3D Magnetoconvection Code

Variables : $\vec{b}$, $\vec{u}$, θ

$$\vec{b} = \nabla \times (g \hat{r}) + \nabla \times \nabla \times (h \hat{r})$$

$$\vec{u} = \nabla \times (e \hat{r}) + \nabla \times \nabla \times (f \hat{r})$$

Expand as

$$g = g_c(r) \, P_l^m(\cos\theta) \cos(m\phi) + g_s(r) \, P_l^m(\cos\theta) \sin(m\phi)$$

$$h = h_c(r) \quad \text{''} \quad \text{''} \quad + h_s(r) \quad \text{''} \quad \text{''}$$

$$e = e_c(r) \quad \text{''} \quad \text{''} \quad + e_s(r) \quad \text{''} \quad \text{''}$$

$$f = f_c(r) \quad \text{''} \quad \text{''} \quad + f_s(r) \quad \text{''} \quad \text{''}$$

$$\theta = \theta_c(r) \quad \text{''} \quad \text{''} \quad + \theta_s(r) \quad \text{''} \quad \text{''}$$

Consider, for example, $g_c(r)\, P_l^m(\cos\theta)\, \cos(m\phi)$.

m runs from 0 to MB.

l then runs from m to LB. $(LB \geq MB)$

(but for $m=0$ l starts at 1)

Also expand $g_c(r) = \sum_{k=1}^{KB2} T_{k-1}(x)$. $(KB2 = KB+2)$

$GC(k,l,m)$

So, dimension $GC(KB2, LB, 0{:}MB)$,

and similarly for GS, HC, HS.

Note, though, that this is wasteful in l, but with "triangular" truncation this can't be avoided.

With these spectral coefficients so defined,

$$g = \sum_{m=0}^{MB} \sum_{l=m}^{LB} \sum_{k=1}^{KB2} GC(k,l,m)\, T_{k-1}(x)\, P_l^m(\cos\theta)\, \cos(m\phi)$$

start at
1 for $m=0$

$+ \ldots$

$\sin(m\phi)$

Induction Equation

$$\frac{\partial}{\partial t}\vec{b} - \nabla^2 \vec{b} = \nabla \times (\vec{u} \times \vec{b})$$

With $\vec{b}$ expanded in terms of g and h, and g and h expanded in terms of spherical harmonics as above, the induction equation becomes:

call these spectral coeffs DHC and DHS

$$\sum_{l,m} \frac{l(l+1)}{r^2}\left[\frac{\partial h_{lm}}{\partial t} - \frac{\partial^2}{\partial r^2} h_{lm} + \frac{l(l+1)}{r^2} h_{lm}\right] P_l^m e^{im\phi} = \hat{r} \cdot \nabla \times (\vec{u} \times \vec{b})$$

DGC and DGS

$$-\sum_{l,m} \frac{l(l+1)}{r^2}\left[\frac{\partial g_{lm}}{\partial t} - \frac{\partial^2}{\partial r^2} g_{lm} + \frac{l(l+1)}{r^2} g_{lm}\right] P_l^m e^{im\phi} = \hat{r} \cdot \nabla \times \nabla \times (\vec{u} \times \vec{b})$$

So, given the spectral coefficients for $\vec{u}$ and $\vec{b}$, we compute the nonlinear term $\vec{u} \times \vec{b}$ at a set of collocation points, then convert back to spectral space and do the curls, to obtain finally the spectral coefficients of the right-hand sides.

Solve equations of the general type $\frac{\partial y}{\partial t} - D^2 y = F(y)$ by a modified 2^{nd} order Runge-Kutta method:

$$D^2 = \frac{\partial^2}{\partial r^2} - \frac{\ell(\ell+1)}{r^2}$$

Given y_n:

