TJ Math 4
Unit 4 Vectors
8.1 Intro to Vectors

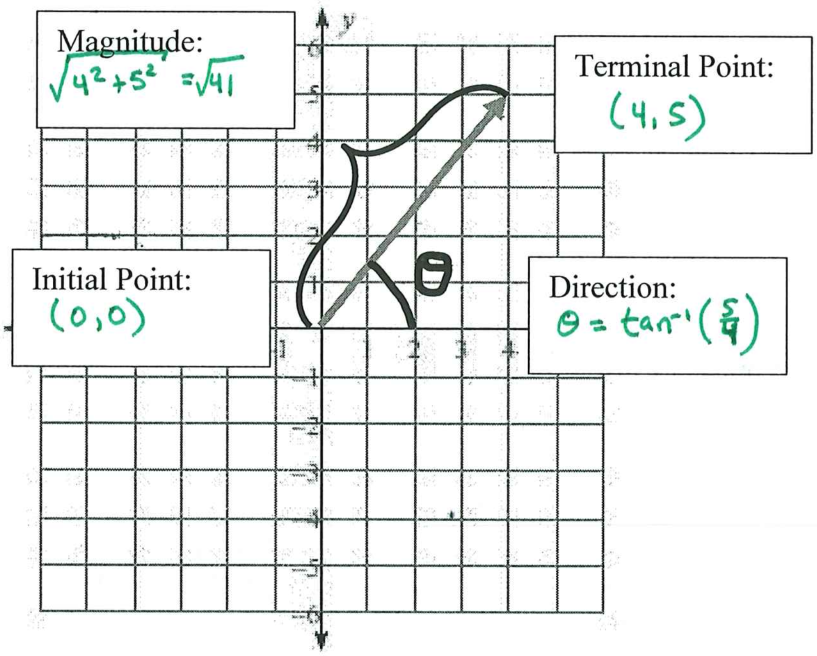
Name____

Many real-world quantities can be described with a single number which would represent the magnitude or size of the quantity. Speed and distance are these types of quantities.

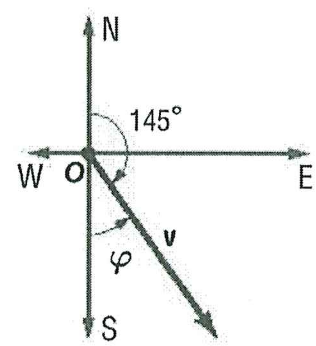
A **vector** represents a quantity that has both **magnitude** and **direction** such as **velocity**. Vectors are either written in bold text, **v**, or need to have an arrow over the letter, $\vec{v}$, to show it is a vector quantity.

Ex: Which of the following describe vector quantities?

a. A car travelling 15 mph 15° east of south
b. A parachutist falling straight down at 12.5 mph
c. A child pulling a sled with a force of 40 newtons



Vectors are said to be in **standard position** if the initial point of the vector is at the origin. Any vector can be drawn in standard position, as long as the magnitude and direction stay the same.



Often in examples and applications, the direction of a vector will be given as a bearing. There are different types of bearings; **quadrant bearing** is always an angle between 0° and 90° that is given as east or west of the north-south line. The vector shown has a quadrant bearing of 35° east of south or S 35° E in short-hand. The angle measure of 145° is the **true bearing** of the vector. True bearing is an angle measure given from true north, increasing clockwise, and is given with three digits (so 035° would be a true bearing of 35°).

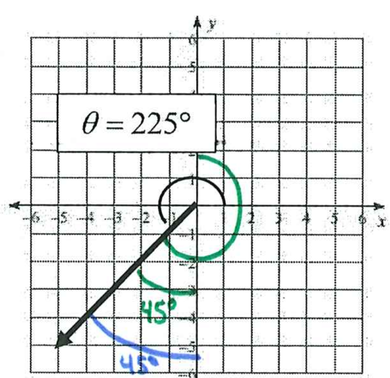
Find the direction of the given vector as

1. A true bearing $180+45 = 225$

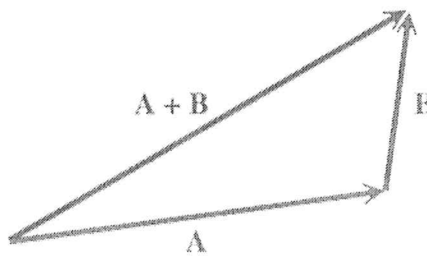
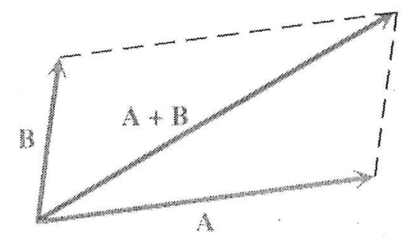
   225°

2. A quadrant bearing

   S 45° W

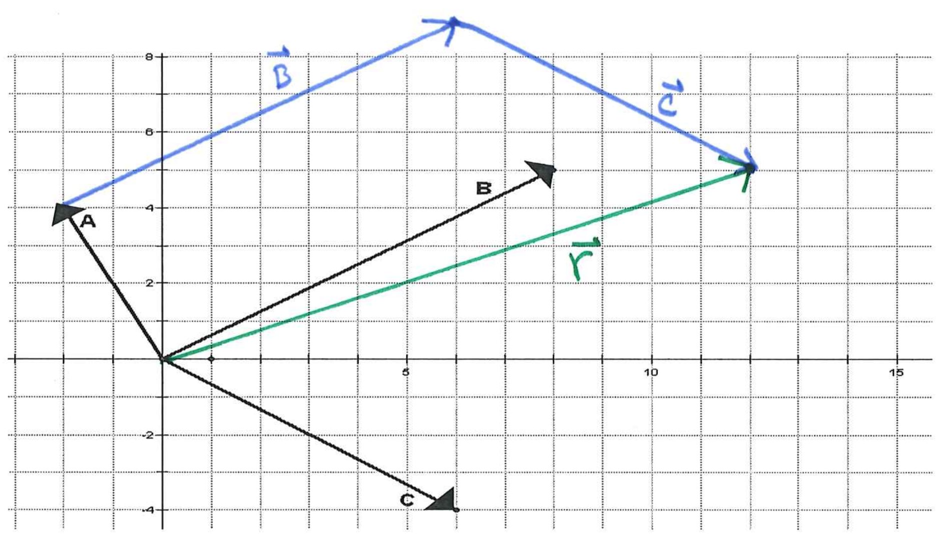


Equivalent vectors have the same magnitude and direction. **Parallel** vectors have the same or opposite direction but do not have to have the same magnitude. **Opposite** vectors have the opposite direction and the same magnitude.

To find the sum of vectors **A** and **B**, written **A** + **B**, place the initial point of vector **B** at the terminal point of vector **A**. The sum is called the **resultant vector.** Its initial point is at the initial point of **A** and its terminal point is at the terminal point of **B.**	Another way to find the sum of two vectors is to use the parallelogram rule. Place vectors **A** and **B** so that their initial points coincide. Then complete a parallelogram that has **A** and **B** as two adjacent sides. The diagonal of the parallelogram with the same initial point as **A** and **B** is the vector sum **A** + **B.**
	

The same process can be used to add multiple vectors. Remember to always go initial point to terminal point. Does the order in which you add the vectors matter?

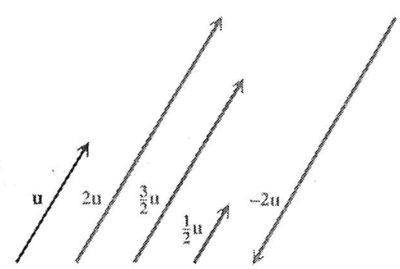
Ex: Given vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ in standard position, sketch the resultant vector of adding all 3.



$\vec{A} + \vec{B} + \vec{C} = \vec{r}$

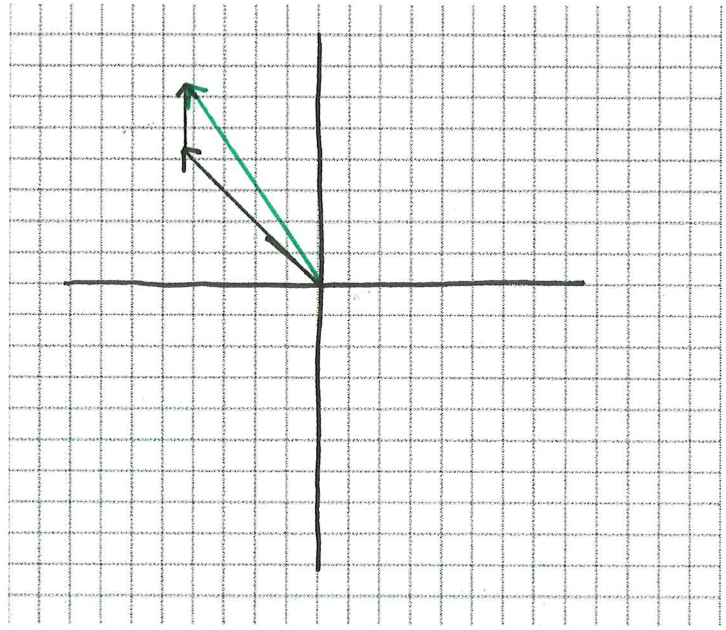
Vector **subtraction** is the same as adding the opposite vector.

The **product** of a real number (or scalar) α and a vector **u** is the vector (α **u**), which has a magnitude |α| times the magnitude of **u** and the same or opposite direction as vector **u**.



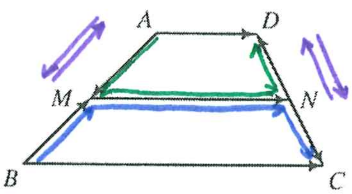
The **zero vector** is the resultant vector of adding a vector and it's opposite. The zero vector is necessary because when you add or subtract vectors your answer must be a vector.

Ex: You walk due N45W for five miles before turning right and walking due north for 2 more miles. Add the vectors graphically to determine how far, and in what direction you have gone from your starting point.



~ N 35° W 6.5 miles

Ex: Complete the following equations given that $\overrightarrow{MN}$ is the median of trapezoid ABCD.



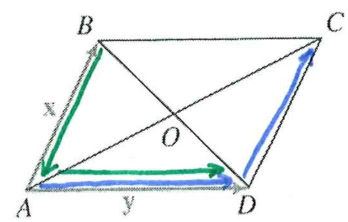
a. $\overrightarrow{AM}+\overrightarrow{MN}+\overrightarrow{ND}=?$ $\overrightarrow{AD}$

b. $\overrightarrow{BM}+\overrightarrow{MN}+\overrightarrow{NC}=?$ $\overrightarrow{BC}$

c. $\overrightarrow{AM}+\overrightarrow{BM}=?$ 0

d. $\overrightarrow{ND}+\overrightarrow{NC}=?$ 0

Ex: In the diagram, ABCD is a parallelogram. If $\overrightarrow{AB}$ = **x** and $\overrightarrow{AD}$ = **y**, express each vector in terms of **x** and **y**.



a. $\overrightarrow{BC}$ $\vec{y}$ (remember, you can move a vector if direction & magnitude don't change)

b. $\overrightarrow{CD}$ $-\vec{x}$

c. $\overrightarrow{AC}$ $\vec{x}+\vec{y}$

d. $\overrightarrow{AO}$ $\frac{1}{2}(\vec{x}+\vec{y})$

e. $\overrightarrow{BD}$ $-\vec{x}+\vec{y}$

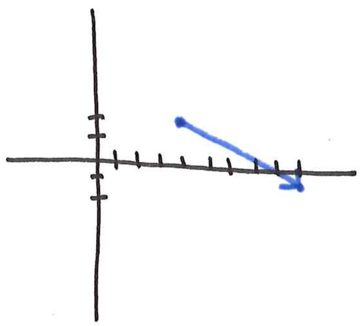
f. $\overrightarrow{BO}$ $\frac{1}{2}(-\vec{x}+\vec{y})$

If **v** is a vector with initial point $P_1 = (x_1, y_1)$ and terminal point $P_2 = (x_2, y_2)$, then **v** can be written **in component form** as $\mathbf{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$.

The **magnitude** of **v** is written as |**v**| or ‖**v**‖ and is the length of **v.** (*Where else have we used this notation? How do we know the difference?*) We can use the distance formula/Pythagorean Theorem to find the magnitude:

$$|\mathbf{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

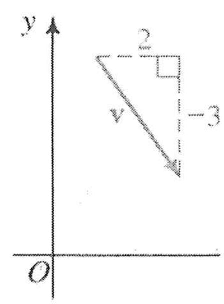
<u>**Ex**</u>**:** Given $P_1 = (4, 2)$ and $P_2 = (9, -1)$, express $\overrightarrow{P_1P_2}$ in component form and find its magnitude.

Sketch the vector	Component Form	Magnitude
	$\overrightarrow{P_1P_2} = \langle 9-4, -1-2 \rangle$ $= \langle 5, -3 \rangle$	$\lvert\overrightarrow{P_1P_2}\rvert = \sqrt{(9-4)^2 + (-1-2)^2}$ $= \sqrt{(5)^2 + (-3)^2}$ $= \sqrt{25+9}$ $= \sqrt{34}$

A vector with a magnitude or length of 1 is called a <u>**unit vector**</u>. To find a unit vector **u** in the same direction of a given nonzero vector **v**, use the following formula: $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$

There are two vectors called the <u>**principal unit vectors**</u>, **i** = <1, 0> and **j** = <0, 1>.

We can represent a vector algebraically as a linear combination of the principal unit vectors, so **v** = <*a, b*> = *a***i** + *b***j**. This is <u>**linear combination**</u> form. In the coordinate system shown at right, the vector **v** consists of a 2-unit change in the positive *x*-direction (2**i**) and a 3-unit change in negative *y*-direction (-3**j**). So, **v** can be written in two ways, <2, -3> or 2**i** – 3**j**.



