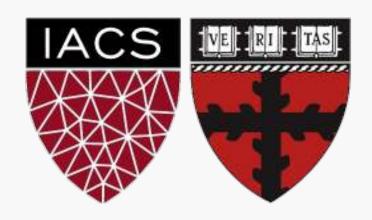
# Variational Auto-Encoders Part One

# CS109B Data Science 2 Pavlos Protopapas, Mark Glickman, and Chris Tanner



#### Outline

- Motivation for Variational Autoencoders (VAE)
- Inference in Neural Networks
  - Bayesian Linear Regression
  - Bayesian Neural Networks
  - Introduction to variational methods
  - Variational Autoencoder as an inference model
- Variational Autoencoders as generative model
  - Separability of VAE
  - Tips & tricks
  - Other generative models

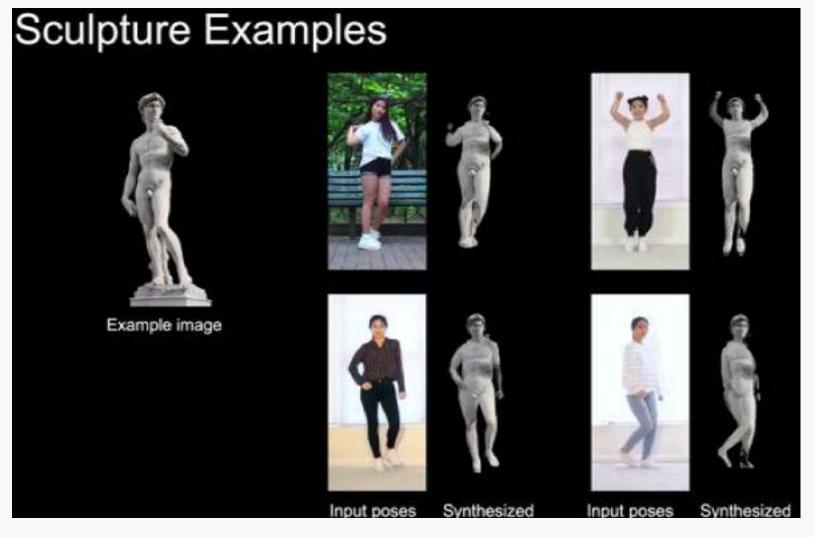


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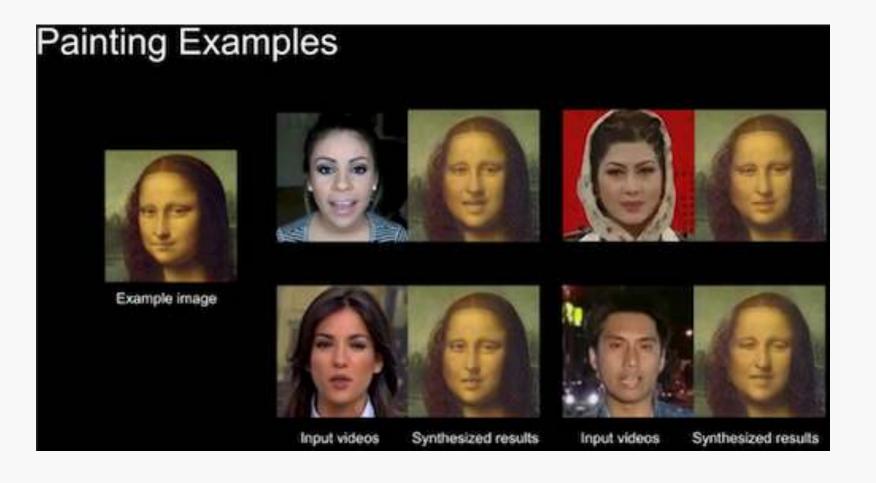
#### State of the Art in AI - sans NLP



https://nvlabs.github.io/few-shot-vid2vid/



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https://github.com/tkarras/progressive growing of gans



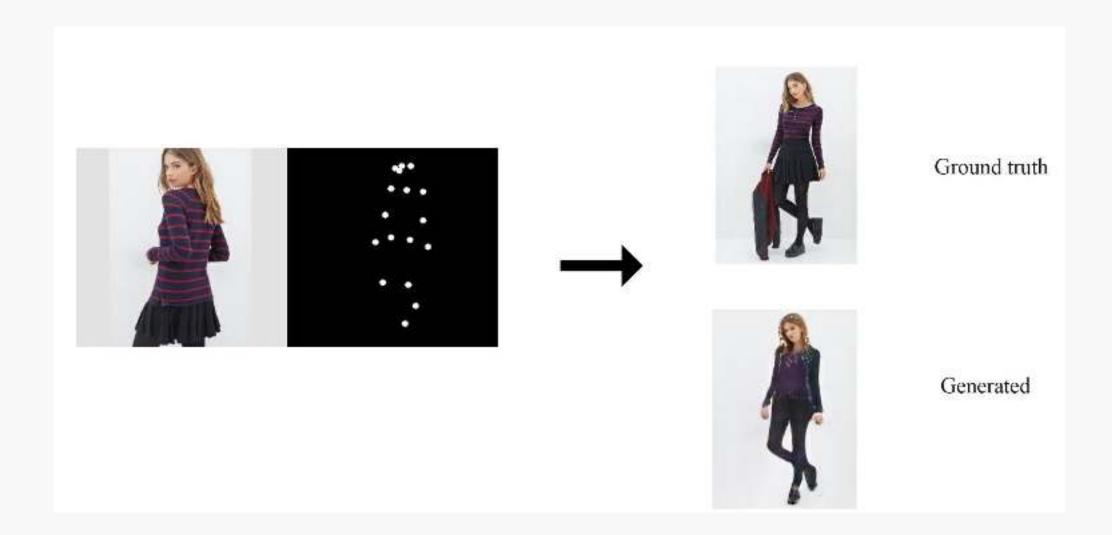


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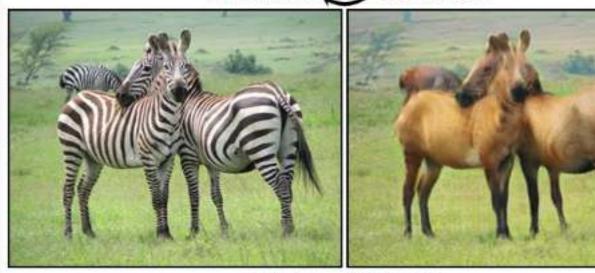






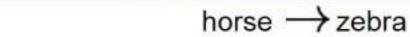


#### Zebras C Horses



zebra → horse







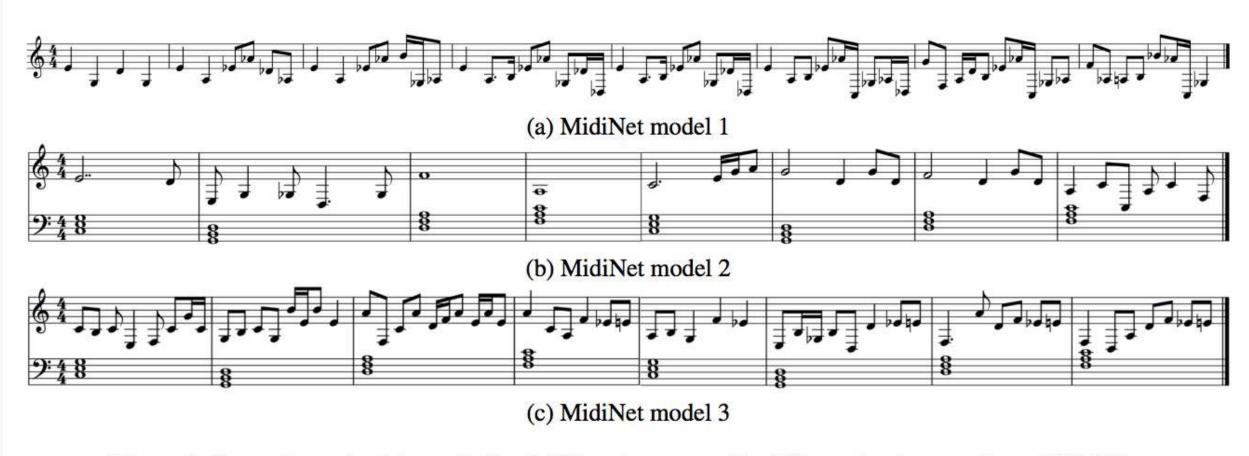


Figure 3. Example result of the melodies (of 8 bars) generated by different implementations of MidiNet.

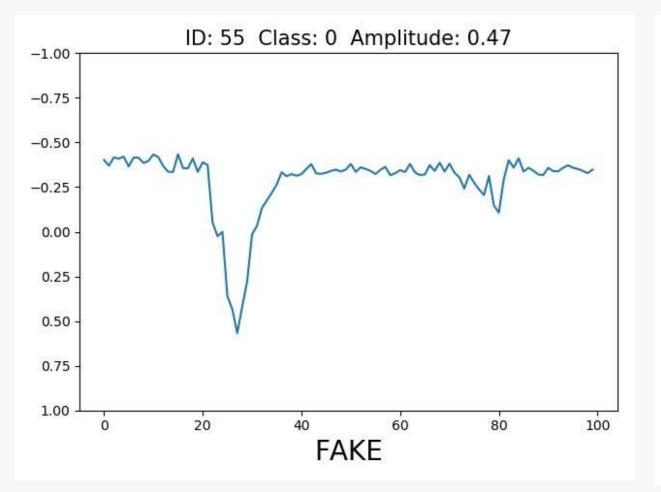


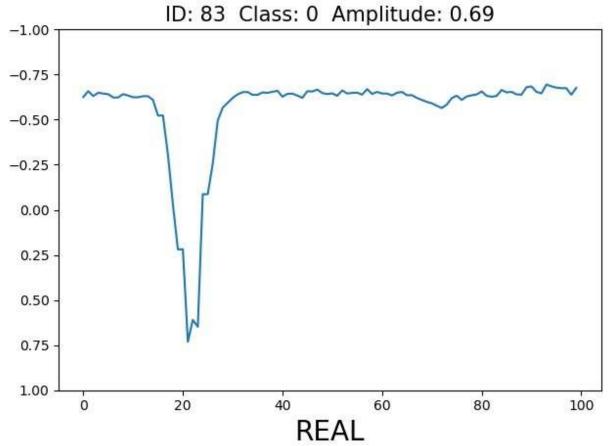
Another use of generating new data is to give us ideas and options. Suppose we're planning a house. We can give the computer the space we have available, and its location. From this, the computer can can give us some ideas.





