Consistency and Pairwise Consistency

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Abstract

We study the finite Harsanyi model where each agent's type is associated with a posterior belief over some fundamental uncertainty and type profiles of other agents. We show that if the agents' posteriors have full support, the model admits a common prior (i.e., is consistent in Harsanyi's sense) if and only if any pair of agents have a common prior. We extend the result to Aumann models where posteriors need not have full support but satisfy a condition called *double irreducibility* which strengthens the irreducibility condition of Samet (1998). Our results imply circumstances where characterizations of consistency such as the no-trade theorems and the convergence of higher-order expectations reduce to the corresponding characterizations for pairwise consistency.

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1 Introduction

The common prior assumption, introduced by Harsanyi (1967/68), plays a central role in the theory of incomplete-information games. This assumption, which Harsanyi coined consistency, states that the agents' beliefs over some fundamental uncertainty are derived from a common distribution via Bayes' rule, conditioning on their respective information. In other words, any difference in the agents' beliefs must be attributed to differences in their information. The common prior assumption has led to a number of remarkable implications; for instance, it is impossible for the agents to agree to disagree (Aumann, 1976) or accept speculative trade (Milgrom and Stokey, 1982); moreover, their iterative expectations must converge (Samet, 1998b)[1]. The common prior assumption also prevails in the epistemic game theory, as part of the epistemic foundation of correlated/Nash equilibrium (Aumann, 1987; Aumann and Brandenburger, 1995).

In this paper, we consider a model of incomplete information with a finite set of agents in which the fundamental uncertainty is represented by finitely many states of nature. In the baseline result, we adopt the Harsanyi model to represent each agent's information by a finite type space. Each type of an agent is associated with a posterior belief over the fundamental uncertainty and the type profiles of other agents. The formulation of a model is exclusively interim and notwithstanding the existence of a common or heterogeneous prior. We show that when the agents' posteriors all have full support, consistency is equivalent to pairwise consistency, namely that the agents have a common prior if and only if any pair of agents have a common prior. It follows that any inconsistent model has a nearby model that is pairwise inconsistent.

To establish consistency from pairwise consistency, we focus on examining consistent *marginal* probabilities over an agent's own types. In particular, we associate each agent with a matrix (called P-matrix) whose rows consist of the posteriors associated with her/his types. For each pair of agents, we compute a new matrix, called R-matrix, by taking the entry-wise ratio of their P-matrices. For a consistent model, an R-matrix of two agents must not depend on other agents' types nor the fundamental uncertainty. When the posteriors have full support, we show that the agents' posteri-

[1]More broadly, this is related to the literature on coordination and public information, for example Morris and Shin (1998, 2002).

ors admit a common prior if and only if every R-matrix has rank one.[2] In particular, from an R-matrix with rank one, we are able to factor out two candidate marginal distributions for the two agents. Moreover, we show that the marginal distribution over an agent's own types must be invariant across pairs with different agents; hence, pairwise consistency implies consistency.

We also extend the results to models in which the posteriors need not have full support. To do so, we formulate our result in the Aumann model where each agent's information is represented by an information partition. We follow Samet (1998b) in using a Markov transition to describe an agent's beliefs, where each row or column is a state, with the (i, j)-th entry representing the posterior belief for state j when the agent is at state i. Two states communicate if they lie in a common partition element of one agent. Samet shows that when a group of agents' partitions is irreducible, i.e. the product their belief matrices is irreducible, there exists at most one common prior. However, we show with an example that his irreducibility condition is insufficient to guarantee consistency from pairwise consistency.

To restore the equivalence between consistency and pairwise consistency, we strengthen Samet's irreducibility to require two states communicate when they simultaneously lie in the same partition element of *two* agents. For a triplet of agents, we iteratively add *virtual* states to the state space to further construct communicating relations. The model is doubly irreducible for the triplet if all states communicate when no more virtual states can be added in this manner. We show that for a model structure being doubly irreducible for each triplet, regardless of the posteriors assigned to the partition, consistency is equivalent to pairwise consistency. We also provide a direct comparison between double irreducibility and Samet's irreducibility condition in Section 3.2.

For Aumann or Harsanyi models to which our equivalence results apply, existing characterizations of consistency reduce to the corresponding characterizations for pairwise consistency. First, it is well known that a common prior exists if and only if there is no speculative trade (see, e.g., Morris (1994); Feinberg (2000); Samet (1998a)); hence, by our result, a common prior exists if and only if no speculative

[2] Harsanyi (1967/68) remarks the necessity of this condition for the consistency assumption, while we show that the condition is also sufficient after a slight strengthening.

trade can arise for any pair of agents. Second, Guarino and Tsakas (2021) show that the existence of an "action-independent" common prior is equivalent to the non-existence of an acceptable "strategic-invariant" bet. Our result then implies that the existence of a "strategic-invariant" bet acceptable to all agents is equivalent to the existence of "strategic-invariant" bets acceptable to any pair of agents. Third, our result together with Samet (1998b) implies that a common prior exists if and only if the higher-order expectations of any pair of agents converge. Fourth, the existence of a common prior is also equivalent to the existence of an interior full-insurance Pareto optimal allocation in, for instance, Billot et al. (2000) and Kajii and Ui (2009). It follows from our result that the existence of such allocations hinges on the existence of interior Pareto optimal allocations for all pairs of agents. Finally, Rodrigues-Neto (2009) and Hellwig (2013) characterizes consistency in terms of checking cycle equations on diagrams in regards to the join and meet of agents' partitions. Our result simplifies their consistency conditions from checking all cycles or cycles that involve four agents to checking cycles that involve only a pair of agents.

The paper is organized as follows. Section 2 studies consistency under the full-support assumption and establishes our baseline result. We then extend the result to models without the full support assumption in Section 3. Section 4 discusses various applications of the result. We relate our findings to the literature and discuss the connections to the information design literature in Section 5.

2 Consistency for Beliefs with Full Support

2.1 The Harsanyi model

There are two commonly used models in incomplete information game theory (Zamir, 2020). Here we introduce the Harsanyi model $(\mathcal{I}, \{(T^i, P^i)\}_{i\in\mathcal{I}}, S)$ which is more practically used in applied game-theoretical analysis. For our main result, we start under the Harsanyi model for the special case of beliefs with full support, which is better structured. In Section 3, we extend our result to its general form within the Aumann model as it better suits the case without full support.

There is a finite set of agents $\mathcal{I} = \{1, 2, ..., I\}$. For each agent $i \in \mathcal{I}$, let $T^i = \{t^i_k : k = 1, 2, ..., K_i\}$ be the set of her types. $T = \times_{i\in\mathcal{I}} T^i$ is the set of type profiles. A finite set of states of nature S describes all payoff-relevant information. $T \times S$ describes all the uncertainty an agent faces. An agent i of type t^i_k holds her (posterior) belief regarding the state of nature s and all other agents' types, which we denote by a $|T^{-i}| \times |S|$ dimensional row vector $P^i_k \in \Delta(T^{-i} \times S)$. We can further write agent i's posteriors at each possible type as a matrix

$$P^i = \begin{pmatrix} P^i_1 \\ \vdots \\ P^i_{K_i} \end{pmatrix},$$

which we call a P-matrix.

Given all the P-matrices (a total I of them), we would like to check whether there exists a common prior $\hat{P} \in \Delta(T \times S)$, upon which each agent i updates her belief given her own type to generate her posteriors P^i. The following definition formalizes the notion.

Definition 1 (Consistency)**.** The agents' posteriors $\{P^i\}_{i\in\mathcal{I}}$ are *consistent* if there exists a probability measure $\hat{P} \in \Delta(T \times S)$ (called the *common prior*[3]) such that for every $i \in \mathcal{I}$ and $t^i_k \in T^i$,

(i) $\hat{P}(t^i_k) > 0$,
(ii) $P^i_k(\cdot) = \hat{P}(\cdot|t^i_k)$.

Equivalently, when there exist *marginals* $\mu^i : T^i \to \mathbb{R}_{>0}$[4] such that $P^i_k(t^{-i}, t)\mu^i(t^i_k) = \hat{P}(t^i_k, t^{-i}, s)$.

Specifically, we are interested in the simple case of *pairwise consistency*, i.e. the consistency between a pair of agents. For this purpose, we introduce a family of

[3]We require the common prior have positive marginals for two reasons. First, this is a regular assumption in the literature with the Harsanyi model (e.g. Morris (2020); Guarino and Tsakas (2021)) as it would avoid generating "artificial" posteriors made up by agents from an zero-probability state and focus on the Bayesian belief updating process. Second, this requirement also corresponds to the "common support" assumption often assumed in papers studying similar problems in the Aumann model (Samet, 1998b; Golub and Morris, 2017). The connection will be elaborated in detail when we take on the Aumann model for extension in the next section.mark

[4]All strictly positive real numbers.

auxiliary matrices, which we call ratio matrices (R-matrices), defined for every pair of agents $a, b \in \mathcal{I}$.