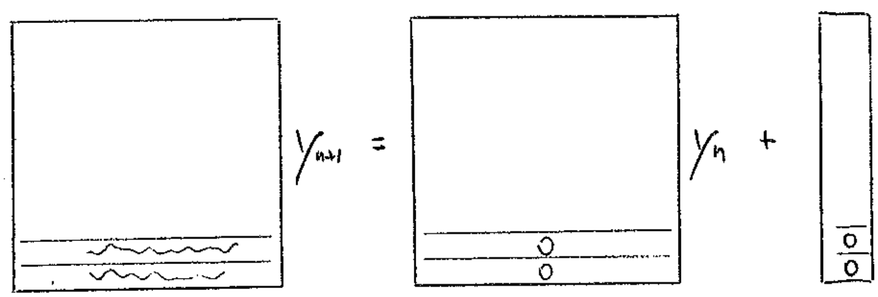
① $f_n = F(y_n)$

② $(\tilde{y}_{n+1} - y_n)/\Delta t - D^2(\tilde{y}_{n+1} + y_n)/2 = f_n$

$$(I_1 - \tfrac{1}{2}\Delta t D^2)\tilde{y}_{n+1} = (I_2 + \tfrac{1}{2}\Delta t D^2) y_n + \Delta t \cdot f_n$$

③ $\tilde{f}_{n+1} = F(\tilde{y}_{n+1})$

④ $(I_1 - \tfrac{1}{2}\Delta t D^2) y_{n+1} = (I_2 + \tfrac{1}{2}\Delta t D^2) y_n + \tfrac{1}{2}\Delta t (f_n + \tilde{f}_{n+1})$



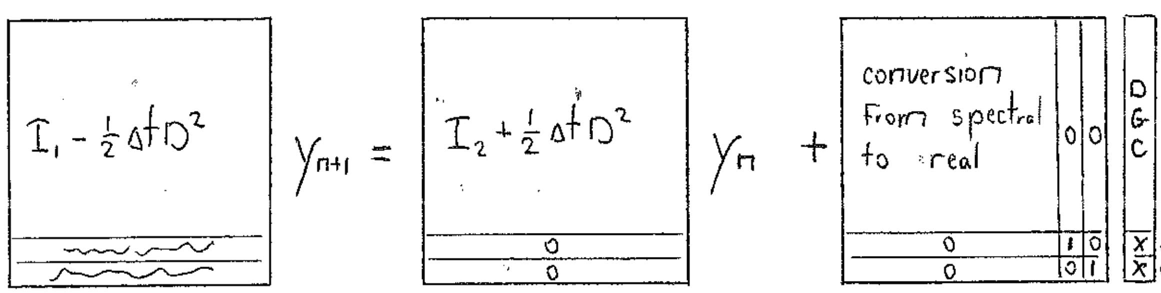
last two rows impose b.c.'s by setting some appropriate linear combination of the spectral coefficients equal to zero

The first KB rows are given by

$$\frac{\ell(\ell+1)}{r_i^2}\left[T_{j-1}(x_i) - \frac{1}{2}\Delta t\left(\frac{d^2}{dr^2}T_{j-1}(x_i) - \frac{\ell(\ell+1)}{r_i^2}T_{j-1}(x_i)\right)\right]$$

where the rows i denote the appropriate collocation points, and the columns j the expansion functions.

```
      DO 10 M=0,MB
       LS=MAX(M,1)
       DO 10 L=LS,LB
        DO 10 IK=1,KB2          <- WGC1=(I_1 - 1/2 Δt D^2)^-1 (I_2 + 1/2 Δt D^2) y_n
         WGC1(IK,L,M)=0.D0
         DO 10 K=1,KB2
          WGC1(IK,L,M)=WGC1(IK,L,M)  +  GMT1(K,IK,L)*GC(K,L,M)
10    CONTINUE
*
*
      CALL NONLIN  <- evaluate f_n
      DO 30 M=0,MB
       LS=MAX(M,1)                 inhomogeneous b.c.'s would be included here
       DO 30 L=LS,LB
        DO 20 IK=1,KB2                 WGC2 = (...)^-1 Δt · f_n        and here
         WGC2(IK,L,M)=0.D0
         DO 20 K=1,KB
          WGC2(IK,L,M)=WGC2(IK,L,M)  +  GMT2(K,IK,L)*DGC(K,L,M)
20      CONTINUE
        DO 30 K=1,KB2
         GC(K,L,M)=WGC1(K,L,M)  +  WGC2(K,L,M) <- obtain ỹ_{n+1}
30    CONTINUE
*
*
      CALL NONLIN  <- evaluate f̃_{n+1}
      DO 60 M=0,MB
       LS=MAX(M,1)
       DO 60 L=LS,LB
        DO 50 IK=1,KB2
         DO 40 K=1,KB                WGC2 = (...)^-1 1/2 Δt · (f_n + f̃_{n+1})
          WGC2(IK,L,M)=WGC2(IK,L,M)  +  GMT2(K,IK,L)*DGC(K,L,M)
40       CONTINUE
         WGC2(IK,L,M)=0.5D0*WGC2(IK,L,M)
50      CONTINUE
        DO 60 K=1,KB2
         GC(K,L,M)=WGC1(K,L,M)  +  WGC2(K,L,M) <- obtain y_{n+1}
60    CONTINUE
```



