when $v^L - v^F \le \sqrt{\frac{\gamma}{s}} - 2$. Since I_L^* is the unique maximum of π_L, so no unilateral deviation is beneficial for SP$_L$.

□

Theorem 25. *If $v^L - v^F \ge 1$, then the unique negative-corner SPNE strategy is:*

(1) $I_L^* = \delta$.
(2) $I_F^* = 0$.
(3) $p_F^* = p_L^* + v^F - v^L$, $c + 1 \le p_L^* \le c + v^L - v^F$.
(4) $n_L^* = 1$, $n_F^* = 0$.

Proof. **Step 1.** We prove $c + 1 \le p_L^* \le c + v^L - v^F$.

From Steps 1, 3 in Lemma 10, $I_F^* = 0$, $t_F^* = 1$ and $p_F^* = p_L^* + v^F - v^L$.

Suppose $p_L^* > c + v^L - v^F$, $p_F^* = p_L^* + v^F - v^L > c$. Now consider a unilateral deviation of SP$_F$, $p_F = p_F^* - \epsilon$, where $0 < \epsilon < 1$, with all the rest the same, then

$$n_L = x_0 = t_F^* + v^L - v^F + p_F - p_L^* = 1 - \epsilon$$
$$n_F = 1 - n_L = \epsilon.$$

Thus,

$$\pi_F - \pi_F^* = n_F(p_F - c) > 0.$$

The last inequality follows because we can choose $0 < \epsilon < 1$ such that $p_F - \epsilon > c$. Thus, $p_L^* > c + v^L - v^F$ can not be SPNE.

Suppose $p_L^* < c + 1$, consider a unilateral deviation of SP$_L$, $p_L = p_L^* + \epsilon$, where $0 < \epsilon < 1$, with all the rest the same, then

$$n_L = x_0 = t_F^* + v^L - v^F + p_F - p_L^* = 1 - \epsilon$$

Thus,

$$\pi_L - \pi_L^* = \epsilon(-p_L^* + c + 1 - \epsilon) > 0$$

The last inequality follows because we can choose $0 < \epsilon < 1$ such that $p_L = p_L^* + \epsilon < c + 1$. Thus, $p_L^* < c + 1$ can not be an SPNE.

In addition, we prove that no unilateral deviation is beneficial for both SPs when $c + 1 \le p_L^* \le c + v^L - v^F$. Consider another unilateral deviation of SP$_F$, $p'_F = p_F^* - \epsilon$, where $0 < \epsilon < 1$, with all the rest the same, then

$$n'_L = x'_0 = t_F^* + v^L - v^F + p'_F - p_L^* = 1 - \epsilon$$
$$n'_F = 1 - n'_L = \epsilon.$$

Since $p_F^* = p_L^* + v^F - v^L$, then $p_F^* \in [c + v^F - v^L + 1, c]$, then

$$\pi'_F - \pi_F^* = n'_F(p'_F - c) < 0,$$

which implies no unilateral deviation is beneficial for SP$_F$.

Consider another unilateral deviation of SP$_L$, $p'_L = p_L^* + \epsilon$, where $0 < \epsilon < 1$, with all the rest the same, then

$$n'_L = x'_0 = t_F^* + v^L - v^F + p_F^* - p'_L = 1 - \epsilon.$$

Thus, note that $c + 1 \le p_L^* \le c + v^L - v^F$,

$$\begin{aligned} \pi'_L - \pi_L^* &= n'_L(p'_L - c) - p_L^* + c \\ &= \epsilon(-p_L^* + c + 1 - \epsilon) \le -\epsilon^2 < 0. \end{aligned}$$

which implies no unilateral deviation is beneficial for SP$_L$.

Step 2. Find $I_F^* = 0$. From Lemma 10, $I_F^* = 0$, and any other I_F^* can not be a SPNE.

Step 3. Find $I_L^* = \delta$.

Since p_L^* is independent of I_L^*, then from (1), $\pi_L = p_L^* - c - \gamma I_L^*$ is a decreasing function of I_L. Note that $I_L \geq \delta$, therefore $I_L^* = \delta$. Since $I_L^* = \delta$ is the unique maximum of π_L, so no unilateral deviation is beneficial for SP$_L$.

