Real Analysis

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June 2024

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I. Real Numbers

Definition. If $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers, and $L \in \mathbb{R}$, we say the sequence converges to L, if for any $\epsilon > 0$, there's an integer $N = N(\epsilon) > 0$, such that

$$|a_n - L| < \epsilon$$

for all $n \geq N$. In this case, we write

$$\lim_{n\to\infty} a_n = L.$$

Example. If $\alpha > 0$, then $\lim_{n\to\infty} n^{-\alpha} = 0$.
Proof. Given $\epsilon > 0$, we need an $N = N(\epsilon) > 0$, such that $|n^{-\alpha}| < \epsilon$. Note if $n \geq N$, then $n^{\alpha} \geq N^{\alpha}$, that is $\frac{1}{N^{\alpha}} \geq \frac{1}{n^{\alpha}}$. We want $\frac{1}{N^{\alpha}} < \epsilon$,that is $N > \epsilon^{-\frac{1}{\alpha}}$ Take $N = N(\epsilon) = \lceil \epsilon^{-\frac{1}{\alpha}} \rceil + 1$ For all $n \geq N$, we have

$$|n^{-\alpha}| < (\lceil \epsilon^{-\frac{1}{\alpha}} \rceil + 1)^{-\alpha} < \epsilon.$$

□

Definition. A sequence $\{a_n\}_{n=1}^{\infty}$ is called a Cauchy sequence if for any $\epsilon > 0$, there's an integer $N > 0$ so that for all $m, n \geq N$,

$$|a_n - a_m| < \epsilon.$$

Theorem A sequence $\{a_n\}_{n=1}^{\infty}$ on $\mathbb{R}$ is a Cauchy sequence if and only if there's an $L \in \mathbb{R}$ with $\lim_{n\to\infty} a_n = L$.

Proof. "$\Leftarrow$" Let $\lim_{n\to\infty} a_n = L$. Given $\epsilon > 0$, we choose $N > 0$ so that for all $n \geq N$, we have

$$|a_n - L| < \frac{\epsilon}{2}.$$

Then if $n, m \geq N$, we have

$$|a_n - a_m| = |(a_n - L) + (L - a_m)| \leq |a_n - L| + |a_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

The inequality holds by triangle inequality. This shows $\{a_n\}_{n=1}^{\infty}$ is Cauchy.
"$\Rightarrow$" Follows directly by principle of real numbers. □

Definition. A subset $S \subseteq \mathbb{R}$ is called complete if every Cauchy sequence $\{a_n\}_{n=1}^{\infty}$ in S converges to a point in S.

Example. $\mathbb{R}$ is complete. $(0, 1) \in \mathbb{R}$ is NOT complete since $\{\frac{1}{n}\}_{n=2}^{\infty}$ has limit 0, so $(a, b) \in \mathbb{R}$ is NOT complete. $\{\frac{1}{n} | n \in \mathbb{N}\}$ is NOT complete but $\{0\} \cup \{\frac{1}{n} | n \in \mathbb{N}\}$ is complete. $\mathbb{Q}$ is NOT complete.
Fact: $a_n = \sum_{k=0}^{n} \frac{1}{k!}$ converges to e, $a_n = \sum_{k=0}^{n} \frac{1}{k^2}$ converges to $\frac{\pi^2}{6}$.

Definition. A set $S \subseteq \mathbb{R}$ is bounded above(bounded below) if there is a real number M such that $s \leq M$ for all $s \in S$.(there is a real number m such that $s \geq m$ for all $s \in S$). We call $M(m)$ an upper(lower) bound for S. A set that is bounded above and below is called bounded.

Definition. Suppose a nonempty subset S of $\mathbb{R}$ is bounded above. Then L is the supremum or least upper bound for S if L is an upper bound for S that is smaller than all other upper bounds, i.e., for all $s \in S, s \leq L$, and if M is another upper bound for S, then $L \leq M$. It is denoted by $\sup S$. Similarly, if S is a nonempty subset of $\mathbb{R}$ which is bounded below, its infimum or greatest lower bound, denoted by $\inf S$, is the number L such that L is an lower bound and whenever M is another lower bound for S, then $L \geq M$.

Theorem Suppose a nonempty set $S \subseteq \mathbb{R}$ is bounded above. Then S has a unique least upper bound, denoted as $\sup(S)$.

Proof.

Definition. A sequence $\{a_n\}_{n=1}^{\infty}$ is called monotone nonincreasing(or nondecreasing) if $a_n \geq a_{n+1}$(or $a_n \leq a_{n+1}$) for all $n \in \mathbb{N}$.

Lemma Suppose a nonempty set $S \subseteq \mathbb{R}$ and $\sup(S) = B$. Then for any $\epsilon > 0$ there's an $s \in S$ with $|s - B| < \epsilon$.

Proof. Suppose not, that is, there is an $\epsilon > 0$ so that for all $s \in S$, we have $|s - B| \geq \epsilon$. Take $B' = B - \frac{\epsilon}{2}$. Then for any $s \in S$, $B' - s = B - \frac{\epsilon}{2} - s \geq \frac{\epsilon}{2}$. So B' is an upper bound for s, and $B' < B$, which leads to a contradiction. □

Theorem (Monotone Convergence Theorem)
Suppose that $\{a_n\}_{n=1}^{\infty}$ is monotone nonincreasing(nondecreasing) and is bounded below(above). Then $\{a_n\}_{n=1}^{\infty}$ converges.