So, precompute $(I_1 - \frac{1}{2}\Delta t D^2)^{-1}(I_2 + \frac{1}{2}\Delta t D^2)$

and $(I_1 - \frac{1}{2}\Delta t D^2)^{-1}\, T$

include inhomogeneous b.c.'s here

$$b_r = l(l+1)\,\frac{1}{r^2}\,h_c \cdot P_l^m \cdot \cos(m\phi)$$
$$+\, l(l+1)\,\frac{1}{r^2}\,h_s \cdot P_l^m \cdot \sin(m\phi)$$

$$b_\theta = \frac{1}{r}\,h_c' \cdot \frac{d}{d\theta} P_l^m \cdot \cos(m\phi) - \frac{1}{r}\,g_c \cdot \frac{1}{s} P_l^m \cdot m\sin(m\phi)$$
$$+ \frac{1}{r}\,h_s' \cdot \frac{d}{d\theta} P_l^m \cdot \sin(m\phi) + \frac{1}{r}\,g_s \cdot \frac{1}{s} P_l^m \cdot m\cos(m\phi)$$

$$b_\phi = -\frac{1}{r}\,h_c' \cdot \frac{1}{s} P_l^m \cdot m\sin(m\phi) - \frac{1}{r}\,g_c \cdot \frac{d}{d\theta} P_l^m \cdot \cos(m\phi)$$
$$+ \frac{1}{r}\,h_s' \cdot \frac{1}{s} P_l^m \cdot m\cos(m\phi) - \frac{1}{r}\,g_s \cdot \frac{d}{d\theta} P_l^m \cdot \sin(m\phi)$$

$$u_r = l(l+1)\,\frac{1}{r^2}\,f_c \cdot P_l^m \cdot \cos(m\phi)$$
$$+\, l(l+1)\,\frac{1}{r^2}\,f_s \cdot P_l^m \cdot \sin(m\phi)$$

$$u_\theta = \frac{1}{r}\,f_c' \cdot \frac{d}{d\theta} P_l^m \cdot \cos(m\phi) - \frac{1}{r}\,e_c \cdot \frac{1}{s} P_l^m \cdot m\sin(m\phi)$$
$$+ \frac{1}{r}\,f_s' \cdot \frac{d}{d\theta} P_l^m \cdot \sin(m\phi) + \frac{1}{r}\,e_s \cdot \frac{1}{s} P_l^m \cdot m\cos(m\phi)$$

$$u_\phi = -\frac{1}{r}\,f_c' \cdot \frac{1}{s} P_l^m \cdot m\sin(m\phi) - \frac{1}{r}\,e_c \cdot \frac{d}{d\theta} P_l^m \cdot \cos(m\phi)$$

Given the spectral coefficients for g and h, compute $\vec{b}$ at a set of collocation points:

$$b_r = \sum_{k l m} HC(k,l,m) \cdot \left(\frac{1}{r^2} T_{k-1}(x)\right) \cdot \left(l(l+1) P_l^m(\cos\theta)\right) \cdot \begin{matrix}\cos(m\phi) \\ \sin(m\phi)\end{matrix}$$

$$+ \dots$$

At the collocation points (x_{ik}, $ik = 1, KN$; θ_{il}, $il = 1, LN$; ϕ_{im}, $im = 1, 2 \cdot MN$) precompute the arrays

$$\left.\begin{aligned} TT1(k, ik) &= \frac{1}{r^2} T_{k-1}(x) \\ TT2(k, ik) &= \frac{1}{r} \frac{d}{dr} T_{k-1}(x) \\ TT3(k, ik) &= \frac{1}{r} T_{k-1}(x) \end{aligned}\right\} \text{ at } x = x_{ik}\text{, the } KN \text{ zeros of } T_{KN}$$