Big networks require big data, and getting high-quality, labeled data is difficult. If we're generating that data our selves, we can make as much of it as we like.





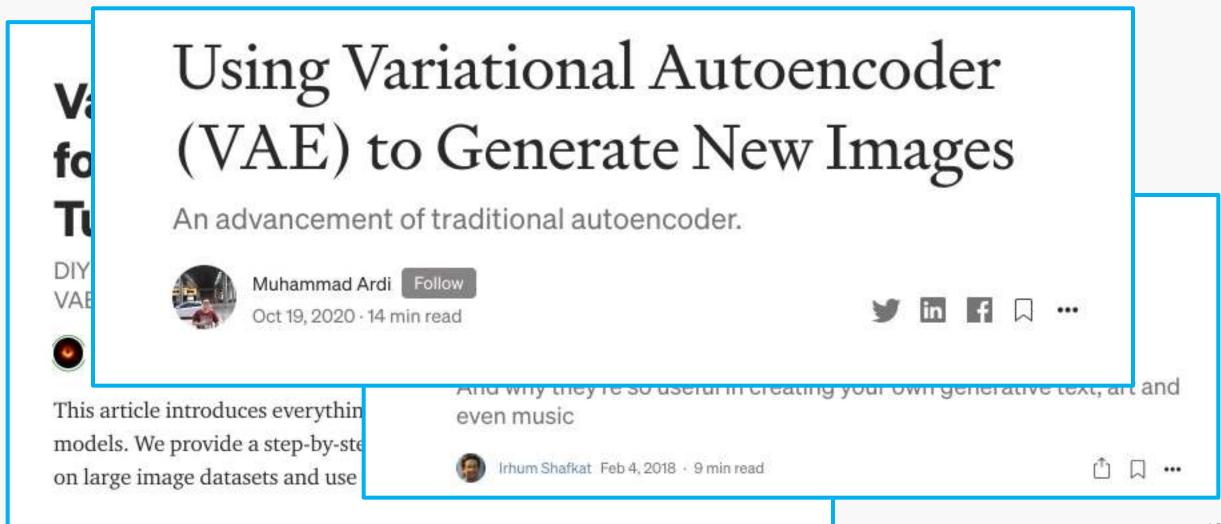


# To summarize: VAEs are a form of generative models

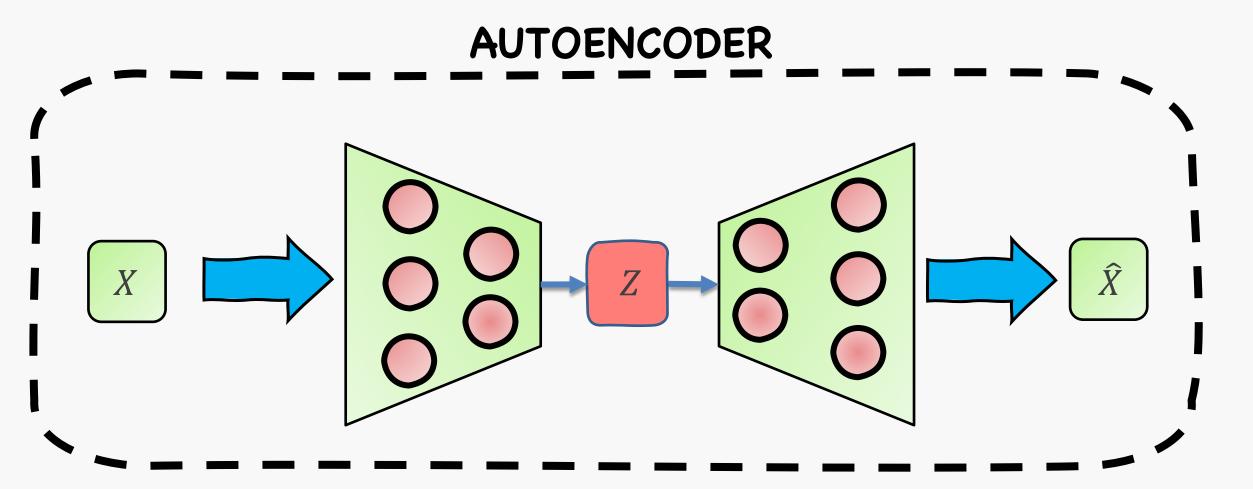


#### Variational AutoEncoder

Popular explanations of VAE as generative models

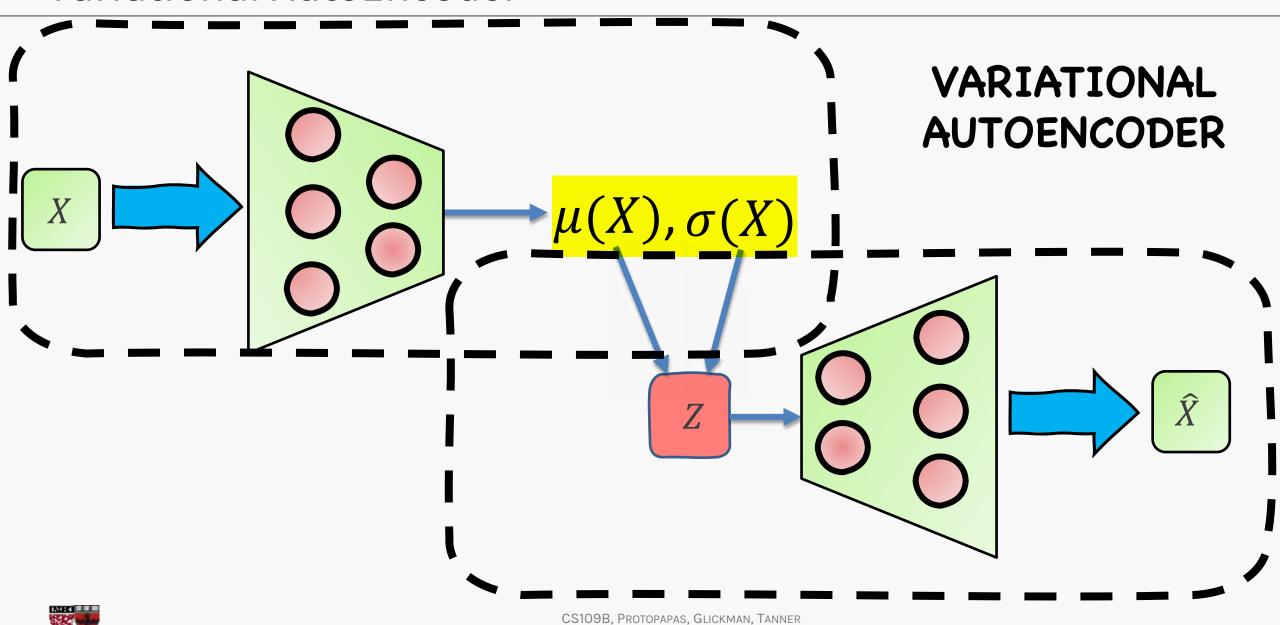


#### RECAP: Vanilla AutoEncoder





### Variational AutoEncoder



#### Variational Autoencoders

#### Typical explanations include:

- We want a continuous latent space representation
- We use a simple Reparameterization "trick"
- We add an Estimated Lower Bound (ELBO) error to reconstruction term



Why are Variational Autoencoders built this way?



# Variational AutoEncoder: Original Paper

Let us consider some dataset  $X = \{x^{(i)}\}_{i=1}^{N}$  consisting of N i.i.d. samples of some continuous or discrete variable x. We assume that the data are generated by some random process, involving an unobserved continuous random variable z.

• • •

We are interested in a general algorithm that works in case of:

1. Intractability:  $p_{\theta}(x) = \int p_{\theta}(z) p_{\theta}(x|z) dz$  is **intract**?

2. Large Dataset: Sampling based solutions eg. Mowen would be too slow

<u>Auto-Encoding Variational Bayes (Diederik P. Kingma et al)</u>



#### Variational Autoencoders as Inference models

The story of Variational Auto-encoders is like Coca-Cola

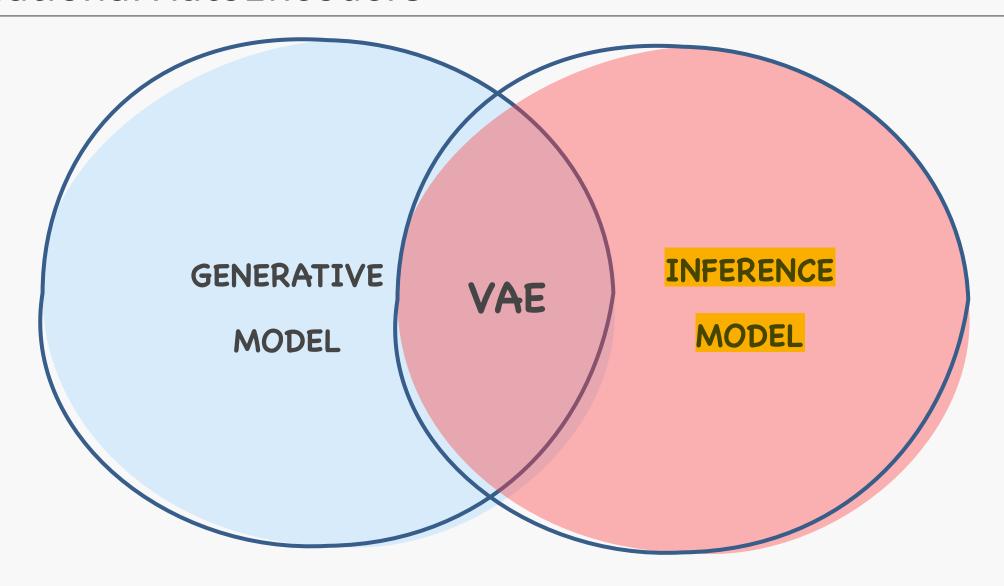
- Originally invented by John Pemberton to counter his own morphine addiction.
- Eventually found commercial success as a sugary drink and now is synonymous with soft beverages.



