Calculation of Kinetic Energy

$$KE = \int \frac{1}{2} u^2 \, dV = \iiint \frac{1}{2} u^2 \, r^2 dr \sin\theta \, d\theta \, d\phi$$

For $m=0$ mode

$$u = \left(\frac{l(l+1)}{r^2} F_c \cdot P_l^0 \,,\; \frac{1}{r} F_c' \cdot \frac{d}{d\theta} P_l^0 \,,\; -\frac{1}{r} e_c \cdot \frac{d}{d\theta} P_l^0 \right)$$

so

$$KE = \cancel{\int}\iint \frac{1}{2} \left(\frac{l(l+1)}{r^2} F_c \cdot P_l^0 \right)^2 r^2 dr \sin\theta \, d\theta \cdot 2\pi$$

$$+ \cancel{\int}\iint \frac{1}{2} \left(\frac{1}{r} F_c' \cdot \frac{d}{d\theta} P_l^0 \right)^2 r^2 dr \sin\theta \, d\theta \cdot 2\pi$$

$$+ \cancel{\int}\iint \frac{1}{2} \left(-\frac{1}{r} e_c \cdot \frac{d}{d\theta} P_l^0 \right)^2 r^2 dr \sin\theta \, d\theta \cdot 2\pi$$

$$= \pi (l(l+1))^2 \int \frac{F_c^2}{r^2} dr \int (P_l^0)^2 \sin\theta \, d\theta$$

$$+ \pi \int F_c'^2 dr \int \left(\frac{d}{d\theta} P_l^0 \right)^2 \sin\theta \, d\theta$$

$$+ \pi \int e_c^2 dr \int \left(\frac{d}{d\theta} P_l^0 \right)^2 \sin\theta \, d\theta$$

$$= \pi \frac{2\ell^2(\ell+1)^2}{2\ell+1} \int \frac{F_c^2}{r^2} dr + \pi \frac{2\ell(\ell+1)}{2\ell+1} \int (F_c'^2 + e_c^2) dr$$

For $m>0$ modes,

$$KE = \iint \frac{1}{2} \left(\frac{\ell(\ell+1)}{r^2}\right)^2 (F_c^2 + F_s^2)(P_\ell^m)^2 r^2 dr \sin\theta d\theta \cdot \pi$$

$$+ \iint \frac{1}{2} \frac{1}{r^2} (F_c'^2 + F_s'^2) \left(\frac{d}{d\theta} P_\ell^m\right)^2 r^2 dr \sin\theta d\theta \cdot \pi$$

$$+ \iint \frac{1}{2} \frac{1}{r^2} (F_c'^2 + F_s'^2) \left(\frac{1}{\sin\theta} P_\ell^m\right)^2 m^2 r^2 dr \sin\theta d\theta \cdot \pi$$

$$+ \iint \frac{1}{2} \frac{1}{r^2} (e_c^2 + e_s^2) \left(\frac{d}{d\theta} P_\ell^m\right)^2 r^2 dr \sin\theta d\theta \cdot \pi$$

$$+ \iint \frac{1}{2} \frac{1}{r^2} (e_c^2 + e_s^2) \left(\frac{1}{\sin\theta} P_\ell^m\right)^2 m^2 r^2 dr \sin\theta d\theta \cdot \pi$$

$$= \frac{\pi}{2} \int \frac{1}{r^2} (F_c^2 + F_s^2) dr \cdot I_1 + \frac{\pi}{2} \int (F_c'^2 + F_s'^2 + e_c^2 + e_s^2) dr \cdot I_2$$

where

$$I_1 = \ell^2(\ell+1)^2 \int (P_\ell^m)^2 \sin\theta d\theta = \frac{2\ell^2(\ell+1)^2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!}$$

$$I_2 = \int \left[\left(\frac{d}{d\theta} P_\ell^m\right)^2 + \left(\frac{m}{\sin\theta} P_\ell^m\right)^2 \right] \sin\theta d\theta$$

$$= \frac{2\ell(\ell+1)}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!}$$