Proof. Suppose that $\{a_n\}_{n=1}^{\infty}$ is monotone nondecreasing(i.e.$a_{n+1} \geq a_n$) and bounded above. Set $L = \sup\{a_n | n \in \mathbb{N}\}$. Given $\epsilon > 0$, using the preceding lemma, there's an $N \in \mathbb{N}$ such that $|a_N - L| < \epsilon$. As L is an upper bound, $a_N \leq L$. Also if $n \geq N$, then $a_N \leq a_n \leq L$. So $|a_n \leq L|$. Similar argument for nonincreasing case. □

Example. Let α be a positive real number, define $\{a_n\}_{n=1}^{\infty}$ by

$$a_1 = \sqrt{\alpha}, \ldots, a_{n+1} = \sqrt{\alpha + a_n}.$$

Write its limit as

$$\lim_{n \to \infty} a_n = \sqrt{\alpha + \sqrt{\alpha + \sqrt{\alpha + \ldots}}}$$

If $\lim_{n\to\infty} a_n = L$ ($L > 0$) exists, then $L = \sqrt{\alpha + L}$. I.e. $L^2 = \alpha + L$,which gives $L = \frac{1}{2} + \frac{\sqrt{1+4\alpha}}{2}$ since $L > 0$. To show the limit exists, we'll show $\{a_n\}_{n=1}^{\infty}$ is bounded above and increasing, then we could use Monotone Convergence Theorem.
Step 1: by induction, we will show that $a_n \leq \sqrt{\alpha} + 1$, for all $n \in \mathbb{N}$.
base case: $a_1 = \sqrt{\alpha} < \sqrt{\alpha} + 1$
inductive case: $a_{n+1} = \sqrt{\alpha + a_n} \leq \sqrt{\alpha + \sqrt{\alpha} + 1} < \sqrt{\alpha + 2\sqrt{\alpha} + 1} = \sqrt{\alpha} + 1$
Step : by induction, we will show that $a_{n+1} \geq a_n$, for all $n \in \mathbb{N}$.
base case: $a_2 = \sqrt{\alpha + \sqrt{\alpha}} > \sqrt{\alpha} = a_1$
inductive case: $a_{n+2} = \sqrt{\alpha + a_{n+1}} > \sqrt{\alpha + a_n} = a_{n+1}$.
By previous theorem, the limit exists. □

Lemma (Nested Interval lemma)
Suppose that $I_n = [a_n, b_n]$ is a sequence of nonempty closed intervals with I_n contains I_{n+1}. That is $I_1 \subseteq \cdots \subseteq I_n \subseteq I_{n+1} \subseteq \ldots$. Then $\cap_{n=1}^{\infty} I_n$ is nonempty.

Proof. Let $\{a_n\}_{n=1}^{\infty}$,$\{b_n\}_{n=1}^{\infty}$ be monotone nondecreasing, nonincreasing sequences, respectively. Also, $a_n \leq b_k$ and $b_n \geq a_k$ for all $n \in \mathbb{N}$, with $I_k = [a_k, b_k]$. By monotone convergence theorem, $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both converges, and call their limits a, b, respectively. Then $a_n \leq a \leq b \leq b_n$ for all $n \in \mathbb{N}$. So $[a, b] \subseteq \cap_{n=1}^{\infty} I_n$. □
Remark. It's important to note that I_n must be closed, since $\cap_{n=1}^{\infty}(0, \frac{1}{n}) = \varnothing$ and $\cap_{n=1}^{\infty}[n, \infty) = \varnothing$, but $\cap_{n=1}^{\infty}[0, \frac{1}{n}] = \{0\}$.

Definition. If $\{a_n\}_{n=1}^{\infty}$ is a sequence, a subsequence of $\{a_n\}_{n=1}^{\infty}$ is a sequence of the from $\{a_{n_k}\}_{k=1}^{\infty}$ with $n_1 < n_2 < \ldots n_k < \ldots$.
Especially, increasing map $\sigma : \mathbb{N} \to \mathbb{N}$, $\{a_{n_{\sigma(k)}}\}_{k=1}^{\infty}$.

Fact: If $\lim_{n\to\infty} a_n = L$, then $\lim_{k\to\infty} a_{n_k} = L$ for every subsequecne.
Example. $a_n = (-1)^{n+1}$, then $\lim_{n\to\infty} a_n$ does not exist. But $\lim_{n\to\infty} a_{2n+1} = 1$, $\lim_{n\to\infty} a_{2n} = -1$.

Theorem (Bolzano-Weierstrass Theorem)
Suppose that $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence(that is both bounded above and below). Then $\{a_n\}_{n=1}^{\infty}$ has a convergent subsequence.

Proof. Let b, B be a(n) lower(uppper) bound of $\{a_n\}_{n=1}^{\infty}$, respectively. Set $d = B - b$. Let

$$J_1 = [b, \frac{B+b}{2}],\ J_2 = [\frac{B+b}{2}, B]$$

At least one of these intervals contains infinitely many elements of $\{a_n\}_{n=1}^{\infty}$. Let I_1 be that interval and $\{a_{1_k}\}_{k=1}^{\infty}$ be the sequence of $\{a_n\}_{n=1}^{\infty}$ which contained in I_1. Inductively, construct I_{m+1} by bisecting I_m in the middle into J_{m+1}, J'_{m+1} and set I_{m+1} be one of these intervals that contains infinitely many elements of the sequence $\{a_{m_k}\}_{k=1}^{\infty}$. Set $\{a_{(m+1)_k}\}_{k=1}^{\infty}$ to be the subsequence of $\{a_{m_k}\}_{k=1}^{\infty}$ contained in I_{m+1}. These I_m's are a sequence of nested intervals with length $2^{-m}d$, so $L \in \cap_{m=1}^{\infty} I_m$ by nested sequence lemma. Next, if we set $\{a_{m_k}\}_{k=1}^{\infty}$ to be the subsequence. We could find $\lim_{k\to\infty} a_{m_k} = L$. □

II. Introduction to Series

II-1 Convergent Series

Definition. For a series $\sum_{n=1}^{\infty} a_n$, we define its partial sum as

$$S_n = \sum_{k=1}^{n} a_k$$

Definition. We say that a series $\sum_{n=1}^{\infty} a_n$ converges if

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k = S,$$

for a number $S \in \mathbb{R}$, and then

$$\sum_{n=1}^{\infty} a_n = S.$$

If the limit does not exist, we say $\sum_{n=1}^{\infty} a_n$ diverges.