$$\left.\begin{aligned} PP1(l, m, il) &= l(l+1) P_l^m \\ PP2(l, m, il) &= \frac{d}{d\theta} P_l^m \\ PP3(l, m, il) &= \frac{1}{\sin\theta} P_l^m \end{aligned}\right\} \text{ at } \theta = \theta_{il}\text{, the } LN \text{ zeros of } P_{LN}^0$$

$$\left.\begin{aligned} CS(m, im) &= \cos(m\phi) \\ SN(m, im) &= \sin(m\phi) \end{aligned}\right\} \text{ at } \phi = \phi_{im}$$

Once these arrays have been precomputed,

$$b_r(x_{ik}, \theta_{il}, \phi_{im})$$

$$= \sum_{klm} HC(k,l,m) \cdot TT1(k,ik) \cdot PP1(l,m,il) \cdot CS(m,im)$$

$$+ \sum_{klm} HS(k,l,m) \cdot TT1(k,ik) \cdot PP1(l,m,il) \cdot SN(m,im)$$

and similarly for b_θ and b_ϕ.

Define work arrays

$$WHC1(ik,l,m) = \sum_k HC(k,l,m) \cdot TT1(k,ik)$$

$$WHS1(ik,l,m) = \ldots$$

then

$$b_r = \sum_{lm} WHC1(ik,l,m) \cdot PP1(l,m,il) \cdot CS(m,im)$$

$$+ \sum_{lm} WHS1(ik,l,m) \cdot \ldots$$

Define more work arrays

$$WHC11(ik,il,m) = \sum WHC1(ik,l,m) \cdot PP1(l,m,il)$$

then

$$b_r = \sum_m WHC11(ik, il, m) \cdot CS(m, im)$$

$$+ WHS11(ik, il, m) \cdot SN(m, im)$$

Actually, the do loops can be rearranged so that these work arrays only need to be subscripted

WHC1(l,m) ...

WHC11(m) ...

so save some memory by doing that.

Given $\vec{b}$ and $\vec{u}$ at the collocation points, compute $\vec{u} \times \vec{b}$, and separate out the different azimuthal modes again (easy using slow Fourier transform).

So, for each azimuthal mode m, we have the three components of $\vec{u} \times \vec{b}$ at the collocation points ik, il. Then compute $\hat{r} \cdot \nabla \times (\vec{u} \times \vec{b})$ and $\hat{r} \cdot \nabla \times \nabla \times (\vec{u} \times \vec{b})$ as follows:

Given $\vec{F} = \vec{u} \times \vec{b}$ at the collocation points, it will have a spectral expansion of the form

$$F1C(ik,il,m) = \sum_{k,l} F1C^*(k,l,m) \cdot T_{k-1}(x_{ik}) \cdot P_l^m(\cos\theta_{il})$$

$$F1S(ik,il,m) = \ldots$$

$$F2C(ik,il,m) = \sum_{k,l} F2C^*(k,l,m) \cdot T_{k-1}(x_{ik}) \cdot \frac{1}{\sin\theta_{il}} P_l^m(\cos\theta_{il})$$

$$\ldots$$

$$F3C(ik,il,m) = \sum_{k,l} F3C^*(k,l,m) \cdot T_{k-1}(x_{ik}) \cdot \frac{1}{\sin\theta_{il}} P_l^m(\cos\theta_{il})$$

$$\ldots$$

(Except for $m=0$, for which F2 and F3 are like $\sin\theta_{il} \cdot P_l^m(\cos\theta_{il})$)

Recalling that we have already done the Fourier transform in m, what we have are precisely the left-hand sides. We can therefore recover the spectral coefficients from

$$F1C^{*}_{klm} = T^{-1}_{k,ik} \cdot F1C_{ik,il,m} \cdot P^{-1}_{il,l,m}$$

$$F2C^{*}_{klm} = T^{-1}_{k,ik} \cdot F2C_{ik,il,m} \cdot s_{il}\, P^{-1}_{il,l,m}$$

$$F3C^{*}_{klm} = T^{-1}_{k,ik} \cdot F3C_{ik,il,m} \cdot s_{il}\, P^{-1}_{il,l,m}$$

where the real-to-spectral conversion matrices

$$T^{-1}_{k,ik} \equiv \text{inverse of } T_{ik,k} \equiv T_{k-1}(x_{ik})$$

$$P^{-1}_{il,l,m} \equiv \text{inverse of } P_{l,il,m} \equiv P^{m}_{l}(\cos\theta_{il})$$

are precomputed and stored.