Source: https://en.wikipedia.org/wiki/Coca-Cola



#### Variational AutoEncoders







# Bayesian Linear Regression



#### Inference Review

Bootstrap combined with Maximum Likelihood can give us the distribution of the coefficients.

#### Such a method is:

- Easy to understand
- Easy to implement
- Statistically equivalent to analytical solution



#### Inference Review

Let us begin by considering the case of linear regression:

$$y = f(x) + \epsilon$$
 where  $f(x) = \beta_0 + \beta_1 x$ 

We interpret the  $\varepsilon$  term to be noise introduced by random variations in nature, or imprecisions of our scientific instruments and everything else.

If we knew the exact form of f(x), for example,  $f(x) = \beta_0 + \beta_1 x$ , and there was no noise in the data, then estimating the  $\hat{\beta}'s$  would have been exact.

Sometimes  $\varepsilon$  is considered as "catch-it-all" term



**However**, two things happen, which result in mistrust of the values of  $\hat{\beta}'s$ :

- observational error is always there this is called aleatoric error, or irreducible error.
- we do not know the exact form of f(x) this is called **misspecification** error and it is part of the epistemic error.

We will put everything into the catch-it-all term ε.

Because of  $\varepsilon$ , every time we measure the response y for a fix value of x, we will obtain a different observation, and hence a different estimate of  $\hat{\beta}'s$ .



So, if we just have one set of measurements of  $\{X,Y\}$ , our estimates of  $\hat{\beta}_0$  and  $\hat{\beta}_1$  are just for this particular realization.

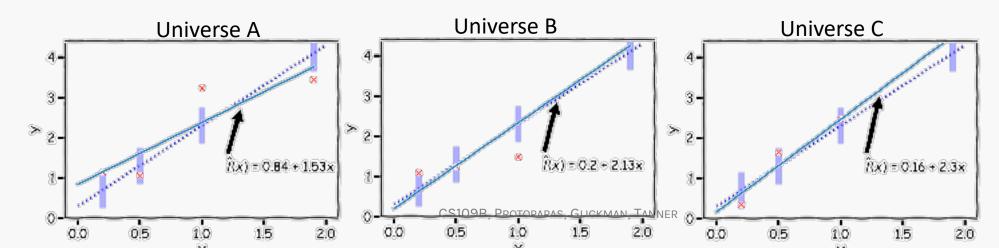
**Question:** If this is just one realization of the reality how do we know the truth? How do we deal with this conundrum?



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**Question:** If this is just one realization of the reality how do we know the truth? How do we deal with this conundrum?

**Imagine** (magic realism) we have parallel universes, and we repeat this experiment on each of the other universes.





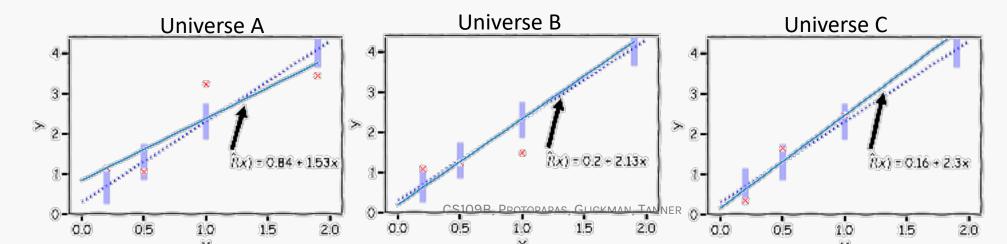
So, if we just have one set of measurements of  $\{X,Y\}$ , our estimates of  $\hat{\beta}_0$  and  $\hat{\beta}_1$  are just for this particular realization

Question: If this is just one realization truth? How do we deal with this

We use bootstrap to create different dataset.

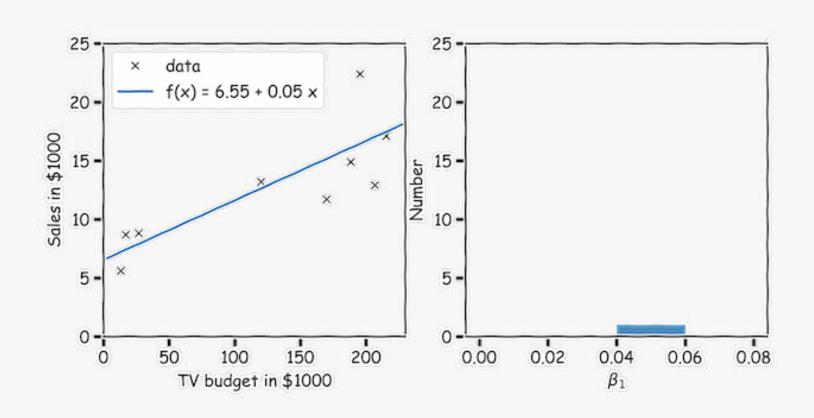
Bootstrap is sampling with replacement. Bootstrapping was covered **extensively** in CS109A.

**Imagine** (magic realism) we have parallel universes, and we repeat this experiment on each of the other universes.



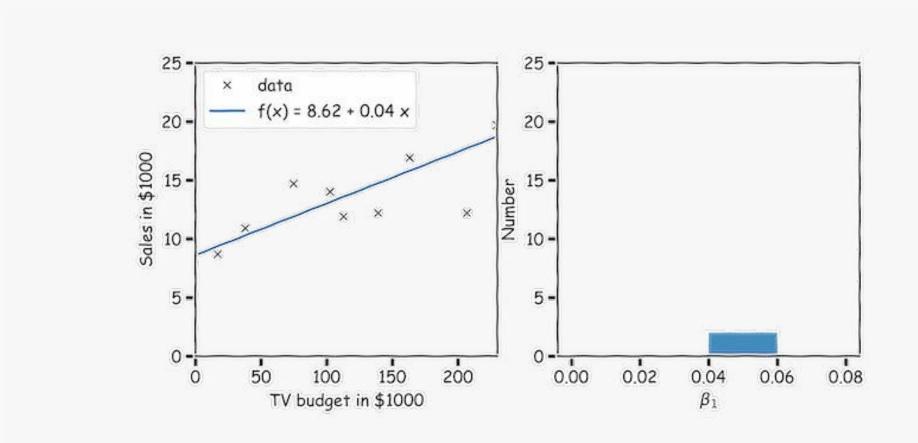


In our magical realisms, we can now sample multiple times. One universe, one sample, one set of estimates for  $\hat{\beta}_0$ ,  $\hat{\beta}_1$ 





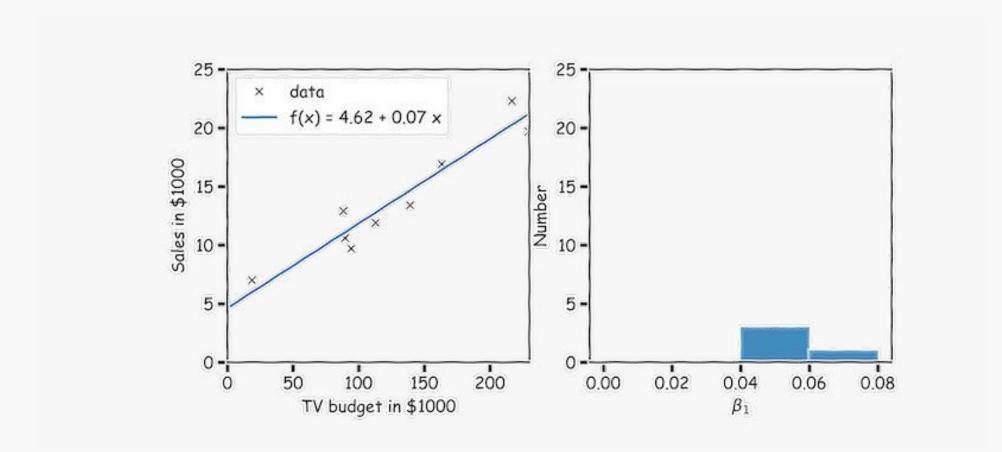
Another sample, another estimate of  $\hat{eta}_0$ ,  $\hat{eta}_1$ 







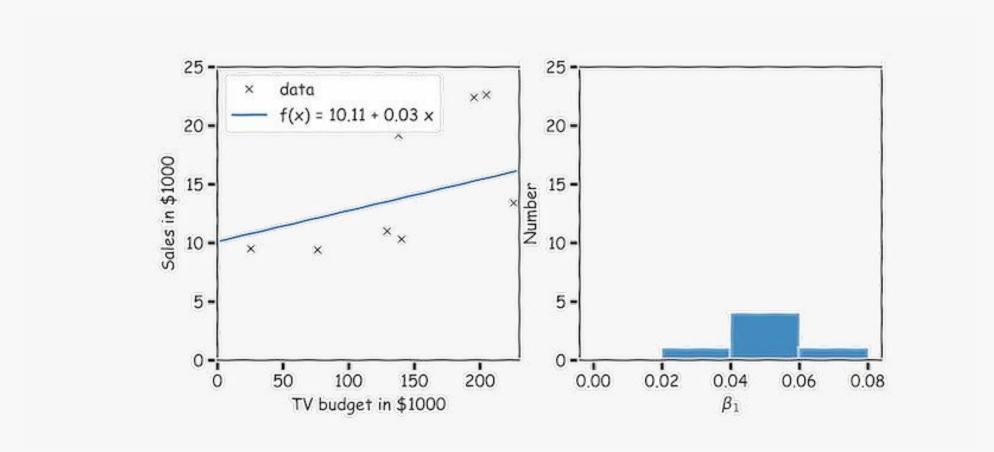
again







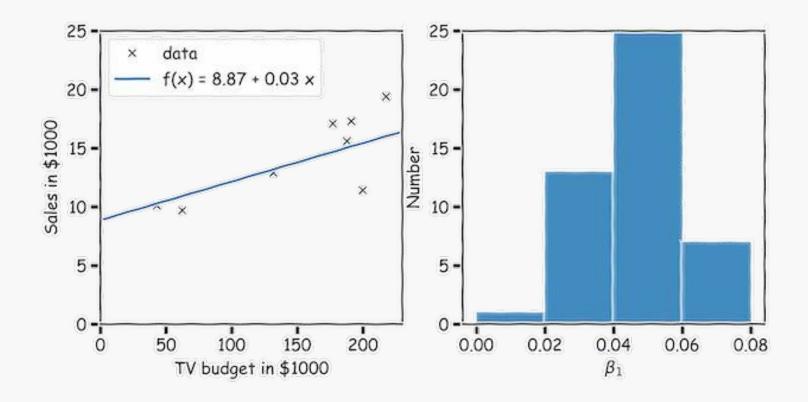
and again







repeat this for 100 times, until we have enough samples of  $\hat{\beta}_0$ ,  $\hat{\beta}_1$ .