To start with, for a pair of agents a, b and state-type profile (t^{-ab}, s), let $R^{ab}(t^{-ab}, s)$ be a $K_a \times K_b$ dimensional matrix, with its entry r_{kl} equal to $P^a_k(t^b_l, t^{-ab}, s)/P^b_l(t^a_k, t^{-ab}, s)$ if both the denominator and the numerator are positive, and 0 otherwise. Now suppose there is a common prior $\hat{P}$. Then, we have

$$P^a_k(t^b_l, t^{-ab}, s) = \frac{\hat{P}(t^a_k, t^b_l, t^{-ab}, s)}{\mu^a(t^a_k)} \quad \text{and} \quad P^b_l(t^a_k, t^{-ab}, s) = \frac{\hat{P}(t^a_k, t^b_l, t^{-ab}, s)}{\mu^b(t^b_l)}.$$

Note that requirement (i) guarantees the positiveness of the marginals μ^a, μ^b. Therefore, for an element in the R-matrix, either both the denominator and the numerator are positive, or neither is; moreover, the ratio $r_{kl} = \mu^b(t^b_l)/\mu^a(t^a_k)$ is *invariant* over other agents' types and state (t^{-ab}, s). Hence we arrive at the following observation.

Observation. The agents' posteriors $\{P^i\}_{i\in\mathcal{I}}$ are consistent only if for every pair of agents (a, b), their R-matrices are invariant over other agents' types and the state of nature: $R^{ab}(t^{-ab}, s) = R^{ab}(\tilde{t}^{-ab}, \tilde{s})$ for all (t^{-ab}, s) and $(\tilde{t}^{-ab}, \tilde{s}) \in T^{-ab} \times S$.

As we are looking for a necessary (and sufficient) condition for belief consistency, we maintain the assumption that this invariance conditions holds for our posteriors. We now define R^{ab}, the R-matrix, for each pair of agents (a, b).

Assumption 1. For every pair of agents (a, b), the matrices $R^{ab}(t^{-ab}, s)$ satisfies $R^{ab}(t^{-ab}, s) = R^{ab}(\tilde{t}^{-ab}, \tilde{s})$ for all (t^{-ab}, s) and $(\tilde{t}^{-ab}, \tilde{s}) \in T^{-ab} \times S$.

Definition 2 (R-matrix)**.** For every pair of agents (a, b), their R-matrix R^{ab} is the $K_a \times K_b$ dimensional matrix with its entry r_{kl} equal to $P^a_k(t^b_l, t^{-ab}, s)/P^b_l(t^a_k, t^{-ab}, s)$ for a fixed $(t^{-ab}, s) \in T^{-ab} \times S$.

Example (Two agents)**.** There are two agents a, b, with 2 and 3 types each. We have the following posterior matrices for agents:

$$P^a = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/5 & 2/5 & 2/5 \end{pmatrix}, \quad P^b = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \\ 0 & 1 \end{pmatrix},$$

where $P^a_{ij} = P^a_i(t^b_j), P^b_{ji} = P^b_j(t^a_i)$. Then, the ratio matrix is $R^{ab} = \begin{pmatrix} 2/3 & 4/3 & 0 \\ 2/5 & 4/5 & 2/5 \end{pmatrix}$.

2.2 Two agents

We first study the case of two agents: when would it be possible for a pair of agents in $\mathcal{I}$ to share a common prior. This will pave the way for the analysis of more involving situations. We now refine our R-matrices at the "undefined" entries due to zeros.

Definition 3 (Completion)**.** A strictly positive matrix $\tilde{R}$ is a completion of a positive matrix R if they have the same dimensions and coincide on all strictly positive entries of R.

Example (Two agents, continued)**.** The ratio matrix $R^{ab} = \begin{pmatrix} 2/3 & 4/3 & 0 \\ 2/5 & 4/5 & 2/5 \end{pmatrix}$ has a following completion: $\tilde{R}^{ab} = \begin{pmatrix} 2/3 & 4/3 & 2/3 \\ 2/5 & 4/5 & 2/5 \end{pmatrix}$

For two agents a, b and their posteriors P^a, P^b given, pairwise consistency turns out to depend on whether we can find a "nice" completion of R^{ab}.

Lemma 1 (Rank-one completion)**.** Under Assumption 1, for two agents a, b with posteriors P^a, P^b, their posteriors are pairwise consistent if and only if there exists a completion of their R-matrix $\tilde{R}^{ab}$ of rank one.

For ease of notation, for a strictly positive vector x, we write $\frac{1}{x}$ as the vector of the same dimension with entries $(\frac{1}{x})_k = \frac{1}{x_k}$. When the posteriors are consistent, we can in fact write $\tilde{R}^{ab} = \left(\frac{1}{\mu^a}\right)' \mu^b$, where the row vectors μ^a and μ^b are the marginals of the corresponding common prior P, namely $\mu^a_k \equiv \mu^a(t^a_k)$ and $\mu^b_l \equiv \mu^b(t^b_l)$.

The necessity can be seen from our argument for Lemma 1. The sufficiency follows from the fact that given a completion of rank one, we can recover (after normalization) the desired marginals μ^a_k, μ^b_l, which further gives the common prior $\hat{P}$ by $\hat{P}(t^a_k, t^b_l, t^{-ab}, s) = P^b_l(t^a_k, t^{-ab}, s)\mu^b(t^b_l)$. This proof is effectively an adaptation of Arnold et al. (2004) and Song et al. (2010).

The condition can be seen as an extension of Harsanyi's necessary condition for belief consistency. In his seminal paper, Harsanyi (1967/68) effectively establishes a necessary condition for the consistency in a full-support finite Harsanyi model, which is later also known as the Harsanyi's condition. In our framework, Harsanyi's condition requires that for every R-matrix, the ratio of two entries in a row k must be the same as that of the two entries of the *same columns* in another row k', i.e. $r_{kl}/r_{kl'} = r_{k'l}/r_{k'l'}$, *if the ratio is valid.* From this perspective, our Proposition 1 strengthens Harsanyi's result by showing that his condition is not only necessary but also sufficient after the "completion". A continuous version of Harsanyi's condition is also mentioned in Hellwig (2022), where a two-agent model is used to help analyse the consistency of beliefs in a large populations incomplete-information game.

Example (Two agents, continued)**.** As the completion $\tilde{R}^{ab} = \begin{pmatrix} 2/3 & 4/3 & 2/3 \\ 2/5 & 4/5 & 2/5 \end{pmatrix}$ has rank one, we know the posteriors admit a common prior. Moreover, $\tilde{R}^{ab} = \begin{pmatrix} 2/3 & 4/3 & 2/3 \\ 2/5 & 4/5 & 2/5 \end{pmatrix} = \begin{pmatrix} 1/4 & 1/2 & 1/4 \end{pmatrix} \begin{pmatrix} 8/3 \\ 8/5 \end{pmatrix}$ gives $\mu^a = \begin{pmatrix} 3/8 & 5/8 \end{pmatrix}$, $\mu^b = \begin{pmatrix} 1/4 & 1/2 & 1/4 \end{pmatrix}$, and thus $\hat{P} = \begin{pmatrix} 1/8 & 1/4 & 0 \\ 1/8 & 1/4 & 1/4 \end{pmatrix}$.

Remark. Note that there are in fact two ratio matrices available for inspection, namely, R^{ab} and R^{ba}. We only need to work with the former because of the simple relation $R^{ab}_{kl} = 1/R^{ba}_{lk}$ (if both are positive) and $\tilde{R}^{ba} = \frac{1}{\mu^b}{}'\mu^a$. Similarly, when we recover the common prior from the marginals, we can actually do it from either $\mu^b(t^b_l)$'s or $\mu^a(t^a_k)$'s, both of which will give the same result due to the relation above. Finally, there could be multiple completions for a given ratio matrix, which lead to multiple common priors.

2.3 More than two agents

When there is one more agent c involved, the conditions for common prior existence become more complicated due to the interactions, but, we find an intuitive description. We now need to work with two ratio matrices at a time. Let $R^{abc} = (R^{ab}, R^{cb})$ be the $(K_a + K_c)$ by K_b composite matrix of the two original R-matrices involving agent b. The following lemma shows that a slight modification of Lemma 1 gives us the necessary and sufficient condition for the three-agent case.

Lemma 2 (Rank-one completion)**.** Under Assumption 1, the posteriors of agents a, b, c are consistent if and only if there exists a completion of their composite ratio matrix $\tilde{R}^{abc}$ of rank one.

As in the previous lemma, for consistent posteriors, we can write $\tilde{R}^{abc} = {\frac{1}{(\mu^a,\mu^c)}}^T \mu^b$, where $\mu^i_k = P(t^i_k)$, $i = a, b, c$ are the vectors of marginals of the common prior.

The necessity trivially follows from Lemma 1. The sufficiency holds due to the similar fact that given a completion of rank one, we can recover (after normalization) the desired marginal probabilities μ^i_k which further gives the common prior by $\hat{P}(t^a_k, t^b_l, t^c_m, s) = P^b_l(t^a_k, t^c_m, s)\mu^b(t^b_l)$. Though there are effectively six (or three if taking into account of symmetry) composite matrices available, namely R^{abc}, R^{bca} and R^{cab} and their transposes, due to the fact that $R^{ac}_{km} = R^{ab}_{kl}/R^{cb}_{ml}$, one composite matrix R^{abc} carries all the necessary information about the other two. Therefore, the lemma only invokes the check on one of them. Similarly, when we recover the common prior from the marginals, starting from any of the three different marginals eventually gives the same outcome.

The intuition that agent b serves as a "bridge" can be extended to the analysis for all agents. Let the composite matrix $R^{\mathcal{I}} = (R^{1b}, R^{2b}, ..., R^{I-1})$ consist of all R-matrices involving a fixed agent b.

Proposition 1. Under Assumption 1, the posteriors of all agents are consistent if and only if all there exists a completion of the grand composite matrix $R^{\mathcal{I}}$ of rank one.