invert separate for each m, of course

in this routine must

So, labelling $T^{-1}_{k,ik} = TINV$, $P^{-1}_{il,l,m} = PINV1$

and $s_{il}\, P^{-1}_{il,l,m} = PINV2$, for each m we have

ferent
- $m=0$

$$F1C^* = TINV \cdot F1C \cdot PINV1 \,, \quad F1S^* = \ldots$$

$$F2C^* = TINV \cdot F2C \cdot PINV2 \,, \quad F2S^* = \ldots$$

$$F3C^* = TINV \cdot F3C \cdot PINV2 \,, \quad F3S^* = \ldots$$

Given these spectral coefficients for $\vec{F}$, at the collocation points $\vec{G} = \nabla \times \vec{F}$ will be given by

$G1C(ik, il, m)$

$$= \sum_{k,l} F3C^*(k,l,m) \cdot \frac{1}{r} T_{k-1}(x) \cdot \frac{1}{s} \frac{d}{d\theta} P_l^m \cdot \cos(m\phi)$$

$$- F2S^*(k,l,m) \cdot \frac{1}{r} T_{k-1}(x) \cdot \frac{m}{s^2} P_l^m \cdot \cos(m\phi)$$

$G1S(ik, il, m)$

$$= \sum_{k,l} F3S^*(k,l,m) \cdot \frac{1}{r} T_{k-1}(x) \cdot \frac{1}{s} \frac{d}{d\theta} P_l^m \cdot \sin(m\phi)$$

$$+ F2C^*(k,l,m) \cdot \frac{1}{r} T_{k-1}(x) \cdot \frac{m}{s^2} P_l^m \cdot \sin(m\phi)$$

G2C(ik,il,m)

$$= \sum_{k,l} F1S^*(k,l,m) \cdot \frac{1}{r} T_{k-1}(x) \cdot \frac{m}{s} P_l^m \cdot \cos(m\phi)$$

$$- F3C^*(k,l,m) \cdot \frac{1}{r}\frac{d}{dr}\left(r T_{k-1}(x)\right) \cdot \frac{1}{s} P_l^m \cdot \cos(m\phi)$$

G2S(ik,il,m)

$$= -\sum_{k,l} F1C^*(k,l,m) \cdot \frac{1}{r} T_{k-1}(x) \cdot \frac{m}{s} P_l^m \cdot \sin(m\phi)$$

$$- F3S^*(k,l,m) \cdot \frac{1}{r}\frac{d}{dr}\left(r T_{k-1}(x)\right) \cdot \frac{1}{s} P_l^m \cdot \sin(m\phi)$$

G3C(ik,il,m)

$$= \sum_{k,l} F2C^*(k,l,m) \cdot \frac{1}{r}\frac{d}{dr}\left(r T_{k-1}(x)\right) \cdot \frac{1}{s} P_l^m \cdot \cos(m\phi)$$

$$- F1C^*(k,l,m) \cdot \frac{1}{r} T_{k-1}(x) \cdot \frac{d}{d\theta} P_l^m \cdot \cos(m\phi)$$

G3S(ik,il,m)

$$= \sum_{k,l} F2S^*(k,l,m) \cdot \frac{1}{r}\frac{d}{dr}\left(r T_{k-1}(x)\right) \cdot \frac{1}{s} P_l^m \cdot \sin(m\phi)$$

$$- F1S^*(k,l,m) \cdot \frac{1}{r} T_{k-1}(x) \cdot \frac{d}{d\theta} P_l^m \cdot \sin(m\phi)$$