Example. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. To show this, we first show a claim that its partial sums satisfies $S_{2^k} \geq \frac{k}{2} + 1$. We use induction to show the claim:
For base case $k = 0$, $S_{2^0} = S_1 = 1 \geq 1$
Inductive step: Assume $S_{2^k} \geq \frac{k}{2} + 1$ and

$$S_{2^{k+1}} = S_{2^k} + \sum_{m=2^k+1}^{2^{k+1}} \frac{1}{m} \geq \frac{k}{2} + 1 + \sum_{m=2^k+1}^{2^{k+1}} \frac{1}{2^{k+1}} = \frac{k}{2} + 1 + 2^k \frac{1}{2^{k+1}} = \frac{k+1}{2} + 1.$$

After showing the claim, we have an unbounded subsequence of $\{S_n\}_{n=1}^{\infty}$ which implies $\lim_{n\to\infty} S_n$ does not exist. I.e., $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. $\square$
Note that in the first Inequality, we use the assumption and since $m \leq 2^{k+1}$, $\frac{1}{m} \geq \frac{1}{2^{k+1}}$. And we write $S_n = 1 + \underbrace{\frac{1}{2}} + \underbrace{(\frac{1}{3} + \frac{1}{4})} + \cdots + \underbrace{(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k})}$, there are for 2^{k-1} terms in each brackets.
Example. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges. First note that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. The partial sums are called the telescope sum

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} (\frac{1}{k} - \frac{1}{k+1}) = 1 - \frac{1}{n+1}.$$

So $\lim_{n\to\infty} S_n = \lim_{n\to\infty}(1 - \frac{1}{n+1}) = 1$ $\square$
Similarly, $\sum_{n=1}^{\infty} \frac{1}{n(n+l)}$ converges for all $l \in \mathbb{N}$.
Example. We let $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, which is called Riemann Zeta function. We already know that $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. And $\zeta(2k)$ are known for $k \in \mathbb{N}$, but $\zeta(2k+1)$ are unknown.

Theorem If a series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof. Suppose $\lim_{n\to\infty} S_n = S$, then $a_n = S_n - S_{n-1}$, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - S_{n-1} = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$$

$\square$

Note. The converse of preceding theorem is not true, since $\lim_{n\to\infty} \frac{1}{n} = 0$, but $\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
Recall: Geometric Series:

$$\sum_{k=1}^{\infty} cr^k = c(r + r^2 + \cdots + r^n) = cr(\frac{1-r^n}{1-r}).$$

Thus if $c \neq 0$, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} cr(\frac{1-r^n}{1-r})$,

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} cr^k = \begin{cases} \frac{cr}{1-r}, & \text{if} |r| < 1 \\ \text{diverges}, & \text{if} |r| \geq 1 \end{cases}$$

Cauchy Criterion for Series
The following are equivalent for a series $\sum_{n=1}^{\infty} a_n$:
(1)The series converges
(2)For any $\epsilon > 0$, there's an $\mathrm{N} \in \mathbb{N}$ so that for all $n \geq \mathrm{N}$, we have $|\sum_{k=n+1}^{\infty} a_k| < \epsilon$. (Implicitly says $\sum_{k=n+1}^{\infty} a_k$ converges)
(3)For any $\epsilon > 0$, there's an $\mathrm{N} \in \mathbb{N}$ so that for all $m, n \geq \mathrm{N}$, we have $|\sum_{k=n+1}^{m} a_k| < \epsilon$. If $m > n$, this is $|S_m - S_n|$. This essentially says that $\{S_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

II-2 Comparison Test

Comparison Test for Series
Suppose that $\sum_{n=1}^{\infty} b_n$ converges, and $\{a_n\}_{n=1}^{\infty}$ is a sequence with $|a_n| \leq b_n$, then $\sum_{n=1}^{\infty} a_n$ converges, and

$$|\sum_{n=1}^{\infty} a_n| \leq \sum_{n=1}^{\infty} b_n.$$

Proof. Given $\epsilon > 0$, and choose $\mathrm{N} \in \mathbb{N}$ such that

$$|\sum_{n=\mathrm{N}+1}^{\infty} b_n| < \epsilon$$

note all $b_n \geq |a_n| \geq 0$, so $|\sum_{n=\mathrm{N}+1}^{\infty} b_n| = \sum_{n=\mathrm{N}+1}^{\infty} b_n$.
Now suppose that $m \geq \mathrm{N}$. Then

$$|\sum_{k=m+1}^{\infty} a_k| \leq \sum_{k=m+1}^{\infty} |a_k| \leq \sum_{n=m+1}^{\infty} b_k \leq \sum_{n=\mathrm{N}+1}^{\infty} b_k < \epsilon$$

For the first step we use triangle Inequality, the seoncd step follows from our assumption, and the last inequality holds since the series converge and all $b_n \geq 0$. By Cauchy criterion for series, series $\sum_{n=1}^{\infty} a_n$ converges $\square$

Proposition. Suppose that $a_n \geq 0$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n$ coverges if and only if the sequence of partial sums is bounded above.

Proof. Write $S_n = \sum_{k=1}^{n} a_k$. As $a_n \geq 0$ for all $n \in \mathbb{N}$, S_n is montone nondecreasing sequence, so that $\lim_{n\to\infty} S_n$ exists if and only if S_n is bounded above. $\square$

Remark. This proof needs all a_n to be same sign.

Example. $\sum_{k=1}^{\infty}(-1)^{k+1}$ is a sequence of partials $-1, 0, -1, 0, \ldots$ which are bounded, but this series does not converge.

> $\liminf$ **and** $\limsup$:
>
> **Definition.** Suppose that $\{a_n\}_{n=1}^{\infty}$ is sequence, we define
>
> $$\lim_{n\to\infty} \sup(a_n) = \lim_{n\to\infty} (\sup\{a_k | k \geq n\}),$$
>
> $$\lim_{n\to\infty} \inf(a_n) = \lim_{n\to\infty} (\inf\{a_k | k \geq n\}),$$
>
> we call $\{a_k | k \geq n\}$ the "tail of sequence".

Proposition. (1) $\lim_{n\to\infty} a_n = L$ if and only if $\lim_{n\to\infty} \sup(a_n) = \lim_{n\to\infty} \inf(a_n) = L$
(2) $\lim_{n\to\infty} \sup(a_n) \leq \lim_{n\to\infty} \inf(a_n)$ $\square$

Example. $\lim_{n\to\infty} \inf(\sin n) = -1$ and $\lim_{n\to\infty} \sup(\sin n) = 1$

Example. $\lim_{n\to\infty} \inf \frac{(-1)^{n+1}}{n} = 0 = \lim_{n\to\infty} \sup \frac{(-1)^{n+1}}{n}$, but $0 = \lim_{n\to\infty} \sup \frac{(-1)^{n+1}}{n}$ is not an upper bound, and $0 = \lim_{n\to\infty} \inf \frac{(-1)^{n+1}}{n}$ is not a lower bound.