□

APPENDIX E
EUs WITH OUTSIDE OPTION: SPNE ANALYSIS

In this section, we let $\delta < \frac{4}{b}$.

Definition 4. *The fraction of EUs with each SP is*

$$\begin{aligned}\tilde{n}_L &= \alpha\big(n_L + \varphi_L(p_L, I_L)\big),\\ \tilde{n}_F &= \alpha\big(n_F + \varphi_F(p_F, I_F)\big),\end{aligned} \tag{54}$$

where

$$\begin{aligned}\varphi_L(p_L, I_L) &= k - p_L + b(I_L - I_F),\\ \varphi_F(p_F, I_F) &= k - p_F + bI_F\end{aligned} \tag{55}$$

and $\alpha > 0$, k and b are constants.

Stage 3: We consider interior NE strategies, i.e., $0 < n_F, n_L < 1$. Using Definition 4, (1), (2) and (33), note that $v^L = v^F$, the payoffs of SPs are:

$$\begin{aligned}\pi_F =& \alpha(t_L + k + p_L - 2p_F + bI_F)(p_F - c) - sI_F^2\\ \pi_L =& \alpha(t_F + k + p_F - 2p_L + bI_L - bI_F)(p_L - c)\\ &+ sI_F^2 - \gamma I_L^2\end{aligned} \tag{56}$$

Then, we characterize the NE of access fees,

Theorem 26. *For given I_F and I_L, the NE strategies of access fees are unique, and are:*

$$\begin{aligned}p_L^* =& \frac{1}{15} + \frac{2c}{3} + \frac{k}{3} + \frac{t_F}{5} - \frac{b}{5}I_F + \frac{4b}{15}I_L,\\ p_F^* =& \frac{1}{15} + \frac{2c}{3} + \frac{k}{3} + \frac{t_L}{5} + \frac{b}{15}I_L + \frac{b}{5}I_F.\end{aligned} \tag{57}$$

if and only if I_L satisfies:

$$I_L < \frac{4}{b}. \tag{58}$$

Proof. In this case, every NE by which $0 \leq x_0 \leq 1$, should satisfy the first order condition. Thus p_L^* and p_F^* should be such that

$$\frac{d\pi_L}{dp_L}|_{p_L^*} = 0,\ \frac{d\pi_F}{dp_F}|_{p_F^*} = 0,$$

note that $t_L + t_F = 1$, then

$$\begin{aligned}p_L^* =& \frac{1}{15} + \frac{2c}{3} + \frac{k}{3} + \frac{t_F}{5} - \frac{b}{5}I_F + \frac{4b}{15}I_L,\\ p_F^* =& \frac{1}{15} + \frac{2c}{3} + \frac{k}{3} + \frac{t_L}{5} + \frac{b}{15}I_L + \frac{b}{5}I_F.\end{aligned}$$

Take the second derivative of π_L with respect to p_L,

$$\frac{d^2\pi_L}{d(p_L^*)^2} = \frac{d^2\pi_F}{d(p_F^*)^2} = -4\alpha < 0,$$

then p_L^* and p_F^* are the unique maximal solutions of π_L and π_F, respectively.

Thus, p_F^* and p_L^* are the unique interior NE strategies if and only if $0 < x_0 < 1$. Substituting (57), $t_L = I_F/I_L$, and $t_F = (I_L - I_F)/I_L$ into (4) yields:

$$x_0 = \frac{4}{5} - \frac{b}{5}I_L + (\frac{2b}{5} - \frac{3}{5I_L})I_F \triangleq \Psi(I_F).$$

Once I_L is fixed, $\Psi(I_F)$ would be a linear function of I_F. Thus, $0 < \Psi(I_F) < 1$ for any values of I_F such that $0 \leq I_F \leq I_L$, if and only if

$$\begin{aligned}0 < \Psi(0) < 1\\ 0 < \Psi(I_L) < 1.\end{aligned}$$

Thus,

$$\begin{aligned}\Psi(I_L) =& \frac{1}{5} + \frac{b}{5}I_L \in (0,1)\\ \Psi(0) =& \frac{4}{5} - \frac{b}{5}I_L \in (0,1)\end{aligned}$$

if and only if $0 < I_L < \frac{4}{b}$. □

Stage 2: Based on the NE strategies of access fees, we obtain the optimum investment level of the MVNO.