2.4 Pairwise consistency to consistency: extension

There is a clear resemblance and connection between Lemma 1 and Proposition 1 in terms of the utilization of R-matrices, suggesting a possible connection between pairwise consistency and consistency. After all, the latter already implies the former by definition. It turns out that the reverse could hold as well through the lemma below, with some constraints on the structure of the type/state space, e.g., the full-support assumption.

Definition 4 (Extension condition)**.** We say the extension condition holds for a triplet $\{a, b, c\} \subset \mathcal{I}$ if, whenever each pair among them are pairwise consistent, the two common priors (and marginals) concerning i are the same for each $i \in \{a, b, c\}$.

Lemma 3 (Extension)**.** Suppose that the *extension* condition holds for every triplet of agents. Then, pairwise consistency implies consistency for all subgroups of $n = 3, 4, \ldots, I$ agents.

Proof. We do induction on the size of the subgroup of agents. The case for $n = 3$ directly follows from the extension condition. For $n \geqslant 4$, let's consider a sample group of agents $N = \{1, 2, ..., n\}$. Let $\hat{P}_{-2}, \hat{P}_{-3}, \hat{P}_{-23}$ be the common prior for groups $N\backslash\{2\}, N\backslash\{3\}, \{2, 3\}$ respectively. Note that $\hat{P}_{-2}, \hat{P}_{-3}, \hat{P}_{-23}$ can be seen as the three common priors for the three pairs of agents $\{1, 3\}, \{1, 2\}, \{2, 3\}$. Since we assume the extension condition holds for the triplet $\{1, 2, 3\}$, the common priors for groups $\hat{P}_{-2}$ and $\hat{P}_{-3}$ must be the same as they respectively include pairs $\{1, 3\}$ and $\{1, 2\}$. Then, this is also the common prior for group $N = \{1, 2, ..., n\}$. $\square$

Assumption 2 (Full-support)**.** For each agent i, each of her posterior $\hat{P}^i_k$ has full support over the marginal T^{-i}, i.e. $\hat{P}^i_k(t^{-i}, s) > 0, \forall t^{-i} \in T^{-i}$ and $s \in S$.

In words, each agent always puts a positive possibility on every type profile of others and the state of nature in her beliefs. Now, we formulate the main result linking consistency and pairwise consistency.

Theorem 1 (Pairwise consistency implies consistency)**.** Under Assumption 2, the posteriors of all agents $\mathcal{I}$ are consistent if and only if there is pairwise consistency between every pair of agents in $\mathcal{I}$.

Before the formal proof, we introduce an equation that plays a key part both in the proof and later in our further extension, which we call the *rotation equation*.

Consider a representative triplet of agents $\{a, b, c\} \subset \mathcal{I}$. Now suppose each pair among them admit a common prior between the two. By Lemma 1, there exist marginal probability distributions $\mu^i, \tilde{\mu}^i$, $i \in \{a, b, c\}$ on the respective supports and corresponding completions of R-matrices, such that

$\tilde{R}^{ab} = \left(\frac{1}{\mu^a}\right)' \tilde{\mu}^b, \tilde{R}^{bc} = \left(\frac{1}{\mu^b}\right)' \tilde{\mu}^c, \tilde{M}^{ca} = \left(\frac{1}{\mu^c}\right)' \tilde{\mu}^a$. By the definition of R-matrix, if the posteriors at (t^a_k, t^b_l, t^c_m) are non-zero (which trivially holds for the full-support case), then

$$\begin{aligned} 1 &= \frac{P^a_k(t^b_l, t^c_m)}{P^b_l(t^a_k, t^c_m)} \times \frac{P^b_l(t^a_k, t^c_m)}{P^c_m(t^a_k, t^b_l)} \times \frac{P^c_m(t^a_k, t^b_l)}{P^a_k(t^b_l, t^c_m)} \\ &= \tilde{R}^{ab}_{kl} \times \tilde{R}^{bc}_{lm} \times \tilde{R}^{ca}_{mk} \\ &= \frac{\tilde{\mu}^b_l}{\mu^a_k} \times \frac{\tilde{\mu}^c_m}{\mu^b_l} \times \frac{\tilde{\mu}^a_k}{\mu^c_m} \\ &= \frac{\tilde{\mu}^a_k}{\mu^a_k} \times \frac{\tilde{\mu}^b_l}{\mu^b_l} \times \frac{\tilde{\mu}^c_m}{\mu^c_m}. \end{aligned} \tag{1}$$

Note that the first equality is a mere rotation over the denominators and numerators. The second line comes from the definition of R-matrix, the next from Lemma 1, while the last line is again a rearrangement of the denominators.

Definition 5 (Rotation equation)**.** For a triplet of agents $\{a, b, c\} \subset \mathcal{I}$, we call (1) the *rotation equation* for type profile (t^a_k, t^b_l, t^c_m):

$$\frac{\tilde{\mu}^a_k}{\mu^a_k} \times \frac{\tilde{\mu}^b_l}{\mu^b_l} \times \frac{\tilde{\mu}^c_m}{\mu^c_m} = 1.$$

Now, if we fix k and m, while letting l go through all possible values, we will see that the ratio $\frac{\tilde{\mu}^b_l}{\mu^b_l}$ is constant across l. Note that both $\tilde{\mu}^b_l$ and μ^b_l are probability densities, which add up to 1 across l. This means $\tilde{\mu}^b_l = \mu^b$ and consequently, the extension lemma applies.

Proof of Theorem 1. One direction is obvious by definition: if all the I agents' posteriors are consistent with a common prior $\hat{P}$, then this $\hat{P}$ is also a pairwise common prior between any pair of agents in $\mathcal{I}$. Now we show the converse.

Suppose every pair of agents have common priors. For any triplet of agents $a, b, c \in \mathcal{I}$, Lemma 1 implies the existence of marginals $\mu^i, \tilde{\mu}^i, i \in \{a, b, c\}$ on the respective supports and corresponding completions of pairwise R-matrices, such that $\tilde{R}^{ab} = \frac{1}{\mu^a}' \tilde{\mu}^b, \tilde{R}^{bc} = \frac{1}{\mu^b}' \tilde{\mu}^c, \tilde{R}^{ca} = \frac{1}{\mu^c}' \tilde{\mu}^a$. Since we assume full-support posteriors, the rotation equation (1) holds for all (k, l, m). By argument above, we must have $\tilde{\mu}^b = \mu^b$ and consequently, the extension lemma applies. The consistency for all agents $\mathcal{I}$ is now established through induction. $\square$

3 Posteriors without Full Support

We now switch to the Aumann model $(\mathcal{I}, \Omega, \{(\Pi^i, p^i(\cdot|\omega)\}_{i\in\mathcal{I}}, S)$. There is a finite set of agents $\mathcal{I} = \{1, 2, ..., I\}$, a finite set of states of the worlds Ω representing the fundamental uncertainty, and a finite set of states of nature S. For each $i \in \mathcal{I}, \Pi^i$ is a partition of Ω acting as signals: after a state of world $\omega \in \Omega$ is chosen, agent i is informed of π^i_ω, the element of her partition that contains ω. Her belief at ω is a probability measure $p^i(\cdot|\omega) \in \Delta\Omega$ supported on π^i_ω that describes the probabilities she believes each state will occur. Finally, a mapping $s : \Omega \to S$ decides the game form containing all payoff relevant information.

The Aumann model can be equivalently transformed into the Harsanyi model in order to apply our matrix analysis and the results (Zamir, 2020). For each agent $i \in \mathcal{I}$, let $T^i = \{\pi^i_\omega : \omega \in \Omega\}$. Let $T = \times_{i\in\mathcal{I}} T^i$ denote the set of type profiles. An agent i of type π^i_ω holds posterior belief $P^i(\cdot|\pi^i_\omega) \in \Delta(T^{-i} \times S)$ such that $P^i(\pi^{-i}, s|\pi^i_\omega) = \sum_{\pi^{-i}_{\omega'} = \pi^{-i} \text{ and } s(\omega')=s} p^i(\omega'|\omega)$. Zero probability is assigned to those points where there are no ω' such that $\pi^{-i}_{\omega'} = \pi^{-i}$ and $s(\omega') = s$, which is also where those beliefs without full support stem from.

The reverse transformation from the Harsanyi model back to the Aumann model is likewise. Let $\Omega = \{\omega = (t, s) : \exists i, k \text{ such that } P^i_k(t^{-i}, s) > 0\}$ be the state space, with $(\cdot)_\omega$ being the mapping between Ω and $S \times T$. Define the mapping $S(\omega) := s_\omega$. The partition for agent i is defined as $\pi^i(\omega) := \{\omega' : t^i_\omega = t^i_{\omega'}\}$, with beliefs $p^i(\omega'|\omega) := P'^i_k((t^{-i}, s)_{\omega'})$ where t^i_k is the type of i corresponding to state ω.

We restrict attention to the case where the meet of partitions $\bigwedge \pi^i$ is singleton, i.e. $\bigwedge \pi^i = \{\Omega\}$. This is without loss for the study of common priors as, otherwise, a common prior on Ω is just the convex combination of the common priors restricted to each element of the meet (Samet, 1998a). We also make the natural assumption that an agent at state ω must deem it "possible",[5] so that the posteriors after the aforementioned transform will be compatible to the condition (ii) in Definition 1.

Assumption 3. For each state $\omega \in \Omega$ and agent $i \in \mathcal{I}$, $p^i(\omega|\omega) > 0$.