Proposition. $\lim_{n\to\infty} \sup(a_n) < \infty$ if and only if the sequence is bounded above;
$\lim_{n\to\infty} \inf(a_n) > -\infty$ if and only if the sequence is bounded below. $\square$

II-3 Root Test

Root Test for Series

Suppose that $a_n \geq 0$ for all $n \in \mathbb{N}$ and let

$$L = \lim_{n\to\infty} \sup \sqrt[n]{a_n},$$

or if $\{a_n\}_{n=1}^{\infty}$ any sequence

$$L = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}.$$

(1)If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges,
(2)If $L > 1$, then $\{a_n\}_{n=1}^{\infty}$ diverges.

Remark. If $L = 1$, the series may or maynot converges

Proof. Suppose $L < 1$, choose an $r \in \mathbb{R}$ such that $L < r < 1$. Then, there's an $N \in \mathbb{N}$ with $0 < \sqrt[n]{a_n} < r$ for all $n \in \mathbb{N}$, define

$$b_n = \begin{cases} a_n, & \text{if } n < N \\ r^n, & \text{if } n \geq N \end{cases},$$

then $b_n \geq a_n$ for all $n \in \mathbb{N}$ since $a_n < r^n$. The tail $\sum_{n=N}^{\infty} b_n = \sum_{n=N}^{\infty} r_n$ is convergent, so $\sum_{n=1}^{\infty} b_n$ is converge and then $\sum_{n=1}^{\infty} a_n$ is convergent by comparison test.
Now, suppose $L > 1$, then there's a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ with $\sqrt[n_k]{a_{n_k}} > 1$, which implies $a_{n_k} > 1$. Then it's impossible for $\lim_{n\to\infty} a_n = 0$. $\Rightarrow$ $\{a_n\}_{n=1}^{\infty}$ diverges. $\square$
Remark. If $a_n \neq 0$ for some $n \in \mathbb{N}$, we can replace the proof with $L = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$. case.

II-4 Alternating Series

Definition. A series $\sum_{n=1}^{\infty} a_n$ is called alternating if

$$(-1)^n a_n \geq 0 \ (\text{or}(-1)^{n+1} a_n \geq 0)$$

for all $n \in \mathbb{N}$.

Theorem Suppose that $|a_n|$ is a monotone nonincreasing sequence and $\sum_{n=1}^{\infty} a_n$ is an alternating series. Then the series converges if and only if

$$\lim_{n\to\infty} a_n = 0$$

Example. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2)$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n}$ doesn't.

Definition. Suppose that $\sum_{n=1}^{\infty} a_n$ converges. We say it is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges, say it is conditionally convergent if it is not absolutely convergent.

Example. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2)$ is conditionally convergent.

Definition. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence, then a rearrangement is a sequence $\{a_{\pi(n)}\}_{n=1}^{\infty}$ $\pi : \mathbb{N} \to \mathbb{N}$ is any bijective function. π is called a permutation.

Theorem If $\sum_{n=1}^{\infty} a_n$ absolutely convergent, and $\pi : \mathbb{N} \to \mathbb{N}$ is any permutation, then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\pi(n)}$$

.

Proof. Suppose $\lim_{n\to\infty} a_n = L$, $L \in \mathbb{R}$, and π be any permutation from $\mathbb{N}$ to $\mathbb{N}$. Given

$\epsilon > 0$, we can find $N \in \mathbb{N}$ such that for all $n \geq N$

$$\sum_{k=n+1}^{\infty} |a_k| < \frac{\epsilon}{2}.$$

Since π map the set $\{1, \ldots, N\}(\subseteq \mathbb{N})$ to $\{\pi(1), \ldots, \pi(N)\}(\subseteq \mathbb{N})$, we could find an $M \in \mathbb{N}$ such that $M = max\{\pi(1), \ldots, \pi(N)\}$. Clearly, $M \geq N$. So for all $n \geq M$, we still have

$$\sum_{k=n+1}^{\infty} |a_k| < \frac{\epsilon}{2}.$$

And

$$|\sum_{k=n+1}^{\infty} a_{\pi(k)} - L| = |(\sum_{k=1}^{n} a_{\pi(k)} - \sum_{k=1}^{n} a_k) - (L - \sum_{k=1}^{n} a_k)| \leq |\sum_{k=1}^{n} a_{\pi(k)} - \sum_{k=1}^{n} a_k| + |\sum_{k=1}^{n} a_k - L|$$

by triangle inequality. Note that

$$|\sum_{k=1}^{n} a_{\pi(k)} - \sum_{k=1}^{n} a_k| \leq \sum_{k=n+1}^{\infty} |a_k|$$

,

$$|\sum_{k=1}^{n} a_k - L| \leq \sum_{k=n+1}^{\infty} |a_k|.$$

Therefore,

$$|\sum_{k=n+1}^{\infty} a_{\pi(k)} - L| \leq 2 \sum_{k=n+1}^{\infty} |a_k| < \epsilon,$$

which shows $\sum_{n=1}^{\infty} a_{\pi(n)}$ also converges to L. □

Theorem If $\sum_{n=1}^{\infty} a_n$ conditionally convergent, and $L \in \mathbb{R}$, L is also possibly infinite(i.e.$L = \pm\infty$), then there's a permutation $\pi : \mathbb{N} \to \mathbb{N}$ so that

$$\sum_{n=1}^{\infty} a_{\pi(n)} = L.$$

Proof.

III. Topology in $\mathbb{R}^n$

III-1 Norms and basis

First, we recall some basic definitions and some important inequalities from linear algebra:

Definition. For $\overrightarrow{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, its (Euclidean) norm is defined as

$$\|\overrightarrow{x}\| = \sqrt{\langle \overrightarrow{x}, \overrightarrow{x} \rangle} = \sqrt{x_1^2 + \cdots + x_n^2},$$

where $\langle , \rangle$ is the inner product of $\overrightarrow{x}$ defined as

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

for any $\overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^n$.