Vector operations with components are simple:

Let $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j} = \langle a_1, b_1 \rangle$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j} = \langle a_2, b_2 \rangle$ be two vectors, and let α be a scalar. Then

$$\mathbf{v} + \mathbf{w} = (a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j} = \langle a_1 + a_2, b_1 + b_2 \rangle \quad \textbf{(2)}$$

$$\mathbf{v} - \mathbf{w} = (a_1 - a_2)\mathbf{i} + (b_1 - b_2)\mathbf{j} = \langle a_1 - a_2, b_1 - b_2 \rangle \quad \textbf{(3)}$$

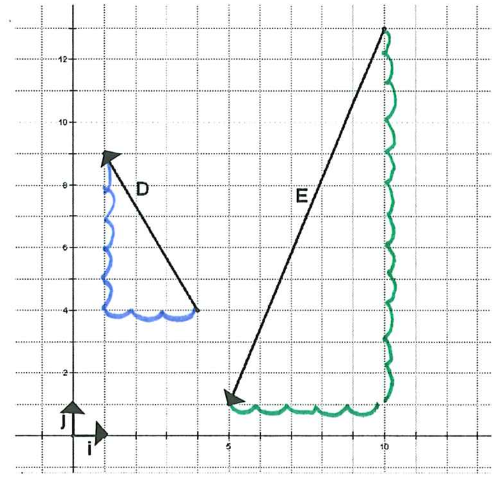
$$\alpha\mathbf{v} = (\alpha a_1)\mathbf{i} + (\alpha b_1)\mathbf{j} = \langle \alpha a_1, \alpha b_1 \rangle \quad \textbf{(4)}$$

$$\|\mathbf{v}\| = \sqrt{a_1^2 + b_1^2} \quad \textbf{(5)}$$

Ex: Find a unit vector in the same direction as $\mathbf{v} = \mathbf{i} - 2\mathbf{j}$

$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 1,-2\rangle}{\sqrt{1+4}} = \langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \rangle$

Ex: Using the given diagram,



a. Express vectors $\vec{D}$ and $\vec{E}$ in terms of the principal unit vectors using linear combination.

$\vec{D} = \langle -3, 5 \rangle = -3\vec{i} + 5\vec{j}$

$\vec{E} = \langle -5, -12 \rangle = -5\vec{i} - 12\vec{j}$

b. Find the sum, $\vec{D} + \vec{E}$.

$\vec{D} + \vec{E} = \langle -3 + -5, 5 + -12 \rangle = \langle -8, -7 \rangle$

c. Find a unit vector in the ***opposite*** direction as vector $\vec{E}$.

Vector opposite $\vec{E} = \langle 5, 12 \rangle$

$|\vec{E}| = \sqrt{25+144} = 13$

$\vec{U}_e = \frac{\langle 5,12 \rangle}{13} = \langle \frac{5}{13}, \frac{12}{13} \rangle$

d. Find a vector having magnitude 7 units in the ***opposite*** direction as vector $\vec{E}$.

$7\vec{u}_e = 7\langle \frac{5}{13}, \frac{12}{13} \rangle = \langle \frac{35}{13}, \frac{84}{13} \rangle$

in linear comb. form: $\frac{35}{13}\vec{i} + \frac{84}{13}\vec{j}$

8.2 Representing Vectors Algebraically

In addition to component form and linear-combination form, vectors can be written in magnitude-direction form. To summarize:		
Component form, (rectangular form) • Indicated by angle brackets. $\langle x, y \rangle$. • x and y represent how far the vector reaches in the x-direction and how far it reaches in the y-direction.	**Linear-combination form** • The vector is written as a linear combination of the two unit vectors $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$.	**Magnitude-direction form.** (polar form) • Indicated by square brackets $[\lvert \mathbf{v} \rvert, \theta]$. • Magnitude is NEVER negative. • Direction, θ, unless otherwise stated, is always from 0° due east (positive x-axis), and increases in the counter-clockwise direction.

Convert vector $\vec{v} = \langle 3, -2 \rangle$, from rectangular (component) form to magnitude direction form:

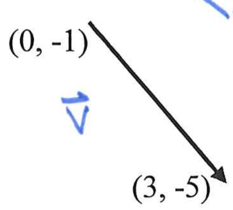
1. Find the magnitude of the vector.	$\lvert \vec{v} \rvert = \sqrt{3^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13}$
2. Find the direction. Include a sketch to be sure you get the correct angle. Use degrees NOT radians	$\tan \theta = \frac{-2}{3}$ $\theta = \tan^{-1}\left(\frac{-2}{3}\right) = -33.69°$
3. Put it together.	$[\sqrt{13}, -33.69°]$

Arctan is used to find the angle because the vertical component is opposite the angle and the horizontal component is adjacent to the angle if the angle is in standard position.

$$\tan \theta = \frac{\lvert \vec{v} \rvert \sin \theta}{\lvert \vec{v} \rvert \cos \theta} = \frac{b}{a} \text{ where } \vec{v} = \langle a, b \rangle = \left\langle \lvert \vec{v} \rvert \cos \theta, \lvert \vec{v} \rvert \sin \theta \right\rangle .$$

Don't forget: arctan has a limited range and will give angles between -90° and 90°. You may need to add 180° to get the correct angle.

$\langle 3-0, -5--1 \rangle$



Given the vector at left, write the vector in:

Component Form: $\langle 3, -4 \rangle$	
Linear-Combination Form: $3\vec{i} - 4\vec{j}$	
Magnitude Direction Form: $[5, -53.13°]$	

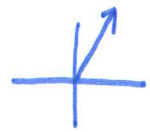
$\lvert \vec{v} \rvert = \sqrt{3^2 + (-4)^2}$

$= \sqrt{9+16} = 5$

$\tan \theta = \frac{-4}{3}$

$\theta = \tan^{-1}\left(\frac{-4}{3}\right) = -53.13°$

Ex: Find the magnitude-direction form of $\vec{v} = \langle 5,12 \rangle$.



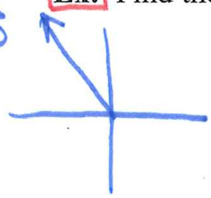
$|\vec{v}| = \sqrt{5^2+12^2} = 13$

$\tan\theta = \frac{12}{5}$

$\theta = \tan^{-1}\left(\frac{12}{5}\right) = 67.38°$

$[13, 67.38°]$

Ex: Find the magnitude-direction form of $\vec{w} = \langle -8,15 \rangle$.



$|\vec{w}| = \sqrt{8^2+15^2} = \sqrt{64+225} = \sqrt{289} = 17$

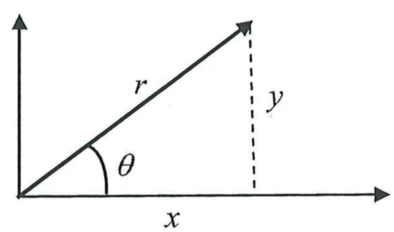
$\tan\theta = -\frac{15}{8}$

$\theta = \tan^{-1}\left(\frac{-15}{8}\right) = -61.928°$

$[17, 118.074°]$

BUT this is not in the correct quadrant. Because of the restricted range of arctan, need to add 180°

Converting from magnitude-direction form to component form is a more straight-forward operation. For any vector $\vec{v} = [r, \theta°]$, the equivalent component form representation is $\vec{v} = \langle r\cos\theta, r\sin\theta \rangle$.



$\cos\theta = \frac{x}{r}$

$x = r\cos\theta$

$\sin\theta = \frac{y}{r}$

$y = r\sin\theta$

Ex: Find the component form of $\vec{v} = [6, 30°]$.

$\vec{v} = \langle 6\cos 30°, 6\sin 30° \rangle$

$= \langle 6 \cdot \frac{\sqrt{3}}{2}, 6 \cdot \frac{1}{2} \rangle$

$= \langle 3\sqrt{3}, 3 \rangle$

Ex: Find the component form of $\vec{w} = [8, 225°]$.

$\vec{w} = \langle 8\cos 225°, 8\sin 225° \rangle$

$= \langle 8 \cdot -\frac{\sqrt{2}}{2}, 8 \cdot -\frac{\sqrt{2}}{2} \rangle$

$= \langle -4\sqrt{2}, -4\sqrt{2} \rangle$

Applications of Adding Vectors

The most common applications with vectors involve multiple vectors in one problem setting. Frequently, vectors represent forces, such as wind or gravity or the power generated by an engine. When multiple forces are acting on an object, the object moves in accordance with the *sum* of those forces. Application problems often involve vectors given in magnitude-direction form.

Adding vectors graphically is a fairly elementary exercise, as demonstrated previously. But adding vectors algebraically is not always as simple, because YOU CANNOT ADD FORCES BY ADDING THEIR MAGNITUDES AND ADDING THEIR DIRECTIONS.*

* In case it wasn't obvious from the all-caps, this is the single-most important sentence of this page.

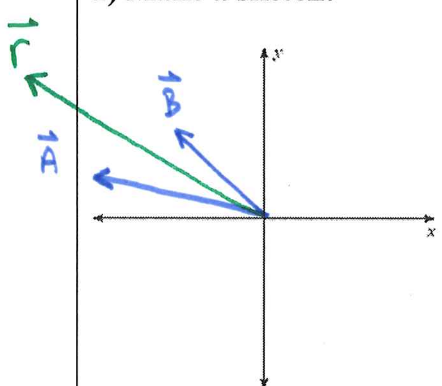
However, we CAN decompose the vectors into component form, add their x-components and add their y-components, and then recompose the resultant vector into magnitude direction form.

The four steps involved in nearly every vector addition problem are:

1) Make a sketch. Do NOT skip this step!!
2) De-Compose (i.e. represent all vectors in component form).
3) Add (by adding x-components together and adding y-components together).
4) Re-Compose (i.e. convert resultant vector to magnitude-direction form).

Ex: Person A pushes on a large rock with a force of 145 pounds in a direction of 170°, while Person B pushes on the rock with a force of 115 pounds in a direction of 140°. Find the direction and magnitude of the resultant force.

1) Make a sketch:



Check how angles are given. Switch to standard angles.

2) De-compose into component form:

$\vec{A} = [145, 170°] = \langle 145\cos 170°, 145 \sin 170° \rangle$

$\vec{B} = [115, 140°]$ $\vec{B} = \langle 115 \cos 140, 115 \sin 140 \rangle$

3) Add the vectors:

$\vec{A} + \vec{B} = \langle 145\cos 170° + 115 \cos 140°, 145 \sin 170° + 115 \sin 140° \rangle$

$\approx \langle -230.892, 99.1 \rangle$

4) Re-compose into magnitude direction form:

$\vec{r} = |\vec{A} + \vec{B}| = 251.261$ lbs

Do NOT use rounded values here. Grab the actual values from your calculator!

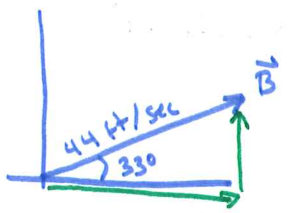
$\tan^{-1}\left(\frac{99.1}{-230.892}\right) \approx -23.229°$

Check diagram. What quadrant will the resultant vector be in?
Need to add 180°

$[251.261, 156.771°]$

Practice Problems

1) A soccer player kicks a ball so that it leaves the ground with a velocity of 44 feet per second at an angle of 33° with the ground. Draw a diagram that shows the resolution of this vector into its component forces and then calculate the horizontal and vertical components.



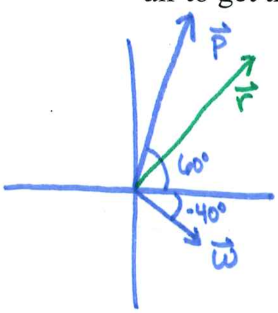
$\vec{B} = [44, 33°]$

$x = 44\cos 33° \approx 36.902$

$y = 44\sin 33° \approx 23.964$

$\vec{B} = \langle 36.902, 23.964 \rangle$

2) A plane is flying on a true bearing of 030° at 300 mph. The wind is blowing in the direction of S50°E at 60 mph. Determine the actual speed and direction of the plane over ground. (Keep in mind that the plane is flying through the air. You must add the wind vector to the plane's course and speed through the air to get the actual course and speed of the plane over the ground).



$\vec{P} = [300, 60°] = \langle 300\cos 60°, 300\sin 60° \rangle$

$\vec{w} = [60, 320°] = \langle 60\cos 320°, 60\sin 320° \rangle$

$\vec{r} = \vec{P} + \vec{w} \approx \langle 195.963, 221.240 \rangle$

$|\vec{r}| = 295.548$ mph $\quad \theta = \tan^{-1}\left(\frac{221.240}{195.963}\right) = 48.467°$

$[295.548, 48.467°]$

Set this up so that either the man or the horse is pulling in the direction of the x-axis

3) A man and a horse are both pulling on a large rock. The man is pulling with a force of 200 pounds and the horse is pulling with a force of 1000 pounds 50° away from the direction of the pull of the man. Find the magnitude of the resultant force and what direction angle that force makes with the force of the man.

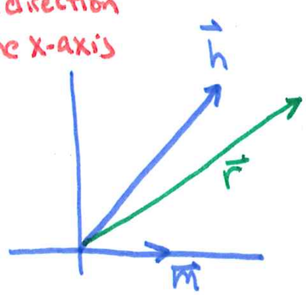
$\vec{h} = [1000, 50°] = \langle 1000\cos 50, 1000\sin 50 \rangle$

$\vec{m} = [200, 0°] = \langle 200, 0 \rangle$

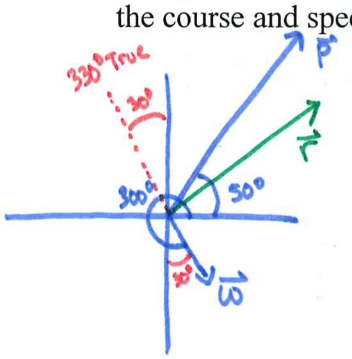
$\vec{r} = \vec{h} + \vec{m} = \langle 200 + 1000\cos 50, 1000\sin 50 \rangle$

$\vec{r} \approx \langle 842.788, 766.044 \rangle$

$= [1138.91, 42.269°]$



4) An airplane is traveling on true bearing 040° with an air speed of 300 miles per hour, and there is a 35 mile per hour wind <u>from</u> a true bearing of 330°. [what do you think "from" means in this context?]. Find the course and speed over ground of the airplane.