Definition 5. $g(I_L) = \frac{b}{15}I_L + \frac{1}{15} - \frac{c}{3} + \frac{k}{3}$, $f(I_L) = \frac{1}{5I_L} + \frac{b}{5} > 0$

Theorem 27. *If $\pi_F(I_F; I_L) \geq 0$, and denote*

$$I_F^0 = \frac{-2\alpha f(I_L)g(I_L)}{2\alpha f^2(I_L) - s}.$$

Then, the unique optimal investment level of SP$_F$, I_F^*, is:

$$I_F^* = \begin{cases} I_F^0 & \text{if} \quad I_L \in \{s > 2\alpha f^2(I_L) + 2\alpha f(I_L)g(I_L)/I_L,\\ & \qquad g(I_L) \geq 0\}\\ I_L & \text{if} \quad I_L \in \{2\alpha f^2(I_L) \leq s \leq 2\alpha f^2(I_L)\\ & \qquad + 2\alpha f(I_L)g(I_L)/I_L, g(I_L) \geq 0\}\\ & \qquad \cup \{2\alpha f^2(I_L) + 4\alpha f(I_L)g(I_L)/I_L \geq s,\\ & \qquad 2\alpha f^2(I_L) > s\}\end{cases} \tag{59}$$

Proof. First, we give the following the lemma

Lemma 11. *The optimum investment level I_F^* is obtained by*

$$\begin{aligned}\max_{I_F} \quad & \pi_F = (2\alpha f^2(I_L) - s)I_F^2\\ & + 4\alpha f(I_L)g(I_L)I_F + 2\alpha g^2(I_L)\\ s.t \quad & 0 \leq I_F \leq I_L.\end{aligned} \tag{60}$$

Proof. Substituting (57) into π_F in (56), we get the objective function. The constraints come from the model assumptions directly. $\square$

We consider different cases. First, we consider the case that $2\alpha f^2(I_L)-s=0$ (Step (i)). Then, we consider the case that $2\alpha f^2(I_L)-s\neq 0$ and π_F is a quadratic function of I_F (Step (ii)). In Step (iii), we prove that $I_F^*\neq 0$. Combining the steps yields the result of the theorem.

Step (i): If $2\alpha f^2(I_L)-s=0$, π_F is linear function of I_F, i.e., $\pi_F=4\alpha f(I_L)g(I_L)I_F+2\alpha g^2(I_L)$ Thus,

$$\begin{cases} I_F^*=0 & \text{if} \quad g(I_L)<0 \\ I_F^*=I_L & \text{if} \quad g(I_L)\geq 0 \end{cases}.$$

Step (ii): Now, consider the case that $2\alpha f^2(I_L)-s\neq 0$ and π_F is a quadratic function of I_F. We characterize the optimum answer in two cases: (a) if $2\alpha f^2(I_L)-s>0$, and (b) if $2\alpha f^2(I_L)-s<0$, $\pi_F(I_F;I_L)$.

For the case that π_F is a quadratic function, we use the solution to the first order condition (I_F^0),

$$\frac{d\pi_F}{dI_F}|_{I_F^0}=0\Rightarrow I_F^0=\frac{-2\alpha f(I_L)g(I_L)}{2\alpha f^2(I_L)-s}.$$