Remark. $\langle \overrightarrow{x}, \overrightarrow{x} \rangle \geq 0$ and the euqality holds if and only if $\overrightarrow{x} = 0$.

Schwartz Inequality:

Proposition. For any $x, y \in \mathbb{R}^n$, we have

$$\|\langle x, y \rangle\| \leq \|x\|\|y\|$$

and the euqality holds if and only if x, y are colinear(i.e., $x = \lambda y$ for some $\lambda \in \mathbb{R}$). $\square$

Remark. If angle between x, y is given by θ, then $\langle x, y \rangle = \|x\|\|y\| \cos\theta$

Triangle Inequality:

Proposition. For any $x, y \in \mathbb{R}^n$,

$$\|x + y\| \leq \|x\| + \|y\|$$

and the quality holds if and only if $x, y = 0$ or $x = \lambda y$ for some $\lambda \in \mathbb{R}_+$. $\square$

Definition. A subset $\{v_1, \ldots, v_n\} \subseteq \mathbb{R}^n$ is called an orthornormal basis if $\langle v_i, v_j \rangle = \delta_{ij}$

Recall some facts about orthornormal basis:

Proposition. Let $\{v_1, \ldots, v_n\}$ be orthornormal basis in $\mathbb{R}^n$.

(1)If $\overrightarrow{a} = \sum_{i=1}^n a_i v_i$, then $\|\overrightarrow{a}\|^2 = \sum_{i=1}^n a_i^2$

(2)For any $x, y \in \mathbb{R}^n$, $x = \sum_{i=1}^n \langle x, v_i \rangle v_i$, $\langle x, y \rangle = \sum_{i=1}^n \langle x, v_i \rangle \langle y, v_i \rangle$. $\square$

III-2 Limit points

Definition. A sequence $\{a_i\}_{i=1}^{\infty}$ in $\mathbb{R}^n$ converges to $a \in \mathbb{R}^n$, if for all $\epsilon > 0$, there's an $\mathrm{N} \in \mathbb{N}$, such that

$$\|a_i - a\| < \epsilon$$

for all $i \geq \mathrm{N}$. Or Equivalently,

$$\lim_{n \to \infty} \|a_n - a\| = 0.$$

Definition. If $A \subseteq \mathbb{R}^n$, a limit point of A is a point $a \in \mathbb{R}^n$, so that there's a sequence $a \in \mathbb{R}^n$, so that there's a sequence $\{a_k\}_{k=1}^{\infty}$ of element of A so that

$$\lim_{k\to\infty} a_k = a$$

Note. Any point $a \in A$ is a limit point of A.
Example. In set $(0,1)$, 0 is a limit point:

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$

1 is a limit point:

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$$

$\frac{1}{2}$ is a limit point:

$$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots$$

$x \in \mathbb{R}^n$ is a limit point of $[0,1]$ if and only if $0 \le x \le 1$.

III-3 Closed and Open Sets

Definition. A set is called closed if it contains all its limit points. if $A \subseteq \mathbb{R}^n$, then its closure, denoted $\overline{A}$ is the set

$$\overline{A} = \{x \in \mathbb{R}^n | x \text{ is a limit point of } A\ \}$$

Example. If $A = (0,1)$, then $\overline{A} = [0,1]$. If $A = [0,1]$, then $\overline{A} = A = [0,1]$.
Note. A set $A \subseteq \mathbb{R}^n$ is closed if and only if $\overline{A} = A$(proved in topology courses). Also, $A \subseteq \overline{A}$ for arbitrary sets .
Example. $\varnothing, \mathbb{R}^n$ are closed, $[0,\infty]$ is closed, singletons are closed
Proposition.
(1) If $A_1, \dots, A_k$ is a finite collection of closed subsets of $\mathbb{R}^n$, then $\cup_{i=1}^{k} A_i$ is also closed
(2) If $\{A_i\}_{i\in I}$ is any collection of closed subsets of $\mathbb{R}^n$, then $\cap_{i\in I} A_i$ is also closed.
Proof. For (1) If $\{a_l\}_{l=1}^{\infty}$ (which converges to a) is a sequence in $\cup_{i=1}^{k} A_i$, then since there are only finite A_i's, we can find an $i \in \{1, \dots, k\}$ such that A_i contains infinitely many elements of the sequence $\{a_l\}_{l=1}^{\infty}$, say $\{a_{l_m}\}_{m=1}^{\infty}$. Since $\{a_{l_m}\}_{m=1}^{\infty}$ is a subsequence of a convergent sequence, it converges to the same limit of $\{a_l\}_{l=1}^{\infty}$, i.e.a. Note that A_i is a closed set and $\{a_{l_m}\}_{m=1}^{\infty}$ is a sequence in A_i, we conclude that its limit $a \in A_i$. But $A_i \subseteq \cup_{i=1}^{k} A_i$, $a \in \cup_{i=1}^{k} A_i$. This proves the limit of a convergent sequence in $\cup_{i=1}^{k} A_i$ is also in $\cup_{i=1}^{k} A_i$. So $\cup_{i=1}^{k} A_i$ is closed.
For (2) If $\{a_k\}_{k=1}^{\infty}$ (which converges to a) is a sequence in $\cap_{i\in I} A_i$, then for each i, $\{a_k\}_{k=1}^{\infty}$ is also a sequence in A_i, so that $a \in A_i$, as A_i is closed. But then $a \in \cap_{i\in I} A_i$. This proves the limit of a convergent sequence in $\cap_{i\in I} A_i$ is also in $\cap_{i\in I} A_i$. So $\cap_{i\in I} A_i$ is closed $\square$
Proposition.
(1) If $A \subseteq \mathbb{R}^n$, then $\overline{A}$ is closed, and $\overline{\overline{A}} = \overline{A}$
(2) If $C \subseteq \mathbb{R}^n$ is any closed subset and $A \subseteq C$, then $\overline{A} \subseteq C$.
Proof. For (1) Let $\{x_k\}_{k=1}^{\infty}$ be a convergent sequence in $\overline{A}$, and let x be its limit. We want

to show that $x \in A$. For each x_k, let $\{a_{k_i}\}_{i=1}^{\infty}$ be a sequence in A that converges to x_k. For each k, choose i_k so that

$$\|x_k - a_{k_{i_k}}\| < \frac{1}{k}$$

Claim: $\{a_{k_{i_k}}\}_{k=1}^{\infty}$ converges to x
Given $\epsilon > 0$, choose $\mathrm{K} \in \mathbb{N}$ so that

$$\|x - x_k\| < \frac{\epsilon}{2}, \forall k \geq \mathrm{K},$$

and also $\frac{1}{k} < \frac{\epsilon}{2}$. If $k \geq \mathrm{K}$,

$$\|x - a_{k_{i_k}}\| \leq \|x - x_k\| + \|x_k - a_{k_{i_k}}\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