The earlier definition of consistency is stated for the Harsanyi model. Given the above mutual transformation, we state an equivalent definition for the Aumann model where $\hat{P}$ and μ^i are defined on the state space Ω and partitions Π^i, respectively. (See, e.g. Samet (1998a,b).[6])

Definition 1* (Consistency)**.** The agents' beliefs are *consistent* if there exists a probability measure $\hat{P} \in \Delta\Omega$ (the *common prior*) such that for every $i \in \mathcal{I}$ and $\omega' \in \pi^i_\omega$, $p^i(\omega'|\omega) = \hat{P}(\omega')/\hat{P}(\pi^i_\omega)$; or equivalently, if there exist *marginals* $\mu^i : \Pi^i \to \mathbb{R}_{>0}$ such that $p^i(\omega'|\omega) \cdot \mu^i(\pi^i_\omega) = \hat{P}(\omega')$.

Now, we separate the model into two parts: (i) the model "structure" which specifies the group of agents $\mathcal{I}$, the state of world Ω, the partitions $\Pi^i, i \in \mathcal{I}$ with the resulted types $T^i, i \in \mathcal{I}$,[7] and (ii) the posterior beliefs p^i built on it. The way we will proceed is "structure-oriented": we want to know what model structure ensures that the "two means all" results like Theorem 1 and Corollary 1 always hold, *regardless of* the posterior beliefs, and correspondingly, what model structure admits posteriors such that "two does not mean all" as the following example shows. For this purpose, we follow the steps in the previous section by first examining the simpler question of when "two means three" and then extend to the general question of when "two means all".

3.1 An example

The following example[8] shows the gap between pairwise consistency and consistency for all agents when posteriors are not full-support.

[5] In literature, it is sometimes called "truth" (Nehring, 2001).

[6] Samet's definition allows for a common prior $\hat{P}$ assigning zero probability at certain states. When $\bigwedge \pi^i = \{\Omega\}$, one can easily verify that it must be $\hat{P}(\pi^i_\omega) > 0$ for all i and ω.

[7] Hellman and Samet (2012) call it the partition profile.

[8] Also used by Nehring (2001) for a similar purpose.

Example (Three agents)**.** There are 3 agents $\mathcal{I} = \{a, b, c\}$ with beliefs $p^i = \{p^i(\cdot|\omega) : \omega \in \Omega\}$ supported on respective elements of $\Pi^i = \{\pi_1^i, \pi_2^i\}$, partitions of the state space $\Omega = \{\omega_1, \omega_2, \omega_3\}$. The details are summarized in Table 1 below: each column represents a state ω, while for each agent i, each row representing the partition element including the state ω, with the dots representing the partition (equivalently, the support of belief $p^i(\cdot|\omega)$).

	ω_1	ω_2	ω_3
π_1^a	$\bullet$		
π_2^a		$\bullet$	$\bullet$
π_1^b		$\bullet$	
π_2^b	$\bullet$		$\bullet$
π_1^c			$\bullet$
π_2^c	$\bullet$	$\bullet$	

Table 1: Example of three agents.

In this example, with a generic set of beliefs, every pair of agents admit a unique common prior. For instance, a prior $\hat{P}$ is common to agents a and b if and only if the following holds (we omit the ω in $p^i(\omega'|\omega)$ for simplicity as it is clear from the partitions):

(i) $\frac{\hat{P}(\omega_2)}{\hat{P}(\omega_3)} = \frac{p^a(\omega_2)}{p^a(\omega_3)}$, which reflects the updating from $\hat{P}$ to p^a through Bayes rule;
(ii) $\frac{\hat{P}(\omega_1)}{\hat{P}(\omega_3)} = \frac{p^b(\omega_1)}{p^b(\omega_3)}$, which reflects the updating from $\hat{P}$ to p^b.

Putting them together, we see ratio $\hat{P}(\omega_1) : \hat{P}(\omega_2) : \hat{P}(\omega_3) = p^a(\omega_3)p^b(\omega_1) : p^a(\omega_2)p^b(\omega_3) : p^a(\omega_3)p^b(\omega_3)$ is fixed. Hence, the p respecting this ratio is the unique common prior to both a and b. The pairwise consistency for the other two pairs can be similarly devised. However, for a common prior $\hat{P}$ to generate *all* $\{p^a, p^b, p^c\}$ through Bayes rule, the ratio $\hat{P}(\omega_1) : \hat{P}(\omega_2) : \hat{P}(\omega_3)$ will have to be separately pinned down by any pair within $\{p^a, p^b, p^c\}$. Therefore, unless the respectively determined priors coincide, the triplet $\{p^a, p^b, p^c\}$ do not admit a common prior.

Note that the example does not satisfy the extension condition: though there is always pairwise consistency, consistency for the triplet is not guaranteed as the marginals for, e.g., agent b in his pairing with a and c could be different. The full support assumption earlier eliminates such possibility.

Samet (1998a) gives a neat matrix representation of agents' beliefs in the Aumann model. Let an $|\Omega| \times |\Omega|$ Markov transition matrix Q^i have entry $Q^i_{kl} = p^i(\omega_l|\omega_k)$. Then, $q \in \mathbb{R}^{|\Omega|}$ is a prior for $i \iff qQ^i = q$, i.e q is a left eigenvector with eigenvalue 1 of Q^i. Note that in our example, the meet of any two Π^i and Π^j is $\{\Omega\}$. Hence, if we assume $p^i_\omega(\omega') > 0$ for each i and $\omega' \in \Pi^i(\omega)$, then Q^iQ^j is irreducible (and aperiodic) for each pair $i \neq j$, and so is $Q^aQ^bQ^c$. Samet shows that when the partitions of a group of agents is irreducible, i.e. the product of their belief (Markov) matrices is irreducible, there exists at most one common prior. By the above example, we see that irreducibility is not enough to guarantee "two means all". An intuition is that for irreducibility, two states are considered as in a same communicating class as long as they appear in a same partition element of Π^i for some agent i, which does not help in generalizing the pairwise common prior to be compatible for a third agent. Below, we propose a new irreducibility-like notion by, roughly, requiring two states ω, ω' be in a same class only if there exist $i \neq j$ such that $\omega' \in \pi^i_\omega \cap \pi^j_\omega$, i.e they appear in a same partition element of two agents. This turns out to be useful for our extension result.

3.2 Double Irreducibility

We first pick a representative triplet of agents $\{a, b, c\}$. We construct a binary relation $\sim$ ("communicating") on the *extended* state space Ω^*, which can be mapped into the set of intersections of partition elements $\Pi^\cap = \left\{(\pi^a_k, \pi^b_l, \pi^c_m)\right\}_{k \in K, m \in M, n \in N}$. For notation simplicity, we write $(\pi^a_k, \pi^b_l, \pi^c_m)$ as (k, l, m) and represent the states in the intersection, and similarly for Π^a, Π^b, Π^c. The following procedure is crucial in extending our main result.

Procedure (Double irreducibility)**.** Set $\Omega^* := \Omega$. The communication relation is constructed iteratively as follows:

(i) For ω, ω'^* that belong to two same partition elements π^i_l, π^j_m with $i \neq j$ (e.g., $i = b$ and $j = c$), set $\omega \sim^{\Omega^*} \omega'$.

(ii) For $\omega \in \Omega^*$, suppose $\omega \in \pi^a_k \cap \pi^b_l \cap \pi^c_m$.

  (a) Find k' such that both $\pi^a_k \cap \pi^b_{l'} \cap \pi^c_{m'}$ and $\pi^a_{k'} \cap \pi^b_{l'} \cap \pi^c_{m'}$ are nonempty for some l', m'.

(b) Set $\Omega^* := \Omega^* \cup \{\omega'\}$ with the partitions updated as $\pi^a_{k'} := \pi^a_{k'} \cup \{\omega'\}$, $\pi^b_l := \pi^b_l \cup \{\omega'\}$, $\pi^c_m := \pi^c_m \cup \{\omega'\}$ while the others left unchanged. Call ω' a *virtual* state and set $\omega \sim^{\Omega^*} \omega'$.

(c) Similarly carry out the process for b, c.

(iii) Stop if no more new virtual state can be created. Otherwise, return to (ii), repeat.

(iv) For the terminal extended space Ω^*, construct a new partition $\Pi^* = \{\pi^* : \omega \sim^{\Omega^*} \omega',\ \forall \omega, \omega'^*\}$.

Definition 6 (Double irreducibility)**.** For triplet $\{a, b, c\}$, the model structure is *doubly irreducible*, if Π^* is a trivial partition, i.e. $\Pi^* = \{\Omega^*\}$.

The virtual state procedure stops in finite iterations as $\Pi^\cap$ is finite. We provide an illustrating graphic example at the end of this subsection.

Double irreducibility is a strengthening of Samet's (1998b) irreducibility condition in the following sense. For irreducibility, two states ω_1, ω_2 initially communicate when they are in the same partition element of *one* agent. This can be seen as a different "extension" of consistency condition within that agent. Then, when the meet of partitions $\bigwedge \pi^i$ is singleton, i.e. $\bigwedge \pi^i = \{\Omega\}$, the product of three Markov matrices is irreducible, and therefore allowing at most one left eigenvector as the common prior. In contrast, double irreducibility requires that when two states communicate, they are in the same partition elements of *two* agents. Correspondingly, the product of Markov matrices induced for the extended space is also irreducible.

The following comparison would be useful: if we relax the common entry $(l, m)/(l', m')$ requirement in step (ii) to only having a common l/l', then double irreducibility reduces to Samet's irreducibility condition. We can in fact mimic the procedure above and define irreducibility *iteratively* for two agents $\{a, b\}$. It is easy to see that the partitions of two agents satisfy irreducibility if and only if the terminal partition $\Pi^+ = \{\Omega^+\}$ is the trivial partition.