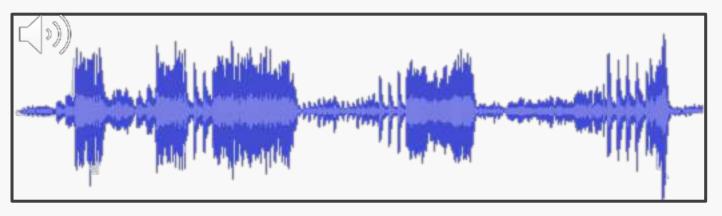
#### **BOOTSTRAP ISSUES?**

- Although easy to implement, it is computationally expensive.
- The output distribution is sensitive to input data.
- As it uses maximum likelihood point estimates, it is not a natural candidate for inference.



## The marriage of Figaro





The composition is a "collection" of notes, as seen by the waveform above



CS109B, PROTOPAPAS, GLICKMAN, TANNER

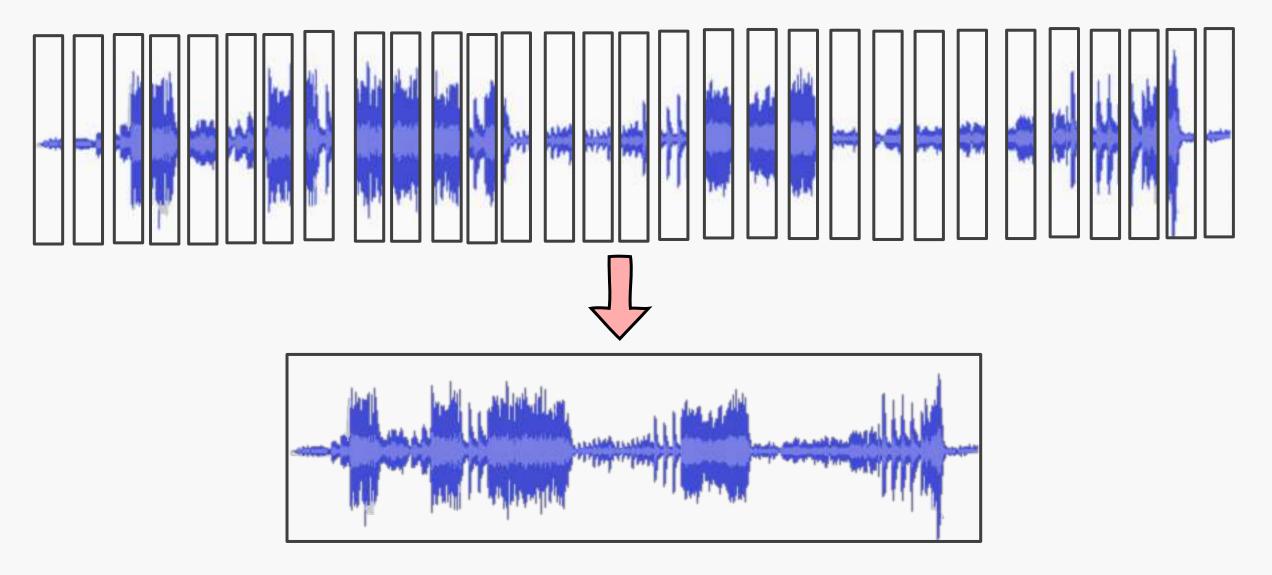
## The Marriage of Figaro – Bootstrap edition



MysteryGuitarMan – <u>The marriage of Figaro</u>



## The Marriage of Figaro – Bootstrap edition





#### What we want

#### **INFERENCE WISHLIST?**

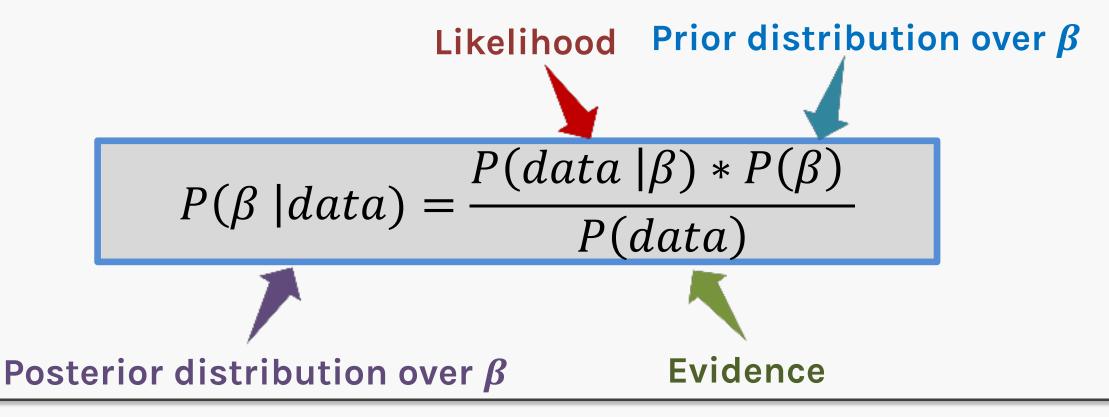
- We want the full probability distribution of the parameters given the data, i.e., let  $\beta = [\beta_0, \beta_1]$ , then we are looking for  $P(\beta \mid data)$ .
- We want the posterior predictive distribution of the output given the data. i.e., P(Y|X).

#### We could use Bayes Theorem

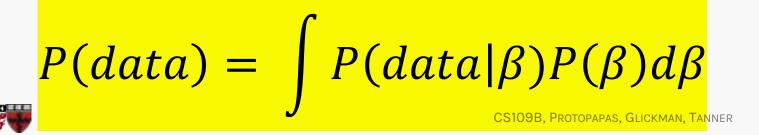
$$P(\beta \mid data) = \frac{P(data \mid \beta) * P(\beta)}{P(data)}$$



#### RECAP: Bayes Theorem

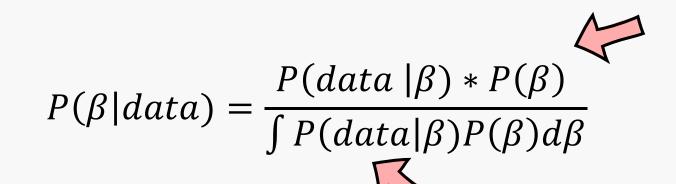


#### **Evidence**

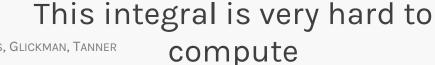


#### ISSUES WITH ANALYTICAL FORM

- Unless for very simple distributions, the **evidence**, P(data) is hard to compute.
- It is difficult to express the posterior as a closed form, which we will need to perform inference.



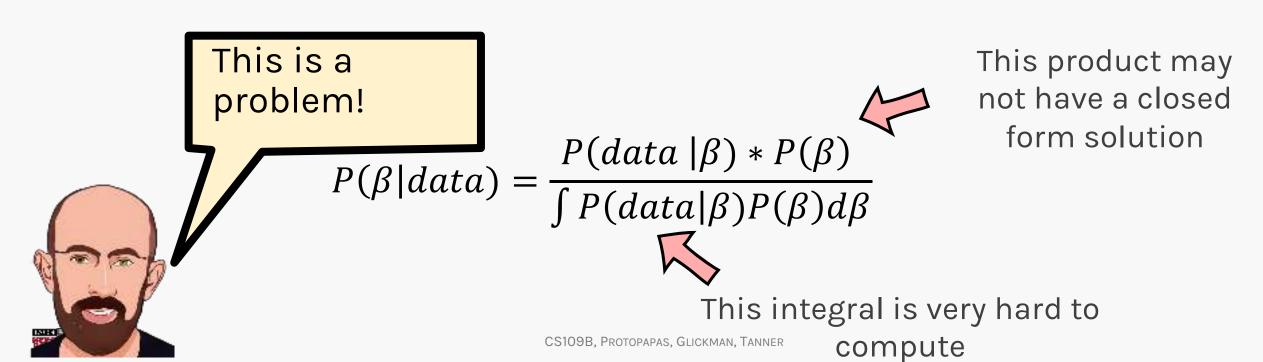
This product may not have a closed form solution





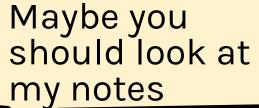
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$$P(\beta|data) = \frac{P(data|\beta) * P(\beta)}{\int P(data|\beta)P(\beta)d\beta}$$



This product may not have a closed form solution



This integral is very hard to CS109B, PROTOPAPAS, GLICKMAN, TANNER compute



#### RECAP: Bayes Theorem

$$P(\beta|data) = \frac{P(data|\beta) * P(\beta)}{\int P(data|\beta)P(\beta)d\beta}$$

$$P(\beta|data) \propto P(data|\beta) * P(\beta)$$

For any **given** value of  $\beta_0$ ,  $\beta_1$  we can find the posterior probability **up to a proportionality constant**:

$$P(\beta|data) = C * P(data|\beta) * P(\beta)$$



#### RECAP: Bayes Theorem

For any **given** value of  $\beta_0$ ,  $\beta_1$  we can find the posterior probability **up to a proportionality constant**:

$$P(\beta|data) = C * P(data|\beta) * P(\beta)$$

#### LINEAR REGRESSION EXAMPLE

let: 
$$y = f(x) + \epsilon$$

where: 
$$f(x) = \beta_0 + \beta_1 x$$

Assume  $\beta_0 \& \beta_1$  have normally distributed priors ~  $\mathcal{N} \big( 0, \sigma_{0,1} \big)$  where  $\sigma_{0,1}$  are arbitrarily large enough variance.