(2) follows from $A \subseteq B \implies \overline{A} \subseteq \overline{B}$ □

Definition. For $a \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$, the open ball of radius r centered at a is the set

$$B_r(a) = \{x \in \mathbb{R}^n | \|a - x\| < r\},$$

and the closed ball of radius r centered at a is the set

$$\overline{B_r(a)} = \{x \in \mathbb{R}^n | \|a - x\| \leq r\}.$$

Definition. A set $A \subseteq \mathbb{R}^n$ is open if whenever $a \in A$, there's an $\epsilon \in \mathbb{R}, \epsilon > 0$ so that

$$B_\epsilon(a) \subseteq A$$

Example. $(0,1), B_r(a)$ are open. $[0,1]$ is closed, since if $r > 0$, $B_r(a) \nsubseteq [0,1]$
Example. $\mathbb{R}^n, \varnothing$ are both clopen sets, but $[0,1)$ is neither closed nor open!

Definition. If $A \subseteq \mathbb{R}$, its complement is the set

$$A^c = \{x \in \mathbb{R}^n | x \nsubseteq A\}$$

Example. $(\mathbb{R}^n)^c = \varnothing$, $\varnothing^c = \mathbb{R}^n$
Fact: $(A^c)^c = A$

Theorem. A set $A \subseteq \mathbb{R}^n$ is open if and only if A^c is closed. Equivalently, A is closed if and only if A^c is open.

Proof. $\Rightarrow$ Let $B = A^c$ and suppose A is open, and let $\{x_k\}_{k=1}^{\infty}$ be a convergent sequence in B, we want to show its limit, say b, is contained in B. Suppose not, then $b \in A$. Since A is open, for any given $\epsilon > 0$, we can find an open ball $B_\epsilon(b) \subseteq A$. Since $\lim_{k\to\infty} x_k = b$, for any given $\epsilon > 0$, we can find an $\mathrm{K} \in \mathbb{N}$, such that for any $\mathrm{k} \geq \mathrm{K}$, we have $\|x_k - b\| < \epsilon$. In other words, $x_k \in B_\epsilon(b) \subseteq A$ for all $\mathrm{k} \geq \mathrm{K}$. This contradicts $\{x_k\}_{k=1}^{\infty}$ is a sequence in B. Hence, we must have $b \in B$. This imples B is closed.

$\Leftarrow$ Write $B = A^c$ and suppose B is closed, and let $a \in A$, then $a \notin B$. As B is closed, there must be an $\epsilon > 0$ so that whenever $b \in B$,

$$\|a - b\| \geq \epsilon,$$

because if not we could pick a sequence $\{b_k\}_{k=1}^{\infty}$ in B with its limit equal to a, which would imply that $a \in B$(as B is closed). Thus, $B_\epsilon(a) \cap B = \varnothing$, so that $B_\epsilon(a) \subseteq A$. This proves A is open. $\square$

Proposition.
(1) If $\{U_i\}_{i \in I}$ is a any collection of open subsets of $\mathbb{R}^n$, then $\cup_{i \in I} U_i$ is also open
(2) If $U_1, \ldots, U_k$ is finite collection of open subsets of $\mathbb{R}^n$, then $\cap_{i=1}^{k} U_i$ is also open.
Example. For $A_i = (-\frac{1}{n}, \frac{1}{n})$, $\cap_{i=1}^{\infty} A_i = \{0\}$.

III-4 Compact Sets and Heine-Borel

Definition. A set $A \subseteq \mathbb{R}^n$ is called compact if whenever $\{a_k\}_{k=1}^{\infty}$ is a sequence in A, there's a subsequence $\{a_{k_l}\}_{l=1}^{\infty}$ that converges to a point $a \in A$

Definition. A set $A \subseteq \mathbb{R}^n$ is called bounded if there's an $R \in \mathbb{R}$ greater than zero so that $\|a\| \leq R$ for all $a \in A$

Theorem (Heine-Borel)
A set $A \subseteq \mathbb{R}^n$ is compact $\iff$ A is closed and bounded.