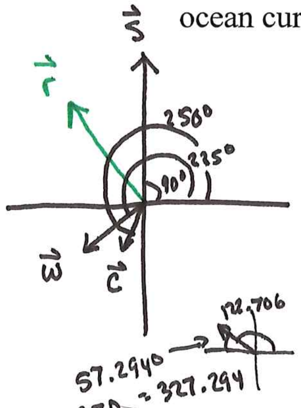
$\vec{P} = [300, 50°] = \langle 192.836, 229.813 \rangle$

$\vec{w} = [35, 300°] = \langle 17.5, -30.311 \rangle$

$\vec{r} = \vec{P} + \vec{w} = \langle 210.336, 199.502 \rangle$

$= [289.901, 43.486°]$

5) (For this problem, all directions are true bearings) A ship steams on a course of 000° at 45 knots (a measure of speed, one knot is one nautical mile/hour). There is a 20-knot wind **from** 045° and the ocean current is 6 knots in the direction of 200°. Find the resultant course and speed of the ship.



$\vec{s} = [45, 90°] = \langle 45\cos 90°, 45\sin 90°\rangle$

$\vec{w} = [20, 225°] = \langle 20\cos 225°, 20\sin 225°\rangle$

$\vec{c} = [6, 250°] = \langle 6\cos 250°, 6\sin 250°\rangle$

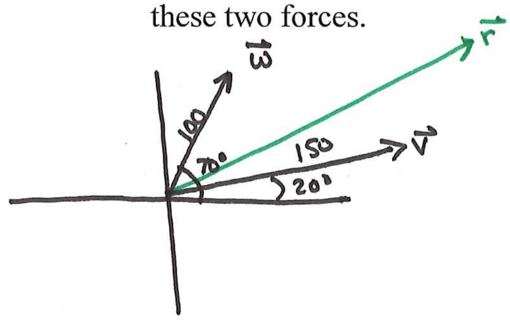
$\vec{r} = \vec{s} + \vec{w} + \vec{c} = \langle -16.194, 25.220\rangle$

$= [29.971, -57.294 + 180] = [29.971, 122.706°]$

Speed = 29.971 knots

Course = 327.294°

6) Two forces, one of 100 pounds and the other of 150 pounds, act on the same object, at angles of 70° and 20°, respectively, with the positive x-axis. Find the direction and magnitude of the resultant vector of these two forces.



$\vec{v} = [150, 20°] = \langle 150\cos 20°, 150\sin 20°\rangle$

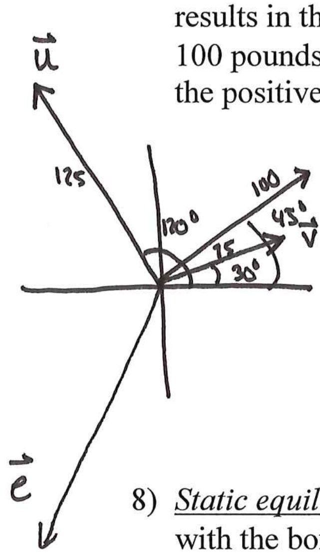
$\vec{w} = [100, 70°] = \langle 100\cos 70°, 100\sin 70°\rangle$

$\vec{r} = \vec{v} + \vec{w} = \langle 175.156, 145.272\rangle$

$= [227.560, 39.672°]$

227.560 lbs at 39.672°

7) An *equilibriant* force is defined to be a force that, when added to the forces already acting on an object, results in the sum of all forces acting on the object to be 0. In this problem, three forces of 75 pounds, 100 pounds, and 125 pounds act on the same object at angles of 30°, 45°, and 120°, respectively, with the positive x-axis. Find the direction and magnitude of the equilibriant of these forces.



$\vec{v} + \vec{w} + \vec{u} = \langle 75\cos 30 + 100\cos 45 + 125\cos 120,\ 75\sin 30 + 100\sin 45 + 125\sin 120\rangle$

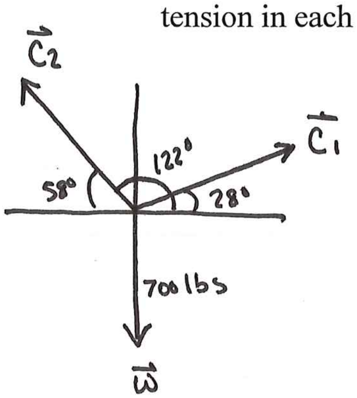
$= \langle 73.163, 216.464\rangle$

Equilibriant vector $= \langle -73.163, -216.464\rangle$

$|\vec{e}| = 228.494$ lbs

direction $= 71.325 + 180 = 251.325°$

8) *Static equilibrium*: Two cables support a box weighing 700 pounds. Cable 1 makes an angle of 28° with the box, while Cable 2 makes an angle of 58° with the box (in opposite directions). Find the tension in each cable.



$\vec{C_1} + \vec{C_2} + \vec{w} = 0$

$\vec{C_1} = \langle C_1\cos 28°, C_1\sin 28°\rangle$

$\vec{C_2} = \langle C_2\cos 122°, C_2\sin 122°\rangle$

$\vec{w} = \langle 0, -700\rangle$

$C_1\cos 28 + C_2\cos 122 = 0$

$C_1\sin 28 + C_2\sin 122 = 700$

$$\left[\begin{array}{cc|c} \cos 28 & \cos 122 & 0 \\ \sin 28 & \sin 122 & 700 \end{array}\right]$$

$C_1 = 371.849$ lbs

$C_2 = 619.573$ lbs

8.3 The Dot Product

To compute the scalar product of **u** and **v**, we multiply each component of **u** by the corresponding component of **v** and add these products. The result is called the **<u>dot product</u>**, written **u • v**.

$$\vec{u} = \langle a, b \rangle, \quad \vec{v} = \langle c, d \rangle$$

$$\vec{u} \bullet \vec{v} = ac + bd$$

Notice that the dot product of two vectors is a **scalar**, not a vector.

Ex: If $\vec{v} = -2\vec{i} + \vec{j}$ and $\vec{w} = 4\vec{i} - 3\vec{j}$ find

(a) $\vec{v} \bullet \vec{w}$ $-8-3=-11$

(b) $\vec{w} \bullet \vec{v}$ $-8-3=-11$

(c) $\vec{w} \bullet \vec{w}$ $16+9=25$

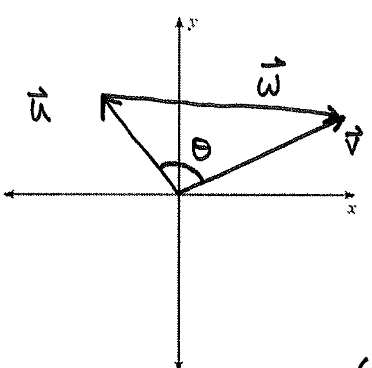
(d) $|\vec{w}|$ $\sqrt{16+9} = \sqrt{25} = 5$

The example illustrated the fact that the dot product is commutative. Other properties of the dot product include:

If **u, v** and **w** are vectors then

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (commutative)
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (distributive)
- $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$ (the dot product of a vector and itself is the square of the magnitude)
- $\vec{0} \cdot \mathbf{v} = 0$

If two vectors **u** and **v** are in standard position, where θ is the angle between them, then we can let vector **w** connect their terminal points. These 3 vectors form a triangle.



If we say that $\mathbf{u} = \langle u_x, u_y \rangle, \mathbf{v} = \langle v_x, v_y \rangle$ then we can use the distance formula to find:

$$\|\mathbf{w}\|^2 = (V_x - U_x)^2 + (V_y - U_y)^2$$

We can also use the law of cosines to find:

$$\|\mathbf{w}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta$$

Set these equal to each other:

$$(V_x - U_x)^2 + (V_y - U_y)^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta$$

Use $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$ and $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$ to simplify:

$$V_x^2 - 2V_xU_x + U_x^2 + V_y^2 - 2V_yU_y + U_y^2 = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - 2\|\vec{u}\|\|\vec{v}\| \cos\theta$$

$$\cancel{V_xV_x} - 2V_xU_x + \cancel{U_xU_x} + \cancel{V_yV_y} - 2V_yU_y + \cancel{U_yU_y} = \cancel{U_xU_x} + \cancel{U_yU_y} + \cancel{V_xV_x} + \cancel{V_yV_y} - 2\|\vec{u}\|\|\vec{v}\| \cos\theta$$

$$-2V_xU_x - 2V_yU_y = -2\|\vec{u}\|\|\vec{v}\| \cos\theta$$

$$V_xU_x + V_yU_y = \|\vec{u}\|\|\vec{v}\| \cos\theta$$

$$\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$$

Write an equation that relates the angle between vectors and their dot product:

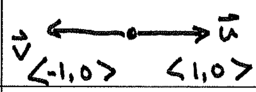
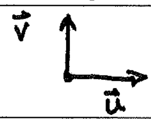
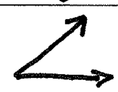
u•v = $\|\vec{u}\|\|\vec{v}\|\cos\theta$	$\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$

If the angle between two vectors is 90°, what will the dot product be?

$\cos 90° = 0$ so dotproduct will be 0

Vectors that have a dot product of 0 are called **orthogonal**. Non-zero orthogonal vectors are perpendicular.

Given two **unit** vectors, $\vec{u}$ and $\vec{v}$, with angle between $\vec{u}$ and $\vec{v} = \theta$, and the formula $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$, fill in the chart:

θ	Sketch:	$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta = \cos\theta$
180°		-1
150°		$-\frac{\sqrt{3}}{2}$
90°		0
45°		$\frac{\sqrt{2}}{2}$
30°		$\frac{\sqrt{3}}{2}$
0°		1

What pattern do you see?

When the vectors are opposite directions, the dot product is -1 times the product of their magnitudes. As the angle between them decreases, the dot product increases to a maximum = product of magnitudes

The dot product shows how much the "same" the two vectors are.

Ex: Find the dot product between the two vectors and then determine if they are orthogonal.

a. $\vec{u} = \langle 3,-2\rangle, \vec{v} = \langle -5,1\rangle$

$\vec{u}\cdot\vec{v} = 3(-5) - 2(1)$
$= -17$

Not orthogonal

b. $\vec{u} = \langle -2,-3\rangle, \vec{v} = \langle 9,-6\rangle$

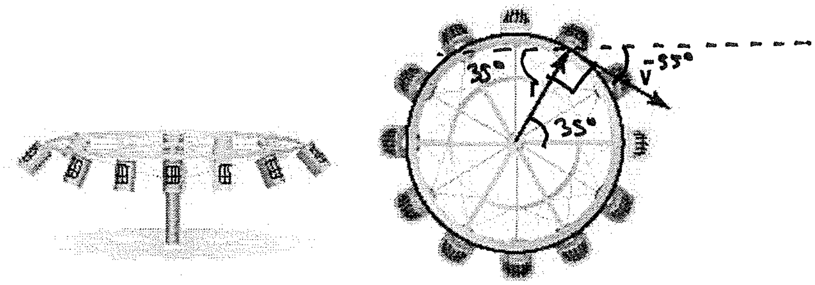
$\vec{u}\cdot\vec{v} = -2(9) - 3(-6)$
$= 0$

yes, orthogonal

since $\cos\theta = \dfrac{0}{\|\vec{u}\|\|\vec{v}\|} = 0$

$\cos\theta = 0$ & $\theta = 90°$

Ex: For a circular amusement park ride, the position vector **r** is perpendicular to the tangent velocity vector **v** at any point on the circle, as shown below.



Front View Top View

a. If the radius of the ride is 20 ft and the speed of the ride is constant at 40 feet per second, write the component forms of the position vector r and the tangent velocity vector v when r is at a directed angle of 35°.

$\|\vec{r}\| = 20$ $\|\vec{v}\| = 40$

$\vec{r} = \langle 20\cos 35°, 20\sin 35° \rangle$

$\vec{v} = \langle 40\cos(-55°), 40\sin(-55°) \rangle$

b. Verify mathematically that the two vectors are perpendicular.

$\vec{r}\cdot\vec{v} = 20\cos 35 \cdot 40\cos(-55) + 20\sin 35 \cdot 40\sin(-55°)$

$= 800\cos 35 \cdot \cos 55 - 800\sin 35\sin 55$

$= 800(\cos(35+55)) = 800\cos(90°)$

$= 0$

Ex: If $\vec{v} = 3\vec{i} + 5\vec{j}$ and $\vec{w} = -2\vec{i} + 6\vec{j}$,

a. find the angle between the two vectors.

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \frac{-6+30}{\sqrt{34}\cdot\sqrt{40}} = \frac{24}{\sqrt{2\cdot17\cdot8\cdot5}} = \frac{24}{4\sqrt{85}} = \frac{6}{\sqrt{85}}$$

$\theta \approx 49.399°$

b. Find a vector parallel to vector **v**.

Parallel to vector $\vec{v}$ means magnitude can be different, so multiply $\vec{v}$ by a scalar ie $2\vec{v} = \langle 6, 10 \rangle$ or $3\vec{v} = \langle 9, 15 \rangle$

c. Find a vector perpendicular to vector **w**. $\vec{w}\cdot\vec{p} = 0$

$\vec{p} = \langle p_x, p_y \rangle$ $\quad -2p_x + 6p_y = 0$

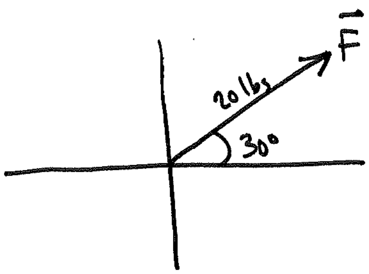
ie $\langle 3, 1 \rangle$ or $\langle 6, 2 \rangle$

In elementary physics, the **work, *W*,** done by a constant force F in moving an object from a point A to a point B is defined as

$$\boxed{W = \vec{F} \bullet \overrightarrow{AB} = |F|\left|\overrightarrow{AB}\right|\cos\theta.}$$

Work is commonly measured in foot-pounds or in newton-meters (joules).

Ex: A wagon is pulled horizontally by exerting a force of 20 pounds on the handle at an angle of 30° with the horizontal. How much work is done in moving the wagon 100 feet?