### **RECAP: Bayes Linear Regression**

We assume that the likelihood  $P(data|\beta)$  is also normal,  $\mathcal{N}(\beta_0 + \beta_1 x, \sigma_D)$ Hence, for given values of parameters,  $\beta_0, \beta_1$ , and for each  $(x_i, y_i) \in data$ 

$$P((x_n, y_n)|\beta) = \frac{1}{\sqrt{2\pi\sigma_D^2}} e^{-\left(\frac{(y_n - (\beta_o + \beta_1 x_n))^2}{2\sigma_D^2}\right)}$$

$$P(data|\boldsymbol{\beta}) = \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi\sigma_D^2}} e^{-\left(\frac{(y_n - (\beta_o + \beta_1 x_n))^2}{2\sigma_D^2}\right)}$$



### **RECAP: Bayes Linear Regression**

Calculating the prior distribution is much easier for the chosen values of  $eta_o \& eta_1$ 

$$P(\beta) = \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{\beta_0^2}{2\sigma_0^2}} * \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{\beta_1^2}{2\sigma_1^2}}$$

#### TECHNICAL DETAIL

Here we assume that the parameters  $\beta_0 \& \beta_1$  are independent random variables, hence the joint distribution  $(\beta_0, \beta_1)$  is the product of the individual distributions



## Putting it all together

$$P(\beta|data) = C * P(data|\beta^i) * P(\beta^i)$$



P(β|data) Now we can crunch some numbers

$$\frac{(y_n - (\beta_o + \beta_1 x_n))^2}{2\sigma_D^2}$$

$$*\frac{1}{\sqrt{2\pi\sigma_0^2}}e^{-\frac{\beta_0^2}{2\sigma_0^2}}*\frac{1}{\sqrt{2\pi\sigma_1^2}}e^{-\frac{\beta_1^2}{2\sigma_1^2}}$$

Proportionality constant





#### **INFERENCE WISHLIST?**

Even if we cannot get the analytical solution for the parameters, we can compare how likely a given value of  $\beta^{(i)}$  is to another value  $\beta^{(j)}$  using the equations from before

$$\frac{P(\beta^{(i)}|data)}{P(\beta^{(j)}|data)} = \frac{P(data|\beta^{(i)}) * P(\beta^{(i)})}{P(data|\beta^{(j)}) * P(\beta^{(j)})}$$

This way, we are still using Bayesian inference, and we can work with equations that are computable.



## A (very) brief recap of sampling

$$\frac{P(\beta^{(i)}|data)}{P(\beta^{(j)}|data)} = \frac{P(data|\beta^{(i)}) * P(\beta^{(i)})}{P(data|\beta^{(j)}) * P(\beta^{(j)})}$$

- Sampling methods using Bayesian inference with relative comparison between candidate points are popular methods to calculate para But how can we distributions.
- Popular sampling methods include Monte Carlo sampling and Markov Chain Monte Carlo (MCMC) sampling methods.
- We will consider a much popular variant of MCMC, Metropolis ak modern walk for our discussion, but other variants can also be used to d posterior distributions.

find the

parameter

distributions by such a

comparison?

## MCMC Sampling



## MCMC - Metropolis Hastings

Metropolis-Hastings is very powerful and a widely-used sampler.

If we are looking for the distribution for some parameter,  $\theta$ , it reduces the problem of sampling from a difficult distribution  $p(\theta|data)$  to:

- making proposals:  $q(\theta^{(j)}|\theta^{(j-1)})$
- evaluating ratios:  $\frac{p(\theta^*|data)/q(\theta^*|\theta^{(j-1)})}{p(\theta^{(j-1)}|data)/q(\theta^{(j-1)})|\theta^*)}$

The proposals can be trivial, for e.g., random walk (choose  $\theta^{(j)}$  from a normal distribution centered at  $\theta^{(j-1)}$ ).

Note: Only efficiency, not correctness is affected by the proposal.



### Metropolis Hastings

#### **TECHNICAL DETAILS**

The Metropolis-Hastings algorithm is outlined below:

- 1. Select an initial value  $\theta_0$
- 2. For i = 1, ..., m repeat:
  - a) Draw a candidate  $\theta^* \sim q(\theta^* | \theta^{(j-1)})$

b) 
$$\alpha = \frac{p(\theta^*|data)/q(\theta^*|\theta^{(j-1)})}{p(\theta^{(j-1)}|data)/q(\theta^{(j-1)})|\theta^*)} = \frac{p(\theta^*|data)q(\theta^{(j-1)})|\theta^*)}{p(\theta^{(j-1)}|data)q(\theta^*|\theta^{(j-1)})}$$

c) If  $\alpha \geq 1$ , accept  $\theta^*$  & set  $\theta^{(j)} \leftarrow \theta^*$ 

Else if  $0 < \alpha < 1$  accept  $\theta^* \& \text{set } \theta^{(j)} \leftarrow \theta^*$  with probability  $\eta$  reject  $\theta^* \& \text{set } \theta^{(j)} \leftarrow \theta^*$  with probability  $1 - \eta$ 



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q(.) is the proposal. For example,  $\theta^* \sim \mathrm{N}(\theta^{(j-1)},\sigma)$ 

How? Draw a random number,  $\rho$ . If  $\rho > (1-\alpha)$  accept.



## Metropolis Hastings - Random Walk

A special case of the Metropolis-Hastings algorithm is the Metropolis where the distribution q is symmetric i.e.,

$$q(\theta^* | \theta^{(j-1)}) = q(\theta^{(j-1)} | \theta^*)$$

This makes the algorithm much simpler, as

$$\alpha = \frac{p(\theta^*|data)q(\theta^{(j-1)})|\theta^*)}{p(\theta^{(j-1)}|data)q(\theta^*|\theta^{(j-1)})}$$



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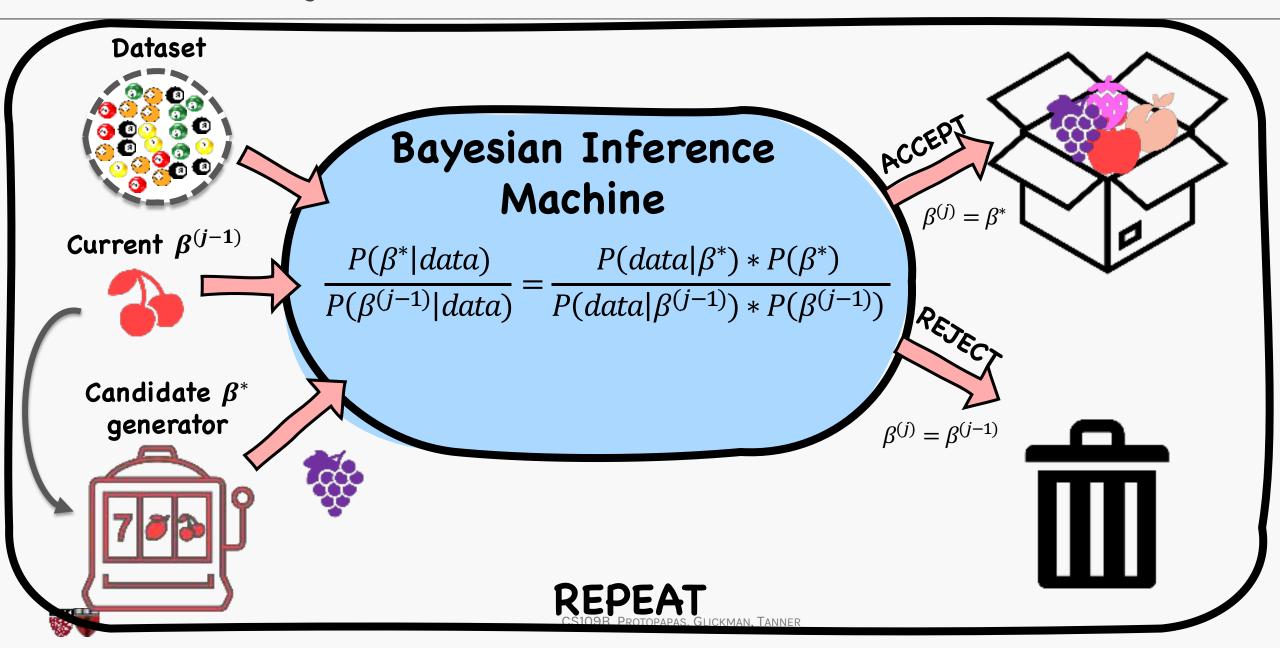
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We can show using Bayes theorem:

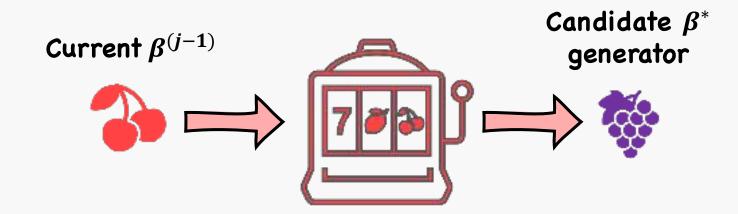
$$\alpha = \frac{p(data|\theta^*)}{p(data|\theta^{(j-1)})} \frac{p(\theta^*)}{p(\theta^{(j-1)})}$$