Procedure (Irreducibility)**.** Set $\Omega^+ := \Omega$. The communication relation is constructed iteratively as follows:

(i) For ω, ω'^+ that belong to one same partition element π^i_l for some $i \in \{a, b\}$, set

$\omega \sim^{\Omega^+} \omega'$.

(ii) For $\omega \in \Omega^+$, suppose $\omega \in \pi_k^a \cap \pi_l^b$.

  (a) Find k' such that both $\pi_k^a \cap \pi_{l'}^b$ and $\pi_{k'}^a \cap \pi_{l'}^b$ are nonempty for some l'.

  (b) Set $\Omega^+ := \Omega^+ \cup \{\omega'\}$ with the partitions updated as $\pi_{k'}^a := \pi_{k'}^a \cup \{\omega'\}$, $\pi_l^b := \pi_l^b \cup \{\omega'\}$, while the others left unchanged. Call ω' a *virtual* state and set $\omega \sim^{\Omega^+} \omega'$.

  (c) Similarly carry out the process for b.

(iii) Stop if no more new virtual state can be created. Otherwise, return to (ii), repeat.

(iv) For the terminal extended space Ω^+, construct a new partition $\Pi^+ = \{\pi^+ : \omega \sim^{\Omega^+} \omega',\ \forall \omega, \omega'^+\}$.

Now, we are ready to state the general version of our "two means all" result.

Theorem 2 (Pairwise consistency implies consistency)**.** If every triplet is doubly irreducible, then pairwise consistency is equivalent to consistency for all agents.

The theorem is again proved by induction through the extension lemma. We only need to verify the extension condition as the building block, which we show below by again applying the rotation equation to the new space Ω^*. The case of full-support posteriors is an obvious example of model structure satisfying double irreducibility, whereas the leading example does not as the procedure stops right at the beginning. Theorem 2 now inherits and generalizes Theorem 1 from both the statement and proof technique perspectives.

One implication of our theorems is the simplification on the procedure checking consistency for a group of agents. Apparently pairwise check is simpler, but it is accompanied by the extra check on of double irreducibility. As we advocate earlier however, our condition is on the *partition structure* of the model, not on specific values of the beliefs. Hence, the clear advantage is that the check on double irreducibility is "one-shot" for a fixed partition profile on state space; once the check is passed, we can rest assured that pairwise check is sufficient. This is useful, for instance, when we apply the result to no-trade theorems as there we are facing the abstract consequence of whether a trade happens, without knowledge on the values of beliefs.

Lemma 4 (Double irreducibility)**.** If the model structure is doubly irreducible for $\{a, b, c\}$, then the extension condition holds.

Proof. Note that rotation equation holds for any partition profile (a_k, b_l, c_m) when $(\pi_k^a \cap \pi_l^b \cap \pi_m^c) \cap \Omega^* \neq \varnothing$. In fact, it holds at the beginning when $\Omega^* = \Omega$ due to Assumption 3[9]. Now, in stage (ii), every newly communicating pair, say, for example $(k, l, m) \sim (k', l, m)$ share two common entries where the latter the newly added virtual state. Let $\mu^i, \tilde{\mu}^i$ be marginals defined as in the definition of rotation equation. Given the criteria for building new $\sim^{\Omega^*}$ links, the existence of (l', m') ensures the following rotation equations hold:

$$\frac{\tilde{\mu}_k^a}{\mu_k^a} \times \frac{\tilde{\mu}_{l'}^b}{\mu_{l'}^b} \times \frac{\tilde{\mu}_{m'}^c}{\mu_{m'}^c} = 1 = \frac{\tilde{\mu}_{k'}^a}{\mu_{k'}^a} \times \frac{\tilde{\mu}_{l'}^b}{\mu_{l'}^b} \times \frac{\tilde{\mu}_{m'}^c}{\mu_{m'}^c}.$$

Hence, we have $\frac{\tilde{\mu}_k^a}{\mu_k^a} = \frac{\tilde{\mu}_{k'}^a}{\mu_{k'}^a}$. Note that rotation equation also holds at (k, l, m):

$$\frac{\tilde{\mu}_k^a}{\mu_k^a} \times \frac{\tilde{\mu}_l^b}{\mu_l^b} \times \frac{\tilde{\mu}_m^c}{\mu_m^c} = 1,$$

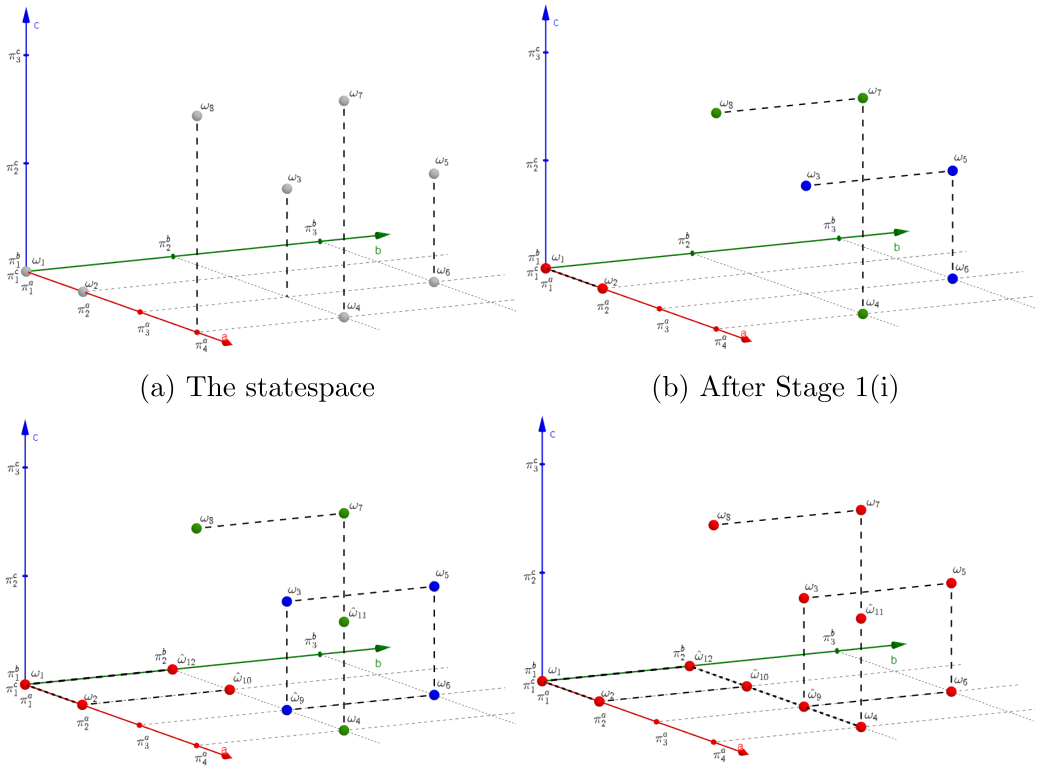
so, by replacing $\frac{\tilde{\mu}_k^a}{\mu_k^a}$ with $\frac{\tilde{\mu}_{k'}^a}{\mu_{k'}^a}$, we see the rotation equation also holds at (k', l, m). Finally, given double irreducibility, the terminal $\Pi^* = \{\Omega^*\}$ means this local property is expanded globally across all (k, l, m) by iterations. Now, as in the proof of Theorem 1, the ratio $\frac{\tilde{\mu}_k^a}{\mu_k^a}$ is constant across k for any fixed $l, m.$, so that this two marginals concerning agent a are the same. Therefore, we find a common prior for $\{a, b, c\}$. □

Proof for Theorem 2. Induction on the size of a subgroup of agents with extension lemma and Lemma 4. □

Example. Consider an Aumann model with 3 agents $\{a, b, c\}$ and 8 states below.

In Figure (a), we depict the partitions in the translated Harsanyi model, where each axis in the graph represents one agent. For example, agent a (the red axis) has 4 information sets $\{\pi_1^a = \{\omega_1\}, \pi_2^a = \{\omega_2\}, \pi_3^a = \{\omega_3, \omega_5, \omega_6\}, \pi_4^a = \{\omega_4, \omega_7, \omega_8\}\}$. The 8 states are colored in white in step 0 (Figure (a)). We only display and color some necessary vertices and $\sim^{\Omega^*}$ links (dashed) that demonstrate double irreducibility

[9] The assumption requires strict positiveness of $p_\omega^i(\omega)$.



(a) The statespace (b) After Stage 1(i)

(c) After Stage 1(ii) (d) After Stage 2(i)

Figure 1: An example of the virtual state procedure

through the resulted singleton partition.

Stage 1. In step (i), $\sim^{\Omega^*}$-links are made between qualified states, and we color those $\sim^{\Omega^*}$-linked states correspondingly (Figure (b)). Then in step (ii), 4 virtual states $\hat{\omega}^9, \ldots, \hat{\omega}^{12}$ are created due to the $\sim^{\Omega^*}$-links made in the last step (Figure (c)). We then enter the next iteration as the stopping criteria are not met yet.

Stage 2. In step (i), a few more $\sim^{\Omega^*}$-links and coloring are made due to the emergence of the virtual states. We only display the communicating relation throughout $\hat{\omega}^9, \ldots, \hat{\omega}^{12}$ (Figure (d)).