$|\vec{F}| = 20$

$|\overrightarrow{AB}| = 100$

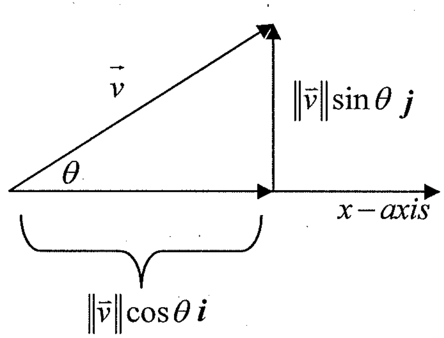
$\text{Work} = 20 \cdot 100 \cdot \cos 30^\circ$
$= 1000\sqrt{3}$ ft·lbs

8.3 Vector Projection

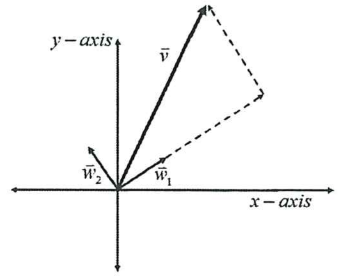
We have already learned vector projection, we just didn't realize it: we learned how to "project" any vector $\vec{v}$ onto the x-axis. That projection was equal to $\|\vec{v}\|\cos\theta$. We also "projected" vector $\vec{v}$ onto an axis that was perpendicular to the x-axis, namely the y-axis, and that projection was $\|\vec{v}\|\sin\theta$. So, you could think of vector projection as asking the question, "how much of this vector points in one particular direction, and how much of this vector points in the direction perpendicular to that first direction?" For our discussion here, that original vector $\vec{v}$ could be written in projected form as:

$$\vec{v} = \left(\|\vec{v}\|\cos\theta\right)\vec{i} + \left(\|\vec{v}\|\sin\theta\right)\vec{j}$$

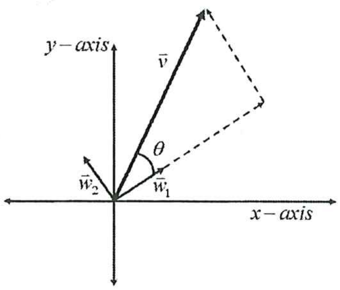
Thus, we can re-write any vector $\vec{v}$ as a linear combination of two unit vectors, one parallel to the x-axis, and one perpendicular to the x-axis, as in the figure below:



But sometimes, we want to re-write a vector $\vec{v}$ as a linear combination of two unit vectors, but rather than those unit vectors being $\vec{i}$ and $\vec{j}$, we want them to be two vectors that are perpendicular to each other, but *not* to the axes. See the figure to the right, where vector $\vec{v}$ is represented as a linear combination of vectors $\vec{w}_1$ and $\vec{w}_2$.



This can be a complicated algebraic problem, but it is helped significantly by our understanding of the dot product. Consider the same figure again, but this time with an angle labeled:



$Proj_{\vec{w}_1}\vec{V}$ is read as "the projection of V onto w_1"

From the diagram at left, what is the **magnitude** of the dotted line that represents the projection of **v** onto **w1**?	$\left\|proj_{\vec{w}_1}\vec{v}\right\| = \|\vec{V}\| \cos\theta$
How could we find $\cos\theta$?	$\cos\theta = \dfrac{\vec{V}\cdot\vec{w}_1}{\|\vec{V}\|\,\|\vec{w}_1\|}$
Substitute to get the magnitude of the projection:	$\left\|proj_{\vec{w}_1}\vec{v}\right\| = \|\vec{V}\| \cdot \dfrac{\vec{V}\cdot\vec{w}_1}{\|\vec{V}\|\,\|\vec{w}_1\|}$
This simplifies to: Remember, this is the magnitude of the projection Note that this value is a **scaler**!	$\left\|proj_{\vec{w}_1}\vec{v}\right\| = \dfrac{\vec{V}\cdot\vec{w}_1}{\|\vec{w}_1\|}$
To find a **vector** with this magnitude parallel to $\vec{w}_1$, we need to find a **unit** vector in the direction of $\vec{w}_1$	$\dfrac{\vec{w}_1}{\|\vec{w}_1\|}$
Now multiply the scaler by the unit vector:	$proj_{\vec{w}_1}\vec{v} = \dfrac{\vec{V}\cdot\vec{w}_1}{\|\vec{w}_1\|^2}\vec{w}_1$
To find the component of $\vec{v}$ that is orthogonal with $\vec{w}_1$, simply subtract, since vector $\vec{v}$ minus the quantity above results in the remaining component.	$proj_{\vec{w}_2}\vec{v} = \vec{V} - \dfrac{\vec{V}\cdot\vec{w}_1}{\|\vec{w}_1\|^2}\vec{w}_1$

Example:

Decompose $\vec{v}$ into two vectors $\vec{v}_1$ and $\vec{v}_2$, where $\vec{v}_1$ is parallel to $\vec{u}$ and $\vec{v}_2$ is perpendicular to $\vec{u}$. So project $\vec{V}$ onto $\vec{u}$

$\vec{v} = 2\vec{i} - 3\vec{j}$ $\quad\quad$ $\vec{u} = \vec{i} - \vec{j}$

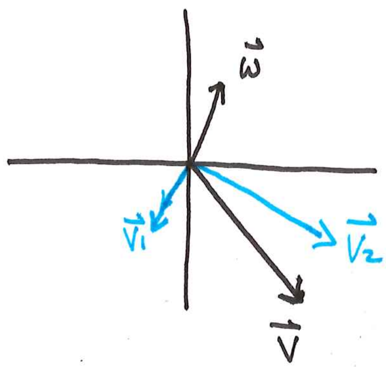
$$\vec{V}_1 = Proj_{\vec{u}}\vec{V} = \frac{\vec{V}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u} = \frac{2(1)-3(-1)}{2}\langle 1,-1\rangle = \frac{5}{2}\langle 1,-1\rangle = \left\langle \frac{5}{2}, -\frac{5}{2}\right\rangle$$

$$\vec{V}_2 = \vec{V} - \vec{V}_1 = \langle 2,-3\rangle - \left\langle \frac{5}{2}, -\frac{5}{2}\right\rangle = \left\langle -\frac{1}{2}, -\frac{1}{2}\right\rangle$$

Exercises:

For each of the following problems, sketch the vectors involved on a coordinate plane, then decompose vector $\vec{v}$ into two vectors $\vec{v}_1$ and $\vec{v}_2$, where $\vec{v}_1$ is parallel to $\vec{u}$, and $\vec{v}_2$ is orthogonal to $\vec{u}$.

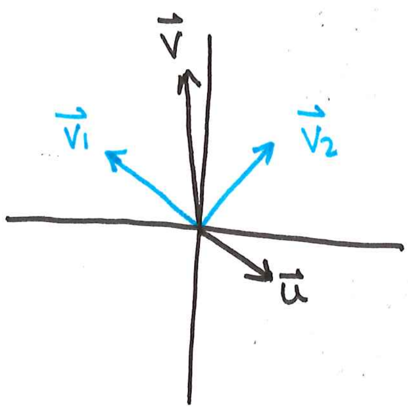
1) $\vec{v} = 3\vec{i} - 4\vec{j}$ $\qquad$ $\vec{u} = \vec{i} + 2\vec{j}$



$$\vec{V}_1 = \text{Proj}_{\vec{\omega}}\vec{V} = \frac{\vec{V}\cdot\vec{\omega}}{\|\vec{\omega}\|^2}\vec{\omega} = \frac{3-8}{5}\langle 1,2\rangle$$
$$= -1\langle 1,2\rangle = \langle -1,-2\rangle$$

$$\vec{V}_2 = \vec{V} - \vec{V}_1 = \langle 3,-4\rangle - \langle -1,-2\rangle$$
$$= \langle 4,-2\rangle$$

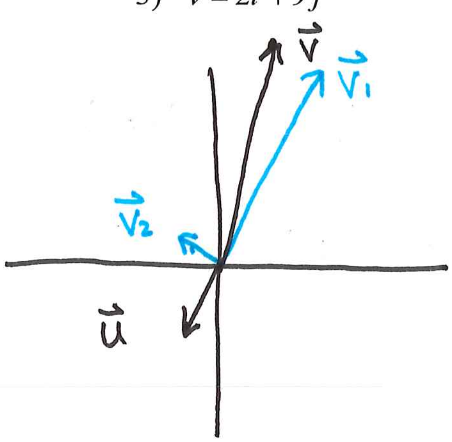
2) $\vec{v} = -\vec{i} + 6\vec{j}$ $\qquad$ $\vec{u} = 3\vec{i} - 2\vec{j}$



$$\vec{V}_1 = \text{Proj}_{\vec{u}}\vec{V} = \frac{\vec{u}\cdot\vec{V}}{\|\vec{u}\|^2}\vec{u} = \frac{-3-12}{13}\langle 3,-2\rangle = \frac{-15}{13}\langle 3,-2\rangle$$
$$= \left\langle \frac{-45}{13}, \frac{30}{13}\right\rangle$$

$$V_2 = \vec{V} - \vec{V}_1 = \langle -1,6\rangle - \left\langle \frac{-45}{13}, \frac{30}{13}\right\rangle = \left\langle \frac{32}{13}, \frac{48}{13}\right\rangle$$

3) $\vec{v} = 2\vec{i} + 9\vec{j}$ $\qquad$ $\vec{u} = -\vec{i} - 3\vec{j}$

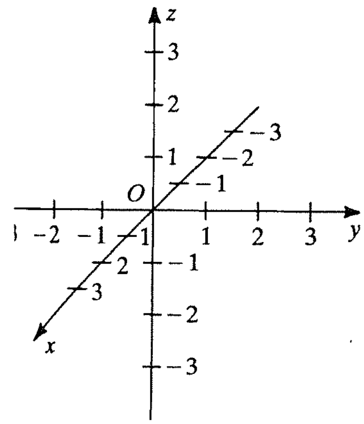


$$\vec{V}_1 = \text{Proj}_{\vec{u}}\vec{V} = \frac{\vec{u}\cdot\vec{V}}{\|\vec{u}\|^2}\vec{u} = \frac{-2-27}{10}\langle -1,-3\rangle$$
$$= \frac{-29}{10}\langle -1,-3\rangle$$
$$= \left\langle \frac{29}{10}, \frac{87}{10}\right\rangle$$

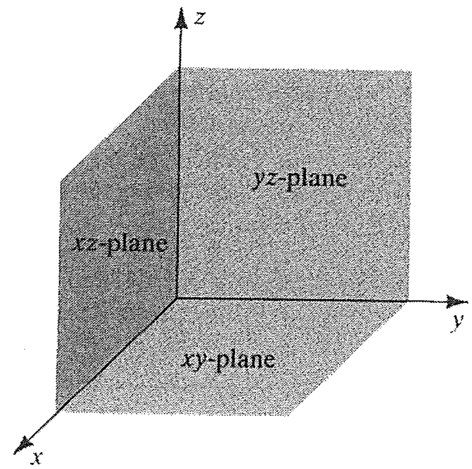
$$\vec{V}_2 = \vec{V} - \vec{V}_1 = \langle 2,9\rangle - \left\langle \frac{29}{10}, \frac{87}{10}\right\rangle$$
$$= \left\langle -\frac{9}{10}, \frac{3}{10}\right\rangle$$

8.4 Vectors in 3-D Space

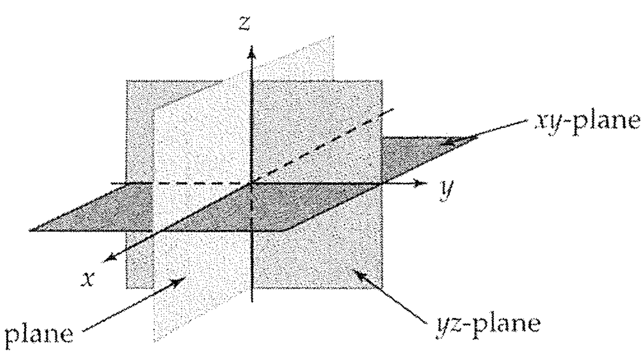
Systems of equations in three variables can be represented by three-dimensional graphs. Just as ordered pairs represent points in a plane, ordered triples (*x, y, z*) represent points in space. The axes in a three –dimensional system are called the *x*-axis, *y*-axis, and *z*-axis. In three-dimensional coordinates, the arrowhead on each axis indicates the positive direction.



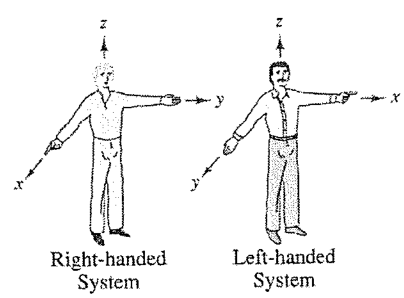
Each pair of axes determines a coordinate plane, which is named by the axes that determine it. There are three coordinate planes, the *xy*-plane, the *yz*-plane, and the *xz*-plane, which divide the three –dimensional space into eight regions, called octants. The only named octant is the entirely positive octant. It is called the first octant. The other seven octants are named with positive and negative signs. For example, the octant immediately below the first octant is called the ++- octant because points have positive *x* and *y* coordinates and negative *z* coordinates.



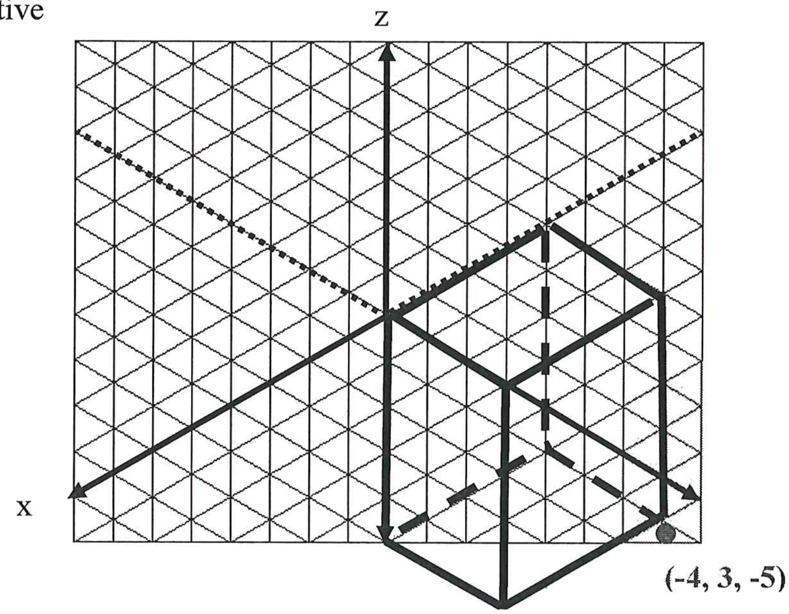
Ex: Name the other six octants.