### **MCMC** Factory



### MCMC Factory - What is a generator



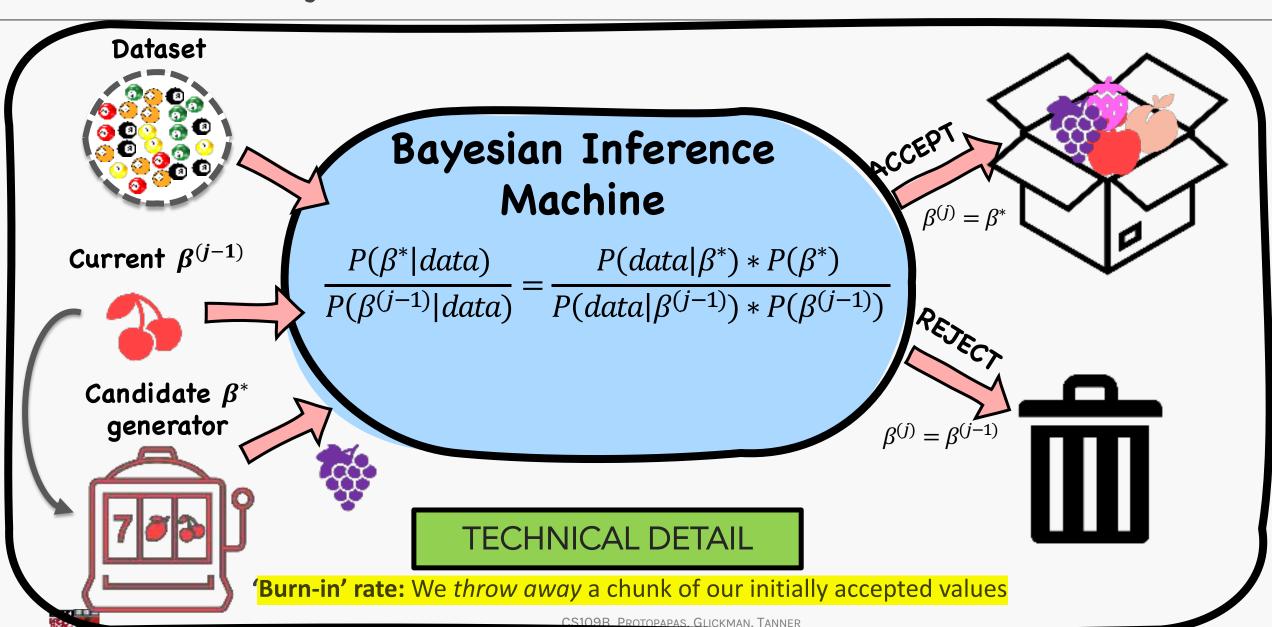
• The candidate  $\pmb{\beta}^*$  generator takes the currently accepted  $\beta^{(j-1)}$  value and draws a value from a distribution, for example:

$$\beta^* \sim \mathcal{N}(\beta^{(j-1)}, \sigma)$$

• This ensures that we don't get wildly crazy candidates, but ones quite similar to the current  $\beta$ .

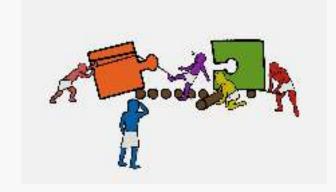


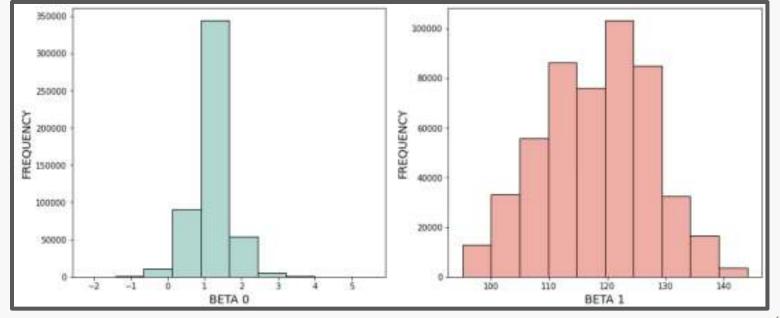
### **MCMC** Factory



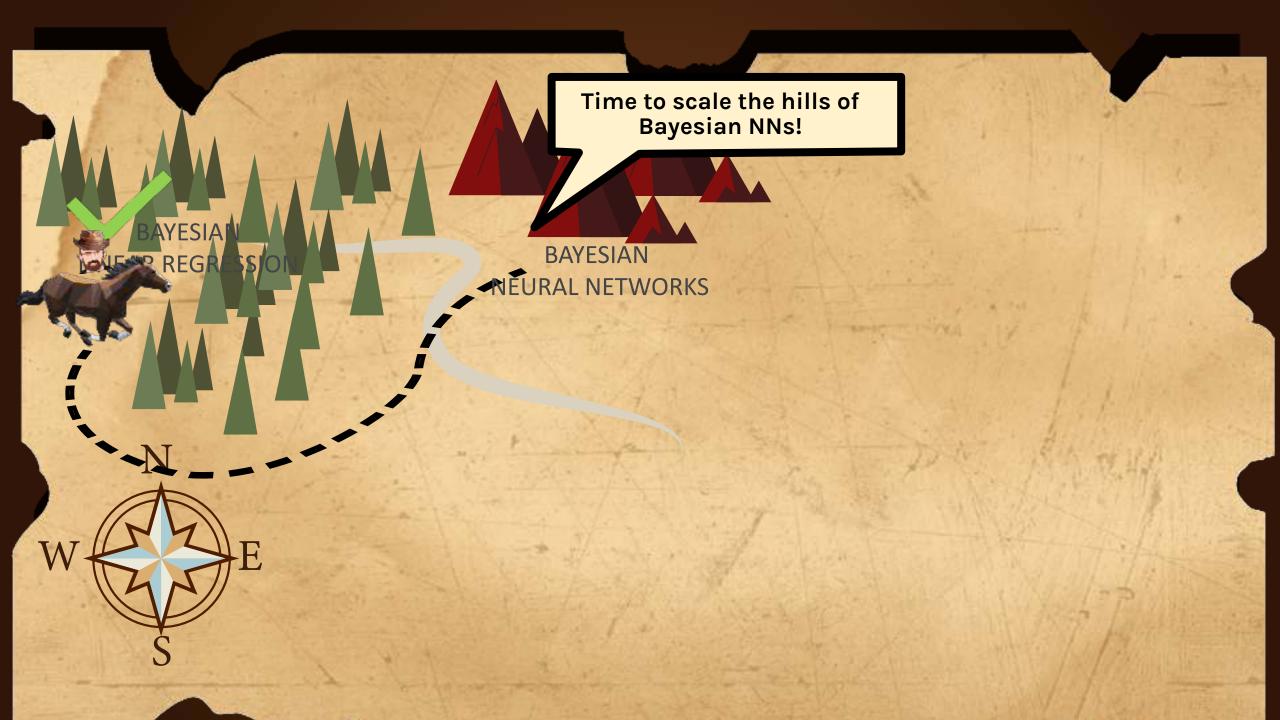
#### Exercise: Linear Regression MCMC from scratch

The aim of this exercise is to perform Monte Carlo Markov Chain (MCMC) from scratch for linear regression. On completing the exercise you should be able to see the following distribution. One for each of the beta value:









## Bayesian Neural Network



## Bayesian Linear Regression

We assume that the likelihood  $P(data|\beta)$  is also normal,  $\mathcal{N}(\beta_0 + \beta_1 x, \sigma_D)$ Hence, for given values of parameters,  $\beta_0, \beta_1$ , and for each  $(x_i, y_i) \in data$ 

$$P(\{x_n, y_n\} | \boldsymbol{\beta}) = \frac{1}{\sqrt{2\pi\sigma_D^2}} e^{-\left(\frac{(y_n - (\beta_o + \beta_1 x_n))^2}{2\sigma_D^2}\right)}$$

$$P(data|\boldsymbol{\beta}) = \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi\sigma_D^2}} e^{-\left(\frac{(y_n - (\beta_o + \beta_1 x_n))^2}{2\sigma_D^2}\right)}$$



## Bayesian Neural Network

We assume that the likelihood  $P(data|\beta)$  is also normal,  $\mathcal{N}(NN_W(x), \sigma_D)$ 

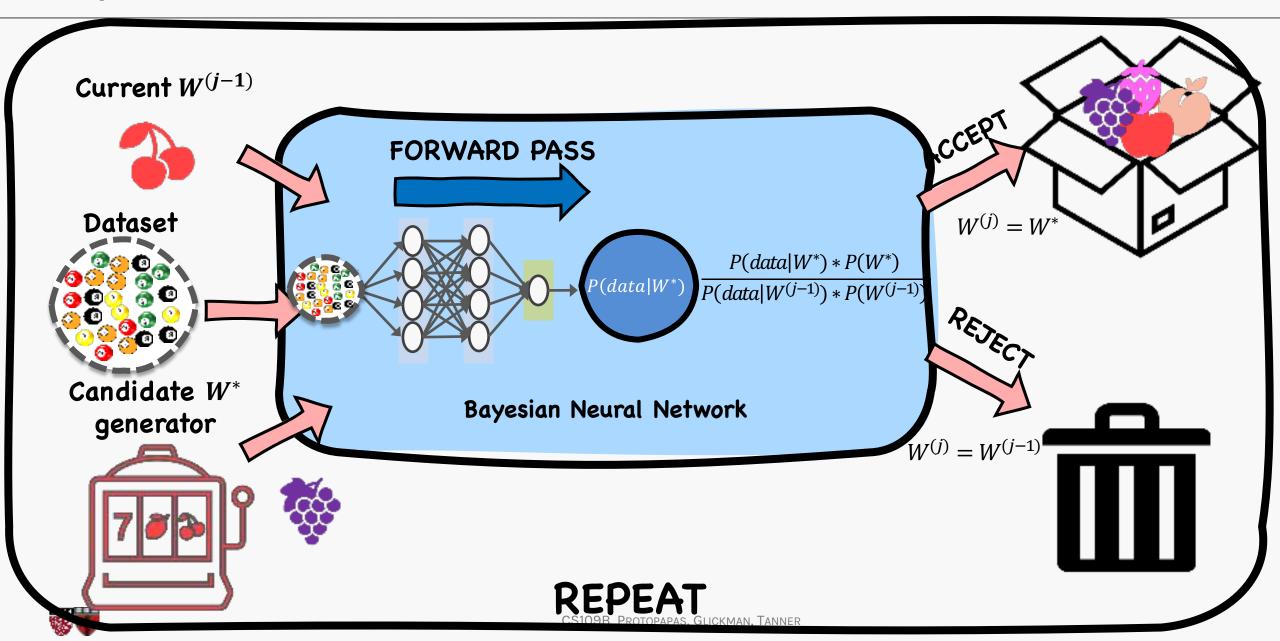
Hence, for given values of parameters, W, and for each  $(x_i, y_i) \in data$ 