From step (ii), we can continue and ultimately add and link together all $4 \times 3 \times 3$ vertices as virtual states. But so far it is already clear from the $\sim^{\Omega^*}$-links through $\hat{\omega}^9, \ldots, \hat{\omega}^{12}$ the terminal Ω^* would be doubly irreducible with $\Pi^* = \{\Omega^*\}$, implying that "two means all" holds for this Aumann model.

3.3 What we exclude: another example

Now we would like to take a step back and see what is lost by restricting to those doubly irreducible model structures. In the leading example, the procedure stops right at the beginning step with no $\sim^{\Omega^*}$ relation constructed as no two states simultaneously lie in a same partition element for at least two agents. So, the terminal extended space is just Ω itself, and by definition this model structure is not doubly irreducible. Theorem 2, or simply Lemma 4, implies two will not necessarily mean three here, which is confirmed by our arguments after the example.

Meanwhile, the following example, also with three agents, serves to justify our restriction from the opposite direction.

Example (Structural admissibility)**.** There are three agents a, b and c, each with three information sets that partition the states $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.

Types	ω_1	ω_2	ω_3	ω_4
π_1^a	$\bullet$	$\bullet$		
π_2^a			$\bullet$	
π_3^a				$\bullet$
π_1^b	$\bullet$			
π_2^b		$\bullet$	$\bullet$	
π_3^b				$\bullet$
π_1^c	$\bullet$			$\bullet$
π_2^c		$\bullet$		
π_3^c			$\bullet$	

Table 2: Example of structural admissibility.

Here, it can be easily checked that the model structure is not doubly irreducible. In fact, we can see from Figure 2 below that the model structure is very "loose" in the sense that only the three ratios between the (potential) prior probabilities at certain pair of states are fixed by posteriors, namely, the ratios $\frac{p(\omega_1)}{p(\omega_2)}, \frac{p(\omega_2)}{p(\omega_3)}$, and $\frac{p(\omega_3)}{p(\omega_4)}$. Such a common prior for all three agents always exists, and of course so does a common prior for any pair within the three. Therefore, the situation here is "unconditionally" consistent for all, rather than the more interesting question of whether "two *means* all" or not. Note that the extension does not hold here either.

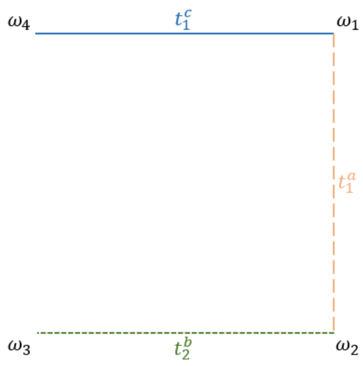


Figure 2: Example of structural admissibility - only ratios between probability on three pairs of states are fixed for the potential common prior and therefore, a compatible one always exists.

4 Applications

4.1 No-trade and common priors

The “two means all” result, viewed together with the no-trade and no-bet literature, generates interesting implications. For example, in a simple setup, Morris (1994) shows that a group of risk neutral agents with given posteriors can bet among themselves and enjoy positive interim payoff if and only if their posteriors do not admit a common prior.

Here we follow the notations from Samet (1998a) and Morris (2020). Consider a group of risk-neutral agents $\mathcal{I}$ in the Aumann model. Let $f = (f^1, f^2, ..., f^n)$ be a trade where $f^i \in \mathbb{R}^\Omega$ is a state contingent random variable specifying for agent i her payoff at state ω, and satisfies $\sum_{i\in\mathcal{I}} f^i = 0$. A trade is acceptable if the interim expected payoff $p^i_\omega \cdot f^i \geqslant 0$ for each agent i and state ω with the inequality being strict for some i' and ω' . Theorem 1 can be applied here to show that in fact we can focus on bipartite trades that involves only two agents a time.[10]

[10]We can tell from the example in Section 3.1 that the equivalence between the two notions are not free. Suppose there is an acceptable trade $f^a = f, f^b = -f$ for agents a and b. Now, π^a_1 requires $f^a(\omega_1) > 0$, so for π^b_2, $f^b(\omega_1) < 0 \implies f^b(\omega_3) > 0$. Then, for π^a_2, $f^a(\omega_3) < 0 \implies f^a(\omega_2) > 0$, so that $f^b(\omega_2) < 0$ which means for b the trade is unacceptable as it generates negative expected payoff at state ω_2. A similar argument shows that there is no acceptable trade between another two pairs. However, generically it is easy to find a trade for all because of the flexibility brought by the

Corollary 1. When the model structure is doubly irreducible, the following are equivalent:

(i) The agents' beliefs are consistent.
(ii) There exists no acceptable trade.
(iii) For any two agents in $\mathcal{I}$, there is no acceptable trade between them.

Remark. Note that results like Corollary 1 can be further applied to other scenarios involving no-trade/betting-like conditions, for instance, Gilboa et al. (2014)[11]. For the financial markets, the corollary sheds light on increasing regulations on bilateral and central clearing trading especially since the 2008 financial crisis (see e.g., Domanski et al. (2015)). For informed traders, when the assumptions and conditions are satisfied, successful central clearing implies the possibility of a direct bilateral trade, as the former means grand inconsistency which suggests the existence of pairwise inconsistency. Thus, we provide a normative support for regulations on central clearing mandate as without such mandate, market players would directly resort to bilateral trading that entails more counterparty risk (see e.g., Acharya and Bisin (2014) for a theoretical analysis).

The Corollary can also help understand the situation where a Harsanyi model is not full-support. Suppose a group of agents in the model have inconsistent beliefs. Then, there is a trade acceptable to all agents. If we can find a set of sufficiently "nearby" beliefs by perturbation, all of which are full-support, such that the trade is still acceptable to all, then, for this "nearby" belief profile, pairwise inconsistency follow from the Corollary. This also connects to the theory of conditional probability system (CPS): if we set the agent's types/partitions as the events in the CPS, then any non-full-support belief profile is a limit of full-support ones (Theorem 1.4, Myerson

sharing effect of a third agent. Our main result and its extension below can be seen as fixing this extra flexibility by restricting the model structure.

[11]Gilboa et al. (2014) propose a refined notion of Pareto dominance which they call no-betting-Pareto dominance, where a shared-belief constraint is levied on candidate improvements upon the traditional Pareto criteria. An allocation f dominates another g if, upon the classic Pareto improvement being satisfied, there exists a common belief for all those *affected* by the shift from f to g such that they would also be willing to accept the new allocation, had this hypothetical belief been their true belief. The intuition is that when agents agree on an collective deviation, it should not be the case where the agreement is reached only based on "irreconcilable" beliefs. Gilboa et al. (2014) show that this additional shared-belief requirement is equivalent to a no-betting condition. If the model structure is doubly irreducible, Theorem 1 and Corollary 1 can be applied to the case of risk-neutral agents and simplify the check/search on this shared-belief.

(1991)).

4.2 Common priors under endogenous uncertainty

We demonstrate here how our result can be directly related to other results that revise the classical Harsanyi model. For example, Guarino and Tsakas (2021) study the existence of common priors in an extended Harsanyi model where extra uncertainty is introduced from agents' beliefs about their opponents' actions. The priors and posteriors are defined as probability measures on the *product* space of types and strategies. Formally, the space $S = \times_{i \in \mathcal{I}} S^i$ now captures the strategy profiles with each S^i being exactly agent i's strategy space.

In particular, they are interested the following restricted subsets of priors and bets. A prior P satisfies *Action Independence (AI)* if it generates posteriors invariant across agents' own strategies for each fixed type, i.e. $P(s^{-i}, t^{-i}|s^i, t^i) = P(s^{-i}, t^{-i}|s'^i, t^i)$ for each agent i and every $s^i, s'^i, t^i, s^{-i}, t^{-i}$ in the respective space. A bet f on the state space satisfies *Strategy Invariance (SI)* if each agent i's interim payoff does not depend her strategy taken, i.e. $\mathbb{E}[f|s^i, t^i] = \mathbb{E}[f|s'^i, t^i]$ for every s^i, s'^i, t^i. To restore the classical no-betting style result, auxiliary Harsanyi types are constructed to translate their model to a standard Harsanyi model *without* endogenous uncertainty. The equivalence is then established between the (non-)existence of *SI* trades and belief consistency in terms of *AI* priors ((i) and (ii) in the following corollary). Theorem 1 can thus be applied to this intermediate step and establish the equivalence between (ii) and (iii).

Corollary 2. Under Assumption 2, the following are equivalent:

(i) There exists an *AI* common prior.
(ii) There exists no acceptable *SI* bet.
(iii) For any two agents in $\mathcal{I}$, there is no mutually acceptable *SI* bet between them.

4.3 Pareto optimal allocations with ex ante/interim trade

Billot et al. (2000) and Kajii and Ui (2009) study the existence of Pareto optimal allocations with uncertainty and the relation to the existence of a common prior among agents in a non-Bayesian utility framework. Consider a two-period pure-exchange economy, where each agent's endowment in period 2 is contingent on the realisation of the state $\omega \in \Omega$. They allow each agent i to have a convex set of priors $\mathcal{P}^i$ over the states Ω. (A common prior exists if the intersection of respective $\mathcal{P}^i$'s are non-empty.) The utility of agent i is in the maxmin expectation form (Gilboa and Schmeidler, 1989): for a consumption bundle $C^i(C^i(\omega))_{\omega\in\Omega}$,

$$V^i(C^i) = \min_{P^i \in \mathcal{P}^i} \mathbb{E}_{P^i} U^i(C^i),$$

where $U^i : \mathbb{R}_+ \to \mathbb{R}$ is the underlying utility for consumption in a particular state.[12]

By Samet (1998a), all priors of an agent i in the Aumann model is exactly the convex hull of her beliefs $\{p^i_\omega\}_{\omega\in\Omega}$. This is also the approach taken by Kajii and Ui (2009) where, with this extra step, they extend the original results for ex-ante trades in Billot et al. (2000) to interim trades and further include Bewley's incomplete information preferences. The following equivalence holds for both maximin expected utility and Bewley's preferences.

