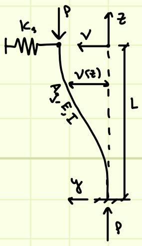
Adam Novak | CE-810 | HW #3 | 09/24/2025

2. 

Where $k_s = \frac{\gamma EI}{L^3}$ + $\gamma \in [0, \infty)$

a) Governing Equation:

- $EI v^{IV} + Pv'' + \alpha v = q$ where $\alpha = q = 0$
  $\therefore EI v^{IV} + P v'' = 0$
- Let $k^2 = P/EI$
  $\therefore v^{IV} + k^2 v'' = 0$

General Solution:

- $v(z) = A + Bz + C\cos(kz) + D\sin(kz)$
- $v'(z) = B - Ck\sin(kz) + Dk\cos(kz)$
  $v''(z) = -Ck^2\cos(kz) - Dk^2\sin(kz)$
  $v'''(z) = Ck^3\sin(kz) - Dk^3\cos(kz)$

Boundary Conditions:

- $v(0) = 0$
- $v'(0) = 0$
- $v''(L) = 0$
- $-EI v'''(L) - Pv'(L) = v(L)\cdot k_s$

Applying BC's:

- 1) $v(0) = A + B(0) + C\cos(0) + D\sin(0) = 0$
  $= A + 0\cdot B + 1\cdot C + 0\cdot D = 0 \Rightarrow A = -C$ (1)

  2) $v'(0) = B - Ck\sin(0) + Dk\cos(0) = 0$
  $= 0\cdot A + 1\cdot B - 0\cdot C + k\cdot D = 0 \Rightarrow B = -Dk$ (2)

  3) $v''(L) = -Ck^2\cos(kL) - Dk^2\sin(kL) = 0$
  $= 0\cdot A + 0\cdot B - k^2\cos(kL)\cdot C - k^2\sin(kL)\cdot D = 0 \Rightarrow C\cos(kL) + D\sin(kL) = 0$ (3)

  4) $-EI v'''(L) - Pv'(L) = v(L)\cdot \gamma EI/L^3$
  $v'''(L) + \frac{P}{EI} v'(L) = \frac{-\gamma}{L^3} v(L)$

  $v'''(L) + \frac{P}{EI} v'(L) + \frac{\gamma}{L^3} v(L) = 0$

  $Ck^3\sin(kz) - Dk^3\cos(kz) + k^2\left(B - Ck\sin(kz) + Dk\cos(kz)\right) + \frac{\gamma}{L^3}\left(A + BL + C\cos(kL) + D\sin(kL)\right) = 0$

  $Ck^3\cancel{\sin(kz)} - Dk^3\cancel{\cos(kz)} + k^2B - Ck^3\cancel{\sin(kz)} + Dk^3\cancel{\cos(kz)} + \frac{\gamma}{L^3}A + \frac{\gamma L}{L^3}\cdot B + \frac{\gamma}{L^3}\cos(kL)\cdot C + \frac{\gamma}{L^3}\cdot\sin(kL)\cdot D = 0$

  $k^2B + \frac{\gamma}{L^3}A + \frac{\gamma}{L^2}B + \frac{\gamma}{L^3}\cos(kz)\cdot C + \frac{\gamma}{L^3}\cdot\sin(kL)\cdot D = 0$ (4)

- To simplify the solution, plug (1) + (2) into (4) + solve a 2x2 system instead of the 4x4 system

$$\left.\begin{aligned} &\frac{\gamma}{L^3}(-C) + k^2(-kD) + \frac{\gamma}{L^2}(-kD) + \frac{\gamma}{L^3}\cos(kL)\,C + \frac{\gamma}{L^3}\sin(kL)\,D = 0 \\ &\left(\frac{\gamma}{L^3}\cos(kL) - \frac{\gamma}{L^3}\right)C + \left(\frac{\gamma}{L^3}\sin(kL) - k^2 - \frac{\gamma k}{L^2}\right)D = 0 \end{aligned}\right\} \text{combine w/ (3)}$$

$$\underbrace{\begin{bmatrix} \cos(kL) & \sin(kL) \\ \frac{g}{L^3}\cos(kL) - \frac{g}{L^3} & \frac{g}{L^3}\sin(kL) - k^2 - \frac{gk}{L^2} \end{bmatrix}}_{[A]} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$[A] \Rightarrow \det(A) = 0$$

$$\therefore \cos(kL)\left(\frac{g}{L^3}\sin(kL) - k^2 - \frac{gk}{L^2}\right) - \sin(kL)\left(\frac{g}{L^3}\cos(kL) - \frac{g}{L^3}\right) = 0$$

$$\cos(kL)\cdot\cancel{\frac{g}{L^3}\sin(kL)} - k^2\cos(kL) - \frac{gk}{L^2}\cos(kL) - \cancel{\sin(kL)\frac{g}{L^3}\cos(kL)} + \frac{g}{L^3}\sin(kL) = 0$$

$$-k^2\cos(kL) - \frac{gk}{L^2}\cos(kL) + \frac{g}{L^3}\sin(kL) = 0$$

$$k^2\cos(kL) + \frac{gk}{L^2}\cos(kL) - \frac{g}{L^3}\sin(kL) = 0$$

Thus the eigenfunction for this system is: $\boxed{k^2\cos(kL) + \frac{gk}{L^2}\cos(kL) - \frac{g}{L^3}\sin(kL) = 0}$

b) When $g \to \infty$:

- $k^2\cos(kL) + \frac{gk}{L^2}\cos(kL) - \frac{g}{L^3}\sin(kL) = 0$

$$\frac{k^2}{g}\cos(kL) + \frac{k}{L^2}\cos(kL) - \frac{1}{L^3}\sin(kL) = 0$$

$$\frac{k^2L^3}{g}\cos(kL) + kL\cos(kL) - \sin(kL) = 0$$

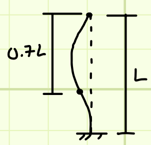
$$\lim_{g\to\infty} = 0\cdot\cos(kL) + kL\cos(kL) - \sin(kL) = 0$$

$$\Rightarrow kL\cos(kL) = \sin(kL)$$

$$\therefore kL = \tan(kL)$$

- Solving for first kL we get: $kL \approx 4.4934$
- Thus $kL = 4.4934 \Rightarrow \sqrt{\frac{P}{EI}} = \frac{4.4934}{L} \Rightarrow \frac{P}{EI} = \frac{20.19064}{L^2} \Rightarrow P_{cr} = \frac{20.19064\,EI}{L^2} \Rightarrow P_{cr} = 2.045\,\frac{\pi^2 EI}{L^2} = \frac{\pi^2 EI}{(0.7L)^2}$

- Buckled Shape:



c) Since $P_{cr} = \frac{\pi^2 EI}{(KL)^2}$, we can simply see that $K = 0.7$ for this case :)