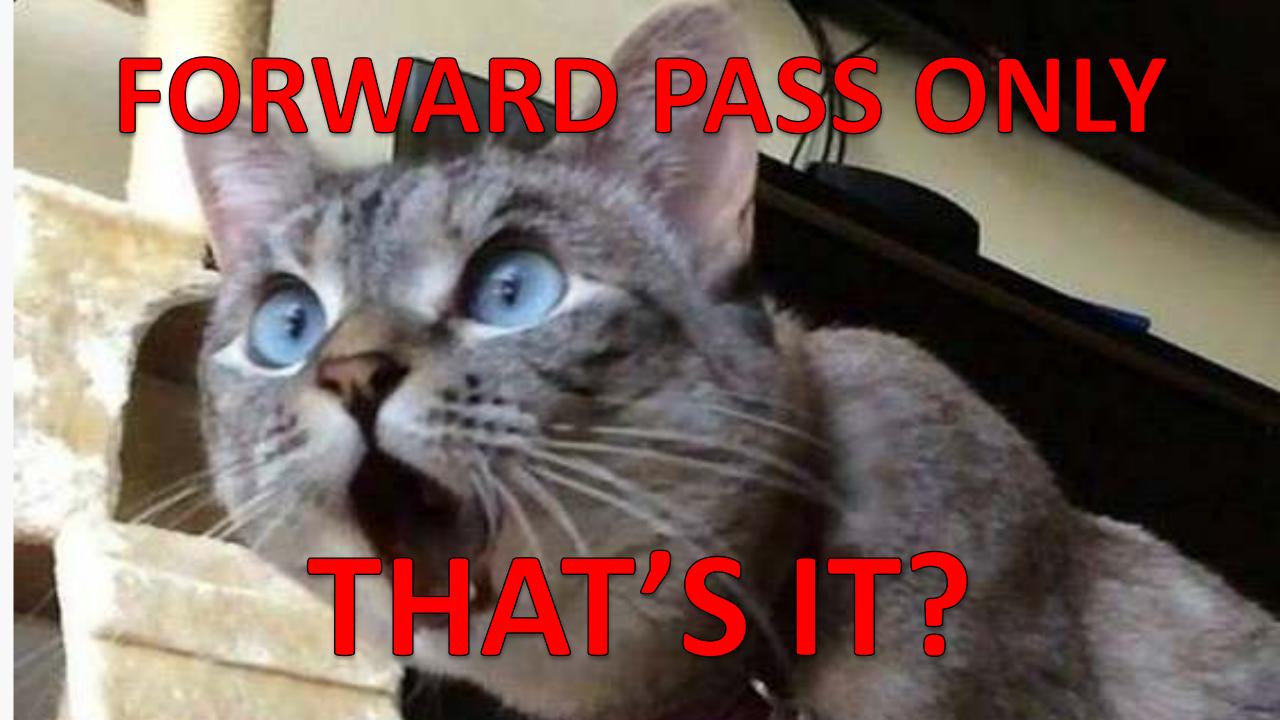
$$P(\{x_n, y_n\}|W) = \frac{1}{\sqrt{2\pi\sigma_D^2}} e^{-\left(\frac{(y_n(x_n))^2}{2}\right)}$$

$$P(data|W) = \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi\sigma_D^2}} e^{-\left(\frac{(y_n - NN_W(x_n))^2}{2}\right)}$$

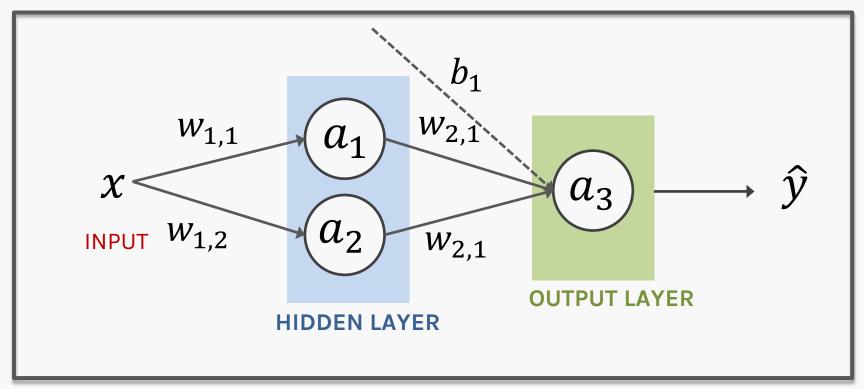


## Bayesian Neural Network

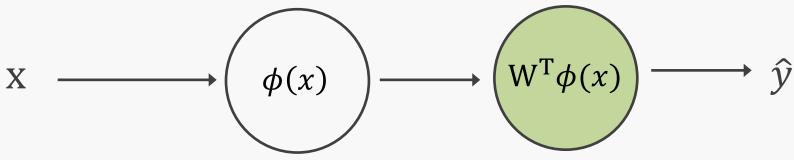




#### Example: A simple Bayesian Neural Network (BNN)

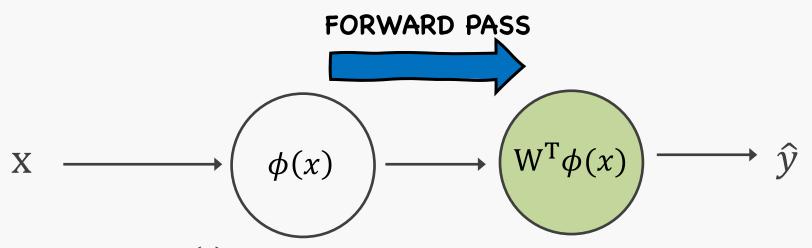


#### Neural Network Architecture





#### Bayesian Neural Network (BNN)



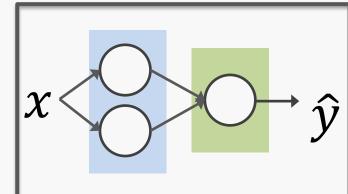
- Select an initial value  $W^{(0)}$  (this represent all weights in the network)
- 2. For i = 1, ..., m repeat:
  - a) Draw a candidate  $W^* \sim q(W^*|W^{(j-1)})$

b) 
$$\alpha = \frac{p(data|W^*)}{p(data|W^{(j-1)})} \frac{p(W^*)}{p(W^{(j-1)})}$$

c) If  $\alpha \geq 1$ , accept  $W^*$  & set  $W^{(j)} \leftarrow W^*$ else if  $0 < \alpha < 1$  accept  $W^*$  & set  $W^{(j)} \leftarrow W^*$  with probability  $\alpha$ 

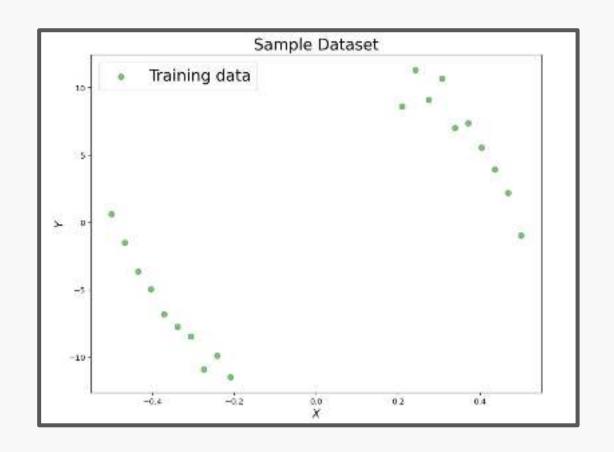






## Bayesian Neural Network

Let us consider the dataset below for regression





## **Exercise:** MCMC from Scratch for Neural Networks

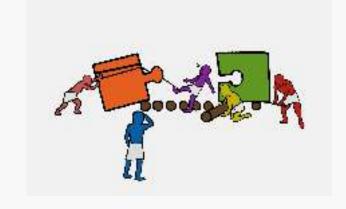
etworks

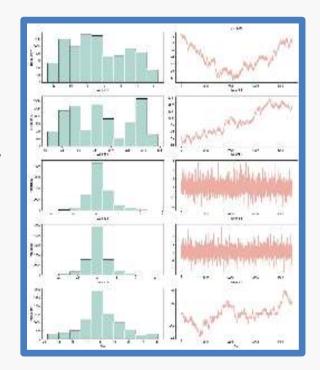
e aim of this exercise is to perform Metropolis Monte

The aim of this exercise is to perform Metropolis Monte Carlo Markov Chain (MCMC) from scratch (as in exercise 1) for a simple neural network.

On completing the exercise, you should be able to see the following distribution. One for each of the beta value:

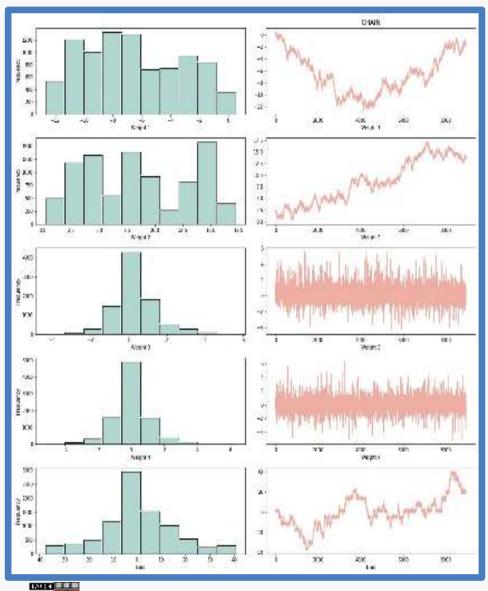
Warning: This is not going to converge unless we start very near the mode of the distribution!







#### **FULL MCMC ISSUES**



MCMC we will not work for NN's with more than a few dozens of parameters.