Case (ii-a): If $2\alpha f^2(I_L)-s>0$, then π_F is convex function. From Lemma 3 (1),

$$\begin{cases} I_F^0-\dfrac{I_L}{2}\leq 0 & \text{if} \quad 2\alpha I_L f^2(I_L)+4\alpha f(I_L)g(I_L)-I_L s\geq 0 \\ I_F^0-\dfrac{I_L}{2}>0 & \text{if} \quad 2\alpha I_L f^2(I_L)+4\alpha f(I_L)g(I_L)-I_L s<0 \end{cases},$$

thus

$$\begin{cases} I_F^*=I_L & \text{if}\, 2\alpha I_L f^2(I_L)+4\alpha f(I_L)g(I_L)-I_L s\geq 0 \\ I_F^*=0 & \text{if}\, 2\alpha I_L f^2(I_L)+4\alpha f(I_L)g(I_L)-I_L s<0 \end{cases}.$$

Case (ii-b): If $2\alpha f^2(I_L)-s<0$, then π_F is a concave function. Thus, from Lemma 3 (2),

$$\begin{cases} I_F^0-0<0 & \text{if} \quad g(I_L)<0 \\ 0\leq I_F^0<I_L & \text{if} \quad 2\alpha I_L f^2(I_L)+2\alpha f(I_L)g(I_L)-I_L s<0, \\ & \quad g(I_L)\geq 0 \\ I_F^0\geq I_L & \text{if} \quad 2\alpha I_L f^2(I_L)+2\alpha f(I_L)g(I_L)-I_L s\geq 0, \\ & \quad g(I_L)\geq 0 \end{cases},$$

Thus,

$$\begin{cases} I_F^*=0 & \text{if} \quad g(I_L)<0 \\ I_F^*=I_F^0 & \text{if} \quad 2\alpha I_L f^2(I_L)+2\alpha f(I_L)g(I_L)-I_L s<0, \\ & \quad g(I_L)\geq 0 \\ I_F^*=I_L & \text{if} \quad 2\alpha I_L f^2(I_L)+2\alpha f(I_L)g(I_L)-I_L s\geq 0, \\ & \quad g(I_L)\geq 0 \end{cases}.$$

Step (iii): We now prove $I_F^*\neq 0$. From Case (ii-a), if $I_F^*=0$, then

$$2\alpha I_L f^2(I_L)+4\alpha f(I_L)g(I_L)-I_L s<0,$$

i.e.,

$$s>2\alpha f^2(I_L)+4\alpha f(I_L)g(I_L)/I_L,$$

which implies $g(I_L)<0$ since $2\alpha f^2(I_L)-s>0$. Thus from Step (i), and Cases (ii-a) and (ii-b), if $I_F^*=0$, then $g(I_L)<0$.

Since $t_L^*=0$ and $t_F^*=1$, when $I_F^*=0$, then

$$p_F^*-c=\frac{1}{15}-\frac{c}{3}+\frac{k}{3}+\frac{b}{15}I_L=g(I_L)<0.$$

For an equilibrium solution p_F^*, $p_F^*\geq c$, otherwise

$$\pi_L^*=\tilde{n}_F^*(p_F^*-c)-s(I_F^*)^2<0.$$

Hence $I_F^*=0$ can not be an equilibrium solution for SP$_F$.

Combining Steps (i), (ii), and (iii), we obtain the desired results. $\square$

Stage 1

Finally, we characterize the optimum investment level of the MNO.

Theorem 28. *The unique optimum investment level of SP$_L$, I_L^*, a solution of the following optimization problem:*

$$\begin{aligned} \max_{I_L} \quad & \pi_L(I_L;I_F^*)=2\alpha(\frac{b}{5}I_L+\frac{1}{5}+g(I_L)-f(I_L)I_F^*)^2 \\ & +s(I_F^*)^2-\gamma I_L^2 \\ s.t \quad & \delta\leq I_L<\frac{4}{b}. \end{aligned} \tag{61}$$

Proof. Substituting (57) into π_L in (56), we get the objective function. The constraints come from the model assumptions directly. $\square$