In this exercise, you will practice drawing coordinate boxes so that it is possible to determine where a point is in space and later you will practice drawing planes. We will use the standard right-handed system in our drawings. We will also use isometric graph paper to help give a good perspective to our drawings.



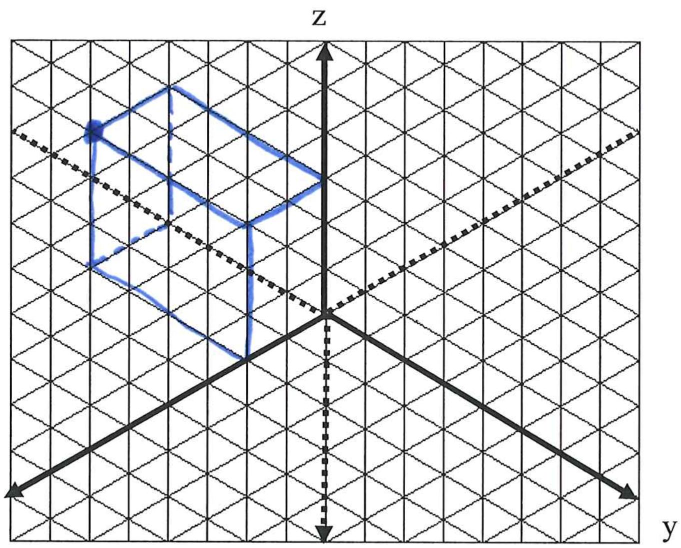
When you draw a coordinate box, you should have 12 edges. The isometric paper shown below helps give the drawings perspective

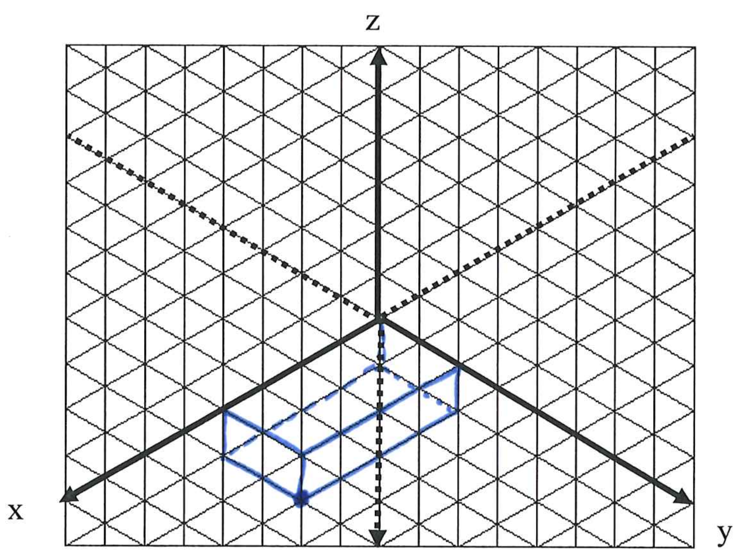


Ex: Draw a coordinate box for each of the following points using the isometric grids below.

a. (2, - 4, 3)

b. (4, 2, -1)





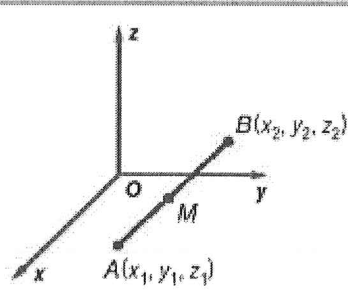
KeyConcept Distance and Midpoint Formulas in Space

The distance between points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

The midpoint M of $\overline{AB}$ is given by

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$



$\frac{1}{2}$ mile = 2640 ft

Ex: Safety regulations require that planes are at least half a mile apart when in the sky. Two planes are flying above Cleveland with the coordinates (300, 150, 30000) and (450, -250, 28000), where the coordinates are given in feet.

a. Are the two planes in violation of the safety regulations? Explain.

$$\sqrt{(450-300)^2+(-250-150)^2+(28{,}000-30{,}000)^2}$$

$$=\sqrt{150^2+(-400)^2+(-2000)^2} \approx 2045.116 \text{ ft} < \tfrac{1}{2} \text{ mile too close}$$

b. If a firework was launched and exploded directly between the two planes, what are the coordinates of the firework explosion?

$$\left(\frac{(300+450)}{2}, \frac{(150-250)}{2}, \frac{(30{,}000+28{,}000)}{2}\right) = (375, -50, 29{,}000)$$

Vectors in space work the same way as they do in 2D. If the terminal point of a vector **v** in standard position is (a,b,c) then we can write **v** = <a,b,c>. The zero vector in 3D is <0,0,0> and the principal unit vectors are **i** = <1,0,0>, **j** = <0,1,0>, and **k** = <0,0,1>. And we can write vector **v** as a linear combination so

v = a**i** + b**j** + c**k**.

To find the components of a vector between points A (a_1, a_2, a_3) and B (b_1, b_2, b_3) if B is the terminal point, we would calculate $\vec{AB} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$ and the magnitude of the vector is $\left|\vec{AB}\right| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$. Vector addition, subtraction, and multiplication by a scalar work the same was as they do in 2D.

Ex: For each pair of points A and B, find an ordered triple that represents $\vec{AB}$. Then write each ordered triple as a linear combination of basic unit vectors. Then find the magnitude of each vector.

a. A(-4, 5, 8), B(7, 2, -9)

$$\vec{AB} = \langle 7--4, 2-5, -9-8 \rangle$$
$$= \langle -11, -3, -17 \rangle$$
$$\vec{AB} = -11\vec{i} - 3\vec{j} - 17\vec{k}$$
$$\|\vec{AB}\| = \sqrt{(-11)^2+(-3)^2+(-17)^2}$$
$$= \sqrt{419} \approx 20.469$$

b. A(6, 8, -5), B(7, -3, 12)

$$\vec{AB} = \langle 7-6, -3-8, 12--5 \rangle$$
$$= \langle 1, -11, 17 \rangle$$
$$\vec{AB} = \vec{i} - 11\vec{j} + 17\vec{k}$$
$$\|\vec{AB}| = \sqrt{1^2+(-11)^2+17^2}$$
$$= \sqrt{411} \approx 20.273$$

Ex: Given vectors **v** = 2**i** - 4**j** + 5**k** and **w** = 6**i** - 8**j** + 9**k**, find

a. **v** + **w**

$(2+6)\vec{i} + (-4-8)\vec{j} + (5+9)\vec{k}$

$= 8\vec{i} - 12\vec{j} + 14\vec{k}$

b. **v** - **w**

$-4\vec{i} + 4\vec{j} - 4\vec{k}$

c. 4**v** + 3**w**

$(8+18)\vec{i} + (-16-24)\vec{j} + (20+27)\vec{k}$

$= 26\vec{i} - 40\vec{j} + 47\vec{k}$

d. 5**v** - 2**w**

$-2\vec{i} - 4\vec{j} + 7\vec{k}$

Ex: Given a vector **v** = 3**i** + 4**j** - 5**k**, and **w** = -2**i** - **j** + 4**k**

a. Find |**v**|. $= \sqrt{3^2+4^2+(-5)^2} = \sqrt{9+16+25} = \sqrt{50} = 5\sqrt{2}$

b. Find the unit vector going in the same direction as **v**.

$$\frac{\vec{v}}{\|\vec{v}\|} = \left\langle \frac{3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{-5}{5\sqrt{2}} \right\rangle = \left\langle \frac{3\sqrt{2}}{10}, \frac{2\sqrt{2}}{5}, \frac{-\sqrt{2}}{2} \right\rangle$$

c. Find a vector of magnitude 3 in the opposite direction of **w**.

Opposite $\vec{w} = \langle 2, 1, -4 \rangle$

$\|\vec{w}\| = \sqrt{4+1+16} = \sqrt{21}$

Vector of magnitude 3 $\longrightarrow$ $\frac{3}{\sqrt{21}}\langle 2,1,-4\rangle$

$= \frac{\sqrt{21}}{7}\langle 2,1,-4\rangle$

$= \left\langle \frac{2\sqrt{21}}{7}, \frac{\sqrt{21}}{7}, \frac{-4\sqrt{21}}{7} \right\rangle$

d. Find $\vec{v} \bullet \vec{w}$.

$\vec{v} \cdot \vec{w} = 3(-2) + 4(-1) - 5(4)$

$= -6 - 4 - 20 = -30$

also $= \|\vec{v}\| \|\vec{w}\| \cos\theta$

What if I wanted to find the dot product of $\langle 2,3 \rangle$ and $\langle 1,4,-2 \rangle$?
The first vector is in the *xy* plane, so I can rewrite it as $\langle 2,3,0 \rangle$

8.5 The Cross Product

The dot product of vectors in space works the same as 2D vectors. Vectors in space are orthogonal if and only if their dot product is zero. The formula for finding the angle between two vectors in space is the same as in 2D as well.

Ex: Use the dot product to determine if $\langle -4,2,1\rangle$ ($\vec{u}$) and $\langle 4,0,3\rangle$ ($\vec{v}$) are orthogonal. If not, find the angle between them.

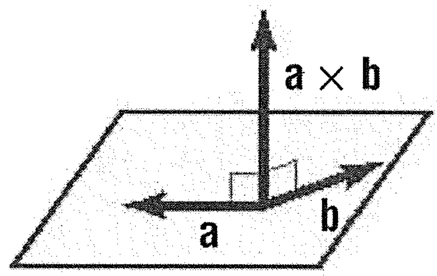
$\vec{u}\cdot\vec{v} = -16+3 = -13$

not orthogonal

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{-13}{\sqrt{21}\cdot\sqrt{25}} = \frac{-13}{5\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{-13}{5\sqrt{21}}\right) = 124.567°$$

The second of the two vector products that we will learn is the **cross product** or **vector product**. Our motivation is the need to calculate such quantities as torque. When we turn a bolt by applying a force to a wrench, the torque is the force that we produce acting along the axis of the bolt to drive the bolt forward. Vector products are also widely used to describe the effects of forces in studies of electricity, magnetism, fluid mechanics, and planetary motion.



Mathematically, the cross product of two vectors **a** and **b** produces a vector perpendicular to the plane containing **a** and **b**.

The vector product or cross product, $\vec{v}\,x\,\vec{w}$ (read "**v** cross **w**") of two vectors , $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, is the vector

$$\boxed{\vec{a}\,x\,\vec{b} = (a_2b_3 - a_3b_2)\vec{i} - (a_1b_3 - a_3b_1)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}\,.}$$

The formula can be remembered as the terms in the expansion of the determinant $\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Ex: Find $\vec{v}\,x\,\vec{w}$ and $\vec{w}\,x\,\vec{v}$ if $\vec{v} = 3\vec{i} - 2\vec{j} + \vec{k}$ and $\vec{w} = 4\vec{i} + \vec{j} + \vec{k}$.

$$\vec{v}\times\vec{w} = \begin{vmatrix} i & j & k \\ 3 & -2 & 1 \\ 4 & 1 & 1 \end{vmatrix} = -3\vec{i} + \vec{j} + 11\vec{k} \qquad \vec{w}\times\vec{v} = \begin{vmatrix} i & j & k \\ 4 & 1 & 1 \\ 3 & -2 & 1 \end{vmatrix} = 3\vec{i} - \vec{j} - 11\vec{k}$$

Unlike the dot product, the cross product is not a commutative operation. Reversing the order of the factors in a cross product reverses the direction of the resulting vector. Thus, $\vec{v}\,x\,\vec{w} = -(\vec{w}\,x\,\vec{v})$.

If two vectors are parallel, what would their cross product be? Hint: think about the determinant.

If the vectors are parallel, one would be a scalar multiple of the other. The determinant would be 0, so the cross product would be the zero vector

The Associative and Distributive Laws

As a rule, cross product multiplication is not associative, but the scalar distributive law

$$(r\vec{v})x(s\vec{w}) = (rs)(\vec{v}\,x\,\vec{w})$$

does hold, as do the vector distributive laws

$$\vec{v}\,x(\vec{w}+\vec{z}) = (\vec{v}\,x\,\vec{w})+(\vec{v}\,x\,\vec{z}) \text{ and}$$
$$(\vec{w}+\vec{z})x\vec{v} = (\vec{w}\,x\vec{v})+(\vec{z}\,x\vec{v})$$

The Test for Parallelism: Nonzero vectors **v** and **w** are parallel if and only if **v** x **w** = **0**.

Ex: Determine if the following pairs of vectors are parallel

(a) $\vec{v} = \vec{i} + 2\vec{j} - \vec{k}$ and $\vec{w} = -2\vec{i} - 4\vec{j} + 2\vec{k}$

$$\vec{v}\times\vec{w} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & -4 & 2 \end{vmatrix} = (4-4)\vec{i} + (2-2)\vec{j} + (-4+4)\vec{k} = \langle 0,0,0\rangle$$

$\vec{v}$ & $\vec{w}$ are parallel

(b) $\vec{v} = \vec{i} + 2\vec{j} - \vec{k}$ and $\vec{w} = -2\vec{i} + 2\vec{j} + 2\vec{k}$

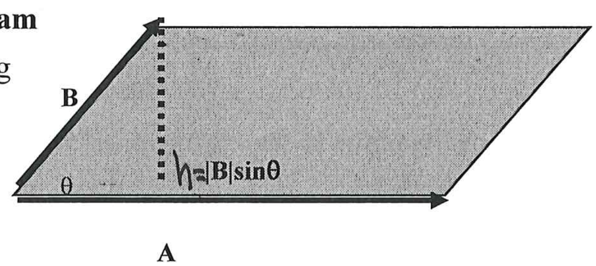
$$\vec{v}\times\vec{w} \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = (4+2)\vec{i} + (2-2)\vec{j} + (2+4)\vec{k} = 6\vec{i} + 6\vec{k}$$

Not parallel

Applications of Vector Products

$|\vec{v}\,x\,\vec{w}|$ is the Area of a Parallelogram

The area of the parallelogram having **v** and **w** as adjacent sides is $|\vec{v}\,x\,\vec{w}|$.



