1. Let (M, d) be a metric space and $C(M)$ the space of continuous functions $f : M \to \mathbb{R}$ with metric
$$D(f, g) = \sup_{x \in M} |f(x) - g(x)|.$$
We say that a set of functions $\mathcal{F} \subset C(M)$ is pointwise equicontinuous if for every $x \in M$ and $\varepsilon > 0$ there is a δ (depending on x and $\varepsilon > 0$) for which
$$|f(x) - f(y)| < \varepsilon$$
for all $f \in \mathcal{F}$ and y satisfying $d(x, y) < \delta$. If M is compact show that a pointwise equicontinuous family of functions is a equicontinuous.

2. Let $\{(X_i, d_i)\}_{i=1}^{\infty}$ be a sequence of metric spaces. Let $X = \prod_{i=1}^{\infty} X_i$ and for any $x = (x_i)_i, y = (y_i)_i \in X$ define,
$$d(x, y) := \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{d_j(x_j, y_j)}{1 + d_j(x_j, y_j)}.$$
   (a) Show that (X, d) is a metric space
   (b) Let $x^{(n)} \in X$ and $x \in X$. Show that $x^{(n)} \to x$ in (X, d) iff $x_i^{(n)} \to x_i$ in (X_i, d_i) for all i.
   (c) Consider the case $X_i = \mathbb{R}$ and $d_i(x, y) := |x - y|$ for all i. Then $\ell^p \subseteq X$ for all $1 \leq p \leq \infty$. Compare convergence in ℓ^p with convergence in (X, d).

3. Let $f : \mathbb{R}_{\geq} \to \mathbb{R}$ be a continuous function s.t. $\lim_{n\to\infty} f(nx) = 0$ for all $x \in \mathbb{R}$ (here $\mathbb{R}_{\geq} := \{x \in \mathbb{R} : x \geq 0\}$). Prove that $\lim_{x\to\infty} f(x) = 0$.

   Hint: apply the Baire Category theorem in the following steps:
   (a) Let $\varepsilon > 0$ and define $E_k := \{x : |f(nx)| \leq \varepsilon, \forall n \geq k\}$. Show that E_k is closed.
   (b) Show that $\mathbb{R}_{\geq} = \bigcup_k E_k$.
   (c) Conclude by the BCT that some E_k contains an interval (a, b).
   (d) Show that $\bigcup_{n\geq k}(na, nb) \supseteq (T, \infty)$ for some $T > 0$.
   (e) Conclude that $\limsup_{t\to\infty} |f(t)| \leq \varepsilon$ and so $\lim_{t\to\infty} f(t) = 0$

4. Let $S \subseteq \ell^{\infty}$ be the set of bounded sequences s.t.
$$\lim_{n\to\infty} |x_n| = 0$$
if $(x_n)_n \in S$.
   (a) Prove that S is closed.
   (b) Let B be the unit ball in ℓ^{∞}. Show that $S \cap B$ is a closed and bounded set but is not compact.
   (c) Fix $x \in S$. Let $\mathcal{F}$ be the set of $y \in \ell^{\infty}$ so that $|y_n| \leq |x_n|$ for all $n \geq 0$. Show that $\mathcal{F}$ is compact.

5. Let $C([0,1])$ be the set of continuous functions $f : [0,1] \to \mathbb{R}$ with

$$D(f,g) := \sup_{x \in [0,1]} |f(x) - g(x)|.$$

For any $f \in C([0,1])$ and $0 < \alpha \leq 1$ define,

$$\|f\|_\alpha := \sup_x |f(x)| + \sup_{x,y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

Let $S := \{f : \|f\|_\alpha \leq 1\}$. Show that S is a compact set.

6. Let $X = [0,1]^n$ be the unit cube in $\mathbb{R}^n$. Show that multivariate polynomials are dense in $C_b(X)$.

7. Let $(b_{ij})_{i,j=1}^\infty$ be a doubly semi-infinite matrix so that

$$\sup_i \sum_{j=1}^\infty |b_{ij}| < \infty$$

(a) For any $x \in \ell^\infty$ define a sequence $F(x) = (F(x)_i)_i$ by

$$F(x)_i = \sum_j b_{ij} x_j^2$$

Show that F is a continuous mapping from ℓ^∞ to ℓ^∞.

(b) Show that there is an $r > 0$ so that for any $a \in \ell^\infty$ with $\|a\|_\infty < r$ the equation

$$x = a + F(x)$$

has a unique solution $x \in \ell^\infty$.

8. Let $C_b(\mathbb{R})$ be the space of bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$ with metric

$$D(f,g) = \sup_{|x| \in \mathbb{R}} |f(x) - g(x)|.$$

(a) Find an equicontinuous, bounded sequence of functions of $C_b(\mathbb{R})$ that contains no subsequence that converges wrt D.

(b) Let $\mathcal{F}$ be an equicontinuous, uniformly bounded family of functions in $C_b(\mathbb{R})$ with the property that for all $\varepsilon > 0$ there is an $N > 0$ so that

$$|f(x)| \leq \varepsilon$$

for all $f \in \mathcal{F}$ and $|x| > N$. Show that sequences of functions from $\mathcal{F}$ have uniformly convergent subsequences.

9. Let X be a Banach space and $F : X \to X$ a contraction. Define $G(x) := x - F(x)$. Show that G is a bijection with continuous inverse.