We define functions $f(I_L)$, $g(I_L)$, $\pi_L(I_F)$ and sets $\mathbb{L}_1$, $\mathbb{L}_2$ as follows:

$$g(I_L)=\frac{b}{15}I_L+\frac{1}{15}-\frac{c}{3}+\frac{k}{3},\ f(I_L)=\frac{1}{5I_L}+\frac{b}{5},$$
$$\theta(y)=2\alpha\big(\frac{b}{5}I_L+\frac{1}{5}+g(I_L)-f(I_L)y\big)^2+sy^2-\gamma I_L^2,$$
$$\begin{aligned}\mathbb{L}_1=&\{s>2\alpha f^2(I_L)+2\alpha f(I_L)g(I_L)/I_L,\, g(I_L)\geq 0,\\ &\delta\leq I_L<\frac{4}{b}\},\end{aligned}$$
$$\begin{aligned}\mathbb{L}_2=&\{0\leq I_L<\frac{4}{b}\}\cap\Big(\{g(I_L)\geq 0,\\ &2\alpha f^2(I_L)\leq s\leq 2\alpha f^2(I_L)+2\alpha f(I_L)g(I_L)/I_L\}\\ &\cup\{2\alpha f^2(I_L)+4\alpha f(I_L)g(I_L)/I_L\geq s,\, 2\alpha f^2(I_L)>s\}\Big).\end{aligned}$$

Collecting results in Stages 1∼4, we have

Corollary 5. *The interior SPNE strategies are:*

(1) I_L^* *is characterized in*

$$I_L^* = \operatorname{argmax}_{I_L} \Big(\max_{I_L \in \mathbb{L}_1} \theta(\frac{-2\alpha f(I_L)g(I_L)}{2\alpha f^2(I_L) - s}), \max_{I_L \in \mathbb{L}_2} \theta(I_L) \Big)$$

(2) I_F^* *is characterized in*

$$I_F^* = \begin{cases} \dfrac{-2\alpha f(I_L)g(I_L)}{2\alpha f^2(I_L) - s} & \text{if } I_L \in \mathbb{L}_1 \\ I_L & \text{if } I_L \in \mathbb{L}_2 \end{cases}$$

(3) $p_L^* = \frac{1}{15} + \frac{2c}{3} + \frac{k}{3} + \frac{I_L^* - I_F^*}{5I_L^*} - \frac{b}{5}I_F^* + \frac{4b}{15}I_L^*$, $p_F^* = \frac{1}{15} + \frac{2c}{3} + \frac{k}{3} + \frac{I_F^*}{5I_L^*} + \frac{b}{15}I_L^* + \frac{b}{5}I_F^*$.

(4) $\tilde{n}_L^* = \frac{I_L^* - I_F^*}{I_L^*} + p_F^* - 2p_L^* + k + bI_L^* - bI_F^*$, $\tilde{n}_F^* = \frac{I_F^*}{I_L^*} + p_L^* - 2p_F^* + k + bI_F^*$

Appendix F
Proof of Corollary 1

Stage 4: Similar with Definition 2, $u_F(x_0) = v - t(2\pi - x_0) - p_F = v - tx_0 - p_L = u_L(x_0)$, thus,

$$x_0 = \pi + \frac{p_F - p_L}{2t}. \tag{62}$$

Since EUs are distributed uniformly along $[0, 2\pi]$, the fraction of EUs with each SP is:

$$n_L = \begin{cases} 0, & \text{if} \quad x_0 \le 0 \\ x_0, & \text{if} \quad 0 < x_0 < 2\pi \\ 2\pi, & \text{if} \quad x_0 \ge 2\pi \end{cases}, \; n_F = 2\pi - n_L, \tag{63}$$

where x_0 is defined in (62) and $n_F = 2\pi - n_L$.

Only "interior" strategies may be SPNE, as:

Theorem 29. *In the SPNE it must be that* $0 < x_0 < 2\pi$.

Proof. Let $(p_L^*, p_F^*, I_L^*, I_F^*)$ be a corner SPNE strategy. Thus, 1) $x_0 \ge 2\pi$, or 2) $x_0 \le 0$. We arrive at a contradiction for 1) **Step 1** and 2) in **Step 2** respectively.

Lemma 12. $\pi_F^* \ge 0$. *If* $n_F^* > 0$, $p_F^* \ge c$.