Define the matrices

$$\left.\begin{aligned} T1_{ik,k} &= \frac{1}{r}\, T_{k-1}(x) \\[2ex] T2_{ik,k} &= \frac{1}{r}\frac{d}{dr}\left(r\, T_{k-1}(x)\right) \end{aligned}\right\} \text{ at } x = x_{ik}$$

fferent r m=0 →

$$\left.\begin{aligned} P1_{l,il,m} &= \frac{1}{s}\frac{d}{d\theta} P_l^m \\[2ex] P2_{l,il,m} &= \frac{m}{s^2}\, P_l^m \\[2ex] P3_{l,il,m} &= \frac{d}{d\theta} P_l^m \end{aligned}\right\} \text{ at } \theta = \theta_{il}$$

then for each m

$$G1C = T1 \cdot F3C^* \cdot P1 - T1 \cdot F2S^* \cdot P2$$

$$G1S = T1 \cdot F3S^* \cdot P1 + T1 \cdot F2C^* \cdot P2$$

$$G2C = T1 \cdot F1S^* \cdot \left(\frac{m}{s} P_l^m\right) - T2 \cdot F3C^* \cdot \left(\frac{1}{s} P_l^m\right)$$

$$G2S = -T1 \cdot F1C^* \cdot \left(\frac{m}{s} P_l^m\right) - T2 \cdot F3S^* \cdot \left(\frac{1}{s} P_l^m\right)$$

$$G3C = T2 \cdot F2C^* \cdot \left(\frac{1}{s} P_l^m\right) - T1 \cdot F1C^* \cdot P3$$

$$G3S = T2 \cdot F2S^* \cdot \left(\frac{1}{s} P_l^m\right) - T1 \cdot F1S^* \cdot P3$$

Finally, remembering that we know the F^*'s in terms of the F's, and in exactly the same way the G^*'s in terms of the G's, we have,

$$G1C^* = TINV \cdot [T1 \cdot (TINV \cdot F3C \cdot PINV2) \cdot P1] \cdot PINV1$$
$$- TINV \cdot [T1 \cdot (TINV \cdot F2S \cdot PINV2) \cdot P2] \cdot PINV1$$

$$G1S^* = \ldots$$

$$G2C^* = TINV \cdot T1 \cdot (TINV \cdot F1S \cdot PINV1) \cdot m$$
$$- TINV \cdot T2 \cdot (TINV \cdot F3C \cdot PINV2)$$

$$G2S^* = \ldots$$

$$G3C^* = TINV \cdot T2 \cdot (TINV \cdot F2C \cdot PINV2)$$
$$- TINV \cdot [T1 \cdot (TINV \cdot F1C \cdot PINV1) \cdot P3] \cdot PINV2$$

$$G3S^* = \ldots$$

So, define the matrices

$$TT1 = TINV \cdot T1 \cdot TINV$$

$$TT2 = TINV \cdot T2 \cdot TINV$$

$$PP1 = PINV2 \cdot P1 \cdot PINV1$$

$$PP2 = PINV2 \cdot P2 \cdot PINV1$$

$$PP3 = PINV1 \cdot P3 \cdot PINV2$$

then

$$G1C^* = TT1 \cdot F3C \cdot PP1 - TT1 \cdot F2S \cdot PP2$$

$$G1S^* = TT1 \cdot F3S \cdot PP1 + TT1 \cdot F2C \cdot PP2$$

$$G2C^* = TT1 \cdot F1S \cdot mPINV1 - TT2 \cdot F3C \cdot PINV2$$

$$G2S^* = -TT1 \cdot F1C \cdot mPINV1 - TT2 \cdot F3S \cdot PINV2$$

$$G3C^* = TT2 \cdot F2C \cdot PINV2 - TT1 \cdot F1C \cdot PP3$$

$$G3S^* = TT2 \cdot F2S \cdot PINV2 - TT1 \cdot F1S \cdot PP3$$

Now, $G1C^*$ and $G1S^*$ are half of what we want, namely the spectral coefficients of $\hat{r} \cdot \nabla \times \vec{F}$. To obtain the spectral coefficients of $\hat{r} \cdot \nabla \times \nabla \times \vec{F}$, just take one more curl,