An allocation is full-insurance if it is constant (almost surely) over the state space Ω. The above papers establish the equivalence between (i) and (ii), while our result adds (iii) to the equivalence.

Corollary 3. Suppose that for each agent i, the uncertainty of priors $\mathcal{P}^i$ is the convex hull of a finite number of beliefs, so that it induces an Aumann model. If the induced Aumann model is doubly irreducible, then the following are equivalent:

(i) There exists a common prior for all agents.
(ii) There exists an interior full-insurance Pareto optimal allocation.
(iii) For any two agents in $\mathcal{I}$, there exists an interior full-insurance Pareto optimal allocation between them.

[12] Billot et al. (2000) only consider the case without aggregate uncertainty in endowment, and the restriction is dropped by Rigotti et al. (2008) by including Billot et al. (2000) as a special case of theirs.

4.4 Iterated expectations

Samet (1998b) establishes the connection between belief compatibility and the convergence of iterated expectations of random variables on Ω in the Aumann model.[13] Formally, a random variable on Ω is a real-valued function on Ω, which is considered as a column vector in $\mathbb{R}^{\Omega}$. Then, agent i's expectation of f, denoted $E^i f$, is the random variable $(E^i f)(\omega) = p^i_\omega \cdot f$. For a subset $\mathcal{I}' \in \mathcal{I}$, we call a sequence $s = (i_1, i_2, \dots)$ composed of elements of $\mathcal{I}'$ an $\mathcal{I}'$-sequence if for each agent $i \in \mathcal{I}'$, $i = i_k$ for infinitely many k's. The iterated expectation of a random variable f on Ω with respect to the $\mathcal{I}'$-sequence s is the sequence of random variables $(E^{i_k} \cdots E^{i_1} f)_{k=1}^{\infty}$. Samet shows that for a fixed f, any iterated expectation regarding a group of agents $\mathcal{I}'$ converges to a same limit if and only if the agents in $\mathcal{I}'$ share a common prior. Our Theorem 1 now implies that the following equivalence result.

Corollary 4. When the model structure is doubly irreducible, the following are equivalent:

(i) The agents' beliefs are consistent.

(ii) For each random variable f, it is common knowledge in each state that the iterated expectations of f, with respect to all $\mathcal{I}$-sequences s, converge to the same limit.

(iii) For any two agents a, b in $\mathcal{I}$ and each random variable f, it is common knowledge in each state that the iterated expectations of f, with respect to all $\{a, b\}$-sequences s, converge to the same limit.

Nehring (2001) further improves Samet's result by consider the following *mutual calibration* for finite sequences of agents. Say the agents $\mathcal{I}$ are mutually calibrated if for any random variable f, there does not exist a finite sequence $s = (i_1, i_2, \dots, i_k)$ such that it is common knowledge in each state that $E^{i_k} \cdots E^{i_1}(f - E_{i_1} f) > 0$, or it is common knowledge in each state that $E^{i_k} \cdots E^{i_1}(f - E_{i_1} f) < 0$. Nehring notices the gap between pairwise mutual calibration and mutual calibration for more than two agents through an example similar to ours. We now have the following.

Corollary 5. When the model structure is doubly irreducible, the following are equivalent:

[13] Golub and Morris (2017) extend the result to the expectations space and apply the results to their network model.

(i) The agents' beliefs are consistent.

(ii) The agents are mutually calibrated.

(iii) Any two agents a, b in $\mathcal{I}$ are mutually calibrated.

4.5 The cycles condition for consistency

Consider the consistency problem for a group of agents $\mathcal{I}$ in the Aumann model. The following graphic description is from Rodrigues-Neto (2009).

An edge is an ordered triple $e = (i, \omega, \omega')$ where both ω, ω' are in the same partition element of i. A cycle is finite sequence of edges, $c = \{e_k^c = (i_k, \omega_k, \omega_k')\}_{k=1}^K$ such that $\omega'^k = \omega^{k+1}$ (in mod K sense). The opposite edge of $e = (i, \omega, \omega')$ is $e^- = (i, \omega', \omega)$, while the opposite cycle c^- is obtained by replacing its edges with their opposite edges and reversing the order. The weight of an edge $e = (i, \omega, \omega')$ is $\theta^e = p^i(\omega'|\omega)$. Hellwig (2013) shows that when the partitions satisfy $\pi^i \cap \pi^j$ for all agents i, j and their partition elements π^i, π^j, the agents' beliefs are consistent if and only if for every cycle with length at most 4, the following cycle equation holds:

$$\Pi_{k=1}^K \theta^{e_k^c} = \Pi_{k=1}^K \theta^{e_k^{c^-}},$$

i.e. the product of weights are equal for the original and the opposite cycles. Rodrigues-Neto (2009) relaxes the assumption on the support but require additional checks on all cycles. The cycle equation naturally connects to our ratio matrix: the ratio $\theta^e/\theta^{e^-} = p^a(\omega'|\omega)/p^b(\omega|\omega')$ in the Aumann model becomes $R^{ab}(t^{-ab}, s) = P_k^a(t_l^b, t^{-ab}, s)/P_l^b(t_k^a, t^{-ab}, s)$ in the transformed Harsanyi model, with ω, ω' transformed into types t_k^a, t_l^b for a, b.

When double irreducibility is satisfied, our Theorem 2 is an even further simplification upon Hellwig's: we "return" to the original Harsanyi's condition, with the check being made on *two* agents a time, while a cycle of length four typically involves four agents.

Corollary 6. When the model is doubly irreducible, the agents' beliefs are consistent if and only if the cycle equation holds for all cycles of length 4 involving two agents.

5 Related Literature and Discussions

The consistency among posterior beliefs is first analysed, naturally, by Harsanyi (1967/68) in his seminal paper on games of incomplete information, where he proposed what we now call the Harsanyi model. Milgrom and Stokey (1982) link the consistency among agents' beliefs to the possibility of trade among them in a general utility setting. Morris (1991, 1994), Feinberg (1996, 2000), and later Samet (1998a) strengthen the link to a series of equivalences, the former stated in a state space model while the latter two in the Harsanyi model.[14,15] Samet simplifies the proof with a clever application of the separation theorem while bringing along a neat geometric interpretation.[16] Along this line of research, Ng (2003) proves the classical results in the more general (continuous) setting by extending the idea in Samet's proof and invoking the duality theory. Ng also introduces the idea of checking consistency in subsets of agents, which is a recurring theme in our analysis. Our main results naturally fit to this literature by clarifying the role of Harsanyi's original condition through an emphasis on pairwise consistency and pairwise trades. It is interesting to see how the analysis is boosted and simplified with the new focus on marginal probabilities induced by a prior.

Recently, there has been a few belief-based study on information design problems. For example, Arieli et al. (2021) examine the feasibility of joint (first-order) posterior beliefs. In the Appendix, we describe their problem in our setting and draw connections between the relevant results. Compared with agents in our original model, the agents here are equipped with "blurred" initial posteriors in the sense that they only hold first order beliefs on events measurable in the payoff uncertainty. We "recover" the state space by reconstructing the virtual states that correspond to intersections of types and the payoff uncertainty. We prove a similar *conditional* "two means all" result, though independent of Theorem 1, again with utilization of marginals and

[14]Morris and Feinberg both independently establish the results in their thesis, though in different models, without knowing the existence of the other until their respective publication.

[15]Heifetz (2006) develops a simple proof for Feinberg (2000) and applies the no-trade results to establish a "positive foundation" for the consistency of beliefs in terms of knowledge among agents. Lehrer and Samet (2014) further clarify different "levels" of consistency by linking three notions of epistemic consistency with corresponding trade consistency, establishing the equivalence in a countable state space. Notably, the three equivalences coincide in the finite case.

[16]Morris (2020) gives an interesting narrative on the development of this strand of literature from his own perspective.

the idea of *extension*. More generally, Corrao and Morris (2021) provides a no-trade interpretation of feasible distributions of belief hierarchies and their "coarsenings" through a duality approach, which they further apply to solving information design problems. Our results provides a simplification in narrowing down the candidate support of joint distributions for such works.

In a different line of research, Samet (1998b) establishes a connection between consistency and the convergence of iterated expectations among agents. As we introduced earlier, our result also fits into this analysis by highlighting the implications of pairwise convergence. Recently, Golub and Morris (2017) study a relaxed problem by focusing on some meaningful subsets of expectations. One of the equivalence result for consistency is then applied to their analysis in networks. Lipman (2003), on the other hand, investigates finite order beliefs and knowledge. He demonstrates that in a finite model, any given finite order beliefs and knowledge up to a given level can be generated by a finite model with a common prior.