The parallelogram determined by **A** and **B** has

$$\boxed{\text{area} = \text{base} * \text{height} = |\vec{A}||\vec{B}|\sin\theta = |\vec{A}\,x\,\vec{B}|}$$

So for a rectangle, sin θ = 1 & area = base x height. as θ decreases to 0°, sin θ decreases to 0 & area decreases to 0.

Ex: Find the area of the parallelogram formed by $\vec{A} = 0\vec{i} + 3\vec{j} + \vec{k}$ and $\vec{B} = -3\vec{i} + 5\vec{j} - 2\vec{k}$

area $= \|\vec{A}\times\vec{B}\|$

$$\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ 0 & 3 & 1 \\ -3 & 5 & -2 \end{vmatrix} = (-6-5)\vec{i} + 3\vec{j} + 9\vec{k} = \langle -11,3,9\rangle$$

$\|\vec{A}\times\vec{B}\| = \sqrt{121+9+81} = \sqrt{211}$

Area $= \sqrt{211}$ units^2

Ex: Find the area of the triangle with vertices $A(1,-1,0)$, $B(2,1,-1)$, and $C(-1,1,2)$.

$\vec{AB} = \langle 1,2,-1\rangle$

$\vec{AC} = \langle -2,2,2\rangle$

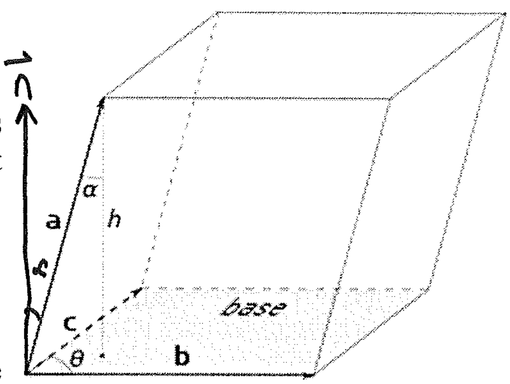
Area $= \frac{1}{2}\|\vec{AB}\times\vec{AC}\|$

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = (4+2)\vec{i} + (2-2)\vec{j} + (2+4)\vec{k} = \langle 6,0,6\rangle$$

$\|\vec{AB}\times\vec{AC}\| = \sqrt{36+36} = 6\sqrt{2}$

$\frac{1}{2}\cdot 6\sqrt{2} = 3\sqrt{2}$ units^2

Building on the parallelogram, we can make a 3-dimensional figure called a **parallelepiped** which is a prism.



If the base of the parallelepiped is a parallelogram then its area is determined by the magnitude of the cross product of the vectors that determine its sides as shown above.

Area of base = $\left|\vec{b}\times\vec{c}\right|$

The third vector that determines the shape of the parallelepiped can be compared to the vector normal to the base in order to get the height of the figure.

Height is then $h = \left|\vec{a}\right|\left|\cos\alpha\right|$

Let $\vec{n} = \vec{b}\times\vec{c}$

Then $\cos\alpha = \dfrac{\vec{a}\bullet\vec{n}}{\left|\vec{n}\right|\left|\vec{a}\right|}$

And $\left|\vec{n}\right|\left|\vec{a}\right|\cos\alpha = \vec{a}\bullet\vec{n}$

We can calculate the volume of the parallelepiped using the formula:

$$\text{Volume} = (\text{area of the base})(\text{height}) = \left|\vec{b}\times\vec{c}\right|\left|\vec{a}\right|\left|\cos\alpha\right| = \left|\vec{a}\bullet\left(\vec{b}\times\vec{c}\right)\right|$$

This combination of the of the dot product and the cross product is called the **scalar triple product.**

Ex: Find the volume of the parallelepiped determined by **a** = <1, 0, 6>, **b** = <2, 3, -8>, and **c** = <8, -5, 6>.

$$\text{Volume} = \left|\vec{a}\cdot(\vec{b}\times\vec{c})\right|$$

$$\vec{b}\times\vec{c} = \begin{vmatrix} i & j & k \\ 2 & 3 & -8 \\ 8 & -5 & 6 \end{vmatrix} = (18-40)\vec{i} + (-64-12)\vec{j} + (-10-24)\vec{k}$$

$$= \langle -22, -76, -34 \rangle$$

$$\langle 1, 0, 6\rangle \cdot \langle -22, -76, -34\rangle = -22+6(-34) = -226$$

$$|-226| = 226 \text{ units}^3$$

Ex: Use the scalar triple product to verify that the vectors **a** = 2**i** + 3**j** + **k**, **b** = **i** – **j**, **c** = 7**i** +3**j** +2**k** are coplanar. (so volume of parallelepiped = 0)

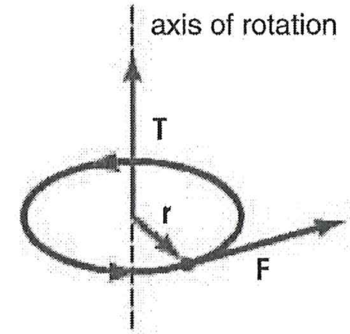
$$\left|\vec{a}\cdot(\vec{b}\times\vec{c})\right| = 0$$

$$\vec{b}\times\vec{c} = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 7 & 3 & 2 \end{vmatrix} = \vec{i}(-2) - \vec{j}(2) + \vec{k}(3+7)$$

$$= \langle -2, -2, 10\rangle$$

$$\langle 2, 3, 1\rangle \langle -2, -2, 10\rangle = -4-6+10 = 0 \quad \checkmark$$

As mentioned before, the cross product can be used to find a vector quantity called <u>torque</u>. Torque measures how effectively a force applied to a lever causes rotation along the axis of rotation. The torque vector **T** is perpendicular to the plane containing the directed distance **r** from the axis of rotation to the point of the applied force and the applied force **F** as shown. Therefore,



$$\boxed{\mathbf{T} = \mathbf{r} \times \mathbf{F} \text{ and is measured in newton-meters.}}$$

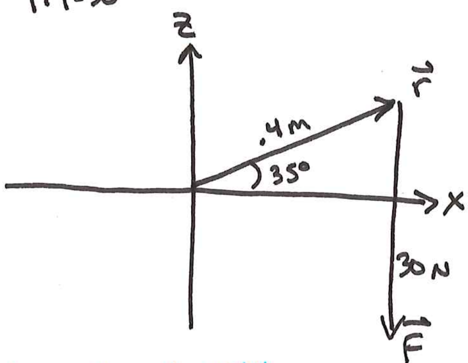
$|\vec{r}| = .4$

Ex: A mechanic uses a 0.4-meter-long wrench to tighten a nut. Find the magnitude and direction of the torque about the nut if the force is 30 newtons straight down to the end of the handle when it is $35°$ above the positive x-axis.

$|\vec{F}| = 30$

So $\|\vec{T}\| = \|\vec{F}\|\|\vec{r}\|\sin\theta$

Meaning, to increase torque you can increase $\|\vec{F}\|$, or increase $\|\vec{r}\|$ <u>or</u> make sure force is applied perpendicular to the radius (so $\sin\theta$ is maximized)



$\vec{r} = \langle .4\cos 35, 0, .4\sin 35 \rangle$

$\vec{F} = \langle 0, 0, -30 \rangle$

$$\vec{T} = \vec{r} \times \vec{F} = \begin{vmatrix} i & j & k \\ .4\cos 35 & 0 & .4\sin 35 \\ 0 & 0 & -30 \end{vmatrix}$$

$= 0\vec{i} - (-12\cos 35°)\vec{j} + 0\vec{k}$

$\vec{T} = 12\cos 35°\,\vec{j} = \langle 0, 12\cos 35°, 0 \rangle$

so 9.83 in direction of y axis

also could find $\|\vec{T}\|$ as $\|\vec{F}\|\|\vec{r}\|\sin\theta$

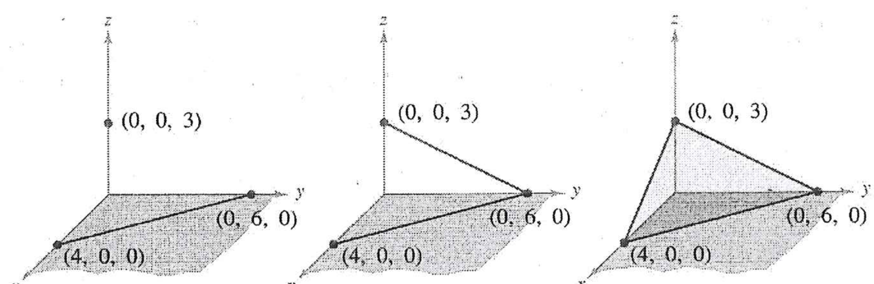
$= 30 \cdot .4 \cdot \sin 55°$

≈ 9.83

8.5 Graphing Planes and 3D Equations

Drawing planes in space involves two things:

- Finding the axis intercepts
- Finding the traces. Traces are lines joining two axis intercepts.



(a) xy-trace ($z = 0$): $3x + 2y = 12$

(b) yz-trace ($x = 0$): $2y + 4z = 12$

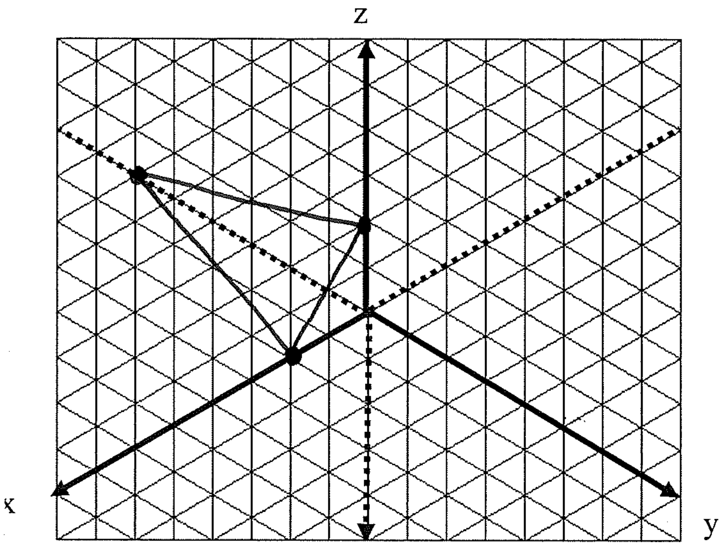
(c) xz-trace ($y = 0$): $3x + 4z = 12$

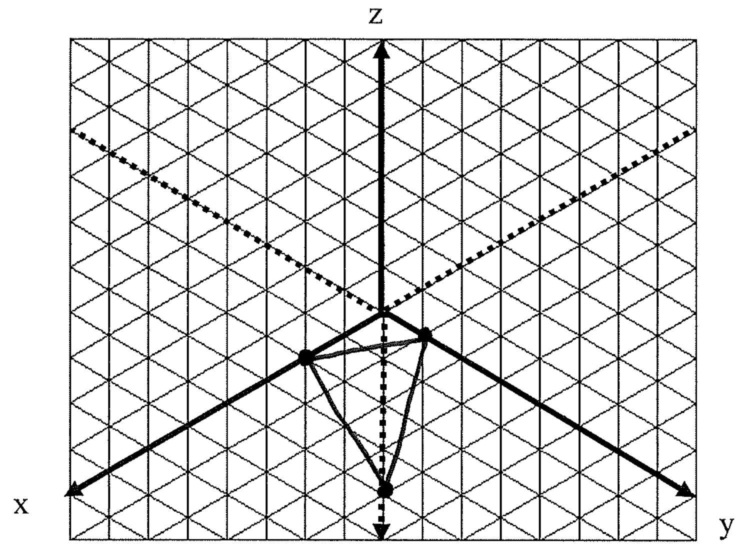
Traces of the Plane: $3x + 2y + 4z = 12$

Ex: Draw each of the following planes. The intercepts and traces should be clear.

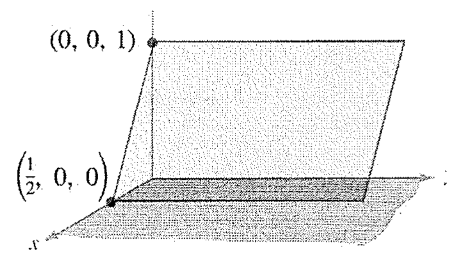
a. $3x - y + 3z = 6$

b. $2x + 4y - z = 4$



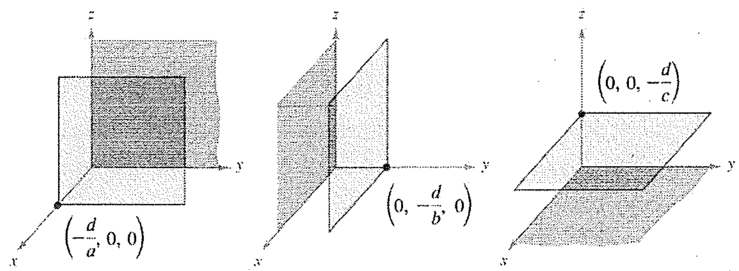


What is also important to realize is that if a variable is missing in the equation then the plane has traces parallel to the axis of that variable. If a plane is missing two variables, then it is parallel to one of the coordinate planes, as shown in the next 3 graphs.



Plane: $2x + z = 1$

Ex:



Plane $ax + d = 0$ is parallel to yz-plane.

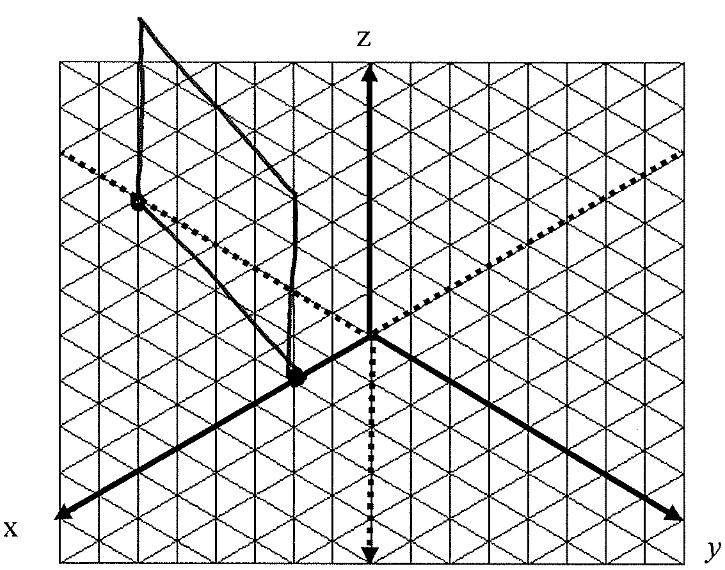
Plane $by + d = 0$ is parallel to xz-plane.