Proof. "$\Rightarrow$" Suppose A is compact.
(i) To show A is closed, we let $\vec{a}$ be a limit point of A, and $(a_k)_{k=1}^{\infty}$ a sequence in A converging to $\vec{a}$. As $\lim_{k\to\infty} \vec{a_k} = \vec{a}$, any subsequence of $(a_{k_l})_{l=1}^{\infty}$ also converges to $\vec{a}$. By definition of compactness, $\vec{a} \in A$.
(ii) To show A is bounded, we argue by contradiction. Suppose A is not bounded, the for $k \in \mathbb{N}$, let $\vec{a_k} \in A$ be an element with $\|\vec{a_k}\| \geq k$. This implies $(a_k)_{k=1}^{\infty}$ has no convergent subsequecnes, thus A is not compact. Contradiction.
"$\Leftarrow$" Let A be a close and bounded set. Let $(a_k)_{k=1}^{\infty}$ be any sequence in A. We'll write $\vec{a_k} = (a_{k,1}, \ldots, a_{k,n})$. Using bounded-ness, there's an R $\in \mathbb{R}$ so that $\|\vec{a_k}\| \leq$ R. Consider sequence $(a_{k,1})_{k=1}^{\infty}$. This is a bounded sequence in A, so it has a convergent sub-sequence $(a_{k_{l_1},1})_{l_1=1}^{\infty}$ converging to a_1. Then for $2 \leq i \leq n$, we inductively create a sub-sequence from $(a_{k_{l_{i-1}}})_{l_{i-1}=1}^{\infty}$ with $(a_{k_{l_{i-1}},j})_{l_{i-1}=1}^{\infty}$ converging to a_j for $1 \leq j \leq i-1$. From $(a_{k_{l_{i-1}},i})_{l_{i-1}=1}^{\infty}$, we obtain a convergent sub-sequence $(a_{k_{l_i},i})_{l_i=1}^{\infty}$ and we call its limit a_i, then $(a_{k_{l_i}})_{l_i=1}^{\infty}$ is a sub-sequence of $(a_{k_{l_{i-1}}})_{l_{i-1}=1}^{\infty}$ so that we also have $\lim_{l_i\to\infty}(a_{k_{l_i},j}) = a_j$ for $2 \leq j \leq i-1$. After the n^{th} step, we have a sub-sequence $(a_{k_{l_n}})_{l_n=1}^{\infty}$ converging to $\vec{a} = (a_1, \ldots, a_n)$. Since A is closed, we have $\vec{a} \in A$ (as $(a_{k_{l_n}})_{l_n=1}^{\infty}$ is a sequence in A). We've shown that $(a_k)_{k=1}^{\infty}$ has a convergent sub-sequence converging to a point in A, and as this sequence is arbitrary. A is compact. $\square$
Remark. idea of this proof: We construct a n subsequences of $(a_k)_{k=1}^{\infty}$, say $(a_{k_{l_1}})_{k_{l_1}=1}^{\infty}$, $(a_{k_{l_2}})_{k_{l_2}=1}^{\infty}$, ..., $(a_{k_{l_n}})_{k_{l_n}=1}^{\infty}$, so that $\lim_{l_1\to\infty} a_{k_{l_1},1} = a_1$, $\lim_{l_1\to\infty} a_{k_{l_1},2} = a_2$, $\lim_{l_1\to\infty} a_{k_{l_1},j} = a_j, \forall 1 \leq j \leq n$.

Example. $[a,b]$ be any closed interval in $\mathbb{R}$ is compact.

$$\overline{B_r(\overrightarrow{a})} = \{\overrightarrow{x} \in \mathbb{R} | \|\overrightarrow{x} - \overrightarrow{a}\| \leq r\}$$

is compact.
Proposition. If $C_1, \ldots, C_k$ are compact sets, $\cup_{i=1}^{\infty} C_i$ is compact, if $\{C_i\}_{i \in I}$ is any collection of compact sets $\cap_{i=1}^{\infty} C_i$ is compact. $\square$
Proposition. If $C_1, C_2 \subseteq \mathbb{R}^n$, and $C_1 \in C_2$ such that C_2 compact and C_1 closed, then C_1 is also compact. $\square$

Proposition. If $C_1 \subseteq \mathbb{R}^m, C_2 \subseteq \mathbb{R}^n$ are both compact, then $C_1 \times C_2 \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is also compact. $\square$

Example. $[a,b]^n \subseteq \mathbb{R}^n$ is compact, $[a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ is also compact.

III-5 The Cantor Set

Theorem(Cantor's Intersection Theorem)
Suppose that $C_1 \supseteq C_2 \supseteq C_3 \supseteq \ldots$ is a decreasing sequence of nonempty compact sets in $\mathbb{R}^n$. Then

$$C = \cap_{i=1}^{\infty} C_i$$

is also nonempty and compact in $\mathbb{R}^n$.

Proof. First, an intersection of compact sets are compact, so C is compact. To show it's nonempty, construct a sequence $(x_i)_{i=1}^{\infty}$ by choosing $\overrightarrow{x_i} \in C$ arbitrarily. Then each $\overrightarrow{x_i} \in C_i$ for any $i \in \mathbb{N}$ since $C \subseteq C_i$for any $i \in \mathbb{N}$. This implies $(\overrightarrow{x_i})_{i=1}^{\infty}$ has a convergent subsequence, say $(\overrightarrow{x_{l_j}})_{j=1}^{\infty}$, converging to some $\overrightarrow{x}$. For each $k \in \mathbb{N}$, there's an ideal j_k with $i_{j_k} \geq k$, then $(\overrightarrow{x_{i_j}})_{j=j_k}^{\infty}$ is a sequence contained in C_k, since $x_{1_j} \in C_{1_j} \subseteq C_k$ for all $i_j \geq k$. We have $\overrightarrow{x} \in C_k$, as C_k is compact. Therefore, $\overrightarrow{x} \in \cap_{k=1}^{\infty} C_k = C$. So C is nonempty. $\square$
The Cantor Set
Define

$$S_0 = [0,1];$$
$$S_1 = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1];$$
$$S_2 = [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1]; \ldots$$

I.e., at each step, we cut the middle one third of each interval.
I get a sequence $S_0 \supseteq S_1 \supseteq S_2 \supseteq \ldots$ of compact subsets of $\mathbb{R}$, with each S_i a union of length $(\frac{1}{3})^i$ each.
Definition. The Cantor Set is defined as $C = \cap_{i=0}^{\infty} S_i$.
Remark. It's compact and nonempty by Cantor's Intersection Theorem.

Proposition.(Properties of Cantor Set C)
(1) C has empty interior, i.e. $Int(C) = \varnothing$. Equivalently, there's no nonempty open set which is contained in C.
(2) C has no isolated points
(3) C has measure zero.
(4) C is uncountable. In particular, there's a bijection between C and $\mathbb{R}$