Proof. Let $\pi_F^* < 0$. Consider a unilateral deviation in which $I_F = 0, p_F \ge c$. From (2), $\pi_F \ge 0$, leading to a contradiction. Now, let $n_F^* > 0$ and $p_F^* < c$. Thus, $\pi_F^* < 0$ which is a contradiction. □

Step 1. Let $x_0^* \ge 2\pi$. Clearly, $n_F^* = 0$ and $n_L^* = 2\pi$. From (2), $\pi_F^* = -sI_F^{*2}$. From Lemma 12, $I_F^* = 0$. Thus, $\pi_F^* = 0$. From (62), $2\pi \le x_0^* = \pi + \frac{p_F^* - p_L^*}{2t}$. Thus, $p_F^* \ge p_L^* + 2\pi t$.

From (1), $\pi_L^* = 2\pi(p_L^* - c) - \gamma I_L^{*2}$. If $p_L^* < c$, then $\pi_L^* < -\gamma\delta^2 < 0$ since $I_L^* \ge \delta$. Consider a unilateral deviation by which $I_L = \delta, p_L = c$, then $\pi_L = -\gamma\delta^2$, which is beneficial for SP_L. Thus, $p_L^* \ge c$.

Now, let $p_L^* > c$. Thus, $p_F^* \ge p_L^* + 2\pi t > c + 2\pi t > c$. Recall that $x_0^* = \pi + \frac{p_F^* - p_L^*}{2t}$. Consider a unilateral deviation by which $p_F = p_L^* + 2\pi t - \epsilon$. Now, by (62), $x_0 < 2\pi$, and hence $n_F > 0$. Now, from (2), $\pi_F > 0 = \pi_F^*$. Thus, (I_F^*, p_F^*) is not SP_F's best response to SP_L's choices (I_L^*, p_L^*), which is a contradiction. Hence, $p_L^* = c$.

Now consider another unilateral deviation of SP_L, $p_L' = p_F^* - 2\pi t + \epsilon$, where $0 < \epsilon < \min(1, t)$, with all the rest the same. Since $p_L^* \le p_F^* - t$, $p_L' > p_L^* = c$.

$$n_L' = x_0' = \pi + \frac{p_F^* - p_L'}{2t} = 2\pi - \frac{\epsilon}{2t}.$$

Then

$$\pi_L' - \pi_L^* = n_L'(p_L' - c) - (p_L^* - c) = (2\pi - \frac{\epsilon}{2t})(p_L' - c) > 0.$$

The last inequality follows because $p_L' > c$ and $\epsilon < \min(1, t)$. Thus, we again arrive at a contradiction.

Step 2. Let $x_0^* \le 0$. Clearly, $n_F^* = 2\pi, n_L^* = 0$. Since $n_F^* > 0$, by Lemma 12, $p_F^* \ge c$. From (4), $x_0^* = \pi + \frac{p_F^* - p_L^*}{2t} \le 0$. Thus, $p_L^* \ge p_F^* + 2\pi t$. Now, from (1),

$$\pi_L^* = sI_F^{*2} - \gamma I_L^{*2}. \tag{64}$$

Consider a unilateral deviation by SP_L, by which $p_L' = 2\pi t + p_F^* - \epsilon$, $0 < \epsilon < \min(1, t)$. Then

$$n_L' = x_0' = \pi + \frac{p_F^* - p_L'}{2t} = \frac{\epsilon}{2t} > 0$$

Therefore, by (64),

$$\pi_L' - \pi_L^* = n_L'(p_L' - c) = \frac{\epsilon}{2t}(p_F^* - \epsilon + 2\pi t - c)$$

Since $p_F^* \ge c$, and $\epsilon < \min(1, t)$. Then, $\pi_L' - \pi_L^* > 0$. We again arrive at a contradiction. □

By Theorem 29 proved above henceforth we only consider interior SPNE in which $0 < x_0^* < 2\pi$. □

Stage 3: SP_L and SP_F determine their access fees for EUs, p_L and p_F, respectively, to maximize their payoffs.