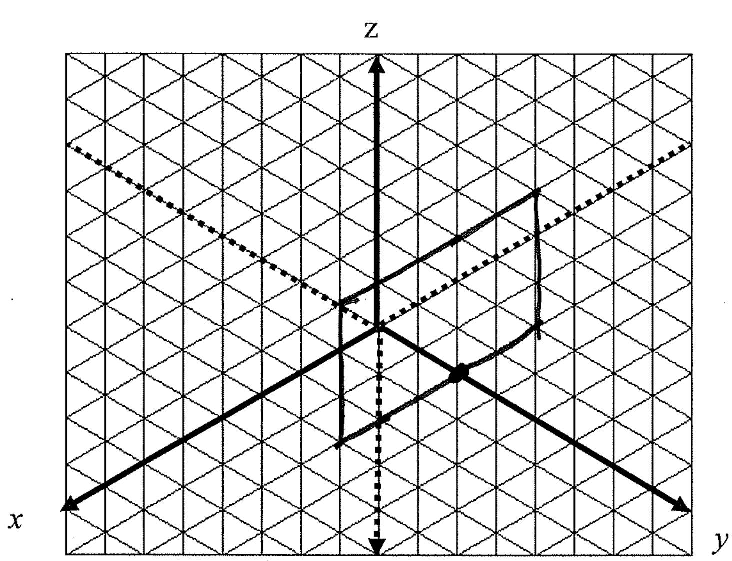
Plane $cz + d = 0$ is parallel to xy-plane.

Draw each of the following planes. The intercepts and traces should be clear.

a. $3x - y = 6$

b. $y = 2$





8.5 Equations in 3-D

In space, a linear equation is a plane, as we saw earlier. If *A, B,* and *C* are not all zero, the equation of the form $Ax + By + Cz + D = 0$ is a plane.

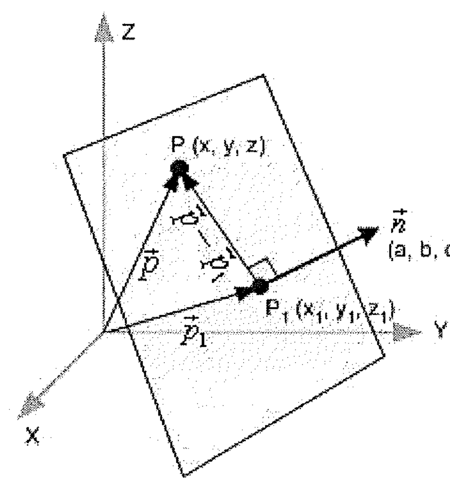
We know from Geometry that three non-collinear points determine a plane. We can use that knowledge with systems of equations to find the equation of a plane. Since the three points lie in the plane, each of them satisfies the general equation of the plane above.

If you are asked to find an equation of the plane passing through the points (2, 1, 3), (1, 3, 2) and (-1, 2, 4), then the general equations would be:

(2, 1, 3)	$2A + B + 3C + D = 0$
(1, 3, 2)	$A + 3B + 2C + D = 0$
(-1, 2, 4)	$-A + 2B + 4C + D = 0$

A system of equations like this could be solved using matrices, but since there are 4 variables and only 3 equations you have to choose something for D to be such as 1.

A second way of finding the equation of a plane is by using vector language. Suppose you are given a nonzero vector $\mathbf{n} = \langle A, B, C \rangle$ that is normal (perpendicular) to the plane and a point on the plane, $P(x, y, z)$, that is not on the vector.



The set of points $P(x, y, z)$ satisfying the equation $\overrightarrow{P_1P} \cdot \mathbf{n} = 0$ is the plane through point P_1 and perpendicular to vector $\mathbf{n}$ (n is used for normal vector). We can set this dot product equal to zero since we know the plane and the normal vector are at right angles. If the dot product is equal to zero then the vectors are perpendicular.
Since every plane contains a point and is perpendicular to some vector, a plane can be characterized this way:

If $\overrightarrow{P_1P} = \langle x - x_1, y - y_1, z - z_1 \rangle$,

Then $\overrightarrow{P_1P} \cdot \mathbf{n} = \langle x - x_1, y - y_1, z - z_1 \rangle \cdot \langle A, B, C \rangle = A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

Ex: Find the equation of the plane through point A=(5, 1, -2) and perpendicular to $\mathbf{n} = \langle 2, 4, 3 \rangle$

For any point $p = (x, y, z)$ on the plane then $\overrightarrow{AP} \cdot \vec{n} = 0$

$\overrightarrow{AP} = \langle x-5, y-1, z+2 \rangle$

$\overrightarrow{AP} \cdot \vec{n} = 2(x-5) + 4(y-1) + 3(z+2) = 0$

$2x - 10 + 4y - 4 + 3z + 6 = 0$

$2x + 4y + 3z - 8 = 0$

or

$2x + 4y + 3z = 8$

We can find the normal vector to the plane if we are given two vectors in the plane or three points that determine the plane. The cross product of the two vectors is another vector perpendicular to both of them and therefore also perpendicular to the plane.

$\vec{n}$

Ex: Find the equation of the plane which passes through A = (1, 1, -2), B = (3, 2, 0) and C = (-1, 0, 1)

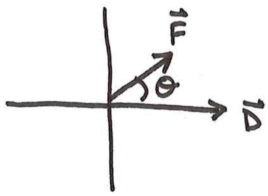
Find two vectors from the 3 points.	$\vec{AB} = \langle 2, 1, 2 \rangle$ $\vec{AC} = \langle -2, -1, 3 \rangle$
Find the cross product of the two vectors from above. This gives a vector, **n**, that is perpendicular to the plane.	$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 2 & 1 & 2 \\ -2 & -1 & 3 \end{vmatrix} = (3+2)\vec{i} - (6+4)\vec{j} + (-2+2)\vec{k}$ $= 5\vec{i} - 10\vec{j} = \langle 5, -10, 0 \rangle$ this vector is normal to the plane
If possible, find a parallel vector with smaller numbers (divide out a common factor)	$\langle 1, -2, 0 \rangle = \vec{n}$
Given a point on the plane P=(x, y, z), then the vector between P and any other point on the plane, say $\overrightarrow{AP}$, $\overrightarrow{AP} \cdot \vec{n}$ must equal 0. Find $\overrightarrow{AP}$	$\vec{AP} = \langle x-1, y-1, z+2 \rangle$
Set $\overrightarrow{AP} \cdot \vec{n} = 0$ and simplify.	$\langle x-1, y-1, z+2 \rangle \cdot \langle 1, -2, 0 \rangle = 0$ $x-1-2(y-1) = 0$ $x-1-2y+2 = 0$ $x-2y+1 = 0$

Vector Application Problems

$\vec{D} = [100, 0°]$

1. I did 2400 foot-pounds of work when I pulled a wagon full of bricks 100 feet. I can pull with a force of 50 pounds. At what angle was I pulling on the wagon?

$\vec{F} = [50, \theta]$

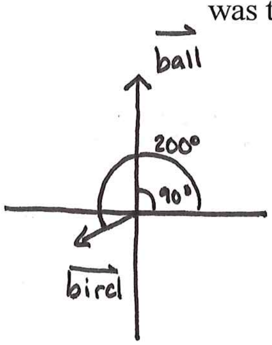


$W = \vec{F} \cdot \vec{D} = \|\vec{F}\|\|\vec{D}\| \cos\theta$

$2400 = 50 \cdot 100 \cdot \cos\theta$

$\cos\theta = \frac{2400}{5000}$ $\quad \theta = \cos^{-1}\left(\frac{24}{50}\right) \approx 61.315°$

2. During a baseball game on March 24th, 2001, an unlucky bird flew directly in the path of a Randy Johnson fastball. The fastball was traveling due north at a speed of 95 mph, while the bird was flying S70°W at 18 mph. If we assume that, upon collision, the bird and the ball fused into one object, what was the resulting speed and direction of the bird-ball?



$\overrightarrow{ball} = [95, 90°]$

$\overrightarrow{bird} \quad [18, 200°]$

$\overrightarrow{ball} + \overrightarrow{bird} = \langle 95\cos 90° + 18\cos 200°, 95\sin 90° + 18\sin 200° \rangle$

$\approx \langle -16.914, 88.844 \rangle$

$\approx [90.439, -79.221 + 180]$

90.439 mph at 100.779°

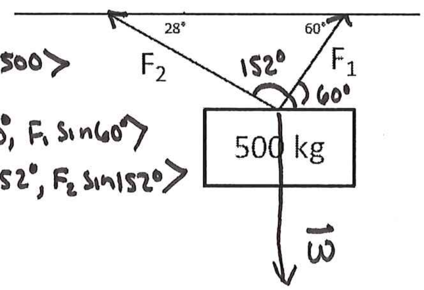
3. In the diagram shown, determine the tensions in cables F_1 and F_2 in order for the box to maintain static equilibrium.

$\vec{F}_1 + \vec{F}_2 + \vec{w} = 0$ or $\vec{F}_1 + \vec{F}_2 = -\vec{w}$

$\vec{w} = \langle 0, -500 \rangle$

$\vec{F}_1 = \langle F_1 \cos 60°, F_1 \sin 60° \rangle$

$\vec{F}_2 = \langle F_2 \cos 152°, F_2 \sin 152° \rangle$



$F_1 \cos 60 + F_2 \cos 152 = 0$

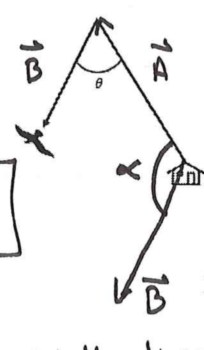
$F_1 \sin 60 + F_2 \sin 152 = 500$

$$\left[\begin{array}{cc|c} \cos 60 & \cos 152 & 0 \\ \sin 60 & \sin 152 & 500 \end{array}\right]$$

$F_1 = 441.743$ kg

$F_2 = 250.152$ kg

★ 4. A bird leaves his birdhouse in the morning and flies to a point 8 miles to the north and 5 miles to the west. The bird then turns and flies to a point 6 miles to the south and 3 miles to the west of its turning point. What was the angle of the bird's turn? See figure to the right. Make sure you provide the angle indicated in the picture!



$\vec{A} = \langle -5, 8 \rangle$

$\vec{B} = \langle -3, -6 \rangle$

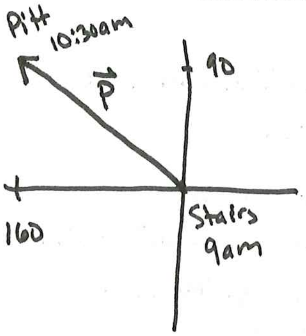
$\cos\alpha = \frac{\vec{A} \cdot \vec{B}}{\|\vec{A}\|\|\vec{B}\|} = \frac{-33}{\sqrt{89} \cdot \sqrt{45}}$

$\alpha \approx 121.430°$

$\theta = 180 - \alpha = \boxed{58.570°}$

★ we use the dot product to find the angle between vectors in standard position (so α in the diagram)
θ is supplementary to α (why?)

5. An airplane was spotted flying directly over the Henry David Thoreau Stairway to Humanities at 9:00 this morning. At 10:30 AM, the plane was spotted flying directly over Pittsburgh, which is 90 miles to the north and 160 miles to the west. Represent the motion of the plane as a vector in magnitude-direction form, where the magnitude represents the plane's speed in miles per hour.



$$\text{distance} = \sqrt{90^2+160^2} \text{ miles}$$

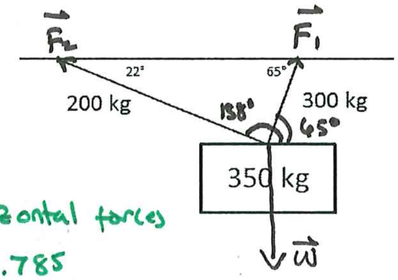
$$\text{speed} = \frac{\text{distance}}{\text{time}} = \frac{\sqrt{33{,}700} \text{ miles}}{1.5 \text{ hours}} \approx 122.384 \text{ mph}$$

$$\tan^{-1}\left(\frac{90}{-160}\right) \approx -29.358$$

$$-29.358^\circ + 180 \approx 150.642^\circ$$

$$\vec{P} = [122.384, 150.642^\circ]$$

6. In the figure at right, cable F1 has a tensile strength of 300 kg, and F2 has a tensile strength of 200 kg. Will the two cables support the object? If not, how much weight needs to be removed in order for the cables to support it?



$\vec{F_1} = \langle 300\cos 65^\circ, 300 \sin 65^\circ \rangle$

$\vec{F_2} = \langle 200 \cos 158^\circ, 200 \sin 158^\circ \rangle$

$\vec{w} = \langle 0, -350 \rangle$

$\vec{F_1} + \vec{F_2} + \vec{w} = \langle -58.651, -3.186 \rangle$

NOT in static equilibrium!

★ First need to balance horizontal forces

$F_{1x} = 300 \cos 65 \approx 126.785$

$F_{2x} = 200 \cos 158 \approx -185.437$

So F_2 has "excess" strength.

F_{2x} must $= -126.785$

so $-126.785 = m \cos 158$

$m = 136.743$ kg

m is max force F_2 can exert, so check y components

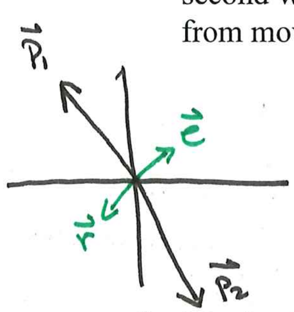
$\rightarrow$ $136.743 \sin 158 + 300 \sin 65$

$= 323.116$ kg

This is max weight

so 26.883 lbs must be removed.

7. Two people are pushing on a desk, one with a force magnitude of 80 N in a direction of 133°, and the second with a magnitude of 95 N in a direction of 290°. If a third person wants to prevent the desk from moving, with what magnitude and direction of force must he/she push?