#### Why?

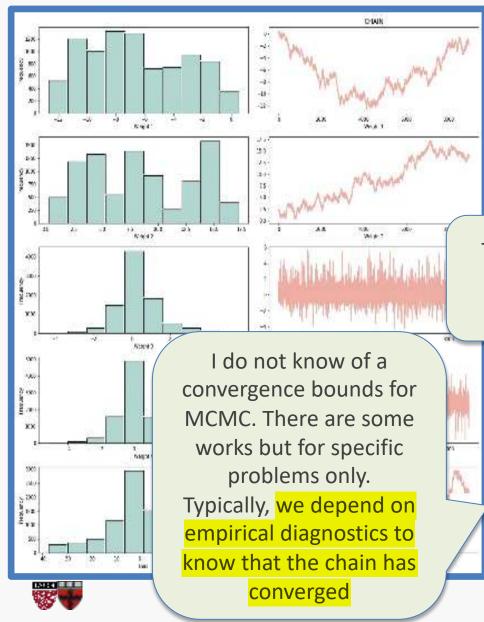
For each parameter (weight) we sample and calculate the likelihood 5\*n times, where n being the length of the chain.

We also throw away a significant number of samples.

Sufficient n also grows with the number of parameters and complexity of the posterior.



#### **FULL MCMC ISSUES**



MCMC we will not work for NN's with more than a few dozens of parameters.

Why?

For each parameter we sample and

Typically, we throw the first 10-20% of the samples for burn in.

Five times because we usually have acceptance rate of  $\sim 20\%$ .

elihood 5 \* n times, where n being chain.

We also throw away a significant number of samples.

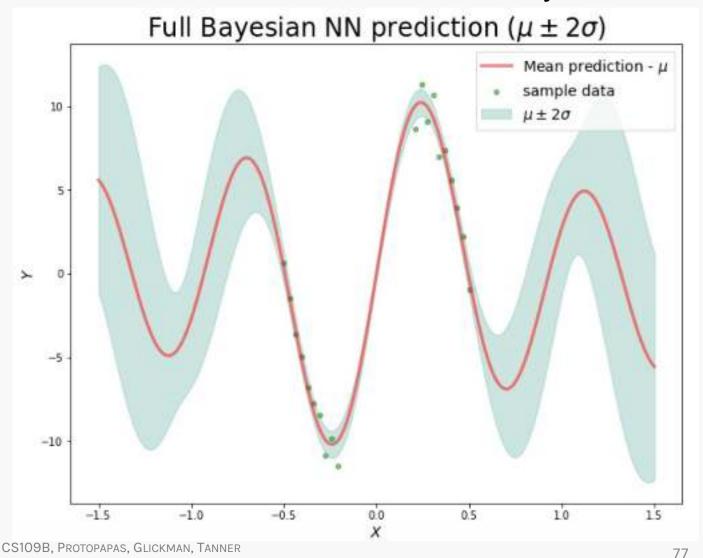
Sufficient n also grows with the number of parameters and complexity of the posterior.

#### Bayesian Neural Network: Full MCMC with HMC

Instead of the simple Metropolis MCMC, we could use a more sophisticated sampler HMC, as we did in HW3 and lab.

Here with Tensorflow Probability

#### POSTERIOR DISTRIBUTION y

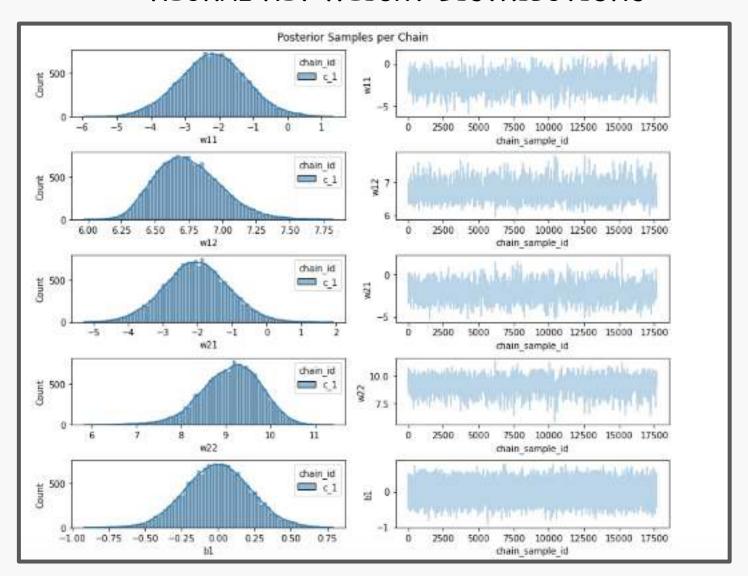




#### Bayesian Neural Network: Full MCMC with HMC

- The first set of plots on the right represents the distribution of the individual weights.
- The second set of plots on the right show the individual values of the weights considered while running the sampling algorithm.

#### NEURAL NET WEIGHT DISTRIBUTIONS



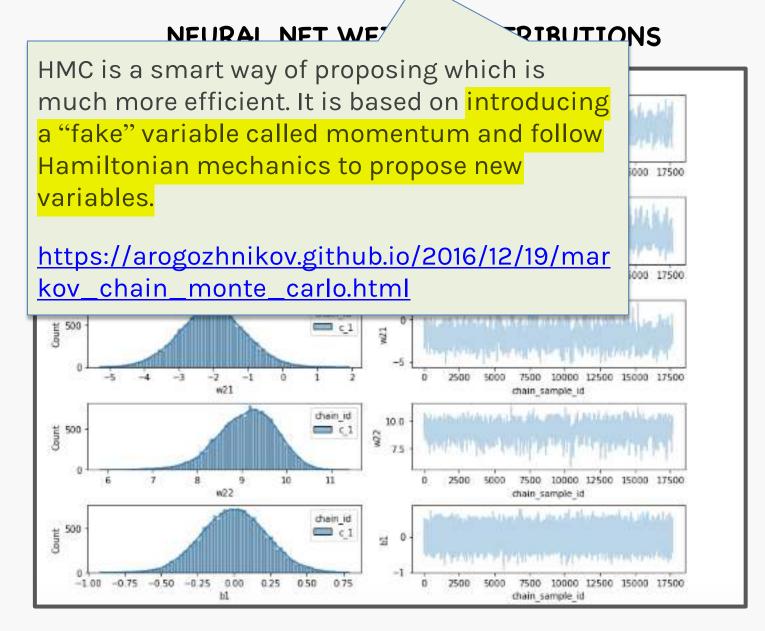


#### Bayesian Neural Network: Full MCMC with HMC

- The first set of plots on the right represents the distribution of the individual weights.
- The second set of plots on the right show the individual values of the weights considered while running the sampling algorithm.

Activation: sin()

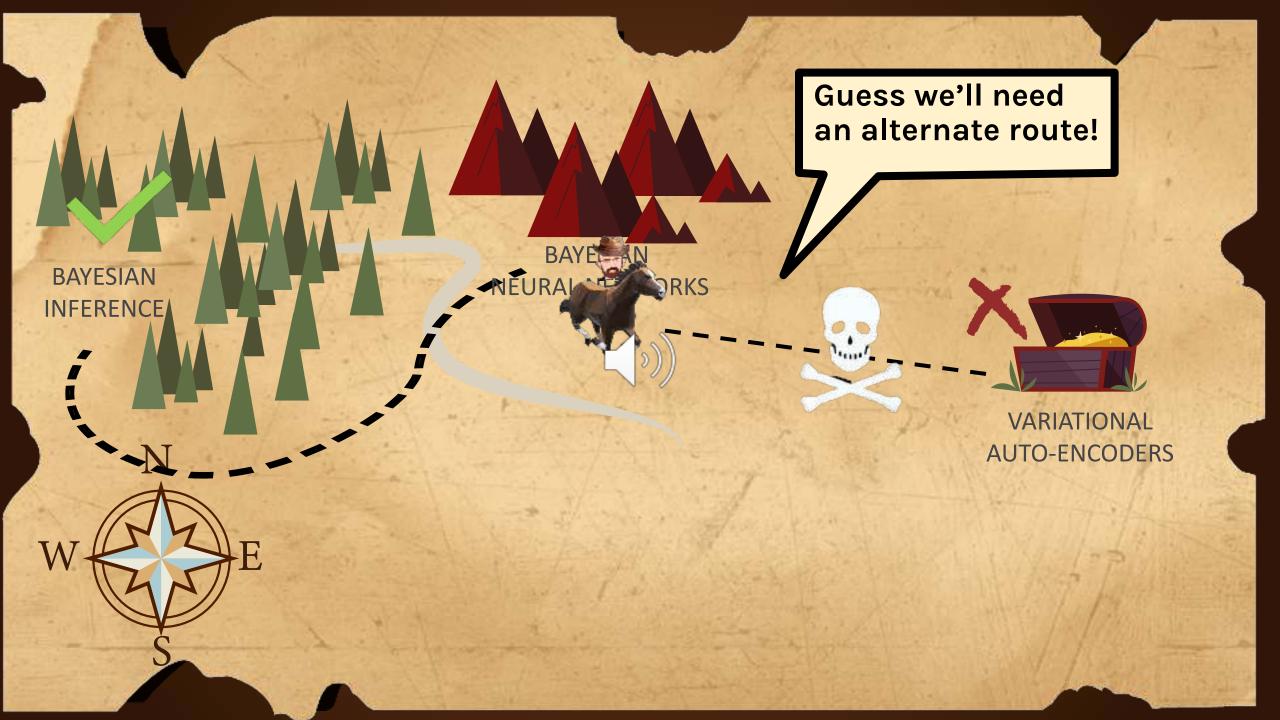
Burn rate: 30%





# Bayesian Neural Network – Roadblocks MCMC doesn't work for higher dimension

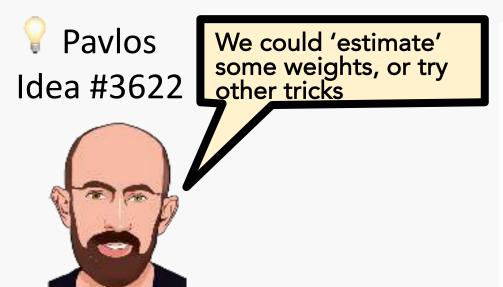




#### End of Part 1



#### Bayesian Neural Network: Hacks



 $W_{11} \& W_{12}$  ML estimate

