ANALYTIC THEOREM FOR PRIME NUMBER THEOREM

Abstract. We give a modification of the proof of the analytic theorem.

1. The theorem

Theorem 1.1. *Let $f : [0, \infty) \to \mathbb{R}$ be a bounded locally integrable function. Suppose that $g(z) := \int_0^\infty f(t)e^{-tz}dt$ (for $\{\mathrm{re}(z) > 0\}$) extends to a holomorphic function over a neighborhood of $\{\mathrm{re}(z) \geq 0\}$. Then $\int_0^\infty f(t)dt$ exists (i.e. f is integrable) and equal to $g(0)$.*

Proof. For $T > 0$, let $g_T(z) = \int_0^T f(t)e^{-tz}dt$. It is well-defined because f is integrable over $[0, T]$ and hence $f(t)e^{-tz}$ is integrable over $[0, T]$. It is also holomorphic everywhere.

Let $R > 0$ large and $\delta > 0$ be small (depending on R). Let C be the boundary of $\{-\delta \leq \mathrm{re}(z) \leq R, -R \leq \mathrm{im}(z) \leq R\}$. By assuming δ is sufficiently small compared to R, we can assume that g is holomorphic inside C up to the boundary (because g is assumed to be holomorphic over a neighborhood of $\{\mathrm{re}(z) \geq 0\}$).

Apply Cauchy theorem to the function $(g(z) - g_T(z))e^{zT}(1 + \frac{z^2}{R^2})$ at $z = 0$, we get

$$g(0) - g_T(0) = \int_C (g(z) - g_T(z))e^{zT}(1 + \frac{z^2}{R^2})\frac{dz}{z}$$

Let $C_+ = C \cap \{\mathrm{re}(z) > 0\}$. We can bound the intergrand over C_+ as follows: let $B = \max_{t \geq 0} |f(t)|$ (recall that f is a bounded function)

$$|g(z) - g_T(z)| = |\int_T^\infty f(t)e^{-tz}dt| \leq B|\int_T^\infty e^{-t\cdot\mathrm{re}(z)}dt| = \frac{Be^{-T\mathrm{re}(z)}}{\mathrm{re}(z)} \tag{1.1}$$

Write $a := \mathrm{re}(z)$ and $b := \mathrm{im}(z)$

$$(1 + \frac{z^2}{R^2})\frac{1}{z} = \frac{1}{z} + \frac{z}{R^2} = \frac{a - bi}{a^2 + b^2} + \frac{a + bi}{R^2} \tag{1.2}$$

$$= \frac{a(R^2 + a^2 + b^2) + b(a^2 + b^2 - R^2)i}{(a^2 + b^2)R^2} \tag{1.3}$$

So we have

$$\left| \int_{C_+ \cap \{|\mathrm{im}(z)| = R\}} (g(z) - g_T(z))e^{zT}(1 + \frac{z^2}{R^2})\frac{dz}{z} \right| \tag{1.4}$$

$$\leq \int_{C_+ \cap \{|\mathrm{im}(z)| = R\}} \left| (g(z) - g_T(z))e^{zT}(1 + \frac{z^2}{R^2})\frac{1}{z} \right| dz \tag{1.5}$$

$$\leq \int_{C_+ \cap \{|\mathrm{im}(z)| = R\}} \frac{Be^{-Ta}}{a} e^{Ta} \left| \frac{a(2R^2 + a^2) \pm R(a^2)i}{(a^2 + R^2)R^2} \right| dz \tag{1.6}$$

$$\leq \int_{C_+ \cap \{|\mathrm{im}(z)| = R\}} \frac{B}{a}\frac{a}{R^2} \left| \frac{(2R^2 + a^2) \pm aRi}{(a^2 + R^2)} \right| dz \tag{1.7}$$

$$\leq \int_{C_+ \cap \{|\mathrm{im}(z)| = R\}} \frac{B}{a}\frac{a}{R^2}\sqrt{5}dz = \frac{2\sqrt{5}B}{R} \tag{1.8}$$

We also have

$$\left|\int_{C_+\cap\{|\mathrm{re}(z)|=R\}}(g(z)-g_T(z))e^{zT}(1+\frac{z^2}{R^2})\frac{dz}{z}\right| \tag{1.9}$$

$$\leq\int_{C_+\cap\{|\mathrm{re}(z)|=R\}}\left|(g(z)-g_T(z))e^{zT}(1+\frac{z^2}{R^2})\frac{1}{z}\right|dz \tag{1.10}$$

$$\leq\int_{C_+\cap\{|\mathrm{re}(z)|=R\}}\frac{Be^{-TR}}{R}e^{TR}\left|\frac{R(2R^2+b^2)\pm R(b^2)i}{(R^2+b^2)R^2}\right|dz \tag{1.11}$$

$$\leq\int_{C_+\cap\{|\mathrm{re}(z)|=R\}}\frac{B}{R}\left|\frac{R(2R^2+R^2)\pm R(R^2)i}{(R^2)R^2}\right|dz \tag{1.12}$$

$$\leq\frac{2\sqrt{10}B}{R} \tag{1.13}$$

Now we consider the contribution over $C_-=C\cap\{\mathrm{re}(z)\leq 0\}$. Note that $g_T(z)e^{zT}(1+\frac{z^2}{R^2})$ is a holomorphic function for $z\in\mathbb{C}$ so

$$\int_{C_-}g_T(z)e^{zT}(1+\frac{z^2}{R^2})\frac{dz}{z}=\int_{\{-R\leq\mathrm{re}(z)\leq R,-R\leq\mathrm{im}(z)\leq R\}\cap\{\mathrm{im}(z)\leq 0\}}g_T(z)e^{zT}(1+\frac{z^2}{R^2})\frac{dz}{z} \tag{1.14}$$

We want to bound the integrand as above. We have

$$|g_T(z)|\leq B\left|\int_{-\infty}^{T}e^{-t\cdot\mathrm{re}(z)}dt\right|=\frac{Be^{-T\mathrm{re}(z)}}{|\mathrm{re}(z)|} \tag{1.15}$$

Similar calculation as above shows that (1.14) is bounded above by $\frac{2(\sqrt{10}+\sqrt{5})B}{R}$

Finally, we want to analyze $\int_{C_-}g(z)e^{zT}(1+\frac{z^2}{R^2})\frac{dz}{z}$. By assumption, $g(z)e^{zT}(1+\frac{z^2}{R^2})$ is well-defined over C_-. For fix R (so C_- is fixed), we have

$$\lim_{T\to\infty}\int_{C_-}g(z)e^{zT}(1+\frac{z^2}{R^2})\frac{dz}{z}=\int_{C_-}\lim_{T\to\infty}g(z)e^{zT}(1+\frac{z^2}{R^2})\frac{dz}{z}=0$$

because $\mathrm{re}(z)\leq 0$.

In conclusion, for any fixed $R>0$, we have

$$\lim_{T\to\infty}|g(0)-g_T(0)|=\lim_{T\to\infty}\left|\int_C(g(z)-g_T(z))e^{zT}(1+\frac{z^2}{R^2})\frac{dz}{z}\right|\leq\frac{4(\sqrt{10}+\sqrt{5})B}{R}$$

Since LHS is independent of R and R is arbitrary, we conclude that $\lim_{T\to\infty}|g(0)-g_T(0)|=0$ so indeed $g(0)=\int_0^\infty f(t)dt$. $\square$