$\vec{P_1} + \vec{P_2} = \vec{r}$

$\vec{e} = -\vec{r}$

$\vec{P_1} + \vec{P_2} = \langle 80 \cos 133 + 95 \cos 290, 80 \sin 133 + 95 \sin 290 \rangle$

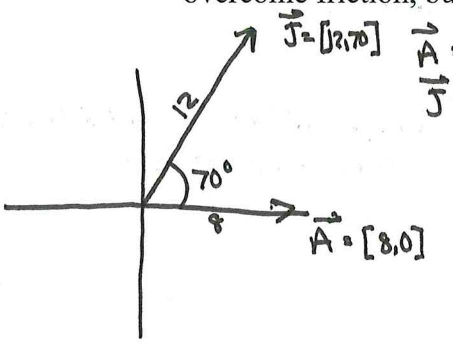
$\vec{r} \approx \langle -22.068, -30.763 \rangle$

$\vec{e} \approx \langle 22.068, 30.763 \rangle$

$= [37.859, 54.346^\circ]$

so 37.859 N at 54.346°

8. Mr. Auerbach's kittens Annie and Josie are at it again. This time, they have found the bag of cat food and are attempting to drag it out of the pantry by their teeth. Annie can pull with a force of 8 pounds and Josie can pull with a force of 12 pounds. It only takes 17 pounds of force in a single direction to overcome friction, but will the bag move if Annie and Josie are pulling at a 70° angle to each other?



$\vec{A} = \langle 8, 0 \rangle$

$\vec{J} = \langle 12 \cos 70^\circ, 12 \sin 70^\circ \rangle$

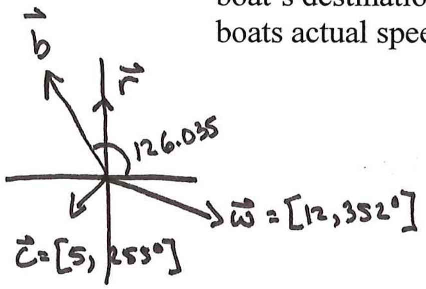
$\vec{A} + \vec{J} \approx \langle 12.104, 11.276 \rangle$

$= [16.543, 42.972^\circ]$

not enough force :(

9. A boat (all directions given in true bearings) is sailing across a lake. The boat's motor can move the boat at a speed of 18 knots (nautical miles per hour) in still water. Today, there is also a current of 5 knots bearing 195°. Additionally, there is a wind blowing 12 knots in the direction of 098° If the boat's destination is due north (000°), what course should the boat actually drive? What will be the boats actual speed?

→ so X component of $\vec{b}+\vec{c}+\vec{w}=0$ *



$\vec{b}=[18,\theta]=\langle 18\cos\theta,\ 18\sin\theta\rangle$

$\vec{c}=[5,255°]=\langle 5\cos255°,\ 5\sin255°\rangle$

$\vec{w}=[12,352°]=\langle 12\cos352°,\ 12\sin352°\rangle$

$18\cos\theta+5\cos255°+12\cos352°=0$

$18\cos\theta=-10.589$

$\cos\theta=-.588$

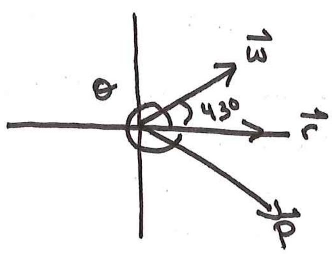
$\theta=126.035$ so true course 323.965°

$18\sin\theta+5\sin255+12\sin352=$ 8.056

So if boat drives true course 323.965° at 18 kts it will trave due North at 8.056 knots

10. An airplane with a top speed of 350 mph needs to maintain a course due east. However, it is a particularly windy day, and the wind speed is 65 mph blowing N47°E. In what direction must the plane navigate to maintain its desired course?

→ so y components = 0



$\vec{w}=[65,43°]$

$\vec{p}=[350,\theta]$

$\vec{w}+\vec{p}=\vec{r}=[x,0°]$

$65\sin43°+350\sin\theta=0$

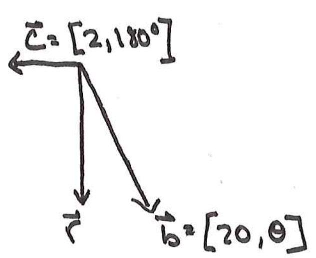
$\sin\theta=\dfrac{-65\sin43}{350}$

$\theta=\sin^{-1}\left(\dfrac{-65\sin43}{350}\right)=-7.276$

$\theta=352.724°$

11. A ship needed to sail across the Mediterranean Sea to get from Marseilles to Algiers, a distance of 500 miles. The ship had a top speed of 20 mph in still water. On the day of departure, the current was 2 mph, flowing directly perpendicular to the path between the two cities. On the return trip, the ship had to return across the Mediterranean, but found the current had increased to 6 mph. How much longer did it take the ship to cross the Mediterranean Sea returning than it did departing?

going:



$\vec{c}=\langle -2,0\rangle$

$\vec{b}:\langle 20\cos\theta,\ 20\sin\theta\rangle$

$\vec{r}=\langle 0,x\rangle$

$-2+20\cos\theta=0$

$\cos\theta=\frac{1}{10}$ $\theta=84.261°$

$360-84.261=275.739°$

$0+20\sin(275.739°)=-19.9$

$\vec{r}=\langle 0,-19.9\rangle$ 19.9mph due south

$\vec{c}$ = current

$\vec{b}$ = boat

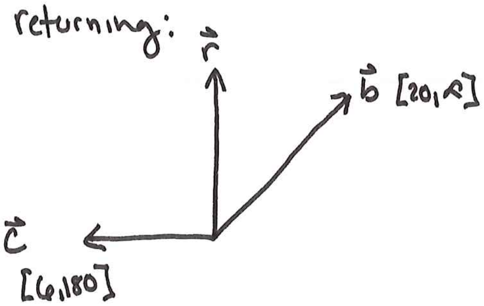
$\vec{r}$ = resultant vector

$\vec{c}+\vec{b}=\vec{r}$

$\vec{r}=\langle 0,x\rangle$

time = $\dfrac{\text{dist}}{\text{speed}}=\dfrac{500}{19.9}=25.126$ hours

returning:



$\vec{c}=\langle -6,0\rangle$

$\vec{b}=\langle 20\cos\alpha,\ 20\sin\alpha\rangle$

$\vec{r}=\langle 0,z\rangle$

$-6+20\cos\alpha=0$

$\cos\alpha=\frac{6}{20}$

$\alpha=72.542°$

$0+20\sin\alpha=19.079$

so $\vec{r}=\langle 0,19.079\rangle$

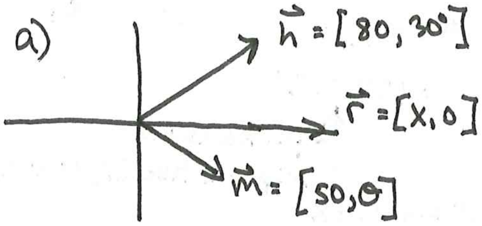
time = $\dfrac{500}{19.079}=26.207$ hours

Diff: $26.207-25.126=1.081$ hours longer on return trip

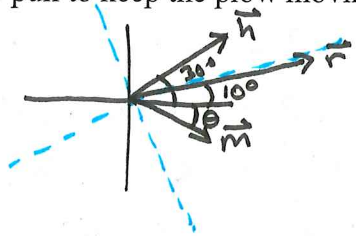
12. I am trying to get my horse to pull a plow across a field. But the horse only wants to pull in the direction of N60°E. Therefore, I need to pull on another line to keep the plow moving in the intended direction. The horse pulls the plow with a force of 80 newtons, while I can pull with a force of 50 newtons.

a) What direction do I need to pull to keep the plow moving due east?
b) What direction do I need to pull to keep the plow moving N80°E?

a)



b)



$\vec{r} = \vec{h} + \vec{m}$

$= \langle 80\cos 30 + 50\cos\theta,\ 80\sin 30 + 50\sin\theta \rangle$

$= \langle X\cos 10°,\ X\sin 10° \rangle$

Too many unknows!

"move" the axis, so that $\vec{r}$ is $[x, 0]$

$\vec{h} = \langle 80\cos 30, 80\sin 30 \rangle$

$\vec{m} = \langle 50\cos\theta, 50\sin\theta \rangle$

$\vec{r} = \langle x, 0 \rangle = \vec{h} + \vec{m}$

$80\sin 30 + 50\sin\theta = 0$

$\sin\theta = \dfrac{-80\sin 30}{50}$

$\theta = -53.130°$

$\vec{h} = [80, 20°] = \langle 80\cos 20°, 80\sin 20° \rangle$

$\vec{m} = [50, \beta] = \langle 50\cos\beta, 50\sin\beta \rangle$

$\vec{r} = \vec{h} + \vec{m} = [x, 0] = \langle x, 0 \rangle$

$80\sin 20° + 50\sin\beta = 0$

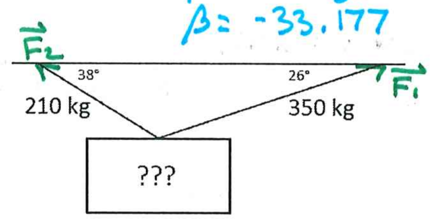
$\sin\beta = \dfrac{-80\sin 20°}{50}$

$\beta = -33.177$

now move axis back

$\theta = -23.177°$

13. In the diagram at right, the numbers listed next to each cable are the maximum amount of weight each cable could support before breaking. What is the maximum weight that could be put in the box without either cable snapping? Hint: the cables can NOT both support their maximum weights at the same time!



1st balance X components

$\vec{F_1} = \langle 350\cos 26°, 350\sin 26° \rangle$

$\vec{F_2} = \langle 210\cos 142°, 210\sin 142° \rangle$

$350\cos 26° = 314.578$

$210\cos 142° = -165.482$

So F_1 cannot exert its full force

$X\cos 26 = 165.482$

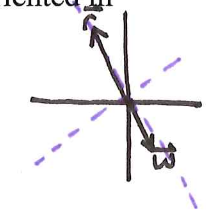
$X = 184.116$ kg

then find max weight possible

$184.116\sin 26° + 210\sin 142° = 210$

max weight $= 210$ lbs

14. An airplane must fly from Kansas City, MO, to Minneapolis, MN. Minneapolis is 650 miles away from Kansas City, in the direction of $N13^\circ W$. The airplane has a top speed of 320 mph, but there is a wind blowing 48 mph in the direction of $S8^\circ E$. In what direction must the plane be oriented in order to reach Minneapolis, and how long will the flight take?



rotate axis 13° ccw so planes desired course is on y axis

$\vec{P} = [320, \theta] \longrightarrow \vec{P} = [320, \alpha] = \langle 320\cos\alpha, 320\sin\alpha \rangle$

$\vec{\omega} = [48, 278^\circ] \longrightarrow \vec{\omega} = [48, 265^\circ] = \langle 48\cos 265, 48\sin 265 \rangle$

$\vec{r} = \vec{P} + \vec{\omega} = [X, 103^\circ] \rightarrow \vec{r} = [X, 90^\circ] = \langle 0, X \rangle$

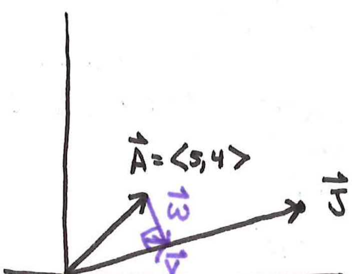
$\theta = \alpha + 13^\circ$
$= 102.251^\circ$
planes course
trip will take 2.388 hours

$\alpha = \theta - 13^\circ$
$\theta = \alpha + 13^\circ$

$320\cos\alpha + 48\cos 265 = 0$
$\cos\alpha = \frac{-48\cos 265}{320}$
$\alpha = 89.251^\circ$

$320\sin\alpha + 48\sin 265 = X$
$X = 272.155$
$\text{time} = \frac{\text{dist}}{X} = \frac{650}{272.155} = 2.388 \text{ hours}$

15. Annie and Josie are located at the origin of a coordinate system. Josie walks to the point (18,7), leaving a trail of catnip as she goes so that Annie can follow her. In her excitement, Annie runs off course and stops at the point (5,4). If she then walks directly to the nearest spot on the trail of catnip, how far does she have to go to get back on course?



$\vec{V} - \vec{\omega} = \vec{A}$

$\vec{V} = \text{Proj}_{\vec{J}} \vec{A} = \frac{\vec{A}\cdot\vec{J}}{\|\vec{J}\|^2}\vec{J} = \frac{118}{373}\langle 18,7 \rangle$
$= \left\langle \frac{2124}{373}, \frac{826}{373} \right\rangle$

$\vec{\omega} = \vec{V} - \vec{A}$

$\vec{\omega} = \left\langle \frac{2124}{373}, \frac{826}{373} \right\rangle - \langle 5,4 \rangle = \langle .694, -1.786 \rangle$

$\|\vec{\omega}\| = 1.916$ units

16. A model rocket was launched from the origin of a coordinate system, and flew 50 feet straight up. At that point, the rocket split into three pieces. The nose followed the vector $\langle 6,-2,3 \rangle$, the body followed vector $\langle -2,4,1 \rangle$, and the tail followed the vector $\langle 7,0,-5 \rangle$. What is the equation of the plane that contains all three parts of the rocket? All vectors started at (0, 0, 50)

$A = (6, -2, 53)$ $\quad \vec{AB} = \langle -8, 6, -2 \rangle$

$B = (-2, 4, 51)$ $\quad \vec{AC} = \langle 1, 2, -8 \rangle$

$C = (7, 0, 45)$ $\quad \vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -8 & 6 & -2 \\ 1 & 2 & -8 \end{vmatrix} = (-48+4)\vec{i} - (64+2)\vec{j} + (-16-6)\vec{k}$
$= -44\vec{i} - 66\vec{j} - 22\vec{k}$

$\vec{n} = \langle 2, 3, 1 \rangle$

$\vec{AP} = \langle x-6, y+2, z-53 \rangle$

$\vec{AP} \cdot \vec{n} = 0 \quad 2(x-6) + 3(y+2) + z - 53 = 0$
$2x - 12 + 3y + 6 + z - 53 = 0$

Here is a picture of Annie and Josie

$2x + 3y + z = 59$