Proof. (1) Argue by contradiction. Suppose there's a nonempty set $(a,b) \subseteq C$, set $\delta = b - a$ and choose $i \in \mathbb{N}$ so that $(\frac{1}{3})^i < \delta$. Then as $(a,b) \subseteq C \subseteq S_i$, but S_i is a union of closed intervals of length $(\frac{1}{3})^i$, of which (a,b) must contained inside of one, but this is not possible as (a,b) has length $\delta = b - a > (\frac{1}{3})^i$.
(2) Suppose $x \in C$, and for each $i \in \mathbb{N}$, choose $x_i \in S_i$ to be a boundary of the interval containing x but not equal to x. Then $\lim_{i\to\infty} x_i = x$ as $|x_i - x| < (\frac{1}{3})^i$.
(3)Given any $\epsilon > 0$, we find S_i can be covered by 2^i open intervals with total length $(\frac{2}{3})^i + \epsilon$. Each interval $[a,b] \subseteq (a-\delta, b+\delta)$ by taking $\delta = \frac{\epsilon}{2^{i+1}}$.
(4) Let B={infinite sequence of 0's and 1's}=$\{C : \mathbb{N} \to \{0,1\}\}$. We'll create a bijection $f : B \to C$. First fix some sequence $\{C_i\}_{i=1}^{\infty} \in B$. We'll construct a sequence of intervals as follows: if $c_1 = 0$ set $I_1 = [0, \frac{1}{3}]$;$c_1 = 1$ set $I_2 = [\frac{2}{3}, 1]$.Then construct I_i from I_{i-1} as follows: $I_{i-1} \cap S_i$ is a disjoint of two intervals $J_1 \cup J_2$, if $c_1 = 0$, take $I_i = J_1$; if $c_1 = 1$, take $I_i = J_2$. Define $f(c) = x$, then $\cap_{i=1}^{\infty} I_i = \{x\}$. $\square$

IV.Real Valued Funtions

IV-1 Limits and Continuity

Notation: Y^X:={function from X to Y}, in particular $\{0,1\}^{\mathbb{N}}$:={function from $\mathbb{N}$ to $\{0,1\}$}

Definition. Suppose $S \subseteq \mathbb{R}^n$, $f : S \to \mathbb{R}^m$ a function, if $\overrightarrow{a}$ is a limit point of $S \setminus \{\overrightarrow{a}\}$ then we say $\lim_{\overrightarrow{x}\to\overrightarrow{a}} f(\overrightarrow{x}) = \overrightarrow{v}$, for some $\overrightarrow{v} \in \mathbb{R}^m$. Equivalently, if for all $\epsilon > 0$, there's a $\delta > 0$, so that if $\overrightarrow{x} \in S \setminus \{\overrightarrow{a}\}$, then

$$0 < \|\overrightarrow{x} - \overrightarrow{a}\| < \delta \Longrightarrow \|f(\overrightarrow{x}) - \overrightarrow{v}\| < \epsilon.$$

Remark. In logic symbol notation: $\forall \epsilon > 0, \exists \delta > 0, \forall x \in S : 0 < \|\overrightarrow{x} - \overrightarrow{a}\| < \delta \Longrightarrow \|f(\overrightarrow{x}) - \overrightarrow{v}\| < \epsilon$
Remark. $f(\overrightarrow{a})$ (if it exists) doesn't depend on $\lim_{\overrightarrow{x}\to\overrightarrow{a}} f(\overrightarrow{x})$.

Definition. Suppose $S \subseteq \mathbb{R}^n$, $f : S \to \mathbb{R}^m$, and $\overrightarrow{a} \in S$. We say that f is continuous at $\overrightarrow{a}$ if for every $\epsilon > 0$, there's a $\delta > 0$, so that $0 < \|\overrightarrow{x} - \overrightarrow{a}\| < \delta \Longrightarrow \|f(\overrightarrow{x}) - f(\overrightarrow{a})\| < \epsilon$ for all $\overrightarrow{x} \in S$.

Remark. In logic symbol notation: $\forall \epsilon > 0, \exists \delta > 0, \forall \overrightarrow{x} \in S : 0 < \|\overrightarrow{x} - \overrightarrow{a}\| < \delta \Longrightarrow \|f(\overrightarrow{x}) - f(\overrightarrow{a})\| < \epsilon$ for all $\overrightarrow{x} \in S$.

Remark. If $\overrightarrow{a} \in S$ is an isolated point, every function is constant at $\overrightarrow{a}$.
Remark. If $\overrightarrow{a} \in S$ is not isolated, then continuity at $\overrightarrow{a}$ is equivalent to $\lim_{\overrightarrow{x} \to \overrightarrow{a}} f(\overrightarrow{x}) = f(\overrightarrow{a})$

Definition. A function $f : S \to \mathbb{R}^m$, with $S \subseteq \mathbb{R}^n$ is called Lipschitz if there's a constant C$\in \mathbb{R}$ so that

$$\|f(x) - f(y)\| \leq \mathrm{C}\|x - y\| \ (1)$$

for all $x, y \in S$. If f is Lipschitz, its Lipschitz constant is the smallest C for which (1) holds.

Example. $y = x^2$ is Lipschitz on $[0,1]$; $y = |x|$ is Lipschitz for all $x \in \mathbb{R}$; but $y = \sqrt{x}$ is NOT Lipschitz for $x \in [0,1]$.

Theorem Every Lipschitz function is continuous.

Proof. Suppose $f : S \to \mathbb{R}^m$, with $S \subseteq \mathbb{R}^n$ is Lipschitz with Lipschitz constant C. Let $a \in S$, and $\epsilon > 0$ be given. Take $\delta = \frac{\epsilon}{\mathrm{C}}$. Then

$$0 < \|x - a\| < \delta \Longrightarrow \|f(x) - f(a)\| \leq \mathrm{C}\|x - a\| < \mathrm{C}\delta = \epsilon.$$

$\square$

Definition. A fucntion $f : S \to \mathbb{R}^m$, with $S \subseteq \mathbb{R}^n$ is discontinuous at $a \in S$ if it is not continuous at a, i.e. there's an $\epsilon > 0$ so that for all $\delta > 0$, there's an $x \in S$, with $0 < \|x - a\| < \delta \Longrightarrow \|f(x) - f(a)\| \geq \epsilon$

Remark. Essentially, there are two ways for a function to be discontinuous at a:
(1) $\lim_{x \to a} f(x)$ does not exist. (Essential Singularity)
(2) $\lim_{x \to a} f(x)$ does exist, but it is not equal to $f(a)$. (Removable Singularity)
Example. Heaviside step function

$$H(x) := \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

and

$$f(x) := \begin{cases} 0, & x = 0 \\ \sin(\frac{1}{x}), & x \neq 0 \end{cases}$$

both not continuous at $x = 0$. But

$$g(x) := \begin{cases} 0, & x = 0 \\ x\sin(\frac{1}{x}), & x \neq 0 \end{cases}$$

is continuous at $x = 0$.