$$H1C = T1 \cdot G3C^* \cdot P1 - T1 \cdot G2S^* \cdot P2$$

$$H1S = T1 \cdot G3S^* \cdot P1 + T1 \cdot G2C^* \cdot P2$$

so

$$\begin{aligned} H1C^* = & \; TINV \cdot [T1 \cdot (TT2 \cdot F2C \cdot PINV2) \cdot P1] \cdot PINV1 \\ & - TINV \cdot [T1 \cdot (TT1 \cdot F1C \cdot PP3) \cdot P1] \cdot PINV1 \\ & + TINV \cdot [T1 \cdot (TT1 \cdot F1C \cdot mPINV1) \cdot P2] \cdot PINV1 \\ & + TINV \cdot [T1 \cdot (TT2 \cdot F3S \cdot PINV2) \cdot P2] \cdot PINV1 \end{aligned}$$

$$\begin{aligned} H1S^* = & \; TINV \cdot [T1 \cdot (TT2 \cdot F2S \cdot PINV2) \cdot P1] \cdot PINV1 \\ & - TINV \cdot [T1 \cdot (TT1 \cdot F1S \cdot PP3) \cdot P1] \cdot PINV1 \\ & + TINV \cdot [T1 \cdot (TT1 \cdot F1S \cdot mPINV1) \cdot P2] \cdot PINV1 \\ & - TINV \cdot [T1 \cdot (TT2 \cdot F3C \cdot PINV2) \cdot P2] \cdot PINV1 \end{aligned}$$

So, define the matrices

$$TT3 = TINV \cdot T1 \cdot TT1$$

$$TT4 = TINV \cdot T1 \cdot TT2$$

$$PP4 = (mPINV1 \cdot P2 - PP3 \cdot P1) \cdot PINV1$$

then finally

$$G1C^* = TT1 \cdot F3C \cdot PP1 - TT1 \cdot F2S \cdot PP2$$

$$G1S^* = TT1 \cdot F3S \cdot PP1 + TT1 \cdot F2C \cdot PP2$$

$$H1C^* = TT4 \cdot F2C \cdot PP1 + TT3 \cdot F1C \cdot PP4$$
$$+ TT4 \cdot F3S \cdot PP2$$

$$H1S^* = TT4 \cdot F2S \cdot PP1 + TT3 \cdot F1S \cdot PP4$$
$$- TT4 \cdot F3C \cdot PP2$$

completes the evaluation of the curls.

Momentum Equation

$$\frac{\partial}{\partial t}\vec{u} - E\nabla^2\vec{u} = -\nabla p - (\vec{u}\cdot\nabla)\vec{u}$$

$$+\ \Lambda(\nabla\times\vec{b})\times\vec{b} + Ra\,\Theta\,\vec{r}$$

With $\vec{u}$ expanded in terms of e and F, and e and F expanded in terms of spherical harmonics as above, the momentum equation becomes:

call these spectral coeffs DEC and DE

$$\frac{l(l+1)}{r^2}\left[\frac{\partial}{\partial t} - E\frac{\partial^2}{\partial r^2} + E\frac{l(l+1)}{r^2}\right] e\,P_l^m = \hat{r}\cdot\nabla\times\vec{F}$$

$$\frac{l(l+1)}{r^2}\left[\frac{\partial}{\partial t} - E\frac{\partial^2}{\partial r^2} + E\frac{l(l+1)}{r^2}\right]\left[-\frac{\partial^2}{\partial r^2} + \frac{l(l+1)}{r^2}\right] F\cdot P_l^m$$

$$= \hat{r}\cdot\nabla\times\nabla\times\vec{F}$$

DFC and DFS

where

$$\vec{F} = \cancel{-(\vec{u}\cdot\nabla)}\vec{u} + \Lambda(\nabla\times\vec{b})\times\vec{b} + Ra\,\Theta\,\vec{r}$$

Temperature Equation

$$\frac{\partial}{\partial t}\Theta - q\nabla^2\Theta = -u\cdot\nabla\Theta$$

Expand as

$$\Theta = \Theta_c(r)\, P_l^m(\cos\theta)\cos(m\phi) + \Theta_s(r)\, P_l^m(\cos\theta)\sin(m\phi)$$

(and remember that here we must include $l=0$)

then the temperature equation becomes

$$\left[\frac{\partial}{\partial t}\Theta - q\left(\frac{\partial^2}{\partial r^2}\Theta + \frac{2}{r}\frac{\partial}{\partial r}\Theta - \frac{l(l+1)}{r^2}\Theta\right)\right]P_l^m = -u\cdot\nabla\Theta$$

DTC and

DTS