Lemma 13. *The payoffs of SPs are:*

$$\begin{aligned} \pi_L &= \frac{1}{2t}(2\pi t + p_F - p_L)(p_L - c) + sI_F^2 - \gamma I_L^2 \\ \pi_F &= \frac{1}{2t}(2\pi t + p_L - p_F)(p_F - c) - sI_F^2 \end{aligned} \tag{65}$$

Proof. From (62) and (63), substitute $(n_L, n_F) = (\pi + \frac{p_F - p_L}{2t}, 2\pi - n_L)$ into (1) and (2), and get (65). □

We next obtain the SPNE p_F^* and p_L^* which maximize the payoffs π_L and π_F of the SPs respectively.

Theorem 30. *The SPNE pricing strategies are:*

$$p_L^* = c + 2\pi t, \quad p_F^* = c + 2\pi t \tag{66}$$

Proof. p_F^* and p_L^* must satisfy the first order condition, i.e., $\frac{d\pi_F}{dp_F} = 0$ and $\frac{d\pi_L}{dp_L} = 0$. Thus, $p_F^* = p_L^* = c + 2\pi t$. p_F^* and p_L^* are the unique SPNE strategies if they yield $0 < x_0 < 2\pi$ and no unilateral deviation is profitable for SPs. We establish these respectively in Parts A and B.

Part A. From (66), $x_0 = \pi + \frac{p_F^* - p_L^*}{2t} = \pi \in (0, 2\pi)$ since $p_L^* = p_F^* = 2\pi t + c$.

Part B. Since $\frac{d^2\pi_F}{dp_F^2} < 0, \frac{d^2\pi_L}{dp_L^2} < 0$, a local maxima is also a global maximum, and any solution to the first order conditions maximize the payoffs when $0 < x_0 < 2\pi$, and no unilateral deviation by which $0 < x_0 < 1$ would be profitable for the SPs. Now, we show that unilateral deviations of the SPs leading to $n_L = 0, n_F = 2\pi$ and $n_L = 2\pi, n_F = 0$ is not profitable. Note that the payoffs of the SPs, (1) and (2), are continuous as $n_L \downarrow 0$, and $n_L \uparrow 2\pi$ (which subsequently yields $n_F \uparrow 2\pi$ and $n_F \downarrow 0$, respectively). Thus, the payoffs of both SPs when selecting p_L and p_F as the solutions of the first order conditions are greater than or equal to the payoffs when $n_L = 0$ and $n_L = 2\pi$. Thus, the unilateral deviations under consideration are not profitable for the SPs. □

Stage 2: SP$_F$ decides on the amount of spectrum to be leased from SP$_L$, I_F, with the condition that $0 \le I_F \le I_L$, to maximize π_F.

Theorem 31. *The SPNE spectrum acquired by SP$_F$ is:* $I_F^* = 0$.

Proof. Substituting p_F and p_L from (66) into (65), SP$_F$'s payoff becomes,

$$\pi_F(I_F; I_L) = 2\pi^2 t - sI_F^2. \tag{67}$$

Since $\pi_F(I_F; I_L)$ is a decreasing function of I_F and $0 \le I_F \le I_L$, so $I_F^* = 0$. □

Stage 1: SP$_L$ chooses the amount of spectrum I_L to lease from the regulator, to maximize π_L.

Theorem 32. *The SPNE spectrum acquired by SP$_F$ is:* $I_F^* = \delta$.

Proof. Substituting p_L and p_F from (66) into (65), SP$_L$'s payoff becomes:

$$\pi_L(I_L; I_F^*) = 2\pi^2 t - \gamma I_L^2. \tag{68}$$

since from Theorem 31, $I_F^* = 0$. Note that π_L is a decreasing function of I_L, and $I_L \ge \delta$, so $I_L^* = \delta$. □

Collecting all SPNE from Stages 1∼4, the unique SPNE strategies are:

$$I_L^* = \delta,\ I_F^* = 0,\ p_L^* = p_F^* = 2t\pi + c,\ n_F^* = n_L^* = \pi.$$