Finally, there is a literature trying to find and simplify necessary and sufficient conditions of belief consistency solely in terms of the given posterior beliefs, as we do in this paper. Apart from the aforementioned work by Rodrigues-Neto (2009) and Hellwig (2013), Rodrigues-Neto (2012, 2014) also looks deep into his cycle conditions and obtain a few results further reducing the number of cycles necessary for check for consistency. As we remarked earlier, while these papers look at subsets of posterior beliefs, we take each agents' posterior beliefs as a whole and compare "pairs" of agents, thus leading to different form of results and generate different implications. A similar question is also asked and answered in a different direction by Hellman and Samet (2012). In their paper, the structure-oriented approach (as we do for the extension) is also taken to study the set of consistent posterior profiles. When the model is finite, the set of consistent posteriors is either very large (including all positive posterior profiles) or very small (nowhere dense), though those large cases are non-generic. The difference stems from the "tightness" of the model structure, a notion to measure the "robustness" of connectivity of the states in the model.[17]

[17] Specifically, in a model where the agents' partitions have a singleton meet (i.e. $\{\Omega\}$), tightness requires any refinement of the partitions will also refine the meet. Roughly speaking, tightness measures the how common certain two states fall in the same partition of an agent, while connectivity looks at the situations when two states simultaneously fall in two or more partitions. It can be shown that our notion of connectivity is parallel with their tightness in the following sense. First,

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a connected model structure could be either tight or non-tight, though the more interesting case is the non-tight one. Second, an unconnected model could also be either tight (our leading example) or non-tight.

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Appendix: Θ-Measurability and Feasible Joint Posteriors

The existence of common prior naturally relates to the question of feasibility of joint posterior beliefs studied by Arieli et al. (2021). In the following, we describe their problem in our setting and draw connections between the relevant results.

The following restatement of the problem is due to Morris (2020). Let there be $I+1$ agents $\mathcal{I} \cup \{0\}$. A payoff relevant parameter $\theta \in \Theta = \{\theta_1, \ldots, \theta_N\}$ plays the role of our s, but we keep Morris' notation for consistency with his definitions. Each agent $i \in \mathcal{I}$ now has a posterior belief $\pi^i(\theta|t^i) \in \Delta\Theta$. Agent 0 is uninformed with a prior $\pi^0(t, \theta) \in \Delta(T \times \Theta)$.

Definition 7 (Θ-consistency, Definition 8 in Morris (2020))**.** Beliefs $(\pi^i)_{i=0}^I$ are Θ-consistent if there exists, for each $i \in \mathcal{I}$, a $\mu^i : T^i \to \mathbb{R}_{++}$ such that

$$\mu^i(t^i)\pi^i(\theta|t^i) = \pi^0(t^i, \theta) \tag{2}$$

for all $i \in \mathcal{I}, t^i \in T^i, \theta \in \Theta$.

Compared with agents in our original model, the agents here are equipped with "blurred" initial posteriors in the sense that they only hold first order beliefs on events measurable in θ. In broad terms, if we consider the following partition of the state space: $T \times \Theta = \bigcup_{\theta \in \Theta} \{\theta\} \times T$, the events here are exactly those partition elements. Our original model, on the other end, corresponds to the case where the state space is trivially partitioned into singletons. Now, for this "blurred" family of beliefs, we introduce the counterpart of our notion of consistency. Effectively, we are "recovering" the state space by reconstructing the virtual states that correspond to intersections of types T^i and the payoff uncertainty Θ. Let $P \in \Delta T$ be a distribution over the posteriors.

Definition 8 (Feasible joint posterior beliefs, Definition 14 of Morris (2020))**.** Distribution P is a feasible joint distribution of the collection of posterior beliefs $(\pi^i(\theta|t^i))_{i \in \mathcal{I}}$ if there exists a $\pi^0 \in \Delta(T \times \Theta)$ such that (1) $P = \text{marg}_T \pi^0$, and (2) beliefs $(\pi^i)_{i=0}^I$ are Θ-consistent, i.e. (2) holds for all $i \in \mathcal{I}, t^i \in T^i,$ and $\theta \in \Theta$.

An alternative but useful definition is following.

Definition 9 (Feasibility through joint signals))**.** Distribution P is a feasible joint distribution of the collection of posterior beliefs $(\pi^i(\theta|t^i))_{i\in\mathcal{I}}$ if there exists a $\bar{\pi} \in \Delta\Theta$ and $\tilde{\pi}(\cdot|\cdot) : \Theta \to \Delta T$ such that beliefs $(\pi^i)_{i=0}^I$ are Θ-consistent, where $\pi^0(t,\theta) = \bar{\pi}(\theta) \cdot \tilde{\pi}(t|\theta)$.

Definition 10 (Pairwise feasibility implies feasibility)**.** We say "pairwise feasibility implies feasibility" holds if the following (i) implies (ii):

(i) For every pair of agents $a, b \in \mathcal{I}$, there exists a $\pi^{ab} \in \Delta(T \times \Theta)$ (resp. $\bar{\pi}^{ab}, \tilde{\pi}^{ab}$) and μ^a_{ab}, μ^b_{ab} such that both $P = \mathrm{marg}_T \pi^{ab}$ and equation (2) hold for $i = a, b$;

(ii) There exists a $\pi^0 \in \Delta(T \times \Theta)$ (resp. $\bar{\pi}, \tilde{\pi}$) such that (1) $P = \mathrm{marg}_T \pi^0$, and (2) beliefs $(\pi^i)_{i=0}^I$ are Θ-consistent (resp. for $\pi^0(t,\theta) = \bar{\pi}(\theta) \cdot \tilde{\pi}(t|\theta)$).

Proposition 2. Pairwise feasibility implies feasibility if we require additionally that the signals $\tilde{\pi}^{ab}$ in (i) have the same univariate marginals (i.e. $\tilde{\pi}^{ab}(t^i|\theta)$) across all pairs (a, b).

Proof. Suppose now we already have pairwise π^{ab}. Note that given a joint distribution P, by equation (2),

$$\mu^i_{ab} = \mathrm{marg}_{T^i} \pi^{ab} = \mathrm{marg}_{T^i}\ \mathrm{marg}_T \pi^{ab} = \mathrm{marg}_{T^i} P \tag{3}$$

is invariant across b. So, we can replace all these μ^i_{ab} with a μ^i indicating this independence. Apply equation 2 to pair a, b, and we have $\pi^{ab}(t^a,\theta) = \mu^a(t^a) \cdot \pi^a(\theta|t^a)$. Adding them up across all t^a, we can see that $\pi^{ab}(\theta)$ is invariant across b. Thus, there exists a $\bar{\pi} \in \Delta\Theta$ such that

$$\bar{\pi} = \mathrm{marg}_\Theta \pi^{ab},\ \forall a, b \in \mathcal{I}. \tag{4}$$

Now, fix a pair $a, b \in \mathcal{I}$, for any $i \in \mathcal{I}\backslash\{a, b\}$,

$$\pi^{ab}(t^i,\theta) = \pi^{ab}(t^i|\theta) \cdot \pi^{ab}(\theta) = \pi^{ai}(t^i|\theta) \cdot \pi^{ai}(\theta) = \pi^{ai}(t^i,\theta) = \mu^i(t^i)\pi^i(\theta|t^i).$$

The first equation is by applying (2) to pair a, b; the second is due to the assumption on common univariate marginals of signals $\tilde{\pi}^{ab}$ and (4); the third is by definition of conditional probability; the last is again by applying (2), but to pair a, i. This means that π^{ab} actually satisfies equation (2) for all $i \in \mathcal{I}$. Note that $P = \text{marg}_T \pi^{ab}$ holds by assumption. We can now conclude that π^{ab} satisfies both (1) and (2) in part (ii) of Definition 8, and therefore two means all holds. $\square$

Proposition 2 bears a similarity in its form with our main theorem. As the agents are holding blurred beliefs and hence "easier" to agree on a common pairwise prior, it is natural to require the equivalence here be built on a stronger requirement on the "two" side. Though the result is proved independent of Theorem 1, a common feature shared in the establishment is the utilization of marginals and the idea of *expansion*. Still, one may ask if we are able to directly apply our Theorem 1 here. A seemingly possible way is to strengthen the conditions on $\pi^i(\theta|t^i)$ into those on full ("un-blurred") posteriors $p^i(t^{-i}, \theta|t^i)$. Formally, if we can find T-full-support posteriors $(p^i(t^{-i}, \theta|t^i))_{i\in\mathcal{I}}$ with $\pi^i(\theta|t^i) = \text{marg}_\Theta p^i(\cdot|t^i)$, such that for every pair $a, b \in \mathcal{I}$, there exists a $p^{ab} \in \Delta(T \times \Theta)$ satisfying $P = \text{marg}_T p^{ab}$ and $\mu^i(t^i)p^i(t^{-i}, \theta|t^i) = p^{ab}(t, \theta)$ for $i = a, b$, then by Theorem (1) and its proof, there exists a $\pi^0(= p^{ab}) \in \Delta(T \times \Theta)$ such that (i) $P = \text{marg}_T \pi^0$, and (ii) beliefs $(\pi^i)_{i=0}^I$ are Θ-consistent. However, this approach turns out to be overshooting, as the assumption in the "if" part is in itself too strong in the sense that the common prior can be immediately checked without consulting Theorem 1.

The "conditional" equivalence between feasibility and pairwise feasibility also corresponds to the negativity result in Arieli et al. (2021), where they show by example that there is indeed a substantial gap between their condition for two agents and the general condition for n agents. The difference can thus be understood as a reflection of the subtle difference in the problem formulation. The feasibility problem in this section is in fact two-fold: first, the posteriors in the support of P must be "consistent" within themselves so that a signal structure π^0 exists in terms of generating the posteriors given the prior probabilities; then, among those signal structures, we need one with its projection on T being exactly the distribution P. Our original common prior existence problem is similar to the first step as both deal with compatibility among the posteriors themselves.